1-SMOOTH PRO-p GROUPS AND BLOCH-KATO PRO-p GROUPS

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Abstract. Let p be a prime. A pro-p group G is said to be 1-smooth if it can be endowed with a homomorphism of pro-p groups $G \to 1 + p\mathbb{Z}_p$ satisfying a formal version of Hilbert 90. By Kummer theory, maximal pro-p Galois groups of fields containing a root of 1 of order p, together with the cyclotomic character, are 1-smooth. We prove that a finitely generated p-adic analytic pro-p group is 1-smooth if, and only if, it occurs as the maximal pro-p Galois group of a field containing a root of 1 of order p. This gives a positive answer to De Clerq-Florence's "Smoothness Conjecture" — which states that the bijectivity of the norm residue homorphism (i.e., the Bloch-Kato Conjecture) follows from 1-smoothness — for the class of finitely generated p-adic analytic pro-p groups.

1. Introduction

For a field $\mathbb{K}$ let $\bar{\mathbb{K}}_s$ denote the separable closure of K, and $G_{\mathbb{K}} = \mathrm{Gal}(\bar{\mathbb{K}}_s/\mathbb{K})$ the absolute Galois group of $\mathbb{K}$. One of the main open questions in modern Galois theory is to describe absolute Galois groups of fields among profinite groups. The description of the maximal pro-p Galois group $G_{\mathbb{K}}(p)$ — i.e., the Galois group of the maximal p-extension $\mathbb{K}(p)/\mathbb{K}$ — among pro-p groups, for a given prime number p, is already a challenging task. One of the oldest known obstructions for the realization of a pro-p group as $G_{\mathbb{K}}(p)$ for some field $\mathbb{K}$ comes from the Artin-Schreier theorem (whose pro-p version is due to E. Becker, see [1]): the only non-trivial finite group which occurs as the absolute Galois group (and maximal pro-p Galois group) of a field is the cyclic group of order two.

The proof of the celebrated Bloch-Kato conjecture, by M. Rost and V. Voevodsky (with C. Weibel's "patch", see [13, 25, 30–32]), provided a description of the Galois cohomology of absolute Galois groups of fields in terms of low degree cohomology. In particular, the Norm Residue Theorem (also called the Rost-Voevodsky Theorem) implies that if $\mathbb{K}$ contains a root of 1 of order p, then $G_{\mathbb{K}}(p)$ is a *Bloch-Kato pro-p group*, i.e., the $\mathbb{Z}/p$-cohomology algebra of every closed subgroup of $G_{\mathbb{K}}(p)$ is a *quadratic* algebra. This led to the achievement of new obstructions for the realization of pro-p groups as maximal pro-p Galois groups (see, e.g., [4, 9, 19, 24]). For instance, one may recover the Artin-Schreier obstruction as consequence of the Bloch-Kato property (see, e.g., [19, p. 796]).

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A pair $\mathcal{G} = (G, \theta)$, consisting of a pro-$p$ group G together with a morphism of pro-p groups $\theta\colon G \to 1+p\mathbb{Z}_p$, is called an *oriented pro-p group* (see [24]) — here $1+p\mathbb{Z}_p$ denotes the multiplicative abelian pro-p group $\{1 + p\lambda \mid \lambda \in \mathbb{Z}_p\}$. Given a field $\mathbb{K}$ containing a primitive p-th root of unity of 1, the maximal pro-p Galois group of $\mathbb{K}$ may be considered naturally as an oriented pro-p group $\mathcal{G}_{\mathbb{K}} = (G_{\mathbb{K}}(p), \theta_{\mathbb{K},p})$, where $\theta_{\mathbb{K},p}\colon G_{\mathbb{K}}(p) \to 1 + p\mathbb{Z}_p$ is the *cyclotomic character*, which describes the action of $G_{\mathbb{K}}(p)$ on the roots of 1 of p-power order lying in $\bar{\mathbb{K}}(p)$ (see [10, § 4]).

The oriented pro-p group $\mathcal{G}_{\mathbb{K}}$ satisfies the following formal version of Hilbert 90. Given an oriented pro-p group $\mathcal{G} = (G, \theta)$, let $\mathbb{Z}_p(\theta)$ denote the continuous G-module which is isomorphic to $\mathbb{Z}_p$ as an abelian pro-p group, and endowed with the left G-action defined by $g.v = \theta(g) \cdot v$ for all $g \in G$ and $v \in \mathbb{Z}_p(\theta)$. The oriented pro-$p$ group $\mathcal{G}$ is said to be *Kummerian* if the morphism

$$H^1(G, \mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta)) \longrightarrow H^1(G, \mathbb{Z}_p(\theta)/p\mathbb{Z}_p(\theta)), \tag{1.1}$$

induced by the epimorphism of G-modules $\mathbb{Z}_p(\theta)/p^n\mathbb{Z}_p(\theta) \to \mathbb{Z}_p(\theta)/p\mathbb{Z}_p(\theta)$, is surjective for every $n \geq 1$; and moreover $\mathcal{G}$ is said to be *1-smooth* if the oriented pro-p group $\mathcal{G}_H = (H, \theta|_H)$ is Kummerian for every closed subgroup $H \subseteq G$. By Kummer theory, the oriented pro-p group $\mathcal{G}_{\mathbb{K}}$ is 1-smooth (see [6, Prop. 14.19] and [24, Thm. 1.1]).

In the paper [6] — motivated by the pursuit of an "explicit" proof of the Bloch-Kato conjecture as an alternative to the proof by Voevodsky — C. De Clerq and M. Florence introduce the 1-smoothness property. In particular, they formulate the "Smoothness Conjecture": namely, that it is possible to deduce the surjectivity part of the Block-Kato conjecture (which is known to be the "hard part" of the conjecture) from the fact that the oriented pro-p group $\mathcal{G}_{\mathbb{K}}$ arising from a field $\mathbb{K}$ containing a root of 1 of order p, is 1-smooth: in other words, they conjecture that a 1-smooth oriented pro-p group yields a *weakly Bloch-Kato* pro-p group (i.e., a pro-p group whose $\mathbb{Z}/p$-cohomology satisfies the aforementioned surjectivity feature, see Definition 3.3). For example, one has that 1-smoothness implies the Artin-Schreier obstruction (see Example 2.5).

Our goal is to prove that in the class of finitely generated *p-adic analytic pro-p groups*, 1-smoothness implies the Bloch-Kato property and the realizability as maximal pro-p Galois group.

