Weak regularization by degenerate Lévy noise and its applications

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1 Stime di Schauder per equazioni stabili degeneri di tipo Kolmogorov

Presentiamo in questo capitolo stime globali di tipo Schauder per una catena di equazioni integro-differenziali parziali (EIDP) controllata da un operatore degenere stabile di tipo Ornstein-Uhlenbeck che può essere perturbato da una componente deterministica, quando i coefficienti appartengono a spazi di Hölder anisotropici adatti. Il nostro metodo segue un approccio perturbativo basato su espansioni di parametrici progressive e, a causa delle proprietà regolarizzanti deboli sulle componenti degeneri e alcuni limiti di integrabilità collegati all'indice di stabilità, sfrutta anche risultati di dualità tra Spazi di Besov. In particolare, il nostro metodo si applica anche ad alcuni casi super-critici. Grazie a queste stime, siamo in grado di mostrare in aggiunta la buona posizione della EIDP considerata in uno spazio funzionale adatto. Questo capitolo ha dato luogo ad una pubblicazione su *Bulletin des Sciences Mathematiques* (cf. [Mar20]).

2 Stime di Schauder per operatori di Lévy degeneri di Ornstein-Uhlenbeck

Stabiliamo qui stime di Schauder globali per equazioni integro-differenziali parziali (EIDP) controllate da un operatore Lévy di tipo Ornstein-Uhlenbeck possibilmente degenere, sia nel contesto ellittico che in quello parabolico, sfruttando spazi di Hölder anisotropici adatti. La classe di operatori che possiamo considerare è composta da una parte lineare più un operatore Lévy che sia comparabile, in un senso adatto, con un operatore stabile che può essere troncato. Include, per esempio, gli operatori stabili relativistici, temperati, a strati o di tipo Lamperti. Il nostro metodo non assume né la simmetria dell'operatore Lévy, nè l'invarianza per dilatazioni per la parte lineare dell'operatore. Grazie a queste stime, possiamo ottenere anche la buona posizione per l'equazione considerata in uno spazio funzionale conveniente. Nella sezione finale, estendiamo alcuni di questi risultati ad operatori più generali che considerino anche una componente non-lineare dipendente dallo spazio e dal tempo. Questo capitolo è stato pubblicato su *Journal of Mathematical Analysis and Applications* (cf. [Mar21]).

3 Buona posizione debole per EDS controllate da processi di Lévy

In questo capitolo, studiamo gli effetti della propagazione di un rumore di Lévy non-degenere attraverso una catena di equazioni differenziali deterministiche i cui coefficienti sono Hölder continui e soddisfano una condizione di tipo Hörmander debole. In particolare, supponiamo la non-degenerazione della deriva rispetto alle componenti che trasmettono il rumore. Per alcune dinamiche specifiche, caratterizziamo inoltre attraverso contro-esempi adatti, gli esponenti di regolarità quasi ottimali che assicurino la buona posizione debole per l'equazione differenziale stocastica (EDS) associata. Come corollario del nostro approccio, deriviamo anche alcune stime di tipo Krylov per la densità delle soluzioni deboli per la EDS considerata. Scritto in collaborazione con Stéphane Menozzi, questo capitolo è ora disponibile come pre-pubblicazione (cf. [MM21]).

4 Sulle constanti ottimali nelle stime di Sobolev e di Schauder per equazioni degeneri di Kolmogorov

In questo capitolo, consideriamo un operatore di Ornstein-Uhlenbeck di tipo Kolmogorov della forma $L = \mathrm{Tr}(BD^2) + \langle Ax, D\rangle$, dove A, B sono matrici che soddisfano una condizione di Kalman che è equivalente alla condizione di ipoellitticità. Mostriamo in particolare il seguente risultato: le stime di Schauder o di Sobolev associate al problema di Cauchy parabolico corrispondente rimangono valide, e con la stessa costante, anche per il problema di Cauchy parabolico associato ad una perturbazione di secondo ordine di L, ovvero per $L + \mathrm{Tr}(S(t)D^2)$, dove $S(t)$ è una $N \times N$ matrice *definita non-negativa* che dipende in maniera continua dal tempo t. Il nostro approccio si basa su una tecnica perturbativa attraverso processi di Poisson introdotta originariamente in [KP17]. Questo capitolo è stato realizzato in collaborazione con Stéphane Menozzi ed Enrico Priola.

Capitolo 1

Introduzione

1 Il modello considerato

La presente tesi di dottorato si concentra sullo studio di fenomeni di regolarizzazione attraverso rumore degenere di tipo Lévy per catene di equazioni differenziali che sono possibilmente mal poste. In particolare, il nostro obbiettivo principale è determinare, sotto condizioni adatte sul sistema degenere, quale sia la regolarità di Hölder minima sui coefficienti che assicuri il carattere ben posto della dinamica stocastica associata.

Più in dettaglio, fissato uno spazio "grande" $\mathbb{R}^N$ e uno "piccolo" $\mathbb{R}^d$ ($d \leq N$), siamo interessati ad equazioni differenziali della seguente forma:

$$dX_t \;=\; F(t, X_t)dt + B\sigma(t, X_{t-})dZ_t, \quad t \geq 0 \tag{1.1}$$

dove $\{Z_t\}_{t\geq 0}$ è un processo di Lévy d-dimensionale su uno spazio di probabilità filtrato $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$ e la deriva $F\colon [0, +\infty) \times \mathbb{R}^N \to \mathbb{R}^N$ e la matrice di diffusione $\sigma\colon [0, +\infty) \times \mathbb{R}^N \to \mathbb{R}^d \otimes \mathbb{R}^d$ sono funzioni Hölder continue, uniformemente in tempo. Sopra, il processo $\{Z_t\}_{t\geq 0}$ viene trasmesso attraverso la matrice di diffusione σ (grazie ad una proprietà di uniforme ellitticità) su $\mathbb{R}^d$ e poi immerso nello spazio grande $\mathbb{R}^N$ attraverso la matrice B in $\mathbb{R}^N \otimes \mathbb{R}^d$ per cui assumiamo, senza perdità di generalità, che $\operatorname{rank} B = d$.

Un'analisi dettagliata sul comportamento ben posto di catene degeneri stocastiche della forma (1.1) non è importante solo per il suo intrinseco interesse matematico ma soprattutto per gli innumerevoli contesti scientifici in cui questo tipo di dinamica viene utilizzata come modello.
Per citarne qualcuno, *diffusioni cinetiche degeneri non-locali* della seguente forma:

$$\begin{cases} dX_t^1 &= F_1(t, X_t^1, X_t^2)dt + dZ_t \\ dX_t^2 &= F_2(t, X_t^1, X_t^2)dt, \end{cases} \tag{1.2}$$

corrispondente all'Equazione (1.1) con sole due componenti ($N = 2d$), $\sigma = 1$ e $B = (I_{d\times d}, 0_{d\times d})^t$, sono spesso utilizzate in meccanica Hamiltoniana per regimi turbolenti o in modelli che considerano fenomeni di diffusione anomala come, per esempio, il variare

del calore tra due materiali differenti a contatto (cf. [BBM01, EPRB99]). Infatti, una dinamica della forma (1.2) con $F_2(t, x_1, x_2) = x_1$ può essere utilizzata per descrivere la dinamica della posizione/velocità di una particella in moto in un particolare sistema, dove solo la componente di velocità X_t^1 viene perturbata casualmente (per esempio a causa di effetti turbolenti dovuti all'aria in regimi ad alta velocità) mentre la variabile di spostamento X_t^2 percepisce il rumore casuale solo attraverso la sua dipendenza dalla prima componente. Questo tipo di modelli cinetici degeneri appaiono anche come limite diffusivo per equazioni di Boltzmann linearizzate se si assume che la funzione di equilibrio sia una distribuzione di tipo Lévy ([Ale12, MMM11, Mel16, CZ18]). Si veda anche [CPKM05] per un'applicazione alla legge di scalabilità di Richardson sulla turbolenza.
Un altro utilizzo naturale delle catene degeneri cinetiche di tipo Lévy appare in finanza ed, in particolare, nella modellizzazione dell'andamento temporale per il prezzo di *opzioni Asiatiche*, un tipo particolare di titolo derivato "esotico" (cf. [BNS01, Bro01, JYC09, BKH10]). Si tratta di opzioni il cui payoff, cioè quanto guadagna il possessore dell'opzione ad una data scadenza, dipende dalla media dei valori assunti dal sottostante durante l'intera vita del contratto e non dal valore del sottostante alla chiusura di esso, come nelle più comuni opzioni europee.
Tutti gli esempi citati sopra si concentrano però solo sul caso cinetico, quando l'Equazione (1.3) è composta da sole due componenti ($N = 2d$). Nel quadro più generale ($N = nd$), modelli del tipo:

$$\begin{cases} dX_t^1 &= F_1(t, X_t^1, \dots, X_t^n)dt + dZ_t \\ dX_t^2 &= F_2(t, X_t^1, \dots, X_t^n)dt; \\ dX_t^3 &= F_3(t, X_t^2, \dots, X_t^n)dt; \\ \vdots \\ dX_t^n &= F_n(t, X_t^{n-1}, X_t^n)dt, \end{cases} \tag{1.3}$$

appaiono per esempio in sismologia quando si considera la propagazione di un'onda d'urto attraverso strutture di diverso materiale. Inoltre, dinamiche della forma in (1.3) vengono spesso utilizzate per rappresentare gli oscillatori elasto-plastici interconnessi, ovvero sistemi di molle collegate tra loro, dove una perturbazione casuale viene applicata solo alla prima di esse (cf. Figura 1). A tal riguardo, si veda per esempio [BT08, BMPT09] in un contesto diffusivo.
Più in generale, modelli che considerano rumori di tipo Lévy appaiono più versatili e realistici visto che permettono la presenza di salti casuali nella dinamica, a differenza del caso Browniano.

Per evidenziare il contesto più generale in cui il nostro problema specifico si inserisce, iniziamo la presente tesi con una breve presentazione sulla teoria della regolarizzazione attraverso rumore.

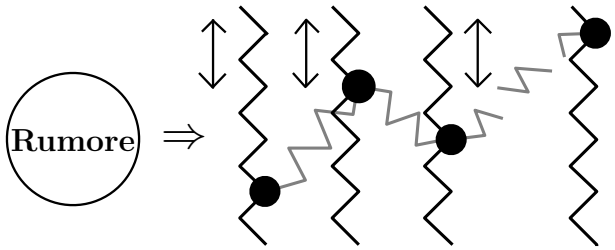

Figura 1.1: Oscillatori Elasto-plastici Interconnessi; da [DM10].

2 Regolarizzazione attraverso rumore

Fissato un tempo iniziale x in $\mathbb{R}^N$, la teoria classica di Cauchy-Lipschitz assicura che un'equazione differenziale ordinaria del tipo:

$$\begin{cases} dX_t \;=\; F(t, X_t)dt, \\ X_0 \;=\; x \end{cases} \tag{2.4}$$

presenti un'unica soluzione globale se la deriva F è sufficientemente regolare, i.e. Lipschitz continua ed a crescita al più lineare.
Successivamente, questo risultato classico è stato poi esteso da diversi autori sotto condizioni sempre più deboli per la regolarità della deriva F. Ricordiamo a tal proposito la famosa teoria dei flussi alla DiPerna-Lyons in [DL89] dove vengono considerate derive solo debolmente Lipschitz continue (i.e. in spazi di Sobolev) e la generalizzazione di tale risultato a derive a variazione limitata (cf. [Amb04, CDL08]). Tutti i lavori citati sopra condividono però una sorta di condizione di limitatezza sulle derivate di F che, almeno nel caso di derive omogenee in tempo, può essere scritta come $\operatorname{div} b$ in $L^\infty(\mathbb{R}^N)$.
Indebolire tale condizione sulla divergenza permette subito l'insorgere di modelli specifici che confutano il carattere ben posto dell'Equazione (2.4).
Un contro-esempio classico è dato dal modello di tipo *Peano* ottenuto a partire da (2.4) imponendo $d = 1$ e $F(s, x) = \operatorname{sgn}(x)|x|^\beta$ per un certo β in $(0, 1)$. Non è difficile mostrare infatti che tale equazione con punto iniziale $x = 0$ presenta un numero infinito di soluzioni della forma:

$$X_s \;=\; (1-\beta)^{\frac{1}{1-\beta}} \mathbb{1}_{[0,T_0]}(s)\,(s - T_0)^{\frac{1}{1-\beta}} \qquad s \geq 0 \tag{2.5}$$

al variare di $T_0 > 0$. Intuitivamente, la presenza della singolarità nell'origine per il sistema, dove la deriva F non è Lipschitz regolare, permette di intrappolare le soluzioni in quel punto per un lasso di tempo qualsiasi. Questo tipo di fenomeno viene spesso chiamato *biforcazione* dei flussi.
Ci siamo soffermati qua solo su un contro-esempio con deriva Hölder regolare e solo sul problema della non-unicità di soluzioni perchè sarà questo il contesto da noi considerato. Citiamo comunque che nel caso a derive discontinue, diversi e più variegati fenomeni possono effettivamente accadere (cf. [Fla11, DL89]), tra cui, per esempio, la *coalescenza* dei flussi, ovvero la collisione di soluzioni a partire da punti iniziali diversi, o addirittura la non-esistenza di soluzioni.

Regolarizzazione attraverso rumore Browniano non-degenere. La situazione cambia drasticamente se si aggiunge al sistema un rumore casuale, ovvero se si considera al posto dell'Equazione deterministica (2.4) la sua controparte stocastica data da

$$\begin{cases} dX_t \;=\; F(t, X_t)dt + \sigma(t, X_t)dW_t, \\ X_0 \;=\; x, \end{cases} \tag{2.6}$$

dove $\{W_s\}_{s\geq 0}$ è un moto Browniano su $\mathbb{R}^N$. Infatti, la presenza di un rumore casuale "sufficientemente" intenso può effettivamente restaurare il carattere ben posto del sistema.
Per dinamiche stocastiche di tipo Peano a rumore evanescente (cf. Equazione (2.6) con $\sigma = \epsilon$ piccolo e $F(t, x) = \mathrm{sgn}(x)|x|^\beta$), questo fenomeno è stato messo in luce da Delarue e Flandoli in [DF14] (si veda anche il lavoro precedente [BB81]): in tempo piccolo, le fluttuazioni del rumore dominano il sistema così che la soluzione possa sfuggire alla singolarità in zero, mentre in tempo lungo, la deriva F domina sul rumore e costringe allora la soluzione a fluttuare intorno ad una delle due soluzioni estremali (cf. Equazione (2.5) con $T_0 = 0$). È importante quindi evidenziare come, in tempo piccolo, sia presente una forte competizione tra l'irregolarità della deriva e le fluttuazioni medie del rumore. Sottolineiamo però che in ambito stocastico particolare accortezza va posta riguardo a cosa si intende esattamente per unicità di soluzione per l'Equazione (2.6).
Una soluzione *debole* della dinamica stocastica in (2.6), sarà una coppia di processi $\{(X_t, W_t)\}_{t\geq 0}$ dove $\{W_t\}_{t\geq 0}$ è un moto Browniano su una base stocastica $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$ mentre $\{X_t\}_{t\geq 0}$ soddisfa l'Equazione (2.6) rispetto al rumore W_t. Intuitivamente, il moto Browniano $\{W_t\}_{t\geq 0}$ in questo caso viene costruito e fa parte esso stesso della nozione di soluzione. Diremo allora che l'unicità in senso *debole* vale per l'Equazione (2.6) quando ad ogni coppia di soluzioni deboli $\{(X_t^1, W_t^1)\}_{t\geq 0}$ e $\{(X_t^2, W_t^2)\}_{t\geq 0}$, le leggi marginali dei processi $\{X_t^1\}_{t\geq 0}$ e $\{X_t^2\}_{t\geq 0}$ coincidono. Sottolineiamo in particolare che questa nozione di unicità non richiede nemmeno che le due soluzioni siano definite sulla stessa base stocastica.
D'altra parte, una soluzione $\{X_t\}_{t\geq 0}$ dell'Equazione (2.6) è detta *forte* se, fissato un rumore $\{W_t\}_{t\geq 0}$, X_t verifica l'equazione data rispetto a W_t ed è adattato alla filtrazione naturale di $\{W_t\}_{t\geq 0}$. Intuitivamente, una soluzione forte $\{X_t\}_{t\geq 0}$ può essere vista ad ogni tempo fissato t come un funzionale del rumore W_t dato come input del sistema.
Un metodo consueto per stabilire l'unicità forte di dinamiche stocastiche come (2.6), ovvero quando una soluzione forte è unica per un certo rumore W_t dato, è sfruttare il teorema di Yamada-Watanabe (cf. [YW71]) che assicura il carattere ben posto in senso forte della dinamica a partire dall'esistenza di soluzioni deboli e dall'unicità traiettoriale, ovvero quando due processi soluzione sono indistinguibili.

In un contesto browniano, è stato mostrato che una dinamica stocastica non-degenere come (2.6) è ben posta in senso debole non appena la deriva F è misurabile e limitata e la matrice di diffusione σ è uniformemente ellittica e solo continua in spazio. Tale risultato è stato ottenuto da Stroock e Varadan [SV79] attraverso la famosa teoria del *problema di martingala*.
Negli ultimi anni, grande attenzione è stata data anche a dinamiche a rumore additivo (i.e. $\sigma = 1$) e derive di tipo distribuzionale in spazio, soprattutto per la loro connessione

con alcuni modelli fisici in teoria dei materiali (cf. [AKQ14, Bro86]). Un primo risultato a riguardo si può trovare in [FIR17] dove viene considerata una deriva F inomogenea in tempo ed appartenente ad uno spazio di Hölder a indice negativo ma strettamente più grande di $-1/2$. Per una presentazione più precisa di tali spazi di Hölder negativi, si rimanda il lettore a [FH14], Sezione 13. Successivamente, Delarue e Diel in [DD16] hanno dimostrato che lo stesso risultato vale anche se si considerano indici di regolarità che siano solo maggiori di $-2/3$, seppure solo per dinamiche mono-dimensionali. Tale lavoro è stato infine esteso al caso multidimensionale in [CC18]. Citiamo inoltre il lavoro di Bass e Chen [BC03] dove viene considerata una dinamica a rumore additivo con deriva indipendente dal tempo ed appartenente alla classe di Kato.

Per quel che riguarda invece l'unicità traiettoriale per soluzioni dell'Equazione (2.6), il primo risultato è stato ottenuto da Zvonkin in [Zvo74] per una dinamica mono-dimensionale con deriva F misurabile limitata ed una matrice di diffusione σ Hölder continua di indice di regolarità strettamente più grande di $1/2$. Tale risultato è stato poi esteso da Veretennikov [Ver81] al caso multi-dimensionale per una matrice di diffusione Lipschitz continua. Come accennato precedentemente, questi lavori sull'unicità traiettoriale permettono poi di mostrare l'esistenza di soluzioni forti per la dinamica, attraverso il teorema di Yamada-Watanabe. Citiamo a riguardo anche l' approccio più diretto per la costruzione di soluzioni forti sotto le stesse assunzioni sulla deriva sopra, ottenuto in [MBP10, MPMBN$^+$13] attraverso il calcolo di Malliavin.
Successivamente, Krylov e Röckner in [KR05] e Zhang in [Zha13b] hanno mostrato il carattere ben posto in senso forte per dinamiche stocastiche come in (2.6) quando il drift è solo integrabile, ovvero se F appartiene a $L^p(0,T;L^q(\mathbb{R}^N))$ sotto la condizione di Prodi-Serrin: $\frac{N}{q}+\frac{2}{p}<1$, $p\geq 2, q\geq 2$, rispettivamente a rumore additivo e a rumore moltiplicativo con matrice di diffusione σ in spazi di Sobolev. Si vedano a riguardo anche i recenti lavori [Kry21] e [RZ21, Nam20] in cui viene affrontato il caso critico additivo (i.e. $\frac{N}{q}+\frac{2}{p}=1$) rispettivamente per deriva omogenee ed inomogenee in tempo. Sottolineiamo però che in [Nam20] è considerata una nozione di integrabilità di tipo Lorentz, leggermente più forte di quella di Lebesgue usuale. Si veda la Definizione 2.1 nell'articolo citato, per maggiori dettagli. Citiamo infine il lavoro di Fedrizzi e Flandoli [FF11], dove sotto le stesse condizioni di Krylov e Röckner, viene mostrata la dipendenza continua delle soluzioni X_t dalla condizione iniziale x e i lavori successivi di Zhang *al.* [Zha15, XZ16] dove viene invece analizzata la differenziabilità in senso debole (i.e. in spazi di Sobolev) sempre rispetto alla condizione iniziale.

I lavori sopra elencati illustrano intuitivamente quello è che stato chiamato, seguendo la terminologia di Flandoli in [Fla11], un fenomeno di *regolarizzazione attraverso rumore*: quando un'equazione differenziale deterministica è mal posta (perché l'esistenza o l'unicità di soluzione falliscono) mentre la dinamica stocastica correlata è ben posta in senso debole o forte. Suggeriamo al lettore interessato di vedere la monografia [Fla11] dove viene presentata un'analisi più generale dell'argomento.

Ci siamo concentrati per ora su dinamiche stocastiche *non-degeneri*, ovvero quando il rumore presenta la stessa dimensione del sistema sottostante su cui agisce (i.e. quando $N=d$ in (1.1)).
Sottolineiamo però che tale condizione non è effettivamente sempre soddisfatta in molti

casi pratici. Si pensi, per esempio, in meccanica Hamiltoniana, alle equazioni di Langevin con una perturbazione sulla componente di velocità, o in finanza, nell'analisi di opzioni asiatiche. È stato inoltre evidenziato ([dCN10, Woy01]) che in molte applicazioni pratiche, le fluttuazioni casuali in sistemi complessi reali sono effettivamente spesso di natura non-Gaussiana.
Il punto iniziale e motivazione principale per questo lavoro è stato allora analizzare i fenomeni di regolarizzazione attraverso rumore introdotti sopra, per catene di equazioni differenziali ordinarie quando la perturbazione casuale non è più un moto Browniano ma un più generale processo di Lévy con proprietà adeguate e che agisce, possibilmente, solo su alcune delle componenti del sistema (i.e. se $d < N$ in (1.1)). Per necessità di tempo, abbiamo deciso però di focalizzarci solo sulla caratterizzazione in senso debole per tale dinamica. Presentiamo ora brevemente una panoramica dei principali risultati già noti in questo ambito.

Regolarizzazione debole per rumore stabile non-degenere. Il carattere ben posto in senso debole per dinamiche stocastiche della seguente forma:

$$X_t = x + \int_0^t F(X_s)ds + Z_t, \quad t \geq 0, \tag{2.7}$$

dove $\{Z_t\}_{t\geq 0}$ è un processo α-stabile simmetrico su $\mathbb{R}^N$, è stato ampiamente studiato negli ultimi decenni. Uno dei primi contributi nel caso monodimensionale è dato dal lavoro di Tanaka *et al.* [TTW74] dove l'unicità in legge viene provata per l'Equazione (2.7) quando la deriva F è limitata e continua ed il simbolo di Lévy Φ associato a $\{Z_t\}_{t\geq 0}$ soddisfa alcune naturali condizioni all'infinito: $\Re\Phi(\xi)^{-1} = O(1/|\xi|)$ se $|\xi| \to \infty$. Un'estensione al caso multi-dimensionale è stata poi ottenuta in [Kom83] assumendo che la deriva F sia continua e limitata e la legge di $\{Z_t\}_{t\geq 0}$ presenti una densità rispetto alla misura di Lebesgue sul $\mathbb{R}^N$. Dinamiche stocastiche come in (2.7) per derive F in spazi L^p adeguati sono state considerate anche in [Jin18].
Al meglio delle nostre conoscenze, il primo lavoro che tratti modelli a derive distribuzionali in spazio e rumori α-stabili è [ABM20] nel caso monodimensionale, dove la deriva F è omogenea in tempo ed appartenente ad uno spazio di Hölder (negativo) di indice strettamente più grande di $(1-\alpha)/2$.
Citiamo inoltre i quasi simultanei lavori [LZ19] e [CdRM20a] dove vengono considerate invece derive F su spazi di Besov generali, sotto condizioni adatte sui parametri, che siano rispettivamente omogenee ed inomogenee in tempo.
I risultati mostrati sopra si basano su una formulazione di tipo Young per dare un senso compiuto alla dinamica. Oltre il regime di Young, citiamo invece [KP20] dove vengono sfruttate tecniche come i prodotti paracontrollati (cf. [GIP15]) per ottenere la buona posizione debole per dinamiche guidate da derive inomogenee in tempo ed appartenenti a spazi di Hölder negativi con indice di regolarità strettamente più grande di $(2-2\alpha)/3$.

Evidenziamo infine che i lavori sopra si concentrano tutti sul caso α-stabile *sotto-critico*, ovvero quando $\alpha > 1$. Sottolineiamo infatti che se $\alpha \leq 1$, dinamiche stocastiche della forma (2.7) sono molto più difficili da trattare visto che il rumore non domina pù il sistema in tempo piccolo. A riguardo, citiamo i lavori [Zha19] e [CdRMP20b] dove gli autori considerano rispettivamente derive che siano $\alpha < 1$, $(1-\alpha)$-Hölder continue e $\alpha = 1$, continue.

Regolarizzazione debole attraverso rumore degenere. In tutti i risultati mostrati precedentemente, il rumore giocava un ruolo fondamentale, permettendo di regolarizzare i coefficienti ad ogni componente del sistema. È allora chiaro che in un contesto a rumore *degenere*, ovvero quando la perturbazione casuale agisce direttamente solo su alcune parti della dinamica, la situazione sembri subito molto più delicata. Consideriamo ad esempio il modello classico di una catena di n equazioni deterministiche ordinarie su $\mathbb{R}^d$ dove solo la prima è perturbata da una diffusione Browniana:

$$\begin{cases} dX_t^1 &= F_1(t, X_t^1, \dots, X_t^n)dt + \sigma(t, X_t^1, \dots, X_t^n)dW_t, \\ dX_t^2 &= F_2(t, X_t^1, \dots, X_t^n)dt; \\ \vdots \\ dX_t^n &= F_n(t, X_t^{n-1}, X_t^n)dt. \end{cases} \tag{2.8}$$

Perchè sia possibile ottenere un fenomeno di regolarizzazione attraverso rumore anche in questo caso, è necessario allora che il rumore W_t agente solo sulla prima linea si propaghi attraverso il sistema, raggiungendo così tutte le componenti del modello. Assunzioni classiche che assicurino questo tipo di fenomeno sono l'uniforme ellitticità della matrice di diffusione σ (cf. [**UE**] in Sezione 6) che garantisce la preservazione del rumore su $\mathbb{R}^d$, e la cosiddetta condizione di Hörmander (cf. [Hör67]) per l'ipoellitticità del sistema.

Sotto una condizione di Hörmander *forte*, ovvero quando i campi vettoriali diffusivi e i loro commutatori generano lo spazio, i lavori principali sono stati ottenuti in ambito diffusivo da Kusuoka e Stroock [KS84, KS85, KS87], sfruttando tecniche di calcolo di Malliavin.

Nella letteratura riguardante modelli degeneri di tipo (2.8), diversi autori hanno invece supposto che ogni componente della deriva F sia differenziabile rispetto alla sua prima componente e che il gradiente risultante sia non-singolare, ovvero si assume che $D_{x_{i-1}}F_i$ (con $i = 2, \dots, n$) abbia rango massimo. Questa assunzione di non-degenerazione per la sotto-diagonale della matrice Jacobiana di F è la ragione per cui questo tipo di condizione è stata spesso chiamata di tipo Hörmander *debole*. In parole povere, considerando una mollificazione dei coefficienti se necessario, la deriva F è effettivamente necessaria per generare lo spazio $\mathbb{R}^N$ attraverso i commutatori di Lie.
Uno dei primi lavori a considerare questo tipo di condizione è stato [Men11] dove l'autore ha mostrato che (2.8) è ben posta in senso debole supponendo che la deriva F sia Lipschitz continua e la matrice di diffusione Hölder continua. Questo risultato è stato poi esteso in [Men18] a matrici di diffusione che siano solo continue in spazio.
Per la caratterizzazione ben posta in senso debole in un contesto cinetico, corrispondente alla dinamica in (2.8) per $n = 2$, citiamo Zhang [Zha18] dove vengono considerate derive F semilineari tali che $F_2(x_1, x_2) = Ax_1$ sotto condizioni di integrabilità locale per F_1 e la continuità del coefficiente di diffusione σ. Sempre nel caso di due oscillatori sotto una condizione di tipo Hörmander debole, Chaudru de Raynal ha mostrato in [CdR18] il carattere ben posto in senso debole dell'equazione non appena la deriva F è Hölder continua rispetto alla variabile degenere x_2 con indice di regolarità strettamente maggiore di $1/3$. In tale lavoro, viene anche mostrato, attraverso adeguati contro-esempi, che la soglia di $1/3$ per la regolarità di Hölder su F è effettivamente "quasi" ottimale. Questo risultato è stato poi esteso in [CdRM20b] al caso più generale di n oscillatori

(cf. Equazione (2.8)).
Intuitivamente, questa soglia minima per la regolarità di Hölder sulla deriva può essere spiegata come il prezzo da pagare per bilanciare la degenerazione del rumore. In riferimento alla discussione precedente sui modelli di tipo Peano in [DF14] (cf. Equazione (2.5)), questa soglia è collegata al fatto che le fluttuazioni del rumore non sono sufficientemente forti per spingere la soluzione lontano dalla singolarità nell'origine nel caso in cui la deriva sia troppo irregolare.

Al meglio della nostra conoscenza, non esistono molti lavori che trattano invece il carattere ben posto in senso debole per dinamiche a rumore degenere e a salti, ovvero quando in (2.8) si sostituisce il moto Browniano $\{W_t\}_{t\geq 0}$ con un processo $\{Z_t\}_{t\geq 0}$ α-stabile. All'interno del contesto della teoria per la regolarizzazione attraverso rumore (ovvero quando la dinamica deterministica associata è mal posta), citiamo solo [HM16] dove gli autori hanno mostrato il carattere ben posto debole per una versione linearizzata della dinamica in (2.8) con $F(t,x) = Ax$ e un coefficiente di diffusione σ Hölder continuo, sotto alcune limitazioni sulle dimensioni d,N.

3 Unicità in legge per catene degeneri

A partire dal lavoro [SV79] nel caso diffusivo non-degenere, è ben noto il collegamento tra le soluzioni del *problema di martingala* e le soluzioni deboli (in senso probabilistico) della dinamica stocastica (1.1). In un certo senso, tale metodo definisce il processo $\{X_t\}_{t\geq 0}$ attraverso il suo generatore infinitesimale L_t. Per riuscire ad introdurlo a modo nel nostro contesto a salti, sia innanzitutto $\mathcal{D}([0,\infty);\mathbb{R}^N)$ lo spazio di tutti i percorsi tra $[0,\infty)$ a $\mathbb{R}^N$ che siano càdlàg, ovvero percorsi continui a destra e con limite finito a sinistra. Skorokhod mostrò in [Sko56] che tale spazio può essere equipaggiato con una metrica naturale in modo tale che diventi uno spazio metrico separabile. Possiamo allora pensare a $\mathcal{D}([0,\infty);\mathbb{R}^N)$ come uno spazio misurabile Boreliano e considerare su di esso delle misure di probabilità.
Introduciamo inoltre il processo canonico, o valore nel punto, $\{y_t\}_{t\geq 0}$ associato allo spazio $\mathcal{D}([0,\infty);\mathbb{R}^N)$ dato da

$$y_t(\omega) \;=\; \omega(t), \quad \omega \in \mathcal{D}([0,\infty);\mathbb{R}^N).$$

Per maggiori dettagli, si veda per esempio [EK86] o [Bas11].
Fissato ancora x in $\mathbb{R}^N$, una misura di probabilità $\mathbb{P}$ su $\mathcal{D}([0,\infty);\mathbb{R}^N)$ è una soluzione del problema di martingala (associato al generatore infinitesimale L_t) con punto iniziale x se:

- $\mathbb{P}\left(y_0 = x\right) \;=\; 1$;
- per ogni funzione ϕ nel dominio $\mathrm{dom}\Big(\partial_t + L_t\Big)$, il processso

$$\left\{\phi(t,y_t) - \phi(0,x) - \int_0^t \Big(\partial_s + L_s\Big)\phi(s,y_s)\,ds\right\}_{t\geq 0} \tag{3.9}$$

  è una $\mathbb{P}$-martingala rispetto alla filtrazione naturale del processo canonico y_t.

Una terza possibile caratterizzazione del processo $\{X_t\}_{t\geq 0}$ è ottenuta attraverso la sua legge marginale μ_t:

$$\begin{cases} \partial_t \mu_t \;=\; L_t^* \mu_t, \quad t > 0 \\ \mu_0 \;=\; x, \end{cases} \tag{3.10}$$

dove L_t^* rappresenta (formalmente) l'operatore aggiunto di L_t. La dinamica sopra viene normalmente chiamata *Equazione di Fokker-Plank progressiva.* Ricordiamo che una famiglia continua di misure $\{\mu_t\}_{t\geq 0}$ è una soluzione di (3.10) se per ogni funzione test ϕ in $C_c^\infty(\mathbb{R}^N)$,

$$\partial_t \int_{\mathbb{R}^N} \phi(y)\, d\mu_t(y) \;=\; \int_{\mathbb{R}^N} L_t\phi(y)\, d\mu_t(y)$$

in un senso distribuzionale in tempo t e la condizione iniziale richiede che μ_t converga (in un senso adatto) alla massa di Dirac δ_x in x. Per una più esaustiva presentazione di questo argomento, rimandiamo il lettore alla monografia [BKRS15].
Un'ampia analisi di questi fenomeni durante gli anni (cf. [SV79, EK86, Kur98, Kur11]) ha ormai messo in luce che i tre modi sopra descritti (equazione stocastica, problema di Martingala ed equazione di Fokker-Plank progressiva) sono, sotto condizioni minime sui coefficienti, effettivamente equivalenti nello specificare una diffusione a salti nel senso che l'esistenza e/o l'unicità per uno implica l'esistenza e/o l'unicità anche negli altri due casi.
Per comprendere meglio come i due primi capitoli della presente tesi si introducano naturalmente, ci concentriamo ora sul problema dell'unicità di soluzione per il problema di martingala associato all'operatore $\partial_t + L_t$.
Siano $\mathbb{P}_1$, $\mathbb{P}_2$ due misure sullo spazio di Skorokhod $\mathcal{D}([0,\infty);\mathbb{R}^N)$ e soluzioni del problema di martingala con punto iniziale x in $\mathbb{R}^N$. Dalla definizione di problema di martingala in (3.9), ha senso introdurre il problema di Cauchy associato ad L_t con condizione terminale nulla. In particolare, fissato un tempo finale $T > 0$ e una sorgente $f\colon [0,T]\times\mathbb{R}^N \to \mathbb{R}$ in una classe di funzioni $\mathcal{F}$ sufficientemente "ricca", consideriamo la seguente equazione alle derivate parziali con condizione al bordo di Cauchy:

$$\begin{cases} \partial_t u(t,x) + L_t u(t,x) \;=\; f(t,x) & \text{su } [0,T)\times\mathbb{R}^N; \\ u(T,x) \;=\; 0 & \text{su } \mathbb{R}^N. \end{cases} \tag{3.11}$$

Assumiamo per il momento che una soluzione $u\colon [0,T]\times\mathbb{R}^N \to \mathbb{R}$ al problema di Cauchy sopra esista. Se inoltre u è sufficientemente regolare, segue immediatamente da (3.9) che

$$\Big\{u(t,y_t) - \int_0^t f(s,y_s)\, ds - u(0,x)\Big\}_{t\in[0,T]} \tag{3.12}$$

è una $\mathbb{P}_i$-martingala per ogni $i \in \{1,2\}$, dove, ricordiamo, abbiamo indicato con $\{y_t\}_{t\geq 0}$ il processo canonico su $\mathcal{D}([0,T];\mathbb{R}^N)$.
Prendendo ora il valore atteso al tempo finale T sopra, possiamo sfruttare la proprietà di martingala in (3.12) e il sistema (3.11) soddisfatto da u (in particolare, $u(T,\cdot) = 0$) per mettere in relazione le due soluzioni del problema nel seguente modo:

$$\mathbb{E}^{\mathbb{P}_1}\left[\int_0^T f(s,y_s)\, ds\right] \;=\; u(0,x) \;=\; \mathbb{E}^{\mathbb{P}_2}\left[\int_0^T f(s,y_s)\, ds\right],$$

dove $\mathbb{E}^{\mathbb{P}_i}$ rappresenta il valore atteso rispetto alla misura di probabilità $\mathbb{P}_i$. Se la classe di funzioni $\mathcal{F}$ considerata è abbastanza ricca, possiamo allora concludere che la legge marginale del processo canonico $\{y_t\}_{t\geq 0}$ è la stessa sotto le due misure considerate, ad ogni tempo fissato t. Sfruttando tecniche di probabilità condizionate regolari, è poi possibile mostrare che il processo $\{y_t\}_{t\geq 0}$ presenta le stesse distribuzioni finito-dimensionali rispetto alle due misura di probabilità, ovvero $\mathbb{P}_1 = \mathbb{P}_2$ (cf. Teorema 4.4.2 in [EK86]) e quindi, che la soluzione del problema di martingala associato all'operatore $\partial_t + L_t$ è unica.
La difficoltà principale nel ragionamento sopra è l'assunzione di regolarità sulla soluzione u del problema di Cauchy (3.11), necessaria a concludere in Equazione (3.12). Infatti, anche se considerassimo una sorgente f liscia in $C_c^\infty([0,T]\times\mathbb{R}^N)$, non sarebbe necessariamente vero che la soluzione u sia anch'essa liscia, a causa della debole regolarità dei coefficienti F e σ considerati, per il nostro modello, solo Hölder continui in spazio.
Quando il rumore è additivo ($\sigma = 1$) e i coefficienti sono abbastanza regolari, la regolarità della soluzione u può essere ottenuta, per esempio, sfruttando tecniche di flussi stocastici (cf. [Kun19]). Inoltre, ragionamenti usuali permettono in questo caso, di dimostrare anche il carattere ben posto in senso forte della dinamica stocastica considerata.
Per applicare il ragionamento sopra nel nostro contesto moltiplicativo e sotto ipotesi di regolarità minima, sarà invece necessario approssimare prima i coefficienti che appaiono nell'operatore L_t con una sequenza di funzioni lisce, per esempio, attraverso un metodo di mollificazione. Sarà poi possibile applicare il metodo sopra descritto per tali coefficienti sufficientemente regolari e concludere che le due soluzioni del problema di martingala "mollificato" siano uguali. Per riottenere infine la dinamica considerata inizialmente, avremo bisogno di una teoria analitica "adatta" associata all'operatore $\partial_t + L_t$ in modo da controllare la convergenza delle soluzioni rispetto al parametro di approssimazione.
Un'ampia letteratura a riguardo (cf. [Pri09, CdR17, CdRHM18b, FGP10]), ha mostrato che un primo passo lungo questa direzione sia stabilire delle particolari stime, dette di *Schauder*, che permettono di controllare le soluzioni approssimate del Problema di Cauchy (3.11) a partire dai coefficienti mollificati, su uno spazio funzionale adatto. Inoltre, è possibile anche dimostrare tramite argomenti di compattezza, che in effetti le stime valgano anche per il limite delle soluzioni approssimate.

3.1 Stime di Schauder per Sistemi Degeneri

Come spiegato alla fine della precedente sezione, siamo ora interessati ad un'analisi dettagliata di un'equazione non-locale di tipo Kolmogorov che si può scrivere nel seguente modo:

$$\begin{cases} \partial_t u(t,x) + L_t u(t,x) \;=\; f(t,x) & \text{on } [0,T)\times\mathbb{R}^N; \\ u(T,x) \;=\; u_T(x) & \text{on } \mathbb{R}^N, \end{cases} \tag{3.13}$$

dove la sorgente $f\colon [0,T)\times\mathbb{R}^N \to \mathbb{R}$ e la condizione terminale $u_T\colon \mathbb{R}^N \to \mathbb{R}$ sono funzioni sufficientemente regolari. Studieremo tale equazione a tempo finito, ovvero per ogni t in $[0,T]$ a un valore finale $T > 0$ fissato. Sopra, L_t rappresenta l'operatore integro-

differenziale dipendente dal tempo che può essere visto come il generatore infinitesimale associato alla diffusione $\{X_t\}_{t\geq 0}$ soluzione di (1.1), ovvero tale che

$$L_t \;=\; \langle F(t,x), D_x\rangle + \mathcal{L}_t, \qquad \text{su } [0,T)\times\mathbb{R}^N, \tag{3.14}$$

dove $\mathcal{L}_t$ è il generatore infinitesimale associato al processo $\{B\sigma Z_t\}_{t\geq 0}$. Sottolineiamo inoltre che similmente alla dinamica stocastica da cui è derivato, tale sistema è degenere nel senso che la componente principale $\mathcal{L}_t$ dell'operatore L_t considera solo alcune direzioni (associate alla matrice B in (1.1)) dello spazio sottostante $\mathbb{R}^N$.

In particolare, i primi due capitoli della presente tesi si concentreranno nello stabilire le stime di Schauder associate alle soluzioni del sistema degenere (3.13).
Caratteristica fondamentale di tali stime è quella di evidenziare quanto una soluzione u del problema di Cauchy (3.13) guadagni in termini di regolarità rispetto alla sorgente f data dal sistema, una nozione usualmente chiamata *bootstrap parabolico* (o effetti regolarizzanti, come riferito più avanti) associato all'operatore L_t.
Illustriamo innanzitutto questo fenomeno in un contesto Gaussiano non-degenere, ovvero quando in (3.14) l'operatore $\mathcal{L}_t$ coincide con la matrice Hessiana D_x^2 agente su tutto lo spazio $\mathbb{R}^N$. In questo caso e per coefficienti limitati e adeguatamente regolari (i.e. F in $C^{\frac{\beta}{2},\beta}$ con le notazioni sotto), Friedman in [Fri64] e Krylov in [Kry96] hanno mostrato il seguente controllo:

$$\|u\|_{C^{\frac{2+\beta}{2},2+\beta}} \;\leq\; C\|f\|_{C^{\frac{\beta}{2},\beta}}, \tag{3.15}$$

dove $\|\cdot\|_{C^{\gamma,\gamma'}}$ denota la classica norma di Hölder sullo spazio-tempo $[0,T)\times\mathbb{R}^N$ di indici γ in tempo e γ' in spazio. Le stime di Schauder in (3.15) suggeriscono allora che ogni soluzione u del problema sia effettivamente Hölder regolare di ordine $\frac{2+\beta}{2}$ in tempo e ordine $2+\beta$ in spazio se supponiamo che la sorgente f sia solo $\frac{\beta}{2}$-Hölder continua in tempo e β-Hölder continua in spazio.
Evidenziamo inoltre che è proprio il carattere limitato dei coefficienti in spazio e la loro regolarità in tempo che permette di considerare il bootstrap parabolico anche rispetto alla variabile temporale.
Nel caso in cui i coefficienti non siano limitati in spazio ma comunque a crescita al più lineare, Krylov e Priola in [KP10] hanno mostrato la seguente variante delle stime di Schauder per il problema:

$$\|u\|_{L^\infty(C^{2+\beta})} \;\leq\; C\|f\|_{L^\infty(C^{\beta})}, \tag{3.16}$$

dove $\|\cdot\|_{L^\infty(C^\gamma)}$ rappresenta la norma di Hölder di ordine γ in spazio, uniformemente in tempo. In pratica, questo tipo di stime considera invece solo il guadagno di regolarità in spazio, di ordine 2 in questo caso, della soluzione u rispetto alla sorgente f. Sottolineiamo comunque che almeno nel caso uniformemente ellittico, è possibile dedurre in un secondo momento la regolarità in tempo per la soluzione u.

Un'applicazione immediata delle stime di Schauder è l'unicità di soluzione per il Problema di Cauchy (3.13) nello spazio funzionale considerato. Infatti, siano u_1, u_2 due soluzioni del sistema considerato. Sfruttando la linearità del problema, possiamo notare che la loro differenza $u_1 - u_2$ è allora soluzione del problema di Cauchy:

$$\begin{cases} (\partial_t + L_t)\,(u_1-u_2)(t,x) \;=\; 0 \\ (u_1-u_2)(T,x) \;=\; 0. \end{cases}$$

Le stime di Schauder presentate, per esempio, in (3.16) con $f = 0$ implicano ora il seguente controllo:

$$\|u_1 - u_2\|_{L^\infty(C^{2+\beta})} \leq 0,$$

che conduce immediatamente all'uguaglianza delle due soluzioni nello spazio funzionale considerato.
Lo studio di Problemi di Cauchy come (3.13) è inoltre fondamentale nell'analisi della dinamica stocastica associata (1.1). Oltre che per la dimostrazione dell'unicità del problema di martingala come spiegato alla fine della precedente sezione, sottolineiamo che le stime di Schauder vengono spesso utilizzate nella trasformazione di Zvonkin per determinare la buona posizione in senso forte dell'equazione stocastica associata. Per maggiori dettagli, si veda [Zvo74, Ver81] in un contesto diffusivo non-degenere rispettivamente mono e multi-dimensionale oppure [CdRHM18b, HWZ20] in quello degenere, rispettivamente diffusivo e α-stabile.
Tali stime appaiono anche in connessione con alcune equazioni stocastiche alle derivate parziali (SPDE). Per esempio, citiamo l'applicazione all'equazione di trasporto stocastica presentata in [FGP10], dove stime di Schauder come in (3.16) sono utilizzate per provare l'esistenza di un flusso stocastico differenziabile per l'equazione caratteristica stocastica associata.
Citiamo infine che un esempio tipico di sistema degenere della forma (3.13) è la seguente equazione cinetica stabile:

$$\partial_t u(t,x) = \mathcal{L}^\alpha u(t,x) - x_1 \cdot \nabla_{x_2} u(t,x) + f(t,x), \quad \text{su } \mathbb{R}^{2d},$$

dove $x = (x_1, x_2)$ in $\mathbb{R}^{2d}$ e $\mathcal{L}^\alpha$ un operatore stabile agente solo su x_1, che appare naturalmente nello studio dell'equazione linearizzata di Boltzmann (cf. [Vil02, CZ18]).

Riassumiamo ora brevemente i principali lavori che riguardino stime di Schauder per sistemi parabolici degeneri.
In un contesto diffusivo Gaussiano, ovvero quando in (3.13) l'operatore L_t si può riscrivere come

$$L_t = \langle F(t,x), D_x \rangle + \frac{1}{2}\mathrm{tr}\left(Ba(t,x)B^* D_x^2\right) =: \langle F(t,x), D_x \rangle + \mathcal{L}_t,$$

per una certa matrice di diffusione $a(t,x)$ in $\mathbb{R}^d \otimes \mathbb{R}^d$, Lunardi in [Lun97] è stata la prima a stabilire stime di Schauder per equazioni di Kolmogorov di tipo Ornstein-Uhlenbeck, i.e. $a(t,x) = 1$ e $F(t,x) = Ax$ con una certa struttura. Tale risultato è stato ottenuto sfruttando spazi di Hölder anisotropici, dove l'indice di Hölder dipende dalla direzione spaziale considerata, proprio per controllare le diverse scale di regolarità causate dalla degenerazione del sistema. Si veda la prossima sezione per una spiegazione più esaustiva di tale fenomeno e una definizione precisa degli spazi anisotropici.
Dopo, in [Lor05] e [Pri09], gli autori hanno ottenuto stime di tipo Schauder per equazioni di Kolmogorov ipoellittiche la cui deriva sia parzialmente non-lineare solo lungo le componenti in cui l'operatore principale $\mathcal{L}_t$ è non-degenere, ovvero:

$$F(x) = Ax + \begin{pmatrix} \tilde{F}_1(x) \\ 0_{(N-d),d} \end{pmatrix}.$$

Infine, il caso diffusivo completamente non-lineare e con una matrice di diffusione non-omogenea in tempo e spazio è stato affrontato per la prima volta in [CdRHM18a]. Il loro risultato è stato ottenuto sotto condizione di regolarità minime e di non-degenerazione sui coefficienti F e a, che conducono a delle ipotesi di tipo Hörmander debole per il sistema. Il metodo di prova in [CdRHM18a] è basato su un approccio perturbativo di tipo progressivo che adatteremo e sfrutteremo anche noi più avanti.
Citiamo infine il lavoro di Di Francesco e Polidoro [DFP06], dove stime di Schauder locali per un sistema degenere lineare con coefficiente di diffusione sono ottenute supponendo una diversa nozione di continuità per il coefficiente di diffusione a che, in un certo senso, tiene conto anche del trasporto associato alla deriva lineare.

In tempi recenti, stime di Schauder per operatori non-locali, soprattutto di tipo stabile, hanno attirato grande interesse nella comunità matematica (cf. [BK15, Bas09, CdRMP20a, DK13, FRRO17, IJS18, Pri12, ROS16, ZZ18]). Quasi tutti i lavori presenti si concentrano però solo sul caso non-degenere.
Al meglio delle nostre conoscenze, l'unico lavoro esistente che tratti il caso degenere non-locale è [HWZ20], dove Zhang e i suoi collaboratori provano stime di Schauder per un sistema cinetico stabile degenere (quando $N = 2d$ e $\operatorname{rank} B = d$ in (3.13)). Il loro metodo è basato su una generalizzazione delle decomposizioni alla Littlewood-Paley già sfruttate in altri lavori dello stesso autore (cf. [ZZ18]) al contesto anisotropico naturale per sistemi degeneri.

In questa tesi di dottorato, analizzeremo in dettaglio due casi specifici che generalizzano i risultati esposti precedentemente:

- un sistema degenere a deriva non-lineare in cui la componente principale dell'operatore è di tipo α-stabile. Questo modello sarà presentato in dettaglio nella Sezione 4;
- un sistema degenere controllato da un operatore di Ornstein-Uhlenbeck di tipo Lévy. Facciamo riferimento alla Sezione 5 per una presentazione più dettagliata del modello.

3.2 Geometria Anisotropica della Dinamica Degenere

In questa sezione, ci concentriamo nel capire quale sia lo spazio funzionale adatto in cui stabilire le nostre stime di Schauder. Come detto già in precedenza, le stime di Schauder nel caso non-degenere sono di solito espresse rispetto a spazi di Hölder "usuali". Vogliamo allora ottenere uno spazio di Hölder rispetto ad una nuova distanza che si adatti alla nostra struttura degenere. In particolare, costruiremo alla fine di questa sezione, degli spazi di Hölder con multi-indici di regolarità dipendenti dalla coordinata considerata.

Per rendere più chiaro al lettore come tale struttura anisotropica della catena degenere appaia naturalmente, presenteremo due possibili approcci: uno analitico, basato su operatori di dilatazione multiscala, ed uno più probabilistico, che sfrutta invece le scale per il tempo caratteristico del processo, soluzione dell'equazione stocastica associata.

