WEAKLY COMPLETE COMPLEX SURFACES

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ABSTRACT. A weakly complete space is a complex space admitting a (smooth) plurisubharmonic exhaustion function. In this paper, we classify those weakly complete complex surfaces for which such exhaustion function can be chosen real analytic: they can be modifications of Stein spaces or proper over a non compact (possibly singular) complex curve or foliated with real analytic Levi flat hypersurfaces which in turn are foliated by dense complex leaves (these we call surfaces of Grauert type). In the last case, we also show that such Levi flat hypersurfaces are in fact level sets of a global proper pluriharmonic function, up to passing to a holomorphic double cover of the space.

Our method of proof is based on the careful analysis of the level sets of the given exhaustion function and their intersections with the *minimal singular set*, i.e the set where every plurisubharmonic exhaustion function has a degenerate Levi form.

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1. Introduction

A *weakly complete* complex space is a non compact, connected complex space X endowed with a smooth plurisubharmonic exhaustion function φ.

Weakly complete subdomains of Stein spaces are Stein; complex spaces which are proper over (i.e. endowed with a proper surjective holomorphic map onto) a Stein space, e.g. modifications of Stein spaces, are weakly complete.

More interesting examples of weakly complete complex spaces which are not Stein are the pseudoconvex subdomains of a complex torus constructed by Grauert (cfr. [16]).

Let us briefly compare these two classes of examples. In the former, $\mathcal{O}(X) \neq \mathbb{C}$ and, if there is a positive dimensional complex space contained in a level set of φ, then it is a compact subspace of X. Such compact subspaces are responsible for the degeneration of the Levi form of φ (or of any other plurisubharmonic function). In the latter, $\mathcal{O}(X) = \mathbb{C}$ and there exists a smooth plurisubharmonic exhaustion function φ whose regular level sets are Levi flat hypersurfaces, foliated by dense complex leaves (along which the Levi form of φ degenerates). In such a situation X is said to be a space of *Grauert type*.

A question naturally arises: are these two phenomena the only possible for a weakly complete space?

Even if this problem has never been explicitly addressed, we find some partial results throughout the literature about the various forms of the Levi problem. For instance, in [18], Ohsawa shows (Proposition 1.4) that a weakly complete surface (i.e. 2-dimensional complex manifold) with a non-constant holomorphic function is holomorphically convex, hence proper on a Stein space, by Remmert's theorem. Later on, Diederich and Ohsawa tackle a weak form of the Levi problem for a domain with real analytic boundary (cfr. [7]). More recently, in [8] Gilligan, Miebach and Oeljeklaus study the case of a pseudoconvex domain of any dimension, spread over a complex homogeneous manifold.

We would like to point out that all these partial results are obtained assuming the existence of some real analytic data: a non-constant holomorphic function, a real analytic boundary, richness of the group of automorphisms; moreover, the only result which holds in arbitrary dimension heavily employs the hypothesis of homogeneity with respect to automorphisms. It is also worth noticing that other results related to this problem, like the classification of holomorphic foliations or Ueda's results (cfr [26]), are fully understood and developed only in dimension 2.

In view of these considerations, while the general problem of classifying weakly complete spaces is surely most interesting, we give, in this paper, a first contribution by solving it for weakly complete *surfaces* which admit a *real analytic* plurisubharmonic exhaustion function.

In this vein Brunella (cfr. [3]) constructed an example of a weakly complete surface which does not admit a real analytic plurisubharmonic exhaustion function. In order to do so, he studied the possible geometries of some special weakly complete surfaces (see also Remark 6.2). In Brunella's example, one can construct a plurisubharmonic exhaustion function which is real analytic outside a compact set (where our results still describe the geometry of the surface, which is there of Grauert-type) and nothing is known about what happens inside that compact, even for that particular example.

In this paper, we prove the following

Main Theorem. *Let X be a weakly complete complex surface, with a real analytic plurisubharmonic exhaustion function α. Then one of the following three cases occurs:*

i) *X is a modification of a Stein space of dimension 2,*
ii) *X is proper over a (possibly singular) open complex curve,*
iii) *X is a Grauert type surface.*

Moreover, in case iii)*, either the critical set* $\mathrm{Crt}(\alpha)$ *of α has dimension ≤ 2 and then*

iii-a) *the absolute minimum set Z of α is a compact complex curve $Z \subset X$ and there exists a proper pluriharmonic function $\chi : X \setminus Z \to \mathbb{R}$ such that every plurisubharmonic function on $X \setminus Z$ is of the form $\gamma \circ \chi$,*

or it is of dimension 3 and then

iii-b) *there exist a double holomorphic covering map $\pi : X^* \to X$ and a proper pluriharmonic function $\chi^* : X^* \to \mathbb{R}$ such that every plurisubharmonic function on X^* is of the form $\gamma \circ \chi^*$.*

In both cases, γ is a convex, increasing real function.

The first part of the Main Theorem is the content of Theorem 4.4, the second one of the Theorems 5.1 and 6.1.

The method we adopted to tackle the problem consists of a careful analysis of the structure of the level sets of α and their behaviour with respect to the *minimal singular set*. This set was introduced in [23] for any weakly complete complex space; let us recall its definition. Given any plurisubharmonic exhaustion function $\varphi \in \mathcal{C}^\infty(X)$, let Σ^1_φ be the minimal closed set such that φ is strictly plurisubharmonic on $X \setminus \Sigma^1_\varphi$, and set $\Sigma^1 = \Sigma^1(X) = \bigcap\limits_{\varphi} \Sigma^1_\varphi$, i.e. $x \in \Sigma^1$ if no plurisubharmonic $\mathcal{C}^\infty$ exhaustion function is strictly plurisubharmonic near x.

A plurisubharmonic exhaustion function φ is called *minimal* if $\Sigma^1 = \Sigma^1_\varphi$.

The following crucial properties were proved in [23] when X is a complex manifold:

a) there exist minimal functions φ (cfr. [23, Lemma 3.1]);
b) if φ is minimal the nonempty level sets $\Sigma^1_c = \{\varphi = c\} \cap \Sigma^1$ have the local maximum property (cfr. [23, Theorem 3.6]);
c) if $\dim_{\mathbb{C}} X = 2$ and c is a regular value of φ, then the (nonempty) level sets Σ^1_c are compact sets foliated by Riemann surfaces (cfr. [23, Lemma 4.1]).

If X is a weakly complete complex surface, it is possible to show that b) and c) hold for any plurisubharmonic exhaustion function and not just for the minimal ones (see Theorem (3.2)).

Thus we are able to link Σ^1 to the Levi flatness of the levels of α, obtaining the first half of the Main Theorem, by studying the complex foliation induced by the degeneracy of the Levi form.

Then, we proceed to construct, for Grauert type surfaces, a proper pluriharmonic function.

These results were announced by the authors in [14].

The paper consists of six sections and an appendix.

In Section 2, we present some relevant examples of the more "exotic" of the three cases, the Grauert type surfaces. Section 3 is devoted to the study of the regular level sets of α which intersect Σ^1. We show that they are Levi flat, provided that they do not contain any compact complex curve, and that the Levi foliation has dense leaves.

The effects of the presence of compact curves is analyzed in Section 4, where, using in a crucial way a theorem of Nishino [17, III.5.B], we prove the Theorem 4.4, i.e. the first part of the Main Theorem, giving the classification of weakly complete surfaces into cases i), ii) and iii).

The proof of parts iii-a), iii-b) of the Main Theorem is given in Sections 5,6 where we are dealing with Grauert type surfaces. Our goal is to construct a proper pluriharmonic function in the Main Theorem.

This is done analyzing the cases $\dim_{\mathbb{R}}\mathrm{Crt}(\alpha) \leq 2$ and $\dim_{\mathbb{R}}\mathrm{Crt}(\alpha) = 3$ separately (see Theorem (5.1) and Theorem 6.1).

The Appendix collects results of various nature which we believe either known or easy to proof, but for which we were not able to find appropriate references.

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2. Examples

Example 2.1. Let $a_1, a_2 \in \mathbb{C}$ with the following properties

$$0 < |a_1| \leq |a_2| < 1, \;\; a_1^k \neq a_2^l$$

for all $(k, l) \in \mathbb{N}^2 \smallsetminus \{(0,0)\}$ and define τ by $|a_1| = |a_2|^\tau$; by hypothesis $\tau \notin \mathbb{Q}$.

Consider on $\mathbb{C}^2 \smallsetminus \{(0,0)\}$ the equivalence relation $\sim$: $(z_1, z_2) \sim (a_1 z_1, a_2 z_2)$. The quotient space $\mathbb{C}^2 \smallsetminus \{(0,0)\}/\sim$ is the Hopf manifold $\mathcal{H}$. Let π denote the projection $\mathbb{C}^2 \smallsetminus \{(0,0)\} \to \mathcal{H}$. The complex lines $\mathbb{C}_{z_1} = \{z_2 = 0\}$, $\mathbb{C}_{z_2} = \{z_1 = 0\}$ project into complex compact curves C_1, C_2 respectively. Consider $X = \mathcal{H} \smallsetminus C_2$. The function

$$\Phi(z_1, z_2) = \frac{|z_2|^{2\tau}}{|z_1|^2}$$

on $\mathbb{C}^2 \smallsetminus \{(0,0)\}$ is $\sim$-invariant and so defines a function $\phi : X \to \mathbb{R}_{\{\geq 0\}}$; ϕ is proper and $\log \phi$ is pluriharmonic on $X \smallsetminus C_1$.The level sets of ϕ contained in $X \smallsetminus C_1$ are the projections of the sets $|z_1| = c|z_2|^\tau$, $c > 0$, and so foliated by the projections of the sets $z_1 = ce^{i\theta} z_2^\tau$ which are everywhere dense leaves, τ being irrational. Observe that C_1 is the minimum set of ϕ. In particular, X is a weakly complete surface of Grauert type and falls into case iii-a.

Example 2.2. With the notation of the previous example, we consider $X_1 = \mathcal{H} \setminus (C_1 \cup C_2)$ with plurisubharmonic exhaustion function $\alpha = (\log \phi)^2$. X_1 is a weakly complete surface, obviously of Grauert type. Here, however, the plurisubharmonic function α_1 has a 3-dimensional minimum set, namely the quotient of the Levi flat surface of $(\mathbb{C}^*)^2$ given by

$$H_0 = \{(z_1, z_2) \in (\mathbb{C}^*)^2 \;:\; |z_2|^\tau = |z_1|\} \,.$$

The pluriharmonic function on X_1 is, obviously, $\log(\phi)$, i.e. a befitting choice of the square root of α_1. Therefore, X_1 falls into case iii-b.

Example 2.3. Another class of example is provided by total spaces of some complex line bundles over compact Riemann surfaces (see also [26]).

Let M be a compact Riemann surface of genus $g > 0$. It is well known that every topologically trivial line bundle can be represented by a flat unimodular cocycle, i.e. an element of $H^1(M, S^1)$.

Consider a line bundle $L \to M$ with trivialization given by the open covering $\{U_j\}_{j=1}^n$ and transition functions $\{\xi_{ij}\}_{i,j}$ which represent a cocycle $\xi \in H^1(M, S^1)$. We can define a function $\alpha : L \to \mathbb{R}$ by defining it on each trivialization as $\alpha_j : U_j \times \mathbb{C}$, $\alpha_j(x, w) = |w|^2$. As $|\xi_{ij}| = 1$, these functions glue into $\alpha : L \to \mathbb{R}$, which is readily seen to be plurisubharmonic and exhaustive.

Now, consider $r > 0$ and the section $f_1 \in \Gamma(U_1, \xi)$ given by $f(x) \equiv r$ for all $x \in U_1$; taking all possible analytic continuations of f_1 as a section of the bundle L, we construct , for every chain $\{U_{j_k}\}_{k\in\mathbb{N}}$ with $j_0 = 1$ and $U_{j_k} \cap U_{j_{k+1}} \neq \varnothing$, the sections $f_k \equiv \xi_{j_k j_{k-1}} \xi_{j_{k-1} j_{k-2}} \cdots \xi_{j_1 j_0} r \in \Gamma(U_{j_k}, \xi)$. Representing ξ as a multiplicative homomorphism $\psi_\xi : \pi_1(M) \to S^1$, it is easy to see that the graphs of such sections glue into a compact complex manifold if and only if $\psi(\pi_1(M))$ is contained in the roots of unity, i.e. if and only if $L^{\otimes n}$ is (analytically) trivial for some n, i.e. if and only if ξ (as an element of the group $H^1(M, S^1)$) is unipotent.

If that is not the case, the graphs of such sections glue into an imbedded, non closed, complex manifold, contained in the Levi flat hypersurface $\alpha^{-1}(r^2)$ and dense in it. The other leaves of the Levi foliation are obtained by the one constructed multiplying it by $e^{i\theta}$.

Finally, we have a pluriharmonic function $\chi : L \setminus M \to \mathbb{R}$ given by $\chi(p) = \log \alpha(p)$.

Therefore, if ξ is not unipotent, its total space gives an example of Grauert type surface for the case iii-a. As with the Hopf surface example, it is not hard to show that $L \setminus M$ is a Grauert type surface and falls in case iii-b.

3. Levels and defining functions

3.1. **Some preliminary remarks.**

From now on we suppose that X is a fixed complex surface that admits a real analytic plurisubharmonic exhaustion function α.

Proposition 3.1. *We can assume α to have the following properties*

$$\begin{cases} \min\limits_X \alpha = 0 \\ \partial\alpha(p) = 0 \text{ if } \partial\bar{\partial}\alpha(p) = 0. \end{cases} \tag{1}$$

Proof. It is enough to replace α with $(\alpha - \min_X \alpha)^2$. $\square$

Let us fix some notations. Given a smooth function $f : W \to \mathbb{R}$ on a complex surface W let us denote $\mathrm{Crt}(f)$ the set of its critical points. If f is real anlytic $\mathrm{Crt}(f)$ is a real analytic set.