Theorem 1.1. *Let G be a finitely generated p-adic analytic pro-p group. The following are equivalent:*

- (i) *G may be completed into a 1-smooth oriented pro-p pair $\mathcal{G} = (G, \theta)$ (with $\mathrm{Im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$, if $p = 2$);*
- (ii) *G is Bloch-Kato (and moreover $\alpha^2 = 0$ for every $\alpha \in H^1(G, \mathbb{Z}/2)$, if $p = 2$).*
- (iii) *G occurs as the maximal pro-p Galois group of a field $\mathbb{K}$ containing a primitive p-th root of 1 (and also $\sqrt{-1}$, if $p = 2$).*

(Observe that if $\mathbb{K}$ is a field containing $\sqrt{-1}$, then it is well-known that $\mathrm{Im}(\theta_{\mathbb{K},2}) \subseteq 1 + 4\mathbb{Z}_2$ and $\alpha^2 = 0$ for every $\alpha \in H^1(G_{\mathbb{K}}(2), \mathbb{Z}/2)$.)

Implication (i)⇒(ii) of Theorem 1.1 gives a positive answer to the Smoothness Conjecture for the class of finitely generated p-adic analytic pro-p groups, as a Bloch-Kato pro-p group is — quite obviously — also weakly Bloch-Kato. Thus, Theorem 1.1 provides a concrete example of a class of pro-p groups for which the (weak) Bloch-Kato property follows from 1-smoothness — other examples are free pro-p groups and Demushkin group. (After the publication of this result, the Smoothness Conjecture has been proved for the class of *right-angled Artin pro-p groups* by I. Snopce and P. Zalesskiĭ, see [28].)

In fact, analytic pro-p groups represent the "upper bound" of the class of Bloch-Kato pro-p groups (i.e., Bloch-Kato pro-p groups for which $H^2(G,\mathbb{Z}/p)$ is as large as possible), while the "lower bound" (i.e., $H^2(G,\mathbb{Z}/p)$ is as small as possible) consists of free pro-p groups and Demushkin groups. Thus, by Theorem 1.1, for the two opposite "pillars" of the class of Bloch-Kato pro-p groups, the Bloch-Kato property follows from 1-smoothness.

Moreover, the structure of torsion-free p-adic analytic Bloch-Kato pro-p groups is extremely rigid, and all such pro-p groups occur as maximal pro-p Galois groups of fields (see, e.g., [5, § 3.1–3.2]). By Theorem 1.1, this rigidity in terms of structure follows also from 1-smoothness: this suggests that 1-smoothness is a very strong and restrictive condition. We believe that a further investigation of 1-smoothness for pro-p groups may lead to the discovery of new obstructions for the structure of maximal pro-p Galois groups — and absolute Galois pro-p groups — of fields (see, e.g., [22]).

Last, but not least, it is worth mentioning that the class of p-adic analytic pro-p groups is an important class of groups to consider — besides the Bloch-Kato property —, for the role such groups play in the *p-adic Langlands program* (see, e.g., [3]).

Remark 1.2. The research carried out in this manuscript was originally made public in the preprint [20], published on arXiv in April 2019 (in particular, Theorem 1.1 was [20, Thm. 1.3]), and submitted to a refereed journal. Subsequently, we decided to change strategy, and to split the original paper: this manuscript is one of the two resulting pieces. In the meanwhile, the research on 1-smooth oriented pro-p groups went on and it lead to other results, such as the aforementioned work by Snopce and Zalesskiĭ [28], and [2, 22]. In particular, the results contained in [20] have been quoted in the subsequent works [21, 22, 27].

2. Oriented pro-p groups and Kummerianity

We work in the category of pro-p groups; by an abuse of notation, "subgroup" will always mean "closed subgroup", and sets of generators of pro-p groups, and presentations, are to be intended in the topological sense. Therefore, sets of generators of pro-p groups, and presentations, are to be intended in the topological sense. Given a pro-p group G, we denote the closed commutator subgroup of G (i.e., the closed normal subgroup generated by commutators $[g,h]=g^{-1}h^{-1}gh$, $g,h\in G$) by G'; the *Frattini subgroup* of G is denoted by $\Phi(G)$ (cf. [7, Prop. 1.13]).

Recall that $1+p\mathbb{Z}_p=\{1+p\lambda \mid \lambda\in\mathbb{Z}_p\}$ is a multiplicative abelian pro-p group. In particular, if p is odd then $1+p\mathbb{Z}_p\simeq\mathbb{Z}_p$ (the latter being considered as an additive pro-p group), and $1+p\mathbb{Z}_p$ is torsion-free; while if $p=2$ then

$$1+2\mathbb{Z}_p=\{\pm1\}\times(1+4\mathbb{Z}_2)=(\mathbb{Z}/2)\oplus\mathbb{Z}_2 \tag{2.1}$$

(the latter being considered as an additive pro-2 group).

Following [24], we call a pair $\mathcal{G}=(G,\theta)$, consisting of a pro-p group G together with a morphism of pro-p groups $\theta\colon G\to 1+p\mathbb{Z}_p$, an *oriented pro-$p$ group*, and the morphism θ is called an *orientation* of G. (In [8, 10], an oriented pro-p group is called a "cyclotomic pro-p pair" — for the motivation of the name "orientation", see the footnote at the end of p. 1885 in [24].) An orientation $\theta\colon G\to 1+p\mathbb{Z}_p$ is said to be *torsion-free* if the group $\mathrm{Im}(\theta)$ is torsion-free (cf. [10, § 2]) — namely, if $p=2$ then by (2.1) we require that $\mathrm{Im}(\theta)\subseteq 1+4\mathbb{Z}_2$.

An oriented pro-p group $\mathcal{G}=(G,\theta)$ has a distinguished continuous pro-p (left) G-module $\mathbb{Z}_p(\theta)$, which is equal to the additive group $\mathbb{Z}_p$, and it is endowed with left G-action given by

$$g\cdot z=\theta(g)\cdot z,\quad \text{for all } g\in G, z\in\mathbb{Z}_p(\theta).$$

The G-module $\mathbb{Z}_p(\theta)/p$ is a trivial G-module isomorphic to $\mathbb{Z}/p$, as $\theta(g)\equiv 1 \bmod p$ for all $g\in G$. Similarly, if $p=2$ and θ is a torsion-free orientation, then $\mathbb{Z}_2(\theta)/4$ is a trivial G-module isomorphic to $\mathbb{Z}/4$, as $\theta(g)\equiv 1 \bmod 4$ for all $g\in G$.