Inoltre, per rendere tale esposizione la più chiara possibile, ci concentreremo su un esempio lineare con sole due componenti ($n = 2$).

Dal punto di vista analitico, siamo interessati all'operatore di Kolmogorov di tipo α-stabile per un certo $\alpha \in (0, 2)$:

$$L^{\mathrm{K}} := \Delta_{x_1}^{\frac{\alpha}{2}} + x_1 \cdot \nabla_{x_2} \quad \text{su } \mathbb{R}^{2d}$$

dove $x = (x_1, x_2)$ è un punto in $\mathbb{R}^{2d}$. Sopra, l'operatore $\Delta_{x_1}^{\frac{\alpha}{2}}$ rappresenta il Laplaciano frazionario di ordine $\alpha/2$ rispetto alla variabile x_1, dato da

$$\Delta_{x_1}^{\frac{\alpha}{2}} \phi(x_1, x_2) = \text{p.v.} \int_{\mathbb{R}^d} [\phi(x_1 + z, x_2) - \phi(x_1, x_2)] \frac{dz}{|z|^{d+\alpha}} \tag{3.17}$$

per ogni funzione $\phi \colon \mathbb{R}^{2d} \to \mathbb{R}$ abbastanza regolare. Nel caso diffusivo, ovvero quando quando $\Delta_{x_1}^{\frac{\alpha}{2}}$ diviene il Laplaciano classico Δ_{x_1} lungo x_1:

$$\Delta_{x_1} \phi(x_1, x_2) = \sum_{i=1}^{d} \partial_{x_1^i} \phi(x_1, x_2),$$

l'equazione associata a questo operatore è stata analizzata precisamente da Kolmogorov [Kol34] ed è stato il primo esempio che ha ispirato la teoria sull'ipoellitticità di Hörmander [Hör67].
Per capire come le differenti componenti della dinamica si comportano una rispetto all'altra, introduciamo un operatore di dilatazione $\delta \colon [0, \infty) \times \mathbb{R}^{2d} \to [0, \infty) \times \mathbb{R}^{2d}$ dato da:

$$\delta(t, x) := (\delta_0 t, \delta_1 x_1, \delta_2 x_2).$$

I valori esatti di $\delta_0, \delta_1, \delta_2$ sono allora da determinare in modo che la dilatazione δ sia invariante per la dinamica

$$\partial_t u(t, x) + L^{\mathrm{K}} u(t, x) = 0 \quad \text{on } (0, \infty) \times \mathbb{R}^{2d}, \tag{3.18}$$

nel senso che trasformi soluzioni dell'equazione sopra in altre soluzioni della stessa.
L'idea di una dilatazione δ che permetta di riassumere il comportamento multi-scala, o anisotropico, della dinamica degenere considerata venne inizialmente introdotta da Lanconelli e Polidoro ([LP94]) per l'analisi dell'equazione di Kolmogorov diffusiva. Da allora, è diventato uno strumento comune per l'analisi della geometria anisotropica per equazioni degeneri, come testimonia la ricca letteratura in cui appare ([Lun97, HMP19, HWZ20, DFP05, MPP02]). Sfruttando ora la proprietà di scalabilità del Laplaciano frazionario, possiamo notare che:

$$\begin{aligned}\left(\partial_t + L^{\mathrm{K}}\right)(u \circ \delta) &= \delta_0(\partial_t u \circ \delta) + \delta_1^{\alpha}(\Delta_{x_1}^{\frac{\alpha}{2}} u \circ \delta) + \delta_2 \left(x_1 \cdot (\nabla_{x_2} u \circ \delta)\right) \\ &= \delta_0(\partial_t u \circ \delta) + \delta_1^{\alpha}(\Delta_{x_1}^{\frac{\alpha}{2}} u \circ \delta) + \delta_1^{-1}\delta_2 \left([x_1 \cdot \nabla_{x_2} u] \circ \delta\right) = 0,\end{aligned}$$

dove, evidenziamo, abbiamo denotato $[x_1 \cdot \nabla_{x_2} u] \circ \delta(t, x) := \delta_1 x_1 \cdot \nabla_{x_2} u(\delta(t, x))$. In modo da ottenere l'omogeneità nei termini sopra, è allora naturale considerare, ad ogni $\lambda > 0$ fissato, l'operatore di dilatazione δ_λ seguente:

$$\delta_\lambda(t, x_1, x_2) := (\lambda^{\alpha} t, \lambda x_1, \lambda^{1+\alpha} x_2). \tag{3.19}$$

In particolare, notiamo che, per la nostra scelta di δ_λ, vale che

$$\Big(\partial_t + L^{\mathrm{K}}\Big)u = 0 \implies \Big(\partial_t + L^{\mathrm{K}}\Big)(u \circ \delta_\lambda) = 0.$$

Riassumendo, l'apparizione di questo fenomeno multiscala è dovuto essenzialmente alla particolare struttura nella dinamica considerata, composta da una parte principale $\Delta_{x_1}^{\alpha/2}$ che fornisce un effetto regolarizzante di ordine α solo sulla prima componente e da un termine di trasporto $x_1 \cdot \nabla_{x_2}$ che permette a tale effetto di trasmettersi anche alla seconda componente, seppur con un'intensità minore (di ordine $\alpha/(1+\alpha)$), come potremo vedere dal bootstrap parabolico associato alle stime di Schauder da noi considerate.
Da un punto di vista più probabilistico, le scale apparse nell'operatore di dilatazione δ possono essere associate agli esponenti dei tempi caratteristici di un processo α-stabile ed il suo integrale in tempo. Infatti il tempo caratteristico di un processo stocastico multi-dimensionale può aiutare a spiegare il rapporto tra le velocità delle diverse componenti del processo stesso.
Iniziamo considerando la controparte stocastica del sistema (3.18) data dalla seguente equazione:

$$\begin{cases} dX_t^1 \;=\; dZ_t, \\ dX_t^2 \;=\; X_t^1 dt, \quad t \geq 0, \end{cases} \tag{3.20}$$

dove, per semplicità, assumiamo che la soluzione parta nell'origine a tempo iniziale. Questa equazione stocastica è associata all'operatore di Kolmogorov introdotto prima nel senso che $\mathcal{L}^{\mathrm{K}}$ è il generatore infinitesimale del processo X_t.
La dinamica stocastica (3.20) può essere ora risolta esplicitamente attraverso un integrazione in tempo:

$$X_t \;=\; (X_t^1, X_t^2) \;=\; \left(Z_t, \int_0^t Z_s\, ds\right). \tag{3.21}$$

Se andiamo ora a controllare i tempi caratteristici associati alle due componenti del processo X_t, notiamo che sono dati da $(t^{\frac{1}{\alpha}}, t^{1+\frac{1}{\alpha}})$. Infatti, è noto che il processo stabile è α-scalabile e il suo integrale in tempo aggiunge semplicemente un ordine in più. Abbiamo effettivamente ritrovato le stesse scale mostrate nell'operatore di diffusione δ, anche se in questo caso, riscalate rispetto al tempo corrente.
Nel caso diffusivo ($\alpha = 2$), questo comportamento multi-scala del processo soluzione X_t è ancora più chiaro. Infatti, grazie all'esistenza dei momenti secondi finiti, è possibile tradurre il ragionamento descritto sopra rispetto alla matrice di covarianza del processo. Se sostituiamo in (3.20) e in (3.21) il processo α-stabile Z_t con un moto Browniano W_t, otteniamo immediatamente che la soluzione $X_t = (X_t^1, X_t^2)$ è un processo Gaussiano con media nulla e covarianza K_t in $\mathbb{R}^{2d} \otimes \mathbb{R}^{2d}$ data da

$$K_t \;=\; \begin{pmatrix} tI_{d\times d} & \frac{t^2}{2} I_{d\times d} \\ \frac{t^2}{2} I_{d\times d} & \frac{t^3}{3} I_{d\times d}. \end{pmatrix}$$

è stata poi mostrata in [KMM10] l'equivalenza, in termini di forme quadratiche associate, tra la matrice di covarianza K_t e una diagonale:

$$\sqrt{K_t} \asymp \begin{pmatrix} t^{\frac{1}{2}} I_{d\times d} & 0_{d\times d} \\ 0_{d\times d} & t^{\frac{3}{2}} I_{d\times d}, \end{pmatrix}$$

dove per due matrici A, B in $\mathbb{R}^{nd} \otimes \mathbb{R}^{nd}$, la notazione $A \asymp B$ indica che esiste una costante $C \geq 1$ tale che $C^{-1}|A\xi|^2 \leq |B\xi|^2 \leq C|A\xi|^2$ per ogni ξ in $\mathbb{R}^{nd}$. Tale proprietà è stata chiamata dagli autori, proprietà di *buona scalabilità*. Si veda anche definizione 3.2 in [DM10] per un estensione alla catena generale ($n > 2$). è chiaro ora come possiamo ritrovare allora sulla diagonale della matrice di scalabilità le stesse scale mostrate precedentemente nel caso diffusivo.

Vogliamo ora introdurre una distanza parabolica $\mathbf{d}_P$ su $[0, \infty) \times \mathbb{R}^N$ che sia omogenea rispetto alla struttura multi-scala della dinamica considerata, nel senso che:

$$\mathbf{d}_P\Big(\delta_\lambda(t,x); \delta_\lambda(s,x')\Big) = \lambda \mathbf{d}_P\Big((t,x);(s,x')\Big).$$

Una scelta naturale per tale distanza è data allora da:

$$\mathbf{d}_P\left((t,x),(s,x')\right) \;=\; |t-s|^{\frac{1}{\alpha}} + |x_1 - x_1'| + |x_2 - x_2'|^{\frac{1}{1+\alpha}}.$$

La distanza d_P appena introdotta può essere vista come una naturale generalizzazione della classica distanza parabolica (cf. [Kry96, Fri64]) alla struttura multiscala della nostra dinamica stabile degenere. Visto che sfrutteremo più avanti quasi sempre solo la parte spaziale di tale distanza, introduciamo anche per futura referenza:

$$\mathbf{d}(x,x') \;:=\; |x_1 - x_1'| + |x_2 - x_2'|^{\frac{1}{1+\alpha}}. \tag{3.22}$$

Per maggiori dettagli sulle metriche omogenee, si veda anche il libro di Stein [Ste93].

Siamo finalmente pronti a introdurre lo spazio funzionale adatto ai nostri scopi: uno spazio di Hölder $C_d^\beta(\mathbb{R}^{2d})$ anisotropico associato alla distanza $\mathbf{d}_P$, nel senso che sia omogeneo rispetto agli operatori di dilatazione δ_λ definiti in (3.19). In termini pratici, la seminorma anisotropica $\|\cdot\|_{C_d^\beta}$ verrà effettivamente considerata componente per componente. Infatti, fissata una coordinata, andremo a calcolare la norma di Hölder standard lungo quella particolare direzione, ma con un indice di regolarità riscalato secondo l'ordine dato dall'operatore di dilatazione δ_λ in (3.19), uniformemente in tempo e rispetto alle altre coordinate. Infine, sommeremo insieme tutti i contributi derivanti dalle diverse componenti.
Più precisamente, possiamo introdurre per una funzione $\phi\colon \mathbb{R}^{2d} \to \mathbb{R}$ e un punto z in $\mathbb{R}^{2d}$, la funzione $\Pi_z^1\phi\colon \mathbb{R}^d \to \mathbb{R}$ data da

$$\Pi_z^1\phi(x_1) \;:=\; \phi(z_1 + x_1, z_2).$$

Una notazione simile viene introdotta anche per $\Pi_z^2\phi$.
Fissato un tempo finale $T > 0$, definiamo lo spazio *omogeneo* $L^\infty(0,T; C_d^\beta(\mathbb{R}^{2d}))$ come la famiglia di funzioni Boreliane $\phi\colon [0,T] \times \mathbb{R}^{2d} \to \mathbb{R}$ tali che la seguente semi-norma di Hölder anisotropica sia finita:

$$\|\phi\|_{L^\infty(C_d^\beta)} \;:=\; \sup_{t,z}\left([\Pi_z^1\phi(t,\cdot)]_{C^\beta(\mathbb{R}^d)} + [\Pi_z^2\phi(t,\cdot)]_{C^{\frac{\beta}{1+\alpha}}(\mathbb{R}^d)}\right) \;\asymp\; \sup_{t,x,x'} \frac{|\phi(t,x) - \phi(t,x')|}{\mathbf{d}(x,x')}.$$

Scelto α in $(0,2)$ tale che $\alpha + \beta$ sia in $(1,2)$, possiamo allora definire anche lo spazio di Hölder anisotropico $L^\infty(0,T; C_d^{\alpha+\beta}(\mathbb{R}^{2d}))$ di ordine $\alpha+\beta$ rispetto alla seguente seminorma:

$$\begin{aligned}\|\phi\|_{L^\infty(C_d^{\alpha+\beta})} \;:=\; & \|D_{x_1}\phi\|_{L^\infty} \\ & + \sup_{t,z}\left([\Pi_z^1 D_{x_1}\phi(t,\cdot)]_{C^{\alpha+\beta-1}(\mathbb{R}^d)} + [\Pi_z^2\phi(t,\cdot)]_{C^{\frac{\alpha+\beta}{1+\alpha}}(\mathbb{R}^d)}\right)\end{aligned} \tag{3.23}$$

dove, ricordiamo che $\|\cdot\|_{L^\infty}$ rappresenta la norma uniforme su $[0,T]\times\mathbb{R}^{2d}$. Nel caso in cui si abbia $\alpha+\beta>2$, una estensione naturale alle derivate di ordine secondo è poi necessaria.
Visto che la sorgente f e la condizione terminale u_T saranno considerate limitate nel nostro modello, introduciamo inoltre la versione inomogenea (con una b in basso) degli spazi di Hölder appena descritti, aggiungendo semplicemente la norma uniforme della funzione stessa. Per esempio,

$$\|\phi\|_{L^\infty(C^{\alpha+\beta}_{b,d})} := \|\phi\|_{L^\infty} + \|\phi\|_{L^\infty(C^{\alpha+\beta}_{d})}.$$

Diremo infine che una funzione ϕ è in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^{2d})$ se ϕ è indipendente dal tempo e la relativa norma di Hölder anisotropica è finita. Questo sarà per esempio il caso della condizione terminale u_T.

4 Stime di Schauder per un sistema degenere non-lineare di tipo stabile

Riassumiamo in questa sezione i risultati presentati nel Capitolo 2 del lavoro presente, poi pubblicati su *Bulletin des Sciences Mathematiques*. Il nostro obiettivo è stabilire stime di Schauder ottimali, nel senso della regolarità minima sui coefficienti, per le soluzioni di un'equazione parabolica integro-differenziale parziale degenere su $\mathbb{R}^{nd}$. La degenerazione in questo contesto viene dal fatto che la parte principale dell'operatore, di tipo α-stabile, agisce solo sulle prime d componenti del sistema.

In particolare, date una sorgente $f\colon [0,T]\times\mathbb{R}^{nd}\to\mathbb{R}$ e una condizione terminale $u_T\colon \mathbb{R}^{nd}\to\mathbb{R}$, siamo interessati ad un problema di Cauchy della seguente forma:

$$\begin{cases} \partial_t u(t,x) + \langle Ax + F(t,x), D_x u(t,x)\rangle + \mathcal{L}_\alpha u(t,x) \;=\; -f(t,x) & \text{su } [0,T)\times\mathbb{R}^{nd};\\ u(T,x) \;=\; u_T(x) & \text{su } \mathbb{R}^{nd}. \end{cases} \tag{4.1}$$

dove $x := (x_1,\dots,x_n)$ è un punto in $\mathbb{R}^{nd}$ con ogni x_i in $\mathbb{R}^d$ e $\langle\cdot,\cdot\rangle$ rappresenta il prodotto interno su $\mathbb{R}^{nd}$. Sopra, $F\colon [0,T]\times\mathbb{R}^{nd}\to\mathbb{R}^{nd}$ è una funzione sufficientemente regolare ed A è una matrice in $\mathbb{R}^{nd}\otimes\mathbb{R}^{nd}$ su cui imporremo condizioni adatte.
L'operatore $\mathcal{L}_\alpha$ considerato è il generatore di un processo α-stabile simmetrico e non-degenere che agisce solo sulla prima componente x_1 del sistema. Più precisamente, l'operatore $\mathcal{L}_\alpha$ può essere rappresentato, per ogni funzione abbastanza regolare $\phi\colon [0,T]\times\mathbb{R}^{nd}\to\mathbb{R}$, nella seguente forma:

$$\mathcal{L}_\alpha\phi(t,x) \;:=\; \text{p.v.}\int_{\mathbb{R}^d}\Big[\phi(t,x+By)-\phi(t,x)\Big]\,\nu_\alpha(dy), \tag{4.2}$$

dove $B := (I_{d\times d}, 0_{d\times d},\dots,I_{d\times d})^*$ è la matrice di immersione da $\mathbb{R}^d$ in $\mathbb{R}^{nd}$ e ν_α è la misura di Lévy simmetrica su $\mathbb{R}^d$ associata ad un processo α-stabile.

Ricordiamo ora che il simbolo di Lévy Φ associato all'operatore $\mathcal{L}_\alpha$ (o più esattamente, al processo di cui è generatore) è normalmente definito attraverso la formula di Lévy-Khintchine che, nel nostro caso simmetrico e stabile, può essere scritta nel seguente

modo (cf. [Sat13]):

$$\Phi(p) \;=\; -\int_{\mathbb{S}^{d-1}} |p\cdot s|^{\alpha}\,\mu(ds),$$

dove "$\cdot$" rappresenta il prodotto scalare sullo spazio "piccolo" $\mathbb{R}^d$. Sopra, μ è una misura sulla sfera $\mathbb{S}^{d-1}$ chiamata usualmente la *misura spettrale* (o sferica) associata ad ν_α, nel senso che un passaggio in coordinate polari $y = \rho s$, dove $(\rho, s) \in (0,\infty)\times\mathbb{S}^{d-1}$, permette di decomporre la misura di Lévy stabile come

$$\nu_\alpha(dy) \;:=\; C_\alpha \mu(ds)\frac{d\rho}{\rho^{1+\alpha}}. \tag{4.3}$$

Per una dimostrazione di questo fatto si veda per esempio il Teorema 14.3 in [Sat13]. In particolare, imporremo che la misura di Lévy stabile ν_α sia simmetrica e *non-degenere* nel senso che la sua misura sferica μ soddisfi la seguente condizione:

[ND] Esiste una costante $\eta \geq 1$ tale che per ogni p in $\mathbb{R}^d$,

$$\eta^{-1}|p|^{\alpha} \;\leq\; \int_{\mathbb{S}^{d-1}} |p\cdot s|^{\alpha}\,\mu(ds) \;\leq\; \eta|p|^{\alpha}. \tag{4.4}$$

Come vedremo più avanti, tale condizione implica in particolare l'esistenza di una soluzione fondamentale per l'operatore $\mathcal{L}_\alpha$, visto che la trasformata di Fourier del processo $\{Z_t\}$ α-stabile associato a $\mathcal{L}_\alpha$ è allora integrabile.
È importante notare inoltre che la famiglia di misure spettrali non-degeneri (nel senso sopra) sia molto ricca e variegata. Infatti, la condizione [**ND**] è soddisfatta per esempio dal Laplaciano frazionario $\Delta_{x_1}^{\frac{\alpha}{2}}$, definito in (3.17), la cui misura spettrale è assolutamente continua rispetto alla misura di Lebesgue sulla sfera ma anche da casi molto singolari, come per esempio, la misura indotta dalla somma di masse di Dirac lungo le coordinate canoniche:

$$\sum_{k=1}^{d} (\partial_{x_k}^2)^{\alpha/2}. \tag{4.5}$$

Questo tipo di operatore viene normalmente chiamato Laplaciano frazionario cilindrico e la sua misura di Lévy ν_α è allora concentrata sugli assi $\{x_1 = 0\}\cup\cdots\cup\{x_d = 0\}$. Per maggiori dettagli, si veda [BC06].
Citiamo infine che in letteratura la condizione [**ND**] per la misura di Lévy ν_α appare spesso anche nelle seguenti formulazioni:

- (supporto minimo) il supporto della misura sferica μ non è contenuto in nessun sottospazio lineare proprio di $\mathbb{R}^d$;
- (Condizione di Picard) Esiste una constante $C := C(\alpha)$ che per ogni $\rho > 0$, u in $\mathbb{S}^{d-1}$, vale che
$$\int_{\{|u\cdot y|\}\leq\rho} |u\cdot y|^2 \nu_\alpha(dy) \;\geq\; C\rho^{2-\alpha}.$$

Per maggiori dettagli ed una dimostrazione dell'equivalenza tra le condizioni, si veda per esempio [Pic96, Szt10b, Pri12].

Come già spiegato in introduzione, l'analisi di dinamiche degeneri dove l'effetto regolarizzante agisce direttamente solo su un sottospazio ($\mathbb{R}^d$) dello spazio ambiente considerato ($\mathbb{R}^{nd}$), richiede alcune assunzioni sul sistema considerato Perché la regolarizzazione si trasmetta effettivamente in tutto il sistema. Nel caso diffusivo ($\alpha = 2$), cioè quando $\mathcal{L}_\alpha = \Delta_{x_1}$, una condizione naturale è data dall'ipoellitticità di Hörmander (cf. [Hör67]) che impone, almeno formalmente, che gli $n-1$ commutatori di Lie iterati associati a ∂_{x_1} e $\langle Ax, D_x\rangle$ generino tutto lo spazio. Nel nostro caso non-locale, sebbene non sembra esistere un teorema di Hörmander in senso generale [KT01], alcune assunzioni naturali (condizioni [**H**] e **ND** sotto) assicurano che il semigruppo di Markov associato all'operatore

$$L^{\text{ou}} := \mathcal{L}_\alpha + \langle Ax, D_x\rangle$$

ammetta una densità sufficientemente regolare [PZ09].
A questo scopo, facciamo inoltre riferimento al lavoro di Cass [Cas09] dove è presentata un'estensione, seppure ancora incompleta, del risultato di Hörmander al caso non-locale, sotto condizioni molto generali.
In pratica, imporremo una particolare forma alla matrice A che assicuri l'ipoellitticità del sistema:

[H] la matrice A ha la seguente struttura sotto-diagonale:

$$A := \begin{pmatrix} 0_{d\times d} & \dots & \dots & \dots & 0_{d\times d} \\ A_{2,1} & 0_{d\times d} & \dots & \dots & 0_{d\times d} \\ 0_{d\times d} & A_{3,2} & 0_{d\times d} & \dots & 0_{d\times d} \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0_{d\times d} & \dots & 0_{d\times d} & A_{n,n-1} & 0_{d\times d} \end{pmatrix} \tag{4.6}$$

e gli elementi $A_{i,i-1}$ in $\mathbb{R}^d \otimes \mathbb{R}^d$ hanno tutti rango massimo d.

La struttura specifica scelta per la matrice A appare inoltre naturale (cf. [LP94]) in quanto invariante per le dilatazioni δ_λ (definite in (3.19) per $n = 2$) intrinseche alla nostra dinamica degenere, nel senso che

$$e^{tA} = \mathbb{M}_t e^A \mathbb{M}_t^{-1}, \tag{4.7}$$

dove $\mathbb{M}_t$ è una matrice in $\mathbb{R}^{nd} \otimes \mathbb{R}^{nd}$ data da

$$\left[\mathbb{M}_t\right]_{i,j} := \begin{cases} t^{i-1} I_{d\times d}, & \text{se } i = j; \\ 0_{d\times d}, & \text{altrimenti.} \end{cases} \tag{4.8}$$

Intuitivamente, $\mathbb{M}_t$ tiene conto della struttura multi-scala associata alla distanza d rispetto al tempo caratteristico $t^{\frac{1}{\alpha}}$. La decomposizione in (4.7) può essere ottenuta facilmente dalla definizione di matrice esponenziale e dall'identità $\mathbb{M}_t A \mathbb{M}_t^{-1} = tA$.
Sottolineiamo comunque che il nostro modello considera solo una particolare specifica struttura tra quelle possibilmente incluse nella teoria generale sull'ipoellitticità sviluppata da Hörmander. Infatti, la non-degenerazione degli elementi sotto-diagonali nella matrice A richiede, ad ogni livello della catena, di sfruttare una sola parentesi di Lie aggiuntiva per generare la direzione corrispondente. è proprio questa proprietà che

permette la trasmissione degli effetti regolarizzanti α-stabili ad ogni componente della catena, come spiegato nella sezione precedente.
Evidenziamo infine che nel nostro contesto non-lineare, la condizione "classica" di Hörmander (cf. [Hör67]) non può essere considerata, a causa della scarsa regolarità della deriva deterministica F che andremo ad assumere, solo Hölder continua in spazio. Questo ci impedisce in particolare di calcolare esplicitamente i commutatori necessari nella condizione di Hörmander.

La geometria anisotropica associata al nostro sistema differenziale degenere su $\mathbb{R}^{nd}$ può essere facilmente intesa come una estensione su n componenti di quella introdotta nella sezione precedente nel caso cinetico ($n = 2$), nel senso che, per esempio, la distanza spaziale $\mathbf{d}$ è ora definita attraverso

$$\mathbf{d}(x, x') \;=\; \sum_{i=1}^{n} |(x - x')_i|^{\frac{1}{1+\alpha(i-1)}}, \quad x, x' \in \mathbb{R}^{nd}. \tag{4.9}$$

In particolare, una funzione ϕ in $C_d^{\beta}(\mathbb{R}^{nd})$ sarà $\beta/(1+\alpha(i-1))$-Hölder continua rispetto alla sua variabile x_i, uniformemente nelle altre variabili x_j ($j \neq i$). La deriva deterministica $F = (F_1, \dots, F_n)$ va intesa come una perturbazione non-lineare del modello di Ornstein-Uhlenbeck degenere L^{ou}. Verrà scelta in modo tale che non vada a distruggere l'ipoellitticità del sistema considerato. Assumeremo inoltre un certo grado di regolarità sulla deriva deterministica F, necessario ai nostri scopi.

[R] per ogni livello i in $[\![1, n]\!]$, F_i dipende solo dal tempo e dalle ultime $n-(i-1)$ variabili, ovvero $F_i(t, x_i, \dots, x_n)$. Inoltre, F_i appartiene allo spazio $L^{\infty}(0, T; C_d^{\gamma_i+\beta}(\mathbb{R}^{nd}))$, dove

$$\gamma_i \;:=\; \begin{cases} 1 + \alpha(i-2), & \text{if } i > 1; \\ 0, & \text{if } i = 1. \end{cases} \tag{4.10}$$

Mettiamo già in evidenza che nessuna condizione di limitatezza è stata però imposta sulla deriva F ma solo una regolarità Hölder con multi-indici crescenti. Sarà infatti questo uno delle principali difficoltà da affrontare nel nostro metodo di dimostrazione. Sottolineiamo inoltre che, a differenza del caso non-degenere (cf. [CdRMP20a]), è necessario imporre qui una regolarità addizionale crescente sulle componenti degeneri ($i > 1$) della perturbazione F_i, rappresentata dal parametro γ_i sopra. Questa assunzione appare però naturale se uno pensa che, a causa della struttura degenere del nostro sistema, l'effetto regolarizzante dell'operatore α-stabile $\mathcal{L}_\alpha$ che agisce solo sulla prima componente si indebolisce via via che discende lungo la catena. In un certo senso, la regolarità aggiuntiva su F appare come il prezzo da pagare per riequilibrare le crescenti singolarità in tempo che appaiono lungo la catena.

Infine, è necessario imporre alcuni limiti ai possibili valori dell'indice di stabilità α in $(0, 2)$ e di quello di Hölder regolatità β in $(0, 1)$. In particolare,

[P] $\alpha + \beta < 2$ e se $\alpha < 1$, vale inoltre che

$$\beta < \alpha, \quad \alpha + \beta > 1, \quad 1 - \alpha < \frac{\alpha - \beta}{1 + \alpha(n-1)}.$$

Alcune considerazioni su tali limitazioni nel caso super-critico ($\alpha < 1$) sono ora necessarie. La condizione $\beta < \alpha$ riflette essenzialmente la debole proprietà di integrabilità (di ordine strettamente minore di α) per un processo α-stabile, possibilmente non-isotropico.
La condizione $\alpha + \beta > 1$ appare naturalmente per dare un significato puntuale al gradiente della soluzione u rispetto alla variabile non-degenere x_1. A riguardo, citiamo inoltre il lavoro di Tanaka *et al.* [TTW74] dove viene mostrato che il carattere ben posto in senso debole, proprietà strettamente interconnessa con le stime di Schauder qui coinvolte, può fallire già per una dinamica stocastica non-degenere con rumore stabile additivo su $\mathbb{R}$ quando $\alpha + \beta < 1$, dove α è l'indice di stabilità del processo e β la regolarità di Hölder della deriva deterministica. Tale controesempio si può intendere come una generalizzazione stocastica dell'esempio di Peano, mostrato in (2.5).
L'ultima assunzione è invece una limitazione tecnica e sembra associata essenzialmente al nostro metodo di prova di tipo perturbativo.
Sottolineiamo infine che nel caso sotto-critico, quando $\alpha \geq 1$, queste condizioni sono sempre soddisfatte.

Data la scarsa regolarità assunta dai coefficienti, è possibile considerare la dinamica (4.1) solo in un senso distribuzionale. Infatti, la regolarità attesa (attraverso il *bootstrap* parabolico in spazio dato dalle stime di Schauder) per una soluzione u in senso classico del problema è nello spazio $L^\infty(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^N))$, non sufficiente per dare un significato puntuale al gradiente $D_x u$ rispetto alla variabili degeneri e quindi dare un senso classico all'equazione.
Sfrutteremo invece qui altre due possibili nozioni di soluzione, più adatte ai nostri scopi. Come accennato prima, per soluzione debole dell'equazione (4.1) intenderemo essenzialmente una soluzione nel senso delle distribuzioni, cioè una funzione $u\colon [0,T] \times \mathbb{R}^{nd} \to \mathbb{R}$ tale che per ogni funzione test ϕ (funzione liscia a supporto compatto) su $(0,T] \times \mathbb{R}^{nd} \to \mathbb{R}$, vale che

$$\int_0^T \int_{\mathbb{R}^{nd}} \Big(-\partial_t + (L_t)^*\Big)\phi(t,y)u(t,y)\,dy + \int_{\mathbb{R}^{nd}} u_T(y)\phi(T,y)\,dy \\ = -\int_0^T \int_{\mathbb{R}^{nd}} \phi(t,y)f(t,y)\,dy,$$

dove $(L_t)^*$ rappresenta (formalmente) l'operatore aggiunto di L_t dato da

$$L_t = \langle Ax + F(t,x), D_x\rangle + \mathcal{L}_\alpha \quad \text{su} \quad \mathbb{R}^{nd}.$$

Invece, una soluzione *mild* del problema (4.1) (nel senso di Stroock e Varadhan [SV79]) sarà una funzione $u\colon [0,T] \times \mathbb{R}^{nd} \to \mathbb{R}$ ottenuta come limite in uno spazio funzionale adeguato della sequenza di soluzioni classiche per delle versioni regolarizzate del Problema di Cauchy considerato. Per maggiori dettagli, si veda Definizione 2.2 nel Capitolo 2 oppure il libro [Kol11].

I principali risultati del Capitolo 2 possono essere ora riassunti nel seguente teorema:

Teorema 4.1. *Sotto le ipotesi descritte sopra, sia f in $L^\infty(0,T;C^{\beta}_{b,d}(\mathbb{R}^{nd}))$ ed u_T in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})$. Allora, esiste un'unica soluzione mild e debole $u\colon [0,T] \times \mathbb{R}^{nd} \to \mathbb{R}$ al*

problema di Cauchy (4.1). *Inoltre, u appartiene a $L^\infty(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd}))$ ed esiste una constante C, indipendente da f e u_T, tale che*

$$\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \leq C\Big[\|f\|_{L^\infty(C^{\beta}_{b,d})} + \|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big]. \tag{4.11}$$

Concludendo questa sezione, sottolineiamo infine che è possibile, con piccole modifiche agli argomenti esposti sotto, considerare dinamiche con una deriva deterministica completamente non-lineare o un coefficiente di diffusione inomogeneo in spazio-tempo in uno spazio funzionale adeguato, come spiegato alla fine del Capitolo 2.

4.1 Guida alla Prova

Presentiamo qui brevemente il metodo di prova da noi usato per determinare le stime di Schauder (4.11) sotto le assunzioni introdotte nella sezione precedente.

Le difficoltà principali da affrontare nel nostro contesto saranno collegate alla degenerazione dell'operatore $\mathcal{L}_\alpha$ α-stabile che agisce solo sulla prima componente ed alla non-limitatezza della perturbazione F. Ricordiamo inoltre che vogliamo ottenere stime di Schauder sotto le regolarità di Hölder minime per i coefficienti dell'equazione. In particolare, non potremo confidare su derivate lungo le componenti degeneri. Per fare ciò, dovremo sfruttare proprietà di dualità tra gli spazi di Hölder e quelli di Besov ed in particolare, stabilire controlli delicati in norma di quest'ultimo.

Il nostro approccio si basa su un metodo perturbativo conosciuto come *tecnica della parametrice*, originariamente introdotta da Levi [Lev07] per l'analisi di equazioni alle derivate parziali lineari ellittiche di ordine pari a coefficienti variabili. In ambito diffusivo non-degenere, citiamo invece i lavori di Friedman [Fri64] e di McKean e Singer in [MS67] che sfruttano tale tecnica per ottenere stime di tipo Aronson (cf. [Aro59, Aro67]) sulla soluzione fondamentale del sistema, rispettivamente sotto le condizioni di Hölder regolarità in spazio e tempo o di misurabilità in tempo e Hölder continuità in spazio, dei coefficienti.
In un contesto degenere più simile a quello presente ma sempre di tipo Gaussiano, tale metodo è stato poi sfruttato in [CdRHM18a] per ottenere stime di Schauder per una catena diffusiva degenere.
Ricordiamo infine che Hadamard in [Had32, Had64] ha esteso tale tecnica anche all'analisi di equazioni di tipo iperbolico.
Un altro metodo, più classico, per ottenere le stime di Schauder è di procedere sfruttando controlli a priori sulla soluzione fondamentale associata al problema. Esistenza ed unicità di una soluzione per l'equazione considerata, in un uno spazio funzionale adatto, sono in questo caso affrontate solo in un secondo momento. Facciamo riferimento a [Fri64] e a [Kry96] per una presentazione chiara di tale approccio a priori o a [KP10] per un'estensione al caso non-degenere diffusivo con coefficienti non-limitati.

L'Operatore Proxy Congelato

L'elemento cruciale nel metodo della parametrice da noi considerato consiste nel scegliere un operatore *proxy* adatto per l'equazione di interesse, ovvero un operatore $\tilde{L}_t$ le

cui proprietà (esistenza e comportamento della densità o del semigruppo Markoviano associato) siano note e che si avvicini, in un determinato senso, all'operatore originale L_t. Potremo poi applicare un'espansione del primo ordine, come una formula di tipo Duhamel, per la soluzione dell'equazione (4.1) attorno all'operatore proxy scelto. Sfruttando le proprietà note sull'operatore proxy, potremo infine ottenere un controllo adatto dell'errore di espansione.

Nel caso di coefficienti limitati, una scelta comune per il proxy è data dall'operatore originario a coefficienti costanti. Quando si considerano invece perturbazioni F potenzialmente illimitate come nel nostro caso, è invece naturale (cf. [KP10, CdRMP20a]) usare un operatore che coinvolga un termine di primo ordine non-nullo, come quello associato al flusso determinato dal termine di transporto $Ax + F$:

$$\begin{cases} d\theta_{\tau,s}(\xi) \;=\; \Big[A\theta_{\tau,s}(\xi) + F(s,\theta_{\tau,s}(\xi))\Big]ds \quad \text{on } [\tau,T]; \\ \theta_{\tau,\tau}(\xi) \;=\; \xi, \end{cases} \tag{4.12}$$

dove (τ,ξ) in $[0,T]\times\mathbb{R}^N$ sono due parametri di "congelamento", rispettivamente in tempo e spazio, il cui valore esatto sarà scelto successivamente.
Notiamo subito però che la soluzione alla dinamica sopra può non essere unica, visto che la tendenza F è solo Hölder continua in spazio. Per questa ragione, andremo a scegliere un particolare flusso, denotato da $\theta_{\tau,s}(\xi)$, e lo considereremo fissato da ora in avanti. L'operatore proxy da noi scelto è allora ottenuto congelando quello originale L_t lungo il flusso $\theta_{\tau,t}(\xi)$ fissato:

$$\tilde{L}_t^{\tau,\xi} \;=\; \mathcal{L}_\alpha + \langle Ax + F(t,\theta_{\tau,t}(\xi)), D_x\rangle.$$

Possiamo ora introdurre il Problema di Cauchy "congelato" associato al proxy scelto:

$$\begin{cases} \Big(\partial_t + \tilde{L}_t^{\tau,\xi}\Big)\,\tilde{u}^{\tau,\xi}(t,x) \;=\; -f(t,x) & \text{su } [0,T)\times\mathbb{R}^{nd}; \\ \tilde{u}^{\tau,\xi}(T,x) \;=\; u_T(x) & \text{su } \mathbb{R}^{nd}. \end{cases} \tag{4.13}$$

Per determinare le proprietà dell'operatore proxy congelato $\tilde{L}^{\tau,\xi}$, introduciamo ora la dinamica stocastica associata. Fissato un punto iniziale (t,x) in $[0,T]\times\mathbb{R}^{nd}$ e un processo α-stabile simmetrico $\{Z_s\}_{s\geq t}$ su $\mathbb{R}^d$ con misura di Lévy data da ν_α, siamo interessati a

$$\begin{cases} \tilde{X}_s^{\tau,\xi} & = \Big[A\tilde{X}_s^{\tau,\xi} + F(s,\theta_{\tau,s}(\xi))\Big]\,ds + BdZ_s; \quad s>t \\ \tilde{X}_t^{\tau,\xi} & = x. \end{cases}$$

Un'integrazione esplicita attraverso la funzione esponenziale matriciale permette poi di riscrivere la dinamica nel seguente modo:

$$\begin{aligned} \tilde{X}_s^{\tau,\xi} \;&=\; e^{A(s-t)}x + \int_t^s e^{A(s-u)}F(u,\theta_{\tau,u}(\xi))\,du + \int_t^s e^{A(s-u)}BdZ_u \\ &=:\; \tilde{m}_{t,s}^{\tau,\xi}(x) + \int_t^s e^{A(s-u)}BdZ_u \end{aligned} \tag{4.14}$$

Grazie alla simmetria di Z_t, possiamo pensare al termine di trasporto $\tilde{m}_{t,s}^{\tau,\xi}(x)$ come alla "media" del processo congelato $\tilde{X}_s^{\tau,\xi}$ affine al punto iniziale x (anche se quando $\alpha<1$,

il valore medio di tale processo non è definito) o almeno come il valore lungo cui il processo fluttua.
L'identità sopra è ora fondamentale per mostrare che la convoluzione stocastica

$$\Lambda_{t,s} := \int_t^s e^{A(s-v)} B dZ_v$$

è a sua volta un processo α-stabile, simmetrico e non-degenere sullo spazio "grande" $\mathbb{R}^{nd}$ ma riscalato a seconda della struttura anisotropica del sistema. Infatti, passando su spazi di Fourier, notiamo che

$$\begin{aligned} \mathbb{E}\Big[\exp\Big(i\langle p, \Lambda_{t,s}\rangle\Big)\Big] &= \exp\Big(-\int_t^s \int_{\mathbb{S}^{d-1}} \Big|\langle p, e^{(s-u)A} B\varsigma\rangle\Big|^\alpha \, \mu(d\varsigma) du\Big) \\ &=: \exp\Big(\Phi_{\Lambda_{t,s}}(p)\Big), \end{aligned} \tag{4.15}$$

ovvero $\Phi_{\Lambda_{t,s}}$ è il simbolo di Lévy associato alla variabile aleatoria $\Lambda_{t,s}$ ad ogni tempo $t < s$ fissato. Il cambio di variabile $v = (s-u)/(s-t)$ ci permette di riscriverlo nel seguente modo

$$\begin{aligned} \Phi_{\Lambda_{t,s}}(p) &= (t-s) \int_0^1 \int_{\mathbb{S}^{d-1}} |\langle p, e^{(s-t)vA} B\varsigma\rangle|^\alpha \, \mu(d\varsigma) dv \\ & (t-s) \int_0^1 \int_{\mathbb{S}^{d-1}} |\langle \mathbb{M}_{s-t} p, e^{vA} B\varsigma\rangle|^\alpha \, \mu(d\varsigma) dv, \end{aligned}$$

sfruttando nell'ultimo passaggio la decomposizione di $e^{A(s-t)v}$ data in (4.7):

$$e^{(s-t)vA} B = \mathbb{M}_{s-t} e^{vA} B,$$

dove la matrice di scalabilità $\mathbb{M}_{s-t}$ è stata definita in (4.8). Inoltre, possiamo ora ri-normalizzare il termine nel prodotto scalare:

$$\begin{aligned} \Phi_{\Lambda_{t,s}}(p) &= (t-s) \int_0^1 \int_{\mathbb{S}^{d-1}} |\langle \mathbb{M}_{s-t} p, \frac{e^{vA} B\varsigma}{|e^{vA} B\varsigma|}\rangle|^\alpha \, |e^{vA} B\varsigma|^\alpha \mu(d\varsigma) dv \\ &=: (t-s) \int_{[0,1]\times\mathbb{S}^{d-1}} |\langle \mathbb{M}_{s-t} p, \frac{e^{vA} B\varsigma}{|e^{vA} B\varsigma|}\rangle|^\alpha \, m_\alpha(d\varsigma, dv). \end{aligned}$$

dove $m_\alpha(dv, ds) := |e^{vA} Bs|^\alpha \mu(ds) dv$ è una misura prodotto su $[0,1] \times \mathbb{S}^{d-1}$. attraverso la funzione di *lift* $l\colon [0,1] \times \mathbb{S}^{d-1} \to \mathbb{S}^{nd-1}$ definita da

$$l(v, s) := \frac{e^{vA} Bs}{|e^{vA} Bs|}.$$

In particolare, definiamo con $\mu_S := \mathrm{Sym}(l_*(m_\alpha))$ la versione simmetrizzata misura immagine di m_α attraverso l. Allora:

$$\mathbb{E}\Big[\exp\Big(i\langle p, \Lambda_{t,s}\rangle\Big)\Big] = \exp\Big(-(s-t) \int_{\mathbb{S}^{d-1}} |\langle \mathbb{M}_{s-t} p, \eta\rangle|^\alpha \, \mu_S(d\eta)\Big). \tag{4.16}$$

Se indichiamo ora con $\{S_u\}_{u\geq 0}$ il processo di Lévy su $\mathbb{R}^{nd}$ il cui simbolo di Lévy Φ_S è dato da

$$\Phi_S(p) = -\int_{S^{nd-1}} |\langle p, \eta\rangle|^\alpha \mu_S(d\eta),$$

otteniamo infine la seguente identità in legge:

$$\Lambda_{s,t} \overset{\text{(legge)}}{=} \mathbb{M}_{s-t} S_t. \tag{4.17}$$

A questo punto, pensiamo sia necessario sottolineare che, seppure la misura spettrale μ_S sia effettivamente non-degenere nel senso di [**ND**], essa è estremamente singolare rispetto alla misura di Lebesgue sulla sfera $\mathbb{S}^{nd-1}$. Infatti, notiamo da come è stata costruita la misura μ_S che il suo supporto è dato dall'immagine del supporto di $|e^{vA}Bs|^\alpha \mu(ds)dv$ attraverso la funzione di lift $(v, \varsigma) \mapsto e^{vA}Bs/|e^{vA}Bs|$ su $\mathbb{S}^{nd-1}$. Assumendo anche che il supporto della misura spettrale μ associato al processo $\{Z_t\}_{t\geq 0}$ sia la sfera $\mathbb{S}^{d-1}$, notiamo che la dimensione del supporto di μ_S sarà solo $d-1+1=d$.

Sotto la condizione di non-degenescenza [**ND**], sappiamo allora che il processo $\{S_t\}_{t\geq 0}$ ammette una densità regolare $p_S(t,z)$. Equazioni (4.14) e (4.17) implicano ora l'esistenza di una densità $\tilde{p}^{\tau,\xi}$ associata all'operatore congelato $\tilde{L}_t^{\tau,\xi}$ data da

$$\tilde{p}^{\tau,\xi}(t,s,x,y) \;=\; \frac{1}{\det(\mathbb{M}_{s-t})} p_S\Big(s-t, \mathbb{M}_{s-t}^{-1}(y - \tilde{m}_{t,s}^{\tau,\xi}(x))\Big). \tag{4.18}$$

In particolare, le stime sulla densità del processo congelato $\tilde{X}_s^{\tau,\xi}$ saranno ottenute a partire da quelle sulla densità di $\{S_u\}_{u\geq 0}$. Per completezza, introduciamo anche il semigruppo Markoviano congelato $\left\{\tilde{P}_{t,s}^{\tau,\xi}\right\}_{t\leq s}$ dato da

$$\tilde{P}_{t,s}^{\tau,\xi}\phi(x) \;:=\; \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t,s,x,y)\phi(y)\,dy,$$

per ogni funzione $\phi\colon \mathbb{R}^{nd} \to \mathbb{R}$ sufficientemente regolare. Vogliamo ora andare ad analizzare quali proprietà di regolarità tale densità congelata $\tilde{p}^{\tau,\xi}(t,s,x,\cdot)$ possiede. Notiamo innanzitutto, che le derivate in spazio possono essere controllare da un'altra densità ma al prezzo di singolarità in tempo aggiuntive. Per ogni i in $[\![1,n]\!]$ e $k \in \{1,2\}$, vale infatti che

$$\Big|D_{x_i}^k \tilde{p}^{\tau,\xi}(t,s,x,y)\Big| \;\leq\; C\frac{\bar{p}\Big(s-t, \mathbb{M}_{s-t}^{-1}(y - \tilde{m}_{t,s}^{\tau,\xi}(x))\Big)}{\det(\mathbb{M}_{s-t})}(s-t)^{-k\frac{1+\alpha(i-1)}{\alpha}}. \tag{4.19}$$

dove $\bar{p}(t,\cdot)$ è una densità su $\mathbb{R}^{nd}$ con proprietà adatte. In parole povere, derivare la densità congelata $\tilde{p}^{\tau,\xi}(t,s,x,\cdot)$ induce una singolarità in tempo la cui intesità dipende dalle scale intrinseche del sistema legate alla direzione di derivazione. Elemento cruciale e ricorrente nella nostra analisi consisterà del ri-equilibrare queste singolarità in tempo attraverso le regolarità in spazio della densità $\bar{p}(t,\cdot)$.
In particolare, mostreremo che essa presenta un effetto regolarizzante in spazio di ordine α, nel senso che, per ogni γ in $[0,\alpha)$, vale che

$$\int_{\mathbb{R}^{nd}} \frac{\bar{p}\Big(s-t, \mathbb{M}_{s-t}^{-1}(y - \tilde{m}_{t,s}^{\tau,\xi}(x))\Big)}{\det(\mathbb{M}_{s-t})} \mathbf{d}^\gamma(y, \tilde{m}_{t,s}^{\tau,\xi}(x))\,dy \;\leq\; C(s-t)^{\frac{\gamma}{\alpha}}, \tag{4.20}$$

dove, ricordiamo, la distanza omogenea $\mathbf{d}$ è stata definita in (3.22) nel caso $n=2$ e poi estesa naturalmente al nostro contesto più generale in (4.9).

Possiamo evidenziare ora una delle principali differenze tra il caso diffusivo, considerato per esempio in [CdRHM18a], e quello α-stabile analizzato qui, caratterizzato da un effetto regolarizzante limitato strettamente dall'ordine di α-stabilità dell'operatore. In termini stocastici, questo problema è essenzialmente legato alle più deboli proprietà di integrabilità associate ai processi di Lévy rispetto al moto browniano.
Le proprietà presentate in (4.19) e (4.20) possono essere infine riassunte attraverso l'effetto regolarizzante associato all'operatore proxy $\tilde{L}_t^{\tau,\xi}$ nel seguente controllo fondamentale sul suo semigruppo congelato:

$$\left|D_{x_i}^k \tilde{P}_{t,s}^{\tau,\xi}\phi(x)\right| \;\leq\; C\|\phi\|_{C_d^\gamma}(s-t)^{\frac{\gamma}{\alpha}-k\frac{1+\alpha(i-1)}{\alpha}}, \quad \forall \phi \in C_d^\gamma(\mathbb{R}^{nd}). \tag{4.21}$$

Controlli come in (4.21) sono ottenuti a partire dalle proprietà regolarizzanti della densità (4.19), utilizzando tecniche di cancellazione. Notando che

$$\int_{\mathbb{R}^{nd}} D_{x_i}\tilde{p}^{\tau,\xi}(t,s,x,y)\,dy \;=\; 0, \tag{4.22}$$

per ogni coppia di parametri congelati (τ,ξ), l'idea sottostante consiste nell'aggiungere un termine all'interno dell'integrale e sfuttare così la regolarità di Hölder della funzione ϕ, ovvero:

$$\begin{aligned}
\left|D_{x_i}^k \tilde{P}_{t,s}^{\tau,\xi}\phi(x)\right| &= \int_{\mathbb{R}^{nd}} D_{x_i}^k \tilde{p}^{\tau,\xi}(t,s,x,y)\phi(y)\,dy \\
&= \int_{\mathbb{R}^{nd}} D_{x_i}^k \tilde{p}^{\tau,\xi}(t,s,x,y)\left[\phi(y)-\phi(\tilde{m}_{t,s}^{\tau,\xi}(x))\right]\,dy \\
&\leq \|\phi\|_{C_d^\gamma}\int_{\mathbb{R}^{nd}}\left|D_{x_i}^k \tilde{p}^{\tau,\xi}(t,s,x,y)\right| d^\gamma(y-\tilde{m}_{t,s}^{\tau,\xi}(x))\,dy \\
&\leq C\|\phi\|_{C_d^\gamma}(s-t)^{\frac{\gamma}{\alpha}-k\frac{1+\alpha(i-1)}{\alpha}}.
\end{aligned} \tag{4.23}$$

Sfruttando attentamente gli effetti regolarizzanti in spazio in modo da bilanciare le singolarità in tempo dovute alle derivate della densità (cf. Controllo (4.19)), è possibile mostrare ora che le stime di Schauder valgono per la soluzione $\tilde{u}^{\tau,\xi}$ della dinamica congelata (4.13):

$$\|\tilde{u}^{\tau,\xi}\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \;\leq\; C\Big[\|f\|_{L^\infty(C_{b,d}^{\beta})} + \|u_T\|_{C_{b,d}^{\alpha+\beta}}\Big]. \tag{4.24}$$

In particolare, tali stime sono valide per ogni valore *fissato* dei parametri di congelamento (τ,ξ) e la costante C sopra è indipendente da essi.
Grazie alla stabilità data dai controlli sopra, è inoltre possibile rappresentare l'unica soluzione $\tilde{u}^{\tau,\xi}$ del problema di Cauchy (4.13) congelato in termini del semigruppo Markoviano $\tilde{P}_{t,s}^{\tau,\xi}$:

$$\tilde{u}^{\tau,\xi}(t,x) \;=\; \tilde{P}_{t,T}^{\tau,\xi}u_T(x) + \int_t^T \tilde{P}_{t,s}^{\tau,\xi}f(s,x)\,ds. \tag{4.25}$$

Vogliamo ora applicare il metodo perturbativo così da ottenere le stime di Schauder per una soluzione u dell'equazione di partenza a partire da quelle appena ottenute in (4.24) per la soluzione $\tilde{u}^{\tau,\xi}$ del problema congelato associato all'operatore proxy.