As anticipated in Section 1, property b) of minimal functions extends to arbitrary smooth plurisubharmonic exhaustion functions:

Theorem 3.2. *Let Σ^1 be the minimal kernel of X and $u : X \to \mathbb{R}$ an arbitrary smooth plurisubharmonic exhaustion function. Then every non-empty*

$$\Sigma^1_u(c) = \{u = c\} \cap \Sigma^1$$

has the local maximum property.

We first prove the following lemmas

Lemma 3.3. *Let Z be a complex manifold, Y_1 a closed subset with the local maximum property and Y_0 a closed subset of Y_1. Assume that $\phi : W \to [-\infty, +\infty)$ is a plurisubharmonic function in (a neighbourhood of) Y_1 such that $\phi = c$ on Y_0 and $\phi < c$ on $Y_1 \setminus Y_0$. Then Y_0 has the local maximum property.*

Proof. Should the local maximum property for Y_0 fail then, by [22, Proposition 2.3] there are $y \in Y_0$ a coordinate ball $B = B(y; r)$ and a strongly plurisubharmonic function u on $\overline{B}$ such that: $u(y) = 0$ and $u(z) < -\epsilon|z - y|^2$, if $z \in B \cap Y_0 \setminus \{y\}$. In particular, there exist a neighbourhood V of $\mathrm{b}B$ and a positive number ϵ such that $u(z) < -\epsilon|z - y_0|^2$ for $z \in B \cap Y_1 \cap \mathrm{b}B$. Then for $m \gg 0$ the plurisubharmonic function $v := u + m(\phi - c)$ satisfies $v(y) = 0$, $v(z) < -\epsilon|z - y|^2$ for $\mathrm{b}B \cap Y_1$. Therefore, since Y_1 has the local maximum property, this inequality must hold on $B \cap Y_1$, a contradiction for $z = y$. $\square$

Lemma 3.4. *If $\psi : X \to \mathbb{R}$ is a smooth plurisubharmonic exhaustion function, $\phi : X \to \mathbb{R}$ a minimal function and c, d real numbers such that the set*

$$Y_0 = \left\{x \in X : x \in \Sigma^1, \psi(x) = c, \phi(x) = d\right\}$$

is non-empty. Then Y_0 has the local maximum property.

Proof. Assume that $v : \mathbb{R}^2 \to \mathbb{R}$ is a smooth, strictly convex function such that for some numbers K, L the functions v_x, v_y are positive on the set

$$\left\{(x, y) \in \mathbb{R}^2 : x \geq K, y \geq L\right\}$$

and that $\min\psi > K$, $\min\phi > L$. Let $\chi(z) = v\big(\psi(z), \phi(z)\big)$, $z \in X$. Then χ is a minimal function. Indeed

$$\mathrm{dd^c}\chi = \begin{bmatrix} v_{xx} & v_{xy} \\ v_{yx} & v_{yy} \end{bmatrix} \begin{bmatrix} \mathrm{d}\phi \\ \mathrm{d}\psi \end{bmatrix} \wedge \begin{bmatrix} \mathrm{d^c}\phi \\ \mathrm{d^c}\psi \end{bmatrix} + v_x \mathrm{dd^c}\phi + v_y \mathrm{dd^c}\psi$$

Then $\mathrm{dd^c}\chi$ is strictly positive definite at all points where $\mathrm{dd^c}\phi$ is and so χ is minimal.

If K, L are fixed numbers such that $\min\psi > K$, $\min\phi > L$ we can take $v(x,y) = (x-L)^2 + (y-K)^2$. Fix such a v and let $\chi = v(\psi, \phi)$. Then χ is minimal. Let now $s = v(c,d)$. Then $\chi_{|Y_0} = s$. Consider

$$Y_1 = \big\{z \in X : z \in \Sigma^1, \chi(z) = s\big\}.$$

By [23, Lemma 3.1] Y_1 has the local maximum property. Take now a linear function $l : \mathbb{R}^2 \to \mathbb{R}$ which defines the tangent plane to v at the point $(c,d,s) \in \mathbb{R}^3$. It has the properties

$l(c,d) = s = v(c,d)$
$l(x,y) < v(x,y)$ for $(x,y) \neq (c,d)$.

Let $\eta(z) = l\big(\psi(z), \phi(z)\big)$, $z \in X$. Then η is a minimal function as well and $\eta_{|Y_0} = s$. However, if $z \in Y_1 \setminus Y_0$, $\big(\psi(z), \phi(z)\big) \neq (c,d)$ and then

$$\eta(z) = l\big(\psi(z), \phi(z)\big) < v\big(\psi(z), \phi(z)\big) = \chi(z).$$

So, η is a plurisubharmonic function on Y_1, $\max\limits_{Y_1} \eta = s$ while

$$\big\{z \in Y_1 : \eta(z) = s\big\} = Y_0.$$

Now Lemma 3.3 implies that Y_0 is a local maximum set. □

Proof of Theorem 3.2. If ϕ is a minimal function then $\psi := u + \phi$ is also minimal. Let c, d be real constants. By Lemma 3.4 the following set $\{\psi = c+d\} \cap \{\phi = d\}$ has the local maximum property if non-empty; since

$$\begin{gathered}\{u = c\} \cap \{\phi = d\} \cap \Sigma^1 = \\ \{\psi = c+d\} \cap \{\phi = d\} \cap \Sigma^1\end{gathered}$$

it follows that $\{u = c\} \cap \{\phi = d\} \cap \Sigma^1$ has the local maximum property if non-empty.

But for fixed c

$$\{u = c\} \cap \Sigma^1 = \bigcup_{d \in \mathbb{R}} \{u = c\} \cap \{\phi = d\} \cap \Sigma^1$$

and so it has the local maximum property as union of sets with the local maximum property (since it is closed, see Proposition 3.5(b) in [22]). □

In the sequel, the function α is assumed to satisfy condition (1).

3.2. **Levi flat levels.** Let W be a complex surface. In the sequel by *complex curve* of W we mean a purely 1-dimensional immersed complex space $C \hookrightarrow W$. We observe that a complex curve lying on a smooth hypersurface of W must be regular (cfr. [23, Proposition 4.2]). For any (complex or real) analytic space Y, we denote by Y_{reg} the regular part of Y.

If $\Sigma^1 \neq \varnothing$, according to [23, Lemma 4.1], some levels of the minimal function ϕ contain complex curves. We can expect that the same is true for the function α. The results of this subsection go in this direction.

Proposition 3.5. *Let X be a complex surface with a real analytic plurisubharmonic exhaustion function $\alpha : X \to \mathbb{R}$ and Y a connected component of a level set $\{\alpha = c\}$ such that $Y \cap \Sigma^1 \neq \varnothing$. Then, for every point $p \in Y_{reg} \cap \Sigma^1$ there exist an open neighbourhood $U \subset X$ and coordinates z, w on U such that $U \cong \Delta_z \times \Delta_w$ and*

$$U \cap Y \cap \Sigma^1 \cong \bigcup_{t \in T} \{(z, f_t(z)) \mid z \in \Delta_z\}$$

where each $f_t : \Delta_z \to \Delta_w$ is a holomorphic function.

Proof. The proof repeats verbatim from [23, Lemma 4.1] once we observe that, by Theorem 3.2, the set $\Sigma^1 \cap Y$ has the local maximum property. □

Theorem 3.6. *Let X be a complex surface with a real analytic plurisubharmonic exhaustion function $\alpha : X \to \mathbb{R}$ and Y a connected component of a level set $\{\alpha = c\}$ such that $Y \cap \Sigma^1 \neq \varnothing$. Assume that Y contains, at worst, isolated critical points of α and no compact complex curve. Then*

a) *Y_{reg} is a real analytic Levi flat hypersurface.*

In general, if Y does not contain local minimum points of α and $Y \subset \Sigma^1$ then

$$\partial\alpha \wedge \bar{\partial}\alpha \wedge \partial\bar{\partial}\alpha = 0,\ \partial\bar{\partial}\alpha \wedge \partial\bar{\partial}\alpha = 0$$

on X and so

b) *$\Sigma^1 = X$, all non-critical level sets of α are compact Levi flat hypersurfaces and the regular parts of the critical level sets have complex foliation.*

Proof. Since, by Theorem 3.2, $Y \cap \Sigma^1 \neq \varnothing$ has the local maximum property it cannot have isolated points hence $Y_{\mathrm{reg}} \cap \Sigma^1 \neq \varnothing$. In view of Proposition 3.5, every point p of $Y_{\mathrm{reg}} \cap \Sigma^1$ has a neighbourhood U in which $Y \cap \Sigma^1$ can be expressed as a union of analytic discs:

$$U \cap Y \cap \Sigma^1 \cong \bigcup_{t \in T(p)} \{(z, f_t(z)) \mid z \in \Delta_z\}$$

where each $f_t : \Delta_z \to \Delta_w$ is a holomorphic function. Let us first suppose that there is a point p such that $T(p)$ is infinite for a fundamental system of neighborhoods of p, take U one of such neighborhoods and let

$$U \cap Y \cap \Sigma^1 \supset \bigcup_{n \in \mathbb{N}} \{(z, f_n(z)) \mid z \in \Delta_z\}$$

with $f_n : \Delta_z \to \Delta_w$ a countable family of holomorphic functions with disjoint graphs. On every such graph we have that $\partial\bar{\partial}\alpha$ degenerates and, those graphs being integral curves for the kernel of such Levi form, we actually have that

$$\partial\bar{\partial}\alpha \wedge \partial\alpha \wedge \bar{\partial}\alpha = 0$$

on each of those graphs. By analyticity, $\partial\bar{\partial}\alpha \wedge \partial\alpha \wedge \bar{\partial}\alpha = 0$ on the whole of Y_{reg}, thus giving a foliation of it in complex curves, hence Levi flatness.

Suppose, at contrary, that for every point $p \in Y_{\mathrm{reg}} \cap \Sigma^1$ there is only a finite number of those analytic disks composing $Y_{\mathrm{reg}} \cap \Sigma^1$ which pass through p; then it is not hard to see that, starting from one of such disks, by analytic continuation, we get a complex curve C embedded in Y_{reg}. The closure $\overline{C}$ of C in Y is a compact complex curve: this is clear if $\overline{C} \subset Y_{\mathrm{reg}}$ otherwise we apply Remmert-Stein theorem (cfr. [15, Chapter VII, Theorem 1]) the critical points in Y being isolated. Since by hypothesis Y has no compact complex curve the only possible case is the previous one. This shows part a).

Assume now that $Y \subset \Sigma^1$ does not contain local minimum points of α. Then, by Theorem 3.2 Y has the local maximum property and so [23, Theorem 3.9] applies: there is an $s < c$ such that, if K is the connected component of the set $\{s \leq \alpha \leq c\}$ containing Y, then its topological boundary bK is contained in $\{\alpha = s\} \cup K$ and furthermore $\overset{\circ}{K}$ is nonempty, the forms

- $(\partial\bar{\partial}\alpha) \wedge \partial\alpha \wedge \bar{\partial}\alpha$
- $(\partial\bar{\partial}\alpha) \wedge \partial\alpha$
- $(\partial\bar{\partial}\alpha) \wedge \bar{\partial}\alpha$
- $\partial\bar{\partial}\alpha \wedge \partial\bar{\partial}\alpha$

vanish on $K \smallsetminus Y$. Now $K \smallsetminus Y$ has nonempty interior in X, and so $\partial\bar{\partial}\alpha \wedge \partial\alpha \wedge \bar{\partial}\alpha = 0$ in X, by real analyticity. This shows that $\Sigma^1 = X$.

Levi flatness of the regular parts of levels follows from Proposition 3.5. □

Lemma 3.7. *Let Y be a (non necessarily compact) real analytic Levi flat hypersurface in a complex surface W. Assume that there is a real analytic plurisubharmonic function $\beta : V \to \mathbb{R}$ on an open neighbourhood V of Y, such that $\{p \in V : \beta(p) = 0\} = Y$ and $\partial\beta(p) \neq 0$ for all $p \in Y$. Then there is an open neighbourhood U of Y, $U \subset V$, and a pluriharmonic function $\chi : U \to \mathbb{R}$ such that $Y = \{p \in U : \chi(p) = 0\}$ and $\partial\chi(p) \neq 0$ for all $p \in Y$.*

Proof. It is well known that a real analytic Levi flat hypersurface M in a complex manifold admits a real analytic defining function which is pluriharmonic in a neighborhood of M if and only its Levi foliation is defined by a nonvanishing real analytic closed 1-form (e.g cfr. [2]).

In our situation since Y is real analytic the foliation of Y extends holomorphically on a neighbourhood of Y (cfr. [20]): there is an atlas for a neighbourhood of Y in X, consisting of distinguished coordinate charts (U_j, z_j, w_j), $p \mapsto (z_j(p), w_j(p))$, $p \in U_j$, $(z_j(p), w_j(p)) \in V_j \times B_j \subset \mathbb{C}^2$, V_j, B_j connected, such that

$$Y \cap U_j = \left\{p \in U_j \colon \operatorname{Im} w_j(p) = 0\right\} .$$

Denote further $u_j = \operatorname{Re} w_j$, $v_j = \operatorname{Im} w_j$. Then $u_j, v_j : U_j \to \mathbb{R}$ are pluriharmonic functions and $U_j \cap Y = \{v_j = 0\}$. By this setup and the assumptions of the Lemma ($\partial\beta \neq 0$ on Y) there are real analytic functions $\rho_j \colon U_j \to \mathbb{R}$ such that: $\beta = v_j \rho_j$ in U_j, $\rho_{j|U_j \cap Y} > 0$.

We draw now some consequences for ρ_j from the plurisubharmonicity of β.