A morphism of oriented pro-p groups $\mathcal{G}_1\to\mathcal{G}_2$, with $\mathcal{G}_i=(G_i,\theta_i)$ for $i=1,2$, is a homomorphism of pro-p groups $\phi\colon G_1\to G_2$ such that $\theta_1=\theta_2\circ\phi$ (cf. [24, § 3, p. 1888]). In the continuation, we will use the following constructions of oriented pro-p groups. Let $\mathcal{G}=(G,\theta)$ be an oriented pro-p group.

(a) If N is a normal subgroup of G contained in $\mathrm{Ker}(\theta)$, one has the oriented pro-p group

$$\mathcal{G}/N=(G/N,\bar\theta), \tag{2.2}$$

where $\bar\theta\colon G/N\to 1+p\mathbb{Z}_p$ is the orientation such that $\bar\theta\circ\pi=\theta$, with $\pi\colon G\to G/N$ the canonical projection.

(b) If A is an abelian pro-p group (written multiplicatively), one has the oriented pro-p pair

$$A\rtimes\mathcal{G}=(A\rtimes G,\tilde\theta), \tag{2.3}$$

with action given by $gag^{-1}=a^{\theta(g)}$ for every $g\in G$, $a\in A$, where the orientation $\tilde\theta\colon A\rtimes G\to 1+p\mathbb{Z}_p$ is the composition of the canonical projection $A\rtimes G\to G$ with θ (this construction was introduced by I. Efrat in [8, § 3]).

Definition 2.1. An oriented pro-p group $\mathcal{G}=(G,\theta)$ is said to be *θ-abelian* if $\mathcal{G}\simeq A\rtimes\mathcal{G}/\mathrm{Ker}(\theta)$ for some free abelian pro-p group A.

An oriented pro-p group $\mathcal{G}=(G,\theta)$ has a distinguished subgroup: the subgroup

$$K(\mathcal{G})=\left\langle\, ghg^{-1}h^{-\theta(g)}\ \mid\ g\in G,\ h\in\mathrm{Ker}(\theta)\,\right\rangle \tag{2.4}$$

(cf. [10, § 3]). The subgroup $K(\mathcal{G})$ is normal in G, and moreover one has

$$\Phi(G)\supseteq K(\mathcal{G})\qquad\text{and}\qquad \mathrm{Ker}(\theta)\supseteq K(\mathcal{G})\supseteq\mathrm{Ker}(\theta)', \tag{2.5}$$

so that $\mathrm{Ker}(\theta)/K(\mathcal{G})$ is an abelian pro-p group. Observe that for every $g\in G$ and $h\in\mathrm{Ker}(\theta)$ one has $ghg^{-1}\equiv h^{\theta(g)}$ modulo $K(\mathcal{G})$, and hence

$$\mathcal{G}/K(\mathcal{G})\simeq\mathrm{Ker}(\theta)/K(\mathcal{G})\rtimes\mathcal{G}/\mathrm{Ker}(\theta) \tag{2.6}$$

in the sense of (2.3). Moreover, if $\mathcal{G}=(G,\theta)$ is a θ-abelian oriented pro-p group, then $K(\mathcal{G})=\{1\}$.

The following result gives a group-theoretic characterization of finitely generated Kummerian oriented pro-p groups (cf. [10, Thm. 5.6 and Thm. 7.1]).

Theorem 2.2. *Let $\mathcal{G}=(G,\theta)$ be an oriented pro-p group, with G finitely generated and $\theta\colon G\to 1+p\mathbb{Z}_p$ a torsion-free orientation. The following are equivalent.*

(i) *$\mathcal{G}$ is Kummerian.*
(ii) $\mathrm{Ker}(\theta)/K(\mathcal{G})$ *is a free abelian pro-p group.*
(iii) $\mathcal{G}/K(\mathcal{G}) = (G/K(\mathcal{G}), \bar{\theta})$ *is a $\bar{\theta}$-abelian oriented pro-p group.*

In particular, by (2.6) and Theorem 2.2, a finitely generated oriented pro-p group $\mathcal{G} = (G, \theta)$, with θ a torsion-free orientation and $K(\mathcal{G}) = \{1\}$, is Kummerian if, and only if, $\mathcal{G}$ is θ-abelian.

Remark 2.3. If $\mathcal{G} = (G, \mathbf{1})$ is an oriented pro-p group with $\mathbf{1}\colon G \to 1 + p\mathbb{Z}_p$ the orientation constantly equal to 1, then $K(\mathcal{G}) = G'$. By Theorem 2.2, the oriented pro-p group $\mathcal{G}$ is Kummerian if, and only if, the abelianization $G/G' = \mathrm{Ker}(\mathbf{1})/K(\mathcal{G})$ of G is a free abelian pro-p group. In particular, if $\mathcal{G}$ is also 1-smooth, then the abelianization of every finitely generated subgroup of G is a free abelian pro-p group.

Example 2.4. (a) Let G be a free pro-p group. Then the oriented pro-p group $\mathcal{G} = (G, \theta)$ is 1-smooth for any orientation θ (cf. [24, § 2.2]).
(b) Let G be a Demushkin group (cf., e.g., [17, Def. 3.9.9]). Then there exists one — and only one — orientation $\theta\colon G \to 1 + p\mathbb{Z}_p$ which completes G into a 1-smooth oriented pro-p group $\mathcal{G} = (G, \theta)$ (cf. [14, Thm. 4] and [24, Cor. 5.7]).

From the following example (cf. [10, Ex. 3.5]), one may recover the Artin-Schreier obstruction as a consequence of 1-smoothness.

Example 2.5. For p odd, let G be a finite p group, and let $\mathcal{G} = (G, \theta)$ be an oriented pro-p group. Then $\theta \equiv \mathbf{1}$, as $1+p\mathbb{Z}_p$ is torsion-free, and thus $\mathrm{Ker}(\theta) = G$ and $K(\mathcal{G}) = \{1\}$ Hence $\mathcal{G}$ is not Kummerian.

Similarly, for $p = 2$ let G a group of order 4, and let $\mathcal{G} = (G, \theta)$ be an oriented pro-2 group. By (2.1) $\mathrm{Ker}(\theta) \neq \{1\}$, while $K(\mathcal{G}) = \{1\}$ (cf. [10, Ex. 3.5–(4)–(5)]). Hence $\mathcal{G}$ is not Kummerian. By contrast, the oriented pro-2 group $\mathcal{G} = (G, \theta)$ with $G \simeq \mathbb{Z}/2$ and $\mathrm{Im}(\theta) = \{\pm 1\}$ is Kummerian (and thus 1-smooth).