Espansione di Tipo Duhamel

Almeno formalmente (in pratica, attraverso una mollificazione dei coefficienti sfruttando la definizione di soluzione mild), si può riscrivere la dinamica originale (4.1) attorno all'operatore $\tilde{L}_\alpha^{\tau,\xi}$ congelato nel seguente modo:

$$\begin{cases} \left(\partial_t + \tilde{L}_t^{\tau,\xi}\right) u(t,x) \;=\; -f(t,x) - \left(L_t - \tilde{L}_t^{\tau,\xi}\right) u(t,x), & \text{on } (0,T)\times\mathbb{R}^{nd}; \\ u(T,x) \;=\; u_T(x) & \text{on } \mathbb{R}^{nd}. \end{cases}$$

L'unicità di soluzione per il problema congelato (4.13) e la rappresentazione di $\tilde{u}^{\tau,\xi}$ in (4.25) implicano ora la seguente formula di Duhamel, che corrisponde ad un espansione della parametrice del primo ordine:

$$u(t,x) \;=\; \tilde{u}^{\tau,\xi}(t,x) + \int_t^T \tilde{P}_{t,s}^{\tau,\xi} R^{\tau,\xi}(s,x)\, ds, \tag{4.26}$$

dove $R^{\tau,\xi}$ è il resto dato da

$$R^{\tau,\xi}(t,x) \;:=\; \langle F(t,x) - F(t,\theta_{\tau,t}(\xi)), D_x u(t,x)\rangle. \tag{4.27}$$

Visto che il controllo adatto (4.24) per la soluzione congelata $\tilde{u}^{\tau,\xi}$ è già stato dimostrato, la rappresentazione di Duhamel (4.26) indica che per ottenere le stime di Schauder per u, il termine principale che rimane da investigare è il resto

$$\int_t^T \tilde{P}_{t,s}^{\tau,\xi} R^{\tau,\xi}(s,x)\, ds, \tag{4.28}$$

che rappresenta proprio l'errore di espansione lungo il proxy.

Fino a questo momento, i parametri di congelamento (τ,ξ) sono stati considerati fissati ma liberi. Verranno ora scelti in maniera appropriata a seconda del controllo che si vuole stabilire. In particolare, seguiremo in questo lavoro un approccio della parametrice di tipo *progressivo*, nel senso che andremo a imporre $(\tau,\xi) = (t,x)$ e quindi il flusso $\theta_{\tau,s}(\xi)$ dato in (1.5) si muoverà progressivamente dal punto iniziale (t,x) verso (s,y), dove y è la variabile di integrazione nella densità congelata. Questo metodo è stato ampiamente utilizzato da Friedman [Fri64] e Il'in et al. [IKO62] per ottenere controlli puntuali sulle derivate della soluzione fondamentale per l'equazione del calore o da Chaudru de Reynal in [CdR17] per derivare la buona posizione in senso forte per una catena diffusiva degenere di tipo cinetico. In particolar modo, il metodo progressivo permette di sfruttare meglio le tecniche di cancellazione in (4.23) che abbiamo visto essere fondamentali per il controllo delle derivate della densità congelata $\tilde{p}^{\tau,\xi}(t,s,x,y)$. Per completezza, citiamo che esiste anche un metodo della parametrice di tipo *retrogrado* (fissando $(\tau,\xi) = (s,y)$), introdotto da McKean e Singer in [MS67]. Sottolineiamo però che in questo caso, la densità $\tilde{p}^{\tau,\xi}(t,s,x,y)$ congelata in $\xi = y$ non è più una vera densità rispetto alla variabile y, visto che il parametro di congelamento svolge il ruolo anche di variabile di integrazione. Questo rende gli effetti regolarizzanti presentati in (4.21) per il semigruppo Markoviano molto più difficili da stabilire. Per maggiori dettagli sul metodo perturbativo a parametrice retrograda, ci riferiamo alla Sezione 6 del presente lavoro.
Ricordiamo infine che nel caso di coefficienti limitati, una scelta naturale è data dal flusso triviale $\theta_{\tau,s}(\xi) = \xi$.

Controllo dell'Errore di Espansione

Per concludere il metodo perturbativo, dobbiamo infine mostrare che l'errore di espansione nella formula di Duhamel apporta un contributo piccolo alle stime di Schauder (4.24) per la soluzione congelata $\tilde{u}^{\tau,\xi}$. Come accennato all'inizio di questa sezione, questo controllo sarà il più delicato da stabilire, a causa della scarsa regolarità della deriva F lungo le componenti degeneri x_i $(i > 1)$. Per mostrare brevemente quali siano effettivamente le difficoltà, ci concentreremo nello stabilire un controllo in norma uniforme sul termine in (4.28). I controlli successivi in norma uniforme per la derivata e quelli rispetto alla semi-norma di Hölder sono ancora più lunghi da stabilire sebbene condividono comunque lo stesso approccio.
Iniziamo decomponendo il resto rispetto alle componenti del sistema:

$$\left|\int_t^T \tilde{P}^{\tau,\xi}_{t,s} R^{\tau,\xi}(s,x)\,ds\right| = \left|\sum_{j=1}^n \int_t^T \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\cdot D_{y_j}u(s,y)\,dyds\right|, \quad (4.29)$$

dove abbiamo chiamato, per semplicità

$$\Delta^{\tau,\xi}F_j(s,y) := F_j(s,y) - F_j(s,\theta_{\tau,s}(\xi)), \quad j \in [\![1,n]\!].$$

Ricordando che la soluzione u è differenziabile rispetto alla componente non-degenere $(j = 1)$, il primo contributo della sommatoria può essere controllato sfruttando ancora gli effetti regolarizzanti della densità congelata in (4.19):

$$\begin{aligned}
&\left|\int_t^T \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_1(s,y)\cdot D_{y_1}u(s,y)\,dyds\right| \\
&\qquad \le C\|D_{y_1}u\|_\infty \|F_1\|_{C^\beta_d} \int_t^T \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t,s,x,y) d^\beta\Big(y,\theta_{\tau,s}(\xi)\Big)\,dyds \\
&\qquad \le C\|D_{y_1}u\|_\infty \|F_1\|_{C^\beta_d} \int_t^T (s-t)^{\frac{\beta}{\alpha}}\,ds \\
&\qquad \le C\|D_{y_1}u\|_\infty \|F_1\|_{C^\beta_d} (T-t)^{\frac{\alpha+\beta}{\alpha}}.
\end{aligned}$$

In particolare, notiamo che imporre $(\tau,\xi) = (t,x)$ sia la scelta naturale per bilanciare la differenza tra le tendenze deterministiche

$$|F_1(s,y) - F_1(s,\theta_{\tau,s}(\xi))|$$

e la struttura anisotropica della densità congelata:

$$\frac{1}{\det(\mathbb{M}_{s-t})}\bar{p}\Big(s-t, \mathbb{M}^{-1}_{s-t}(y - \tilde{m}^{\tau,\xi}_{t,s}(x))\Big)$$

e poter sfruttare così le proprietà regolarizzanti multi-scala delle densità congelata in (4.20). Infatti, è facile controllare dalle dinamiche (4.12) e (4.14) che per la nostra scelta di parametri congelati vale che $\tilde{m}^{t,x}_{t,s}(x) = \theta_{t,s}(x)$.

Possiamo ora concentrarci sui contributi degeneri $(j > 1)$ nella decomposizione in (4.29). Visto che u non è differenziabile in y_j se $j > 1$ (infatti è solo $(\alpha+\beta)/(1+\alpha(j-1))$-Hölder continua in quella variabile), iniziamo spostando la derivata sugli altri termini attraverso un'integrazione per parti:

$$\left|\int_t^T \int_{\mathbb{R}^{nd}} D_{y_j}\cdot\Big\{\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\Big\}u(s,y)\,dyds\right|. \quad (4.30)$$

L'idea sarebbe allora di controllare $D_{y_j} \cdot \left\{\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\right\}$ sfruttando la regolarità della soluzione u in $L^\infty(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd}))$. A priori, tale controllo però non sembra immediato visto che il termine $\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)$ non è differenziabile in y_j a causa della scarsa regolarità della deriva F.
Per ottenerlo, sarà infatti necessario applicare un ragionamento di dualità su spazi di Besov. Infatti, ricordiamo che per un $\tilde{\gamma}$ generico in $\mathbb{R}$, vale la seguente identificazione

$$C^{\tilde{\gamma}}_b(\mathbb{R}^d) \,=\, B^{\tilde{\gamma}}_{\infty,\infty}(\mathbb{R}^d), \tag{4.31}$$

dove per p,q in $[1,\infty]$, $B^{\tilde{\gamma}}_{p,q}(\mathbb{R}^{nd})$ rappresenta uno spazio di Besov su $\mathbb{R}^{nd}$ di indici $(\tilde{\gamma},p,q)$. Inoltre, è noto (si veda per esempio Proposizione 3.6 in [LR02]) che $B^{\tilde{\gamma}}_{\infty,\infty}(\mathbb{R}^d)$ e $B^{-\tilde{\gamma}}_{1,1}(\mathbb{R}^d)$ sono in dualità, nel senso che

$$\Big|\int_{\mathbb{R}^d} fg\,dx\Big| \,\leq\, C\|f\|_{B^{\tilde{\gamma}}_{\infty,\infty}}\|g\|_{B^{-\tilde{\gamma}}_{1,1}}, \tag{4.32}$$

per ogni f in $B^{\tilde{\gamma}}_{\infty,\infty}(\mathbb{R}^d)$ ed ogni g in $B^{-\tilde{\gamma}}_{1,1}(\mathbb{R}^d)$.
Ricordiamo inoltre che esistono diversi modi per definire tali spazi di Besov (attraverso modulo di continuità, decomposizione alla Littlewood-Paley, ecc.) ma la caratterizzazione termica, attraverso la convoluzione con un nucleo del calore frazionario, sembra essere la più naturale per i nostri scopi. Per un'analisi più dettagliata sull'argomento, suggeriamo di vedere Sezione 2.6.4 in Triebel [Tri92].
Definiamo allora lo spazio di Besov con indici $(\tilde{\gamma},p,q)$ su $\mathbb{R}^d$ come:

$$B^{\tilde{\gamma}}_{p,q}(\mathbb{R}^d) := \{f \in \mathcal{S}'(\mathbb{R}^d)\colon \|f\|_{\mathcal{H}^{\tilde{\gamma}}_{p,q}} < +\infty\},$$

dove $\mathcal{S}(\mathbb{R}^d)$ è la classe di Schwartz su $\mathbb{R}^d$. La norma $\|\cdot\|_{\mathcal{H}^{\tilde{\gamma}}_{p,q}}$ è poi data da

$$\|f\|_{\mathcal{H}^{\tilde{\gamma}}_{p,q}} \,:=\, \|(\phi_0\hat{f})^\vee\|_{L^p} + \left(\int_0^1 v^{-\frac{\tilde{\gamma}}{\alpha}}\|\partial_v p_h(v,\cdot) * f\|^q_{L^p}\,dv\right)^{\frac{1}{q}}, \tag{4.33}$$

dove ϕ_0 è una funzione test su $C^\infty_c(\mathbb{R}^d)$ tale che $\phi_0(0) \neq 0$ e p_h è il nucleo del calore α-stabile isotropo su $\mathbb{R}^d$, ovvero la densità su $\mathbb{R}^d$ il cui simbolo di Lévy è equivalente a $-|\lambda|^\alpha$.
Il vantaggio principale della caratterizzazione termica è proprio quello di permettere il passaggio della derivata D_{y_j} sulla densità isotropa $p_h(s-t,y)$ aggiuntiva e di sfruttare quindi la regolarità di Hölder della deriva F_j.

Possiamo allora sfruttare l'identificazione in (4.31) e la proprietà di dualità in (4.32) lungo la componente y_j per stimare la quantità (4.30):

$$\begin{aligned}
&\Big|\int_t^T\int_{\mathbb{R}^{nd}} D_{y_j}\cdot\left\{\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\right\}u(s,y)\,dyds\Big| \\
&\leq \|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\int_t^T\int_{\mathbb{R}^{(n-1)d}}\left\|y_j \to D_{y_j}\cdot\left\{\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\right\}\right\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}dy_{\setminus j}ds,
\end{aligned}$$

dove abbiamo denotato, per semplicità, con $y_{\setminus j}$ la variabile in $\mathbb{R}^{d(n-1)}$ senza la componente y_j.

A questo punto, rimarrebbe allora da controllare in modo opportuno l'integrale della norma Besov sopra. Mostreremo in particolare nella Sezione 5.1 del Capitolo 2 la seguente stima ottimale:

$$\int_{\mathbb{R}^{(n-1)d}} \left\| y_j \to D_{y_j} \cdot \left\{ \tilde{p}^{\tau,\xi}(t,s,x,y_{\setminus j},\cdot)\Delta^{\tau,\xi} F_j(s,y) \right\} \right\|_{B_{1,1}^{-(\alpha_j+\beta_j)}} dy_{\setminus j} \\ \leq C\|F_j\|_{L^\infty(C_d^{\gamma_j+\beta})}(s-t)^{\frac{\beta}{\alpha}},$$

dove, ricordiamo, abbiamo preso $(\tau,\xi) = (t,x)$.

Argomento Circolare e Conclusione della Prova

Sfruttando i vari controlli riassunti sopra, saremo infine in grado di mostrare, a partire dalla formula di Duhamel in (4.26), la seguente stima per una soluzione u in $L^\infty(0,T;C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ della dinamica (4.1) originale:

$$\|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \leq C\Big[\|u_T\|_{C_{b,d}^{\alpha+\beta}} + \|f\|_{L^\infty(C_{b,d}^{\beta})}\Big] + C\sup_i \|F_i\|_{L^\infty(C_d^{\gamma_i+\beta})}\|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})}, \quad (4.34)$$

dove la costante C è indipendente da f, u_T e da F.
Evidenziamo in particolare che lo studio della norma completa associata farà perdere, come in (4.34), la dipendenza da $(s-t)$ in tempo piccolo e non consentirà infatti di applicare direttamente un argomento di tipo circolare per spostare il termine $\|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})}$ a sinistra del controllo sopra.
Se supponiamo allora che la norma di Hölder per la deriva F sia sufficientemente piccola, cioè, per esempio, tale che

$$C\sup_i \|F_i\|_{L^\infty(C_d^{\gamma_i+\beta})} \leq \frac{1}{2}, \quad (4.35)$$

possiamo invece sfruttare un argomento circolare per concludere che le stime di Schauder valgono anche per u:

$$\|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})} \leq 2C\Big[\|u_T\|_{C_{b,d}^{\alpha+\beta}} + \|f\|_{L^\infty(C_{b,d}^{\beta})}\Big]. \quad (4.36)$$

Nel caso generale, sarà invece necessario applicare all'inizio un procedimento di riscalamento dei coefficienti in modo da imporre un condizione simile a (4.35).

Sottolineiamo infine che il procedimento descritto in questa sezione può essere, in realtà, applicato efficacemente solo se l'intervallo di tempo considerato è abbastanza piccolo. Intuitivamente, questo appare naturale visto che l'errore di espansione da controllare, su cui si basa il metodo perturbativo, richiede proprio che l'operatore originale $\mathcal{L}_\alpha$ e quello proxy $\mathcal{L}_\alpha^{\tau,\xi}$ non siano troppo distanti l'uno dall'altro. Per ottenere allora le stime di Schauder per un tempo finale arbitrario ma finito, dovremo allora iterare il ragionamento sopra molteplici volte su ogni intervallo di tempo sufficientemente piccolo.

5 Stime di Schauder per un sistema degenere lineare di tipo Lévy

Presentiamo ora brevemente i risultati esposti nel Capitolo 3, pubblicati poi su *Journal of Mathematical Analysis and Applications*. Sebbene ci limiteremo qui ad una dinamica lineare, questo lavoro può essere inteso come un estensione del precedente capitolo sotto diversi punti di vista.
Fissato uno spazio "grande" $\mathbb{R}^N$, siamo interessati ad un'analisi del seguente operatore di Ornstein-Uhlenbeck:

$$L^{\mathrm{ou}} := \mathcal{L} + \langle Ax, D_x \rangle \quad \text{su } \mathbb{R}^N, \tag{5.1}$$

dove $\langle \cdot, \cdot \rangle$ denota il prodotto interno Euclideo su $\mathbb{R}^N$, A è una matrice in $\mathbb{R}^N \otimes \mathbb{R}^N$ e $\mathcal{L}$ è un operatore di Lévy possibilmente degenere, nel senso che potrebbe agire in maniera non-degenere solo su un sottospazio di $\mathbb{R}^N$.
Più in dettaglio, fissato un intero $d \leq N$ ed una matrice B in $\mathbb{R}^N \otimes \mathbb{R}^d$ tale che $\mathrm{rank}(B) = d$, l'operatore $\mathcal{L}$ può essere rappresentato per ogni funzione abbastanza liscia $\phi\colon \mathbb{R}^N \to \mathbb{R}$ attraverso

$$\begin{aligned}\mathcal{L}\phi(x) := {} & \frac{1}{2}\mathrm{Tr}\Big(B\Sigma B^* D_x^2\phi(x)\Big) + \langle Bb, D_x\phi(x)\rangle \\ & + \text{p.v.} \int_{\mathbb{R}_0^d} \Big[\phi(x+Bz) - \phi(x) - \langle D_x\phi(x), Bz\rangle \mathbb{1}_{B(0,1)}(z)\Big]\, \nu(dz),\end{aligned} \tag{5.2}$$

dove b è un vettore in $\mathbb{R}^d$, Σ è una matrice simmetrica, definita non-negativa in $\mathbb{R}^d \otimes \mathbb{R}^d$ e ν è una misura di Lévy su $\mathbb{R}_0^d$.
Evidenziamo già che per questo modello non supporremo più la *simmetria* della misura ν come invece fatto nel Capitolo 2. È proprio per questo motivo che non possiamo cancellare ora il termine di ordine primo $\langle D_x\phi(x), Bz\rangle$ nella definizione dell'operatore $\mathcal{L}$ in (5.2) (cf. Equazione (4.2) nella sezione precedente).

Siamo interessati qui a stabilire la buona posizione e le stime di Schauder associate per equazioni ellittiche e paraboliche che coinvolgano l'operatore L^{ou} sotto condizioni *minime* di regolarità di Hölder sui coefficienti. In particolare, fissato $\lambda > 0$, considereremo la seguente equazione ellittica:

$$\lambda u(x) - L^{\mathrm{ou}} u(x) = g(x), \quad x \in \mathbb{R}^N, \tag{5.3}$$

e, per un tempo finale fissato $T > 0$, il seguente problema di Cauchy:

$$\begin{cases} \partial_t u(t,x) = L^{\mathrm{ou}} u(t,x) + f(t,x), & (t,x) \in (0,T) \times \mathbb{R}^N; \\ u(0,x) = u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{5.4}$$

dove f, g, u_0 sono funzioni date. Visto che il nostro obbiettivo è in particolare stabilire risultati di regolarità ottimali, fisseremo un $\beta \in (0,1)$ ed assumeremo, nel problema ellittico (1.5), che la sorgente g appartenga allo spazio di Hölder anisotropico $C^{\beta}_{b,d}(\mathbb{R}^N)$ mentre per quello parabolico (5.4), che u_0 sia in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)$ e f in $L^\infty(0,T; C^{\beta}_{b,d}(\mathbb{R}^N))$. Una definizione esatta degli spazi di Hölder anisotropici $C^{\gamma}_{b,d}(\mathbb{R}^N)$ in questo contesto sarà data solo in un secondo momento, quando introdurremo le condizioni sul sistema.

Un'altra differenza rispetto al Capitolo 2 è che per questo modello lineare, assumeremo soltanto che A e B soddisfino una condizione di tipo Hörmander debole, detta del *rango di Kalman*, che assicuri l'ipoellitticità del sistema e che per un certo $\alpha < 2$, l'operatore di Lévy $\mathcal{L}$ sia comparabile, in un senso adatto, ad un operatore α-stabile possibilmente troncato e non-degenere sullo stesso sottospazio ($B\mathbb{R}^N \sim \mathbb{R}^d$) di $\mathbb{R}^N$.
Più in dettaglio, la misura di Lévy associata alla parte integro-differenziale dell'operatore $\mathcal{L}$ sarà controllata dal basso con la misura di Lévy di un operatore α-stabile possibilmente troncato. Ricordando da (4.3) che ogni misura di Lévy α-stabile ν_α può essere decomposta in una parte sferica μ ed una radiale $r^{-(1+\alpha)}dr$, imporremo in particolare che la misura ν soddisfi la seguente condizione, usualmente detta di *dominazione stabile*:

[DS] esistono $r_0 > 0$, α in $(0,2)$ ed una misura μ finita e non-degenere (nel senso di (4.4)) sulla sfera $\mathbb{S}^{d-1}$ tale che

$$\nu(C) \geq \int_0^{r_0} \int_{\mathbb{S}^{d-1}} \mathbb{1}_C(r\theta)\, \mu(d\theta) \frac{dr}{r^{1+\alpha}}, \quad C \in \mathcal{B}(\mathbb{R}^d_0). \tag{5.5}$$

Intuitivamente, la condizione [**DS**] assicura l'esistenza di una densità associata all'operatore $\mathcal{L}$ con proprietà regolarizzanti di ordine (almeno) α. Infatti, sono proprio i salti piccoli (a raggio r_0 piccolo) associati alla misura di Lévy che permettono, se sufficientemente intensi, di generare una densità per il processo associato. Sotto la condizione [**DS**], sappiamo in particolare che i contributi associati ai salti piccoli di ν sono controllati da sotto con quelli di una misura α-stabile non-degenere, la cui assoluta continuità è ben nota in questo contesto.
Chiaramente, la condizione di non-degenerazione [**ND**] assunta nel capitolo precedente può essere intesa qui come un caso speciale di [**DS**] quando $r_0 = +\infty$ e vale un'uguaglianza in (5.5). In particolare, sottolineiamo che la classe di operatori di Lévy su $\mathbb{R}^d$ che soddisfano [**DS**] sia molto ricca e variegata e comprenda alcuni di quelli di tipo quasi-stabile che non potevano essere considerati precedentemente in letteratura. Un possibile esempio su $\mathbb{R}^2$ è dato dal Laplaciano frazionario relativistico $\Delta_{\text{rel}}^{\alpha/2}$ agente solo sulla prima componente, che è definito da:

$$\Delta_{\text{rel}}^{\alpha/2}\phi(x) := \text{p.v.} \int_{\mathbb{R}} \Big[\phi\begin{pmatrix} x_1 + z \\ x_2 \end{pmatrix} - \phi\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}\Big] \frac{1 + |z|^{\frac{d+\alpha-1}{2}}}{|z|^{d+\alpha}} e^{-|z|}\, dz,$$

dove $x = (x_1, x_2)$ in $\mathbb{R}^2$. Questo tipo di operatore, chiamato anche operatore di Schrödinger relativistico, viene spesso considerato per le sue connessioni con lo studio della stabilità relativistica della materia. Per maggiori dettagli, si veda per esempio [BB99, CMS90, Fef86, Lie90] e le referenze all'interno.

Come visto precedentemente, condizioni come [**DS**] assicurano un effetto regolarizzante di ordine (minimo) α associato all'operatore $\mathcal{L}$ che però agisce solo su uno sottospazio di $\mathbb{R}^N$. Per ottenere un effetto globale su tutto lo spazio $\mathbb{R}^N$, è allora necessario che tale proprietà si diffonda in tutto il sistema. Per questo motivo, assumeremo che le matrici A, B soddisfino la seguente *conditione di Kalman*:

[K] vale che $N = \operatorname{rank}\Big[B, AB, \ldots, A^{N-1}B\Big]$,

dove $\Big[B, AB, \dots, A^{N-1}B\Big]$ è la matrice in $\mathbb{R}^N \otimes \mathbb{R}^{dN}$ le cui colonne sono date da $B, AB, \dots, A^{N-1}B$. è noto (si veda per esempio [Zab92]) che esiste un'equivalenza tra la condizione [**K**] e la seguente affermazione:

$$\det K_t \,:=\, \det \int_0^t e^{sA} BB^* e^{sA^*}\, ds \,>\, 0, \quad \forall t > 0. \tag{5.6}$$

Almeno nel caso diffusivo ($\alpha = 2$ ed $\mathcal{L} = BB^*\Delta_x$), K_t è la matrice di covarianza per il processo soluzione della dinamica stocastica associata. A sua volta, l'equazione (5.6) può essere mostrata equivalente all'ipoellitticità nel senso di Hörmander ([Hör67]) dell'operatore di Ornstein-Uhlenbeck L^{ou}, che assicura, in particolare, l'esistenza e la regolarità di una soluzione in senso distribuzionale dell'equazione

$$L^{\mathrm{ou}}u(x) \,=\, \mathcal{L}u(x) + \langle Ax, Du(x)\rangle \,=\, \phi(x), \quad x \text{ in } \mathbb{R}^N$$

per ogni funzione $\phi\colon \mathbb{R}^N \to \mathbb{R}$ sufficientemente regolare. Si veda anche il libro di Ishikawa [Ish16], Capitolo 3.6, per maggiori dettagli nel caso non-degenere.
Evidenziamo infine che la condizione [**K**] è ben nota in teoria dei controlli. Infatti, essa venne introdotta proprio da Kalman (cf. [Kal60a, Kal60b]) come condizione sufficiente (poi effettivamente equivalente) per la *controllabilità da zero* di sistemi lineari del tipo:

$$\dot{x}_t \,=\, Ax_t + Bu_t, \tag{5.7}$$

ovvero, Perché ad ogni stato finale x in $\mathbb{R}^N$, esista un controllo $t \mapsto u_t$ in $\mathbb{R}^d$ tale che $t \mapsto x_t$ soluzione di (5.7) parta da 0 e raggiunga x in un tempo finito. Per maggiori dettagli su questo argomento, si veda, per esempio, [KHN63] o [Zab92].

Attraverso la condizione [**K**], possiamo ora spiegare più precisamente la distanza anisotropica **d** e gli spazi di Hölder $C^{\beta}_{b,d}(\mathbb{R}^N)$ associati in questo contesto.
Fissiamo intanto n come il più piccolo intero tale che la condizione di Kalman [**K**] valga, ovvero:

$$n \,=\, \min\{r \in \mathbb{N}\colon N = \operatorname{rank}\Big[B, AB, \dots, A^{r-1}B\Big]\}.$$

Da un punto di vista più probabilistico, ovvero considerando la seguente dinamica stocastica:

$$dX_t \,=\, AX_t dt + BdZ_t, \quad t \geq 0$$

dove $\{Z_t\}$ è un processo di Lévy su $\mathbb{R}^d$ con tripletta di Lévy (b, Σ, ν), l'intero n può essere inteso come il numero minimo di applicazioni di A che permettono di trasmettere il rumore, localizzato su $B\mathbb{R}^N$, a tutto lo spazio sottostante $\mathbb{R}^N$.
Più esattamente, fissato i in $[\![1, n]\!]$, definiamo V_i come lo spazio immagine raggiunto dai primi $i-1$ commutatori iterati tra A e B:

$$V_i \,:=\, \begin{cases} \operatorname{Im}(B), & \text{se } i = 1, \\ \bigoplus_{k=1}^{i} \operatorname{Im}(A^{k-1}B), & \text{altrimenti.} \end{cases} \tag{5.8}$$

Visto che chiaramente $V_1 \subset V_2 \subset \dots V_n = \mathbb{R}^N$, ha senso denotare ora

$$W_i \,:=\, \begin{cases} V_1, & \text{se } i = 1, \\ (V_{i-1})^{\perp} \cap V_i, & \text{altrimenti.} \end{cases}$$

Intuitivamente, ogni spazio W_i caratterizza quanto un'applicazione successiva del commutatore su V_i permetta di aggiungere in termini di spazio coperto.
Possiamo infine introdurre le proiezioni ortogonali $E_i \colon \mathbb{R}^N \to \mathbb{R}^N$ da $\mathbb{R}^N$ su W_i. Notando che $\dim E_1(\mathbb{R}^N) = \dim B\mathbb{R}^N = d$, ha senso fissare $d_1 := d$ e scrivere

$$d_i \;:=\; \dim E_i(\mathbb{R}^N), \qquad \text{per } i > 1. \tag{5.9}$$

Considerando un cambio di variabili se necessario, supporremo d'ora in avanti che lo spazio $\mathbb{R}^N$ sia decomponibile come $x = (x_1, \dots, x_n)$ tale che $E_i x = x_i$ ed x_i sia in $\mathbb{R}^{d_i}$, per ogni i in $[\![1, n]\!]$. Questa decomposizione esplicita permette ora di estendere facilmente anche a questo contesto la misura anisotropica $\mathbf{d}$, definita come in (4.9), e gli spazi di Hölder associati all'operatore di dilatazione

$$\delta_\lambda(t, x_1, \dots, x_n) \;=\; (\lambda^\alpha, \lambda x_1, \dots, \lambda^{1+\alpha(i-1)} x_n). \tag{5.10}$$

In particolare, una funzione ϕ in $C^\gamma_d(\mathbb{R}^N)$ sarà tale che $x_i \to \phi(x_1, \dots, x_i, \dots, x_n)$ sia in $C^{\frac{\gamma}{1+\alpha(i-1)}}(\mathbb{R}^{d_i})$, uniformemente nelle altre variabili x_j, $(j \neq i)$.

Citiamo infine che la struttura lineare analizzata qui è più generale di quella considerata (sulla matrice A) nel capitolo precedente. Infatti, è stato dimostrato da Lanconelli e Polidoro in [LP94] che la condizione di Kalman [**K**] su A e B impone la seguente forma alle matrici:

$$B \;=\; \begin{pmatrix} B_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \text{e} \quad A \;=\; \begin{pmatrix} * & * & \dots & \dots & * \\ A_2 & * & \ddots & \ddots & \vdots \\ 0 & A_3 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ 0 & \dots & 0 & A_n & * \end{pmatrix} \tag{5.11}$$

dove B_0 è una matrice non-degenere in $\mathbb{R}^{d_1} \otimes \mathbb{R}^{d_1}$, A_i è una matrice in $\mathbb{R}^{d_i} \otimes \mathbb{R}^{d_{i-1}}$ tale che $\operatorname{rank}(A_i) = d_i$ per ogni i in $[\![2, n]\!]$ e gli elementi $*$ possono essere non-nulli. Inoltre, vale che $d_1 \geq d_2 \geq \dots \geq d_n \geq 1$.
Inoltre, la presenza di elementi $*$ diversi da zero nella matrice A aggiunge una difficoltà ulteriore al nostro procedimento di dimostrazione. Infatti, è stato mostrato (cf. [LP94]) che la matrice A è invariante sotto le dilatazioni δ_λ (definite in (5.10)) se e solo se A ha elementi non-nulli solo sulla sotto-diagonale. In particolare, decomposizioni esplicite come in (4.7) non saranno ora più disponibili.

Le ipotesi [**DS**] e [**K**] sopra descritte permettono allora di mostrare le stime di Schauder associate all'operatore di Ornstein-Uhlenbeck L^{ou}. Per coerenza, sottolineiamo inoltre che similmente al capitolo precedentemente, considereremo solo soluzioni *deboli* per il problema ellittico o parabolico nel senso delle distribuzioni. Possiamo ora riassumere i principali risultati ottenuti sia nel contesto ellittico che in quello parabolico.

Teorema 5.1 (Caso Ellittico)**.** *Fissato $\lambda > 0$, sia g in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)$. Allora, esiste un'unica soluzione debole u dell'equazione ellittica* (5.3)*. Inoltre, u appartiene a $C^\beta_{b,d}(\mathbb{R}^N)$ ed esiste una constante $C > 0$ tale che*

$$\|u\|_{C^{\alpha+\beta}_{b,d}} \;\leq\; C\Big(1 + \frac{1}{\lambda}\Big)\|g\|_{C^\beta_{b,d}}. \tag{5.12}$$

Come nel caso parabolico, le stime di Schauder in un contesto ellittico sono spesso utilizzate in relazione alla dinamica stocastica associata. Per esempio in [CC07], tali stime per operatori degeneri del secondo ordine su domini non-lisci sono uno strumento fondamentale per ottenere l'unicità del problema di martingala correlato. In un caso α-stabile sotto-critico ($\alpha \geq 1$) non-degenere, citiamo anche le stime globali ottenute da Priola in [Pri12] e [Pri18] e le rispettive applicazioni alla buona posizione in senso forte e l'unicità alla Davie per l'equazione stocastica associata.

Teorema 5.2 (Caso Parabolico)**.** *Fissato $T > 0$ e β in $(0,1)$, siano u_0 in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)$ e f in $L^\infty\Big(0,T;C^{\beta}_{b,d}(\mathbb{R}^N)\Big)$. allora, Allora, esiste un'unica soluzione debole u del Problema di Cauchy* (5.4)*. Inoltre u appartiene a $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)\Big)$ ed esiste una constante $C := C(T) > 0$ tale che*

$$\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \leq C\Big[\|u_0\|_{C^{\alpha+\beta}_{b,d}} + \|f\|_{L^\infty(C^{\beta}_{b,d})}\Big]. \tag{5.13}$$

Nel caso parabolico, i risultati presentati in questa sezione possono essere inoltre estesi naturalmente a Problemi di Cauchy con coefficienti A_t, B_t dipendenti dal tempo, sotto ipotesi aggiuntive naturali, come la limitatezza della matrice A_t e l'uniforme ellitticità di B_t sullo spazio piccolo $\mathbb{R}^d$. Facciamo riferimento alla Sezione 6 del Capitolo 3 per maggiori dettagli.

5.1 Guida alla Prova

Inspirati dal lavoro di Priola [Pri09], dove analoghe stime di Schauder vengono dimostrare nel caso diffusivo degenere, abbiamo deciso di seguire un approccio con la teoria dei semigruppi per il nostro problema. Questo metodo, introdotto originariamente da Da Prato e Lunardi in [DPL95], consiste nello stabilire controlli a priori del semigruppo P_t associato all'operatore di Ornstein-Uhlenbeck L^{ou} su spazi funzionali adatti e di ricavare da questi, le stime di Schauder paraboliche attraverso una rappresentazione mild (o di Duhamel) della soluzione u rispetto al semigruppo (cf. Equazione (4.25)).
Sottolineiamo di già che tale approccio nel contesto parabolico si concentra però solo sulla regolarità in spazio per le soluzioni u. Questo si riflette, per esempio, anche sulla nostra definizione degli spazi di Hölder anisotropici $L^\infty(0,T;C^{\gamma}_{b,d}(\mathbb{R}^N))$, solo uniformi in tempo. In particolare, le stime di Schauder paraboliche in (5.13) non presentano nessun effetto bootstrap in tempo rispetto alla condizione iniziale u_0.
Citiamo infine che un'altro possibile metodo per l'analisi degli operatori di Ornstein-Uhlenbeck come L^{ou} è stato introdotto da Manfredini in [Man97] e sfrutta un ragionamento più astratto in termini dei gruppi di Lie associati al sistema differenziale. In particolare, le stime di Schauder in questo contesto sono costruite rispetto a spazi di Hölder intrinsechi, nel senso che tengono in considerazione la regolarità congiunta tra spazio e tempo delle funzioni coinvolte. Per maggiori dettagli sull'argomento, si veda per esempio [Pas03].

Come abbiamo già detto, l'elemento fondamentale del nostro metodo consiste in un'analisi a priori delle proprietà del semigruppo Markoviano $\{P_t\}_{t\geq 0}$ (formalmente) associato all'operatore di Ornstein-Uhlenbeck L^{ou}. Dopodichè, la formula della trasformata di

Laplace permetterà di scrivere ogni soluzione u del problema ellittico (1.5) in termini di P_t:

$$u(x) \;=\; \int_0^\infty e^{-\lambda t}\Big[P_t g\Big](x)\,dt \;=:\; \int_0^\infty e^{-\lambda t} P_t g(x)\,dt. \tag{5.14}$$

Nel caso parabolico, sarà invece possibile usare la formula della variazione delle costanti per mostrare un'analoga rappresentazione per una soluzione u del problema di Cauchy (5.4):

$$u(t,x) \;=\; P_t u_0(x) + \int_0^t \Big[P_{t-s} f(s,\cdot)\Big](x)\,ds \;=:\; P_t u_0(x) + \int_0^t P_{t-s} f(s,x)\,ds. \tag{5.15}$$

Per stabilire stime come quelle di Schauder per una soluzione u (sia nel caso ellittico che in quello parabolico), è chiaro allora che sia prima necessario ottenere dei controlli analoghi sul semigruppo associato P_t. Soprattutto, siamo interessati a capire come l'operatore P_t si comporti sugli spazi di Hölder anisotropici $C^\gamma_{b,d}(\mathbb{R}^N)$ da noi considerati. Evidenziamo inoltre che le tecniche usuali per ottenere questo tipo di controlli in ambito Gaussiano sono però difficilmente estendibili al nostro contesto non-locale. Per esempio, strategie di dimostrazione come in [Lun97], attraverso formule esplicite sulla densità del semigruppo, in [Lor05] o [Sai07] nel caso $n = 2$, attraverso stime a priori di tipo Bernstein combinate con metodi di interpolazione o in [Pri09], attraverso il calcolo di Malliavin per rappresentazioni probabilistiche del semigruppo P_t, non possono più essere seguite qui, principalmente a causa della natura non-locale o la scarsa integrabilità associata all'operatore $\mathcal{L}$.
Per sopperire a tale difficoltà, sfrutteremo invece un metodo di tipo *perturbativo* che ci permetterà di considerare l'operatore di Lévy $\mathcal{L}$ come una perturbazione, in un certo senso adatto, di un operatore α-stabile le cui proprietà sono invece ben conosciute.
Citiamo infine che queste tecniche di decomposizione vennero originariamente introdotte in [SSW12] nell'ambito dello studio delle proprietà di coupling per processi di Lévy e in [SW12], in relazione alla generalizzazione di alcuni teoremi di tipo Liouville per operatori non-locali di Ornstein-Uhlenbeck.

Proprietà Regolarizzanti Associate all'Operatore di Ornstein-Uhlenbeck

Come già visto nella sezione precedente, per determinare le proprietà sul semigruppo associato all'operatore L^{ou}, è conveniente considerare inizialmente la sua controparte stocastica. Fissato uno spazio di probabilità $(\Omega, \mathcal{F}, \mathbb{P})$, introduciamo allora il processo di Lévy $\{Z_t\}_{t\geq 0}$ determinato (in legge) dal seguente simbolo di Lévy:

$$\Phi(p) \;=\; ib\cdot p - \frac{1}{2} p\cdot \Sigma p + \int_{\mathbb{R}^d_0} \Big(e^{ip\cdot z} - 1 - ip\cdot z \mathbb{1}_{B(0,1)}(z)\Big)\,\nu(dz), \quad p\in\mathbb{R}^d,$$

dove, ricordiamo, la tripletta (b, Σ, ν) è la stessa che appare nella definizione dell'operatore $\mathcal{L}$ ed, in particolare, la misura di Lévy ν soddisfa l'ipotesi di dominazione stabile [**DS**]. Evidenziamo ora che tale processo $\{Z_t\}_{t\geq 0}$ è associato all'operatore $\mathcal{L}$ nel senso che il generatore infinitesimale del processo degenere $\{BZ_t\}_{t\geq 0}$ esteso su $\mathbb{R}^N$ è allora dato da $\mathcal{L}$.
Fissato un punto x in $\mathbb{R}^N$, siamo allora interessati al processo di Ornstein-Uhlenbeck

$\{X_t\}_{t\geq 0}$ su $\mathbb{R}^N$ controllato da BZ_t, ovvero l'unica soluzione (in senso forte) della seguente equazione differenziale stocastica:

$$\begin{cases} dX_t^x \;=\; AX_t^x dt + BZ_t, \quad t > 0; \\ X_0^x \;=\; x. \end{cases}$$

Integrando direttamente l'equazione sopra attraverso la funzione esponenziale matriciale $e^{(t-s)A}$, è inoltre possibile mostrare una rappresentazione esplicita del processo $\{X_t\}_{t\geq 0}$:

$$X_t^x \;=\; e^{tA}x + \int_0^t e^{(t-s)A} B\, dZ_s, \qquad t \geq 0.$$

Il semigruppo Markoviano associato ad $\{X_t^x\}_{t\geq 0}$ è ora definito come la famiglia di contrazioni lineari $\{P_t \colon t \geq 0\}$ su $C_b(\mathbb{R}^N)$, lo spazio delle funzioni continue e limitate su $\mathbb{R}^N$ a valori reali, tali che

$$P_t\phi(x) \;=\; \mathbb{E}\Big[\phi(X_t^x)\Big], \quad x \in \mathbb{R}^N. \tag{5.16}$$

Citiamo infine che il semigruppo P_t è generato dall'operatore L^{ou} nel senso che il suo generatore infinitesimale coincide con L^{ou} sullo spazio delle funzioni test $C_c^\infty(\mathbb{R}^N)$.

Ragionamenti sugli spazi di Fourier, simili a quelli svolti nel sezione precedente in (4.15)-(4.16), permettono di mostrare in questo caso che la parte casuale di X_t soddisfi di nuovo l'assunzione di dominazione stocastica [**DS**] su $\mathbb{R}^N$, anche se ri-scalata a seconda della struttura anisotropica della dinamica (cf. matrice $\mathbb{M}_t$ in (4.8)). Analogamente a (4.17), è possibile ottenere infatti che

$$X_t \;\overset{\text{(legge)}}{=}\; e^{tA}x + \mathbb{M}_t S_t^t, \tag{5.17}$$

dove ad ogni parametro fissato t, $\{S_u^t\}_{u\geq 0}$ è un processo di Lévy su $\mathbb{R}^N$ con proprietà adatte. Soprattutto, la sua misura di Lévy $\tilde{\nu}^t$ soddisfa di nuovo l'assunzione di dominazione stabile [**DS**] estesa su $\mathbb{R}^N$, ovvero:

$$\tilde{\nu}^t(C) \;\geq\; \int_0^{R_0} \int_{\mathbb{S}^{N-1}} \mathbb{1}_C(r\theta)\, \tilde{\mu}^t(d\theta) \frac{dr}{r^{1+\alpha}} \;=:\; \tilde{\nu}_\alpha^t(C), \quad C \in \mathcal{B}(\mathbb{R}_0^N), \tag{5.18}$$

per un certo $R_0 > 0$ ed una famiglia $\{\tilde{\mu}^t \colon t \geq 0\}$ di misure finite e non-degeneri sulla sfera $\mathbb{S}^{N-1}$.
Evidenziamo ora che la dipendenza dal parametro t per il processo $\{S_u^t\}_{u\geq 0}$ appare proprio a causa degli elementi non-nulli $*$ nella rappresentazione di A data nell'equazione (5.11). Come già sottolineato prima, la matrice A in questo caso non è più infatti invariante sotto gli operatori di dilatazione anisotropica δ_λ dati in (5.10). Infatti, decomposizioni come in (4.7), sfruttate nella sezione precedente proprio per questo tipo di risultati, non sono ora più valide ma devono essere modificate con delle versioni "approssimate" della forma:

$$e^{tA} \;=\; \mathbb{M}_t R_t \mathbb{M}_t, \quad t \ll 1,$$

dove R_t è una matrice (dipendente dal tempo) localmente limitata e non-degenere in $\mathbb{R}^N \otimes \mathbb{R}^N$. Citiamo inoltre che questo è uno dei motivi principali per cui i controlli a priori che vogliano stabilire saranno validi solo in un intervallo di tempo piccolo.

In termini più analitici, l'identità in (5.17) suggerisce che il semigruppo generato dall'operatore di Ornstein-Uhlenbeck L^{ou} coincide con un semigruppo non-degenere seppure "moltiplicato" dalla matrice $\mathbb{M}_t$ che tiene conto della degenerazione originale dell'operatore considerato. Più precisamente, possiamo stabilire a partire dall'identità in legge in (5.17) una prima rappresentazione del semigruppo Markoviano P_t:

$$P_t\phi(x) \;=\; \int_{\mathbb{R}^N} \phi(e^{tA}x + \mathbb{M}_t y)\, \mathbb{P}_{S_t^t}(dy), \quad \phi \in C_b(\mathbb{R}^N), \tag{5.19}$$

dove, per una variabile aleatoria X qualunque, $\mathbb{P}_X$ denota la legge di X.
Per determinare le proprietà regolarizzanti associate al semigruppo P_t è ora chiaro che sia necessaria un'analisi più approfondita della misura $\mathbb{P}_{S_t^t}$.
Notiamo innanzitutto che l'assunzione [**DS**] permette di vedere il processo di Lévy $\{S_u^t\}_{u\geq 0}$ come una *perturbazione* del processo α-stabile possibilmente troncato $\{Y_u^t\}_{t\geq 0}$, associato alla misura di Lévy $\tilde{\nu}_\alpha^t$ definita in (5.18). Considerazioni sulla misura $\mathbb{P}_{S_t^t}$ possono essere allora espresse a partire dalle proprietà, ben più note, di quella stabile.
In realtà, spingeremo tale metodo perturbativo ancora oltre, considerando invece del processo stabile *completo* solo il contributo associato ai suoi salti piccoli che, come è noto, permettono l'esistenza della densità del processo. Questo passo aggiuntivo permetterà infatti di ottenere controlli ancora più precisi su tale densità e le sue derivate in spazio.
Più in dettaglio, sia $(\tilde{\Sigma}^t, \tilde{b}^t, \tilde{\nu}^t)$ la tripletta di Lévy associata al processo $\{S_u^t\}_{u\geq 0}$ ad ogni tempo fissato t, che, ricordiamo, caratterizza il simbolo di Lévy Φ_{S^t} attraverso la formula di Lévy-Khintchine su $\mathbb{R}^N$:

$$\Phi_{S^t}(\xi) \;:=\; i\langle \tilde{b}^t, \xi\rangle - \frac{1}{2}\langle \xi, \tilde{\Sigma}^t\xi\rangle + \int_{\mathbb{R}_0^d} \Big(e^{i\langle \xi, z\rangle} - 1 - i\langle \xi, z\rangle \mathbb{1}_{B(0,1)}(z)\Big)\, \tilde{\nu}^t(dz).$$

In un intervallo sufficientemente piccolo, in pratica supponendo che $t^{\frac{1}{\alpha}} < R_0 \wedge 1$, possiamo troncare la misura di Lévy $\tilde{\nu}_\alpha^t$ al tempo caratteristico tipico associato al processo α-stabile al tempo t, ovvero possiamo troncare al tempo $t^{1/\alpha}$. In particolare, introduciamo ora il simbolo di Lévy Φ_t^{tr} associato ai salti piccoli del processo α-stabile $\{Y_u^t\}_{u\geq 0}$, definito da

$$\Phi_t^{\mathrm{tr}}(\xi) \;:=\; \int_{|z|\leq t^{\frac{1}{\alpha}}} \Big[e^{i\langle \xi, z\rangle} - 1 - i\langle \xi, z\rangle\Big]\, \tilde{\nu}_\alpha^t(dz).$$

Sarà inoltre necessario considerare anche il termine di resto che appare, intuitivamente, dall'errore di aver considerato il processo $\{S_u^t\}_{u\geq 0}$ come una perturbazione del processo α-stabile troncato $\{Y_u^t\}_{u\geq 0}$. Più precisamente, introduciamo ora il simbolo di Lévy Φ_t^{err} definito da

$$\Phi_t^{\mathrm{err}}(\xi) := \Phi_{S^t}(\xi) - \Phi_t^{\mathrm{err}}(\xi), \quad \xi \in \mathbb{R}^N,$$

ed associato alla tripletta $(\tilde{\Sigma}^t, \tilde{b}^t, \tilde{\nu}^t - \mathbb{1}_{B(0,t^{1/\alpha})}\tilde{\nu}_\alpha^t)$.
Sottolineiamo che è proprio la condizione di dominazione stabile [**DS**] su $\{S_u^t\}_{u\geq 0}$ che permette di concludere che Φ_t^{err} sia effettivamente un simbolo di Lévy, visto che assicura la positività della misura $\tilde{\nu}^t - \mathbb{1}_{B(0,t^{1/\alpha})}\tilde{\nu}_\alpha^t$ associata al termine di resto.
Denotiamo ora rispettivamente con $\{\mathbb{P}_t^{\mathrm{tr}}\}_{t\geq 0}$ e $\{\pi_t\}_{t\geq 0}$ le famiglie di probabilità infinitamente divisibili associate ai simboli di Lévy Φ_t^{tr} e Φ_t^{err}. Dalla rappresentazione in termini di funzione caratteristica delle misure di probabilità, possiamo allora disintegrare la probabilità $\mathbb{P}_{S_t^t}$ nel seguente modo:

$$\mathbb{P}_{S_t^t} \;=\; \mathbb{P}_t^{\mathrm{tr}} * \pi_t, \quad t > 0, \tag{5.20}$$

dove $*$ rappresenta l'operazione di convoluzione tra misure di probabilità.