By calculations on β, ρ_j in the local coordinates $z_j, w_j = u_j + iv_j$, we obtain that at points $(z, u, 0) \in U_j \cap Y$ the Levi form reduces to (we drop the index j to simplify the notation)

$$\begin{aligned} \partial\bar\partial\beta(z,u,0) &= +\frac{i}{2}\rho_z(z,u,0)dz \wedge d\bar w - \\ &- \frac{i}{2}\rho_{\bar z}(z,u,0)dw \wedge d\bar z + \\ &+ \frac{i}{2}(\rho_w - \rho_{\bar w})(z,u,0)dw \wedge d\bar w. \end{aligned}$$

Since it is positive semidefinite in U_j and the term $dz \wedge d\bar z$ is missing, we must have

$$\rho_z(z,u,0) = 0 = \rho_{\bar z}(z,u,0),$$

$(z, u, 0) \in U_j \cap Y$ and since V_j, $j \in I$ are assumed connected $\rho_j(z, u, 0)$ is constant in z. We denote it by $\rho_j^*(u) = \rho_j(z, u, 0)$; $\rho_j : B_j \cap \mathbb{R} \to \mathbb{R}$ is a positive real analytic function.

Since we have

$$\partial\beta = v\rho_z dz + \left(-\frac{i}{2}\rho + v\rho_w\right) dw,$$

we obtain eventually

$$\eta := i\partial\beta_{|U_j \cap Y} = \frac{1}{2}\rho_j^*(u_j(p))du_j(p),$$

for $p \in U_j \cap Y$. Therefore η is a nonvanishing real analytic closed 1-form which defines the Levi foliation of Y. This ends the proof of the lemma. $\square$

We conclude this section by showing that, in absence of compact complex curves, the regular level sets of α intersecting Σ^1 are foliated by dense complex curves.

Corollary 3.8. *Let X be a complex surface with a plurisubharmonic real analytic exhaustion function $\alpha : X \to \mathbb{R}$ and Y a connected component of the regular level set $\{\alpha = c\}$ such that $Y \cap \Sigma^1 \neq \varnothing$. Assume that Y does not contain compact complex curves. Then Y is Levi flat, the leaves of the Levi foliation are dense in Y and $Y \subseteqq \Sigma^1$.*

Proof. Y is Levi flat in view of Theorem 3.6 so, by Lemma 3.7, there are a neighbourhood U of Y and a pluriharmonic function $\chi : U \to \mathbb{R}$ such that $\{\chi = 0\} = Y$ and $d\chi \neq 0$ on Y. The maximal complex subspace of T_pY, for $p \in Y$, is given by $\ker d\chi \cap \ker d^c\chi$; therefore, on the manifold Y, the foliation induced by Levi flatness is given by the 1-form $d^c\chi$. Such a form is closed, because $dd^c\chi = 0$ by pluriharmonicity, so every leaf of the foliation has trivial holonomy; hence, by a result of Sacksteder (cfr [4, Notes to Chapter V, (2), p. 109]), the foliation has no exceptional minimal set then, as there are no compact (i.e. closed) leaves, all leaves are dense in Y (we refer to [4], in particular to Chapter III, for the relevant definitions and results related to exceptional minimal sets of foliations).

This easily implies the last part of the statement: let ψ be any plurisubharmonic function on X and let $p_0 \in Y$ be such that $\psi(p_0) = \max_Y \psi$; then ψ is constant on the leaf which passes through p_0, but then, by density, it is constant on Y. Therefore $Y \subseteqq \Sigma^1$. $\square$

4. Propagation of compact complex curves

We observe the following fact. If $\chi : W \to \mathbb{R}$ is a nonconstant pluriharmonic function on a complex surface W such that the level set $S = \{p \in W : \chi(p) = 0\}$ is nonempty, compact and connected, then $Y := S \cap \mathrm{Crt}(\chi)$ is a complex, compact analytic subset. Indeed, we know that Y is a compact real analytic subset. Suppose that it is positive dimensional at some point $x \in Y$ and let B an open ball centered at x and $\tau : B \to \mathbb{R}$ a pluriharmonic conjugate of $\chi_{|B}$. Since $\chi_{|B\cap Y} = 0$ and the points of Y are critical, the function τ is constant on each connected component Y_0 of $B \cap Y$ and we may assume $\tau_{|Y_0} = 0$. If $F : B \to \mathbb{C}$ denotes the holomorphic function $\chi + i\tau$ we then have $Y_0 = \{p \in B : \partial F(p) = 0\}$.

Lemma 4.1. *Let $\chi\colon W \to \mathbb{R}$ be a pluriharmonic function on a complex surface W with no critical points in W. Assume that W contains a connected compact complex curve C. Then there exist a neighborhood V of C and proper holomorphic function $G : V \to \mathbb{C}$ such that $C = \{G = 0\}$.*

Proof. The restriction of χ to C has to be a constant value, say d.

Choose a good covering $\{V_j\}_{j=1}^n$ of C as in Lemma 7.2. Consider $\chi_{|V_j}$, j fixed. Assuming $r > 0$ small enough so that V_j is contained in a topological ball in W, we conclude that χ has a pluriharmonic conjugate in this ball, and so in V_j, say $\tau'_j\colon V_j \to \mathbb{R}$. Since $\chi_{|C} = c$, a constant, we conclude that its harmonic conjugate on $C \cap V_j$ is locally constant. Since $C \cap V_j$ is connected $\tau'_{j|C\cap V_j}$ is constant. Subtracting the latter constant from τ'_j we obtain a function $\tau_j\colon V_j \to \mathbb{R}$ such that

- τ_j is a pluriharmonic conjugate of $\chi_{|V_j}$;
- $\tau_{|C\cap V_j} \equiv 0$.

Consider now two intersecting neighborhoods V_j, V_k and define $V_0 := V_j \cap V_k \neq \varnothing$. Since V_0 is connected, $\tau_j - \tau_k = a$ constant in V_0 and so $\tau_j - \tau_k \equiv 0$ in V_0, because $\tau_{j|V_0\cap C} \equiv 0 \equiv \tau_{k|V_0\cap C}$. (Note that $V_0 \cap C \neq \varnothing$ if $V_0 \neq \varnothing$.) Thus $\tau_j(p) = \tau_k(p)$, whenever $p \in V_j \cap V_k$. Consequently, the family $\{\tau_j\}_{j=1}^n$ defines a single-valued pluriharmonic function $\tau : V \to \mathbb{R}$, where $V = \bigcup_{j=1}^n V_j$, such that $F(p) = \chi(p) + i\tau(p)$, $p \in V$, is holomorphic.

Therefore, there exists $F \in \mathcal{O}(V)$, a non constant holomorphic function, such that $F_{|C} = d$. As d is a regular value of χ, C is non singular, so is a connected component of $\{F = d\}$. It follows that there exists a neighborhood V of C such that $\{F = d\} \cap V = C$ and consequently we set $G = F - d$. As $G^{-1}(0)$ is compact and 0 is a regular value, all the preimages of nearby points are compact, hence, shrinking V if necessary, we have that G is proper. □

Theorem 4.2. *Let X be a weakly complete complex surface, $W \subset X$ a domain and $\chi : W \to \mathbb{R}$ be a pluriharmonic function. Suppose that a regular level of χ contains a compact complex curve. Then X is proper over a (possibly singular) complex curve.*

Proof. Without loss of generality, suppose C connected. In view of Lemma 4.1 there is exist a neighborhood V of C and a holomorphic function $G : V \to \mathbb{C}$ such that $C = \{G = 0\}$. Then the family $\mathfrak{F}_0 = \{G = \zeta, |\zeta| < \epsilon\}$, for some $\epsilon > 0$, consists of compact complex curves so, by [17, III.5.B], $\mathfrak{F}_0$ extends to a family $\mathfrak{F}$ globally defined on X by a holomorphic map $\Phi : X \to R$, where R is an open Riemann surface. In particular, X admits a non constant holomorphic function. In view of [18], X is holomorphically convex hence proper over a Stein space (cfr. [5]). □

We have the following fundamental corollary

Corollary 4.3. *Let X be a complex surface, α a real analytic plurisubharmonic exhaustion function, Y a compact connected component of the regular level set $\{\alpha = c\}$. Assume that Y is Levi flat and contains a non closed leaf. Then there exist a pluriharmonic function $\chi : V \to \mathbb{R}$ on a connected neighbourhood of Y and $\epsilon_0 > 0$ such that*

a) $V \cap \{\alpha = c\} = Y,\ Y = \{\chi = 0\}$

b) *the set*
$$H = \{p \in V : 0 < \chi(p) < \epsilon_0\}\ ,$$
is relatively compact in V, does not contain a critical point of α or χ, and does not contain any compact complex curve;

c) *if $0 < \epsilon < \epsilon_0$, the Levi flat hypersurface $\{\chi = \epsilon\}$ is foliated by dense complex leaves and α is constant on it;*

d) $\partial\bar\partial\alpha \wedge \partial\bar\partial\alpha = 0$ *and* $\partial\alpha \wedge \bar\partial\alpha \wedge \partial\bar\partial\alpha = 0$ *on the whole of X;*

e) *there is a real analytic function $\mu : H \to \mathbb{R}$ such that*
$$\partial\bar\partial\alpha = \mu\partial\alpha \wedge \bar\partial\alpha \quad \textit{and} \quad d\mu \wedge d\alpha = 0\ .$$

Proof. a) It follows from Lemma 3.7.

b) By hypothesis, α has no singular point on Y, therefore there exists a neighborhood of it where no other singular points for α appear; on the other hand, the set $\mathrm{Crt}(\chi)$ of the critical points of χ has complex analytic structure so χ is constant on its connected component. This implies that also χ has no critical point on Y. Therefore, there exists ϵ_0 such that the set H does not contain critical points of either α or χ.

Let us consider the open set

$$N = \{p \in V \ : \ -\epsilon_1 < \chi(p) < \epsilon_0\}$$

with ϵ_1 so small that no critical point of α or χ is contained in N.

If $N \setminus Y$ contains a compact complex curve C, then

by Theorem 4.2 X is union of compact complex curves. This is absurd, as we supposed that Y contained a non-closed complex leaf. Therefore, $H \subsetneqq N \setminus Y$ does not contain any compact curve.

c) Consider the Levi flat level $Y_\epsilon = \{\chi = \epsilon\}$, for $0 < \epsilon < \epsilon_0$; by the previous point, Y_ϵ does not contain complex curves, so by Corollary 3.8, the leaves of the Levi foliation are dense in Y_ϵ. Now, let $p^* \in Y_\epsilon$ be such that $\max \alpha_{|Y_\epsilon} = \alpha(p^*)$ and consider the leaf F of the Levi foliation passing through p^*; the function $\alpha_{|F}$ is a plurisubharmonic function on F, which is a complex immersed curve, attaining maximum in an interior point, hence it is constant on F. But, F being dense in Y_ϵ, α has to be constant on Y_ϵ.

d) Since Y_ϵ is a connected component of $\{\alpha = A_\epsilon\})$ and is Levi flat we have on Y_ϵ

$$\begin{cases} \partial\alpha \wedge \bar\partial\alpha \wedge \partial\bar\partial\alpha = 0 \\ \partial\bar\partial\alpha \wedge \partial\bar\partial\alpha = 0 \end{cases} \tag{2}$$

But since the connected Levi flat components of level set of α cover N, the equations (2) hold on N, a set with non empty interior. In view of the real analiticity of the forms (2) we conclude that the equations(2) hold on the whole of X.

e) By part c), α is constant on every level set of χ in H, and so there is a real-valued function $\varphi : (0, \epsilon_0) \to \mathbb{R}$ such that $\alpha|H = \varphi \circ \chi_0$. Since α and χ_0 are both real analytic and $\partial\alpha$, $\partial\chi_0$ do not vanish in H by part b), it is clear that φ is real analytic and $\varphi'(t) \neq 0$ in $(0, \epsilon_0)$. Now, by direct computation

$$\partial\alpha \wedge \bar\partial\alpha = (\varphi' \circ \chi_0)^2 \partial\chi_0 \wedge \bar\partial\chi_0,$$

in H,

$$\partial\bar\partial\alpha = (\varphi'' \circ \chi_0)^2 \partial\chi_0 \wedge \bar\partial\chi_0,$$

in H.

Thus we set $\mu = (\varphi/\varphi')^2 \circ \chi_0$, which is constant on the level sets of χ in H and therefore on those of $\alpha_{|H}$ as well. Then $d\mu \wedge d\alpha = 0$ in H. □

We conclude this section with a description of the geometric structure of weakly complete surfaces.

Theorem 4.4. *Let X be a weakly complete complex surface and $\alpha : X \to \mathbb{R}$ a real analytic plurisubharmonic exhaustion function. Then three cases can occur:*

1) *X is a modification of a Stein space;*
2) *X is proper over a (possibly singular) complex curve;*
3) *the connected components of the regular levels of α are foliated with dense complex curves, i.e. X is of Grauert type.*

Proof. Let us suppose that there exists a sequence of real numbers $\{c_n\}_{n\in\mathbb{N}} \subset \mathbb{R}$ tending to $+\infty$ such that

$$\{\alpha = c_n\} \cap \Sigma^1 = \emptyset \ ;$$

as Σ^1 is a closed set and the levels of α are compact, for every $n \in \mathbb{N}$ there exists $\delta_n > 0$ such that for every $c \in (c_n - \delta_n, c_n + \delta_n)$ the intersection $\{\alpha = c\} \cap \Sigma^1$ is empty.

Let φ be a minimal plurisubharmonic smooth exhaustion function for X; for every n there is $\epsilon_n > 0$ small enough such that

$$\{\alpha + \epsilon_n \varphi = c_n\} \subset \{c_n - \delta_n < \alpha < c_n + \delta_n\}$$

and then

$$\{\alpha + \epsilon_n \varphi = c_n\} \cap \Sigma^1 = \emptyset \ .$$

The function $\alpha + \epsilon_n \phi$ is minimal as well, hence $X_n = \{\alpha + \epsilon_n \phi < c_n\}$ is a relatively compact strictly pseudoconvex domain (which we can suppose smoothly bounded up to some small perturbation of c_n), hence it is a modification of a Stein surface. Moreover, X_n is Runge in X_{n+1}, therefore X itself is holomorphically convex and, possessing a plurisubharmonic function which is strictly plurisubharmonic at some point, it has to be a modification of a Stein space as well: this is the case 1).

If such a sequence does not exist, then there is $c_0 \in \mathbb{R}$ such that for every $c > c_0$ the intersection $\{\alpha = c\} \cap \Sigma^1$ is not empty.