Remark 2.6. In the original definition given in [10, Def. 3.4], an oriented pro-p group $\mathcal{G} = (G, \theta)$ is said to be Kummerian if the quotient $\mathrm{Ker}(\theta)/K(\mathcal{G})$ is torsion-free. By Theorem 2.2 this original definition and the "cohomological" definition given in the Introduction — i.e., the morphism (1.1) is surjective for every $n \geq 1$ —, which we use throughout the paper, are equivalent if G is finitely generated. In [23, Thm. 1.2] it is shown that these two definitions of Kummerianity are equivalent also in the non-finitely generated case.

Finally, note that in [24], the orientation θ of a 1-smooth oriented pro-p group $\mathcal{G} = (G, \theta)$ is said to be *1-cyclotomic*.

3. Bloch-Kato pro-p groups and the Smoothness conjecture

Here all graded algebras $\mathbf{A}_\bullet = \bigoplus_{n\in\mathbb{Z}} A_n$ over a field $\mathbb{F}$ are assumed to be locally finite-dimensional with $A_n = 0$ for $n < 0$ and $A_0 = \mathbb{F}$. A graded algebra $\mathbf{A}_\bullet$ is called a *quadratic algebra* if it is 1-generated — i.e., every element is a combination of products of elements of degree 1 —, and its relations are generated by homogeneous relations of degree 2 (cf. [18, Ch. 1, § 2]). In other words, one has an isomorphism of graded algebras $\mathbf{T}_\bullet(A_1)/I \xrightarrow{\sim} \mathbf{A}_\bullet$, where $\mathbf{T}_\bullet(A_1) = \bigoplus_{n\geq 0} A_1^{\otimes n}$ is the tensor $\mathbb{F}$-algebra generated by A_1, and I is a two-sided ideal of $\mathbf{T}_\bullet(A_1)$ generated as a two-sided ideal by a subset of $A_1^{\otimes}$.

Example 3.1. Let V be a finite-dimensional vector space over $\mathbb{Z}/p$.

(a) The tensor $\mathbb{Z}/p$-algebra $\mathbf{T}_\bullet(V)$ is quadratic.

(b) The exterior algebra $\mathbf{\Lambda}_\bullet(V)$ is quadratic, as $\mathbf{T}_\bullet(V)/I \simeq \mathbf{\Lambda}_\bullet(V)$, with I the two-sided ideal generated by $\{v \otimes v \mid v \in V\} \subseteq V^{\otimes 2}$.

Remark 3.2. If $\mathbf{A}_\bullet = \bigoplus_{n\geq 0} A_n$ is a quadratic algebra such that $a^2 = 0$ for every $a \in A_1$, then one has an epimorphism of quadratic algebras $\mathbf{\Lambda}_\bullet(A_1) \twoheadrightarrow \mathbf{A}_\bullet$.

Definition 3.3. Let G be a pro-p group, and let $n \geq 1$. Cohomology classes in the image of the natural cup-product

$$H^1(G, \mathbb{Z}/p) \times \ldots \times H^1(G, \mathbb{Z}/p) \xrightarrow{\cup} H^n(G, \mathbb{Z}/p)$$

are called *symbols* (relative to $\mathbb{Z}/p$).

(i) If for every open subgroup $U \subseteq G$ every element $\alpha \in H^n(U, \mathbb{Z}/p)$, for every $n \geq 1$, can be written as

$$\alpha = \mathrm{cor}^n_{V_1,U}(\alpha_1) + \ldots + \mathrm{cor}^n_{V_r,U}(\alpha_r), \quad r \geq 1,$$

where $\alpha_i \in H^n(V_i, \mathbb{Z}/p)$ is a symbol and

$$\mathrm{cor}^n_{V_i,U} \colon H^n(V_i, \mathbb{Z}/p) \longrightarrow H^n(U, \mathbb{Z}/p)$$

is the *corestriction map* (cf. [17, Ch. I, § 5]), for some open subgroups $V_i \subseteq U$, then G is called a *weakly Bloch-Kato pro-p group* (cf. [6, Def. 14.23]).

(ii) If for every subgroup $H \subseteq G$, the $\mathbb{Z}/p$-cohomology algebra

$$\mathbf{H}^\bullet(H, \mathbb{Z}/p) := \coprod_{n\geq 0} H^n(H, \mathbb{Z}/p),$$

endowed with the cup-product, is a quadratic algebra over $\mathbb{Z}/p$, then G is called a *Bloch-Kato pro-p group* (cf. [19]).

Clearly, a Bloch-Kato pro-p group is also weakly Bloch-Kato.

Examples 3.4. (a) A free pro-p group G is Bloch-Kato, as $H^n(G, \mathbb{Z}/p) = 0$ for $n \geq 2$, and also every subgroup $H \subseteq G$ is a free pro-p group (cf. [26, Ch. I, § 4.2, Cor. 2–3]).

(b) A Demushkin group is Bloch-Kato (cf. [24, Thm. 6.8]). In particular, every open subgroup of G is again a Demushkin group (cf. [17, Thm. 3.9.15]), while every closed non-open subgroup of G is a free pro-p group (cf. [26, Ch. I, § 4.5, Ex. 5–(b)])

Let $\mathbb{K}$ be a field containing a primitive p-th root of 1. By the Norm Residue Theorem, the $\mathbb{Z}/p$-cohomology algebra $\mathbf{H}^\bullet(G_{\mathbb{K}}, \mathbb{Z}/p)$ of the absolute Galois group $G_{\mathbb{K}}$ is quadratic. By the Hochschild-Serre exact sequence associated to the short exact sequence of profinite groups

$$\{1\} \longrightarrow \mathrm{Gal}(\bar{\mathbb{K}}_s/\mathbb{K}(p)) \longrightarrow G_{\mathbb{K}} \longrightarrow G_{\mathbb{K}}(p) \longrightarrow \{1\}\ ,$$

one has an isomorphism of graded $\mathbb{Z}/p$ algebras $\mathbf{H}^\bullet(G_{\mathbb{K}}(p), \mathbb{Z}/p) \simeq \mathbf{H}^\bullet(G_{\mathbb{K}}, \mathbb{Z}/p)$ (cf., e.g., [19, § 2]), so that also $\mathbf{H}^\bullet(G_{\mathbb{K}}(p), \mathbb{Z}/p)$ is quadratic. Thus, $G_{\mathbb{K}}(p)$ is a Bloch-Kato pro-p group.

The following is the pro-p version of the Smoothness Conjecture formulated by C. De Clerq and M. Florence (cf. [6, Conj. 14.25]).