Dalle considerazioni sopra, possiamo ora concentrarci sulla famiglia di misure $\{\mathbb{P}_t^{\mathrm{tr}}\}_{t\geq 0}$ che, ricordiamo, sono associate ai salti piccoli di un processo α-stabile.
Sfruttando risultati noti sul simbolo di Lévy associato ad un processo α-stabile non-degenere, come la condizione di Hartman-Wintner e alcune assunzioni di controllabilità in spazi di Fourier, possiamo ottenere l'esistenza di una densità regolare in spazio per la misura di probabilità $\mathbb{P}_t^{\mathrm{tr}}$ e dei controlli adatti sulle sue derivate, almeno in un intervallo di tempo sufficientemente piccolo. Più in dettaglio, mostreremo che esiste un tempo finale $T_0 := T_0(N) > 0$ tale che su $(0, T_0]$, la probabilità $\mathbb{P}_t^{\mathrm{tr}}$ ammette una densità $p^{\mathrm{tr}}(t,\cdot)$ che è 3-volte differenziabile con derivata continua su $\mathbb{R}^N$. Inoltre, per ogni k in $[\![0,3]\!]$, vale che

$$|\partial_y^k p^{\mathrm{tr}}(t,y)| \;\leq\; Ct^{-\frac{N+k}{\alpha}}\Big(1+\frac{|y|}{t^{1/\alpha}}\Big)^{-(N+3)} \;=:\; Ct^{-\frac{k}{\alpha}}\bar{p}(t,y), \tag{5.21}$$

per ogni (t,y) in $(0,T_0]\times\mathbb{R}^N$, dove $C>0$ è una costante dipendente solo da N.
Sottolineiamo comunque che, alle spese di ridurre ancora l'intervallo in tempo, è possibile esibire che la densità $p^{\mathrm{tr}}(t,\cdot)$ sia ancora più regolare e la densità associata $\bar{p}(t,\cdot)$ presenti proprietà regolarizzanti più forti ma che la scelta in (5.21) è quella minima per i nostri scopi. In particolare, l'esponente $N+3$ nella densità $\bar{p}(t,\cdot)$ è quello minimo necessario per poter integrare i contributi associati alle funzioni ϕ in $C^{\beta}_{b,d}(\mathbb{R}^N)$ di indice $\beta < 1+\alpha < 3$ (cf. Equazione (5.27) sotto).
Attraverso l'identità (5.20) possiamo ora riscrivere la rappresentazione per il semigruppo Markoviano P_t in (5.19) nel seguente modo:

$$\begin{aligned} P_t\phi(x) \;&=\; \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi\Big(e^{tA}x + \mathbb{M}_t(y_1+y_2)\Big)p^{\mathrm{tr}}(t,y_1)\,dy_1\pi_t(dy_2) \\ &=\; \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi\Big(y_1+\mathbb{M}_t y_2\Big)\frac{p^{\mathrm{tr}}(t,\mathbb{M}_t^{-1}(y_1-e^{tA}x))}{\det\mathbb{M}_t}\,dy_1\pi_t(dy_2). \end{aligned} \tag{5.22}$$

Inoltre, la forma esplicita della densità $\bar{p}(t,\cdot)$ e i controlli associati in (5.21) permettono facilmente di esibire gli effetti regolarizzanti associati alla densità "parziale" $p^{\mathrm{tr}}(t,\cdot)$, seppur solo lungo la componente y_1. In particolare, fissato γ in $(0,3)$, k in $[\![0,2]\!]$, i in $[\![1,n]\!]$ e t abbastanza piccolo, abbiamo che

$$\int_{\mathbb{R}^{nd}} \frac{|D^k_{x_i}p^{\mathrm{tr}}(t,\mathbb{M}_t^{-1}(y_1-e^{tA}x))|}{\det\mathbb{M}_t}\mathbf{d}^{\gamma}(y_1,e^{tA}x)\,dy_1 \;\leq\; C(s-t)^{\frac{\gamma}{\alpha}-k\frac{1+\alpha(i-1)}{\alpha}}. \tag{5.23}$$

A questo punto, è chiaro che la differenza principale con le tecniche presentate nella Sezione 4 (per le stime di Schauder sul proxy) consiste nel fatto che qui potremo sfruttare solo gli effetti regolarizzanti sulla variabile y_1, associata ai contributi dei salti piccoli del processo α-stabile troncato, visto che del termine di resto (associato alla variabile di integrazione y_2) non conosciamo particolari proprietà. Sappiamo infatti solo che la probabilità π_t è a misura totale finita, uniformemente in t.
Mentre questa peculiarità introduce sicuramente ulteriori difficoltà aggiuntive in alcuni passaggi della prova, per esempio sulle tecniche di cancellazione per i controlli sulle norme di Hölder, evidenziamo che sfruttare solo gli effetti regolarizzanti associati ai salti piccoli permette di evitare qui alcune assunzioni (cf. ipotesi [**P**]) sui parametri α,

β necessarie nel Capitolo 2 proprio perché consideravamo la densità associata ad un processo α-stabile completo.

Controlli sul Semigruppo di Markov Associato

Gli effetti regolarizzanti della densità $p^{\mathrm{tr}}(t,\cdot)$ mostrati in (5.23) implicano ora naturalmente i primi controlli su P_t e le sue derivate quando il semigruppo agisce sullo spazio $C_b(\mathbb{R}^N)$. Più in dettaglio, vale che:

$$\|D^k_{x_i} P_t\phi\|_\infty \,\leq\, C\|\phi\|_\infty\Big(1+t^{-k\frac{1+\alpha(i-1)}{\alpha}}\Big), \quad t>0, \tag{5.24}$$

dove k è in $[\![0,3]\!]$, i in $[\![1,n]\!]$ e ϕ una funzione in $C_b(\mathbb{R}^N)$.
In particolare, dalle stime sopra è ora possibile mostrare la continuità del semigruppo P_t, a tempo $t>0$ fissato, come operatore sullo spazio $C_b(\mathbb{R}^N)$ a valori nello stesso o nello spazio di Zygmund-Hölder anisotropico $C^3_{b,d}(\mathbb{R}^N)$, una naturale generalizzazione ad indici interi degli spazi di Hölder da noi considerati. Tale risultato sarà poi esteso agli spazi di Hölder anisotropici notando che ogni spazio $C^\gamma_{b,d}(\mathbb{R}^N)$ (con γ in $(0,3)$) può essere visto come spazio di interpolazione tra $C_b(\mathbb{R}^N)$ e $C^3_{b,d}(\mathbb{R}^N)$. Infatti, sfrutteremo la seguente identità:

$$\Big(C_b(\mathbb{R}^N), C^3_{b,d}(\mathbb{R}^N)\Big)_{\gamma,\infty} \,=\, C^\gamma_{b,d}(\mathbb{R}^N), \tag{5.25}$$

con l'equivalenza tra le rispettive norme, dove dati due spazi di Banach X, Y, il simbolo $(X,Y)_{\gamma,\infty}$ rappresenta usualmente lo spazio di *interpolazione reale* a norma infinita tra X e Y. Per maggiori dettagli sull'argomento in generale, facciamo riferimento ai libri di Triebel [Tri92] e di Lunardi [Lun18] o al Teorema 2.2 in [Lun97] in un contesto anisotropico diffusivo.
Grazie a tecniche di interpolazione simili a quelle in (5.25), mostreremo allora che il semigruppo P_t è continuo come operatore su $C_b(\mathbb{R}^N)$ a valori nello spazio di Hölder anisotropico $C^\gamma_{b,d}(\mathbb{R}^N)$ per ogni γ in $[0,1+\alpha)$:

$$\|P_t\|_{\mathcal{L}(C_b,C^\gamma_{b,d})} \,\leq\, C\Big(1+t^{-\frac{\gamma}{\alpha}}\Big), \quad t>0, \tag{5.26}$$

dove $\|\cdot\|_{\mathcal{L}(X,Y)}$ denota l'usuale norma operatoriale tra due spazi di Banach generici X e Y.

Abbiamo per ora considerato solo il comportamento del semigruppo P_t sullo spazio $C_b(\mathbb{R}^N)$. Per estendere tale analisi anche su $C^\beta_{b,d}(\mathbb{R}^N)$, il primo passo è quello di esibire dei controlli analoghi a quelli in (5.24) quando P_t agisce invece su funzioni ϕ che siano Hölder regolari. Similmente al capitolo precedente, utilizzeremo in questo caso delle tecniche di cancellazione per sfruttare la regolarità della funzione ϕ ed ottenere così i controlli voluti. Evidenziamo però che la principale difficoltà in questo caso sarà legata al fatto che gli effetti regolarizzanti relativi alla densità p^{tr} saranno solo rispetto alla variabile "buona" y_1. Dovremo allora considerare tecniche di cancellazione *parziale* che permettano infatti di isolare solo le componenti di ϕ lungo y_1.
Più in dettaglio e soffermandoci solo sul caso $\beta<1$, $k=1$ per semplicità, l'idea fondamentale sarà di sfruttare, similmente a (4.23) che

$$\int_{\mathbb{R}^N} \phi(\mathbb{M}_t y_2 + e^{tA}x) D_{x_i} \int_{\mathbb{R}^N} \frac{p^{\mathrm{tr}}(t, \mathbb{M}_t^{-1}(y_1 - e^{tA}x))}{\det \mathbb{M}_t}\, dy_1 \pi_t(dy_2) \,=\, 0,$$

per aggiungere il termine $\phi(\mathbb{M}_t y_2 + e^{tAx})$ nel controllo di $|D_{x_i} P_t|$ e concludere come in (4.23) grazie all'effetto regolarizzante associato alla densità $p^{\mathrm{tr}}(t,\cdot)$ dato in (5.23).
Ragionamenti simili a quello accennato ora permetteranno in particolare di mostrare che per ogni β in $[0,3)$, i in $[\![1,n]\!]$ e k in $[\![0,3]\!]$, vale che

$$\|D^k_{x_i} P_t \phi\|_\infty \,\leq\, C\|\phi\|_{C^\beta_{b,d}} \Big(1 + t^{\frac{\beta}{\alpha} - k\frac{1+\alpha(i-1)}{\alpha}}\Big), \quad t > 0, \tag{5.27}$$

Sfruttando ancora tecniche di interpolazione simili a quelle in (5.25) a partire dai controlli sulle derivate di $P_t\phi$ in (5.27), potremo infine mostrare la continuità del semigruppo P_t come operatore tra spazi di Hölder anisotropici. Più precisamente, otterremo che per ogni $\beta < \gamma$ in $[0, 1+\alpha)$, vale il controllo seguente:

$$\|P_t\|_{\mathcal{L}(C^\beta_{b,d}, C^\gamma_{b,d})} \,\leq\, C\Big(1 + t^{\frac{\beta-\gamma}{\alpha}}\Big), \quad t > 0. \tag{5.28}$$

Tali stime sembrano essere nuove in ambito Lévy degenere e di interesse indipendente dagli scopi da noi qui considerati.
Evidenziamo inoltre che la possibilità di un effetto regolarizzante indipendente dall'ordine α in questo caso (a differenza della sezione precedente) rispecchia essenzialmente il fatto che la densità è in questo caso associata solo al contributo dei salti piccoli, mentre sono le code, correlate ai salti grandi del processo, ad imporre le condizioni di integrabilità viste precedentemente.

Stime di Schauder nel Caso Ellittico

Una volta che i necessari controlli a priori sul semigruppo P_t sono stati mostrati, le stime di Schauder nel contesto ellittico (5.12) e in quello parabolico (5.13) sono allora ottenute a partire dalle relative rappresentazioni per la soluzione u in termini del semigruppo Markoviano P_t (cf. equazioni (5.14) e (5.15)).
Per dare un'idea al lettore del metodo da noi seguito, presentiamo ora brevemente solo il caso ellittico.
Data una soluzione u dell'equazione ellittica (5.3), sappiamo dalla formula di Laplace in (5.14) che è allora necessario controllare il seguente termine:

$$u(t,x) \,=\, \int_0^\infty e^{-\lambda t}\Big(P_t g\Big)(z)\, dt,$$

nello spazio di Hölder anisotropico di ordine $\alpha+\beta$ rispetto alla norma di g in $C^\beta_{b,d}(\mathbb{R}^N)$.
Per questo tipo di problema, è in realtà comodo sfruttare una norma equivalente a quella Hölder introdotta in (3.23) che non richiede di considerare le derivate lungo ogni direzione ma solo le differenze finite di ordine 3 per la funzione u. Più in dettaglio, introduciamo per un punto iniziale x_0 in $\mathbb{R}^N$ ed z in $E_i(\mathbb{R}^N)$,

$$\Delta^3_{x_0}\phi(z) \,:=\, \phi(x_0+3z) - 3\phi(x_0+2z) + 3\phi(x_0+z) - \phi(x_0).$$

è stato infatti mostrato in [Lun97] che una funzione ϕ è in $C^\gamma_b(E_i(\mathbb{R}^N))$ se e solo se

$$\sup_{x_0\in\mathbb{R}^N} \sup_{z\in E_i(\mathbb{R}^N); z\neq 0} \frac{\Big|\Delta^3_{x_0}\phi(z)\Big|}{|z|^\gamma} \,<\, \infty.$$

Per concludere, è allora necessario mostrare che ad ogni i fissato in $[\![1,n]\!]$, vale che

$$|\Delta^3_{x_0}u(z)| \;=\; \left|\int_0^\infty e^{-\lambda t}\Delta^3_{x_0}\Big(P_t g\Big)(z)\,dt\right| \;\le\; C\|g\|_{C^\beta_{b,d}}|z|^{\frac{\alpha+\beta}{1+\alpha(i-1)}}, \tag{5.29}$$

per una certa costante $C>0$ indipendente da x_0 in $\mathbb{R}^N$ e da z in $E_i(\mathbb{R}^N)$.

Come spesso accade nella dimostrazione di stime in norma di Hölder, dovremo innanzitutto dividere l'analisi a seconda di tre regimi possibili, rispetto al rapporto tra il punto spaziale z in $E_i(\mathbb{R}^N)$ e il tempo t alla scala intrinseca del sistema lungo la direzione i-esima considerata.
Da una parte, il regime *macroscopico* apparirà quando $|z|\ge 1$ e sarà il più facile da trattare. Infatti, richiederà solo di mostrare la limitatezza della soluzione u a partire da quella di g, visto che in questo caso, $\|g\|_\infty \le \|g\|_\infty |z|^\gamma$ per ogni $\gamma>0$.
D'altra parte, diremo di essere in un regime *fuori-diagonale* se $t^{\frac{1+\alpha(i-1)}{\alpha}} \le |z| \le 1$. In questo caso, la distanza spaziale lungo la componente i-esima sarà maggiore del tempo caratteristico associato. Infine, un regime *diagonale* apparirà quando $t^{\frac{1+\alpha(h-1)}{\alpha}} \ge |z|$ e il punto spaziale sarà invece minore dell'intensità tipica del tempo caratteristico.
In particolare, denoteremo con t_0 il tempo di transizione tra i regimi diagonali e fuori diagonali, rispetto alle scale di dilatazione δ_λ (definite in (5.10)) lungo la direzione i-esima del sistema, ovvero:

$$t_0 \;:=\; |z|^{\frac{\alpha}{1+\alpha(i-1)}}.$$

Come già accennato, il controllo cercato nel regime macroscopico ($|z|\ge 1$) segue immediatamente dalla proprietà di contrazione del semigruppo P_t su $C_b(\mathbb{R}^N)$:

$$\left|\int_0^\infty e^{-\lambda t}\Delta^3_{x_0}\Big(P_t g\Big)(z)\,dt\right| \;\le\; 3\int_0^\infty e^{-\lambda t}\|P_t g\|_\infty\,dt \;\le\; C\|g\|_\infty|z|^{\frac{\alpha+\beta}{1+\alpha(i-1)}}. \tag{5.30}$$

Per analizzare separatamente il regime diagonale e quello fuori-diagonale, spezzeremo, come in [Pri09], il termine $\Delta^3_{x_0}u(z)$ in due componenti $R_1(z)+R_2(z)$, dove

$$\begin{aligned} R_1(z) \;&:=\; \int_0^{t_0} e^{-\lambda t}\Delta^3_{x_0}\Big(P_t g\Big)(z)\,dt;\\ R_2(z) \;&:=\; \int_{t_0}^\infty e^{-\lambda t}\Delta^3_{x_0}\Big(P_t g\Big)(z)\,dt. \end{aligned}$$

Nel caso fuori-diagonale associato alla prima componente R_1, dovremo alla fine integrare su $[0,t_0]$ e sarà quindi importante porre attenzione a non introdurre singolarità in tempo che non siano poi integrabili. In pratica, sfrutteremo la continuità del semigruppo P_t sullo spazio $C^\beta_{b,d}(\mathbb{R}^N)$ in sé stesso per controllare il termine R_1 nel seguente modo:

$$\begin{aligned} |R_1(z)| \;&\le\; \int_0^{t_0}|\Delta^3_{x_0}\Big(P_t g\Big)(z)|\,dt\\ &\le\; |z|^{\frac{\beta}{1+\alpha(h-1)}}\int_0^{t_0}\|P_t g\|_{C^\beta_{b,d}}\,dt\\ &\le\; C\|g\|_{C^\beta_{b,d}}|z|^{\frac{\alpha+\beta}{1+\alpha(h-1)}}. \end{aligned} \tag{5.31}$$

D'altra parte, il regime diagonale relativo al contributo R_2 non presenterà invece problemi di integrabilità in tempo. Anzi, visto che il punto z è, in questo caso, vicino

allo zero rispetto alle scale del tempo caratteristico, ha senso applicare iterativamente un'espansione di Taylor su $\Delta^3_{x_0}\Big(P_t\phi\Big)$ così da far apparire una derivata di ordine terzo lungo la componente i-esima interessata. Più precisamente,

$$\begin{aligned}
&\Big|\Delta^3_{x_0}\Big(P_tg\Big)(z)\Big| \\
&\quad= \Big|\int_0^1 \langle D_{x_i}P_tg(x_0+\lambda z) - 2D_{x_i}P_tg(x_0+z+\lambda z) + D_{x_i}P_tg(x_0+2z+\lambda z), z\rangle\, d\lambda\Big| \\
&\quad\le \Big|\int_0^1\int_0^1 \langle\Big[D^2_{x_i}P_tg(x_0+(\lambda+\mu)z) - D^2_{x_i}P_tg(x_0+z+(\lambda+\mu)z)\Big]z, z\rangle\, d\lambda d\mu\Big| \\
&\quad\le \Big|\int_0^1\int_0^1\int_0^1 \langle\Big[D^3_{x_i}P_tg(x_0+(\lambda+\mu+\nu)z)\Big](z,z), z\rangle\, d\lambda d\mu d\nu\Big|,
\end{aligned}$$

dove sopra, abbiamo identificato l'operatore differenziale $D^3_{x_i}P_t\phi$ con il 3-tensore associato. I controlli a priori in (5.27) sulle derivate del semigruppo P_t implicano ora che

$$\Big|\Delta^3_{x_0}\Big(P_tg\Big)(z)\Big| \le C\|D^3_{x_i}P_tg\|_\infty|z|^3 \le C\|g\|_{C^\gamma_{b,d}}\Big(1+t^{\frac{\gamma-3(1+\alpha(i-1))}{\alpha}}\Big)|z|^3.$$

Possiamo allora concludere anche nel caso diagonale che vale la seguente stima:

$$\begin{aligned}
|R_2(z)| &\le \int_{t_0}^\infty e^{-\lambda t}|\Delta^3_{x_0}\Big(P_tg\Big)(z)|\, dt \\
&\le C\|g\|_{C^\beta_{b,d}}|z|^3\int_{t_0}^\infty e^{-\lambda t}\Big(1+t^{\frac{\beta-3(1+\alpha(i-1))}{\alpha}}\Big)\, dt \\
&\le C\|g\|_{C^\beta_{b,d}}|z|^3\Big(\lambda^{-1}+|z|^{\frac{\alpha+\beta-3(1+\alpha(i-1))}{1+\alpha(i-1)}}\Big) \\
&\le C\|g\|_{C^\beta_{b,d}}|z|^{\frac{\alpha+\beta}{1+\alpha(i-1)}},
\end{aligned} \tag{5.32}$$

dove, nell'ultimo passaggio, abbiamo sfruttato di nuovo che $|z|\le 1$.

Riunendo infine i contributi (5.30), (5.31) e (5.32) associati ai tre regimi considerati, potremo concludere che l'equazione (5.29) vale e che in particolare, le stime di Schauder (5.12) nel caso ellittico sono valide.

Le stime di Schauder nel caso parabolico saranno ottenute seguendo un procedimento analogo. Sottolineiamo inoltre che questo tipo di decomposizione in regimi diagonale e fuori-diagonale appaiono, seppur in una forma meno esplicita, anche nei controlli di Hölder per le stime di Schauder del Capitolo 2.

Vogliamo infine citare che avremmo potuto usare il metodo esposto qui anche nel Capitolo 2 per mostrare le stime di Schauder (4.24) per la soluzione $\tilde{u}^{\tau,\xi}$ associata all'operatore proxy, sotto la condizione, più generale, di dominazione stabile [**DS**]. Per ottenere però le stime in (4.11) sulla soluzione u del sistema non-lineare originale (4.1) attraverso il metodo perturbativo esposto prima, la parte delicata sarebbe stata dimostrare, sotto le ipotesi più generali, l'effettiva indipendenza della constante dai parametri di congelamento (τ,ξ) usati. Tale difficoltà apparirà ancora più chiaramente nelle ipotesi del modello nella prossima sezione, dove considereremo in aggiunta un rumore moltiplicativo.

6 Buona posizione debole per catene stocastiche degeneri di tipo Lévy

Presentiamo ora brevemente gli elementi principali del Capitolo 4 della presente tesi. Scritto in collaborazione con il mio relatore, Prof. Stéphane Menozzi, questo lavoro è apparso da poco come pre-pubblicazione [MM21].

Vogliamo studiare qui l'effetto della propagazione di un rumore non-degenere di tipo Lévy attraverso una catena di oscillatori interconnessi tra loro, dove il suddetto rumore agisce solo sul primo di essi, come illustrato in in Figura 1.
Più precisamente, siamo interessati ad una dinamica stocastica su $\mathbb{R}^N$ della forma:

$$\begin{cases} dX_s \;=\; G(s, X_s)ds + B\sigma(s, X_{s-})dZ_s, \quad s \geq t, \\ X_t \;=\; x, \end{cases} \tag{6.1}$$

per un certo punto iniziale (t, x) in $[0, \infty) \times \mathbb{R}^N$, dove la deriva deterministica $G\colon [0, \infty) \times \mathbb{R}^N \to \mathbb{R}^N$ e la matrice di diffusione $\sigma\colon [0, \infty) \times \mathbb{R}^N \to \mathbb{R}^d \otimes \mathbb{R}^d$ sono due coefficienti dati.
L'equazione in (6.1) è degenere nel senso che il rumore $\{Z_s\}_{s\geq 0}$ di tipo Lévy sullo spazio "piccolo" $\mathbb{R}^d$ agirà sul sistema in $\mathbb{R}^N$ (dopo essere stato preservato dal coefficiente di diffusione σ) attraverso la matrice di immersione B in $\mathbb{R}^N \otimes \mathbb{R}^d$, data da:

$$B := \Big(I_{d\times d}, \quad 0_{d\times(N-d)}\Big)^t.$$

Assumeremo in particolare che $\{Z_s\}_{s\geq 0}$ sia un processo di salti puro, ovvero che $\Sigma = 0$ nella tripletta di Lévy (b, Σ, ν) associata al processo.
Per sottolineare l'effettiva dipendenza dal punto iniziale (t, x) scelto, indicheremo d'ora in poi con $\{X_s^{t,x}\}_{s\geq 0}$ un generico processo soluzione della dinamica stocastica in (6.1) esteso, per comodità, fino al tempo zero, ovvero imponendo $X_s^{t,x} = x$ se s in $[0, t]$.

In [CdRM20b], gli autori hanno caratterizzato per una catena degenere come in (6.1) perturbata però da un moto Browniano $\{Z_t\}_{t\geq 0}$, la regolarità di Hölder minima sulla deriva G che assicuri la buona posizione in senso debole per la dinamica stocastica considerata. L'obbiettivo iniziale di questo lavoro era di estendere tale risultato alla dinamica in (6.1) il cui rumore fosse solo di Lévy, sotto la stessa assunzione [**DS**] di dominazione stabile presentata nella sezione precedente (cf. Equazione (5.5)).
Siamo effettivamente riusciti ad ottenere solo una generalizzazione parziale dei risultati in [CdRM20b]. Infatti, mostreremo più avanti l'esistenza di una soglia ottimale per la regolarità di Hölder della deriva G, ma solo per un tipo particolare di struttura deterministica diagonale. Inoltre, il metodo di prova perturbativo da noi seguito, attraverso un proxy retrogrado, ha in pratica richiesto di rafforzare la condizione di dominazione stabile [**DS**] e di aggiungere alcune assunzioni ulteriori. Riassumiamo brevemente in questa sezione le motivazioni naturali che ci hanno portato a tali considerazioni.

La struttura deterministica della dinamica stocastica (cf. (6.1) per $\sigma = 0$) sarà simile a quella presentata nei primi due capitoli. In particolare, supporremo di nuovo di poter decomporre lo spazio "grande" $\mathbb{R}^N$ in n sottospazi $\mathbb{R}^{d_i}$, $i \in [\![1, n]\!]$ tali che $d_1 = d$ e

$d_1 + \cdots + d_n = N$, come mostrato nella Sezione 5. Inoltre, la deriva deterministica G presenterà, rispetto a tale decomposizione, una particolare struttura "sopra diagonale" e i suoi elementi sulla sotto-diagonale saranno considerati non-degeneri e lineari. In pratica, imporremo che G abbia la seguente forma:

$$G(s,x) \;:=\; A_s x + F(s,x), \tag{6.2}$$

dove $A\colon [0,\infty) \to \mathbb{R}^N \otimes \mathbb{R}^N$ e $F\colon [0,\infty) \times \mathbb{R}^N \to \mathbb{R}^N$ sono due funzioni tali che

[H]
- per ogni livello i in $[\![1,n]\!]$, F_i dipende solo dal tempo e dalle ultime $n-(i-1)$ variabili, ovvero $F_i(s,x_i,\ldots,x_n)$;
- $A\colon [0,\infty) \to \mathbb{R}^N \otimes \mathbb{R}^N$ è limitata e

$$[A_s]^{i,j} \;=\; \begin{cases} \text{è non-degenere, uniformemente in } s, & \text{se } j = i-1; \\ 0, & \text{se } j < i-1. \end{cases}$$

Come già accennato precedentemente, questa assunzione nel caso lineare a rumore additivo (i.e. $F = 0$ e $\sigma = 1$) può essere intesa come una condizione di tipo Hörmander, o equivalentemente del rango di Kalman [**K**], che assicuri l'ipoellitticità del generatore infinitesimale associato al processo $\{X_s^{t,x}\}_{s\geq 0}$ soluzione dell'Equazione (6.1).

Sotto l'ipotesi [**H**], notiamo allora che la matrice A può essere riscritta come una versione "dipendente dal tempo" di quella apparsa nella Sezione 5, Equazione (5.11). Inoltre, tale condizione permette di rappresentare la dinamica stocastica in (6.1) nella seguente, più esplicita, forma:

$$\begin{cases} dX_t^1 \;=\; \left[A_t^{1,1}X_t^1 + \cdots + A_t^{1,n}X_t^n + F_1(t,X_t^1,\ldots,X_t^n)\right]dt + \sigma(t,X_{t-}^1,\ldots,X_{t-}^n)dZ_t, \\ dX_t^2 \;=\; \left[A_t^{2,1}X_t^1 + \cdots + A_t^{2,n}X_t^n + F_2(t,X_t^2,\ldots,X_t^n)\right]dt, \\ dX_t^3 \;=\; \left[A_t^{3,2}X_t^2 + \cdots + A_t^{3,n}X_t^n + F_3(t,X_t^3,\ldots,X_t^n)\right]dt, \\ \vdots \\ dX_t^n \;=\; \left[A_t^{n,n-1}X_t^{n-1} + A_t^{n,n}X_t^n + F_n(t,X_t^n)\right]dt, \end{cases}$$

dove abbiamo decomposto $X_t = (X_t^1,\ldots,X_t^n)$ tale che X_t^i sia in $\mathbb{R}^{d_i}$, per ogni i in $[\![1,n]\!]$.

Visto che l'obbiettivo primario di questo capitolo sarà determinare la regolarità di Hölder ottimale sulla deriva F che assicuri la buona posizione della dinamica stocastica e tali soglie minime coinvolgeranno solo le componenti degeneri ($i > 1$) di F, possiamo enunciare ora a parte le condizioni sugli altri coefficienti del sistema. In particolare, supporremo che:

[R] esistono un indice β^1 in $(0,1)$ e una constante $K > 0$ tali che
- $\sigma(t,\cdot)$ è β^1-Hölder continua, uniformemente in t;
- $F_1(t,x)$ è β^1-Hölder continua, uniformemente in t;
- per ogni i in $[\![1,n]\!]$, vale che

$$|F_i(t,0)| \;\leq\; K, \quad t \in [0,T].$$

Perché un effetto regolarizzante minimo (di ordine α) sia trasportato dal rumore, supporremo che $\{Z_t\}_{t\geq 0}$ sia riconducibile ad un processo α-stabile di tipo "temperato". Sottolineiamo però che tale classe include ma non comprende solo i processi α-stabili temperati classici. Più in dettaglio, imporremo che la misura di Lévy ν associata al processo $\{Z_s\}_{s\geq 0}$ è *simmetrica* e

[ND'] esiste una funzione $Q\colon \mathbb{R}^d \to \mathbb{R}$ Borel misurabile tale che

- Q è positiva e limitata, i.e. $Q \geq 0$ e $\sup_{z\in\mathbb{R}^d} Q(z) < \infty$;
- Q è lontana da zero e Lipschitz regolare in un intorno dell'origine, ovvero esiste $r_0 > 0$ e $c > 0$ tali che $Q(z) \geq c$ e Q Lipschitz continua in $B(0, r_0)$;
- esiste $\alpha \in (1, 2)$ e una misura μ finita e non-degenere (nel senso di (4.4)) su $\mathbb{S}^{d-1}$ tale che

$$\nu(\mathcal{A}) \;=\; \int_0^\infty \int_{\mathbb{S}^{d-1}} \mathbb{1}_{\mathcal{A}}(\rho s) Q(\rho s)\, \mu(ds) \frac{d\rho}{\rho^{1+\alpha}}, \quad \mathcal{A} \in \mathcal{B}(\mathbb{R}^d_0).$$

Tenendo a mente la decomposizione classica della misura di Lévy associata ad un processo α-stabile data, per esempio, in (4.3), la condizione di *non-degenerazione* [**ND'**] intuitivamente enuncia che la misura di Lévy di $\{Z_t\}_{t\geq 0}$ è assolutamente continua rispetto a quella di un processo α-stabile non-degenere e che la sua derivata di Radon-Nikodym è data da una funzione "temperante" Q con proprietà adeguate.
Chiaramente, una possibilità naturale è data da $Q = c$ costante ovvero, quando il processo $\{Z_t\}_{t\geq 0}$ può essere ricondotto ad un processo simmetrico α-stabile usuale. Evidenziamo però che la classe di rumori considerati qui comprende anche diversi processi *quasi-stabili* tra cui, per esempio, il processo simmetrico α-stabile relativistico o quello di Lamperti (si veda Capitolo 4, Sezione 1.1 per maggiori dettagli).

Un'assunzione naturale quando si considerano equazioni dipendenti da rumori moltiplicativi è l'*uniforme ellitticità* della componente non-degenere associata alla matrice di diffusione $\sigma(t, x)$ ad ogni punto dello spazio-tempo fissato. Assumeremo infatti che:

[UE] esiste una costante $\eta > 1$ tale che per ogni $s \geq 0$ ed ogni x in $\mathbb{R}^N$,vale che

$$\eta^{-1}|\xi|^2 \;\leq\; \sigma(s, x)\xi \cdot \xi \;\leq\; \eta|\xi|^2, \quad \xi \in \mathbb{R}^d,$$

dove "$\cdot$", ricordiamo, indica il prodotto scalare sullo spazio "piccolo" $\mathbb{R}^d$.
Intuitivamente, l'assunzione di uniforme ellitticità assicura che il coefficiente di diffusione σ preservi efficacemente il rumore Z_t su tutto lo spazio $\mathbb{R}^d$, uniformemente in tempo e spazio.

Quando considereremo la dinamica (6.1) regolarizzata attraverso un rumore moltiplicativo, ovvero in presenza di una matrice di diffusione σ che sia anche dipendente da x, dovremo inoltre assumere che la misura ν sia assolutamente continua rispetto alla misura di Lebesgue su $\mathbb{R}^d$ e che la sua derivata di Radon-Nykodim sia Lipschitz regolare. Più precisamente, imporremo la seguente condizione:

[**AC**] se $\sigma(t,\cdot)$ non è costante per un certo $t \geq 0$, allora esiste una funzione Lipschitz continua $g\colon \mathbb{S}^{d-1} \to \mathbb{R}$ tale che

$$\nu(dz) \;=\; Q(z)\frac{g\left(\frac{z}{|z|}\right)}{|z|^{d+\alpha}}dz.$$

Seppure la condizione [**AC**] riduca sensibilmente la classe di misure di Lévy, e quindi processi, che possiamo includere come rumore nell'equazione stocastica (6.1), evidenziamo che tale assunzione sembra essere necessaria nel nostro contesto, almeno rispetto all'approccio considerato (cf. Equazione (6.32) più avanti) e naturale, visto che è apparsa in altri lavori passati che trattavano argomenti simili (cf. [HM16, FKM21]).
Sottolineiamo comunque che almeno nel caso additivo, o più in generale, quando la matrice di diffusione σ non dipende dallo spazio, la condizione di non-degenerazione [**ND**]. considerata nel Capitolo 2 per le stime di Schauder in ambito stabile, può essere compresa come un caso speciale di [**ND**'] assunta qui. In particolare, anche in questo caso non abbiamo imposto nessuna regolarità, nel caso additivo, sulla misura di Lévy ν del processo $\{Z_t\}_{t\geq 0}$ che, a priori, potrebbe presentare un supporto molto singolare sullo spazio $\mathbb{R}^d$. Un esempio è dato dalla misura spettrale associata al processo α-stabile cilindrico, il cui generatore infinitesimale è stato presentato in (4.5), che possiede infatti un supporto concentrato solo sugli assi di $\mathbb{R}^d$.

Possiamo ora riassumere i principali risultati che andremo a provare nel Capitolo 4. Iniziamo mostrando che sotto condizioni di regolarità di Hölder minime sulla deriva deterministica F, è possibile mostrare il carattere ben posto in senso debole dell'equazione stocastica in (6.1).

Teorema 6.1. *Per ogni j in $[\![2,n]\!]$, sia β^j un indice in $(0,1)$ tale che*

- *$x_j \to F_i(t,x_i,\dots,x_j,\dots,x_n)$ è β^j-Hölder continua, uniformemente in tempo e nelle altre variabili spaziali, per ogni i in $[\![1,j]\!]$.*

Allora l'equazione stocastica (6.1) *è ben posta in senso debole se*

$$\beta^j \;>\; \frac{1+\alpha(j-2)}{1+\alpha(j-1)}, \quad j \geq 2. \tag{6.3}$$

Per ottenere tale risultato, sfrutteremo l'equivalenza, spiegata in Sezione 3, tra la buona posizione in senso debole della dinamica stocastica e la relativa buona posizione per il problema di martingala associato all'operatore $\partial_s + L_s$, dove L_s è (formalmente) il generatore infinitesimale del processo $\{X_s^{t,x}\}_{s\geq 0}$ soluzione di (6.1).
Più precisamente, ricordando che la tripletta di Lévy associata al processo $\{Z_s\}_{s\geq 0}$ è, per il nostro modello, $(b,0,\nu)$, l'operatore L_s può essere rappresentato per ogni funzione ϕ abbastanza regolare attraverso

$$\begin{aligned} L_s\phi(s,x) \;&:=\; \langle G(s,x), D_x\phi(x)\rangle + \mathcal{L}_s\phi(s,x) \\ &:=\; \langle A_s x + F(s,x), D_x\phi(x)\rangle + \text{p.v.}\int_{\mathbb{R}^d}\Big[\phi(x+B\sigma(s,x)z)-\phi(x)\Big]\,\nu(dz), \end{aligned} \tag{6.4}$$

dove, per comodità, abbiamo assorbito il termine che coinvolge b all'interno dell'espressione di F e, similmente al Capitolo 2, abbiamo sfruttato la simmetria della misura di Lévy ν per cancellare il termine di ordine primo $\langle D_x\phi(x), B\sigma(t,x)z\rangle$ nell'integrale.

Il nostro metodo di dimostrazione richiederà inoltre, come tappa intermedia per ottenere la buona posizione, di esibire un particolare tipo di stime, dette di Krylov, per il processo soluzione della dinamica in (6.1). Tali controlli devono il loro nome a N.V. Krylov che le provò per primo in [Kry71] per diffusioni di Itô. Da allora, sono diventate uno strumento versatile per molti ambiti diversi, dalla dimostrazione della buona posizione in senso debole o forte per dinamiche stocastiche fino ad applicazioni alla teoria dei controlli o problemi di filtrazione non-lineare.
In un contesto moltiplicativo α-stabile ($\alpha > 1$) non-degenere dove tali stime vengono sfruttate, citiamo, per esempio, [Kur08] in cui viene mostrata l'esistenza di soluzioni deboli per una dinamica stocastica mono-dimensionale con rumore moltiplicativo e deriva solo misurabile e limitata, o [Zha13a], dove viene provato il carattere ben posto (in senso forte) per un'equazione stocastica a rumore additivo controllata da una deriva singolare in spazi di Sobolev appropriati (sotto condizioni simili a (6.5) per $n = 1$).
Per altri lavori simili in ambito moltiplicativo a salti non-degenere, si vedano anche i seguenti lavori: [AP77, Mel83, LM76].
Per enunciare precisamente le stime di tipo Krylov nel nostro contesto, dovremo però imporre alcune condizioni sugli indici di integrabilità sullo spazio $L^p(0,T;L^q(\mathbb{R}^N))$ delle funzioni f considerate. Intuitivamente, questa soglia garantirà l'integrabilità necessaria ai nostri scopi rispetto alle scale intrinseche date dalla natura degenere del sistema considerato.
Per semplicità, diremo che due numeri reali p, q in $(1,+\infty)$ soddisfano la condizione di integrabilità $(\mathscr{C})$ se:

$$\Big(\frac{1-\alpha}{\alpha}N + \sum_{i=1}^{n} id_i\Big)\frac{1}{q} + \frac{1}{p} \;<\; 1. \tag{$\mathscr{C}$}$$

Tale soglia diviene in realtà più chiara se si considera il caso omogeneo, ovvero quando tutte le componenti $\mathbb{R}^{d_i}$ del sistema presentano la stessa dimensione ($d_i = d$ e $N = nd$). Infatti in questo caso, la condizione $(\mathscr{C})$ può essere allora riscritta come

$$\left(\frac{2+\alpha(n-1)}{\alpha}\right)\frac{nd}{q} + \frac{2}{p} \;<\; 2. \tag{6.5}$$

In questa forma, tale soglia può essere intesa più naturalmente come un'estensione delle condizioni imposte in [CdRM20b], per ottenere lo stesso tipo di stime nel caso diffusivo degenere ($\alpha = 2$). Citiamo inoltre che la condizione $(\mathscr{C})$ appare anche in [KR05] in un contesto diffusivo non-degenere ($\alpha = 2$ e $n = 1$) come l'assunzione (di integrabilità su f) necessaria per integrare una funzione f contro la densità Gaussiana (cf. Equazione (3.2) nella prova del Lemma 3.2 in [KR05]).

Teorema 6.2. *Sotto le assunzioni del Teorema 6.1, siano $T > 0$ e p, q in $(1,+\infty)$ tali che la condizione $(\mathscr{C})$ valga. Allora, esiste una constante $C := C(T,p,q)$ tale che per ogni f in $L^p\Big(0,T;L^q(\mathbb{R}^N)\Big)$,*

$$\Big|\mathbb{E}\Big[\int_t^T f(s,X_s^{t,x})\,ds\Big]\Big| \;\le\; C\|f\|_{L^p_tL^q_x}, \quad (t,x)\in[0,T]\times\mathbb{R}^N, \tag{6.6}$$

dove $\{X_s^{t,x}\}_{s\geq 0}$ *è l'unica soluzione debole della dinamica* (6.1) *con condizione iniziale* (t,x).

Questo tipo di stime enfatizza inoltre che il processo soluzione $\{X_s\}_{s\geq 0}$ possieda in effetti una densità con proprietà di integrabilità adatte fino ad una certa soglia determinata dalla condizione $(\mathscr{C})$.

In [CdR18], Chaudru de Reynal ha caratterizzato, attraverso contro-esempi adatti, la regolarità di Hölder ottimale per la buona posizione debole di un'equazione stocastica cinetica a rumore diffusivo degenere (cf. Equazione (6.1) con $\alpha = 2$ e $n = 2$). Attraverso un'estensione di tali contro-esempi alla Peano, saremo in grado di esibire anche noi un risultato di non-unicità per la catena degenere in (6.1). Ricordiamo che $\{e_i \colon i \in [\![1,n]\!]\}$ è la base canonica sullo spazio $\mathbb{R}^N$.

Teorema 6.3. *Fissati* j *in* $[\![2,n]\!]$ *e* i *in* $[\![2,j]\!]$, *esiste* $F(t,x) = e_i sgn(x_j)|x_j|^{\beta_i^j}$ *con*

$$\beta_i^j < \frac{1+\alpha(i-2)}{1+\alpha(j-1)},$$

per cui l'unicità in legge fallisce nella dinamica stocastica (6.1).

A differenza della catena Gaussiana degenere analizzata in [CdRM20b], non è stato possibile però mostrare in questo caso che le regolarità di Hölder sui coefficienti F_i siano effettivamente quelle minime necessarie per determinare la buona posizione della dinamica stocastica. Come spiegato meglio più avanti, tale problema è intrinsecamente collegato alla natura stabile del processo $\{Z_s\}_{s\geq 0}$, al carattere degenere della dinamica in (6.1) ed in particolare, alla geometria, possibilmente molto singolare, della misura spettrale associata al processo proxy sottostante, inteso come il modello linearizzato (i.e. processo di tipo Ornstein-Uhlenbeck) attorno a cui "svilupperemo".
Sottolineiamo comunque che almeno per la catena degenere perturbata da una deriva non-lineare F solo sulla diagonale, i nostri risultati esibiscono le soglie ottimali auspicate.
Si consideri ora un sistema della forma:

$$\begin{cases} dX_t^1 &= F_1(t,X_t^1,\dots,X_t^n)dt + \sigma(t,X_{t-}^1,\dots,X_{t-}^n)dZ_t, \\ dX_t^2 &= [A_t^2 X_t^1 + F_2(t,X_t^2)]\,dt, \\ dX_t^3 &= [A_t^3 X_t^2 + F_3(t,X_t^3)]\,dt, \\ \vdots \\ dX_t^n &= \left[A_t^n X_t^{n-1} + F_n(t,X_t^n)\right]dt, \end{cases} \tag{6.7}$$

ovvero, tale che la funzione F dipenda solo dal livello corrente della catena. Notiamo allora che i Teoremi 6.1 e 6.3 insieme presentano una caratterizzazione (quasi) completa della buona posizione in senso debole per dinamiche stocastiche degeneri come (6.7), rispetto alla regolarità di Hölder dei loro coefficienti. Infatti, sappiamo che:

- se $\beta^j > \frac{1+\alpha(j-2)}{1+\alpha(j-1)}$ per ogni $j \geq 0$, la buona posizione in senso debole della dinamica in (6.1) deriva dal Teorema 6.1;
- se esiste $j \geq 2$ tale che $\beta^j = \beta_j^j < \frac{1+\alpha(j-2)}{1+\alpha(j-1)}$, sono presenti dei contro-esempi (cf. Teorema 6.3) per cui l'unicità in legge fallisce per la dinamica in (6.1).

Citiamo infine che il caso critico, associato agli esponenti

$$\overline{\beta}_j^j = \frac{1+\alpha(j-2)}{1+\alpha(j-1)}, \quad j \in [\![2,n]\!],$$

rimane da investigare e sembra essere un problema delicato già in un contesto Browniano cinetico (cf. [Zha18]).

6.1 Guida alla Prova

Abbiamo deciso di seguire qui un approccio perturbativo *retrogrado* come introdotto originariamente da McKean-Singer in [MS67] in un contesto diffusivo non-degenere e poi esteso al caso degenere con derive illimitate in [DM10, Men18]. Questa terminologia viene dal fatto che il processo proxy sottostante sarà associato ad un flusso retrogrado in tempo, ovvero, fisseremo $(\tau, \xi) = (s, y)$ nella dinamica in (4.12). Questo metodo si è dimostrato particolarmente utile nell'analisi dell'unicità debole in un contesto degenere su spazi $L_t^p - L_x^q$ (cf. [CdRM20b] nel caso Browniano). Infatti, richiede euristicamente di ottenere solo delle stime sui gradienti (in senso debole) delle soluzioni del problema di Cauchy associato, così da applicare la tecnica dell'inversione dell'operatore, come sfruttato originariamente in [SV79] nel caso diffusivo.

Nonostante quanto suggerito inizialmente nella Sezione 3, abbiamo deciso alla fine di non sfruttare le stime di Schauder, mostrate nelle due sezioni precedenti, per dimostrare l'unicità in legge della dinamica stocastica in (6.1). Siamo comunque convinti che l'approccio perturbativo attraverso il proxy progressivo presentato nella Sezione 4 sarebbe potuto essere stato esteso qui per esibire le stime di Schauder per la classe di processi considerati ora. Da esse, avremmo potuto effettivamente dimostrare poi il carattere ben posto in senso debole dell'equazione stocastica, attraverso un ragionamento di tipo Zvonkin. Tale metodo è, nell'immaginario collettivo, strettamente correlato alla dimostrazione per la buona posizione forte di dinamiche stocastiche ma sottolineiamo che è stato usato anche per provare il carattere ben posto in senso debole per catene degeneri, come fatto, per esempio, in [CdR18]. È a quest'ultimo tipo di ragionamento a cui faremo riferimento come ragionamento di tipo Zvonkin più avanti. Questo metodo però appare subito molto lungo e complicato visto che avrebbe richiesto di estendere le stime di Schauder, presentate in (4.11) sotto la regolarità ottimale aspettata in $C_{b,d}^{\beta}(\mathbb{R}^N)$ sulla sorgente f, ad un più generico spazio di Hölder a multi-indici di regolarità slegati tra loro, del tipo $C_{b,d}^{\gamma}(\mathbb{R}^N)$ con $\gamma = (\gamma_1, \dots, \gamma_d)$ in $\mathbb{N}^d$, proprio a causa del metodo di Zvonkin. In particolare, avrebbe richiesto di stabilire stime puntuali sulle derivate del primo ordine della soluzione del problema di Cauchy associato anche rispetto alle componenti degeneri del sistema. Per far questo, avremmo avuto bisogno soprattutto di estendere i nostri ragionamenti di dualità su spazi di Besov anche rispetto alle derivate degeneri e di considerare una sorgente f con la stessa regolarità del drift F in uno spazio $C_{b,d}^{\gamma}(\mathbb{R}^N)$ con multi-indici di regolarità. Un altro possibile vantaggio del metodo perturbativo retrogrado è che permette di ottenere facilmente, come tappa intermedia della nostra prova, le stime di Krylov in (6.6) sul processo $\{X_s^{t,x}\}_{s\geq 0}$ soluzione della dinamica stocastica in (1.3). Questo tipo di controlli sembra essere nuovo per catene stocastiche degeneri controllate da rumori quasi-stabili e sarebbe state molto difficile

da ottenere invece a partire dalle stime di Schauder.
Sottolineiamo infine che il nostro metodo, a confronto con l'approccio attraverso la trasformazione di Zvonkin come in [CdR18], permette di ottenere un'analisi più precisa della catena degenere almeno lungo la prima componente deterministica, nel senso che possiamo evidenziare qui che non siano richieste soglie minime sulla regolarità spaziale di F_1. Intuitivamente, il metodo attraverso la trasformazione di Zvonkin richiede di mettere come sorgente ogni componente F_i della deriva. Questo porta in particolare alle stesse soglie ottimali globali sulla regolarità di Hölder per F ad ogni livello della catena (cf. Equazione (6.3) con $j \geq 1$). Come mostrato per esempio in [CdR17, FFPV17, CdRM20b] nel caso diffusivo, questo approccio sembra essere più adeguato quando si vuole esibire il carattere ben posto *forte* della dinamica stocastica.

Ai fini di questa presentazione, ci limiteremo a considerare una matrice A_t in (6.2) indipendente dal tempo e tale che solo i suoi elementi nella sotto-diagonale siano non-nulli, ovvero, supporremo che A sia data in (4.6).
Infatti, come spiegato nella precedente sezione, le difficoltà aggiuntive date dalla presenza di elementi non-nulli sopra la sotto-diagonale possono essere facilmente risolte attraverso un ragionamento in tempo piccolo. Inoltre, la possibile dipendenza dal tempo richiederebbe essenzialmente di introdurre, al posto della matrice esponenziale $e^{A(s-t)}$, la risolvente $\mathcal{R}_{s,t}$ associata alla matrice A_t. In pratica, $\mathcal{R}_{s,t}$ è la matrice in $\mathbb{R}^N \otimes \mathbb{R}^N$ dipendente dal tempo soluzione della seguente equazione differenziale matriciale:

$$\begin{cases} \partial_s \mathcal{R}_{s,t} \;=\; A_s \mathcal{R}_{s,t}, \quad s \in [t,T]; \\ \mathcal{R}_{t,t} \;=\; \mathrm{Id}_{N\times N}\,. \end{cases}$$

Citiamo infine che decomposizioni simili a quelle presentate in (4.7), si sono mostrate valide anche per la risolvente $\mathcal{R}_{s,t}$. Si veda per esempio [HM16], Lemmi 5.1 e 5.2 oppure [DM10], Proposizione 3.7.

Metodo della Parametrice Retrograda

Come già spiegato nella Sezione 4, l'elemento cruciale del metodo perturbativo consiste nello scegliere attentamente un operatore proxy adeguato con proprietà e controlli noti, attorno a cui espandere il generatore infinitesimale L_s, alle spese di un aggiuntivo errore di espansione da controllare.
Quando la deriva F è sufficientemente regolare, per esempio globalmente Lipschitz continua, è stato mostrato per esempio in [DM10, Men11, Men18] che un proxy adeguato è dato dalla linearizzazione della dinamica stocastica (6.1) attorno al flusso deterministico associato alla dinamica (i.e. quando $\sigma = 0$ in (6.1)), che ha portato, nei lavori sopra in un contesto browniano degenere, a considerare un processo Gaussiano multi-scala come proxy. La naturale generalizzazione al contesto considerato qui porterà invece a scegliere come proxy un processo multi-scala di tipo Lévy il cui simbolo sarà dipendente dal tempo.
Più precisamente, fissati dei parametri di congelamento (s,y) in $[0,T]\times\mathbb{R}^N$, sia $\theta_{t,s}(y)$ una delle possibili soluzioni della seguente equazione:

$$\theta_{t,s}(y) \;=\; y - \int_t^s \Big[A\theta_{u,s}(y) + F(u,\theta_{u,s}(y))\Big]\,du. \tag{6.8}$$

Visto che il punto di congelamento y sarà poi anche la variabile di integrazione (si veda per esempio la definizione del nucleo di Green in (6.18)), è importante sottolineare fin da ora che è sempre possibile, tra i possibili flussi $\theta_{t,s}(y)$ soluzione di (6.8), sceglierne uno che sia misurabile rispetto a (s, y) in $[0, T] \times \mathbb{R}^N$ (cf. Lemma 2.13 nel Capitolo 4). Supporremo d'ora in avanti di aver fissato proprio tale versione di $\theta_{t,s}(y)$.