Suppose that there exists $c_1 > c_0$, regular value for α, such that there is a connected componet Y of $\{\alpha = c_1\}$ which does not contain compact complex curves and such that $Y \cap \Sigma^1 \neq \emptyset$; we apply Theorem 3.6, obtaining that Y is Levi flat and by Corollary 4.3, part d), we get that $\partial\alpha \wedge \bar\partial\alpha \wedge \partial\bar\partial\alpha = 0$ on the whole of X, hence every regular level of α is Levi flat and by Lemma 3.7 every such level has a neighbourhood where it is given as the zero of a pluriharmonic function Therefore, no regular level can contain a compact complex curve, otherwise, by Theorem 4.2, every level would, so all the regular levels are Levi flat and containing no complact complex curves, hence by Corollary 4.3 part c), their connected components are foliated with dense complex leaves. This is case 3)

If every regular level $\{\alpha = c\}$, for $c > c_0$, contains a compact complex curve, then X contains uncountably many compact complex curves; by [17, Proposition 9 and 7] there exist V a neighbourhood of one of these curves and $f : V \to \mathbb{C}$ a holomorphic function which induces on V a foliation in compact curves. Applying Theorem 4.2 with $V = W$ and $\chi = \mathsf{Re} f$, after shrinking V, if needed, to avoid critical points for f, we conclude that X is proper over a non-compact (possibly singular) complex curve, which is case 2). □

It is easy to show that in case 1), Σ^1 is a (at most) countable union of compact complex curves (the exceptional divisor of the modification), whereas in case 2) and 3) it is obvious that $\Sigma^1 = X$. We note that we can tell apart these two cases quite easily by looking at global holomorphic functions: in case 3), $\mathcal{O}(X) = \mathbb{C}$, wherease in case 2) there always exist global non-constant holomorphic functions.

Corollary 4.5. *Let X be a weakly complete 2-dimensional normal complex space and $\alpha\colon X \to \mathbb{R}$ a real analytic plurisubharmonic exhaustion function. Then three cases can occur:*

1) *X is a modification of a Stein space;*
2) *X is proper over a (possibly singular) complex curve C;*
3) *X is of Grauert type.*

Proof. Desingularize X (cfr. [12]) and apply Theorem 4.4 to the desingularization. □

5. Existence of proper pluriharmonic functions - I

Trough Sections 5, 6 we assume that X is a Grauert type surface, i.e. X satisfies conditions of case 3) of Theorem 4.4.

We want to prove the following

Theorem 5.1. *Let X be a Grauert type surface with a real analytic plurisubharmonic exhaustion function α. In particular $\Sigma^1 = X$ and*

$$\partial\alpha \wedge \bar{\partial}\alpha \wedge \partial\bar{\partial}\alpha = \partial\bar{\partial}\alpha \wedge \partial\bar{\partial}\alpha = 0.$$

If $\dim_{\mathbb{R}}\mathrm{Crt}(\alpha) \leq 2$, let Z be the absolute minimum set of α, then

1) *Z is the union of finitely many complex curves;*
2) *there exists an increasing convex function λ such that $\chi = \lambda \circ (\alpha_{|X\setminus Z})$ is pluriharmonic and proper.*

3) *for every plurisubharmonic function $f : X \smallsetminus Z \to \mathbb{R}$ (not necessarily smooth), there exists a real function $\gamma : \mathbb{R} \to \mathbb{R}$ such that $f = \gamma \circ \chi$.*

From the existence of χ immediately follows that

Corollary 5.2. *The function α does not have local minimum points other then absolute minimum ones. The level sets of α, except absolute minimum ones, have pure dimension* 3*.*

In the remainder of this section we prove some auxiliary lemmas.

Lemma 5.3. . *Let W be a complex surface, and $\beta : W \to \mathbb{R}$ a real analytic function, such that*

$$\partial\beta \wedge \bar\partial\beta \wedge \partial\bar\partial\beta = 0, \;\; \partial\bar\partial\beta \wedge \partial\bar\partial\beta = 0$$

in W. Then there is a real analytic function $\mu : W \smallsetminus \mathrm{Crt}(\beta) \to \mathbb{R}$, such that

$$\partial\bar\partial\beta = \mu\, \partial\beta \wedge \bar\partial\beta$$

in $W \smallsetminus \mathrm{Crt}(\beta)$.

Proof. Observe that we do not assume that β is plurisubharmonic. The real-analiticity property of μ is obviuos, once we establish the elementary fact that for every p such that $\partial\beta(p) \neq 0$ there is $\mu(p) \in \mathbb{R}$ satisfying

$$\partial\bar\partial\beta(p) = \mu(p)(\partial\beta \wedge \bar\partial\beta)(p).$$

We sketch the details. Point p is fixed. If $\tilde z, \tilde w$ are any complex local coordinates near p, the complex Hessian of β can be diagonalized by a linear change of coordinates $\tilde z, \tilde w$ into some z, w, so that

$$\partial\bar\partial\beta(p) = a(dz \wedge d\bar z)(p) + c(dw \wedge d\bar w)(p),$$

with a, c are real. We have

$$0 = (\partial\bar\partial\beta \wedge \partial\bar\partial\beta)(p) = ac(dz \wedge d\bar z \wedge dw \wedge d\bar w)(p),$$

$$\partial\beta(p) = (rdz + sdw)(p), r, s \in \mathbb{C}, (r, s) \neq (0, 0),$$

$$0 = (\partial\bar\partial\beta \wedge \partial\bar\partial\beta)(p) = (r\bar r c + s\bar s a)(dz \wedge d\bar z \wedge dw \wedge d\bar w)(p),$$

so $ac = 0$ and $(r\bar r c + s\bar s a) = 0$.

If $a = 0, b \neq 0$ then $r\bar r c = 0$, $r = 0$. Thus

$$(\partial\beta \wedge \bar\partial\beta)(p) = s^2(dw \wedge d\bar w)(p),\; s \neq 0,$$

and so:

$$(\partial\bar\partial\beta)(p) = c(dw \wedge d\bar w)(p) = \mu(\partial\beta \wedge \bar\partial\beta)(p)$$

with $\mu(p) = c/s^2$.

The case $a \neq 0$, $c = 0$ is completely analogous, with roles of z, w being interchanged.

If $a = c = 0$, take $\mu(p) = 0$. □

Lemma 5.4. *Let W be a connected complex surface and $\beta : W \to \mathbb{R}$, $\mu : W \smallsetminus \mathrm{Crt}(\beta) \to \mathbb{R}$ two real analytic functions such that*

1) $\dim_{\mathbb{R}} \mathrm{Crt}(\beta) \leq 2$;
2) $\partial\bar{\partial}\beta = \mu \partial\beta \wedge \bar{\partial}\beta$, *on* $W \smallsetminus \mathrm{Crt}(\beta)$;
3) $d\mu \wedge d\beta = 0$ *on* $W \smallsetminus \mathrm{Crt}(\beta)$;
4) $\beta(W) = \beta(W \smallsetminus \mathrm{Crt}(\beta))$.

Then there are a nonconstant pluriharmonic function $\chi : W \to \mathbb{R}$ and a real analytic function $\theta : \beta(W) \to \mathbb{R}$ such that

i) $\theta'(t) > 0$ *for* $t \in \beta(W)$;
ii) $\chi = \theta \circ \beta$ *(and so* $d\chi \wedge d\beta = 0$ *in* W*)*;
iii) *any pluriharmonic function* $\chi^* : W \to \mathbb{R}$ *such that* $\mathrm{d}\chi^* \wedge \mathrm{d}\beta = 0$ *on* W, *must be of the form* $\chi^* = c\chi + c_1$, *where* c, c_1 *are real constants.*

Proof. The following is well known.

Assertion 1. If $d\mu \wedge d\beta = 0$ in $W_0 = W \smallsetminus \mathrm{Crt}(\beta)$, then μ is constant on each connected component of every level set of β.

Assertion 2. If $\mathrm{Crt}(\beta)$ has topological dimension ≤ 2, then there is a real analytic function $m : \beta(W_0) \to \mathbb{R}$ such that $\mu(p) = m(\beta(p))$, $p \in W_0$. If, in addition, β does not have local minimum or local maximum points in W, then $\beta(W) = \beta(W_0)$ and $\partial\bar{\partial}\beta = (m \circ \beta)\partial\beta \wedge \bar{\partial}\beta$ in W.

Observe first, that since W_0 does not contain critical points, and since it is connected due to the fact that $\dim_{\mathbb{R}} (\mathrm{Crt}(\beta)) \leq 2$, $\beta(W_0)$ is an open (perhaps unbounded) interval, say $\beta(W_0) = (a, b) \subseteq (-\infty, +\infty)$. Fix a point $p_0 \in W_0$. Consider the family of open intervals $\mathcal{I} \subset (a, b)$ such that there is a real analytic function $m_{\mathcal{I}} : \mathcal{I} \to \mathbb{R}$ satisfying $m_{\mathcal{I}} \circ \beta_{|W_{\mathcal{I}}} = \mu_{|W_{\mathcal{I}}}$, where $W_{\mathcal{I}}$ is the connected open component of the open set $\{p \in W_0 : \beta(p) \in \mathcal{I}\}$ that contains p_0. It is evident that for fixed $\mathcal{I}$ function $m_{\mathcal{I}}$ is unique, and so if $\mathcal{I}_1 \subset \mathcal{I}_2 \subset (a, b)$ then $m_{\mathcal{I}_2|\mathcal{I}_1} = m_{\mathcal{I}_1}$. Hence, by Zorn Lemma, there is a maximal interval with this property, assuming that there is any nonempty interval. Denote it by $\mathcal{I}^*(a^*, b^*)$ and $\mathcal{I}^*(b^*, b^*) = \varnothing$ where $b^* = \beta(p_0)$ if no interval $\mathcal{I}$ exists.

We claim that $(a^*, b^*) = (a, b)$. Suppose not and assume, without loss of generality, that $b^* < b$. (The case $a^* > a$ is analogous). (This also covers the case when $a^* = b^*$, $\mathcal{I}^* = \varnothing$.) Choose a point $p^* \in \{\beta = b^*\} \cap \overline{W_{\mathcal{I}}} \cap W_0$, in particular $p^* \in b_{W_0} W_{\mathcal{I}}$ (the boundary of $W_{\mathcal{I}}$ in W_0). In case $\mathcal{I}^* = \varnothing$, we choose $p^* = p_0$.

Since $\partial\beta(p^*) \neq 0$, we can select a local (real) coordinate system at p^*, say $(X_1(p), X_2(p), X_3(p), X_4(p))$ on a neighbourhood N_0 such that $X_1(p) = \beta(p)$ and $X_1(p^*) = X_2(p^*) = X_3(p^*) = X_4(p^*) = 0$. We can select an $\varepsilon > 0$ and a smaller neighbourhood N, $p^* \in N \subset N_0$, such that

$$\{(X_1(p), X_2(p), X_3(p), X_4(p)) : p \in N\} = (b^* - \epsilon, b^* + \epsilon) \times (-\epsilon, \epsilon)^3.$$

Since $\{\beta = t\} \cap N = \{t\} \times (-\epsilon, \epsilon)^3$ is connected, and $d\mu \wedge d\beta = 0$, we obtain by Assertion 1 that $\mu_{|\{\beta=t\}\cap N}$ is constant and so $\mu_{|\{\beta=t\}\cap N} = m_\epsilon(t)$. Since, in these coordinates $m_\epsilon(t) = \mu(t, 0, 0, 0)$, we obtain m_ϵ is analytic on $(b^* - \varepsilon, b^* + \varepsilon)$ and $m_\epsilon \circ \beta_{|N} = \mu_{|N}$.

In case $\mathcal{I}^* = \varnothing$ and $a^* = b^*$, $p^* = p_0$ thus yields a nonempty interval $\mathcal{I}^* = (b^* - \epsilon, b^* + \epsilon)$. Since $m_\epsilon \circ \beta = \mu$ on N, and $\mu_\epsilon \circ \beta$, μ are analytic functions, the identity must hold on $W_{(b^*-\epsilon, b^*+\epsilon)}$ as well. In case $a^* < b^*$ we obtain $m_{\mathcal{I}} \cap N = (b^* - \epsilon, b^* + \epsilon) \times (-\epsilon, \epsilon)^3$ (using the local coordinate system), and so $m_{\mathcal{I}}(t) = \mu(t, 0, 0, 0) = m_\epsilon(t)$ for $t \in \mathcal{I}^* \cap (b^* - \epsilon, b^* + \epsilon) = (b^* - \epsilon, b^*)$. Thus $m_{\mathcal{I}^*}$ and m_ϵ define consistently a real analytic function, call it $m_{\mathcal{I}_1} : \mathcal{I}_1 \to \mathbb{R}$, where $\mathcal{I}_1 = (a^*, b^* + \epsilon)$. Since $m\mathcal{I}_1 \circ \beta_{|W_{\mathcal{I}}} = \mu_{|W_{\mathcal{I}}}$, and both $m\mathcal{I}_1$ and μ are defined and real analytic in $W_{\mathcal{I}_1}$, that contains $W_{\mathcal{I}}$, we obtain $m\mathcal{I}_1 \circ \beta = \mu$ in $W_{\mathcal{I}_1}$.

We conclude that $\mathcal{I}^* = (a, b)$. Since $W_{(a,b)} = W_0$, we have $\mu = m \circ \beta$ in W_0, the first of the Assertion 2.

Furthermore, if $\beta(W) = \beta(W_0)$, $m \circ \beta$ is defined and real analytic in W, and since $\partial\bar{\partial}\beta = (m \circ \beta)\partial\beta \wedge \bar{\partial}\beta$ in W_0 as already shown, the same identity must hold in W (just by continuity).

This completes the proof of Assertion 2.