Conjecture 3.5. *Let $\mathcal{G} = (G, \theta)$ be a 1-smooth oriented pro-p group, with θ a torsion-free orientation. Then G is a weakly Bloch-Kato pro-p group.*

A positive answer to the Smoothness Conjecture would provide a new proof of the "1-generation half" of the Bloch-Kato conjecture (cf. [6, § 1.1]), alternative to the proof by Rost and Voevodsky. Indeed, by Milnor K-theory one has that the weak Bloch-Kato property of the maximal pro-p group $G_{\mathbb{K}}(p)$ of a field $\mathbb{K}$, containing a primitive p-th root of 1, implies that the algebra $\mathbf{H}^\bullet(G, \mathbb{Z}/p)$ is 1-generated (cf. [6, Rem. 14.26]).

4. Locally uniform pro-p groups

We recall the following definition.

Definition 4.1. Let G be a pro-p group.

(a) G is *powerful* if G' is contained in the subgroup of G generated by $\{g^{p^\epsilon} \mid g \in G\}$, where $\epsilon = 2$ if $p = 2$, $\epsilon = 1$ otherwise.
(b) If G is finitely generated, then G is *uniformly powerful* (or simply *uniform*) if G is powerful and torsion-free.
(c) G is *locally uniform* if every finitely generated subgroup of G is uniform.

(For a detailed account on powerful and uniform pro-p groups and their properties we refer to [7, Ch. 3–4].)

By Lazard's work [15], if G is a uniform pro-p group one has an isomorphism of quadratic $\mathbb{Z}/p$-algebras

$$\mathbf{\Lambda}_\bullet\left(H^1(G, \mathbb{Z}/p)\right) \stackrel{\sim}{\longrightarrow} \mathbf{H}^\bullet(G, \mathbb{Z}/p) \tag{4.1}$$

(cf., e.g., [29, Thm. 5.1.5]). Therefore, a finitely generated locally uniform pro-p group is Bloch-Kato. Moreover, for locally uniform pro-p groups one has the following (cf. [19, Thm. A] and [5, Prop. 3.5]).

Proposition 4.2. *A pro-p group G is locally uniform if, and only if, there exists a torsion-free orientation $\theta \colon G \to 1 + p\mathbb{Z}_p$ such that the oriented pro-p group $\mathcal{G} = (G, \theta)$ is θ-abelian.*

Consequently, a locally uniform pro-p group may complete into a Kummerian oriented pro-p group, as a θ-abelian oriented pro-p group is Kummerian by Theorem 2.2. In fact, locally uniform pro-p groups are the only uniform pro-p groups which can do this.

Proposition 4.3. *Let G be a uniform pro-p group. Then G may complete into a Kummerian oriented pro-p group $\mathcal{G} = (G, \theta)$ if, and only if, G is locally uniform.*

Proof. By Proposition 4.2, it is enough to prove the following implication: if G may complete into a Kummerian oriented pro-p group $\mathcal{G} = (G, \theta)$, then G is locally uniform.

If $\mathcal{G}$ is Kummerian, then by Theorem 2.2 the oriented pro-p group $\mathcal{G}/K(\mathcal{G}) = (G/K(\mathcal{G}), \bar{\theta})$ is $\bar{\theta}$-abelian, and thus $G/K(\mathcal{G})$ is locally uniform by Proposition 4.2. So, both G and $G/K(\mathcal{G})$ are uniform, and by (4.1) one has

$$\begin{aligned} H^2(G, \mathbb{Z}/p) &\simeq \Lambda_2\left(H^1(G, \mathbb{Z}/p)\right), \\ H^2(G/K(\mathcal{G}), \mathbb{Z}/p) &\simeq \Lambda_2\left(H^1(G/K(\mathcal{G}), \mathbb{Z}/p)\right). \end{aligned} \tag{4.2}$$

On the other hand, the canonical projection $G \to G/K(\mathcal{G})$ induces maps

$$\inf^n_{G,K(\mathcal{G})} \colon H^n(G/K(\mathcal{G}), \mathbb{Z}/p) \longrightarrow H^n(G, \mathbb{Z}/p)$$

for every $n \geq 1$ such that

$$\inf^1_{G,K(\mathcal{G})}(\alpha) \cup \inf^1_{G,K(\mathcal{G})}(\alpha') = \inf^2_{G,K(\mathcal{G})}(\alpha \cup \alpha')$$

for every $\alpha, \alpha' \in H^1(G/K(\mathcal{G}), \mathbb{Z}/p)$ (cf. [17, Prop. 1.5.3]). Moreover, $\inf^1_{G,K(\mathcal{G})}$ is an isomorphism, as $K(\mathcal{G}) \subseteq \Phi(G)$ (cf. [26, Ch. I, § 4.2, Remark]). Therefore, also $\inf^2_{G,K(\mathcal{G})}$ is an isomorphism, and by the 5-term exact sequence in cohomology

$$0 \longrightarrow H^1(G/K(\mathcal{G}), \mathbb{Z}/p) \xrightarrow{\inf^1_{G,K(\mathcal{G})}} H^1(G, \mathbb{Z}/p) \xrightarrow{\mathrm{res}^1_{G,K(\mathcal{G})}} H^1(K(\mathcal{G}), \mathbb{Z}/p)^G \xrightarrow{\mathrm{trg}} H^2(G/K(\mathcal{G}), \mathbb{Z}/p) \xrightarrow{\inf^2_{G,K(\mathcal{G})}} H^2(G, \mathbb{Z}/p)$$

(cf. [17, Prop. 1.6.7]) one has $H^1(K(\mathcal{G}), \mathbb{Z}/p)^G = 0$. Since G is a pro-p group and $H^1(K(\mathcal{G}), \mathbb{Z}/p)$ is a p-elementary abelian group, this implies that $H^1(K(\mathcal{G}), \mathbb{Z}/p) = 0$, i.e., $K(\mathcal{G})$ is trivial, and $G \simeq G/K(\mathcal{G})$ is locally uniform. □

Remark 4.4. It is well-known that a finitely generated locally uniform pro-p group may be realized as the maximal pro-p Galois group of a field (cf. [8, Rem. 3.4]). For example, let ℓ is a prime number, $\ell \neq p$, and for $k \geq 1$ set $\mathbb{F} = \mathbb{F}_\ell(\xi)$, with $\xi \in (\bar{\mathbb{F}}_\ell)_s$ a root of 1 of order p^k. Let $\mathbb{K} = \mathbb{F}_{\ell^n}((X_1, \ldots, X_d))$ be the field of Laurent series in the indeterminates $X_1, \ldots, X_d$, $d \geq 1$, and with coefficients in $\mathbb{F}$. Then

$$\mathcal{G}_{\mathbb{K}} = (G_{\mathbb{K}}(p), \theta_{\mathbb{K},p}) \simeq \mathbb{Z}_p^d \rtimes \mathcal{G}_{\mathbb{K}}/\operatorname{Ker}(\theta_{\mathbb{K},p}),$$

and $\operatorname{Im}(\theta_{\mathbb{K},p}) = 1 + p^k\mathbb{Z}_p$ (cf. [19, Ex. 4.10]).