Il passaggio successivo sarà di introdurre la dinamica stocastica *linearizzata* attorno al flusso retrogrado $\theta_{t,s}(y)$. Più precisamente, considereremo per ogni punto iniziale (t, x) in $[0, s] \times \mathbb{R}^N$, il processo proxy $\{\tilde{X}_u^{t,x,s,y}\}_{u\geq 0}$ soluzione della seguente equazione stocastica:

$$\begin{cases} d\tilde{X}_u^{t,x,s,y} \;=\; \left[A\tilde{X}_u^{t,x,s,y} + \tilde{F}_u^{s,y}\right] du + B\tilde{\sigma}_u^{s,y}\, dZ_u, \quad u \in [t, T], \\ \tilde{X}_t^{t,x,s,y} \;=\; x, \end{cases} \tag{6.9}$$

dove abbiamo denotato per semplicità, $\tilde{F}_u^{s,y} := F(u, \theta_{u,s}(y))$ e $\tilde{\sigma}_u^{s,y} := \sigma(u, \theta_{u,s}(y))$. Un calcolo diretto permette di ottenere poi una rappresentazione integrale del processo proxy:

$$\tilde{X}_s^{t,x,s,y} \;=\; \tilde{m}_{s,t}^{s,y}(x) + \int_t^s e^{A(s-u)} B\tilde{\sigma}_u^{s,y} dZ_u, \tag{6.10}$$

dove, analogamente a (4.14), il termine di trasporto congelato $\tilde{m}_{s,t}^{s,y}(x)$ è dato da

$$\tilde{m}_{s,t}^{s,y}(x) \;=\; e^{A(s-t)}x + \int_t^s e^{A(s-u)} \tilde{F}_u^{s,y}\, du.$$

Ragionamenti sugli spazi di Fourier, simili a quelli svolti nella Sezione 4 in (4.15)-(4.16), permettono di mostrare anche in questo caso la seguente fondamentale identità in legge:

$$\tilde{X}_s^{t,x,s,y} \overset{\text{(legge)}}{=} \tilde{m}_{s,t}^{s,y}(x) + \mathbb{M}_{s-t}\tilde{S}_{s-t}^{s,y}, \tag{6.11}$$

dove ad ogni parametro di congelamento (s, y) fissato, $\{\tilde{S}_u^{s,y}\}_{u\geq 0}$ è riconducibile ad un processo α-stabile non-degenere su $\mathbb{R}^N$ nel senso indicato in [**ND**']. Evidenziamo inoltre che la dipendenza dai parametri di congelamento, difficoltà cruciale in questa parte della dimostrazione, è essenzialmente legata alla presenza della matrice di diffusione congelata $\tilde{\sigma}^{s,y}$ nella convoluzione stocastica in (6.10). Infatti, tale dipendenza sparirebbe nel caso di rumore additivo (i.e. $\sigma(t, x) = 1$) o, più generalmente, per un coefficiente di diffusione omogeneo in spazio.
Come già spiegato nella Sezione 4, la non-degenerazione della misura spettrale del processo $\{\tilde{S}_u^{s,y}\}_{u\geq 0}$ assicura in particolare l'esistenza di una densità $p_{\tilde{S}^{s,y}}(u, \cdot)$ sufficientemente regolare (in spazio) per tale processo. L'identità in (6.11) implica allora che la seguente funzione

$$\tilde{p}^{s,y}(t, s, x, y) \;:=\; \frac{p_{\tilde{S}^{s,y}}(s-t, \mathbb{M}_{s-t}^{-1}(y - \tilde{m}_{t,s}^{s,y}(x)))}{\det \mathbb{M}_{s-t}},$$

sia invece la "densità" associata al processo proxy $\{\tilde{X}_s^{t,x,s,y}\}_{s\geq 0}$ congelato al punto terminale (s, y).

Similmente alle sezioni precedenti, ci focalizzeremo poi nel determinare le proprietà regolarizzanti associate a $\tilde{p}^{s,y}(t, s, x, y)$. In particolare, mostreremo che le derivate della

"densità" congelata sono controllate dall'alto da un'altra densità al prezzo di singolarità aggiuntive in tempo e soprattutto, che tale controllo vale *uniformemente* nei parametri di congelamento (s, y). Più precisamente, varrà per ogni k in $[\![0, 2]\!]$ ed ogni i in $[\![1, n]\!]$, che:

$$|D^k_{x_i}\tilde{p}^{s,y}(t, s, x, y)| \leq C\frac{\overline{p}\left(1, \mathbb{T}^{-1}_{s-t}(y - \tilde{m}^{s,y}_{s,t}(x))\right)}{\det \mathbb{T}_{s-t}}(s-t)^{-k\frac{1+\alpha(i-1)}{\alpha}}, \tag{6.12}$$

dove abbiamo denotato per semplicità $\mathbb{T}_u = u^{\frac{1}{\alpha}}\mathbb{M}_u$. Questo tipo di controllo può essere visto essenzialmente come un analogo di quello ottenuto in (4.19), dove per comodità abbiamo già riscalato il sistema al tempo unitario $t = 1$, sfruttando l'α-autosimilarità associata alla densità α-stabile $\overline{p}(u, \cdot)$, ovvero:

$$\overline{p}(u, z) \;=\; u^{-N/\alpha}\overline{p}(1, u^{-1/\alpha}z), \quad (u, z) \in [0, \infty) \times \mathbb{R}^N.$$

Visto che il punto di congelamento y servirà più avanti anche come variabile di integrazione (cf. definizione di $\tilde{G}_\epsilon f$ in (6.18)), evidenziamo in particolare quanto sia importante ottenere un controllo dall'alto con una densità che sia indipendente da questo parametro.
Stime sulle derivate della densità come in (6.12) o in (4.19) sono spesso ottenute attraverso la decomposizione di Itô-Lévy della variabile aleatoria $\tilde{S}^{s,y}_u$ al tempo caratteristico stabile corrispondente. Più precisamente, siano $\tilde{M}^{s,y}_u$ e $\tilde{N}^{s,y}_u$ le due variabili aleatorie indipendenti associate ai contributi per i salti piccoli e grandi del processo $\{\tilde{S}^{s,y}_u\}$ troncato al tempo $u^{1/\alpha}$.
Tale troncazione permetterà in particolare di riscrivere la densità $p_{\tilde{S}^{s,y}}(u, z)$ di $\tilde{S}^{s,y}_u$ nel seguente modo:

$$p_{\tilde{S}^{s,y}}(u, z) \;=\; \int_{\mathbb{R}^N} p_{\tilde{M}^{s,y}}(u, z - w)\, P_{\tilde{N}^{s,y}_u}(dw) \tag{6.13}$$

dove $p_{\tilde{M}^{s,y}}(u, \cdot)$ è la densità generata da $\tilde{M}^{s,y}_u$ e $P_{\tilde{N}^{s,y}_u}$ è la legge di $\tilde{N}^{s,y}_u$.
Mentre ragionamenti come quelli svolti nella Sezione 5 precedente (cf. Equazione (5.21)) sulla densità troncata p^{tr} possono essere applicati anche in questo caso conducendo alle seguenti stime:

$$\left|D^k_z p_{\tilde{M}^{s,y}}(u, z)\right| \;\leq\; Cu^{-(N+k)/\alpha}\left(\frac{u^{1/\alpha}}{u^{1/\alpha} + |z|}\right)^{N+3} \;=:\; Cu^{-\frac{k}{\alpha}}p_{\bar{M}}(u, z), \tag{6.14}$$

dove C è indipendente dai parametri di congelamento (s, y); sarà molto più delicato ottenere un controllo uniforme sulla misura di probabilità $P_{\tilde{N}^{s,y}_u}(dy)$ del tipo:

$$P_{\tilde{N}^{s,y}_u}(\mathcal{A}) \;\leq\; C\overline{P}_u(\mathcal{A}), \quad \mathcal{A} \in \mathcal{B}(\mathbb{R}^N), \tag{6.15}$$

dove $\{\overline{P}_u\}_{u\geq 0}$ è una famiglia di misure di probabilità che conserva le stesse proprietà di integrazione delle code di un processo α-stabile.
Questa difficoltà è il motivo principale per cui non abbiamo potuto considerare, come in [CdRM20b], una deriva completamente non-lineare, i.e. $G_i(t, x) = G_i(t, x_{i-1}, \cdots, x_n)$ con una dipendenza non-lineare nella variabile x_{i-1} che trasmette il rumore, ma solo una versione semi-lineare di G, data in (6.2). Infatti, il modello più generale analizzato in [CdRM20b] avrebbe richiesto in particolare di linearizzare la dinamica stocastica

(6.1) lungo la sotto-diagonale della matrice jacobiana di G congelata in (s, y), ovvero di considerare il seguente processo proxy:

$$d\tilde{X}_u^{t,x,s,y} = \left[\tilde{A}_u^{s,y}\tilde{X}_u^{t,x,s,y} + \tilde{F}_u^{s,y}\right] du + B\tilde{\sigma}_u^{s,y}\, dZ_u,$$

dove $\tilde{A}_u^{s,y}$ è una matrice dipendente dai parametri di congelamento tale che

$$[\tilde{A}_u^{s,y}]_{i,j} = \begin{cases} D_{x_{i-1}}G_i(s, \theta_{t,s}(y)), & \text{se } j = i-1, \\ 0, & \text{altrimenti.} \end{cases}$$

Per questo tipo di modello, non siamo stati in grado di mostrare effettivamente un controllo come in (6.15).
Sottolineiamo inoltre che è sempre per questo motivo che non abbiamo potuto considerare una misura di Lévy ν associata al processo $\{Z_t\}_{t\geq 0}$ che sia asimmetrica, come fatto per esempio nel Capitolo 3 della presente tesi. Infatti, il metodo perturbativo a parametrice retrograda richiede proprietà regolarizzanti più delicate associate agli operatori coinvolti e soprattutto, una certa compatibilità tra il proxy e il processo originale.
Intuitivamente, la simmetria della misura di Lévy sottostante non è un vincolo forte per il contributo in (6.14) associato ai salti piccoli e questo ha permesso infatti nella sezione precedente di considerare anche operatori asimmetrici. Nel nostro modello, in cui si richiede invece un effetto regolarizzante globale per la densità, sembra essere allora naturale imporre la simmetria della misura spettrale corrispondente per il controllo delle code del processo, come si vedrà nella prova della stima in (6.15) (cf. Lemma 5.2 nel Capitolo 4).

Proprietà Analitiche della Densità Congelata lungo la Condizione Terminale

Evidenziamo ora che non è immediato determinare per quale tipo di problema di Cauchy la "densità" $\tilde{p}^{s,y}(t, s, x, y)$ congelata al punto terminale (s, y) sia effettivamente una soluzione fondamentale. Infatti, la presenza del punto di congelamento y anche come variabile di integrazione (per esempio in (6.18)) rende molto più delicato determinarne le proprietà analitiche e soprattutto, provarne la convergenza alla massa di Dirac δ_x quando il tempo t tende a zero.
Sia $\tilde{L}_t^{s,y}$ il generatore infinitesimale associato al processo congelato $\{\tilde{X}_s^{s,y,t,x}\}_{s\geq 0}$. Più precisamente, scriviamo per ogni funzione $\phi\colon \mathbb{R}^N \to \mathbb{R}$ abbastanza regolare, che:

$$\begin{aligned} \tilde{L}_t^{s,y}\phi(x) &:= \langle Ax + \tilde{F}_t^{s,y}, D_x\phi(x)\rangle + \tilde{\mathcal{L}}_t^{s,y}\phi(x) \\ &:= \langle Ax + \tilde{F}_t^{s,y}, D_x\phi(x)\rangle + \text{p.v.}\int_{\mathbb{R}_0^d}\left[\phi(x + B\tilde{\sigma}_t^{s,y}w) - \phi(x)\right]\nu(dw), \end{aligned} \tag{6.16}$$

dove, ricordiamo, abbiamo denotato $\tilde{F}_t^{s,y} := F(t, \theta_{t,s}(y))$ e $\tilde{\sigma}_t^{s,y} := \sigma(t, \theta_{t,s}(y))$.
Se fissiamo ora il parametro di congelamento y e facciamo variare solo la variabile di integrazione in $\tilde{p}^{s,y}(t, s, x, \cdot)$, la funzione diviene una densità a tutti gli effetti e non è allora difficile mostrare, attraverso un calcolo diretto, che:

$$\left(\partial_t + \tilde{L}_t^{s,y}\right)\tilde{p}^{s,y}(t, s, x, z) = 0, \quad (t, z) \in [0, s) \times \mathbb{R}^N, \tag{6.17}$$

per ogni (s, x, y) in $[0, T] \times \mathbb{R}^{2N}$ fissati.

Per determinare quale tipo di sistema parabolico è risolto dalla densità congelata $(s,y) \mapsto \tilde{p}^{s,y}(t,s,x,y)$, consideriamo ora un operatore $\tilde{G}_\epsilon$ che può essere inteso come il nucleo di Green associato alla "densità" $\tilde{p}^{s,y}(t,s,x,y)$ e localizzato fuori dal tempo iniziale t. In particolare, fissato $\epsilon > 0$ sufficientemente piccolo per i nostri scopi,

$$\tilde{G}_\epsilon f(t,x) := \int_{t+\epsilon}^{T} \int_{\mathbb{R}^N} \tilde{p}^{s,y}(t,s,x,y) f(s,y)\, dyds, \quad (t,x) \in [0,T) \times \mathbb{R}^N, \tag{6.18}$$

per ogni funzione $f\colon [0,T) \times \mathbb{R}^N \to \mathbb{R}$ che sia sufficientemente regolare ed a supporto compatto.
L'espressione in (6.18) è ben definita, visto che sappiamo che la densità congelata $\tilde{p}^{s,y}(t,s,x,y)$ è misurabile in (s,y).
La localizzazione in ϵ per distanziare l'integrale da t è fondamentale visto che assicura l'effetto regolarizzante voluto per il nucleo di Green $\tilde{G}_\epsilon$. Evidenziamo infatti che, nel caso limite (i.e. $\epsilon \to 0$), la regolarità della funzione f non è una condizione sufficiente a derivare la regolarità per il nucleo di Green $\tilde{G}_0 f$. Questa ulteriore difficoltà è data dalla dipendenza del proxy dalla variabile di integrazione y.

Se introduciamo ora anche la seguente quantità:

$$\tilde{M}_\epsilon f(t,x) := \int_{t+\epsilon}^{T} \int_{\mathbb{R}^N} \tilde{L}_t^{s,y} \tilde{p}^{s,y}(t,s,x,y) f(s,y)\, dyds, \quad (t,x) \in [0,T) \times \mathbb{R}^N,$$

possiamo infine derivare dall'Equazione (6.17) che lo pseudo-nucleo di Green $\tilde{G}_\epsilon$ risolve la seguente equazione parabolica:

$$\partial_t \tilde{G}_\epsilon f(t,x) + \tilde{M}_\epsilon f(t,x) = -I_\epsilon f(t,x), \quad (t,x) \in [0,T) \times \mathbb{R}^N. \tag{6.19}$$

Anche in questo caso, la localizzazione in ϵ è stata cruciale per ottenere (6.19) direttamente dall'Equazione (6.17) sfruttando argomenti classici di convergenza dominata.
Sopra, l'operatore I_ϵ per ogni funzione $f\colon [0,T) \times \mathbb{R}^N \to \mathbb{R}$ abbastanza regolare, può essere rappresentato attraverso:

$$I_\epsilon f(t,x) := \int_{\mathbb{R}^N} f(t+\epsilon, y) \mathbb{1}_{[0,T-\epsilon]}(t) \tilde{p}^{t+\epsilon,y}(t, t+\epsilon, x, y)\, dy. \tag{6.20}$$

Evidenziamo ora che I_ϵ può essere inteso come una versione localizzata fuori da t dell'operatore identità. In particolare, ne otterremo nel Capitolo 4 (cf. Lemmi 2.18 e 2.19), la convergenza alla massa di Dirac concentrata in (t,x) sugli spazi funzionali da noi considerati:

$$\lim_{\epsilon \to 0} \|I_\epsilon f - f\|_\infty = 0, \quad \lim_{\epsilon \to 0} \|I_\epsilon f - f\|_{L_t^p L_x^q} = 0. \tag{6.21}$$

Seppure a prima vista le proprietà di convergenza sopra sembrino essere immediate, sottolineiamo che la presenza della variabile di integrazione y anche come parametro di congelamento impedisce di ottenere le stime in (6.21) direttamente dalla convergenza in legge del processo congelato $\{\tilde{X}_s^{s,y,t,x}\}_{s \geq 0}$ alla massa di Dirac (cf. Equazione (6.17)).

Unicità del Problema di Martingala e Stime di Krylov Associate

Possiamo presentare ora i passaggi principali della nostra dimostrazione per l'unicità del problema di martingala associato a $\partial_t + \mathcal{L}_t$. Come già accennato precedentemente, la teoria analitica associata al processo proxy spiegata sopra sarà lo strumento fondamentale per il nostro metodo di prova.
Esibiremo inizialmente le stime di tipo Krylov in (6.6) sfruttando il metodo perturbativo col proxy retrogrado, supponendo però che gli indici p, q siano sufficientemente grandi ma finiti.
Più precisamente, fissata una funzione f abbastanza regolare per i nostri scopi ed una soluzione $\{X_s^{t,x}\}_{s\geq 0}$ della dinamica stocastica in (6.1), il primo passo del nostro metodo consiste nell'applicare la formula di Itô al nucleo di Green congelato $\tilde{G}_\epsilon f$ rispetto al processo $\{X_s^{t,x}\}_{s\geq 0}$:

$$\mathbb{E}\left[\tilde{G}_\epsilon f(t,x) + \int_t^T (\partial_s + L_s)\tilde{G}_\epsilon f(s, X_s^{t,x})ds\right] \;=\; 0. \tag{6.22}$$

Ricordando che lo pseudo-nucleo di Green $\tilde{G}_\epsilon f$ risolve l'equazione in (6.19), possiamo riscrivere (6.22) nel seguente modo:

$$\begin{aligned}\mathbb{E}\Big[\int_t^T \int_{\mathbb{R}^N} I_\epsilon f(s, X_s^{t,x})\, ds\Big] \;&=\; \tilde{G}_\epsilon f(t,x) + \mathbb{E}\Big[\int_t^T \Big[L_s\tilde{G}_\epsilon f - \tilde{M}_\epsilon f\Big](s, X_s^{t,x})\, ds\Big] \\ &=:\; \tilde{G}_\epsilon f(t,x) + \mathbb{E}\Big[\int_t^T \tilde{R}_\epsilon f(s, X_s^{t,x})\, ds\Big].\end{aligned} \tag{6.23}$$

Per ottenere allora le stime di Krylov in (6.6) dall'equazione sopra, dovremo esibire dei controlli del tipo:

$$\|\tilde{G}_\epsilon f\|_\infty \;\leq\; C\|f\|_{L_t^p L_x^q}, \quad \|\tilde{R}_\epsilon f\|_\infty \;\leq\; C\|f\|_{L_t^p L_x^q}. \tag{6.24}$$

In particolare, la necessità di imporre in un primo tempo una soglia minima sugli indici p, q, è proprio dovuta al controllo puntuale del termine di resto $\tilde{R}_\epsilon f$, valida solo per p e q sufficientemente grandi.
Dopo aver esibito stime come in (6.24), potremo allora mostrare dall'equazione (6.23) che

$$\left|\mathbb{E}\left[\int_t^T I_\epsilon f(s, X_s^{t,x})\, ds\right]\right| \;\leq\; C\|f\|_{L_t^p L_x^q}.$$

Sfruttando ora le proprietà di convergenza puntuale di I_ϵ in (6.21) per far tendere ϵ a zero, e un ragionamento di approssimazione regolare sullo spazio di funzioni $L^p(0,T;L^q(\mathbb{R}^N))$, sarà infine possibile concludere che stime di Krylov della forma:

$$\left|\mathbb{E}\left[\int_t^T f(s, X_s^{t,x})\, ds\right]\right| \;\leq\; C\|f\|_{L_t^p L_x^q}, \tag{6.25}$$

valgono per ogni f in $L^p(0,T;L^q(\mathbb{R}^N))$, seppure sotto la condizione aggiuntiva che p e q siano abbastanza grandi.

Il risultato "parziale" ottenuto sopra è però sufficiente per mostrare l'unicità del problema di martingala associato all'operatore $\partial_t + L_t$. Attraverso un ragionamento di

dualità sugli spazi $L^p_t - L^q_x$, la stima in (6.25) implicherà in particolare l'esistenza di una "densità" per il processo $\{X^{t,x}_s\}_{s\geq 0}$ soluzione della dinamica stocastica (6.1) con punto iniziale (t,x), definita però solo per quasi ogni (s,y).
Chiamata $p(t,s,x,y)$ tale densità, segue facilmente dall'Equazione (6.23) che

$$\tilde{G}_\epsilon f(t,x) \;=\; \int_t^T \int_{\mathbb{R}^N} (I_\epsilon - \tilde{R}_\epsilon) f(s,y) p(t,s,x,y)\, dyds.$$

A questo punto, utilizzeremo la tecnica dell'inversione dell'operatore $I_\epsilon - \tilde{R}_\epsilon$ sullo spazio funzionale $L^p(0,T;L^q(\mathbb{R}^N))$. Per fare ciò, sarà necessario esibire in particolare delle stime sul termine di resto $\tilde{R}_\epsilon$ della seguente forma:

$$\|\tilde{R}_\epsilon\|_{L^p_t L^q_x} \;\leq\; C_T \|f\|_{L^p_t L^q_x}, \tag{6.26}$$

per una certa costante $C_T > 0$ indipendente da ϵ e tale che $C_T \to 0$ quando T tende a zero.
Scegliendo un intervallo di tempo sufficientemente piccolo, potremo supporre allora che la norma di $\tilde{R}_\epsilon$ come operatore su $L^p(0,T;L^q(\mathbb{R}^N))$ sia più piccola di 1 e quindi, che l'operatore $I_\epsilon - \tilde{R}_\epsilon$ sia effettivamente invertibile:

$$\mathbb{E}\left[\int_t^T f(s, X^{t,x}_s)\, ds\right] \;=\; \tilde{G}_\epsilon \circ (I_\epsilon - \tilde{R}_\epsilon)^{-1} f(t,x).$$

Sfruttando di nuovo le proprietà di convergenza dell'operatore I_ϵ in (6.21), questa volta nello spazio $L^p(0,T;L^q(\mathbb{R}^N))$, potremo concludere che

$$\mathbb{E}\left[\int_t^T f(s, X^{t,x}_s)\, ds\right] \;=\; \tilde{G} \circ (I - \tilde{R})^{-1} f(t,x).$$

L'identità sopra implica immediatamente l'unicità del problema di martingala su un intervallo temporale sufficientemente piccolo. Un argomento di concatenamento in tempo permetterà di estendere all'unicità globale per il problema di martingala, concludendo così la prova del Teorema 6.1.Per maggiori dettagli su questo tipo di localizzazione, si veda per esempio [EK86] Sezione 4.6.

Mostreremo solo in un secondo momento che le stime di Krylov in (6.6) valgono effettivamente per ogni coppia (p,q) che soddisfi la condizione di integrabilità $(\mathscr{C})$, attraverso un ragionamento di mollificazione stocastica.
Più in dettaglio, fisseremo un processo α-stabile isotropo $\{\overline{Z}_s\}_{s\geq 0}$ e un parametro di regolarizzazione δ piccolo e andremo a considerare una versione regolarizzata del processo $\{X^{t,x}_s\}_{s\geq 0}$ soluzione della dinamica stocastica in (6.1), data da

$$\overline{X}^{t,x,\delta}_s \;:=\; X^{t,x}_s + \delta \mathbb{M}_{s-t} \overline{Z}_{s-t}. \tag{6.27}$$

Intuitivamente, l'idea di regolarizzare il processo soluzione $\{X^{t,x}_s\}_{s\geq 0}$ serve ad ottenere, a livello delle densità associate, una controllabilità nello spazio funzionale duale a $L^p_t - L^q_x$ fino alla soglia cercata (cf. condizione di integrabilità $(\mathscr{C})$).
Infatti, se denotiamo rispettivamente con p', q' gli esponenti coniugati di p e q, sappiamo dalle stime di Krylov "parziali" in (6.25) che la densità $p(t,s,x,y)$ ha norma finita nello

spazio $L^{p'}(0,T,L^{q'}(\mathbb{R}^N))$ per valori di p,q abbastanza grandi.
Inoltre, se chiamiamo con $p^\delta(t,s,x,\cdot)$ la densità associata alla variabile casuale $\overline{X}_s^{t,s,\delta}$, l'identità in (6.27) implica immediatamente che

$$p^\delta(t,s,x,y) \;=\; \Big[p(t,s,x,\cdot) * q^\delta(s-t,\cdot)\Big](y), \tag{6.28}$$

dove $q^\delta(t,\cdot)$ rappresenta la densità associata al processo $\{\delta\mathbb{M}_s\overline{Z}_s\}_{s\geq 0}$.
Sfruttando ora disuguaglianze di convoluzione in (6.28), otterremo in particolare che per ogni coppia (p,q) che soddisfi la condizione $(\mathscr{C})$, la quantità $\|p^\delta\|_{L_t^{p'}L_x^{q'}}$ è finita, seppure possibilmente esplosiva per δ che tende a zero.
Tale controllo permetterà di provare allora che il processo mollificato $\overline{X}_s^{t,x,\delta}$ soddisfa le stime di Krylov in (6.6) per tutti gli indici p, q nell'intervallo interessato ma per una constante C possibilmente dipendente dal parametro di mollificazione δ (ed esplosiva rispetto a tale parametro).
Riproducendo l'analisi perturbativa svolta nella prima parte della dimostrazione, mostreremo infine che i controlli sulla densità mollificata $p^\delta(t,s,x,y)$, e quindi la constante nelle stime di Krylov, non dipendono effettivamente da δ.
Facendo infine tendere δ a zero, potremo esibire le stime di Krylov sotto le condizioni richieste su p q per la soluzione $X_s^{t,x}$ della dinamica stocastica originale (6.1).

Sulle stime Fondamentali per il Nucleo di Green ed il Termine di Resto

Come abbiamo appena visto, il nostro metodo si poggia effettivamente su controlli fondamentali sul nucleo di Green $\tilde{G}_\epsilon f$ e il termine di resto $\tilde{R}_\epsilon f$ dati in (6.24) e (6.26).
La dimostrazione di tali stime sarà piuttosto lunga e complessa e riempirà, insieme alla dimostrazione della convergenza di I_ϵ in (6.21), buona parte della sezione tecnica del Capitolo 4.
Visto che non pensiamo sia possibile riassumerli per questa introduzione in una maniera coerente per il lettore, abbiamo deciso invece di evidenziare solo alcuni dei passaggi più salienti del ragionamento che permettano di mostrare la necessità di alcune delle assunzioni da noi fatte.

Per esempio, le soglie nella condizione di integrabilità $(\mathscr{C})$ appaiono evidenti nella dimostrazione delle stime puntuali in (6.24) sul nucleo di Green $\tilde{G}_\epsilon$.
Infatti, ricordando la definizione di $\tilde{G}_\epsilon f$ data in (6.18), possiamo inizialmente spezzare l'integrale attraverso una disuguaglianza di Hölder:

$$\begin{aligned} |\tilde{G}_\epsilon f(t,x)| \;&\leq\; C\|f\|_{L_t^pL_x^q}\left(\int_{t+\epsilon}^T\left(\int_{\mathbb{R}^N}|\tilde{p}^{s,y}(t,s,x,y)|^{q'}\,dy\right)^{\frac{p'}{q'}}ds\right)^{\frac{1}{p'}} \\ &=:\; C\|f\|_{L_t^pL_x^q}\left(\int_{t+\epsilon}^T(|\mathfrak{J}(t,s,x)|)^{\frac{p'}{q'}}\,ds\right)^{\frac{1}{p'}}, \end{aligned} \tag{6.29}$$

dove, ricordiamo, abbiamo rispettivamente denotato con p', q' gli esponenti coniugati di p e q.
Sfruttando i controlli sulla densità $\tilde{p}^{s,y}(t,s,x,\cdot)$ in (6.12), è possibile ora controllare il

termine $\mathcal{J}(t,s,x)$ nel seguente modo:

$$|\mathcal{J}(t,s,x)| \le C\,(\det \mathbb{T}_{s-t})^{-q'} \int_{\mathbb{R}^N} \left[\bar{p}\left(1, \mathbb{T}_{s-t}^{-1}(y-\tilde{m}_{s,t}^{s,y}(x))\right)\right]^{q'} dy \le C\,(\det \mathbb{T}_{s-t})^{1-q'}.$$

Ne segue allora che

$$|\tilde{G}_\epsilon f(t,x)| \le C\|f\|_{L_t^p L_x^q} \left(\int_{t+\epsilon}^T (\det \mathbb{T}_{s-t})^{(1-q')\frac{p'}{q'}}\, ds\right)^{\frac{1}{p'}}. \tag{6.30}$$

Per concludere, è necessario mostrare che l'integrale in tempo sia finito. Dalla definizione $\mathbb{T}_t := t^{1/\alpha}\mathbb{M}_t$, notiamo allora che

$$\det \mathbb{T}_{s-t} = (s-t)^{\frac{1}{\alpha}+\sum_{i=1}^n d_i(i-1)} = (s-t)^{\sum_{i=1}^n d_i \frac{1+\alpha(i-1)}{\alpha}}.$$

Visto che $(1-q')\frac{p'}{q'} = -\frac{p'}{q}$, possiamo infine concludere che l'integrale in tempo in (6.30) è finito se

$$\Big(\sum_{i=1}^n d_i \frac{1+\alpha(i-1)}{\alpha}\Big)\frac{p'}{q} < 1 \Leftrightarrow \Big(\sum_{i=1}^n d_i \frac{1+\alpha(i-1)}{\alpha}\Big)\frac{1}{q} + \frac{1}{p} < 1.$$

La disuguaglianza a destra è proprio la condizione di integrabilità ($\mathscr{C}$) che abbiamo assunto nel Teorema 6.2. Intuitivamente, possiamo allora spiegare la soglia in ($\mathscr{C}$) come la condizione di integrabilità in tempo necessaria per controllare le diverse scale intrinseche associate al sistema degenere, quando si considerano stime in norma $L_t^p - L_x^q$.

Ci concentriamo ora sulla stima puntuale del termine di resto $\tilde{R}_\epsilon$ data in (6.24). Dalla definizione in (6.23), possiamo decomporre $\tilde{R}_\epsilon$ in due parti:

$$\begin{aligned}\tilde{R}_\epsilon f(t,x) &= \int_{t+\epsilon}^T \int_{\mathbb{R}^N} (\mathcal{L}_t - \tilde{\mathcal{L}}_t^{s,y})\tilde{p}^{s,y}(t,s,x,y) f(s,y)\, dyds \\ &\qquad + \int_{t+\epsilon}^T \int_{\mathbb{R}^N} \langle F(t,x) - \tilde{F}_t^{s,y}, D_x \tilde{p}^{s,y}(t,s,x,y)\rangle f(s,y)\, dyds \\ &=: \tilde{R}_\epsilon^0 f(t,x) + \tilde{R}_\epsilon^1 f(t,x),\end{aligned} \tag{6.31}$$

dove, ricordiamo, gli operatori $\mathcal{L}_s$ e $\tilde{\mathcal{L}}_s^{s,y}$ sono la componente non-locale del generatore infinitesimale originale e di quello congelato, definiti in (6.4) e (6.16), rispettivamente. Evidenziamo, in particolare, che il primo termine delle decomposizione $\tilde{R}_\epsilon^0 f$ è non-nullo solo se la matrice di diffusione $\sigma(t,\cdot)$ non è costante in spazio. Controllare direttamente la differenza dei due generatori in $\tilde{R}_\epsilon^0 f$, data da:

$$\begin{aligned}&(\mathcal{L}_t - \tilde{\mathcal{L}}_t^{s,y})\tilde{p}^{s,y}(t,s,\cdot,y)(x) \\ &\qquad = \int_{\mathbb{R}_0^d} \left[\tilde{p}^{s,y}(t,s,x+B\sigma(t,x)z,y) - \tilde{p}^{s,y}(t,s,x+B\tilde{\sigma}_t^{s,y}z,y)\right] \nu(dz)\end{aligned} \tag{6.32}$$

appare subito delicato, soprattutto perché non possiamo sfruttare direttamente la regolarità della densità congelata $\tilde{p}^{s,y}(t,s,x,y)$ in x, visto che i suoi effetti regolarizzanti

nella variabile y saranno necessari successivamente per stimare l'integrale esterno (rispetto a y). Nel caso in cui la condizione di assoluta continuità [**AC**] valga, sappiamo invece che

$$\nu(dz) \;=\; Q(z)\frac{g\left(\frac{z}{|z|}\right)}{|z|^{d+\alpha}}dz,$$

per una certa funzione Lipschitz continua su $\mathbb{S}^{d-1}$.
Inoltre, la condizione di uniforme ellitticità [**UE**] implica in particolare che $\det\sigma(t,x)\neq 0$. Supponendo allora, senza perdita di generalità, che $\det\sigma(t,x)>0$, i cambi di variabile $\tilde{z}=\sigma(t,x)z$ nell'integrale all'interno di $\mathcal{L}_s$ e $\tilde{z}=\tilde{\sigma}_t^{s,y}z$ in quello per $\tilde{\mathcal{L}}_s^{s,y}$ permettono di riscrivere la differenza dei generatori infinitesimali nel seguente modo:

$$\left(\mathcal{L}_t-\tilde{\mathcal{L}}_t^{s,y}\right)\tilde{p}^{s,y}(t,s,\cdot,y)(x) \;=\; \int_{\mathbb{R}_0^d}\left[\tilde{p}^{s,y}(t,s,x+Bz,y)-\tilde{p}^{s,y}(t,s,x,y)\right]\tilde{H}_{t,x}^{s,y}(z)\,\frac{dz}{|z|^{d+\alpha}},$$

dove abbiamo denotato, per semplicità,

$$\tilde{H}_{t,x}^{s,y}(z) := Q(\sigma^{-1}(t,x)z)\frac{g\left(\frac{\sigma^{-1}(t,x)z}{|\sigma^{-1}(t,x)z|}\right)}{\det\sigma(t,x)|\sigma^{-1}(t,x)\frac{z}{|z|}|^{d+\alpha}}-Q((\tilde{\sigma}_t^{s,y})^{-1}z)\frac{g\left(\frac{(\tilde{\sigma}_t^{s,y})^{-1}z}{|(\tilde{\sigma}_t^{s,y})^{-1}z|}\right)}{\det\tilde{\sigma}_t^{s,y}|(\tilde{\sigma}_t^{s,y})^{-1}\frac{z}{|z|}|^{d+\alpha}}$$

Per ottenere i controlli voluti, potremo allora sfruttare che la funzione $\tilde{H}_{t,x}^{s,y}$ sia limitata e Hölder regolare in un intorno dell'origine, proprietà ereditate dalle funzioni g,Q ed utilizzare solo in un secondo momento (per l'integrale rispetto a y) gli effetti regolarizzanti associati alla densità $\tilde{p}^{s,y}(t,s,x,\cdot)$. Intuitivamente, la condizione [**AC**] permette proprio di trasferire l'errore da stimare sulle funzioni temperanti Q,q, così da poterne sfruttare le proprietà.
Citiamo inoltre che per il controllo della differenza tra gli operatori infinitesimali $\mathcal{L}_t-\tilde{\mathcal{L}}_t^{s,y}$ sarà fondamentale il fatto che $\tilde{H}_{t,x}^{s,y}$ sia pari, grazie alla simmetria della misura di Lévy ν, in quanto consentirà di aggiungere i termini di ordine primo, necessari alle espansioni di Taylor, a qualsiasi livello di taglio.

Nel controllo del termine di errore $\tilde{R}_\epsilon^1 f$ in (6.31), è invece possibile mostrare, almeno euristicamente, che le soglie sulle regolarità di Hölder per F date nel Teorema 6.3 di non-unicità, sono effettivamente naturali e che sarebbe quindi attendibile ottenere una caratterizzazione (quasi) ottimale della buona posizione della dinamica stocastica (6.1) in termini della regolarità dei coefficienti.
Supponiamo allora, come nel teorema sopra citato, che $x_j\to F_i(t,x)$ sia β_i^j-Hölder continuo, uniformemente in tempo e nelle altre variabili spaziali.
Un calcolo ricorrente che dovremo affrontare nella stima di $\tilde{R}_\epsilon^1 f$ sarà di mostrare che quantità della seguente forma:

$$\tilde{R}_\epsilon^i f(t,x) \;:=\; \int_{\mathbb{R}^N}\left[F_i(s,x)-F_i(s,\theta_{t,s}(y))\right]D_{x_i}\tilde{p}^{s,y}(t,s,x,y)\,dy$$

generino una singolarità in tempo che sia integrabile.
Sfruttando le proprietà sulla densità congelata $\tilde{p}^{s,y}(t,s,x,y)$ in (6.12), possiamo allora

scrivere che:

$$
\begin{aligned}
|\tilde{R}^i_\epsilon f(t,x)| &= \sum_{j=i}^n \int_{\mathbb{R}^N} \frac{\left|(x-\theta_{t,s}(y))_j\right|^{\beta^j_i}}{(s-t)^{\frac{1+\alpha(i-1)}{\alpha}}} \frac{\bar{p}(1,\mathbb{T}^{-1}_{s-t}(y-\tilde{m}^{s,y}_{s,t}(x))}{\det \mathbb{T}_{s-t}}\, dy \\
&= \sum_{j=i}^n (s-t)^{-\zeta^j_i} \int_{\mathbb{R}^N} \left(\frac{\left|(x-\theta_{t,s}(y))_j\right|}{(s-t)^{\frac{1+\alpha(i-1)}{\alpha}}} \right)^{\beta^j_i} \frac{\bar{p}(1,\mathbb{T}^{-1}_{s-t}(y-\tilde{m}^{s,y}_{s,t}(x))}{\det \mathbb{T}_{s-t}}\, dy \\
&\leq C \sum_{j=i}^n (s-t)^{-\zeta^j_i} \int_{\mathbb{R}^N} \left|\mathbb{T}^{-1}_{s-t}\left(x-\theta_{t,s}(y)\right)\right|^{\beta^j_i} \frac{\bar{p}(1,\mathbb{T}^{-1}_{s-t}(y-\tilde{m}^{s,y}_{s,t}(x))}{\det \mathbb{T}_{s-t}}\, dy
\end{aligned}
$$

dove abbiamo definito, per comodità,

$$
\zeta^j_i \;:=\; \frac{1+\alpha(i-1)}{\alpha} - \beta^j \frac{1+\alpha(j-1)}{\alpha}.
$$

Notiamo intanto che la scelta di congelamento al punto finale $(\tau,\xi)=(s,y)$ è necessaria per avere un'omogeneità tra la differenza delle derive in $x-\theta_{t,s}(y)$ e l'argomento della densità in $y-\tilde{m}^{s,y}_{s,t}(x)$. In particolare, vale che

$$
y-\tilde{m}^{s,y}_{t,s}(x) \;=\; \tilde{m}^{s,y}_{t,s}(y)-x \;=\; \theta_{t,s}(y)-x.
$$

Supponendo per il momento che a partire dalle proprietà regolarizzanti associate alla densità $\bar{p}(t,\cdot)$, si possa ottenere il seguente controllo:

$$
\int_{\mathbb{R}^N} \left|\mathbb{T}^{-1}(\theta_{t,s}(y)-x)\right|^{\beta^j_i} \frac{\bar{p}(1,\mathbb{T}^{-1}(\theta_{t,s}(y)-x))}{\det \mathbb{T}_{s-t}}\, dy \;<\; +\infty, \tag{6.33}
$$

ne segue subito che

$$
\begin{aligned}
|\tilde{R}^i_\epsilon f(t,x)| &\leq C \sum_{j=i}^n (s-t)^{\zeta^j_i} \int_{\mathbb{R}^N} \left|\mathbb{T}^{-1}_{s-t}\left(x-\theta_{t,s}(y)\right)\right|^{\beta^j_i} \frac{\bar{p}(1,\mathbb{T}^{-1}_{s-t}(x-\theta_{t,s}(y))}{\det \mathbb{T}_{s-t}}\, dy \\
&\leq C \sum_{j=i}^n (s-t)^{-\zeta^j_i}.
\end{aligned}
$$

Perché allora il modulo di $\tilde{R}^i_\epsilon f$ dia una singolarità integrabile in tempo, le soglie *naturali* sulla regolarità di Hölder su F_i dovrebbero essere date da:

$$
\zeta^j_i < 1 \Leftrightarrow \beta^j_i \;>\; \frac{1+\alpha(i-2)}{1+\alpha(j-1)}. \tag{6.34}
$$

Nel caso in cui $\alpha=2$, cioè in un contesto Browniano, tali soglie si ritrovano effettivamente nel lavoro [CdRM20b].

Il tipo di controlli esibito sopra permette anche di mostrare in maniera chiara perché, in un primo tempo, dovremo assumere che gli indici p, q siano sufficientemente grandi nella stima puntuale del termine di resto $\tilde{R}_\epsilon f$.

Infatti, un ragionamento simile a quello in (6.29) permette controllare il termine $\tilde{R}^1_\epsilon f$ associato alla deriva F in (6.31) nel seguente modo:

$$|\tilde{R}^1_\epsilon f(t,x)| \leq C\|f\|_{L^p_t L^q_x} \sum_{j=i}^{n} \left(\int_t^T (s-t)^{-\zeta_i^j p'} \left(\det \mathbb{T}_{s-t}\right)^{-\frac{p'}{q}} ds \right)^{\frac{1}{p'}},$$

dove, ricordiamo, p' e q' sono, rispettivamente, gli esponenti coniugati di p e q. Allora, per ottenere una quantità integrabile dentro l'integrale in tempo sarà necessario imporre una soglia minima su p e q così che p' e q' siano sufficentemente piccoli da assicurare che

$$\zeta_i^j p' + \Big(p' - \frac{p'}{q'}\Big)\Big(\sum_{i=1}^{n} d_i \frac{1+\alpha(i-1)}{\alpha}\Big) < 1.$$

In conclusione, spieghiamo ora brevemente perché non siamo riusciti ad ottenere il risultato auspicato, ovvero la buona posizione in senso debole della dinamica stocastica (6.1) rispetto alle soglie naturali in (6.34) sulla regolarità della deriva F, ma abbiamo invece dovuto assumere la stessa regolarità β^j per ogni componente F_i lungo la variabile x_j.
L'elemento cruciale risiede proprio nel riuscire a dimostrare l'Equazione (6.33) nella stima puntuale del termine di resto $\tilde{R}^1_\epsilon f$. Evidenziamo subito che per il nostro modello, la scarsa regolarità del flusso $y \to \theta_{t,s}(y)$ impedisce di derivare direttamente (6.33) attraverso il cambio di variabili $\tilde{y} = \theta_{t,s}(y) - x$.
Infatti, l'approccio più naturale, sfruttato per esempio anche in [CdRM20b], è di spostare il flusso sulla variabile x attraverso una proprietà di Lipschitz "approssimata" del tipo:

$$|\mathbb{T}^{-1}_{s-t}(\theta_{t,s}(y) - x)| \asymp (1 + |\mathbb{T}^{-1}_{s-t}(y - \theta_{s,t}(x))|). \tag{6.35}$$

Il problema principale nel nostro caso è che però non siamo riusciti a stabilire, in completa generalità, che:

$$\overline{p}(1, \mathbb{T}^{-1}_{s-t}(\theta_{t,s}(y) - x)) \leq C\check{p}(1, \mathbb{T}^{-1}_{s-t}(y - \theta_{s,t}(x)), \tag{6.36}$$

per una certa densità $\check{p}$ che possieda le stesse proprietà regolarizzanti di $\bar{p}$.
Citiamo che tale difficoltà è intrinsecamente legata alla natura α-stabile degenere del nostro sistema. Infatti, un controllo come in (6.36) è assolutamente diretto nel caso Gaussiano a partire dall'espressione esplicita della densità $\bar{p}$ e del controllo in (6.35).
Per ottenere una stima puntuale, come in (6.36), la difficoltà nel caso stabile consiste proprio nel riuscire ad ottenere una descrizione sufficientemente precisa del comportamento delle code (associate ai salti grandi) che, com'è noto, sono associate alla geometria della misura spettrale corrispondente. Citiamo a riguardo i lavori di Watanabe [Wat07] nel caso stabile e quello di Sztonyk [Szt10a] per un estensione al caso stabile temperato.
In particolare, la parte delicata appare quando si considera il comportamento della misura di Poisson (salti grandi) associata alla densità $\bar{p}$ in un regime fuori-diagonale, ovvero se $|\mathbb{T}^{-1}_{s-t}(\theta_{t,s}(y) - x)| > K$. In questo caso, avremmo da (6.13), (6.14) e (6.15),

che:

$$\begin{aligned}\overline{p}(1,\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)) &\leq C\int_{\mathbb{R}^N}\frac{1}{(1+|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)-w|)^{N+3}}P_{\bar{N}_1}(dw)\\ &\leq C\int_0^1 P_{\bar{N}_1}(\{w\in\mathbb{R}^N:(1+|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)-w|)^{-(N+3)}|>u\})du\\ &\leq C\int_0^1 P_{\bar{N}_1}(B(\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x),u^{-1/(N+3)})du. \qquad (6.37)\end{aligned}$$

Ricordando dagli argomenti sotto (4.17) che il supporto della misura spettrale su $\mathbb{S}^{N-1}$ del processo $\{\tilde{S}_u^{s,y}\}_{u\geq 0}$ ha effettivamente dimensione d, Watanabe in [Wat07] Lemma 3.1 ha mostrato che esiste una costante $C>0$ tale che, per ogni z in $\mathbb{R}^N$ e $r>0$:

$$P_{\bar{N}_1}(B(z,r))\leq Cr^{d+1}(1+r^\alpha)|z|^{-(d+1+\alpha)}. \qquad (6.38)$$

In altre parole, il peggior decadimento nella stima globale è proprio dato dalla dimensione del supporto della misura spettrale associata. Evidenziamo inoltre che tali stime sono in un certo senso, ottimali, almeno lungo alcune direzioni del sistema (cf. Lemma 3.1 in [Wat07]). A riguardo si veda anche [PT69]. Sfruttando ora il controllo in (6.38) nella stima principale in (6.37), otterremmo che

$$\begin{aligned}\overline{p}(1,\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)) &\leq C|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)|^{-(d+1+\alpha)}\int_0^1 u^{-\frac{d+1}{N+3}}(1+u^{-\frac{\alpha}{N+3}})du\\ &\leq C(1+|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)|)^{-(d+1+\alpha)}.\end{aligned}$$

La proprietà di Lipschitz approssimata in (6.35) implicherebbe infine che

$$\begin{aligned}\overline{p}(1,\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)) &\leq C(1+|\mathbb{T}_{s-t}^{-1}(y-\theta_{s,t}(x))|)^{-(d+1+\alpha)}\\ &=: C\check{p}(1,\mathbb{T}_{s-t}^{-1}(y-\theta_{s,t}(x))).\end{aligned} \qquad (6.39)$$

Un controllo di questo tipo sarebbe effettivamente sufficiente per i nostri scopi ma imporrebbe condizioni molto forti sulle dimensioni d,n dello spazio perché la densità $\check{p}(t,\cdot)$ sia integrabile. Questo fenomeno è apparso anche in [HM16] ed impose in quel caso di limitare l'analisi al caso $d=1,n=3$ per dimostrare la buona posizione del problema di martingala associato ad una catena degenere a deriva lineare con rumore moltiplicativo stabile isotropo.
Citiamo infine che questa difficoltà si presenterebbe pure nel caso temperato "classico", ovvero se imponessimo ulteriori condizioni sulla funzione Q. Il vantaggio in questo contesto sarebbe stato di mantenere la funzione temperante Q anche all'interno della densità $\bar{p}$, così da sfruttare i vantaggi del temperamento all'infinito, e recuperare così i problemi di concentrazione in (6.39). Come analizzato in [Szt10a] Corollario 6, avremmo allora ottenuto controlli della forma:

$$\tilde{p}^{\tau,\xi}(t,s,x,y)\leq C(1+|\mathbb{T}_{s-t}^{-1}(y-\theta_{s,t}(x))|)^{-(d+1+\alpha)}Q\left(|\mathbb{M}_{s-t}^{-1}(y-\theta_{s,t}(x))|\right)$$

che chiaramente migliorano l'integrabilità in spazio ma allo stesso tempo, deteriorano quella in tempo, visto che non è più possibile sfruttare l'α-autosimilarità della densità stabile sottostante.

Questo tipo di difficoltà sarebbe apparso anche se avessimo considerato solo il caso troncato, ovvero quando $Q(z) = \mathbb{1}_{B(0,r_0)}$ per un certo $r_0 > 0$. Per maggiori dettagli, si veda per esempio, [CKK08] in un contesto non-degenere.

Per risolvere questa difficoltà, seguiremo allora nel Capitolo 4 un ragionamento alternativo.
Ricordando che il cambio di variabili naturale $\tilde{y} := \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y) - x)$ in (6.33) non è direttamente possibile nel nostro contesto perchè i coefficienti non sono abbastanza regolari, introdurremo un flusso regolarizzato $\theta_{t,s}^{\delta}(y)$ associato ad una versione mollificata dei coefficienti coinvolti. Applicando allora il cambio di variabili auspicato (rispetto al flusso regolarizzato), sarà possibile ottenere la stima in (6.33) rispetto a $\theta_{t,s}^{\delta}(y)$, e controllare la differenza tra i flussi similmente a come avremo già fatto per stabilire la condizione di Lipschitz approssimata (cf. Equazione (6.35)).
Per concludere, rimarrà infine solo da controllare $\det(\nabla\theta_{t,s}^{\delta}(y))$, uniformemente rispetto al parametro di mollificazione. Sarà proprio quest'ultimo controllo ad obbligarci a rinforzare la soglia naturale di Hölder regolarità su F in (6.34) ed assumere che ogni componente F_i, ($i \in [\![2, n]\!]$) presenti la stessa regolarità lungo la variabile x_j ($j \in [\![2, n]\!]$), uniformemente in spazio e nelle altre variabili spaziali (cf. Equazione (6.3)).