By Assertion 2, any (pluriharmonic) function $\chi^* : W \to \mathbb{R}$ such that $\mathrm{d}\chi^* \wedge d\beta = 0$ in W must be of the form $\chi^* = \theta_1 \circ \beta$, θ_1 real analytic on $\beta(W)$. We look now for a condition for θ_1 so that $\theta_1 \circ \beta$ be pluriharmonic.

$$\begin{aligned} 0 = \partial\bar{\partial}(\theta_1 \circ \beta) &= (\theta_1'' \circ \beta)\partial\beta \wedge \bar{\partial}\beta + (\theta_1' \circ \beta)\partial\bar{\partial}\beta = \\ &= (\theta_1'' \circ \beta)\partial\beta \wedge \bar{\partial}\beta + (m \circ \beta)(\theta_1' \circ \beta)\partial\beta \wedge \bar{\partial}\beta \end{aligned}$$

i.e. $[(\theta_1'' \circ \beta)(p) + (m \circ \beta)(p)\theta_1'(\beta(p))]\,(\partial\beta \wedge \bar{\partial}\beta)(p) = 0$.

Since $(\partial\beta)(p) \neq 0$ for $p \in W \smallsetminus \mathrm{Crt}(\beta) = W_0$, we obtain the condition for $t = \beta(p)$,

$$\theta_1''(t) + m(t)\theta_1'(t) = 0\ . \tag{3}$$

Since

$$\{t = \beta(p),\, p \in W \smallsetminus \mathrm{Crt}(\beta)\} = \beta(W) = (a, b)$$

we have to solve (3) on (a, b). Applying standard techniques we obtain, fixing a point $t_0 \in (a, b)$

$$\theta_1(t) = c \int_{t_0}^{t} \exp\left(-\int_{t_0}^{\tau} m(\sigma)\, d\sigma\right) d\tau + c_1.$$

Choosing as θ solution with $c = 1$ and $c_1 = 0$ we have evidently $\theta'(t) > 0$ on (a, b), $\chi = \theta \circ \beta$ satisfies $\mathrm{d}\chi \wedge \mathrm{d}\beta = 0$ and is pluriharmonic, and any other pluriharmonic solution χ^* satisfying $d\chi^* \wedge d\beta = 0$, is equal to

$$\chi^* = \theta_1 \circ \beta = (c\theta + c_1) \circ \beta = c\chi + c_1,$$

as required. □

Lemma 5.5. *Let X be a complex surface, α a real analytic plurisubharmonic exhaustion function. Let $U \subset X$ be a domain such that* $\dim_{\mathbb{R}} \mathrm{Crt}(\alpha) \cap U \leq 2$. *If $\alpha_{|U}$ has an absolute minimum value A_0, let $Z_0 = \{p \in U : \alpha(p) = A_0\}$ (otherwise $Z_0 = \varnothing$), then*

i) *α does not have local minimum points on $U \setminus Z_0$;*
ii) *there is a proper pluriharmonic function $\chi : U \smallsetminus Z_0 \to \mathbb{R}$ such that $\alpha = \lambda \circ \chi$ for some increasing convex function λ;*
iii) *if $Z_0 \neq \varnothing$*

$$\lim_{U \ni p \to Z_0} \chi(p) = -\infty.$$

Proof. Let $W = U \setminus Z_0$ and $W_0 = U \setminus \mathrm{Crt}(\alpha)$. Of course $\dim_{\mathbb{R}} Z_0 \leq 2$ and Z_0 is real analytic. Thus, U being connected and $\dim_{\mathbb{R}} U = 4$, Z_0 cannot separate U and $\mathrm{Crt}(\alpha)$ cannot separate W, so W and and W_0 are connected; it follows that

$$\alpha(W_0) = \alpha(W) = (A_0, A^*) \qquad \text{where } A^* \leq +\infty\ . \tag{4}$$

(Indeed, the only possible values in $\alpha(W) \smallsetminus \alpha(W_0)$ could be one of the local minimum values but since $\alpha(W_0)$ is a connected interval and $\inf\, \alpha(W_0) = \inf\, \alpha(W) = A_0$, we have equality.)

Let Y be a connected component of $\{\alpha = c\}$. As $\Sigma^1 = X$, in view of Theorem 3.6, Y is Levi flat so, by Lemma 3.7, it is the zero set of a pluriharmonic function near Y so the fundamental Corollary 4.3 applies. Then, saving the same notations $V \supset Y$, $\chi : V \to \mathbb{R}$, $\overline{N} \subset V$, by part d)

$$\partial\bar{\partial}\alpha \wedge \partial\bar{\partial}\alpha = 0,\ \partial\alpha \wedge \bar{\partial}\alpha \wedge \partial\bar{\partial}\alpha = 0$$

and hence, by Lemma 5.3 there is a real analytic function $\mu : W_0 \to \mathbb{R}$ such that

$$\partial\bar{\partial}\alpha = \mu \partial\alpha \wedge \bar{\partial}\alpha\ . \tag{5}$$

By part e) of Corollary 4.3, $d\mu \wedge d\alpha = 0$ in N. But then by real analyticity,

$$d\mu \wedge d\alpha = 0 \qquad \text{in } X \setminus \mathrm{Crt}(\alpha)\ . \tag{6}$$

In view of (4), (5), (6), the function α and the sets W, W_0 satsfy all the conditions of Lemma 5.4 with $\beta = \alpha$ and so there exist real analytic functions $\theta : (A_0, A^*) \to \mathbb{R}$, such that $\theta'(t) > 0$, for all t, $\chi = \theta \circ \alpha : W \to \mathbb{R}$ is pluriharmonic, i.e. $\chi = \theta \circ \alpha : W \to \mathbb{R}$.

Let now $\lambda = \theta^{-1} : \chi(W) \to \mathbb{R}$. Of course, λ is real analytic, $\lambda'(t) > 0$ and $\alpha = \lambda \circ \chi$ in W. Since $\partial\bar\partial\alpha = (\lambda'' \circ \chi_0)\partial\chi \wedge \bar\partial\chi$, and α is plurisubharmonic, $\lambda'' \geq 0$, i.e. λ is convex.This shows ii).

Observe now that, since χ is pluriharmonic on W (by definition), χ does not have any local minimum points in W, and since $\alpha = \lambda \circ \chi$, λ strictly increasing, also α cannot have any local minimum points in W.

The set $\chi(W)$ is an open interval, say $\chi(W) = (c_0, c^*)$, with $c_0 \geq -\infty$, $c^* \leq +\infty$. Since $\alpha(p) \to 0$ as $p \to Z_0$, and $\chi = \theta \circ \alpha$, we obtain

$$\lim_{p \to Z_0} \chi(p) = c_0\ . \tag{7}$$

We prove now that $c_0 = -\infty$. Suppose , to the contrary, that $c_0 > -\infty$; we will show that Z_0 is empty. Consider any regular point $p \in Z_0$ and its neighbourhood V with complex coordinates z, w. Without loss of generality, $V \subset \mathbb{C}^2$. Then p is either an isolated point, or $T_p(Z_0)$, the tangent space, is a line or a plane in $\mathbb{C}^2$. Whatever of these three cases may be, there is a complex line L through p transversal to $T_p(Z_0)$, hence p is an isolated point of $L \cap Z_0 \cap V$. The pluriharmonic function χ restricted to $L \cap Z_0 \cap V$ is a bounded harmonic function with p as its isolated singular point and so extends to a harmonic function $\tilde\chi : L \cap \{|(z,w) - p|^2 < \delta\} \to \mathbb{R}$,which must have value c_0 at p, by (7), and so a strict minimum there (as p is an isolated point of $Z_0 = \{\alpha = \min \alpha\}$). This is a contradiction. Thus $c_0 = -\infty$, which proves iii). $\square$

Proof of Theorem 5.1. The existence of χ follows from Lemma 5.5 applied to $U = X$. The absolute minimum set Z of α is a compact complex pluripolar set in X defined by the plurisubharmonic function

$$\psi(p) = \begin{cases} \chi(p) & \text{if } x \in X \smallsetminus Z \\ -\infty & \text{if } p \in Z. \end{cases}$$

(i.e. $Z = \{\psi = -\infty\}$). Since, in addition, Z is compact and real analytic from Lemma 7.7 (see Appendix), it follows that Z is the union of finitely many compact complex curves.

Now, let $f : X \smallsetminus Z \to \mathbb{R}$ be a plurisubharmonic function. For every $c \in \mathbb{R}$, the level set $\{\chi = c\}$ is compact, hence f attains a maximum on it, say at p; therefore, f attains a maximum on the complex leaf through p of the foliation given by the Levi flatness, but then f is constant on such a leaf. The leaf being dense, this implies that f is constant on $\{\chi = c\}$, so $f = \gamma \circ \chi$. Theorem 5.1 is completely proved. □

6. Existence of proper pluriharmonic functions - II

In this section, we prove the last part of the Main Theorem, namely the following

Theorem 6.1. *Let X be a Grauert type surface and $\alpha : X \to \mathbb{R}$ a real analytic plurisubharmonic exhaustion. Assume that* $\dim_{\mathbb{R}} \mathrm{Crt}(\alpha) = 3$. *Then either there is a proper pluriharmonic function $\chi : X \to \mathbb{R}$, or there is a double holomorphic covering map $\pi : X^* \to X$ and a proper pluriharmonic function $\chi^* : X^* \to \mathbb{R}$.*

Moreover, if $f : X \to \mathbb{R}$ (resp. $f : X^* \to \mathbb{R}$) is any plurisubharmonic function (not necessarily smooth), then $f = \gamma \circ \chi$ (resp. $f = \gamma \circ \chi^*$) for a suitable real function γ.

In order to present the proof of this result, we need a careful analysis of the critical level sets of α, to which is devoted the reminder of this section. All the results before the proof of Theorem 6.1 assume implicitly its hypotheses.

6.1. Geometrical structure of the set of critical points of α.

The critical set $\mathrm{Crt}(\alpha)$ is a real-analytic subset of X, and so, its regular part is dense in it (see Theorem 1 in Chapter 3, Section 1 of [15]). The regular part is the union of (at most) countably many pairwise disjoint locally closed real-analytic submanifols of X.

Denote by $\{M^i\}_{i \in I}$ the collection of 3-dimensional components of the regular part of $\mathrm{Crt}(\alpha)$ and, for each $i \in I$, let $A^i \in \mathbb{R}$ be the value of α on M^i; we set

$$\widetilde{M} = \bigcup_{i \in I} \overline{M}^i .$$

Since we assume that $\dim_{\mathbb{R}} \mathrm{Crt}(\alpha) = 3$, $\widetilde{M} \neq \emptyset$.

Proposition 6.2. *Let U be a connected component of $X \setminus \widetilde{M}$, then*

1) *U is open*
2) *if $M^i \cap \mathrm{b}U \neq \emptyset$, then $M^i \subseteq \mathrm{b}U$*
3) *$\alpha|_{\mathrm{b}U}$ is constant and equal to $\min_{\overline{U}} \alpha$.*

Proof. We note that, α being proper, $\{\overline{M}^i\}_{i \in I}$ is a locally finite family of compact sets, hence their union $\widetilde{M}$ is a closed subset of X. Thus 1) follows.

To prove 2), we remark that each M^i is relatively open in $\mathrm{Crt}(\alpha)$, so $M^i \cap \overline{M}^j = \emptyset$ whenever $i \neq j$; moreover, for each $i \in I$, there exists an open set $V^i \subsetneqq X$ such that $\widetilde{M} \cap V^i = M^i$ and $V^i \setminus M^i$ has *at most* 2 connected components. Up to shrinking the open sets V^i, we can assume that $V^i \cap V^j = \emptyset$ if $i \neq j$; therefore

$$V^i \cap \widetilde{M} = M^i \quad \text{and} \quad V^i \setminus M^i \subsetneqq X \setminus \widetilde{M} \ .$$

If, for some i, $M^i \cap \mathrm{b}U \neq \emptyset$, then $V^i \cap U \neq \emptyset$; therefore there exists a connected component V^i_+ of $V^i \setminus M^i$ which is contained, by connectedness, in U. Since the boundary of V^i_+ contains M^i, $M^i \subsetneqq \overline{U}$.

We now show 3). Let

$$A = \min_{\overline{U}} \alpha \ .$$

Now, suppose that there exists a face M^i of U such that $A^i > A$. By Theorem 5.5, there exists a real-analytic concave function $\theta : (A, A^*) \to \mathbb{R}$, with $A^* = \sup_U \alpha \in (A, +\infty]$, such that $\chi := \theta \circ (\alpha|_{U\setminus \mathrm{Min}(\alpha)})$ is a pluriharmonic function in $U \setminus \mathrm{Min}(\alpha)$.

By the previous part of this proof, at least one component V^i_+ of $V^i \setminus M^i$ intersects U; let H be a connected component of $V^i_+ \setminus \{\alpha = A^i\}$. If $\alpha|_H < A^i$, by Hopf Lemma [23, Lemma 3.8], $bH \cap \widetilde{M} = \emptyset$, but this cannot happen for all the connected components of $V^i_+ \setminus \{\alpha = A^i\}$, as $bV^i_+ \cap M^i \neq \emptyset$. Therefore we have that $\alpha|_H > A^i$ for some H, hence $A^* > A^i$.

Let $c = \theta(A^i)$, then $\chi|_{V^i_+} > c$ and $\chi|_{M^i} = c$. By Lemma 7.6, there exist a domain $\widetilde{V} \supseteq V^i_+ \cup M^i$ and a pluriharmonic function $\widetilde{\chi} : \widetilde{V} \to \mathbb{R}$ extending χ. We can assume, without loss of generality, that $\widetilde{V}$ is a connected open subset of $\{x \in X \ \alpha(x) > A\}$

Consider now the function $\widetilde{\alpha} = \theta^{-1} \circ \widetilde{\chi} : \widetilde{V} \to \mathbb{R}$; such a function is real-analytic and coincides with α on the non-empty open set V^i_+, therefore they are equal on $\widetilde{V}$, by connectedness.

Since $\chi|_{V^i_+} > c$, $d\widetilde{\chi}(x) \neq 0$ for $x \in M^i$, but then

$$d\alpha(x) = (\theta^{-1})'(x) d\widetilde{\chi}(x) \neq 0 \ ,$$

as θ^{-1} is increasing.