5. p-adic analytic pro-p groups

For a pro-p group G let $\mathrm{d}(G)$ denote the minimal number of generators of G, i.e., $\mathrm{d}(G) = \dim(G/\Phi(G))$, and let the *rank* of G be the supremum of all $\mathrm{d}(H)$ with H running through all closed subgroups of G (cf. [7, § 3.2]). Then every finitely generated powerful pro-p group has finite rank (cf. [7, Thm. 3.13]).

The following result defines finitely generated *p-adic analytic pro-p groups* (cf. [7, Thm. 8.32 and Cor. 8.33]).

Theorem 5.1. *Let G be a finitely generated pro-p group. The following are equivalent:*
- (i) *G is a p-adic analytic manifold and the map $(x, y) \mapsto x^{-1}y$ is analytic;*
- (ii) *G contains an open subgroup which is uniformly powerful;*
- (iii) *G has finite rank.*

A finitely generated pro-p groups satisfying the above properties is a p-adic analytic pro-p group.

Hence, a subgroup of a finitely generated p-adic analytic pro-p group has finite rank, and thus is p-adic analytic. Moreover, if N is a normal subgroup of a p-adic analytic pro-p group G, then also G/N has finite rank, and thus it is p-adic analytic (cf. [7, Exercise 3.1]).

The *dimension* $\dim(G)$ of a p-adic analytic pro-p group G is the minimal number of generators $\mathrm{d}(U)$ of a uniform subgroup U of G (by [7, Lemma 4.6] $\dim(G)$ does not depend on the choice of the uniform subgroup). One has the following (cf. [7, Thm. 4.8]).

Proposition 5.2. *Let G be a p-adic analytic pro-p group, and let $N \subseteq G$ be a normal subgroup of G. Then*

$$\dim(G) = \dim(N) + \dim(G/N). \tag{5.1}$$

Example 5.3. (a) A finitely generated abelian pro-p group G is p-adic analytic. In particular, if $G \simeq \mathbb{Z}_p^n \oplus A$, with A a finite abelian p-group, then $\dim(G) = n$.
(b) If G is a finitely generated locally powerful pro-p group, then G is p-adic analytic by Theorem 5.1, and $\dim(G) = \mathrm{d}(G)$.

Example 5.4. Let p be a odd prime. The *Heisenberg group over* $\mathbb{Z}_p$ is the group G of upper uni-triangular matrices over $\mathbb{Z}_p$, and it is a torsion-free p-adic analytic pro-p group of dimension 3 (cf. [11, Thm. 7.4–(2)]). In particular, G has a presentation

$$G = \langle\, x, y, z \;\mid\; [x, y] = z,\ [x, z] = [y.z] = 1 \,\rangle,$$

and one has $G/G' \simeq \mathbb{Z}_p^2$ and $G' = \langle z\rangle \simeq \mathbb{Z}_p$. Thus, the oriented pro-p group $\mathcal{G}_{\mathbf{1}} = (G, \mathbf{1})$ is Kummerian by Remark 2.3. Set $t = x^p$, and let U be the subgroup of G generated by t, y, z. Then

$$U = \langle\, t, y, z \;\mid\; [t, y] = z^p,\ [t, z] = [y, z] = 1 \,\rangle$$

(cf. [12, Ex. 7.2]). Hence, U is uniform, and consequently $\dim(U) = \mathrm{d}(U) = 3$. Yet, U is not locally uniform, and therefore U cannot complete into a Kummerian oriented pro-p group by Proposition 4.3. Altogether, G cannot complete into a 1-smooth oriented pro-p group.

Proposition 5.5. *Let G be a finitely generated p-adic analytic pro-p group, and suppose that the oriented pro-p group $\mathcal{G} = (G, \mathbf{1})$, with $\mathbf{1}\colon G \to 1+p\mathbb{Z}_p$ the orientation constantly equal to 1, is 1-smooth. Then G is a free abelian pro-p group.*

Proof. Since G is p-adic analytic, every subgroup of G is finitely generated by Theorem 5.1. Thus, by Remark 2.3 every subgorup of G has torsion-free abelianization, i.e., G is an *absolutely torsion-free* pro-p group (absolutely torsion free pro-p groups were introduced by T. Würfel in [33]).

Let $G^{(n)}$, $n \geq 1$, denote the derived series of G, i.e., $G^{(1)} = G$ and $G^{(n+1)} = [G^{(n)}, G^{(n)}]$. Since G is a finitely generated p-adic analytic pro-p group, also the subgroups $G^{(n)}$ and the quotients $G^{(n)}/(G^{(n)})' = G^{(n)}/G^{(n+1)}$ are finitely generated p-adic analytic pro-p groups. Moreover, since G is absolutely torsion-free, one has

$$G^{(n)}/G^{(n+1)} = G^{(n)}/(G^{(n)})' \simeq \mathbb{Z}_p^{\mathrm{d}(G^{(n)})} \qquad \text{for all } n \geq 1. \tag{5.2}$$

Consequently, $\dim(G^{(n)}/G^{(n+1)}) = \mathrm{d}(G^{(n)})$. From Proposition 5.2 and from (5.2), one deduces

$$\dim(G^{(n+1)}) = \dim(G^{(n)}) - \mathrm{d}(G^{(n)}). \tag{5.3}$$

Since $\dim(G)$ is finite, one has $\dim(G^{(n)}) = 0$ for some n. Again by Proposition 5.2, this implies that $\dim(G^{(n)}/(G^{(n)})') = \mathrm{d}(G^{(n)}) = 0$, i.e., $G^{(n)} = \{1\}$. This proves that G is a solvable pro-p group. By [33, Prop. 2], an absolutely torsion-free solvable pro-p group is a free abelian pro-p group, and this concludes the proof. □

Proposition 5.6. *Let $\mathcal{G} = (G, \theta)$ be a 1-smooth oriented pro-p group with θ a torsion-free orientation. If $\mathrm{Ker}(\theta)$ is abelian, then $\mathcal{G}$ is θ-abelian.*

Proof. If the orientation θ is constantly equal to 1, then $\mathrm{Ker}(\theta) = G$. Thus, by Remark 2.3 $G = G/G'$ is a free abelian pro-p group, so that $\mathcal{G}$ is θ-abelian.