Contro-esempi alla Peano per l'unicità in senso debole

Spieghiamo ora brevemente i ragionamenti euristici dietro la dimostrazione per il risultato di non-unicità (cf. Teorema (6.3)). L'idea è di adattare i contro-esempi alla Peano presentati in (2.5) e sfruttati anche in [CdRM20b], al nostro contesto di Lévy.
Se vogliamo testare per esempio la soglia β_i^j associata all'esponente di Hölder critico per la componente i-esima della deriva F rispetto alla variabile x_j (con $j \geq i > 1$), considereremo il seguente modello (per $d_1 = \cdots = d_n = 1$ e $N = n$):

$$\begin{cases} dX_t^1 &= dZ_t, & \text{se } k = 1; \\ dX_t^k &= X_t^{k-1}dt, & \text{se } k \in [\![2, i-1]\!]; \\ dX_t^i &= X_t^{i-1}dt + \operatorname{sgn}(X_t^j)|X_t^j|^{\beta_i^j}dt, & \text{se } k = i; \\ dX_t^k &= X_t^{k-1}dt, & \text{se } k \in [\![i+1, n]\!], \end{cases} \tag{6.40}$$

dove $\{Z_t\}_{t\geq 0}$ è un processo α-stabile simmetrico reale. Non è difficile notare che l'Equazione si può scrivere nella forma di (6.1) imponendo $\sigma = 1$ e $G(t, x) = Ax + e_i\operatorname{sgn}(x_j)|x_j|^{\beta_i^j}$ dove e_i è l'i-esimo elemento della base canonica su $\mathbb{R}^N$ e A è la matrice in $\mathbb{R}^N \otimes \mathbb{R}^N$ data da:

$$A := \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

Se ci concentriamo in particolare sulla componente i-esima della dinamica (6.40) sopra, essa può essere riscritta in forma integrale come:

$$X_t^j = \int_0^t \operatorname{sgn}\Big(I_t^{j-i}(X^j)\Big)\Big|I_t^{j-i}(X^j)\Big|^{\beta_i^j}dt + I_t^{i-1}(Z), \quad t \geq 0, \tag{6.41}$$

dove, per ogni percorso càdlàg $y\colon [0,\infty) \to \mathbb{R}$, la notazione $I_t^k(y)$ rappresenta l'integrale iterato k-volte al tempo t. Come già spiegato precedentemente, perché la regolarizzazione attraverso rumore avvenga è necessario che, almeno in tempo piccolo, le fluttuazioni medie della perturbazione casuale dominino l'irregolarità della deriva deterministica. Più precisamente per il nostro modello, possiamo allora confrontare le fluttuazioni del rumore $I_t^{i-1}(Z)$ di ordine $i-1+\frac{1}{\alpha}$ con le soluzioni estremali deterministiche ottenute senza perturbazione (i.e. $B=0$ above), così da ottenere:

$$t^{i-1+\frac{1}{\alpha}} > t^{\frac{(j-1)\beta_i^j-1}{1-\beta_i^j}}.$$

Visto che tale condizione deve valere per t piccolo, possiamo allora concludere che la condizione

$$i-1+\frac{1}{\alpha} < \frac{(j-1)\beta_i^j-1}{1-\beta_i^j} \;\Leftrightarrow\; \beta_i^j > \frac{1+\alpha(i-1)}{1+\alpha(j-2)}$$

sia la relazione euristica che garantisce che il rumore domini sul sistema e che si presenti effettivamente una regolarizzazione attraverso rumore.

7 Sulle costanti ottimali nelle stime di Sobolev e di Schauder per operatori di Kolmogorov degeneri

Presentiamo ora brevemente i principali risultati del Capitolo 5. Tale lavoro, scritto in collaborazione con i miei relatori di tesi, Prof. Stéphane Menozzi e Prof. Enrico Priola, è da poco apparso in pre-pubblicazione.
Siamo interessati qui a studiare gli effetti di una perturbazione di secondo ordine su un operatore di Ornstein-Uhlenbeck degenere diffusivo. In particolare, vogliamo determinare in che modo le costanti di alcune stime "note" per questa classe di operatori, come per esempio le stime di Schauder presentate nelle Sezioni 4 e 5, dipendano effettivamente dalla perturbazione considerata. Il nostro metodo di prova si baserà su una trasformazione dello spazio adeguata, che permetta di cancellare il termine di trasporto di ordine primo, e sul metodo perturbativo attraverso processi di Poisson, introdotto in [KP17] ed adattato al nostro contesto.

Più precisamente, fissato un intero positivo N, considereremo la seguente famiglia di operatori di Ornstein-Uhlenbeck diffusivi:

$$L^{\mathrm{ou}} := \mathrm{Tr}(BD_z^2) + \langle Az, D_z\rangle, \qquad \text{su } \mathbb{R}^N, \tag{7.1}$$

dove $\langle\cdot,\cdot\rangle$ denota ancora il prodotto scalare su $\mathbb{R}^N$ e A, B sono due matrici in $\mathbb{R}^N\otimes\mathbb{R}^N$ tali che B è simmetrica.
Supporremo, come nel Capitolo 3, che le due matrici A, B soddisfino la condizione del rango di Kalman che assicura l'ipoellitticità del sistema:

[K] esiste un intero non-negativo n tale che

$$\mathrm{rank}[B, AB, \cdots, A^{n-1}B] = N,$$

dove, ricordiamo, $[B, AB, \ldots, A^{n-1}B]$ è la matrice in $\mathbb{R}^N\otimes\mathbb{R}^{Nn}$ i cui blocchi sono $B, AB, \cdots A^{n-1}B$.

Come già accennato nella Sezione 5, la condizione [**K**] di Kalman permette di decomporre lo spazio $\mathbb{R}^N$ a seconda dello spazio immagine raggiunto dalle iterazioni successive dei commutatori tra A e B (cf. Equazione (5.8)).
Supponendo che $\operatorname{rank}(B) = d$ per un certo $d > 0$, sappiamo in particolare che esistono $\{d_1, \dots, d_n\}$ interi non-negativi tali che $d_1 = d$, $\sum_{i=1}^n d_i = N$ e le due matrici A, B siano riscrivibili, dopo un possibile cambio di coordinate, nella seguente più esplicita forma (cf. Equazione (5.11)):

$$B = \begin{pmatrix} B_0 & 0 & \dots & 0 \\ 0 & 0 & \ddots & \vdots \\ \vdots & \ddots & \ddots & \vdots \\ 0 & \dots & \dots & 0 \end{pmatrix} \quad \text{e} \quad A = \begin{pmatrix} * & * & \dots & \dots & * \\ A_2 & * & \ddots & \ddots & \vdots \\ 0 & A_3 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ 0 & \dots & 0 & A_n & * \end{pmatrix}$$

dove B_0 è una matrice non-degenere in $\mathbb{R}^d \otimes \mathbb{R}^d$, A_i è una matrice in $\mathbb{R}^{d_i} \otimes \mathbb{R}^{d_{i-1}}$ tale che $\operatorname{rank}(A_i) = d_i$ per ogni i in $[\![2, n]\!]$ e gli elementi $*$ possono essere non-nulli.

Siamo allora interessati alle soluzioni del seguente Problema di Cauchy:

$$\begin{cases} \partial_t u(t,z) \;=\; L^{\mathrm{ou}} u(t,z) + f(t,z) & \text{su } (0,T) \times \mathbb{R}^N; \\ u(0,z) \;=\; 0, & \text{su } \mathbb{R}^N. \end{cases} \tag{7.2}$$

Proveremo l'esistenza e l'unicità di soluzioni limitate e regolari per l'Equazione (7.2) assumendo, come in [KP17], che la sorgente f appartenga allo spazio $B_b\left(0, T; C_0^\infty(\mathbb{R}^N)\right)$. Tale spazio può essere inteso come la famiglia di funzioni che sono misurabili, limitate in tempo e lisce, a supporto compatto in spazio, uniformemente in tempo. Per una definizione precisa di tali spazi, rimandiamo il lettore alla Sezione 1.2 del Capitolo 5.
Evidenziamo che non abbiamo potuto considerare la sorgente f in una classe più "usuale" di funzioni, come per esempio $C_c^\infty([0,T] \times \mathbb{R}^N)$ proprio perchè il metodo perturbativo in [KP17] attraverso i processi di Poisson, che sfrutteremo anche noi, richiederà di considerare sorgenti f in (7.2) che siano possibilmente discontinue in tempo (cf. Sezione 2 in [KP17]).

A causa della scarsa regolarità in tempo di f, il sistema (7.2) sarà inteso solo in senso integrale, ovvero una funzione limitata e continua $u\colon [0,T] \times \mathbb{R}^N \to \mathbb{R}$ sarà una soluzione dell'Equazione (7.2) se $u(t, \cdot)$ appartiene a $C^2(\mathbb{R}^N)$ per ogni t fissato e

$$u(t,x) \;=\; \int_0^t \Big[\operatorname{Tr}(B D_z^2 u(s,z)) + \langle Az, D_z u(s,z) \rangle + f(s,z) \Big] \, ds. \tag{7.3}$$

Fissata una funzione continua $t \mapsto S(t)$ tale che $S(t)$ è una matrice simmetrica e non-negativa in $\mathbb{R}^N \otimes \mathbb{R}^N$, saremo allora interessati anche alla perturbazione di ordine secondo associata ad $S(t)$ per l'operatore di Ornstein-Uhlenbeck, ovvero il seguente operatore:

$$L_t^{\mathrm{ou},S} \;:=\; L^{\mathrm{ou}} + \operatorname{Tr}(S(t) D_z^2) \;=\; \operatorname{Tr}\Big([B + S(t)] \, D_z^2 \Big) + \langle Az, D_z \rangle, \qquad \text{su } \mathbb{R}^N,$$

ed al problema di Cauchy *perturbato* ad esso associato:

$$\begin{cases} \partial_t u_S(t,z) \;=\; \mathrm{Tr}\left([B+S(t)]\, D_z^2 u_S(t,z)\right) + \langle Az, D_z u_S(t,z)\rangle + f(t,z); \\ u_S(0,z) \;=\; 0. \end{cases} \tag{7.4}$$

In questo lavoro, ci concentreremo principalmente su due tipi di stime per le soluzioni u del Problema di Cauchy (7.2): stime di tipo Sobolev, ovvero controlli in norma L^p sulla prima componente non-degenere del gradiente della soluzione u e le stime di Schauder rispetto agli spazi di Hölder con multi-indice di regolarità, già incontrati nelle Sezioni 4 e 5 della presente tesi.
In particolare, Bramanti *et al.* hanno mostrato in [BCLP10], Teorema 3 che per ogni p in $(1,\infty)$, esiste una costante $C_p > 0$, indipendente da f, tale che:

$$\|B^{1/2} D^2 u\, B^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N)} \;\leq\; C_p \|f\|_{L^p((0,T)\times\mathbb{R}^N)}, \tag{7.5}$$

Evidenziamo però che le stime in (7.5) vengono in realtà provate in [BCLP10] supponendo che la sorgente f sia liscia in spazio e tempo. Attraverso alcune proprietà esplicite sul nucleo del calore Gaussiano sottostante, mostreremo nella Sezione 2.3 nel Capitolo 5 che in effetti tali stime possano essere estese a considerare anche la classe di sorgenti f nel nostro contesto.

Sotto la condizione **[K]** di Kalman, Lunardi ha invece mostrato in [Lun97] Teorema 1.2 che per ogni β in $(0,1)$ esiste una constante C_β, indipendente da f, tale che:

$$\|u\|_{L^\infty((0,T),C^{2+\beta}_{b,d})} \;\leq\; C_\beta \|f\|_{L^\infty((0,T),C^{\beta}_{b,d})}, \tag{7.6}$$

dove, ricordiamo, gli spazi di Hölder anisotropici $C^{\gamma}_{b,d}(\mathbb{R}^N)$ sono definiti esattamente come in Sezione 5.

Possiamo ora riassumere i principali risultati del Capitolo 5 nel seguente teorema:

Teorema 7.1. *Sia f in $B_b\left(0,T;C_0^\infty(\mathbb{R}^N)\right)$. Allora, esiste un'unica soluzione integrale u_S del Problema di Cauchy* (7.4) *tale che, per ogni p in $(1,+\infty)$ e β in $(0,1)$ vale che*

$$\|B^{1/2} D^2 u_S\, B^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N)} \;\leq\; C_p \|f\|_{L^p((0,T)\times\mathbb{R}^N)} \tag{7.7}$$

$$\|u_S\|_{L^\infty(C^{2+\beta}_{b,d})} \;\leq\; C_\beta \|f\|_{L^\infty(C^{\beta}_{b,d})}, \tag{7.8}$$

*con le **stesse** costanti C_p, C_β apparse rispettivamente in* (7.5) *e* (7.6)*. In particolare, le costanti C_p, C_β non dipendono dalla matrice $S(t)$.*

Al di là della proprietà di preservazione delle costanti mostrata nel Teorema 7.1 sopra, le stime di Sobolev in (7.7) sembrano, al meglio della nostra conoscenza, essere nuove per un operatore come $L_t^{\mathrm{ou},S}$ e di interesse indipendente. Citiamo a riguardo anche il recente lavoro di Fornaro *et al.* [FMPS21] in cui si delinea una descrizione completa dello spettro di operatori di Ornstein-Uhlenbeck ipoellittici in spazi L^p.
Evidenziamo inoltre che se si considera una matrice S indipendente dal tempo, risultati analoghi al Teorema 7.1 possono essere ottenuti anche per le stime ellittiche corrispondenti, seguendo il metodo mostrato nel Corollario 3.5 di [KP17].

Sottolineiamo infine che nel Capitolo 5 mostreremo in aggiunta che stime L^p più generali, che considerano anche le direzioni degeneri, sono indipendenti da perturbazioni del secondo ordine, seppure solo nel caso di matrici A invarianti per dilatazioni (cf. Equazione (4.6) nella Sezione 4). Per maggiori dettagli a riguardo, si veda la Sezione 4 del Capitolo 5.

7.1 Guida alla prova

Per dare un'idea al lettore del metodo di prova da noi usato, illustriamo ora brevemente i passaggi principali della dimostrazione sulle stime L^p in (7.7). Come già accennato all'inizio della sezione, uno strumento fondamentale sarà una nota (cf. [DPL95]) trasformazione dello spazio $\mathbb{R}^N$ (ad ogni tempo fissato) che permetterà precisamente di sbarazzarsi del termine di drift $\langle Az, D_z u\rangle$ nel problema di Cauchy (1.10).
Più in dettaglio, fissata una soluzione limitata u del Problema di Cauchy (1.10), introdurremo la funzione $v\colon [0,T]\times\mathbb{R}^N$, data da

$$v(t,z) \;:=\; u(t, e^{-tA}z).$$

Infatti, ricordando che u risolve il Problema di Cauchy in (1.10) e notando che $u(t,z) = v(t, e^{tA}z)$, non è difficile controllare che

$$\begin{aligned} f(t,z) \;&=\; \partial_t u(t,z) - L^{\mathrm{ou}}u(t,z) \\ &=\; v_t(t,e^{tA}z) + \langle Dv(t,e^{tA}z), Ae^{tA}z\rangle - \mathrm{Tr}\Big(e^{tA}Be^{tA^*}D^2v(t,e^{tA}z)\Big) \\ &\qquad - \langle Dv(t,e^{tA}z), Ae^{tA}z\rangle \\ &=\; v_t(t,e^{tA}z) - \mathrm{Tr}\Big(e^{tA}Be^{tA^*}D^2v(t,e^{tA}z)\Big). \end{aligned}$$

per ogni (t,z) in $(0,T)\times\mathbb{R}^N$. Denotando per semplicità $\tilde f(t,z) := f(t,e^{-tA}z)$, segue immediatamente dai conti sopra che v è allora una soluzione del seguente Problema di Cauchy:

$$\begin{cases} \partial_t v(t,z) \;=\; \mathrm{Tr}\Big(e^{tA}Be^{tA^*}D^2v(t,z)\Big) + \tilde f(t,z) & \text{on } (0,T)\times\mathbb{R}^N; \\ v(0,z) \;=\; 0 & \text{on } \mathbb{R}^N. \end{cases}$$

Inoltre, le stime in (7.5) possono essere riscritte in termini di v come

$$\|B^{1/2}e^{-tA^*}D^2v(t,e^{tA}\cdot)\,e^{tA}B^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N)} \;\le\; C_p\|\tilde f(t,e^{tA}\cdot)\|_{L^p((0,T)\times\mathbb{R}^N)}. \tag{7.9}$$

Attraverso un cambio di variabili negli integrali e denotando con $L^p((0,T)\times\mathbb{R}^N, m)$ lo spazio L^p usuale rispetto alla misura m data da

$$m(dt,dx) \;:=\; \det(e^{-At})dtdx,$$

è immediato notare allora che il controllo in (7.9) è equivalente alla stima seguente:

$$\|B^{1/2}e^{tA^*}D^2v(t,\cdot)\,e^{tA}B^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N,m)} \le C_p\|\tilde f\|_{L^p((0,T)\times\mathbb{R}^N,m)}. \tag{7.10}$$

Ci possiamo concentrare ora sul Problema di Cauchy perturbato dalla matrice $S(t)$:

$$\begin{cases} \partial_t w(t,z) + \mathrm{Tr}\Big(e^{tA} B e^{tA^*} D^2 w(t,z)\Big) + \mathrm{Tr}\Big(e^{tA} S(t) e^{tA^*} D^2 w(t,z)\Big) \;=\; \tilde{f}(t,z); \\ w(0,z) \;=\; 0. \end{cases} \tag{7.11}$$

Sfruttando argomenti di tipo probabilistico, mostreremo in particolare che esiste un'unica soluzione $w\colon [0,T] \times \mathbb{R}^N \to \mathbb{R}$ al problema sopra.

Adattando ora alcuni degli argomenti presentati in [KP17], potremo derivare che le stime in norma L^p in (7.10) valgono anche per la soluzioni w del problema perturbato in (7.11), indipendentemente dalla perturbazione data dalla matrice $S(t)$. Più precisamente, otterremo che

$$\|B^{1/2} e^{tA^*} D^2 w(t,\cdot)\, e^{tA} B^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N,m)} \le C_p \|\tilde{f}(t,\cdot)\|_{L^p((0,T)\times\mathbb{R}^N,m)}, \tag{7.12}$$

vale con la stessa costante C_p apparsa in (7.10).
L'elemento cruciale nella prova in [KP17] è introdurre una piccola perturbazione casuale sulla sorgente f attraverso un processo di tipo Poisson adeguato e di analizzare le proprietà associate all'equazione corrispondente. Prendendo poi il valore atteso nella formulazione integrale dell'equazione, i contributi associati ai salti del processo generano, per un'intensità appropriata del processo di Poisson sottostante, un operatore alle differenze finite. In particolare, le stime iniziali rimangono preservate per il sistema risolto dal valore atteso della soluzione e che coinvolge anche l'operatore alle differenze finite. Argomenti di compattezza permettono infine di concludere che le stime iniziale valgono anche al limite, scambiando l'operatore alle differenze finite con il corrispondente operatore differenziale di ordine due.

Per concludere, dovremo allora tornare indietro al nostro modello originale di Ornstein-Uhlenbeck, applicando la trasformazione inversa rispetto alla variabile spaziale. In particolare, introdurremo $\tilde{u}(t,z) := w(t, e^{tA} z)$ che risolve, per definizione, la seguente equazione:

$$\begin{cases} \partial_t \tilde{u}(t,z) + L_t^{\mathrm{ou},S} \tilde{u}(t,z) \;=\; f(t,z), & \text{su } (0,T)\times\mathbb{R}^N, \\ \tilde{u}(0,z) = 0, & \text{su } \mathbb{R}^N. \end{cases}$$

Sfruttando la seguente identità:

$$D^2 w(t,\cdot) = D^2[\tilde{u}(t, e^{-tA}\,\cdot)] = e^{-tA^*} D^2 \tilde{u}(t, e^{-tA}\cdot) e^{-tA}$$

potremo allora concludere dal Controllo (7.12) che le seguente stime valgono:

$$\|B^{1/2} D^2 \tilde{u}\, B^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N)} \;\le\; C_p \|f\|_{L^p((0,T)\times\mathbb{R}^N)}.$$

Riassumendo, i ragionamenti mostrati sopra permettono in effetti di costruire una soluzione $\tilde{u}$ del problema di Cauchy in (1.14) che soddisfa le stime L^p in (1.15) con la *stessa* costante C_p apparsa nelle stime analoghe per l'operatore *proxy* L^{ou}. Notando infine che il principio del massimo vale anche rispetto all'operatore perturbato di Ornstein-Uhlenbeck $L_t^{\mathrm{ou},S}$, potremo mostrare anche l'unicità di tale soluzione $\tilde{u}$.

In conclusione, evidenziamo che per applicare il metodo perturbativo brevemente riassunto sopra, solo poche proprietà specifiche sono in effetti richieste sulle semi-norme

sottostanti. Intuitivamente, il ragionamento presentato in [KP17] sfrutta solo l'invarianza per traslazione delle semi-norme e una specie di proprietà di commutatività tra le norme (o una funzione della norma come nel caso L^p) e l'operatore di valore atteso. Sembra infatti naturale che questo approccio possa essere allora esteso ad una classe molto più generale di stime su altri spazi funzionali, come, per esempio, spazi di Besov (cf. Equazione (4.33)).
Inoltre, questo tipo di controlli sembra essere promettente per un'analisi più dettagliata del carattere ben posto di alcune classi di equazioni stocastiche correlate. Questi aspetti saranno oggetto di ricerca nel prossimo futuro.

Chapter 2

Schauder estimates for degenerate stable Kolmogorov equations

Abstract: We provide here global Schauder-type estimates for a chain of integro-partial differential equations (IPDE) driven by a degenerate stable Ornstein-Uhlenbeck operator possibly perturbed by a deterministic drift, when the coefficients lie in some suitable anisotropic Hölder spaces. Our approach mainly relies on a perturbative method based on forward parametrix expansions and, due to the low regularizing properties on the degenerate variables and to some integrability constraints linked to the stability index, it also exploits duality results between appropriate Besov Spaces. In particular, our method also applies in some super-critical cases. Thanks to these estimates, we show in addition the well-posedness of the considered IPDE in a suitable functional space.

1 Introduction

For a fixed time horizon $T > 0$ and two integers n, d in $\mathbb{N}$, we are interested in proving global Schauder estimates for the following parabolic integro-partial differential equation (IPDE):

$$\begin{cases} \partial_t u(t,x) + \langle Ax + F(t,x), D_x u(t,x)\rangle + \mathcal{L}_\alpha u(t,x) \;=\; -f(t,x) & \text{on } [0,T]\times\mathbb{R}^{nd}; \\ u(T,x) \;=\; u_T(x) & \text{on } \mathbb{R}^{nd}. \end{cases} \tag{1.1}$$

where $x := (x_1, \dots, x_n)$ is in $\mathbb{R}^{nd}$ with each x_i in $\mathbb{R}^d$ and $\langle \cdot, \cdot \rangle$ represents the inner product on $\mathbb{R}^{nd}$. We consider a symmetric, α-stable operator $\mathcal{L}_\alpha$ acting non-degenerately only on the first d variables and a matrix A in $\mathbb{R}^{nd} \otimes \mathbb{R}^{nd}$ with the following sub-diagonal structure:

$$A := \begin{pmatrix} 0_{d\times d} & \dots & \dots & \dots & 0_{d\times d} \\ A_{2,1} & 0_{d\times d} & \dots & \dots & 0_{d\times d} \\ 0_{d\times d} & A_{3,2} & 0_{d\times d} & \dots & 0_{d\times d} \\ \vdots & \ddots & \ddots & \vdots & \vdots \\ 0_{d\times d} & \dots & 0_{d\times d} & A_{n,n-1} & 0_{d\times d} \end{pmatrix}. \tag{1.2}$$

We will assume moreover that it satisfies a Hörmander-like condition, allowing the smoothing effect of $\mathcal{L}_\alpha$ to propagate into the system.
Above, the source $f\colon [0,T]\times\mathbb{R}^{nd}\to\mathbb{R}$ and the terminal condition $u_T\colon \mathbb{R}^{nd}\to\mathbb{R}$ are assumed to be bounded and to belong to some suitable anisotropic Hölder space.
The additional drift term $F(t,x)=\Big(F_1(t,x),\dots,F_n(t,x)\Big)$ can be seen as a perturbation of the Ornstein-Ulhenbeck operator $\mathcal{L}_\alpha+\langle Ax,D_x\rangle$ and it has structure "compatible" with A, i.e. at level i, it depends only on the super diagonal entries:

$$F_i(t,x)\ :=\ F_i(t,x_i,\dots,x_n).$$

It may be unbounded but we assume it to be Hölder continuous with an index depending on the level of the chain.

Related Results. A large literature on the topic of Schauder estimates in the α-stable non-local framework has been developed in the recent years (see e.g. Lunardi and Röckner [LR21] for an overview of the field), mainly in the non-degenerate setting and assuming that $\alpha\geq 1$, the so called sub-critical case. We mention for instance the stable-like setting, corresponding to time-inhomogeneous operators of the form

$$\bar{L}_t\phi(x)\ =\ \int_{\mathbb{R}^{nd}}\Big[\phi(x+y)-\phi(x)-\mathbb{1}_{1\leq\alpha<2}\langle y,D_x\rangle\Big]m(t,x,y)\frac{dy}{|y|^{d+\alpha}}\\ +\mathbb{1}_{1\leq\alpha<2}\langle F(t,x),D_xu(t,x)\rangle \quad (1.3)$$

where the diffusion coefficient m is bounded from above and below, Hölder continuous in the spatial variable x and even in y if $\alpha=1$. Under these conditions and assuming the drift F to be bounded and Hölder continuous in space, Mikulevicius and Pragarauskas in [MP14] obtained parabolic Schauder type bounds on the whole space and derived from those estimates the well-posedness of the corresponding martingale problem. We notice however that for the super-critical case (when $\alpha<1$), the drift term in (1.3) is set to zero. This is mainly due to the fact that in the super-critical case, $\mathcal{L}_\alpha$ is of order α (in the Fourier space) and does not dominate the drift term F which is roughly speaking of order one.
In the non-degenerate, driftless framework (i.e. when $Ax+F=0$ and $n=1$ in (1.1)), Bass [Bas09] was the first to derive elliptic Schauder estimates for stable like operators. We can refer as well to the recent work of Imbert and collaborators [IJS18] concerning Schauder estimates for stable-like operator (1.3) with $\alpha=1$ and some related applications to non-local Burgers equations. Eventually, still in the driftless case, Ros-Oton and Serra worked in [ROS16] for interior and boundary elliptic-regularity in a general, symmetric α-stable setting, assuming that the Lévy measure ν_α associated with $\mathcal{L}_\alpha$ writes in polar coordinates $y=\rho s$, $(\rho,s)\in[0,\infty)\times\mathbb{S}^{d-1}$ as

$$\nu_\alpha(dy)\ =\ \tilde{\mu}(ds)\frac{d\rho}{\rho^{1+\alpha}}$$

where $\tilde{\mu}$ is a non-degenerate, symmetric measure on the sphere $\mathbb{S}^{d-1}$. Related to the above, we can mention also the associated work of Fernandez-Real and Ros-Oton [FRRO17] for parabolic equations.

In the elliptic setting, when $\alpha \in [1, 2)$ and $\mathcal{L}_\alpha$ is a non-degenerate, symmetric α-stable operator and for bounded Hölder drifts, global Schauder estimates were obtained by Priola in [Pri12] or in [Pri18] for respective applications to the strong well-posedness and Davie's uniqueness for the corresponding SDE. We notice furthermore that in the sub-critical case, elliptic Schauder estimates can be proven for more general, translation invariant, Lévy-type generators for following [Pri18] (see Section 6, and Remark 5 therein).

In the super-critical case, parabolic Schauder estimates were established by Chaudru de Raynal, Menozzi and Priola in [CdRMP20a] under similar assumptions to [ROS16]. An existence result is also provided therein.
We mention as well the work of Zhang and Zhao [ZZ18] who address through probabilistic arguments the parabolic Dirichlet problem for stable-like operators of the form (1.3) with a non-trivial bounded drift, i.e. getting rid of the indicator function for the drift. They also obtain interior Schauder estimates and some boundary decay estimates (see e.g. Theorem 1.5 therein).

As we have seen, most of the literature is focused on the non-degenerate case. In the degenerate diffusive setting, Lunardi [Lun97] was the first one to prove Schauder estimates for linear Kolmogorov equations under weak Hörmander assumptions, exploiting anisotropic Hölder spaces (where the Hölder index depends on the variable considered), in order exactly to control the multiple scales appearing in the different directions, due to the degeneracy of the system.
After, in [Lor05] and [Pri09], the authors established Schauder-like estimates for hypoelliptic Kolmogorov equations driven by partially nonlinear smooth drifts. On the other hand, let us also mention [CdRHM18a] where the authors first establish Schauder estimates for nonlinear Kolmogorov equations under some weak Hörmander-type assumption. Their method is based on a perturbative approach through proxies that we here adapt and exploit. In the degenerate, stable setting, we have to refer also to a recent work of Zhang and collaborators [HWZ20] who show Schauder estimates for the degenerate kinetic dynamics ($n = 2$ above) extending a method based on Littlewood-Paley decompositions already used in other works by Zhang (see e.g. [ZZ18]), to the degenerate, multi-scaled framework. Even with different approaches and frameworks, we consider here a generic d-level chain and we exploit thermic characterizations of Besov norms, our and their works bring to the same results in the intersecting cases, at least to the best of our knowledge. About a different but correlated argument, we mention that the L^p-maximal regularity for degenerate non-local Kolmogorov equations with constant coefficients was also obtained in [CZ19] for the kinetic dynamics ($n = 2$ above) and in [HMP19] for the general n-levels chain.

In the diffusive setting, Equation (1.1) appears naturally as a microscopic model for heat diffusion phenomena (see [RBT00]) or, in the kinetic case ($n = 2$), it can be naturally associated with speed/position (or Hamiltonian) dynamics where the speed component is noisy. It can be found in many fields of application from physics to finance, see for example [HN04] or [BPV01]. When noised by stable processes, it can be used to model the appearance of turbulence (cf. [CPKM05]) or some abnormal diffusion phenomena. Moreover, the Schauder estimates will be a fundamental first step in order to study the

weak and strong well-posedness for the following stochastic differential equation (SDE):

$$\begin{cases} dX_t^1 &= F_1(t, X_t^1, \dots, X_t^n)dt + dZ_t \\ dX_t^2 &= A_{2,1}X_t^1 + F_2(t, X_t^2, \dots, X_t^n)dt \\ \vdots \\ dX_t^n &= A_{n,n-1}X_t^{n-1} + F_n(t, X_t^n)dt \end{cases} \tag{1.4}$$

where Z_t is a symmetric, $\mathbb{R}^d$-valued α-stable process with non-degenerate Lévy measure ν_α on some filtered probability space $(\Omega, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$. The complete operator $\mathcal{L}_\alpha + \langle Ax+F(t,x), D_x\rangle$ then corresponds to the infinitesimal generator of the process $\{X_t\}_{t\geq 0}$, solution of Equation (1.4).

Mathematical Outline. In this work, we will establish global Schauder estimates for the solution of the IPDE (1.1) exploiting the perturbative approach firstly introduced in [CdRHM18a] to derive such estimates for degenerate Kolmogorov equations. Roughly speaking, the idea is to perform a first order parametrix expansion, such as a Duhamel-type representation, to a solution of the IPDE (1.1) around a suitable proxy. The main idea behind consists in exploiting this easier framework in order to subsequently obtain a tractable control on the error expansion. When applying such a strategy, we basically have two ways to proceed.
On the one hand, one can adopt a backward parametrix approach, as introduced by McKean and Singer [MS67] in the non-degenerate, diffusive setting. This technique has been extended to the degenerate Brownian case involving unbounded perturbation, and successfully exploited for handling the corresponding martingale problem in [CdRM20b]. Anyway, this approach does not seem very adapted to our framework especially because it does not allow to deal easily with point-wise gradient estimates which will, at least along the non-degenerate variable x_1, be fundamental to establish our result.
On the other hand, the so-called forward parametrix approach has been successfully used by Friedman [Fri64] or Il'in et al. [IKO62] in the non-degenerate, diffusive setting to obtain point-wise bounds on the fundamental solution and its derivatives for the corresponding heat-type equation or in [CdR17] to derive strong uniqueness for the associated SDE (1.4) (i.e. $n = 2$ with the previous notations). Especially, this approach is better tailored to exploit cancellation techniques that are crucial when derivatives come in, as opposed to the backward one.

The main difficulties to overcome in order to prove Schauder estimates in our framework will be linked to the degeneracy of the operator $\mathcal{L}_\alpha$ that acts only on the first d variables, as well as the unboundedness of the perturbation F. Concerning this second issue, let us also mention that Schauder estimates for unbounded non-linear drift coefficients in the non-degenerate diffusive setting were obtained under mild smoothness assumptions by Krylov and Priola [KP10] who heavily used an auxiliary, deterministic flow associated with the transport term in (1.1), i.e. for a fixed couple (t, x),

$$\begin{cases} \partial_s\theta_s(x) = A\theta_s(x) + F(s, \theta_s(x)); \quad \text{if } s > t \\ \theta_t(x) \;=\; x, \end{cases} \tag{1.5}$$

to precisely get rid of the unbounded terms.

The drawback of this approach is that we will need at first to establish Schauder estimates in a small time interval. This seems quite intuitive since the expansion along the chosen proxy on which the method relies is precisely designed for small times because it requires that the original operator and the proxy are "close" enough in a suitable sense. To obtain the result for an arbitrary but finite time, we will then iterate the reasoning, which is quite natural since Schauder estimates provide a sort of stability in the considered functional space. We are therefore far from the optimal constants for the Schauder estimates established in the non-degenerate, diffusive setting for time dependent coefficients by Krylov and Priola [KP17].

On the other hand, we want to establish the Schauder estimates in the sharpest possible Hölder setting for the coefficients of the IPDE (1.1). To do so, we will need to establish some subtle controls, in particular we have no true derivatives of the coefficients. This is the reason why we will heavily rely on duality results on Besov spaces (see Section 4.1 below, Chapter 3 in [LR02] or [Tri92] for a more complete survey of the argument). However, in contrast with the non-degenerate case (cf. [CdRMP20a]), we will need to ask for the perturbation F some additional regularity, represented by parameter γ_i in assumption [**R**] below, on the degenerate entries F_i ($i > 1$). This assumption seems quite natural if we think that, due to the degenerate structure of the system (cf. Section 2.2 below), the more we descend on the chain, the lower the smoothing effect of $\mathcal{L}_\alpha$ will be. The additional smoothness on F can be then seen as the "price" to pay to re-equilibrate the increasing time singularities appearing along the chain.

Organization of the paper. The article is organized as follows. We state our precise framework and give our main results in the following Section 2. Section 3 is then dedicated to the perturbative approach which is the central argument to derive our estimates. In particular, we obtain therein some Schauder estimates for drifted operators along the inhomogeneous flow $\theta_{s,t}$ defined above in (1.5), as well as the key Duhamel representation for solutions. Since the arguments to show the Schauder estimates will be quite long and involved, we postpone the proofs of these results in the next Sections 4 and 5. The existence results are then established in Section 6. In the last Section 7, we are going to explain briefly how the perturbative approach presented before could be applied with slight modifications to prove Schauder-type estimates for a class of completely non-linear, locally Hölder continuous drifts with an additional "diffusion" coefficient.
Finally, the proof of some technical results concerning the stability properties of Hölder flows are postponed to the Appendix.

2 Setting and main results

2.1 Main operators considered

The operator $\mathcal{L}_\alpha$ we consider is the generator of a non-degenerate, symmetric, stable process and it acts only on the first d coordinates of the system. More precisely, $\mathcal{L}_\alpha$

can be represented for any sufficiently regular $\phi\colon [0,T]\times\mathbb{R}^{nd}\to\mathbb{R}$ as

$$\mathcal{L}_\alpha\phi(t,x) \;:=\; \text{p.v.}\int_{\mathbb{R}^d}\Big[\phi(t,x+By)-\phi(t,x)\Big]\,\nu_\alpha(dy), \quad \text{where} \quad B \;:=\; \begin{bmatrix} I_{d\times d}\\ 0_{d\times d}\\ \vdots\\ 0_{d\times d}\end{bmatrix}$$

and ν_α is a symmetric, stable Lévy measure on $\mathbb{R}^d$ of order α that we assume to be non-degenerate in a sense that we are going to specify below.
Passing to polar coordinates $y=\rho s$ where $(\rho,s)\in[0,\infty)\times\mathbb{S}^{d-1}$, it is well-known (see for example Chapter 3 in [Sat13]) that the stable Lévy measure ν_α can be decomposed as

$$\nu_\alpha(dy) \;:=\; \tilde{\mu}(ds)\frac{d\rho}{\rho^{1+\alpha}}, \tag{2.6}$$

where $\tilde{\mu}$ is a symmetric measure on $\mathbb{S}^{d-1}$ which represents the spherical part of ν_α.
We remember now that the Lévy symbol associated with $\mathcal{L}_\alpha$ is defined through the Levy-Khitchine formula (see, for instance [Jac01]) as:

$$\Phi(p) \;:=\; \int_{\mathbb{R}^d}\Big[e^{ip\cdot y}-1\Big]\,\nu_\alpha(dy), \qquad \text{for any } p \text{ in } \mathbb{R}^d,$$

where "$\cdot$" represents the inner product on the smaller space $\mathbb{R}^d$. In the current symmetric setting, it can be rewritten (cf. Theorem 14.10 in [Sat13]) as

$$\Phi(p) \;=\; -\int_{\mathbb{S}^{d-1}}|p\cdot s|^\alpha\,\mu(ds), \tag{2.7}$$

where $\mu=C_{\alpha,d}\tilde{\mu}$ is usually called the spherical measure associated with ν_α . Following [Kol00], we then say that ν_α is non-degenerate if the associated Lévy symbol Φ is equivalent, up to some multiplicative constant, to $|p|^\alpha$. More precisely, we suppose that μ is non-degenerate if

[ND] there exists a constant $\eta\geq 1$ such that for any p in $\mathbb{R}^d$.

$$\eta^{-1}|p|^\alpha \;\leq\; \int_{\mathbb{S}^{d-1}}|p\cdot s|^\alpha\,\mu(ds) \;\leq\; \eta|p|^\alpha. \tag{2.8}$$

It is important to remark that such a condition does not restrict our model too much. Indeed, there are many different kind of spherical measures μ that are non-degenerate in the above sense, from the stable-like case, i.e. measures that are absolutely continuous with respect to the Lebesgue measure on $\mathbb{S}^{d-1}$, to very singular ones such that the spherical measure induced by the sum of Dirac masses along the canonical directions:

$$\sum_{i=1}^{d}(\partial^2_{x_k})^{\alpha/2}.$$

We can introduce now the complete Ornstein-Uhlenbeck operator L^{ou}, defined for any sufficiently regular $\phi\colon\mathbb{R}^{nd}\to\mathbb{R}$ as

$$L^{\text{ou}}\phi(x) \;:=\; \langle Ax, D_x\phi(x)\rangle + \mathcal{L}_\alpha\phi(x), \tag{2.9}$$

where A is the matrix in $\mathbb{R}^{nd}\times\mathbb{R}^{nd}$ defined in Equation (1.2). We assume that A satisfies the following Hörmander-like condition of non-degeneracy:

[H] $A_{i,i-1}$ is non-degenerate (i.e. it has full rank d) for any i in $[\![2,n]\!]$.

Above, $[\![2,n]\!]$ denotes the set of all the integers in the interval. It is well known (see for example [Sat13]) that under these assumptions, the operator L^{ou} generates a convolution Markov semigroup $\{P_t^{\mathrm{ou}}\}_{t\geq 0}$ on $B_b(\mathbb{R}^{nd})$, the family of all the bounded and Borel measurable functions on $\mathbb{R}^{nd}$, defined by

$$\begin{cases} P_t^{\mathrm{ou}}\phi(x) \;=\; \int_{\mathbb{R}^{nd}} \phi(x+y)\,\mu_t(dy); \\ P_0^{\mathrm{ou}}\phi(x) \;=\; \phi(x), \end{cases}$$

where $\{\mu_t\}_{t>0}$ is a family of Borel probability measures on $\mathbb{R}^{nd}$. In particular, the function $P_t^{\mathrm{ou}}\phi$ provides the classical solution to the Cauchy problem

$$\begin{cases} \partial_t u(t,x) + \mathcal{L}_\alpha u(t,x) + \langle Ax, D_x u(t,x)\rangle \;=\; 0 & \text{on } (0,\infty)\times\mathbb{R}^{nd}; \\ u(0,x) \;=\; \phi(x) & \text{on } \mathbb{R}^{nd}. \end{cases} \tag{2.10}$$

Moving to the stochastic counterpart if necessary, it is readily derived from [PZ09] that the semigroup $(P_t^{\mathrm{ou}})_{t\geq 0}$ admits a smooth density $p^{\mathrm{ou}}(t,\cdot)$ with respect to the Lebesgue measure on $\mathbb{R}^{nd}$. Moreover, such a density p^{ou} has the following useful representation:

$$p^{\mathrm{ou}}(t,x,y) \;=\; \frac{1}{\det \mathbb{M}_t} p_S\big(t, \mathbb{M}_t^{-1}\big(e^{At}x - y\big)\big), \tag{2.11}$$

where p_S is the density of $\{S_t\}_{t\geq 0}$, a stable process in $\mathbb{R}^{nd}$ whose Lévy measure satisfies the assumption [**ND**] above on $\mathbb{R}^{nd}$ and $\mathbb{M}_t$ is a diagonal matrix on $\mathbb{R}^{nd}\times\mathbb{R}^{nd}$ given by

$$\big[\mathbb{M}_t\big]_{i,j} \;:=\; \begin{cases} t^{i-1} I_{d\times d}, & \text{if } i=j; \\ 0_{d\times d}, & \text{otherwise.} \end{cases} \tag{2.12}$$

We remark already that the appearance of the matrix $\mathbb{M}_t$ in Equation (2.11) and its particular structure reflect the multi-scaled structure of the dynamics considered (cf. Paragraph (2.2) below for a more precise explanation).
Moreover, the density p_S shows a useful property we will call the *smoothing effect* since it will be fundamental to reduce the singularities appearing when working with time integrals. Fixed γ in $[0,\alpha)$, there exists a constant $C := C(\gamma)$ such that for any l in $[\![0,3]\!]$,

$$\int_{\mathbb{R}^{nd}} |y|^\gamma |D_y^l p_S(t,y)|\,dy \;\leq\; C t^{\frac{\gamma-l}{\alpha}} \quad \text{for any } t>0. \tag{2.13}$$

These results can be proven following the arguments of Proposition 2.3 and Lemma 4.3 in [HMP19]. We will provide however a complete proof in the Appendix for the sake of completeness.

2.2 Intrinsic time scale and associated Hölder spaces

In this section, we are going to choose which is the most suitable functional space in which to state our Schauder estimates.

To answer this question, we need firstly to understand how the system typically behaves. We focus for the moment on the Ornstein-Uhlenbeck case:

$$\Big(\partial_t + L^{\mathrm{ou}}\Big)u(t,x) \;=\; -f(t,x) \qquad \text{on } (0,\infty)\times\mathbb{R}^{nd},$$

and search for a dilation operator $\delta_\lambda\colon (0,\infty)\times\mathbb{R}^{nd}\to(0,\infty)\times\mathbb{R}^{nd}$ that is invariant for the considered dynamics, i.e. a dilation that transforms solutions of the above equation into other solutions of the same equation.
Due to the structure of A and the α-stability of ν, we can consider for any fixed $\lambda>0$, the following

$$\delta_\lambda(t,x) := (\lambda^\alpha t, \lambda x_1, \lambda^{1+\alpha}x_2, \dots, \lambda^{1+\alpha(n-1)}x_n),$$

i.e. with a possible slight abuse of notation, $\Big(\delta_\lambda(t,x)\Big)_0 := \lambda^\alpha t$ and for any i in $[\![1,n]\!]$, $\Big(\delta_\lambda(t,x)\Big)_i := \lambda^{1+\alpha(i-1)}x_i$. It then holds that

$$\Big(\partial_t + L^{\mathrm{ou}}\Big)u = 0 \implies \Big(\partial_t + L^{\mathrm{ou}}\Big)(u\circ\delta_\lambda) = 0.$$

The previous reasoning suggests us to introduce a parabolic distance $\mathbf{d}_P$ that is homogenous with respect to the dilation δ_λ, so that $\mathbf{d}_P\Big(\delta_\lambda(t,x);\delta_\lambda(s,x')\Big) = \lambda\mathbf{d}_P\Big((t,x);(s,x')\Big)$. Precisely, following the notations in [HMP19], we set for any s,t in $[0,T]$ and any x,x' in $\mathbb{R}^{nd}$,

$$\mathbf{d}_P\Big((t,x),(s,x')\Big) \;:=\; |s-t|^{\frac{1}{\alpha}} + \sum_{j=1}^{n} |(x-x')_j|^{\frac{1}{1+\alpha(j-1)}}. \tag{2.14}$$

The idea of a dilation δ_λ that summarizes the multi-scaled behaviour of the dynamics was firstly introduced by Lanconelli and Polidoro in [LP94] for degenerate Kolmogorov equations in the diffusive setting. Since then, it has become a "standard" tool in the analysis of degenerate equations (see for example [Lun97], [HMP19] or [HWZ20]).
Since we will quite always use only the spatial part of the distance $\mathbf{d}_P$, we denote for simplicity

$$\mathbf{d}(x,y) \;=\; \sum_{j=1}^{n} |(x-x')_j|^{\frac{1}{1+\alpha(j-1)}}. \tag{2.15}$$

Technically speaking, $\mathbf{d}_P$ (and thus, $\mathbf{d}$) does not however induce a norm on $[0,T]\times\mathbb{R}^{nd}$ in the usual sense since it lacks of linear homogeneity. We remark anyhow again that for any $\lambda>0$, it precisely holds that $\mathbf{d}\Big(\delta_\lambda(t,x);\delta_\lambda(s,x')\Big) = \lambda\mathbf{d}\Big((t,x);(s,x')\Big)$. As it can be seen, $\mathbf{d}_P$ is an extension of the standard parabolic distance in the stable case, adapted to respect the multi-scaled nature of our dynamics. Indeed, the exponents appearing in (2.14) are those which make each space component homogeneous to the characteristic time scale $t^{1/\alpha}$.
The appearance of this kind of phenomena is due essentially by the particular structure of the matrix A (cf. Equation (1.1)) that allows the smoothing effect of $\mathcal{L}_\alpha$, acting only on the first variable, to propagate in the system, as it can be seen in the following lemma:

Lemma 2.1 (Scaling Lemma)**.** *Let i be in $[\![1,n]\!]$. Then, there exist $\{C_j\}_{j\in[\![1,n]\!]}$ positive constants, depending only from A and i, such that*

$$D_{x_i}p^{ou}(t,x,y) \;=\; -\sum_{j=i}^{n} C_j t^{j-i} D_{y_j}p^{ou}(t,x,y)$$

for any $t > 0$ and any x, y in $\mathbb{R}^{nd}$.

Proof. Recalling the representation of p^{ou} in Equation (2.11), it is easy to see that

$$D_{x_i} p^{\mathrm{ou}}(t,x,y) \;=\; \frac{1}{\det \mathbb{M}_t} D_z p_S(t,\cdot)\Big(\mathbb{M}_t^{-1}(e^{At}x - y)\Big)\mathbb{M}_t^{-1} D_{x_i}\Big[e^{At}x - y\Big].$$

Hence, in order to conclude, we need to show that

$$D_{x_i}\Big[e^{At}x - y\Big] \;=\; -\sum_{j=i}^{n} C_j t^{j-i} D_{y_j}\Big[e^{At}x - y\Big]. \tag{2.16}$$

To prove the above equality, we need to analyze more in depth the structure of the resolvent e^{At}. Recalling from Equation (1.2) that A has a sub-diagonal structure, we notice that for any i, j in $[\![1,n]\!]$,

$$\Big[e^{At}\Big]_{i,j} \;=\; \begin{cases} C_{i,j} t^{j-i}, & \text{if } j \geq i; \\ 0, & \text{otherwise,} \end{cases} \tag{2.17}$$

for a family of constants $\{C_{i,j}\}_{i,j\in[\![1,n]\!]}$ depending only from A. It then follows that for any x, y in $\mathbb{R}^{nd}$, it holds that

$$\Big[e^{At}x - y\Big]_i \;=\; \sum_{k=1}^{i} C_{i,k} t^{i-k} x_k - y_i. \tag{2.18}$$

Equation (2.16) then follows immediately. For a more detailed proof of this result, see also [HM16] or [HMP19]. □

We finally remark the link with the stochastic counterpart of equation (1.1). From a more probabilistic point of view, the exponents in Equation (2.14) can be related to the characteristic time scales of the iterated integrals of an α-stable process.

We are now ready to define the suitable Hölder spaces for our estimates. We start recalling some useful notations we will need below. Fixed k in $\mathbb{N} \cup \{0\}$ and β in $(0,1)$, we follow Krylov [Kry96], denoting the usual *homogeneous* Hölder space $C^{k+\beta}(\mathbb{R}^d)$ as the family of functions $\phi\colon \mathbb{R}^d \to \mathbb{R}$ such that

$$\|\phi\|_{C^{k+\beta}} \;:=\; \sum_{i=1}^{k} \sup_{|\vartheta|=i} \|D^{\vartheta}\phi\|_{L^\infty} + \sup_{|\vartheta|=k} [\mathbf{d}^{\vartheta}\phi]_{\beta} \;<\; \infty,$$

where

$$[\mathbf{d}^{\vartheta}\phi]_{\beta} \;:=\; \sup_{x\neq y} \frac{|\mathbf{d}^{\vartheta}\phi(x) - \mathbf{d}^{\vartheta}\phi(y)|}{|x-y|^{\beta}}.$$

Additionally, we are going to need the associated subspace $C_b^{k+\beta}(\mathbb{R}^d)$ of bounded functions in $C^{k+\beta}(\mathbb{R}^d)$, equipped with the norm

$$\|\cdot\|_{C_b^{k+\beta}} \;=\; \|\cdot\|_{L^\infty} + \|\cdot\|_{C^{k+\beta}}.$$

We can now define the anisotropic Hölder space with multi-index of regularity associated with the distance d. For sake of brevity and readability, we firstly define for a function $\phi\colon \mathbb{R}^{nd} \to \mathbb{R}$, a point z in $\mathbb{R}^{d(n-1)}$ and i in $[\![1,n]\!]$, the function

$$\Pi^i_z\phi\colon x \in \mathbb{R}^d \mapsto \phi(z_1,\dots,z_{i-1},x,z_{i+1},z_n) \in \mathbb{R},$$

with the obvious modifications if $i=1$ or $i=n$. Intuitively speaking, the function $\Pi^i_z\phi$ is the restriction of ϕ on its i-th d-dimensional variable while fixing all the other coordinates in z. The space $C^{k+\beta}_d(\mathbb{R}^{nd})$ is then defined as the family of all the function $\phi\colon \mathbb{R}^{nd} \to \mathbb{R}$ such that

$$\|\phi\|_{C^{k+\beta}_d} := \sum_{i=1}^{n} \sup_{z\in\mathbb{R}^{d(n-1)}} \|\Pi^i_z\phi\|_{C^{\frac{k+\beta}{1+\alpha(i-1)}}} < \infty. \tag{2.19}$$

The modification to the bounded subspace $C^{k+\beta}_{b,d}(\mathbb{R}^{nd})$ is straightforward.
Roughly speaking, the anisotropic norm works component-wise, i.e. we firstly fix a coordinate and then calculate the standard Hölder norm along that particular direction, but with index scaled according to the dilation of the system in that direction, uniformly over time and the other space components. We conclude summing the contributions associated with each component.
We highlight however that it is possible to recover the expected joint regularity for the partial derivatives, when they exist. In such a case, they actually turn out to be Hölder continuous in the pseudo-metric $\mathbf{d}$ with order one less than the function. (cf. Lemma 8.4 in the Appendix for the case $i=1$).