This contradicts the fact that $M^i \subsetneqq \mathrm{Crt}(\alpha)$. Therefore, $A^i = A$.

□

We call any connected component U of $X \setminus \widetilde{M}$ a *cell*, and any M^i a *face*. If $M^i \subset \overline{U}$, we call M^i a *face* of U. To every face M^i we associate an open neighbourhood V^i, as described in the proof of Proposition 6.2.

We note that, for every cell U, $\bar{U} \cap \widetilde{M} \neq \emptyset$, otherwise $U = \overline{U}$, i.e. $U = X$ by connectedness.

If M^i is a face of U and $V^i \setminus M^i \subseteqq U$, we call M^i an *internal face* of U and set

$$W^i = U \cup M^i \ .$$

Such W^i is again an open set, as $W^i = U \cup V^i$, and it is uniquely determined by M^i, as $V^i \setminus M^i$ can be contained in a unique cell.

On the other hand, if $V^i \setminus M^i = V^i_+ \cup V^i_-$ and $V^i_+ \subseteqq U$, $V^i_- \subseteqq U'$ with $U \neq U'$, we say that M^i is a *connecting face* between U and U' and we set

$$W^i = U \cup M^i \cup U' \ .$$

Again, $W^i = U \cup V^i \cup U'$, so it is an open set and it is uniquely determined by M^i.

Theorem 6.3. $\widetilde{M} = \{x \in X \ : \ \alpha(x) = \min_X \alpha\}$

Proof. By part 3) of Proposition 6.2, α is constant on bU for every cell U.

Let $\bar{A}$ be the smallest element of the set $\{A^i\}_{i\in I}$. By properness, there is a finite number of faces contained in $\{\alpha = \bar{A}\}$; let us change the indexing so that these faces are $M^1, \ldots, M^N$.

We set

$$W = \bigcup_{i\in I} W^i = X \setminus \bigcup_{i\in I} \left(\overline{M}^i \setminus M^i\right)$$

and we remark that, the family of compact sets $\{\overline{M}^i \setminus M^i\}_{i\in I}$ being locally finite, W is open in X and, as $\dim(\overline{M}^i \setminus M^i) \leq 2$ (as a semi-analytic set), W is also connected.

We can also write

$$W = \bigcup_{i\in I} M^i \cup (X \setminus \widetilde{M}) \ .$$

Now, define $H \subseteqq W$ as

$$H := W^1 \cup \ldots \cup W^N \ .$$

It is clear that H is open in W; moreover,

$$H = (M^1 \cup \ldots \cup M^N) \cup H_0$$

where H_0 is a union of finitely many cells U, all such that $\alpha|_{bU} = \bar{A}$.

Obviously, the relative boundary of such a cell U in W is contained in $M^1 \cup \ldots \cup M^N$ and the closure of H_0 in W is H. Therefore, H is both closed and open in W and, since W is connected, $H = W$. Therefore,

$\alpha|_{M^i} = \bar{A}$ for every $i \in I$; by continuity we have that α is constant on $\widetilde{M}$.

Now, let U be a cell. Suppose $\{\alpha = \overline{A}\}$ intersects U, and let Y be a connected component of $\{\alpha = \overline{A}\}$ such that $Y \cap U \neq \varnothing$. Since $Y \subset \mathrm{Crt}(\alpha)$ and $\dim_{\mathbb{R}}(Y \setminus \bigcup_i M^i) \leq 2$, we have $\dim_{\mathbb{R}} Y \cap U \leq 2$. Applying Lemma 5.5 to the domain U, we obtain the existence of a pluriharmonic function $\chi : U \setminus Y \to \mathbb{R}$, of the form $\chi = \lambda \circ (\alpha|_{U\setminus Y})$, with λ an increasing convex function, such that

$$\lim_{x \to U\cap Y} \chi(x) = -\infty\ .$$

In particular

$$\lim_{t \to \overline{A}^+} \lambda(t) = -\infty\ . \tag{8}$$

Let M be a face adjacent to U and $q \in M$ a point. Choose a connected open neighborhood H of q such that $H \setminus M$ has two connected components, at least one of them, denoted by H^+, contained in U. Define now a function $\chi_1 : H \to [-\infty, +\infty)$ by

$$\chi_1(x) = \begin{cases} \chi(x) & x \in H^+ \\ -\infty & x \in H \setminus H^+\ . \end{cases}$$

Then

$$\lim_{H \ni x \to M\cap H} \chi_1(x) = \lim_{H \ni x \to M\cap H} \lambda \circ \alpha(x) = \lim_{t \to \overline{A}^+} \lambda(t) = -\infty\ ,$$

by (8). Thus χ_1 is plurisubharmonic in H and equal to $-\infty$ in $H \setminus H^+$, a set with non-empty interior, which is a contradiction.

Therefore $\{\alpha = \overline{A}\} \cap U = \varnothing$. □

We immediately obtain the following

Corollary 6.4. *There are only finitely many faces and finitely many cells.*

Proof. The faces are the 3-dimensional components of the regular part of $\alpha^{-1}(\bar{A})$, which are finitely many, as α is proper. Moreover, every face may belong to at most two cells, hence there is only a finite number of cells too. □

6.2. **Construction of pluriharmonic functions on W.** Without loss of generality, we suppose that the value of α on $\widetilde{M}$ is 0. Let U be a cell. Due to the standing assumptions, $\partial\alpha \wedge \bar{\partial}\alpha \wedge \partial\bar{\partial}\alpha = 0$ in U, and so, by Lemma 5.3 there is a real analytic function $\mu_1 : U \setminus \mathrm{Crt}(\alpha) \to \mathbb{R}$ such that $\partial\bar{\partial}\alpha = \mu_1 \partial\alpha \wedge \bar{\partial}\alpha$ in $U \setminus \mathrm{Crt}(\alpha)$. Clearly, part e) of Corollary 4.3 applies in the present situation, giving $\mathrm{d}\mu_1 \wedge \mathrm{d}\alpha = 0$ in $U \setminus \mathrm{Crt}(\alpha)$.

Proposition 6.5. *Let M^i be a face of the cell U and consider, for a point $p \in M^i$, a neighbourhood V_p containing p such that $V_p \setminus M^i$ has exactly two connected components, denoted by V_p^+ and V_p^-. Then there exist an integer k, a pluriharmonic function $\widetilde{\chi} : V_p \to \mathbb{R}$ and a real-analytic, strictly increasing function $\lambda : \widetilde{\chi}(V_p) \to \mathbb{R}$ such that*

$$\alpha(x) = (\lambda \circ \widetilde{\chi}(x))^{2k}$$

for every $x \in V_p$.

Proof. Set $M_p = V_p \cap M$. By Proposition 7.3, there exists $k \in \mathbb{N}$ such that α is flat of order $2k-1$ on $M_p \setminus T_p$ where T_p is a real analytic subset of M_p, $\dim T_p \leq 2$. By Proposition 7.4, the function

$$\beta(x) = \begin{cases} 0 & \text{if} \quad x \in M_p \setminus T_p \\ \alpha(x)^{1/2k} & \text{if} \quad x \in V_p^+ \\ -\alpha(x)^{1/2k} & \text{if} \quad x \in V_p^- \end{cases}$$

is a real analytic function on $V_p \setminus T_p$, without critical points on $M_p \setminus T_p$; on $V_p \setminus M_p$, the critical points of β are the same as those of α, so $\dim(\mathrm{Crt}(\beta)) \leq 2$.

Let $W_p = V_p \setminus T_p$; since $\beta = \pm\alpha^{1/2k}$ outside M_p, direct computation shows that $\partial\beta \wedge \bar{\partial}\beta \wedge \partial\bar{\partial}\beta = 0$ and $\partial\bar{\partial}\beta \wedge \partial\bar{\partial}\beta = 0$ on $V_p^+ \cup V_p^-$ and hence, by analyticity, on W_p. By Lemma 5.3, there is a real analytic function $\mu_p : W_p \setminus \mathrm{Crt}(\beta) \to \mathbb{R}$ such that $\partial\bar{\partial}\beta = \mu_p \partial\beta \wedge \bar{\partial}\beta$.

We have, by direct computation,

$$\mu_p = \frac{1-2k}{\beta} + 2k\beta^{2k-1}\mu_1$$

and since $\mathrm{d}\mu_1 \wedge \mathrm{d}\alpha = 0$ in U, $\mathrm{d}\mu_p \wedge \mathrm{d}\beta = 0$ in $U \cap V_p$ hence in $W_p \setminus \mathrm{Crt}(\beta)$ by analyticity.

We are now in the position to apply Lemma 5.4: there exists a real analytic function $\theta : \beta(W_p) \to \mathbb{R}$, strictly increasing on the open interval $\beta(W_p)$ and such that $\chi := \theta \circ \beta$ is pluriharmonic on W_p, with $\mathrm{d}\chi \wedge \mathrm{d}\beta = 0$ in W_p and so $\mathrm{d}\chi \wedge \mathrm{d}\alpha = 0$ on W_p.

As $0 \in \beta(W_p)$ and $\lim_{x \to T_p} \beta(x) = 0$, we extend χ by continuity to $\widetilde{\chi} : V_p \to \mathbb{R}$, which is pluriharmonic on $W_p = V_p \setminus T_p$. By Lemma 7.5, $\widetilde{\chi}$ is indeed pluriharmonic on V_p.

We obtain the claim by setting $\lambda = \theta^{-1}$. □

We thus constructed a pluriharmonic function on V_p; up to now, such a function is not uniquely determined by our construction. In fact, we have two "degrees of freedom", the choice of V_p^+ and V_p^- and the choice of the function θ.

The latter issue is solved by Lemma 5.4, iii), where it is proved that, once we choose such a function θ, any other possible function is of the form $a\theta + b$; therefore, we can fix a particular θ_0 by requiring that $\theta_0(0) = 0$ and $\theta_0'(0) = 1$ (as $0 \in \beta(W_p)$). The pluriharmonic function $\widetilde{\chi}$ constructed on V_p with this normalized function θ_0 is denoted by χ_p.

The following result is immediate.

Corollary 6.6. *Let M^i be a face of the cell U and consider, for a point $p \in M^i$, a neighbourhood V_p containing p such that $V_p \setminus M^i$ has exactly two connected components, denoted by V_p^+ and V_p^-. Then there exists a unique pluriharmonic function $\chi_p : V_p \to \mathbb{R}$, positive in V_p^+, such that*

$$\lim_{V_p^+ \ni x \to M^i} \frac{(\chi_p(x))^{2k}}{\alpha(x)} = 1$$

where $2k - 1$ is the order of flatness of α along $M^i \cap V_p$.

On the other hand, the sign issue cannot be satisfactorily solved, particularly if we take into account that some face could be internal. Therefore, for every point $p \in M^i$, we associate a *pair* of germs of pluriharmonic functions $\{(\chi_p, V_p), (-\chi_p, V_p)\}$, since an exchange between V_p^+ and V_p^- will only produce a change in the sign of β and hence of χ_p.

This is indeed a well defined germ on every point of M^i: by Corollary 6.6, if we take another neighbourhood V_p' of p, different from V_p, but still such that $V_p' \setminus M^i$ has two connected components, the pair of germs we obtain in the end of the construction will be the same, i.e. we will have $\{(\chi_p', V_p'), (-\chi_p', V_p)\}$ with $\chi_p = \pm\chi_p'$ on $V_p \cap V_p'$.

Proposition 6.7. *The function α has the same order of flatness along all the faces, including internal ones.*

Proof. Let M^i be a connecting face between the cells U^+ and U^-. For any $p \in M^i$, we can take $V_p = W^i$, because $\dim_{\mathbb{R}}(W^i \cap \mathrm{Crt}(\beta)) \leq 2$ and $\beta(W^i) = \beta(W^i \setminus \mathrm{Crt}(\beta))$ (as β does not have absolute maximum or minimum points in W^i); therefore, we can apply Lemma 5.4 to the domain W^i and the function β.

By Corollary 6.6, we have a pluriharmonic function $\chi^i := \chi_p$, canonical up to sign, such that there exists a strictly increasing function $\lambda := \theta^{-1}$ so that

$$\alpha(x) = (\lambda \circ \chi^i)^{2k}(x) \tag{9}$$

for $x \in W^i$. We also ask that $\chi^i > 0$ on U^+ and $\chi^i < 0$ on U^-.

We note that, as $\alpha|_{bW^i} = 0$, also $\chi|_{bW^i} = 0$. Now, let $M^j \subsetneqq bW^i$ be a face; as $\chi^i > 0$ on U^+ and $\chi^i < 0$ on U^-, by Hopf Lemma [23, Lemma

3.8], we have that $\mathrm{d}\chi^i(x) \neq 0$ for $x \in M^j$. From Equation (9), it follows that the the order of flatness of α on M^j is $2k-1$, for every M^j adjacent to either U^+ or U^-.

Now, the set

$$H' = H \setminus \{\text{internal faces}\}$$

is open and connected, because every internal face is contained in the closure of a unique cell. Obviously every open set W^i is connected and

$$H' = \bigcup_{M^i \text{ connecting}} W^i \ ,$$

so, for every two cells U^1 and U^2 there exists a chain $W^{i_1}, \dots, W^{i_m}$, with $W^{i_h} \cap W^{i_{h+1}} \neq \emptyset$ for $h = 1, \dots, m-1$, connecting them inside H'. The order of flatness of α on the faces in $\mathrm{b}W^{i_1}$ (internal or connecting) will be the same as the order of flatness of α on the faces of bW^{i_2} (internal or connecting), because there are connecting faces belonging to both, and so on, proving our claim. $\square$

Corollary 6.8. *Let U be a cell and M^i a connecting face between the cells U and U'. For any face M^j of U and any $p \in M^j$, let V_p be a neighbourhood of p such that $V_p \setminus M^j$ has exactly two components. Then the pairs $\{(\chi^i, W^i), (-\chi^i, W^i)\}$ and $\{(\chi_p, V_p), (-\chi_p, V_p)\}$ induce the same germs in every $q \in V_p \cap W^i$.*

Proof. This is a simple consequence of Corollary 6.6, once we know, from Proposition 6.7, that the order of flatness of α is the same along every face. $\square$

Up to this point, we have given an open cover of W, constituted by the open sets W^i corresponding to connecting faces and by the open sets V_p, for $p \in M^j$, as M^j ranges among internal faces; on each of these open sets, we have constructed an *unordered* pair of pluriharmonic functions, which differ just by sign, i.e. of the form $\pm\chi$.