Suppose now that $\theta \not\equiv \mathbf{1}$. We assume first that $p \neq 2$. Pick two arbitrary elements $x, y \in G$ such that $\theta(x) \neq 1$ and $y \in \mathrm{Ker}(\theta)$, and put $z = [x, y]$ and $t = y^p$. Clearly, $z, t \in \mathrm{Ker}(\theta)$. Since $z, y \in \mathrm{Ker}(\theta)$, which is abelian by hypothesis, one has $z^y = z$, and hence commutator calculus yields

$$[x, t] = [x, y^p] = z \cdot z^y \cdots z^{y^{p-1}} = z^p. \tag{5.4}$$

Let H be the subgroup of G generated by x, y, and let U be the subgroup of H generated by x, t, z. Then the oriented pro-p groups $\mathcal{G}_H = (H, \theta|_H)$ and $\mathcal{G}_U = (U, \theta|_U)$ are 1-smooth.

Put $\lambda = 1 - \theta(x)^{-1}$. Then $0 \neq \lambda \in p\mathbb{Z}_p$, as $1 \neq \theta(x)^{-1} \in 1 + p\mathbb{Z}_p$. By definition, $[x, t] \cdot t^{-\lambda} \in K(\mathcal{G}_U)$. Since t and z commute, from (5.4) one deduces

$$\left(zt^{-\lambda/p}\right)^p = z^p t^{-\frac{\lambda}{p}p} = z^p t^{-\lambda} = [x, t]t^{-\lambda} \in K(\mathcal{G}_U). \tag{5.5}$$

Moreover, $zt^{-\lambda/p} \in \mathrm{Ker}(\theta|_U)$. Since $\mathcal{G}_U$ is 1-smooth (and thus Kummerian), by Theorem 2.2 the quotient $\mathrm{Ker}(\theta|_U)/K(\mathcal{G}_U)$ is a free abelian pro-p group, and therefore (5.5) implies that also $zt^{-\lambda/p}$ is an element of $K(\mathcal{G}_U)$.

Since $K(\mathcal{G}_U) \subseteq \Phi(U)$, one has $z \equiv t^{\lambda/p} \bmod \Phi(U)$. Then by [7, Prop. 1.9], U is generated by x and t. Since $[x, t] \in U^p$ by (5.4), the pro-p group U is powerful — and hence uniformly powerful, as it is torsion-free (cf. Example 2.5). Therefore, $\mathcal{G}_U$ is θ_U-abelian by Proposition 4.3. In particular, $K(\mathcal{G}_U) = \{1\}$ by (2.6) and Theorem 2.2, and thus

$$[x, y] = z = t^{\lambda/p} = y^{1-\theta(x)^{-1}}. \tag{5.6}$$

Since $\mathrm{Ker}(\theta)$ is abelian by hypothesis, and since $x \in G \smallsetminus \mathrm{Ker}(\theta)$ and $y \in \mathrm{Ker}(\theta)$ were arbitrarily chosen, (5.6) implies that $\mathcal{G} \simeq \mathrm{Ker}(\theta) \rtimes \mathcal{G}/\mathrm{Ker}(\theta)$ in the sense of (2.3). Since $\mathrm{Ker}(\theta)$ is torsion-free (cf. Example 2.5), $\mathcal{G}$ is θ-abelian.

Finally, assume that $\theta \not\equiv \mathbf{1}$ and $p = 2$. Since $\mathcal{G}$ is torsion-free, $\mathrm{Im}(\theta) \subseteq 1 + 4\mathbb{Z}_2$, and the above argument works verbatim if one replaces p with 4: indeed, one has $0 \neq \lambda \in 4\mathbb{Z}_2$, as $1 \neq \theta(x)^{-1} \in 1 + 4\mathbb{Z}_2$, and $[x, t] \in U^4$, so that the pro-2 group U is powerful also in this case. Hence, $\mathcal{G}$ is a θ-abelian oriented pro-2 group. □

Theorem 5.7. *Let $\mathcal{G} = (G, \theta)$ be an oriented pro-p group with G a finitely generated p-adic analytic pro-p group and θ a torsion-free orientation. If $\mathcal{G}$ is 1-smooth, then it is θ-abelian.*

Proof. Since G is p-adic analytic, also $\mathrm{Ker}(\theta)$ is p-adic analytic. Since the oriented pro-p group $\mathcal{G}_{\mathrm{Ker}(\theta)} = (\mathrm{Ker}(\theta), \mathbf{1})$ is 1-smooth, Proposition 5.5 implies that $\mathrm{Ker}(\theta)$ is a free abelian pro-p group. Thus, Proposition 5.6 implies the claim. □

Let $p = 2$, and let G be a pro-2 group. Also, let $\mathbb{Z}/4$ be a trivial G-module. The short exact sequence of trivial G-modules

$$0 \longrightarrow \mathbb{Z}/2 \xrightarrow{2\cdot} \mathbb{Z}/4 \xrightarrow{\pi} \mathbb{Z}/2 \longrightarrow 0$$

induces an exact sequence in cohomology

$$\begin{array}{l} H^1(G,\mathbb{Z}/2) \longrightarrow H^1(G,\mathbb{Z}/4) \xrightarrow{\pi^*} H^1(G,\mathbb{Z}/2) \xrightarrow{\mathfrak{b}} \\ H^2(G,\mathbb{Z}/2) \longrightarrow H^2(G,\mathbb{Z}/4) \longrightarrow \cdots \end{array} \tag{5.7}$$

and the connecting homomorphism $\mathfrak{b}$ is called the *Bockstein morphism*. Clearly, the map $\mathfrak{b}$ is trivial if, and only if, the map $\pi^*\colon H^1(G,\mathbb{Z}/4) \to H^1(G,\mathbb{Z}/2)$ is surjective. Moreover, the map $\mathfrak{b}$ is trivial if, and only if $\alpha^2 = 0$ for every $\alpha \in H^1(G,\mathbb{Z}/2)$ (cf. [9, Lemma 2.4]).

Remark 5.8. Set $p = 2$.

(i) Let $\mathbb{K}$ be a field containing $\sqrt{-1}$. Then $\mathrm{Im}(\theta_{\mathbb{K},2}) \subseteq 1 + 4\mathbb{Z}_2$, and $\mathbb{Z}_2(\theta_{\mathbb{K},2})/4$ is isomorphic to $\mathbb{Z}/4$ as a (trivial) $G_{\mathbb{K}}(2)$-module. Since the oriented pro-2 group $\mathcal{G}_{\mathbb{K}} = (G_{\mathbb{K}}(2), \theta_{\mathbb{K},2})$ is Kummerian, the map

$$\pi^*\colon H^1(G_{\mathbb{K}}(2),\mathbb{Z}/4) \longrightarrow H^1(G_{\mathbb{K}}(2),\mathbb{Z}/2)$$

is surjective, and thus $\mathfrak{b}$ is trivial.