Since we are working with evolution equations, the functions we consider will quite often depend on time, too. For this reason, we denote by $L^\infty\Big(0,T;C^{k+\beta}_d(\mathbb{R}^{nd})\Big)$ (respectively, $L^\infty\Big(0,T;C^{k+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$) the family of functions $\psi\colon [0,T]\times\mathbb{R}^{nd} \to \mathbb{R}$ with finite $C^{k+\beta}_d$-norm (respectively, $C^{k+\beta}_{b,d}$-norm), uniformly in time. It is endowed with the following norm:

$$\|\phi\|_{L^\infty(C^{k+\beta}_d)} := \sup_{t\in[0,T]} \|\phi(t,\cdot)\|_{C^{k+\beta}_d}, \tag{2.20}$$

with a straightforward modification for the bounded subspace $L^\infty\Big(0,T;C^{k+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$.

2.3 Assumptions and main results

From this point further, we consider two fixed numbers α in $(0,2)$ and β in $(0,1)$ such that α will represent the index of stability of the operator $\mathcal{L}_\alpha$ while β will stand for the index of Hölder regularity of the coefficients.

We will assume the following:

[S] assumptions [**ND**] and [**H**] are satisfied and the drift $F=(F_1,\dots,F_n)$ is such that for any i in $[\![1,n]\!]$, F_i depends only on time and on the last $n-(i-1)$ components, i.e. $F_i(t,x_i,\dots,x_n)$;

[P] α is a number in $(0,2)$, β is in $(0,1)$ such that $\alpha+\beta\in(1,2)$ and if $\alpha<1$ (super-critical case),

$$\beta<\alpha, \quad 1-\alpha < \frac{\alpha-\beta}{1+\alpha(n-1)};$$

[**R**] Recalling the notations in (2.19)-(2.20), the source f is in $L^\infty(0,T;C^\beta_{b,d}(\mathbb{R}^{nd}))$, the terminal condition u_T is in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})$ and for any i in $[\![1,n]\!]$, the drift F_i belongs to $L^\infty(0,T;C^{\gamma_i+\beta}_d(\mathbb{R}^{nd}))$, where

$$\gamma_i := \begin{cases} 1+\alpha(i-2), & \text{if } i>1; \\ 0, & \text{if } i=1. \end{cases} \tag{2.21}$$

From now on, we will say that assumption [**A**] holds when the above conditions [**S**], [**P**] and [**R**] are in force.

Remark 2.1 (About the Assumptions)**.** We remark that the constraints [**P**] we are imposing in the super-critical case ($\alpha<1$) seem quite natural for our system. The condition $\beta<\alpha$ reflects essentially the low integrability properties of the stable density p_S (cf. Equation (2.13)). Even if one is interested only on the fractional Laplacian case, i.e. $\mathcal{L}_\alpha=\Delta^{\alpha/2}$, such a condition cannot be dropped in general, since it does not refer to the integrability property of p_α and its derivatives but instead to those of its "projection" p_S on the bigger space $\mathbb{R}^{nd}$ (cf. Equation (2.11)).
The second condition $\alpha+\beta>1$ is necessary to give a point-wise definition of the gradient of a solution u with respect to the non-degenerate variable x_1. Moreover, there is a famous counterexample of Tanaka and his collaborators [TTW74] that shows that even in the scalar case, weak uniqueness (a direct consequence of Schauder estimates) may fail for the associated SDE if $\alpha+\beta$ is smaller than one.
The last assumption is indeed a technical constraint and it is necessary to work properly with the perturbation F at any level $i=1,\dots,n$. In particular, it seems the minimal threshold that allows us to exploit the smoothing effect of the density (see for example Equation (5.32) in the proof of Lemma 5.2 for more details). We conclude highlighting that these assumptions are always fulfilled if $\alpha\geq 1$ (sub-critical case).

At this stage, it should be clear that under our assumptions [**A**], IPDE (1.1) will be understood in a *distributional* sense. Indeed, we cannot hope to find a "classical" solution for Equation (1.1), since for such a function u in $L^\infty(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd}))$, the total gradient D_xu is not defined point-wise.
Let us denote for any function $\phi\colon[0,T]\to\mathbb{R}^{nd}$ regular enough, the complete operator L_t as

$$L_t\phi(t,x) := \langle Ax+F(t,x), D_xu(t,x)\rangle + \mathcal{L}_\alpha u(t,x). \tag{2.22}$$

We will say that a function u in $L^\infty(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd}))$ is a distributional (or weak) solution of IPDE (1.1) if for any ϕ in $C^\infty_0((0,T]\times\mathbb{R}^{nd})$, it holds that

$$\int_0^T\int_{\mathbb{R}^{nd}}\Big(-\partial_t+L^*_t\Big)\phi(t,y)u(t,y)\,dy+\int_{\mathbb{R}^{nd}}u_T(y)\phi(T,y)\,dy = -\int_0^T\int_{\mathbb{R}^{nd}}\phi(t,y)f(t,y)\,dy, \tag{2.23}$$

where $\mathcal{L}^*_\alpha$ denotes the formal adjoint of L_t. On the other hand, denoting from now on,

$$\|F\|_H := \sup_{i\in[\![1,n]\!]}\|F_i\|_{L^\infty(C^{\gamma_i+\beta}_d)}, \tag{2.24}$$

we will quite often use the following other notion of solution:

Definition 2.2. A function u is a mild solution in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ of Equation (1.1) if for any triple of sequences $\{f_m\}_{m\in\mathbb{N}}$, $\{u_{T,m}\}_{m\in\mathbb{N}}$ and $\{F_m\}_{m\in\mathbb{N}}$ such that

- $\{f_m\}_{m\in\mathbb{N}}$ is in $C^\infty_b((0,T)\times\mathbb{R}^{nd})$ and f_m converges to f in $L^\infty\Big(0,T;C^{\beta}_{b,d}(\mathbb{R}^{nd})\Big)$;
- $\{u_{T,m}\}_{m\in\mathbb{N}}$ is in $C^\infty_b(\mathbb{R}^{nd})$ and $u_{T,m}$ converges to u_T in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})$;
- $\{F_m\}_{m\in\mathbb{N}}$ is in $C^\infty_b((0,T)\times\mathbb{R}^{nd};\mathbb{R}^{nd})$ and $\|F_m-F\|_H$ converges to 0,

there exists a sub-sequence $\{u_m\}_{m\in\mathbb{N}}$ in $C^\infty_b\Big((0,T)\times\mathbb{R}^{nd}\Big)$ such that

- u_m converges to u in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$;
- for any fixed m in $\mathbb{N}$, u_m is a classical solution of the following "regularized" IPDE:

$$\begin{cases}\partial_t u_m(t,x)+\mathcal{L}_\alpha u_m(t,x) & \\ \qquad\qquad\qquad +\langle Ax+F_m(t,x),D_x u_m(t,x)\rangle \;=\; -f_m(t,x), & \text{on } (0,T)\times\mathbb{R}^{nd};\\ u_m(T,x)\;=\;u_{T,m}(x) & \text{on } \mathbb{R}^{nd}.\end{cases} \tag{2.25}$$

We can now state our main result:

Theorem 2.3. *(Schauder Estimates) Let u be a mild solution in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ of IPDE* (1.1)*. Under [**A**], there exists a constant $C:=C(T)$ such that*

$$\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\;\le\;C\Big[\|f\|_{L^\infty(C^{\beta}_{b,d})}+\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big]. \tag{2.26}$$

Associated with an existence result we will exhibit in Section 6, we will eventually derive the well-posedness for Equation (1.1).

Theorem 2.4. *Under [**A**], there exists a unique mild solution u of IPDE* (1.1) *belonging to $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$. Moreover, such a function u is a weak solution, too.*

In the following, we will denote for sake of brevity

$$\alpha_i\;:=\;\frac{\alpha}{1+\alpha(i-1)}\quad\text{and}\quad\beta_i\;:=\;\frac{\beta}{1+\alpha(i-1)}\quad\text{for any } i \text{ in } [\![1,n]\!]. \tag{2.27}$$

Clearly, these quantities were introduced to reflect exactly the relative scale of the system at every considered level i (cf. Section 2.2 above).
In the following, as well as in Theorem 2.3 above, C denotes a generic constant that may change from line to line but depending only on the parameters in assumption [**A**]. Other dependencies that may occur are explicitly specified.

3 Proof through Perturbative Approach

As already said in the introductive section, our method of proof relies on a perturbative approach introduced in [CdRHM18a] for the degenerate, Kolmogorov, diffusive setting. Roughly speaking, we will firstly choose a suitable proxy for the equation of interest,

i.e. an operator whose associated semigroup and density are known and that is close enough to the original one:

$$\mathcal{L}_\alpha + \langle Ax + F(t,x), D_x\rangle.$$

Furthermore, we will exhibit suitable regularization properties for the proxy and in particular, we will show that it satisfies the Schauder estimates (2.26). This will be the purpose of Sub-section 3.1.
In Sub-section 3.2 below, we will then expand a solution u of IPDE (1.1) along the chosen proxy through a Duhamel-type formula and eventually show that the expansion error only brings a negligible contribution, so that the Schauder estimates still hold for u. Due to our choice of method, this will be possible only adding some more assumptions on the system. Namely, we will assume in addition to be in a small time interval, so that the proxy and the original operator do not differ too much.
The last Sub-section 3.3 will finally show how to remove the additional assumption in order to prove the Schauder estimates (Theorem 2.3) through a scaling argument.

3.1 Frozen semigroup

The crucial element in our approach consists in choosing wisely a suitable proxy operator along which to expand a solution u in $L^\infty(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd}))$ of IPDE (1.1). In order to deal with potentially unbounded perturbations F, it is natural to use a proxy involving a non-zero first order term associated with a flow representing the dynamics driven by $Ax+F$, the transport part of Equation (1.1) (see e.g. [KP10] or [CdRMP20a]).

Remembering that we assume F to be Hölder continuous, we know that there exists a solution of

$$\begin{cases} d\theta_{s,\tau}(\xi) \;=\; \Big[A\theta_{s,\tau}(\xi) + F(s,\theta_{s,\tau}(\xi))\Big]ds \quad \text{on } [\tau,T]; \\ \theta_{\tau,\tau}(\xi) \;=\; \xi, \end{cases}$$

even if it may be not unique. For this reason, we are going to choose one particular flow, denoted by $\theta_{s,\tau}(\xi)$, and consider it fixed throughout the work.
More precisely, given a freezing couple (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$, the flow will be defined on $[\tau,T]$ as

$$\theta_{s,\tau}(\xi) \;=\; \xi + \int_\tau^s \Big[A\theta_{v,\tau}(\xi) + F(v,\theta_{v,\tau}(\xi))\Big]\,dv. \tag{3.1}$$

We can now introduce the "frozen" IPDE on $(0,T)\times\mathbb{R}^{nd}$, associated with the chosen proxy:

$$\begin{cases} \partial_s\tilde{u}^{\tau,\xi}(s,x) + \mathcal{L}_\alpha\tilde{u}^{\tau,\xi}(s,x) + \langle Ax + F(s,\theta_{s,\tau}(\xi)), D_x\tilde{u}^{\tau,\xi}(s,x)\rangle \;=\; -f(s,x); \\ \tilde{u}^{\tau,\xi}(s,x) \;=\; u_T(x). \end{cases} \tag{3.2}$$

Noticing that the proxy operator $\mathcal{L}_\alpha+\langle Ax+F(s,\theta_{s,\tau}(\xi)), D_x\rangle$ can be seen as an Ornstein-Ulhenbeck operator with an additional time-dependent component $F(s,\theta_{s,\tau}(\xi))$, it is clear that under assumption [**A**], it generates a two parameters semigroups we will denote by $\{\tilde{P}^{\tau,\xi}_{s,t}\}_{t\le s}$. Moreover, it admits a density given by

$$\tilde{p}^{\tau,\xi}(t,s,x,y) \;=\; \frac{1}{\det\mathbb{M}_{s-t}} p_S\Big(s-t, \mathbb{M}^{-1}_{s-t}(y-\tilde{m}^{\tau,\xi}_{s,t}(x))\Big), \tag{3.3}$$

remembering Equation (2.11) for the definition of p_S and with the following notation for the "frozen shift" $\tilde{m}^{\tau,\xi}_{s,t}(x)$:

$$\tilde{m}^{\tau,\xi}_{s,t}(x) \;=\; e^{A(s-t)}x + \int_t^s e^{A(s-v)}F(v,\theta_{v,\tau}(\xi))\,dv. \tag{3.4}$$

We point out already the following important property of the shift $\tilde{m}^{\tau,\xi}_{s,t}(x)$:

Lemma 3.1. *Let $t < s$ in $[0,T]$ and x a point in $\mathbb{R}^{nd}$. Then,*

$$\tilde{m}^{\tau,\xi}_{s,t}(x) \;=\; \theta_{s,\tau}(\xi), \tag{3.5}$$

taking $\tau = t$ and $\xi = x$.

Proof. We start noticing that by construction, $\tilde{m}^{\tau,\xi}_{s,t}(x)$ satisfies

$$\tilde{m}^{\tau,\xi}_{s,t}(x) \;=\; x + \int_t^s \Big[A\tilde{m}^{t,x}_{v,t}(x) + F(v,\theta_{v,\tau}(\xi))\Big]\,dv.$$

It then holds that

$$|\tilde{m}^{t,x}_{s,t}(x) - \theta_{s,t}(x)| \;\le\; \int_t^s A|\tilde{m}^{t,x}_{v,t}(x) - \theta_{v,t}(x)|\,dv.$$

The above Equation (3.5) then follows immediately applying the Grönwall lemma. □

Moreover, we can extend the smoothing effect (2.13) of p_S to the frozen density $\tilde{p}^{\tau,\xi}$ through the representation (3.3):

Lemma 3.2 (Smoothing effects of the frozen density)**.** *Under [**A**], let ϑ, ϱ be two multi-indexes in $\mathbb{N}^n$ such that $|\varrho + \vartheta| \le 3$ and γ in $[0,\alpha)$. Then, there exists a constant $C := C(\vartheta,\varrho,\gamma)$ such that*

$$\int_{\mathbb{R}^{nd}} |D^{\varrho}_y D^{\vartheta}_x \tilde{p}^{\tau,\xi}(t,s,x,y)|\mathbf{d}^{\gamma}\Big(y,\tilde{m}^{\tau,\xi}_{s,t}(x)\Big)\,dy \;\le\; C(s-t)^{\frac{\gamma}{\alpha}-\sum_{i=k}^{n}\frac{\vartheta_k+\varrho_k}{\alpha_k}} \tag{3.6}$$

for any $t < s$ in $[0,T]$, any x in $\mathbb{R}^{nd}$ and any frozen couple (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$. In particular, if $|\vartheta| \neq 0$, it holds for any ϕ in $C^{\gamma}_d(\mathbb{R}^{nd})$ that

$$\Big|D^{\vartheta}_x \tilde{P}^{\tau,\xi}_{s,t}\phi(x)\Big| \;\le\; C\|\phi\|_{C^{\gamma}_d}(s-t)^{\frac{\gamma}{\alpha}-\sum_{k=1}^{n}\frac{\vartheta_k}{\alpha_k}}. \tag{3.7}$$

Proof. Let us assume that $|\vartheta| = 1$ and $|\varrho| = 1$. The other cases can be treated in a similar way.
Since p_S is the density of an α-stable process, we remember that the following α-scaling property

$$p_S(t,y) \;=\; t^{-\frac{nd}{\alpha}}p_S(1,t^{-\frac{1}{\alpha}}y) \tag{3.8}$$

holds for any $t > 0$ and any y in $\mathbb{R}^{nd}$. Fixed i in $[\![1,n]\!]$, we then denote for simplicity

$$\mathbb{T}_{s-t} \;:=\; (s-t)^{\frac{1}{\alpha}}\mathbb{M}_{s-t}$$

and we calculate the derivative of $\tilde{p}^{\tau,\xi}$ with respect to x_i through

$$\begin{aligned}
|D_{x_i}\tilde{p}^{\tau,\xi}(t,s,x,y)| &= \Big|\frac{1}{\det \mathbb{M}_{s-t}} D_{x_i}\Big[p_S\Big(s-t, \mathbb{M}_{s-t}^{-1}(\tilde{m}_{s,t}^{\tau,\xi}(x)-y)\Big)\Big]\Big| \\
&= \Big|\frac{1}{\det \mathbb{T}_{s-t}} D_{x_i}\Big[p_S\Big(1, \mathbb{T}_{s-t}^{-1}(\tilde{m}_{s,t}^{\tau,\xi}(x)-y)\Big)\Big]\Big| \\
&= \Big|\frac{1}{\det(\mathbb{T}_{s-t})}\langle D_z p_S\Big(1,\cdot\Big)(\mathbb{T}_{s-t}^{-1}(\tilde{m}_{s,t}^{\tau,\xi}(x)-y)); \mathbb{T}_{s-t}^{-1} D_{x_i}(\tilde{m}_{s,t}^{\tau,\xi}(x))\rangle\Big|,
\end{aligned}$$

where in the second equality we exploited the α-scaling Property (3.8). From Equation (2.17) in the Scaling Lemma 2.1, we now notice that

$$\begin{aligned}
\Big|\mathbb{T}_{s-t}^{-1} D_{x_i}(\tilde{m}_{s,t}^{\tau,\xi}(x))\Big| &= \Big|\mathbb{T}_{s-t}^{-1} D_{x_i}\Big(e^{A(t-s)}(x)\Big)\Big| \\
&= (s-t)^{-\frac{1}{\alpha}} \sum_{k=i}^{n} C_k (s-t)^{-(k-1)}(s-t)^{k-i} \\
&\leq C(s-t)^{-\frac{1+\alpha(i-1)}{\alpha}}
\end{aligned}$$

and we use it to show that

$$|D_{x_i}\tilde{p}^{\tau,\xi}(t,s,x,y)| \leq C(s-t)^{-\frac{1+\alpha(i-1)}{\alpha}} \frac{1}{\det(\mathbb{T}_{s-t})}\Big|D_z p_S\Big(1,\cdot)(\mathbb{T}_{s-t}^{-1}(\tilde{m}_{s,t}^{\tau,\xi}(x)-y)\Big)\Big|.$$

Similarly, if we fix j in $[\![1,n]\!]$, it holds that

$$|D_{y_j} D_{x_i}\tilde{p}^{\tau,\xi}(t,s,x,y)| \leq C(s-t)^{-\frac{1}{\alpha_i}-\frac{1}{\alpha_j}} \frac{1}{\det(\mathbb{T}_{s-t})}\Big|D_z^2 p_S\Big(1,\cdot)(\mathbb{T}_{s-t}^{-1}(\tilde{m}_{s,t}^{\tau,\xi}(x)-y)\Big)\Big|.$$

It is then easy to show by iteration of the same argument that

$$\begin{aligned}
&|D_y^{\varrho} D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)| \\
&\qquad \leq C(s-t)^{-\sum_{k=1}^{n}\frac{\varrho_k+\vartheta_k}{\alpha_k}} \frac{1}{\det(\mathbb{T}_{s-t})}\Big|\mathbf{d}_z^{|\varrho+\vartheta|} p_S\Big(1,\cdot)(\mathbb{T}_{s-t}^{-1}(\tilde{m}_{s,t}^{\tau,\xi}(x)-y)\Big)\Big|. \quad (3.9)
\end{aligned}$$

Control (3.6) immediately follows from the analogous smoothing effect for p_S (cf. Equation (2.13)) and the change of variables $z = \mathbb{T}_{s-t}^{-1}(\tilde{m}_{s,t}^{\tau,\xi}(x)-y)$. Indeed,

$$\begin{aligned}
\int_{\mathbb{R}^{nd}} |D_y^{\varrho} D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)|\mathbf{d}^{\gamma}\Big(y, \tilde{m}_{s,t}^{\tau,\xi}(x)\Big)\,dy &\leq C(s-t)^{-\sum_{k=1}^{n}\frac{\varrho_k+\vartheta_k}{\alpha_k}} \\
&\quad \times \int_{\mathbb{R}^{nd}} \frac{1}{\det(\mathbb{T}_{s-t})}\Big|\mathbf{d}_z^{|\varrho+\vartheta|} p_S\Big(1,\cdot\Big)(\mathbb{T}_{s-t}^{-1}(\tilde{m}_{s,t}^{\tau,\xi}(x)-y))\Big|\mathbf{d}^{\gamma}\Big(y, \tilde{m}_{s,t}^{\tau,\xi}(x)\Big)\,dy \\
&= (s-t)^{-\sum_{k=1}^{n}\frac{\varrho_k+\vartheta_k}{\alpha_k}} \int_{\mathbb{R}^{nd}} \Big|\mathbf{d}_z^{|\varrho+\vartheta|} p_S(1,z)\Big|\mathbf{d}^{\gamma}\Big(\mathbb{T}_{s-t}z + \tilde{m}_{s,t}^{\tau,\xi}(x), \tilde{m}_{s,t}^{\tau,\xi}(x)\Big)\,dy.
\end{aligned}$$

To conclude, we notice that

$$\mathbf{d}^{\gamma}\Big(\mathbb{T}_{s-t}z + \tilde{m}_{s,t}^{\tau,\xi}(x), \tilde{m}_{s,t}^{\tau,\xi}(x)\Big) \leq C\sum_{i=1}^{n} |(s-t)^{\frac{1+\alpha(i-1)}{\alpha}} z_i|^{\frac{\gamma}{1+\alpha(i-1)}} = (s-t)^{\frac{\gamma}{\alpha}} \sum_{i=1}^{n} |z_i|^{\frac{\gamma}{1+\alpha(i-1)}}$$

and use it to write that

$$\int_{\mathbb{R}^{nd}} \Big| D_y^{\varrho} D_x^{\vartheta} \tilde{p}^{\tau,\xi}(t,s,x,y) \Big| \mathbf{d}^{\gamma}\Big(y, \tilde{m}_{s,t}^{\tau,\xi}(x)\Big)\, dy$$
$$\leq C(s-t)^{\frac{\gamma}{\alpha} - \sum_{k=1}^{n} \frac{\varrho_k + \vartheta_k}{\alpha_k}} \sum_{i=1}^{n} \int_{\mathbb{R}^{nd}} \Big| \mathbf{d}_z^{|\varrho+\vartheta|} p_S(1,z) \Big| |z_i|^{\frac{\gamma}{1+\alpha(i-1)}}\, dy \leq C(s-t)^{\frac{\gamma}{\alpha} - \sum_{k=1}^{n} \frac{\varrho_k + \vartheta_k}{\alpha_k}},$$

where in the last passage we used the smoothing effect for p_S (Equation (2.13)), recalling that for any i in $[\![1,n]\!]$, it holds that

$$\frac{\gamma}{1+\alpha(i-1)} \leq \gamma < \alpha$$

and we have thus the required integrability.
To prove instead Inequality (3.7), we use a cancellation argument to write

$$\begin{aligned} \Big| D_x^{\vartheta} \tilde{P}_{s,t}^{\tau,\xi} \phi(x) \Big| &= \Big| \int_{\mathbb{R}^{nd}} D_x^{\vartheta} \tilde{p}^{\tau,\xi}(t,s,x,y) \Big[\phi(y) - \phi(\tilde{m}_{s,t}^{\tau,\xi}(x)) \Big]\, dy \Big| \\ &\leq \int_{\mathbb{R}^{nd}} |D_x^{\vartheta} \tilde{p}^{\tau,\xi}(t,s,x,y)|\, |\phi(y) - \phi(\tilde{m}_{s,t}^{\tau,\xi}(x))|\, dy. \end{aligned}$$

But since we assume ϕ to be in $C_d^{\gamma}(\mathbb{R}^{nd})$, we can control the last expression as

$$\begin{aligned} \Big| D_x^{\vartheta} \tilde{P}_{s,t}^{\tau,\xi} \phi(x) \Big| &\leq \|\phi\|_{C_d^{\gamma}} \int_{\mathbb{R}^{nd}} \mathbf{d}^{\gamma}\Big(y, \tilde{m}_{s,t}^{\tau,\xi}(x)\Big) |D_x^{\vartheta} \tilde{p}^{\tau,\xi}(t,s,x,y)|\, dy \\ &\leq C \|\phi\|_{C_d^{\gamma}} (s-t)^{\frac{\gamma}{\alpha} - \sum_{k=1}^{n} \frac{\vartheta_k}{\alpha_k}}, \end{aligned}$$

where in the last passage we used Equation (3.6). □

We can define now our candidate to be the mild solution of the "frozen" IPDE. If it exists and it is smooth enough, such a candidate appears to be a representation of the solution of Equation (3.2) obtained through the Duhamel principle. For this reason, the following expression

$$\tilde{u}^{\tau,\xi}(t,x) := \tilde{P}_{T,t}^{\tau,\xi} u_T(x) + \int_t^T \tilde{P}_{s,t}^{\tau,\xi} f(s,x)\, ds \quad \text{for any } (t,x) \text{ in } [0,T] \times \mathbb{R}^{nd}, \tag{3.10}$$

will be called the *Duhamel representation of the proxy.* As it seems, under our assumption [**A**] such a representation is robust enough to satisfy Schauder estimates similar to (2.26). Since the proof of this result is quite long, we will postpone it to Section 4.2 for clarity.

Proposition 3.3. *(Schauder Estimates for Proxy) Under [**A**], there exists a constant $C := C(T)$ such that*

$$\|\tilde{u}^{\tau,\xi}\|_{L^{\infty}(C_{b,d}^{\alpha+\beta})} \leq C\Big[\|f\|_{L^{\infty}(C_{b,d}^{\beta})} + \|u_T\|_{C_{b,d}^{\alpha+\beta}} \Big] \tag{3.11}$$

for any freezing couple (τ,ξ) in $[0,T] \times \mathbb{R}^{nd}$.

We conclude this section showing that the function $\tilde{u}^{\tau,\xi}$ is indeed a mild solution in $L^\infty(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd}))$ of the "frozen" IPDE (3.2). Moreover, the converse statement is also true. If regular enough, any solution of Equation (3.2) corresponds to the Duhamel Representation (3.10).

Proposition 3.4. *Let us assume to be under assumption [**A**]. Then,*

- *the function $\tilde{u}^{\tau,\xi}$ defined in* (3.10) *is a mild solution in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ of the "frozen" IPDE* (3.2) *for any freezing couple (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$;*
- *Fixed (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$, let $\tilde{v}^{\tau,\xi}$ be a mild solution in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ of IPDE* (3.2)*. Then,*

$$\tilde{v}^{\tau,\xi}(t,x) \;=\; \tilde{P}^{\tau,\xi}_{T,t}u_T(x) + \int_t^T \tilde{P}^{\tau,\xi}_{s,t}f(s,x)\,ds.$$

Proof. The first assertion is quite straightforward. Let us consider three sequences $\{f_m\}_{m\in\mathbb{N}}$, $\{u_{T,m}\}_{m\in\mathbb{N}}$ and $\{F_m\}_{m\in\mathbb{N}}$ of smooth and bounded coefficients such that f_m converges to f in $L^\infty\Big(0,T;C^{\beta}_{b,d}(\mathbb{R}^{nd})\Big)$, $u_{T,m}$ to u_T in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})$ and $\|F_m-F\|_H\to 0$. Denoting now by $\{\tilde{P}^{m,\tau,\xi}_{s,t}\}_{t\le s}$ the semigroup associated with the "regularized" operator

$$\mathcal{L}_\alpha + \langle Ax + F_m(t,\theta_{t,\tau}(\xi)), D_x\rangle,$$

it is not difficult to show that for any fixed m in $\mathbb{N}$, the following

$$\tilde{u}^{\tau,\xi}_m \;:=\; \tilde{P}^{m,\tau,\xi}_{T,t}u_{T,m}(x) + \int_t^T \tilde{P}^{m,\tau,\xi}_{s,t}f_m(s,x)\,ds$$

is a classical solution of the "frozen" IPDE (3.2) with regularized coefficients $f_m, u_{T,m}$ and F_m. A detailed guide of this result can be found, even if in the diffusive setting, in Lemma 3.3 in [KP10]. Using now the Schauder Estimates (3.11) for the regularized solutions $\tilde{u}^{\tau,\xi}_m$, it follows immediately that $\tilde{u}^{\tau,\xi}_m\to\tilde{u}^{\tau,\xi}$ in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ and thus, that $\tilde{u}^{\tau,\xi}$ is a mild solution of (3.2) in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$.
To prove the second statement, we start fixing a freezing couple (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$ and consider three sequences $\{f_m\}_{m\in\mathbb{N}}$, $\{u_{T,m}\}_{m\in\mathbb{N}}$ and $\{F_m\}_{m\in\mathbb{N}}$ of bounded and smooth coefficients such that $f_m\to f$ in $L^\infty\Big(0,T;C^{\beta}_{b,d}(\mathbb{R}^{nd})\Big)$, $u_{T,m}\to u_T$ in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})$ and $\|F_m-F\|_H\to 0$. They can be constructed through mollification.
Since $\tilde{v}^{\tau,\xi}$ is a mild solution of the "frozen" IPDE (3.2), we know that there exists a sequence $\{\tilde{v}^{\tau,\xi}_m\}_{m\in\mathbb{N}}$ of classical solutions of the "regularized frozen" IPDE (3.2) with coefficients $f_m, u_{T,m}$ and F_m such that $\tilde{v}^{\tau,\xi}_m\to\tilde{v}^{\tau,\xi}$ in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$. Fixed m in $\mathbb{N}$, we then denote

$$h_m(t,x) \;:=\; \tilde{v}^{\tau,\xi}_m\Big(t, x - \int_t^T e^{A(t-s)}F_m(s,\theta_{s,\tau}(\xi))\,ds\Big)$$

for any t in $[0,T]$ and any x in $\mathbb{R}^{nd}$. Direct calculations imply that

$$D_xh_m(t,x) \;=\; D_x\tilde{v}^{\tau,\xi}_m\Big(t, x - \int_t^T e^{A(t-s)}F_m(s,\theta_{s,\tau}(\xi))\,ds\Big);$$
$$\mathcal{L}_\alpha h_m(t,x) \;=\; \mathcal{L}_\alpha\tilde{v}^{\tau,\xi}_m\Big(t, x - \int_t^T e^{A(t-s)}F_m(s,\theta_{s,\tau}(\xi))\,ds\Big)$$

and

$$\begin{aligned}\partial_t h_m(t,x) \;=\; & \partial_t \tilde{v}_m^{\tau,\xi}\Big(t, x - \int_t^T e^{A(t-s)} F_m(s,\theta_{s,\tau}(\xi))\,ds\Big) \\ & + \Big\langle F_m(t,\theta_{t,\tau}(\xi)), D_x \tilde{v}_m^{\tau,\xi}\Big(t, x - \int_t^T e^{A(t-s)} F_m(s,\theta_{s,\tau}(\xi))\,ds\Big)\Big\rangle \\ & - \Big\langle A\int_t^T e^{A(t-s)} F_m(s,\theta_{s,\tau}(\xi))\,ds, D_x \tilde{v}_m^{\tau,\xi}\Big(t, x - \int_t^T e^{A(t-s)} F_m(s,\theta_{s,\tau}(\xi))\,ds\Big)\Big\rangle.\end{aligned}$$

Remembering that $\tilde{v}_m^{\tau,\xi}$ is a classical solution of Equation (3.2) replacing therein f, u_T and F with coefficients $f_m, u_{T,m}$ and F_m, it follows immediately that the function h_m solves for any m in $\mathbb{N}$ the following:

$$\begin{cases}\partial_t h_m(t,x) + \mathcal{L}_\alpha h_m(t,x) + \langle Ax, D_x h_m(t,x)\rangle \;=\; -l_m(t,x); \\ h_m(T,x) \;=\; u_{T,m}(x)\end{cases} \tag{3.12}$$

where $l_m(t,x) := f_m\Big(t, x - \int_t^T e^{A(t-s)} F_m(s,\theta_{s,\tau}(\xi))\,ds\Big)$.
Since we are going to exploit reasonings in Fourier spaces, we need however to have integrability properties on the solution h_m. For this reason, we introduce now a family $\{\rho_R\}_{R>0}$ of smooth functions such that any ρ_R is equal to 1 in $B(0,R)$ and vanishes outside $B(0,R+1)$. We then denote for any $R>0$,

$$h_{m,R}(t,x) \;:=\; h_m(t,x)\rho_R(x).$$

It is then straightforward that $h_{m,R}$ solves

$$\begin{cases}\partial_t h_{m,R}(t,x) + \mathcal{L}_\alpha h_{m,R}(t,x) + \langle Ax, D_x h_{m,R}(t,x)\rangle \;=\; -\tilde{l}_{m,R}(t,x); \\ h_{m,R}(T,x) \;=\; g_{m,R}(x),\end{cases} \tag{3.13}$$

where $g_{m,R}(x) = u_{T,m}(x)\rho_R(x)$ and

$$\begin{aligned}\tilde{l}_{m,R}(t,x) \;=\; & \rho_R(x) l_m(t,x) + h_m(t,x)\mathcal{L}_\alpha \rho_R(x) \\ & + \int_{\mathbb{R}^d} \Big[h_m(t,x+By) - h_m(t,x)\Big]\Big[\rho_R(x+By) - \rho_R(x)\Big]\,\nu_\alpha(dy).\end{aligned}$$

Noticing now that $\tilde{l}_{m,R}$ is integrable with integrable Fourier transform, we can apply the Fourier transform in space to Equation (3.13) in order to write that

$$\begin{cases}\partial_t \widehat{h}_{m,R}(t,p) + \mathcal{F}_x\Big(\Big[\mathcal{L}_\alpha + \langle Ax, D_x\rangle\Big] h_{m,R}\Big)(t,p) \;=\; -\widehat{\tilde{l}}_{m,R}(t,p); \\ \widehat{h}_{m,R}(T,p) \;=\; \widehat{u_T}_{m,R}(p).\end{cases}$$

We remember in particular that the above operator $\mathcal{L}_\alpha + \langle Ax, D_x\rangle$ has an associated Lévy symbol $\Phi^{\mathrm{ou}}(p)$ and, following Section 3.3.2 in [App09], it holds that

$$\mathcal{F}_x\Big(\Big[\mathcal{L}_\alpha + \langle Ax, D_x\rangle\Big] h_{m,R}\Big)(t,p) \;=\; \Phi^{\mathrm{ou}}(p)\widehat{h}_{m,R}(t,p).$$

We can then use it to show that $\widehat{h}_{m,R}$ is a classical solution of the following equation:

$$\begin{cases}\partial_t \widehat{h}_{m,R}(t,p) + \Phi^{\mathrm{ou}}(p)\widehat{h}_{m,R}(t,p) \;=\; -\widehat{\tilde{l}}_{m,R}(t,p); \\ \widehat{h}_{m,R}(T,p) \;=\; \widehat{u_T}_{m,R}(p).\end{cases}$$

The above equation can be easily solved by integration in time, giving the following representation of $\widehat{h}_{m,R}(t,p)$:

$$\widehat{h}_{m,R}(t,p) \;=\; e^{(T-t)\Phi^{\mathrm{ou}}(p)}\widehat{u_T}_{m,R}(p) + \int_t^T e^{(s-t)\Phi^{\mathrm{ou}}(p)}\widehat{\tilde{l}}_{m,R}(s,p)\,ds.$$

In order to go back to $\tilde{v}_m^{\tau,\xi}$, we apply now the inverse Fourier transform to write that

$$h_{m,R}(t,x) \;=\; P^{\mathrm{ou}}_{T-t}g_{m,R}(x) + \int_t^T P^{\mathrm{ou}}_{s-t}\tilde{l}_{m,R}(s,x)\,ds,$$

remembering that $\{P_t^{\mathrm{ou}}\}_{t\geq 0}$ is the convolution Markov semigroup associated with the Ornstein-Uhlenbeck operator $\mathcal{L}_\alpha + \langle Ax, D_x\rangle$. Letting m go to ∞, it then follows immediately that $g_{m,R} \to u_{T,m}$, $h_{m,R} \to h_m$ and $\tilde{l}_{m,R} \to l_m$. A change of variable allows us to show the Duhamel representation, at least in the regularized setting:

$$\begin{aligned}\tilde{v}_m^{\tau,\xi}(t,y) =\; & P^{\mathrm{ou}}_{T-t}u_{T,m}\Big(y + \int_t^T e^{A(t-s)}F_m(s,\theta_{s,\tau}(\xi))\,ds\Big)\\ & + \int_t^T P^{\mathrm{ou}}_{s-t}f_m\Big(s, y + \int_t^s e^{A(t-u)}F_m(u,\theta_{\tau,u}(\xi))\,du\Big)\,ds.\end{aligned}$$

Letting m goes to zero and remembering that $\tilde{v}_m^{\tau,\xi} \to \tilde{v}^{\tau,\xi}$, $f_m \to f$, $u_{T,m} \to u_T$ and $F_m \to F$ in the right functional spaces, we can conclude that $\tilde{v}^{\tau,\xi} = \tilde{u}^{\tau,\xi}$. □

3.2 Expansion along the proxy

We are going to use now the "frozen" IPDE (3.2) in order to derive appropriate quantitative controls of a solution u of Equation (1.1). Up to now, the freezing parameters (τ,ξ) were set free but they will be later chosen appropriately depending on the control we aim to establish.

The main idea is to exploit the Duhamel formula (Proposition 3.4) for the proxy to expand any solution u of the original IPDE (1.1) along the proxy. To make things more precise, let u be a mild solution in $L\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ of IPDE (1.1). Mollifying if necessary, it is possible to construct three sequences $\{f_m\}_{m\in\mathbb{N}}$, $\{u_{T,m}\}_{m\in\mathbb{N}}$ and $\{F_m\}_{m\in\mathbb{N}}$ of bounded and smooth functions with bounded derivatives such that $f_m \to f$ in $L^\infty\Big(0,T;C^{\beta}_{b,d}(\mathbb{R}^{nd})\Big)$, $u_{T,m} \to u_T$ in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})$ and $\|F_m - F\|_H \to 0$. Since u is a mild solution of (1.1), we know that there exists a smooth sequence $\{u_m\}_{m\in\mathbb{N}}$ converging to u in $L\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ and such that for any fixed m in $\mathbb{N}$, u_m solves in a classical sense the "regularized" IPDE (2.25).

Exploiting now that F_m is bounded and smooth, we can define the "regularized" flow $\theta^m_{\cdot,\tau}(\xi)$ as the *unique* flow satisfying

$$\theta^m_{t,\tau}(\xi) \;=\; \xi + \int_\tau^t \Big[A\theta^m_{s,\tau}(\xi) + F_m(s,\theta^m_{s,\tau}(\xi))\Big]\,ds, \quad t \in [\tau,T]. \tag{3.14}$$

It is then easy to notice that u_m is also a classical solution in $L\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ of

$$\partial_t u_m(t,x) + \mathcal{L}_\alpha u_m(t,x) + \langle Ax + F_m(t,\theta^m_{t,\tau}(\xi)), D_x u_m(t,x)\rangle \;=\; -\Big[f_m(t,x) + R^{\tau,\xi}_m(s,x)\Big]$$

on $(0,T)\times\mathbb{R}^{nd}$ with terminal condition $u_{T,m}$. Above, we have denoted

$$R_m^{\tau,\xi}(t,x) \;:=\; \langle F_m(t,x) - F_m(t,\theta_{t,\tau}^m(\xi)), D_x u_m(t,x)\rangle. \tag{3.15}$$

Since clearly, $R_m^{\tau,\xi}$ is in $L^\infty\Big(0,T;C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd})\Big)$, we can use the Duhamel Formula (Proposition 3.4) for the proxy to write that

$$u_m(t,x) \;=\; \tilde{P}_{T,t}^{m,\tau,\xi}u_{T,m}(x) + \int_t^T \tilde{P}_{s,t}^{m,\tau,\xi}\Big[f_m(s,x) + R_m^{\tau,\xi}(s,x)\Big]\,ds, \quad (t,x)\in(0,T)\times\mathbb{R}^{nd},$$

where $\{\tilde{P}_{s,t}^{m,\tau,\xi}\}_{t\le s}$ is the semigroup associated with the operator

$$\mathcal{L}_\alpha + \langle Ax + F_m(t,\theta_{t,\tau}^m(\xi)), D_x\rangle.$$

The reasoning above is summarized in the following Duhamel-type formula that allows to expand any classical solution u_m of the "regularized" IPDE (2.25) along the "regularized frozen" proxy.

Proposition 3.5 (Duhamel Type Formula)**.** *Let (τ,ξ) a freezing couple in $[0,T]\times\mathbb{R}^{nd}$. Under [**A**], any classical solution u_m of the "regularized" IPDE* (2.25) *can be represented as*

$$u_m(t,x) \;=\; \tilde{u}_m^{\tau,\xi}(t,x) + \int_t^T \tilde{P}_{s,t}^{m,\tau,\xi}R^{m,\tau,\xi}(s,x)\,ds, \quad (t,x)\in(0,T)\times\mathbb{R}^{nd} \tag{3.16}$$

where $R_m^{\tau,\xi}$ is as in (3.15) *and $\tilde{u}_m^{\tau,\xi}$ is defined through the Duhamel Representation* (3.10) *with the "regularized" coefficients f_m, $u_{T,m}$.*

Thanks to the above representation (Equation (3.16)), we know that, since we have already shown the suitable control for the frozen solution $u_m^{\tau,\xi}$ (namely, Proposition 3.3 with $f_m, u_{T,m}$), the main term which remains to be investigated in order to show the Schauder Estimates (Theorem 2.3) is the remainder

$$\int_t^T \tilde{P}_{s,t}^{m,\tau,\xi}R_m^{\tau,\xi}(s,x)\,ds, \tag{3.17}$$

that represents exactly the error in the expansion along the proxy.

To be precise, we could have passed to the limit in Equation (3.16) in order to obtain a similar Duhamel-type formula for a mild solution u in $L^\infty\Big([0,T];C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd})\Big)$. However, a problem appears when trying to give a precise meaning at the limit for the remainder contribution (3.17). We already know that the limit exists point-wise by difference, but for our approach to work, we need to establish precise quantitative controls on this term. Such estimates could be obtained through duality techniques in Besov spaces (cf. Section 5.1) but only at the expense of fixing already the freezing couple as $(\tau,\xi)=(t,x)$. The drawback of this method is that it does not allow to differentiate Equation (3.16), which is needed to estimate $D_{x_1}u$.

In order to show the suitable estimates for Expression (3.17), we will need at first an additional constraint on the behaviour of the system. In particular, we will say to be under assumption [**A'**] when assumption [**A**] is considered and if moreover,

[ST] we assume to be in a small time interval, i.e. $T \leq 1$.

Under these stronger assumptions, we will then be able to show in Section 5 below that the following control holds:

Proposition 3.6 (A Priori Estimates)**.** *Let u be a mild solution of IPDE* (1.1) *in $L^\infty\Big([0,T]; C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$. Under [**A'**], there exists a constant $C \geq 1$ such that*

$$\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \leq C c_0^{\frac{\beta-\gamma_n}{\alpha}} \Big[\|f\|_{L^\infty(C^{\beta}_{b,d})} + \|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big] + C\Big(c_0^{\frac{\beta-\gamma_n}{\alpha}} \|F\|_H + c_0^{\frac{\alpha+\beta-1}{1+\alpha(n-1)}}\Big)\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}, \quad (3.18)$$

where $c_0 \in (0,1)$ is assumed to be fixed but chosen later.

We remark already that in the above control, the constants multiplying $\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}$ have to be small if one wants to derive the expected Schauder estimates. If c_0 is small enough, then $Cc_0^{\frac{\alpha+\beta-1}{1+\alpha(n-1)}}$ can be made smaller than $1/4$. Anyhow, for this chosen small c_0, the quantity $c_0^{\frac{\beta-\gamma_n}{\alpha}}$ becomes large and therefore, it needs to be balanced with $C\|F\|_H$. Namely, we can conclude if for instance, $Cc_0^{\frac{\beta-\gamma_n}{\alpha}}\|F\|_H < 1/4$ that implies in particular that $\|F\|_H$ has to be small with respect to c_0.

3.3 Conclusion of proof

In the first part of this section, we prove the Schauder estimates (Theorem 2.3) from the A Priori estimates (Proposition 3.6) through a suitable scaling procedure. Roughly speaking, the idea is to start from a general dynamics and then use the scaling procedure to make the Hölder norm $\|F\|_H$ small enough in order to make a *circular* argument work. Again, if c_0 and $\|F\|_H$ are small enough in Equation (3.18), the $L^\infty\Big(0,T; C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$-norm of u on the right-hand side can be absorbed by the left-hand one. Once the Schauder estimates (2.26) hold in the scaled dynamics, we will conclude going back to the original IPDE through the inverse scaling procedure, even if for a small final time horizon T.
The second part of the section focuses on showing how to drop the additional assumption [**A'**]. The key point here is to proceed through iteration up to an arbitrary, but finite, given time T thanks to the stability of a solution u in the space $L^\infty\Big(0,T; C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$.

Scaling Argument

Under [**A**], we start considering a mild solution u of IPDE (1.1) on $[0,T]$ for some final time $T \leq 1$ to be fixed later. For a scaling parameter λ in $(0,1]$ to be chosen later, we would like to analyze IPDE (1.1) under the change of variables

$$(t,x) \mapsto (\lambda t, \mathbb{T}_\lambda x), \quad (3.19)$$

where $\mathbb{T}_\lambda := \lambda^{1/\alpha}\mathbb{M}_\lambda$. Again, the scaling is performed accordingly to the homogeneity induced by the distance $\mathbf{d}_P$ in (2.14).

To this purpose, we firstly introduce the scaled solution u_λ defined by

$$u_\lambda(t,x) := u(\lambda t, \mathbb{T}_\lambda x).$$

It then follows immediately that u_λ is a mild solution of the following Equation:

$$\lambda^{-1}\partial_t u_\lambda(t,x) + \lambda^{-1}\mathcal{L}_\alpha u_\lambda + \Big\langle A\mathbb{T}_\lambda x + F(\lambda t, \mathbb{T}_\lambda x), \mathbb{T}_\lambda^{-1} D_x u_\lambda(t,x)\Big\rangle \\ = -f(\lambda t, \mathbb{T}_\lambda x) \text{ on } (0,T_\lambda)\times\mathbb{R}^{nd},$$

with terminal condition $u_\lambda(T_\lambda, x) = u_T(\mathbb{T}_\lambda x)$, where $T_\lambda := T/\lambda$. Since we want the scaled dynamics to satisfy assumption $(\mathbf{A}')$, we choose now T so that $T_\lambda \leq 1$. It is important to notice that this is possible since we assumed λ to be fixed, even if we have not chosen it yet. Denoting now

$$\begin{aligned} f_\lambda(t,x) &:= \lambda f(\lambda t, \mathbb{T}_\lambda x);\\ u_{T,\lambda}(x) &:= u_T(\mathbb{T}_\lambda x);\\ A_\lambda &:= \lambda \mathbb{T}_\lambda^{-1} A \mathbb{T}_\lambda;\\ F_\lambda(t,x) &:= \lambda \mathbb{T}_\lambda^{-1} F(\lambda t, \mathbb{T}_\lambda x), \end{aligned}$$

we can rewrite the scaled dynamics as:

$$\begin{cases} \partial_t u_\lambda(t,x) + \Big\langle A_\lambda x + F_\lambda(t,x), D_x u_\lambda(t,x)\Big\rangle + \mathcal{L}_\alpha u_\lambda(t,x) = -f_\lambda(t,x);\\ u_\lambda(T_\lambda, x) = u_{T,\lambda}(x). \end{cases} \tag{3.20}$$

To continue, we need the following lemma that shows how the scaling procedure reflects on the norms of the coefficients. Recalling Equation (2.24) for the definition of $\|\cdot\|_H$, a direct calculation on the norms leads to the following result:

Lemma 3.7 (Scaling Homogeneity of Norms)**.** *Under [**A**], it holds that*

$$\begin{aligned} \|F_\lambda\|_H &= \lambda^{\beta/\alpha}\|F\|_H;\\ \lambda^{\frac{\alpha+\beta}{\alpha}}\|f\|_{L^\infty(C^\beta_{b,d})} &\leq \|f_\lambda\|_{L^\infty(C^\beta_{b,d})} \leq \|f\|_{L^\infty(C^\beta_{b,d})};\\ \lambda^{\frac{\alpha+\beta}{\alpha}}\|u_T\|_{C^{\alpha+\beta}_{b,d}} &\leq \|u_{T,\lambda}\|_{C^{\alpha+\beta}_{b,d}} \leq \|u_T\|_{C^{\alpha+\beta}_{b,d}};\\ \lambda^{\frac{\alpha+\beta}{\alpha}}\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})} &\leq \|u_\lambda\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \leq \|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}. \end{aligned} \tag{3.21}$$

Since the scaled dynamics in (3.20) satisfies assumption [**A'**], we know from Proposition 3.6 that the scaled solution u_λ satisfies the a priori Estimates (3.18):

$$\|u_\lambda\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \leq C c_0^{\frac{\beta-\gamma_n}{\alpha}}\Big[\|f_\lambda\|_{L^\infty(C^\beta_{b,d})} + \|u_{T,\lambda}\|_{C^{\alpha+\beta}_{b,d}}\Big] \\ + C\Big(c_0^{\frac{\beta-\gamma_n}{\alpha}}\|F_\lambda\|_H + c_0^{\frac{\alpha+\beta-1}{1+\alpha(n-1)}}\Big)\|u_\lambda\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \tag{3.22}$$

for some constant c_0 in $(0,1]$ to be chosen later.
We would like now to exploit a circular argument in order to bring to the left-hand side

of Equation (3.22) the term involving u_λ on the right-hand one. To do that, we need to choose properly λ and c_0 in order to have

$$C\Big(c_0^{\frac{\beta-\gamma_n}{\alpha}}\|F_\lambda\|_H + c_0^{\frac{\alpha+\beta-1}{1+\alpha(n-1)}}\Big) \,<\, 1.$$

This is true if, for example, we choose firstly c_0 such that

$$Cc_0^{\frac{\alpha+\beta-1}{1+\alpha(n-1)}} \,=\, \frac{1}{4}$$

and fixed c_0, we choose λ so that

$$Cc_0^{\frac{\beta-\gamma_n}{\alpha}}\lambda^{\beta/\alpha}\|F\|_H \,=\, Cc_0^{\frac{\beta-\gamma_n}{\alpha}}\|F_\lambda\|_H \,=\, \frac{1}{4}.$$

With this choice, it then follows from Equation (3.22) that

$$\|u_\lambda\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \,\leq\, 2Cc_0^{\frac{\beta-\gamma_n}{\alpha}}\Big[\|f_\lambda\|_{L^\infty(C^{\beta}_{b,d})} + \|u_{T,\lambda}\|_{C^{\alpha+\beta}_{b,d}}\Big].$$

We can finally use Lemma 3.7 to go back to the original dynamics and write that

$$\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \,\leq\, \lambda^{-\frac{\alpha+\beta}{\alpha}}\|u_\lambda\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \,\leq\, \overline{C}\Big[\|f\|_{L^\infty(C^{\beta}_{b,d})} + \|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big]$$

for some constant $\overline{C} > 0$ defined by

$$\overline{C} \,:=\, 2\lambda^{-\frac{\alpha+\beta}{\alpha}}Cc_0^{\frac{\beta-\gamma_n}{\alpha}}.$$

Schauder Estimates for General Time

Up to this point, we have assumed to be in a small enough final time horizon (i.e. $T \leq 1$) to let our procedure work. We are going now to extend the Schauder estimates (Equation (2.26)) to an arbitrary but fixed final time $T_0 > 0$. Our proof will consist essentially in a backward iterative procedure through a chain of identical differential dynamics on different, small enough, time intervals. We recall indeed that the Schauder estimates precisely provide a stability result in the chosen functional space.