On the intersection of two such open sets, the pairs coincide, i.e. their restrictions to the intersection give the same pair of pluriharmonic functions; therefore, we may also describe what we have obtained so far as a subsheaf $\mathfrak{X}$ of the sheaf of pluriharmonic functions on W, whose stalk at any point of W is made exactly of two germs, which differ just by sign.

6.3. **Extension to critical components of lower dimension.** We define the set

$$S := X \setminus W = \bigcup_{i \in I} (\overline{M}^i \setminus M^i) \ ,$$

consisting of the singular points of $\widetilde{M}$.

Remark 6.1. S is of real dimension 2 or lower. Given $p \in S$ and a neighbourhood V of it, if $\chi \in \Gamma(V \cap W, \mathfrak{X})$, then there is a unique continuous extension of χ to $\widetilde{\chi} : V \to \mathbb{R}$, obtained by setting $\widetilde{\chi}|_{S \cap V} \equiv 0$. By Lemma 7.5, $\widetilde{\chi}$ is then pluriharmonic on V.

Moreover, if $\Gamma(V \cap W, \mathfrak{X}) \neq \emptyset$, then it necessarily has two elements, of opposite sign, both extending to pluriharmonic functions on V. Therefore, in order to extend our subsheaf $\mathfrak{X}$ to the whole of X, we only need to show that for any point $p \in S$ there is a neighbourhood V such that $\Gamma(W \cap V, \mathfrak{X}) \neq \emptyset$.

The existence of sections depends *prima facie* on the topological properties of $V \setminus S$ and of the combinatorial properties of the open cover of it given by the sets W^i and V_p defined above, where we have sections. The combinatorics of such a cover is ultimately determined by the topology of $V \setminus \widetilde{M}$, which we propose to examine in the next pages.

To this aim, we employ the theory of stratified spaces, developed by Whitney, Mather, Thom and others in the 60s and 70s (see [13] for the original paper, [19, 25] for more detailed explanations); in what follows, we will just recall the main concepts and the results we need, avoiding many technical, although important, details that the interested reader can find in [25]. Precise references will be given in the next pages, as we state the relevant definitions and theorems.

Definition 6.9 (Definition 1.1 in [25])**.** Let Z be a closed subset of a real analytic manifold X. A *real analytic stratification of* Z is a filtration by closed subsets $Z = Z_d \supset Z_{d-1} \supset \cdots \supset Z_1 \supset Z_0$ such that each difference $Z_i \setminus Z_{i-1}$ is a real analytic submanifold of X and is of dimension i, or empty. Each connected component of $Z_i \setminus Z_{i-1}$ is called a *stratum*. Thus Z is a disjoint union of strata.

Many pathologies can be found among stratified spaces. To avoid them, at least to some extent, additional conditions are usually required to hold, namely the *frontier condition* and *Whitney's conditions* (a) *and* (b).

The frontier condition asks that, whenever S and T are two strata such that $S \cap \overline{T} \neq \emptyset$, we have $S \subsetneqq \overline{T}$ (cfr [25, Definition 1.2]). Whitney's conditions are more involved and impose some restrictions on the behaviour of tangent spaces when going from one stratum to another adjacent to it. For the precise statements see [25, Definition 2.1].

Definition 6.10. A locally finite (i.e whose strata are a locally finite family) stratification satisfying the frontier condition and Whitney's conditions (a) and (b) is called a *Whitney stratification.*

In 1965, Whitney proved the following (see [25, Theorem 2.1] and the references given therein).

Theorem 6.11. *Every real analytic variety admits a Whitney stratification whose strata are real analytic manifolds*

In particular, as a Whitney stratification is locally finite, a compact analytic variety admits a Whitney stratification with a finite number of strata. Whitney's theorem allows us to apply to real analytic varieties the following *local topological triviality result* for stratified spaces (see [25, Theorem 3.2]).

Theorem 6.12 (Thom-Mather Tubular Neighbourhood Theorem)**.** *Let*

$$Z = Z_d \supset Z_{d-1} \supset \cdots \supset Z_1 \supset Z_0$$

be a Whitney stratified subset of a real analytic manifold X. Then for every stratum Y and each point $y_0 \in Y$ there is a "tubular" neighborhood G of y_0 in X, a stratified set (a "link") $L \subset S^{k-1}$ (a $(k-1)$-dimensional sphere) and a homeomorphism (of stratified spaces)

$$h : \big(G, G \cap Z, G \cap Y\big) \longrightarrow \big(G \cap Y\big) \times \big(B^k, \mathrm{c}(L), O_k\big)$$

where $k = \mathrm{codim}_{\mathbb{R}} Y$ in X, B^k is the open k-ball, $\mathrm{c}(L)$ is the cone on the link L with vertex O_k, the center of B^k, with $h(y_0) = O_k$.

As a link, L is stratified,

$$L = L_{d-k-1} \supset L_{d-k-2} \supset \cdots \supset L_1 \supset L_0,$$

and this obviously induces a stratification of the cone

$$\mathrm{c}(L) = \mathrm{c}(L)_{d-k-1} \supset \mathrm{c}(L)_{d-k-2} \supset \cdots \supset \mathrm{c}(L)_1 \supset \mathrm{c}(L)_0 \supset \{O_k\} .$$

We also assume, without loss of generality, that $\overline{G} \cap Y = \overline{G} \cap Z_{d-k}$, so that, in particular, $\overline{G}$ intersects only one stratum of Z_{d-k}.

We apply this theory to the set $\widetilde{M}$, which is analytic by Theorem 6.3 ; we have a stratification

$$\widetilde{M} = Z_3 \supset Z_2 \supset Z_1 \supset Z_0$$

where $S \subsetneqq Z_2$. As $\widetilde{M}$ is compact, the number of strata is finite.

Proposition 6.13. *Let $Y \subsetneqq Z_2 \setminus Z_1$ be a 2-dimensional stratum which is contained in S. For every point $p \in Y$ there exists a neighbourhood V_p such that*

$$\Gamma(V_p \cap W, \mathfrak{X}) \neq \emptyset .$$

Proof. We apply Theorem 6.12 to $p \in Y$; we note that, in this case, $d = 3$, $k = 2$, so L will be a (compact) link in S^1, i.e. a finite number m of points.

The cone over L in B^2 is the union of m radii $T_1, \dots, T_m$, connecting the points of L to the origin and dividing B^2 in m connected open sectors $S_1, \dots, S_m$, indexed so that $T_i, T_{i+1} \subset \overline{S}_i$, with $T_{m+1} = T_1$.

Let G be the neighbourhood given by Theorem 6.12 and h the homeomorphism of stratified spaces; we assume that $G \cap Y = G_0$ is a topological disc.

The sets $N_j := h^{-1}(G_0 \times T_j)$, for $j = 1, \dots, m$, are open connected subsets of $Z_3 \setminus Z_2$, i.e. of the faces in $\widetilde{M}$; likewise, $E_j := h^{-1}(G_0 \times S_j)$ are open connected subsets of $X \setminus Z_3 = X \setminus \widetilde{M}$, i.e. of the cells.

We define, for $j = 1, \dots, m$,

$$V_j = E_j \cup N_{j+1} \cup E_{j+1} \ ,$$

where $N_{m+1} = N_1$ and $E_{m+1} = E_1$; we apply Corollary 6.6 to each V_j, with $V_j^+ = E_j$, obtaining a (unique) pluriharmonic function χ_j.

Obviously, by the uniqueness of χ_j, we have that

$$\chi_j|_{E_{j+1}} = -\chi_{j+1}|_{E_{j+1}} \ ,$$

for each $j = 1, \dots, m$.

Therefore, if m is even, χ_1 and $(-1)^{m-1}\chi_m = -\chi_m$ have the same sign on E_1, hence coincide. So we can glue together the functions $(-1)^{j-1}\chi_j$, for $j = 1, \dots, m$ into a pluriharmonic function

$$\chi : \bigcup_{j=1}^m V_j \to \mathbb{R}$$

which is obviously a section of $\mathfrak{X}$ on

$$\bigcup_{j=1}^m V_j = W \cap G \ .$$

Setting $V_p = G$ proves the claim.

In order to end the proof we have to show that m is even. This fact is a consequence of Sullivan's Theorem on the local Euler characteristic of an analytic variety (cfr. [24, Corollary 2]): a real analytic space is locally homeomorphic to the cone over a polyhedron with even Euler characteristic.

Indeed, represent the neighborhood $G \cap Z$ of y_0 in Z as an open cone over some polyhedron K and compute $\chi(K)$, the Euler characteristic of K. Obviously, $G \cap Z = G_0 \times \mathrm{c}(L_0)$, where G_0 is topologically a disk. It is clear that $G \cap Z$ is topologically an open cone with vertex at

$(y_0, 0)$ over its relative boundary in Z, denoted K. We consider G_0 as a single (open) triangle $(ABC) = \Delta$. Then $G_0 \times \mathrm{c}(L_0)$ is the union of m prisms $\Delta \times (O, P_i) \simeq \Delta \times (0, 1)$ and of a single simplex $\Delta\{O\}$, which is the common boundary of all prisms (and so belongs to the open set $G_0 \times \mathrm{c}(L_0)$). Compute now the Euler characteristic $\underline{\chi}(K) = c_0 - c_1 + c_2$, where c_i is the number of i-dimensional simplexes in K. We have

$$c_0 = 3m + 3$$

(vertices (A, P_i), (B, P_i), (C, P_i), $i = 0, 1, \ldots, m-1$, and (A, O), (B, O), (C, O)).

We divide each of the three side squares of the m-prisms into two simplexes in whatever way so

$$\begin{aligned} c_1 = \ & 3m \ \text{(from the tops of prisms)} \\ & + \ 2 \cdot 3 \cdot m \ \text{(from the non horizontal side square)} \\ & + \ 3 \ \text{(from the common base)} \\ & = \ 9m + 3. \end{aligned}$$

Finally

$$\begin{aligned} c_2 = \ & m \ \text{(triangles from the tops of prisms)} \\ & + \ 3 \cdot 2 \cdot m \ \text{(non horizontal, from the side squares)} \\ & + \ 0 \ \text{(none from the base)} \\ & = \ 7m \end{aligned}$$

so

$$\underline{\chi}(K) = 3m + 3 - (9m + 3) + 7m = m.$$

The proof of Proposition 6.13 is now complete. □

Proposition 6.14. *The sheaf $\mathfrak{X}$ extends to the whole X.*

Proof. By Proposition 6.13 and Remark 6.1, we can extend the sheaf $\mathfrak{X}$ to $X \setminus Z_1$.

Suppose now that $p \in Z_1 \setminus Z_0$ and denote by τ the 1-dimensional stratum with $p \in \tau$. By Theorem 6.12, we have a neighbourhood G and a homeomorphism h such that $h(G \setminus \tau) = (G \cap \tau) \times (B^3 \setminus \{O_3\})$. Therefore, $G \cap (X \setminus Z_1) = G \setminus \tau$ is simply connected, which is enough to say that

$$\Gamma(G \cap (X \setminus Z_1), \mathfrak{X}) \neq \emptyset \, .$$

Therefore, by Remark 6.1, we can extend $\mathfrak{X}$ on $X \setminus Z_0$. Let $p \in Z_0$ and consider a topological ball B around p. Obviously, $B \setminus \{p\}$ is simply connected, hence we can repeat the previous argument and show that

$$\Gamma(G \cap (X \setminus Z_0), \mathfrak{X}) \neq \emptyset \, .$$

Therefore, we obtain an extension of $\mathfrak{X}$ to the whole X. □

We are now in the position to prove Theorem 6.1.

Proof of Theorem 6.1. Let the extension of $\mathfrak{X}$ to X, given by Proposition 6.14, be denoted again by $\mathfrak{X}$. Let X^* be the total space of $\mathfrak{X}$, together with the projection $\pi : X^* \to X$, which sends the germ $(h)_x \in \mathfrak{X}_x$ to x, for every $x \in X$.

We give X^* the natural complex structure, so that π is a 2-to-1 locally trivial holomorphic covering map, open and proper. We define the function

$$\chi^* : X^* \to \mathbb{R}$$

by $\chi^*((h)_x) = h(x)$, for $(h)_x \in \mathfrak{X}_x$. It is easy to check that χ^* is a proper pluriharmonic function on X^*.

It may happen that X^* is disconnected, then each of the two connected components is biholomorphic to X and χ^* descends to a proper pluriharmonic function $\chi : X \to \mathbb{R}$.

Now, let $f : X \to \mathbb{R}$ (resp. $f : X^* \to \mathbb{R}$) be a plurisubharmonic function. For every $c \in \mathbb{R}$, the level set $\{\chi = c\}$ (resp. $\{\chi^* = c\}$ is compact, hence f attains a maximum on it, say at p; therefore, f attains a maximum on the complex leaf through p of the foliation given by the Levi flatness, but then f is constant on such a leaf. The leaf being dense, this implies that f is constant on $\{\chi = c\}$ (resp. $\{\chi^* = c\}$, so $f = \gamma \circ \chi$ (resp. $f = \gamma \circ \chi^*$). □

Remark 6.2. Together with Theorem 5.1, the previous result proves the second part of the Main Theorem; from the existence of a global pluriharmonic function defining the foliation, we deduce that such a foliation is indeed holomorphic. Hence, both case ii and case iii of the Main Theorem are examples of holomorphic foliations; from such observation, we can proceed as Brunella did in [3], noticing that there are examples of weakly complete surfaces which do not admit holomorphic foliations, but for which $\Sigma^1 = X$, and concluding that such surfaces do not admit real analytic plurisubharmonic exhaustion functions.