(ii) Let G be a pro-2 group. If $\mathbf{H}^\bullet(G,\mathbb{Z}/2)$ is a quadratic $\mathbb{Z}/2$-algebra and the Bockstein morphism $\mathfrak{b}$ is trivial, then by Remark 3.2 one has an epimorphism of quadratic $\mathbb{Z}/2$-algebras

$$\mathbf{\Lambda}_\bullet\left(H^1(G,\mathbb{Z}/2)\right) \longrightarrow\!\!\!\!\!\rightarrow \mathbf{H}^\bullet(G,\mathbb{Z}/2)\ .$$

Hence, $\mathrm{cd}(G) \leq \dim(H^1(G,\mathbb{Z}/2))$ (here $\mathrm{cd}(G)$ denotes the cohomological dimension, cf. [17, Def. 3.3.1]). Consequently, G is torsion-free, as a pro-p group with non-trivial torsion has infinite cohomological dimension.

Corollary 5.9. *Let G be a finitely generated p-adic analytic pro-p group. The following are equivalent.*

(i) *G may be completed into a 1-smooth oriented pro-p group $\mathcal{G} = (G,\theta)$ with θ a torsion-free orientation.*

(ii) *G is a Bloch-Kato pro-p group, and the Bockstein morphism $\mathfrak{b}$ is trivial if $p = 2$.*

(iii) *G occurs as the maximal pro-p Galois group of a field $\mathbb{K}$ containing a primitive p-th root of 1 (and also $\sqrt{-1}$ if $p = 2$).*

Proof. Let G be a finitely generated p-adic analytic pro-p group. First, we show that each of the three conditions implies that G may be completed into a θ-abelian oriented pro-p group $\mathcal{G} = (G,\theta)$ with θ a torsion-free orientation. Then, we show that if $\mathcal{G} = (G,\theta)$ is a θ-abelian oriented pro-p group with θ torsion-free, then all three conditions (i), (ii), (iii) hold.

If G may be completed into a 1-smooth oriented pro-p group $\mathcal{G} = (G,\theta)$ with θ a torsion-free orientation, then $\mathcal{G}$ is θ-abelian by Theorem 5.7. On the other hand, if G is a Bloch-Kato pro-p group (satisfying the further condition $\mathfrak{b} \equiv \mathbf{0}$ if $p = 2$), then G may be completed into a θ-abelian oriented pro-p group $\mathcal{G} = (G,\theta)$ by [19, Thm. 4.6] if $p \neq 2$ and by [19, Thm. 4.11] if $p = 2$ (note that in this case G is torsion-free by Remark 5.8–(ii)). Moreover, if $G \simeq G_{\mathbb{K}}(p)$ for some field containing a primitive p-th root of 1 (and $\sqrt{-1}$ if $p=2$) then G is a Bloch-Kato pro-p group by the Norm Residue Theorem (and by Remark 5.8–(i) $\mathfrak{b}\equiv\mathbf{0}$ if $p=2$), so that (iii) implies (ii).

Conversely, if $\mathcal{G}=(G,\theta)$ is a θ-abelian oriented pro-p group with θ torsion-free, then G is a finitely generated locally uniform pro-p group by Proposition 4.2. Therefore: (i) for every subgroup H of G, the oriented pro-p group $\mathcal{G}_H=(H,\theta|_H)$ is Kummerian by Theorem 2.2, and thus $\mathcal{G}$ is 1-smooth; (ii) G is a Bloch-Kato pro-p group by (4.1) — and moreover $\mathfrak{b}\equiv\mathbf{0}$ as $H^2(G,\mathbb{Z}/2)\simeq\Lambda_2(H^1(G,\mathbb{Z}/2))$, if $p=2$; (iii) G occurs as the maximal pro-p Galois group of a field containing a primitive p-th root of 1 by Remark 4.4. □

Corollary 5.9 implies Theorem 1.1. As mentioned in the Introduction, this result is particularly relevant because p-adic analytic Bloch-Kato pro-p groups are the "upper bound" of the class of Bloch-Kato pro-p groups, in the following sense: if a finitely generated (non-trivial) pro-p group G is Bloch-Kato, then by [19, Prop. 4.1] for the cohomological dimension $\mathrm{cd}(G)$ and the number of defining relations $\mathrm{r}(G)$ — the latter being equal to $\dim H^2(G,\mathbb{Z}/p)$ (cf. [26, Ch. I, § 4.3]) — one has bounds

$$1\leq \mathrm{cd}(G)\leq \mathrm{d}(G)\qquad\text{and}\qquad 0\leq \mathrm{r}(G)\leq\binom{\mathrm{d}(G)}{2}.$$

The lower bounds occur if G is a free pro-p group (and thus G is 1-smooth, cf. Example 2.4–(a)). The upper bounds occur when G is p-adic analytic. In particular, if G is a finitely generated Bloch-Kato pro-p group (satisfying $\mathfrak{b}\equiv\mathbf{0}$, if $p=2$), the following three conditions are equivalent: (i) $\mathrm{cd}(G)=\mathrm{d}(G)$; (ii) $\mathrm{r}(G)=\binom{\mathrm{d}(G)}{2}$; (iii) G is p-adic analytic (cf. [19, Cor. 4.8]).

We conclude with the following remark, which states two open questions on 1-smooth oriented pro-p groups.

Remark 5.10. (i) Bloch-Kato pro-p groups satisfy the following *Tits' alternative*: if a Bloch-Kato pro-p group G is not locally uniform, then it contains a non-abelian free subgroup (cf. [19, Thm. B]). In [21], we conjecture that 1-smooth oriented pro-p groups satisfy the same alternative: namely, if a 1-smooth oriented pro-p group $\mathcal{G}=(G,\theta)$ is not θ-abelian, then G contains a non-abelian free subgroup.

(ii) Torsion-free p-adic analytic pro-p groups G are *Poincaré duality pro-p groups* of cohomological dimension $\mathrm{cd}(G)=\dim(G)$ (cf. [29, § 5]). On the opposite side there are Poincaré duality pro-p groups of cohomological dimension $\mathrm{cd}(G)=2$, namely, infinite Demushkin groups, which are both 1-smooth and Bloch-Kato by Examples 2.4–(b) and 3.4–(b). This raises the following sub-question of Conjecture 3.5: are 1-smooth Poincaré duality pro-p groups (weakly) Bloch-Kato?

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