7. Appendix.

As a consequence of the Hopf Lemma in a weak form (cfr. [23, Lemma 3.8]) we have the following

Proposition 7.1. *Let W be a complex surface and $\beta : W \to \mathbb{R}$ a real analytic plurisubharmonic function. Then every 3-dimensional connected component of* $\mathrm{Crt}(\beta) \setminus \mathrm{Sing}(\mathrm{Crt}(\beta))$ *consists of local minimum* *points of β and so it is contained in $\widetilde{M}$.* $\mathrm{Crt}(\beta) \smallsetminus \widetilde{M}$ *is a real analytic subset of W and* $\dim_{\mathbb{R}} \mathrm{Crt}(\beta) \smallsetminus \widetilde{M} \leq 2$.

The following "good covering lemma" is probably well known, so we just give a brief idea of the proof.

Lemma 7.2. *Let W be a 2-dimensional complex manifold and C a smooth compact complex curve in W. Let U be a neighborhood of C. Then there is a finite covering $\{V_j\}_j^m$ of C by open subsets of W, such that*

i) $C \subset \bigcup_{j=1}^{m} V_j \subset U$;
ii) *every V_j is simply connected;*
ijj) $V_j \cap C$ *is connected for* $j = 1, 2, \ldots, m$;
iv) *whenever $V_j \cap V_k \neq \varnothing$, then $V_j \cap V_k \cap C \neq \varnothing$ and both sets $V_j \cap V_k$ and $V_j \cap V_k \cap C$ are connected, $1 \leq j, k \leq m$.*

Proof. Being smooth, C possesses a tubular neighbourhood in W which is homeomorphic to the product $C \times \Delta$, where Δ is the unit disc. It is now enough to cover C with finitely many connected, simply connected open sets $\{\Omega_j\}_{j=1}^n$ such that their pairwise intersections are either empty or connected, e.g. convex balls for some Riemannian metric on C; the desired covering will be the preimage in the tubular neighbourhood of the products $\{\Omega_j \times \Delta\}_{j=1}^n$. □

7.1. **Flatness.** Let M be a C^∞-smooth hypersurface in a differentiable manifold W and β a C^∞-smooth function defined in a neighbourhood of M. We recall that β is said *flat* of order N along M in X, if

i) $\beta_{|M} = C$ (constant);
ii) $\frac{\partial^k \beta}{\partial \nu^k}(p) = 0$, for $1 \leq k \leq N$ and $\frac{\partial^{N+1}\beta}{\partial \nu^{N+1}}(p) \neq 0$, for every $p \in M$ ($\partial/\partial\nu$ normal derivative).

The normal derivative in question can be taken with respect to any Riemannian metric on M, and the property does not depend on the choice. If $x_1, \ldots, x_n$ are local coordinates on a neighbourhood V of p_0 in W chosen in such a way that $x_n|_{V\cap M} = 0$, then β is flat of order N if and only if $\beta_{|V\cap M}$ is constant, $\frac{\partial^k \beta}{\partial x_n^k}(p) = 0$, for $1 \leq k \leq N$ and $\frac{\partial^{N+1}\beta}{\partial x_n^{N+1}}(p) \neq 0$, $p \in V \cap M$.

Proposition 7.3. . *Let M be a real analytic hypersurface in a real analytic manifold W,and α a real analytic function defined in a neighbourhood of M such that α has local minimum at each point of M.*

Then there is an integer $k \geq 1$ and a real analytic subset $T \subset M$, such that α is flat of order $2k-1$ along $M \smallsetminus T$.

Proof. (Sketch). The fact being essentially local, we can assume without loss of generality that: W is a neighbourhood of zero in $\mathbb{R}^n$, $M = W \cap \{x_n = 0\}$, $\alpha \geq c$ in W and, since α has local minimum at each point of M, that $\alpha_{|M} = c$, $c \in \mathbb{R}$. Developping α in power series with respect to x_n we obtain:

$$\alpha(x', x_n) = c + \sum_{k=1}^{+\infty} \alpha_k(x') x_n^k,$$

where $\alpha_s(x')$, real analytic function of $x' = (x_1, x_2, \dots, x_k, \dots, x_{n-1})$. Let N be the smallest k such that $\alpha_k(x')$ is not identically 0. Denote

$$T = \{(x', 0) \in M : \alpha_k(x') = 0\}.$$

Then T is a nowhere dense real-analytic subset of M. Observe that since $\alpha - c \geq 0$ in W, N cannot be odd and so $N = 2k$, $k \geq 1$. It follows that α is flat of order $2k-1$ along M. □

Proposition 7.4. . *Let W be a connected real analytic manifold and M a closed real analytic hypersurface. Let $\beta : W \to [0, +\infty)$ be a non negative real analytic function on W, identically 0 on M and positive on $W \smallsetminus M$, flat of order $(2k-1)$ with $k \geq 1$, on $M \smallsetminus T$, where T is a real analytic subset of M, nowhere dense in M. Assume that M separates W into two connected open sets W^+ and W^-. Let*

$$\gamma(x) = \begin{cases} 0 & if\ x \in M \smallsetminus T \\ \beta(x)^{1/2k} & if\ x \in W^+ \\ -\beta(x)^{1/2k} & if\ x \in W^-. \end{cases}$$

Then $\gamma : W \smallsetminus T \to \mathbb{R}$ is a real-analytic function without critical points on $M \smallsetminus T$. Clearly $\beta = \gamma^{2k}$ on $W \smallsetminus T$.

Proof. Since the function β is well defined everywhere in $W \smallsetminus T$ and is, obviously, analytic on $W \smallsetminus M$ (note the assumption $\beta > 0$ on $W \smallsetminus M$), it is enough to verify its properties at points of $M \smallsetminus T$. Since they are local, assume, without loss of generality, the setting and notation of the proof of last proposition. That is $p \in M \smallsetminus T$, $M \subset \{x_n = 0\} \subset \mathbb{R}^n$,

$$\beta(x', x_n) = \sum_{s=2k}^{+\infty} x_n^s \alpha_s(x') = x_n^{2k} \rho(x', x_n),$$

for all $(x', x_n) \in V$, a small neighbourhood, where

$$\rho(x', x_n) = \sum_{l=0}^{+\infty} x_n^l \alpha_{2k+l}(x').$$

It is clear that ρ is analytic and $\rho > 0$ in V (V intersects M), and that $(\beta_{|V})(x) = x_n \rho(x)^{1/2k}$. Since $\rho(x) > 0$, $\rho(x)^{1/2k}$ is real analytic. In addition, $\partial\beta/\partial x_n(x', 0) = \rho(x', 0)^{1/2k} > 0$. □

7.2. **Local maximum sets.**

Lemma 7.5. (Removable singularities). *Let W be a complex surface and A a real analytic subset of dimension ≤ 2. Let $\chi : W \to \mathbb{R}$ be a continuous function such that $\chi_{|W \setminus A}$ is pluriharmonic.*

Then χ is pluriharmonic on W.

Proof. The problem is local so we can assume that W is an open subset of $\mathbb{C}^2$. We recall that in view of [11] if $E \subset W$ is a closed subset of 2-Hausdorff measure 0 every plurisubharmonic function on $W \setminus E$ extends to W by a (unique) plurisubharmonic function. It follows that a continuous function on W which is pluriharmonic on $W \setminus E$ is pluriharmonic. This fact reduces the proof of Lemma 7.5 to the case when A is non-singular of pure dimension 2: indeed, the singular set S of A is a semianalytic subset 2-Hausdorff measure 0 ([9, Remark 3.1, p. 27]).

Thus we assume that A is a 2-dimensional connected real analytic submanifold of W.

Let C be the subset of the complex points of A: C is a real analytic subset so either $C = A$ or it is of dimension ≤ 1. In the first case A is a complex curve hence a pluripolar set so, by [6, Theorem (5.24)] and what is preceding we obtain that χ is pluriharmonic. In the second one, away from a proper, closed real analytic subset N, A is a totally real surface, hence in an open neighbourhood V of $p \in A \setminus N$ there exist local holomorphic coordinates $z = x + iy, w = u + iv$ such that $A \cap V = \{y = v = 0\}$. In the latter, away from a proper, closed real analytic subset N, A is a totally real surface so in an open neighbourhood $V \subset W$ of $p \in A \setminus N$ exist local holomorphic coordinates $z = x + iy, w = u + iv$ such that $z(p) = w(p) = 0$, $A \cap V = \{y = v = 0\}$. Let $\rho = y^2 + v^2 + v$. If V is sufficiently small the hypersurface $\{\rho = 0\}$ is smooth, contains $A \cap V$ and the domain $V \cap \{\rho > 0\}$ is simply connected with strongly Levi concave boundary. Then $\chi_{|V \cap \{\rho > 0\}}$ as real part of a holomorphic function extends through $A \cap V$. Thus χ is pluriharmonic on $V \setminus N$ whence is pluriharmonic in V since N has 2-Hausdorff measure 0 □

Lemma 7.6. (Reflection principle for pluriharmonic functions). *Let W be domain in a complex surface X and $M \subset \overline{W} \setminus W$ a real analytic submanifold of X of dimension 3. Let $\chi : W \to \mathbb{R}$ be a non constant pluriharmonic function. Assume that*

$$\lim_{\substack{x \to M \\ x \in W}} \chi(x) = c \in \mathbb{R}.$$

Then

a) *there exist an open set $\widetilde{W}$ containing $W \cup M$ and a pluriharmonic function $\widetilde{\chi} : \widetilde{W} \to \mathbb{R}$ extending χ.*
b) *If, in addition, $\chi(x) > c$ for $x \in W$, then $\mathrm{d}\widetilde{\chi}(x) \neq 0$ for $x \in M$.*

Proof. Let us prove that M is Levi flat. Fix an open neighborhood $U \Subset X$ of a point $a \in M$ and any local defining function φ for M at a (i.e. $U \cap W = \{p \in U : \varphi(p) < 0\}$). Assume, by contradiction, that the Levi form L_α of φ at a is not vanishing when restricted to the complex tangent line to M at a i.e. $M \cap U$ is either strongly pseudoconvex or strongly pseudoconcave at a as boundary of $W \cap U$. In the first case, near a, $W \cap U$ is filled by a family $\{D_\epsilon\}_\epsilon$ of analytic discs attached to M. Since $\chi_{|D_\epsilon}$ is harmonic in D_ϵ and constant on the boundary, it is constant near a and consequently on the whole of W, χ being analytic: contradiction. The proof in the pseudoncave case is similar in view of the fact that then χ locally extends through M by a pluriharmonic function. This proves that M is Levi flat.

Since M is real analytic and Levi flat, it is locally biholomorphic to a real hyperplane so we may assume that X is the open ball $B = \{|z|^2 + |w|^2 < 1\}$, $z = x + iy, w = u + iv$, $W = \{(z, w) \in B \ : \ v > 0\}$. This reduces the proof of the lemma to the classical "reflection principle" for harmonic functions. In view of [1, Theorem 4.12], χ extends to a harmonic function $\widetilde{\chi} : B \to \mathbb{R}$, which is real analytic and pluriharmonic on W, therefore pluriharmonic on W. This shows part a).

Part b) is an application of Hopf Lemma [23, Lemma 3.8]. $\square$

Lemma 7.7. *Let X be a complex surface and Z a compact real analytic set of dimension ≤ 2. Suppose that Z has the local maximum property. Then Z is the union of finitely many compact complex curves.*

Proof. The proof is based on the following classical result of Hartogs [10]. Let U, V be domains in $\mathbb{C}$ and $Y \subset U \times V$ be a closed set, finitely sheeted over U, in the sense that

i) $\overline{Y} \cap \mathrm{b}V = \varnothing$
ii) $Y \cap (\{z\} \times \mathbb{C})$ is finite, for every $z \in U$.

Assume that Y is pseudoconcave in $U \times V$. Then Y is a complex analytic variety in $U \times V$.

By Hartogs' result, it is enough to show that every point $p \in Z$ has a neighborhood $W = U \times V$, which, (with respect to some local holomorphic coordinate system at p) satisfies i) and ii), $X \setminus Z$ being pseudoconvex (see [21, Theorem 2]).

If p is any point in a 2-dimensional connected component of the regular set of Z, and L is a complex line through p, transversal to the 2-dimensional (real) tangent plane $T_p(Z)$, then a small neighborhood of p in Z will do.

If p is not a regular point of Z, consider a coordinate neighborhood W of p with real-analytic boundary and a stratification of $W \cap Z$. By the last paragraph, the two-dimensional strata of $W \cap Z$ (which are contained in Z_{reg}) are complex varieties, and so any complex line L either contains such stratum, or intersects it at most finely many points. Since there are only finely many strata, there are finely many complex lines $L_1, \dots, L_k$, such that any other complex line L intersects every two-dimensional stratum of $W \cap Z$ at finitely many points at most.

By a similar argument, with the exception of finitely many complex lines $L_{k+1}, \dots, L_n$, every complex line L different from them intersects every one dimensional stratum of $W \cap Z$ at finitely many points at most (L either contains a real analytic arc γ, or intersect it at finitely many points). Finally, there are only finitely many zero-dimensional strata.

It follows that there is a complex line $\widetilde{L}$ through p which is not parallel to any of $\{L_1, \dots, L_n\}$, and so for any complex line L parallel to $\widetilde{L}$, $L \cap W \cap Z$ is finite.

Choose now a complex line $\bar{L}$ through p that is transversal to $\widetilde{L}$ and introduce new coordinate system (z, w) in W so that p is the origin and $\bar{L} \cap W$ and $\widetilde{L} \cap W$ are the $z-$axis and the w-axis.

Since $\widetilde{L} \cap W \cap Z$ is finite, one can choose a neighborhood V of 0 in $\mathbb{C}$, such that $(\{0\} \times bV) \cap W \cap Z = \emptyset$ and $0 \times \bar{V} \subseteq W$. Then there is a neighborhood U of 0 in $\mathbb{C}$, such that $(U \times bV) \cap W \cap Z = \emptyset$ and $\bar{U} \times \bar{V} \subseteq W$. It follows that U, V satisfy conditions i), ii), and so $(U \times V) \cap Z$ is a complex variety by Hartogs' theorem. $\square$

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