Proposition 3.8. *Under [**A**], let $T_0 > T$ and u a mild solution in $L^\infty\Big(0, T_0; C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ of IPDE (1.1) on $[0, T_0]$ that satisfies the Schauder Estimates (2.26) on $[0, T]$. Then, there exists a constant $C_0 := C_0(T_0)$ such that*

$$\|u\|_{L^\infty(0,T_0;C^{\alpha+\beta}_{b,d})} \,\leq\, C_0\Big[\|f\|_{L^\infty(0,T_0;C^{\beta}_{b,d})} + \|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big].$$

Proof. Fixed $N = \lceil \frac{T_0}{T} \rceil$, we are going to consider a system of N Cauchy problems:

$$\begin{cases} \partial_t u_k(t,x) + \Big\langle Ax + F(t,x), D_x u_k(t,x)\Big\rangle + \mathcal{L}_\alpha u_k(t,x) \,=\, -f(t,x); \\ u_k((1-\frac{k-1}{N})T_0, x) \,=\, u_{k-1}((1-\frac{k-1}{N})T_0, x), \end{cases}$$

on $((1-\frac{k}{N})T_0,(1-\frac{k-1}{N})T_0)\times\mathbb{R}^{nd}$, for $k=1,\dots,N$ with the notation that $u_0(T_0,x)=u_T(x)$. Reasoning iteratively, we find that any mild solution of IPDE (1.1) on $[0,T_0]$ is also a mild solution of any of the equations of the system. Moreover, since any solution u_k is defined on $[(1-\frac{k}{N})T_0,(1-\frac{k-1}{N})T_0]$ and

$$(1-\frac{k-1}{N})T_0-(1-\frac{k}{N})T_0 \;=\; \frac{k}{N}T_0-\frac{k-1}{N}T_0 \;=\; \frac{1}{N}T_0 \;\le\; T,$$

the Schauder estimates (Equation (2.26)) hold for any solution u_k with terminal condition $u_{k-1}((1-\frac{k-1}{N})T_0,\cdot)$. In particular,

$$\begin{aligned}
&\|u_k\|_{L^\infty((1-\frac{k}{N})T_0,(1-\frac{k-1}{N})T_0;C^{\alpha+\beta}_{b,d})}\\
&\qquad\le\; C\Big[\|f\|_{L^\infty((1-\frac{k}{N})T_0,(1-\frac{k-1}{N})T_0;C^{\beta}_{b,d})}+\|u_{k-1}((1-\frac{k-1}{N})T_0,\cdot)\|_{C^{\alpha+\beta}_{b,d}}\Big]\\
&\qquad\le\; C^2\Big[\|f\|_{L^\infty((1-\frac{k}{N})T_0,(1-\frac{k-1}{N})T_0;C^{\beta}_{b,d})}+\|f\|_{L^\infty((1-\frac{k-1}{N})T_0,(1-\frac{k-2}{N})T_0;C^{\beta}_{b,d})}\\
&\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad+\|u_{k-2}((1-\frac{k-2}{N})T_0,\cdot)\|_{C^{\alpha+\beta}_{b,d}}\Big]\\
&\qquad\le\; C^2\Big[\|f\|_{L^\infty((1-\frac{k}{N})T_0,(1-\frac{k-2}{N})T_0;C^{\beta}_{b,d})}+\|u_{k-2}((1-\frac{k-2}{N})T_0,\cdot)\|_{C^{\alpha+\beta}_{b,d}}\Big],
\end{aligned}$$

exploiting that u_{k-1} satisfies the Schauder estimates with source f and terminal condition $u_{k-2}((1-\frac{k-2}{N})T_0,\cdot)$. Applying the same procedure recursively, we finally find that

$$\|u_k\|_{L^\infty((1-\frac{k}{N})T_0,(1-\frac{k-1}{N})T_0;C^{\alpha+\beta}_{b,d})}\;\le\; C^k\Big[\|f\|_{L^\infty((1-\frac{k}{N})T_0,T_0;C^{\beta}_{b,d})}+\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big].$$

Hence,

$$\|u\|_{L^\infty(0,T_0;C^{\alpha+\beta}_{b,d})}\;\le\; C^N\Big[\|f\|_{L^\infty(0,T_0;C^{\beta}_{b,d})}+\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big]$$

and we have concluded the proof. □

4 Schauder Estimates for the Proxy

The aim of this section is to show how to properly control a solution $\tilde{u}^{\tau,\xi}$ of the "frozen" IPDE (3.2) in order to prove the Schauder estimates (Proposition 3.3) for the proxy. We recall the definition of $\tilde{u}^{\tau,\xi}$ through the Duhamel Representation (3.10). Namely, for any freezing couple (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$, it holds that

$$\tilde{u}^{\tau,\xi}(t,x)\;=\;\tilde{P}^{\tau,\xi}_{T,t}u_T(x)+\tilde{G}^{\tau,\xi}_{T,t}f(t,x) \tag{4.1}$$

where we have denoted for simplicity with $\{\tilde{G}^{\tau,\xi}_{r,v}\}_{t>v\ge0}$ the family of Green kernels associated with the frozen density $\tilde{p}^{\tau,\xi}$. More in details, we have for any $v<r$ in $[0,T]$ that

$$\tilde{G}^{\tau,\xi}_{r,v}f(t,x)\;:=\;\int_v^r\int_{\mathbb{R}^{nd}}\tilde{p}^{\tau,\xi}(t,s,x,y)f(s,y)\,dy\,ds. \tag{4.2}$$

We can then differentiate the above equation with respect to x_1 in order to obtain an analogous Duhamel type representation for the derivative $D_{x_1}\tilde{u}^{\tau,\xi}$:

$$D_{x_1}\tilde{u}^{\tau,\xi}(t,x) \;=\; D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x) + D_{x_1}\tilde{G}^{\tau,\xi}_{T,t}f(t,x). \tag{4.3}$$

It is then clear that in order to control $\tilde{u}^{\tau,\xi}(t,x)$ in the norm $\|\cdot\|_{L^\infty(C^{\alpha+\beta}_{b,d})}$, we can analyze separately the contributions appearing from the frozen semigroup $\tilde{P}^{\tau,\xi}_{T,t}u_T(x)$ and those from the frozen Green kernel $\tilde{G}^{\tau,\xi}_{T,t}f(t,x)$.

4.1 First Besov control

We focus for the moment on the contribution in the Duhamel Representation (4.1) associated with the source u_T that is, as it will be seen, the more delicate to treat. In the non-degenerate setting (i.e. with respect to x_1), it precisely write:

$$D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x) \;=\; \int_{\mathbb{R}^{nd}} D_{x_1}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)\,dy.$$

Looking at the particular structure of $\tilde{p}^{\tau,\xi}$ (cf. Equation (3.3)), it can be seen from Lemma 2.1 that

Lemma 4.1. *Let i in $[\![1,n]\!]$. Then, there exist constants $\{C_j\}_{j\in[\![i,n]\!]}$ such that*

$$D_{x_i}\tilde{p}^{\tau,\xi}(t,s,x,y) \;=\; \sum_{j=i}^{n} C_j(s-t)^{j-i}D_{y_j}\tilde{p}^{\tau,\xi}(t,s,x,y) \tag{4.4}$$

for any $t<s$ in $[0,T]$, any x,y in $\mathbb{R}^{nd}$ and any freezing couple (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$.

We can now use Wquation (4.4) to rewrite $D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x)$ as

$$\begin{aligned}\left|D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x)\right| \;&=\; \left|\int_{\mathbb{R}^{nd}} D_{x_1}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)\,dy\right| \\ &\le\; C\sum_{j=1}^{n}(s-t)^{j-1}\left|\int_{\mathbb{R}^{nd}} D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)\,dy\right|.\end{aligned} \tag{4.5}$$

Remembering that u_T is in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})$ for $\alpha+\beta>1$ by hypothesis, we know that it is differentiable with respect to the first (non-degenerate) variable x_1. Then, the above expression can be controlled easily for $j=1$ as

$$\begin{aligned}\left|\int_{\mathbb{R}^{nd}} D_{y_1}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)\,dy\right| \;&=\; \left|\int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t,T,x,y)D_{y_1}u_T(y)\,dy\right| \\ &\le\; \|D_{y_1}u_T\|_{L^\infty} \\ &\le\; \|u_T\|_{C^{\alpha+\beta}_{b,d}},\end{aligned}$$

using integration by parts formula. We can then focus on the degenerate components in (4.5), i.e.

$$\left|\int_{\mathbb{R}^{nd}} D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)\,dy\right| \tag{4.6}$$

for some $j > 1$. Since u_T is not differentiable with respect to y_j if $j > 1$, we cannot apply the same reasoning above but we will need a more subtle control. Our main idea will be to use the duality in Besov spaces to derive bounds for Expression (4.6). Namely, we introduce for a given y in $\mathbb{R}^d$,

$$y_{\setminus j} := (y_1, \dots, y_{j-1}, y_{j+1}, \dots, y_n) \in \mathbb{R}^{(n-1)d}.$$

With this definition at hand, we then denote for any function ϕ on $\mathbb{R}^{nd}$, the function $\phi(y_{\setminus j}, \cdot)$ on $\mathbb{R}^d$ with a slight abuse of notation as

$$\phi(y_{\setminus j}, z) := \phi(y_1, \dots, y_{j-1}, z, y_{j+1}, \dots, y_n). \tag{4.7}$$

The key point now is to control the Hölder modulus of $u_T(y_{\setminus j}, \cdot)$ on $\mathbb{R}^d$, uniformly in $y_{\setminus j} \in \mathbb{R}^{(n-1)d}$. To do so, we will need the identification $C_b^{\alpha_j+\beta_j}(\mathbb{R}^d) = B_{\infty,\infty}^{\alpha_j+\beta_j}(\mathbb{R}^d)$ with the usual notations for the Besov spaces.

We recall now some useful definitions/characterizations about Besov spaces $B_{p,q}^{\tilde{\gamma}}(\mathbb{R}^d)$. For a more detailed analysis of this argument, we suggest the reader to see Section 2.6.4 of Triebel [Tri92]. For $\tilde{\gamma}$ in $(0,1)$, q, p in $(0, +\infty]$, we define the Besov space of indexes $(\tilde{\gamma}, p, q)$ on $\mathbb{R}^d$ as:

$$B_{p,q}^{\tilde{\gamma}}(\mathbb{R}^d) := \{f \in \mathcal{S}'(\mathbb{R}^d) \colon \|f\|_{\mathcal{H}_{p,q}^{\tilde{\gamma}}} < +\infty\},$$

where $\mathcal{S}(\mathbb{R}^d)$ denotes the Schwartz class on $\mathbb{R}^d$ and

$$\|f\|_{\mathcal{H}_{p,q}^{\tilde{\gamma}}} := \|(\phi_0 \hat{f})^\vee\|_{L^p} + \left(\int_0^1 v^{-\frac{\tilde{\gamma}}{\alpha}} \|\partial_v p_h(v, \cdot) * f\|_{L^p}^q \, dv\right)^{\frac{1}{q}}, \tag{4.8}$$

with ϕ_0 a function in $C_0^\infty(\mathbb{R}^d)$ such that $\phi_0(0) \neq 0$ and p_h the isotropic α-stable heat kernel on $\mathbb{R}^d$, i.e. the stable density on $\mathbb{R}^d$ whose Lévy symbol is equivalent to $|\lambda|^\alpha$.
We point out that the quantities in (4.8) are well-defined for any $q \neq +\infty$. The modifications for $q = +\infty$ are obvious and can be written passing to the limit. The previous definition of $B_{p,q}^{\tilde{\gamma}}(\mathbb{R}^d)$ is known as the stable thermic characterization of Besov spaces and it is particularly adapted to our framework. By a little abuse of notation, we will write $\|f\|_{B_{p,q}^{\tilde{\gamma}}} := \|f\|_{\mathcal{H}_{p,q}^{\tilde{\gamma}}}$ when this quantity is finite.

For the heat-kernel p_h, it is possible to show an improvement of the smoothing effect (cf. Equation (2.13)), due essentially to its better decay at infinity. Namely, we are no more bounded to the condition $\gamma < \alpha$ but we can integrate up to an order γ strictly smaller than $1 + \alpha$.

Lemma 4.2 (Smoothing Effect of the Isotropic Stable Heat-Kernel)**.** *Let l be in $\{1, 2\}$ and γ in $[0, 1 + \alpha)$. Then, there exists a positive constant $C := C(\gamma)$ such that*

$$\int_{\mathbb{R}^d} |y|^\gamma |\partial_v D_y^l p_h(v, y)| \, dy \leq C t^{\frac{\gamma - l}{\alpha} - 1}. \tag{4.9}$$

A proof of the above result can be derived using the estimates of Kolokoltsov [Kol00] (see also [BJ07]).

As already indicated before, it can be seen from the Thermic Characterization (4.8) that

$$C_b^{\tilde{\gamma}}(\mathbb{R}^d) \,=\, B^{\tilde{\gamma}}_{\infty,\infty}(\mathbb{R}^d). \tag{4.10}$$

Moreover, it is well known (see for example Proposition 3.6 in [LR02]) that $B^{\tilde{\gamma}}_{\infty,\infty}(\mathbb{R}^d)$ and $B^{-\tilde{\gamma}}_{1,1}(\mathbb{R}^d)$ are in duality. Namely, it holds

$$\Big|\int_{\mathbb{R}^d} fg\,dx\Big| \,\leq\, C\|f\|_{B^{\tilde{\gamma}}_{\infty,\infty}}\|u_T\|_{B^{-\tilde{\gamma}}_{1,1}}, \tag{4.11}$$

for any f in $B^{\tilde{\gamma}}_{\infty,\infty}(\mathbb{R}^d)$ and any u_T in $B^{-\tilde{\gamma}}_{1,1}(\mathbb{R}^d)$.

With these definitions and properties at hand, we can now go back at Expression (4.6) to write that

$$\begin{aligned}\Big|\int_{\mathbb{R}^{nd}} D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)\,dy\Big| \,&\leq\, \int_{\mathbb{R}^{(n-1)d}}\Big|D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)dy_j\Big|dy_{\smallsetminus j}\\ &\leq\, \int_{\mathbb{R}^{(n-1)d}}\Big\|D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y_{\smallsetminus j},\cdot)\Big\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}\Big\|u_T(y_{\smallsetminus j},\cdot)\Big\|_{B^{\alpha_j+\beta_j}_{\infty,\infty}}dy_{\smallsetminus j}\\ &\leq\, \|u_T\|_{C^{\alpha+\beta}_{b,d}}\int_{\mathbb{R}^{(n-1)d}}\Big\|D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y_{\smallsetminus j},\cdot)\Big\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}\,dy_{\smallsetminus j}.\end{aligned}$$

In order to control the above quantities, we will then need a control on the integral of the Besov norms of the derivatives of the proxy. Since however an additional derivative with respect to x_1 will often appear, for example in Equation (4.24) below, we state the following result in a more general way.

Lemma 4.3 (First Besov Control)**.** *Let j be in $[\![2,n]\!]$ and $l\in\{0,1\}$. Under [**A**], there exists a constant $C:=C(j,l)$ such that*

$$\int_{\mathbb{R}^{(n-1)d}}\Big\|D_{y_j}D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\cdot)\Big\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}\,dy_{\smallsetminus j} \,\leq\, C(s-t)^{\frac{\alpha+\beta}{\alpha}-\frac{1}{\alpha_j}-\frac{l}{\alpha}},$$

for any $t<s$ in $[0,T]$, any x in $\mathbb{R}^{nd}$ and any frozen couple (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$.

Proof. To control the Besov norm in $B^{-(\alpha_j+\beta_j)}_{1,1}(\mathbb{R}^d)$, we are going to use the Thermic Characterization (4.8) with $\tilde{\gamma}=-(\alpha_j+\beta_j)$. We start considering the second term in the characterization, i.e.

$$\int_0^1 v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}\partial_v p_h(v,z-y_j)D_{y_j}D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y)\,dy_j\Big|\,dzdv.$$

Fixed a constant $\delta_j\geq 1$ to be chosen later, we split the integral with respect to v in two components:

$$\begin{aligned}&\|D_{y_j}D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\cdot)\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}\\ &\quad=\int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}\partial_v p_h(v,z-y_j)D_{y_j}D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y)\,dy_j\Big|\,dzdv\\ &\qquad+\int_{(s-t)^{\delta_j}}^1 v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}\partial_v p_h(v,z-y_j)D_{y_j}D^l_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y)\,dy_j\Big|\,dzdv\\ &\quad=:\,\Big(I_1+I_2\Big)(y_{\smallsetminus j}).\end{aligned}$$

The second component I_2 has no time-singularity and can be easily controlled by

$$I_2(y_{\setminus j}) \;=\; \int_{(s-t)^{\delta_j}}^1 v^{\frac{\alpha_j+\beta_j}{\alpha}} \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} D_z \partial_v p_h(v, z-y_j) \otimes D^l_{x_1} \tilde{p}^{\tau,\xi}(t,s,x,y)\, dy_j \Big|\, dz dv,$$

using integration by parts formula and noticing that $D_{y_j} p_h(v, z-y_j) = -D_z p_h(v, z-y_j)$. Then,

$$I_2(y_{\setminus j}) \;\leq\; \int_{(s-t)^{\delta_j}}^1 v^{\frac{\alpha_j+\beta_j}{\alpha}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |D_z \partial_v p_h(v, z-y_j)|\, |D^l_{x_1} \tilde{p}^{\tau,\xi}(t,s,x,y)|\, dy_j\, dz dv.$$

We can then use Fubini theorem to separate the integrals and apply the smoothing effect of the heat-kernel p_h (Lemma 4.2) to show that

$$\begin{aligned} I_2(y_{\setminus j}) \;&\leq\; \int_{(s-t)^{\delta_j}}^1 v^{\frac{\alpha_j+\beta_j}{\alpha}} \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} |D_z \partial_v p_h(v, z-y_j)|\, dz \Big) |D^l_{x_1} \tilde{p}^{\tau,\xi}(t,s,x,y)|\, dy_j dv \\ &\leq\; C \Big(\int_{(s-t)^{\delta_j}}^1 v^{\frac{\alpha_j+\beta_j-1}{\alpha}-1}\, dv \Big) \Big(\int_{\mathbb{R}^d} |D^l_{x_1} \tilde{p}^{\tau,\xi}(t,s,x,y)|\, dy_j \Big) \\ &\leq\; C(s-t)^{\frac{\delta_j(\alpha_j+\beta_j-1)}{\alpha}} \int_{\mathbb{R}^d} |D^l_{x_1} \tilde{p}^{\tau,\xi}(t,s,x,y)|\, dy_j. \end{aligned}$$

Using the smoothing effect (Equation (3.6)) of the frozen density $\tilde{p}^{\tau,\xi}$, we have thus found that

$$\begin{aligned} \int_{\mathbb{R}^{(n-1)d}} I_2(y_{\setminus j})\, dy_{\setminus j} \;&\leq\; (s-t)^{\frac{\delta_j(\alpha_j+\beta_j-1)}{\alpha}} \int_{\mathbb{R}^{nd}} |D^l_{x_1} \tilde{p}^{\tau,\xi}(t,s,x,y)|\, dy \\ &\leq\; C(s-t)^{\frac{\delta_j(\alpha_j+\beta_j-1)-l}{\alpha}}. \end{aligned} \tag{4.12}$$

On the other hand, the term I_1 needs a more delicate treatment in order to avoid time-integrability problems. We start using a cancellation argument with respect to the derivative $\partial_v p_h$ of the heat-kernel to rewrite I_1 as

$$\begin{aligned} I_1(y_{\setminus j}) \;&=\; \int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j+\beta_j}{\alpha}} \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} \partial_v p_h(v, z-y_j) \\ &\qquad \times \Big[D_{y_j} D^l_{x_1} \tilde{p}^{\tau,\xi}(t,s,x,y) - D_{y_j} D^l_{x_1} \tilde{p}^{\tau,\xi}(t,s,x,y_{\setminus j},z) \Big]\, dy_j \Big|\, dz dv \\ &=\; \int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j+\beta_j}{\alpha}} \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} D_z \partial_v p_h(v, z-y_j) \\ &\qquad \otimes \Big[D^l_{x_1} \tilde{p}^{\tau,\xi}(t,s,x,y) - D^l_{x_1} \tilde{p}^{\tau,\xi}(t,s,x,y_{\setminus j},z) \Big]\, dy_j \Big|\, dz dv, \end{aligned}$$

where in the second passage we used again integration by parts formula to move the derivative to p_h and the equality $D_{y_j} p_h(v, z-y_j) = -D_z p_h(v, z-y_j)$. We can then apply a Taylor expansion with respect to variable y_j in order to write that

$$\begin{aligned} I_1(y_{\setminus j}) \;&=\; \int_0^{(s-t)^{\delta_i}} v^{\frac{\alpha_j+\beta_j}{\alpha}} \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} D_z \partial_v p_h(v, z-y_j) \\ &\qquad \times \int_0^1 D_{y_j} D^l_{x_1} \tilde{p}^{\tau,\xi}(t,s,x,y_{\setminus j}, y_j + \lambda(z-y_j)) \cdot (z-y_j)\, d\mu dy_j \Big|\, dz dv \\ &\leq\; \int_0^{(s-t)^{\delta_i}} v^{\frac{\alpha_j+\beta_j}{\alpha}} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_0^1 |D_z \partial_v p_h(v, z-y_j)| \\ &\qquad \times |D_{y_j} D^l_{x_1} \tilde{p}^{\tau,\xi}(t,s,x,y_{\setminus j}, y_j + \lambda(z-y_j))|\, |z-y_j|\, d\lambda dy_j dz dv. \end{aligned}$$

We can then use the Fubini theorem and the changes of variables $\tilde{z} = z - y_j$ (fixed y_j) and $\tilde{y}_j = y_j + \lambda\tilde{z}$ (considering $\tilde{z}$ and λ fixed) to separate the integrals so that

$$I_1(y_{\smallsetminus j}) \leq \int_0^{(s-t)^{\delta_i}} v^{\frac{\alpha_j+\beta_j}{\alpha}} \Big(\int_{\mathbb{R}^d} |D_z \partial_v p_h(v,\tilde{z})|\, |\tilde{z}|\, dz \Big) \\ \times \Big(\int_{\mathbb{R}^d} |D_{y_j} D^l_{x_1} \tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\tilde{y}_j)|\, dy_j \Big)\, dv.$$

The smoothing effect of the heat-kernel p_h (Lemma 4.2) allows now to control the first term:

$$\begin{aligned} I_1(y_{\smallsetminus j}) &\leq C\Big(\int_0^{(s-t)^{\delta_i}} v^{\frac{\alpha_j+\beta_j-1}{\alpha}}\, dv \Big) \Big(\int_{\mathbb{R}^d} |D_{y_j} D^l_{x_1} \tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j}, z+\lambda(y_j-z))|\, dy_j \Big) \\ &\leq C(s-t)^{\delta_j \frac{\alpha_j+\beta_j}{\alpha}} \int_{\mathbb{R}^d} |D_{y_j} D^l_{x_1} \tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j}, z+\lambda(y_j-z))|\, dy_j. \end{aligned}$$

It then follows using the smoothing effect of the frozen semigroup (Lemma 3.2) that

$$\begin{aligned} \int_{\mathbb{R}^{(n-1)d}} I_1(y_{\smallsetminus j})\, dy_{\smallsetminus j} &\leq C(s-t)^{\delta_j \frac{\alpha_j+\beta_j}{\alpha}} \int_{\mathbb{R}^{nd}} |D_{y_j} D^l_{x_1} \tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j}, z+\lambda(y_j-z))|\, dy \\ &\leq C(s-t)^{\delta_j \frac{\alpha_j+\beta_j}{\alpha} - \frac{l}{\alpha} - \frac{l}{\alpha_j}}. \end{aligned} \tag{4.13}$$

Going back to equations (4.12) and (4.13), we notice that we need δ_j to be such that

$$\delta_j \Big[\frac{\alpha_j+\beta_j}{\alpha} \Big] = \frac{\alpha+\beta}{\alpha} \quad \text{and} \quad \delta_j \Big[\frac{\alpha_j+\beta_j-1}{\alpha} \Big] = \frac{\alpha+\beta}{\alpha} - \frac{1}{\alpha_j}.$$

Recalling Equation (2.27) for the relative definitions, we can thus conclude choosing

$$\delta_j = (\alpha+\beta)/(\alpha_j+\beta_j) = 1 + \alpha(j-1).$$

Reproducing the previous computations, we can also write for a test function in ϕ_0 in $C_0^\infty(\mathbb{R}^d)$,

$$\begin{aligned} &\int_{\mathbb{R}^{(n-1)d}} \Big\| \Big(\phi_0 \Big(D_{y_j} D^l_{x_1} \tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\cdot) \Big)\hat{} \Big)^\vee \Big\|_{L^1} dy_{\smallsetminus j} \\ &\qquad = \int_{\mathbb{R}^{(n-1)d}} \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} D_{y_j} \hat{\phi}_0(z-y_j) \cdot D^l_{x_1} \tilde{p}^{\tau,\xi}(t,s,x,y)\, dy_j \Big|\, dz dy_{\smallsetminus j} \\ &\qquad \leq C \int_{\mathbb{R}^{nd}} |D^l_{x_1} \tilde{p}^{\tau,\xi}(t,s,x,y)|\, dy \\ &\qquad \leq C(s-t)^{-\frac{l}{\alpha}}. \end{aligned}$$

The proof is thus concluded. □

4.2 Proof of Proposition 3.3

Thanks to the first Besov control (Lemma 4.3), we are now ready to prove the Schauder estimates for the proxy (Proposition 3.3). Such a proof will be divided in three parts: the estimates for the supremum norms of the solution and its non-degenerate gradient are stated in Lemma 4.4 while the controls of the Hölder moduli of the solution and its gradient with respect to the non-degenerate variable are given in Lemmas 4.5 and 4.6, respectively.

Lemma 4.4. *(Controls on Supremum Norm) Under [**A**], there exists a constant $C := C(T) \geq 1$ such that for any freezing couple (τ, ξ) in $[0,T] \times \mathbb{R}^{nd}$, any t in $[0,T]$ and any x in $\mathbb{R}^{nd}$,*

$$|\tilde{u}^{\tau,\xi}(t,x)| + |D_{x_1}\tilde{u}^{\tau,\xi}(t,x)| \leq C\Big[\|f\|_{L^\infty(C^\beta_{b,d})} + \|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big].$$

Proof. We start noticing that $\tilde{P}^{\tau,\xi}_{T,t}u_T(x)$ and $\tilde{G}^{\tau,\xi}_{T,t}f(t,x)$ can be easily bounded using the supremum norm of f and u_T, respectively.
Moreover, we can use the estimates on the frozen semigroup (Equation (3.7)) to control $D_{x_1}\tilde{G}^{\tau,\xi}_{T,t}f(t,x)$. Indeed,

$$\begin{aligned}\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{T,t}f(t,x)\Big| &\leq \int_t^T \Big|D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}f(s,x)\Big|\,ds \\ &\leq C(T-t)^{\frac{\alpha+\beta-1}{\alpha}}\|f\|_{L^\infty(C^\beta_{b,d})} \\ &\leq CT^{\frac{\alpha+\beta-1}{\alpha}}\|f\|_{L^\infty(C^\beta_{b,d})},\end{aligned}$$

remembering in the last inequality that $\alpha+\beta-1>0$ by hypothesis [**P**].
It remains to control $D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x)$. As done in the previous Sub-section 4.1, we start using the scaling lemma 4.1 to write that

$$\begin{aligned}\Big|D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x)\Big| &= \Big|\int_{\mathbb{R}^{nd}} D_{x_1}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)\,dy\Big| \\ &\leq C\sum_{j=1}^n (T-t)^{j-1}\Big|\int_{\mathbb{R}^{nd}} D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)\,dy\Big| \\ &=: C\sum_{j=1}^n (T-t)^{j-1}J_j.\end{aligned}$$

Since u_T is differentiable in the first, non-degenerate variable x_1, the contribution J_1 can be easily bounded using integration by parts formula:

$$J_1 = \Big|\int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t,T,x,y)D_{y_1}u_T(y)\,dy\Big| \leq \|D_{y_1}u_T\|_{L^\infty} \leq \|u_T\|_{C^{\alpha+\beta}_{b,d}}. \tag{4.14}$$

To control the other terms J_j for $j>1$, we use instead the duality in Besov spaces (Equation (4.11)) and Identification (4.10), so that

$$\begin{aligned}J_j &\leq C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\int_{\mathbb{R}^{(n-1)d}}\|D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y_{\setminus j},\cdot)\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}\,dy_{\setminus j} \\ &\leq C\|u_T\|_{C^{\alpha+\beta}_{b,d}}(T-t)^{\frac{\alpha+\beta}{\alpha}-\frac{1}{\alpha_j}},\end{aligned} \tag{4.15}$$

where in the last inequality we applied the first Besov control (Lemma 4.3).
Looking back at Equations (4.14)-(4.15), it finally holds that

$$\begin{aligned}\Big|D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x)\Big| &\leq C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big(1+\sum_{j=2}^n (T-t)^{j-1}(T-t)^{\frac{\alpha+\beta}{\alpha}-\frac{1}{\alpha_j}}\Big) \\ &\leq C\Big(1+T^{\frac{\alpha+\beta-1}{\alpha}}\Big)\|u_T\|_{C^{\alpha+\beta}_{b,d}},\end{aligned}$$

where in the last passage we used again that $\alpha+\beta-1>0$ by hypothesis [**P**]. □

Before starting with the calculations on the Hölder modulus, we will need to distinguish two cases. Fixed (t, x, x') in $[0, T] \times \mathbb{R}^{2nd}$, we will say that the *off-diagonal regime* holds if $T - t \leq c_0 \mathbf{d}^\alpha(x, x')$ for a constant c_0 to be specified but meant to be smaller than 1. This means in particular that the spatial distance is larger than the characteristic time-scale up to the prescribed constant c_0 which will be useful further on in the computations for a circular argument.
On the other hand, we will say that a *global diagonal regime* is in force if $T - t \geq c_0 \mathbf{d}^\alpha(x, x')$ and the spatial points are instead closer than the typical time-scale magnitude. In particular, when a time integration is involved (for example, in the control of the frozen Green kernel), the same two regime appears even in a local base. Considering a variable s in $[t, T]$, there are again a *local off-diagonal regime* if $s - t \leq c_0 \mathbf{d}^\alpha(x, x')$ and a *local diagonal regime* when $s - t \geq c_0 \mathbf{d}^\alpha(x, x')$. In particular, we will denote with t_0 the critical time at which a change of regime occurs in the globally diagonal regime. Namely,

$$t_0 := \Big(t + c_0 \mathbf{d}^\alpha(x, x')\Big) \wedge T. \tag{4.16}$$

We highlight however that this approach was already used in [CdRHM18a] to obtain Schauder estimates for degenerate Kolmogorov equations and can be adapted in the current setting.

Moreover, it is important to notice that the norm $\|\cdot\|_{C_d^{\alpha+\beta}}$ is essentially defined as the sum of the norms $\|\cdot\|_{C^{\frac{\alpha+\beta}{1+\alpha(i-1)}}}$ with respect to the i-th variable and uniformly on the other components. Thus, there is a big difference between the case $i = 1$ where $\alpha + \beta$ is in $(1, 2)$ and we have to deal with a proper derivative and the other situations $(i > 1)$ where instead $(\alpha + \beta)/(1 + \alpha(i - 1)) < 1$ and the norm is calculated directly on the function. For this reason, we are going to analyze the two cases separately. Lemma 4.5 will work on the non-degenerate setting $(i = 1)$ while Lemma 4.6 will concern the degenerate one $(i > 1)$.

Lemma 4.5 (Controls on Hölder Moduli: Non-Degenerate)**.** *Let x, x' be in $\mathbb{R}^{nd}$ such that $x_j = x'_j$ for any $j \neq 1$. Under [**A**], there exists a constant $C \geq 1$ such that for any t in $[0, T]$ and any freezing couple (τ, ξ) in $[0, T] \times \mathbb{R}^{nd}$, it holds that*

$$\Big|D_{x_1}\tilde{u}^{\tau,\xi}(t, x) - D_{x_1}\tilde{u}^{\tau,\xi}(t, x')\Big| \leq C c_0^{\frac{\alpha+\beta-2}{\alpha}} \Big(\|u_T\|_{C_{b,d}^{\alpha+\beta}} + \|f\|_{L^\infty(C_{b,d}^\beta)}\Big) \mathbf{d}^{\alpha+\beta-1}(x, x').$$

Before proving the above result, we point out the control on the Hölder modulus of $\tilde{u}^{\tau,\xi}$ with respect to the degenerate variables $(i > 1)$:

Lemma 4.6 (Controls on Hölder Moduli: Degenerate)**.** *Let i be in $[\![2, n]\!]$ and x, x' in $\mathbb{R}^{nd}$ such that $x_j = x'_j$ for any $j \neq i$. Under [**A**], there exists a constant $C := C(i)$ such that for any t in $[0, T]$ and any freezing couple (τ, ξ) in $[0, T] \times \mathbb{R}^{nd}$, it holds that*

$$\Big|\tilde{u}^{\tau,\xi}(t, x) - \tilde{u}^{\tau,\xi}(t, x')\Big| \leq C c_0^{\frac{\beta-\gamma_i}{\alpha}} \Big(\|u_T\|_{C_{b,d}^{\alpha+\beta}} + \|f\|_{L^\infty(C_{b,d}^\beta)}\Big) \mathbf{d}^{\alpha+\beta}(x, x').$$

Proof of Lemma 4.5 *Controls on frozen semigroup.* Let us consider firstly the off-diagonal regime, i.e. the case $T - t \leq c_0 \mathbf{d}^\alpha(x, x')$. Using the scaling lemma 4.1, it holds

that

$$\begin{aligned} D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x) &= \int_{\mathbb{R}^{nd}} D_{x_1}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)\,dy \\ &= \sum_{j=1}^{n} C_j(T-t)^{j-1}\int_{\mathbb{R}^{nd}} D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y)u_T(y)\,dy. \end{aligned}$$

It then follows that

$$\begin{aligned} &\Big|D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x) - D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x')\Big| \\ &\qquad \le C\sum_{j=1}^{n}(T-t)^{j-1}\Big|\int_{\mathbb{R}^{nd}}\Big[D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y) - D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x',y)\Big]u_T(y)\,dy\Big| \\ &\qquad =: C\sum_{j=1}^{n}(T-t)^{j-1}I_j^{od}. \end{aligned} \tag{4.17}$$

We are going to treat separately the cases $j=1$ and $j>1$ for the *off-diagonal* contributions $\{I_j^{od}\}_{j\in[\![1,n]\!]}$. Indeed, the function u_T is differentiable only with respect to the first component y_1. In this first case, we can apply integration by parts formula to move the derivative on u_T, so that

$$I_1^{od} = \Big|\int_{\mathbb{R}^{nd}}\Big[\tilde{p}^{\tau,\xi}(t,T,x,y) - \tilde{p}^{\tau,\xi}(t,T,x',y)\Big]D_{y_1}u_T(y)\,dy\Big|.$$

Noticing that $D_{y_1}u_T$ is in $C^{\alpha+\beta-1}_{b,d}(\mathbb{R}^{nd})$ thanks to the reverse Taylor expansion (Lemma 8.4), the last expression can be then rewritten as

$$\begin{aligned} I_1^{od} \le\ & \Big|\int_{\mathbb{R}^{nd}}\tilde{p}^{\tau,\xi}(t,T,x,y)\Big[D_{y_1}u_T(y) \pm D_{y_1}u_T(\tilde{m}^{\tau,\xi}_{T,t}(x))\Big] \\ &\qquad - \tilde{p}^{\tau,\xi}(t,T,x',y)\Big[D_{y_1}u_T(y) \pm D_{y_1}u_T(\tilde{m}^{\tau,\xi}_{T,t}(x'))\Big]\,dy\Big| \\ \le\ & C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big\{\int_{\mathbb{R}^{nd}}\Big[\tilde{p}^{\tau,\xi}(t,T,x,y)\mathbf{d}^{\alpha+\beta-1}(y,\tilde{m}^{\tau,\xi}_{T,t}(x))\Big]\,dy \\ &\qquad + \int_{\mathbb{R}^{nd}}\Big[\tilde{p}^{\tau,\xi}(t,T,x',y)\mathbf{d}^{\alpha+\beta-1}(y,\tilde{m}^{\tau,\xi}_{T,t}(x'))\Big]\,dy \\ &\qquad + \mathbf{d}^{\alpha+\beta-1}(\tilde{m}^{\tau,\xi}_{T,t}(x),\tilde{m}^{\tau,\xi}_{T,t}(x'))\Big\}. \end{aligned} \tag{4.18}$$

Now, we use the smoothing effect of $\tilde{p}^{\tau,\xi}$ (Equation (3.6)) to control the two integrals in the last expression, so that

$$I_1^{od} \le C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big[(T-t)^{\frac{\alpha+\beta-1}{\alpha}} + \mathbf{d}^{\alpha+\beta-1}(\tilde{m}^{\tau,\xi}_{T,t}(x),\tilde{m}^{\tau,\xi}_{T,t}(x'))\Big].$$

We can then conclude the case $j=1$ recalling that the mapping $x\to\tilde{m}^{\tau,\xi}_{T,t}(x)$ is affine (see Equation (3.4) for definition of $\tilde{m}^{\tau,\xi}_{T,t}(x)$) in order to show that

$$I_1^{od} \le C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big[(T-t)^{\frac{\alpha+\beta-1}{\alpha}} + \mathbf{d}^{\alpha+\beta-1}(x,x')\Big]. \tag{4.19}$$

Let us consider now the case $j > 1$. Using Duality (4.11) in Besov spaces and Identification (4.10), we can write from Equation (4.17) that

$$\begin{aligned} I_j^{od} &\leq C\|u_T\|_{C_{b,d}^{\alpha+\beta}} \int_{\mathbb{R}^{(n-1)d}} \|D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y_{\smallsetminus j},\cdot) - D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x',y_{\smallsetminus j},\cdot)\|_{B_{1,1}^{-(\alpha_j+\beta_j)}}\, dy_{\smallsetminus j} \\ &\leq C\|u_T\|_{C_{b,d}^{\alpha+\beta}} \int_{\mathbb{R}^{(n-1)d}} \|D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x,y_{\smallsetminus j},\cdot)\|_{B_{1,1}^{-(\alpha_j+\beta_j)}} \\ &\qquad\qquad + \|D_{y_j}\tilde{p}^{\tau,\xi}(t,T,x',y_{\smallsetminus j},\cdot)\|_{B_{1,1}^{-(\alpha_j+\beta_j)}}\, dy_{\smallsetminus j} \\ &\leq C\|u_T\|_{C_{b,d}^{\alpha+\beta}} (T-t)^{\frac{\alpha+\beta}{\alpha}-\frac{1}{\alpha_j}}, \end{aligned} \tag{4.20}$$

where in the last inequality we applied the first Besov control (Lemma 4.3). Going back at Equations (4.19)-(4.20), we finally conclude that

$$\begin{aligned} &\Big|D_{x_1}\tilde{P}_{T,t}^{\tau,\xi}u_T(x) - D_{x_1}\tilde{P}_{T,t}^{\tau,\xi}u_T(x')\Big| \\ &\qquad \leq C\|u_T\|_{C_{b,d}^{\alpha+\beta}}\Big[(T-t)^{\frac{\alpha+\beta-1}{\alpha}} + \mathbf{d}^{\alpha+\beta-1}(x,x') + \sum_{j=2}^{n}(T-t)^{j-1}(T-t)^{\frac{\alpha+\beta}{\alpha}-\frac{1}{\alpha_j}}\Big] \\ &\qquad \leq C\|u_T\|_{C_{b,d}^{\alpha+\beta}}\Big[(T-t)^{\frac{\alpha+\beta-1}{\alpha}} + \mathbf{d}^{\alpha+\beta-1}(x,x')\Big] \\ &\qquad \leq C\|u_T\|_{C_{b,d}^{\alpha+\beta}}\mathbf{d}^{\alpha+\beta-1}(x,x'), \end{aligned} \tag{4.21}$$

where in the last passage we used that $T-t \leq c_0\mathbf{d}^{\alpha}(x,x')$ for some $c_0 \leq 1$.
We focus now on the diagonal regime, i.e. when $T-t > c_0\mathbf{d}^{\alpha}(x,x')$. Remembering that we assumed that $x_j = x'_j$ for any j in $[\![2,n]\!]$, we start using a Taylor expansion on the density $\tilde{p}^{\tau,\xi}$ with respect to the first, non-degenerate variable x_1. Namely,

$$\begin{aligned} D_{x_1}\tilde{P}_{T,t}^{\tau,\xi}u_T(x) - D_{x_1}\tilde{P}_{T,t}^{\tau,\xi}u_T(x') &= \int_{\mathbb{R}^{nd}}\Big[D_{x_1}\tilde{p}^{\tau,\xi}(t,T,x,y) - D_{x_1}\tilde{p}^{\tau,\xi}(t,T,x',y)\Big]u_T(y)\,dy \\ &= \int_{\mathbb{R}^{nd}}\int_0^1 D_{x_1}^2\tilde{p}^{\tau,\xi}\Big(t,T,x'+\lambda(x-x'),y\Big)(x-x')_1 u_T(y)\,d\lambda dy. \end{aligned}$$

Moreover, from the Scaling Lemma 4.1, it holds that

$$D_{x_1}^2\tilde{p}^{\tau,\xi}\Big(t,T,x'+\lambda(x-x'),y\Big) = \sum_{j=1}^{n} C_j(T-t)^{j-1}D_{y_j}D_{x_1}\tilde{p}^{\tau,\xi}\Big(t,T,x'+\lambda(x-x'),y\Big)$$

and we can use it to write

$$\begin{aligned} &\Big|D_{x_1}\tilde{P}_{T,t}^{\tau,\xi}u_T(x) - D_{x_1}\tilde{P}_{T,t}^{\tau,\xi}u_T(x')\Big| \\ &\qquad \leq C|(x-x')_1|\sum_{j=1}^{n}(T-t)^{j-1}\Big|\int_0^1\int_{\mathbb{R}^{nd}} D_{y_j}D_{x_1}\tilde{p}^{\tau,\xi}\Big(t,T,x'+\lambda(x-x'),y\Big)u_T(y)\,dyd\lambda\Big| \\ &\qquad =: C|(x-x')_1|\sum_{j=1}^{n}(T-t)^{j-1}I_j^d. \end{aligned} \tag{4.22}$$

Similarly to the off-diagonal regime, we are going to treat separately the cases $j=1$ and $j>1$ for the *diagonal* contributions $\{I_j^d\}_{j\in[\![1,n]\!]}$. In the first case, we can apply integration by parts formula to show that

$$I_1^d = \Big|\int_0^1\int_{\mathbb{R}^{nd}} D_{x_1}\tilde{p}^{\tau,\xi}\Big(t,T,x'+\lambda(x-x'),y\Big)\otimes D_{y_1}u_T(y)\,dyd\lambda\Big|.$$

A cancellation argument with respect to $D_{x_1}\tilde{p}^{\tau,\xi}$ then leads to

$$\begin{aligned} I_1^d &= \Big| \int_0^1 \int_{\mathbb{R}^{nd}} D_{x_1}\tilde{p}^{\tau,\xi}(t,T,x'+\lambda(x-x'),y) \\ &\qquad\qquad \otimes \Big[D_{y_1}u_T(y) - D_{y_1}u_T(\tilde{m}_{T,t}^{\tau,\xi}(x'+\lambda(x-x'))) \Big] \, dy d\lambda \Big| \\ &\leq C\|u_T\|_{C_{b,d}^{\alpha+\beta}} \int_0^1 \int_{\mathbb{R}^{nd}} |D_{x_1}\tilde{p}^{\tau,\xi}(t,T,x'+\lambda(x-x'),y)| \\ &\qquad\qquad \times \mathbf{d}^{\alpha+\beta-1}\Big(y, \tilde{m}_{T,t}^{\tau,\xi}(x'+\lambda(x-x'))\Big) \, dy d\lambda. \end{aligned}$$

Since $\alpha+\beta-1<\alpha$ by hypothesis [**P**], we can conclude using the smoothing effect of $\tilde{p}^{\tau,\xi}$ (Lemma 3.2) to show that

$$I_1^d \leq C\|u_T\|_{C_{b,d}^{\alpha+\beta}}(T-t)^{\frac{\alpha+\beta-2}{\alpha}}. \tag{4.23}$$

For the case $j>1$, we use instead the duality in Besov spaces (Equation (4.11)) and Identification (4.10) to write

$$\begin{aligned} I_j^d &\leq \int_0^1 \int_{\mathbb{R}^{(n-1)d}} \|D_{y_j}D_{x_1}\tilde{p}^{\xi}(t,T,x'+\lambda(x-x'),y_{\smallsetminus j},\cdot)\|_{B_{1,1}^{-(\alpha_j+\beta_j)}} \, dy_{\smallsetminus j} d\lambda \\ &\leq C\|u_T\|_{C_{b,d}^{\alpha+\beta}}(T-t)^{\frac{\alpha+\beta}{\alpha}-\frac{1}{\alpha_j}-\frac{1}{\alpha}}, \end{aligned} \tag{4.24}$$

where in the last passage we applied the first Besov control (Lemma 4.3). From Equations (4.22), (4.23) and (4.24), it is possible to conclude that

$$\begin{aligned} \Big|D_{x_1}\tilde{P}_{T,t}^{\tau,\xi}u_T(x) - D_{x_1}\tilde{P}_{T,t}^{\tau,\xi}u_T(x')\Big| &\leq C\|u_T\|_{C_{b,d}^{\alpha+\beta}}|(x-x')_1| \sum_{j=1}^n (T-t)^{j-1}(T-t)^{\frac{\alpha+\beta-1}{\alpha}-\frac{1}{\alpha_j}} \\ &\leq C\|u_T\|_{C_{b,d}^{\alpha+\beta}}|(x-x')_1|(T-t)^{\frac{\alpha+\beta-2}{\alpha}} \\ &\leq Cc_0^{\frac{\alpha+\beta-2}{\alpha}}\|u_T\|_{C_{b,d}^{\alpha+\beta}}\mathbf{d}^{\alpha+\beta-1}(x,x'), \end{aligned}$$

where in the last passage we used that $|(x-x')_1| = \mathbf{d}(x,x')$ and since $\frac{\alpha+\beta-2}{\alpha}<0$, that

$$|(x-x')_1|(T-t)^{\frac{\alpha+\beta-2}{\alpha}} \leq c_0^{\frac{\alpha+\beta-2}{\alpha}}\mathbf{d}^{\alpha+\beta-1}(x,x').$$

Remembering that c_0 is considered fixed and bigger then zero, the searched control follows immediately.

Controls on frozen Green kernel. We recall that, in order to preserve the previous terminology of off-diagonal/diagonal regime for the frozen semigroup, we have introduced the transition time t_0, defined in (4.16). Then, while integrating in s from t to T, we will say that the local off-diagonal regime holds for $\tilde{G}^{\tau,\xi}$ if s is in $[t,t_0]$ and that the local diagonal regime holds if s is in $[t_0,T]$. With the notations of (4.2) in mind, it seems quite natural now to decompose the derivative of the frozen Green kernel with respect to t_0, i.e.

$$D_{x_1}\tilde{G}_{T,t}^{\tau,\xi}f(t,x) = D_{x_1}\tilde{G}_{t_0,t}^{\tau,\xi}f(t,x) + D_{x_1}\tilde{G}_{T,t_0}^{\tau,\xi}f(t,x).$$

We remark however that the globally off-diagonal regime is considered in the above decomposition, too. Indeed, when $T-t \leq c_0\mathbf{d}^\alpha(x,x')$, t_0 coincides with T and the second term on the right-hand side vanishes.
We start considering the off-diagonal regime represented by

$$\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x')\Big|.$$

It holds that

$$\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x')\Big| \leq \int_t^{t_0}\Big[\Big|D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}f(s,x)\Big| + \Big|D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}f(s,x')\Big|\Big]\,ds.$$

We then use the control on the frozen semigroup (Equation (3.7)) to find that

$$\begin{aligned}\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x')\Big| &\leq C\|f\|_{L^\infty(C^\beta_{b,d})}\int_t^{t_0}(s-t)^{\frac{\beta-1}{\alpha}}\,ds\\ &\leq C\|f\|_{L^\infty(C^\beta_{b,d})}(t_0-t)^{\frac{\beta+\alpha-1}{\alpha}}.\end{aligned}$$

Our choice of t_0 (cf. Equation (4.16)) allows then to conclude that

$$\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x')\Big| \leq C\|f\|_{L^\infty(C^\beta_{b,d})}\mathbf{d}^{\alpha+\beta-1}(x,x'),$$

remembering that $c_0 \leq 1$ by assumption.
We can focus now on the diagonal regime represented by

$$\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x')\Big|.$$

We start applying a Taylor expansion on the derivative of the semigroup $\tilde{P}^{\tau,\xi}f(t,x)$ so that

$$\begin{aligned}\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x')\Big| &= \Big|\int_{t_0}^T\Big[D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}f(s,x) - D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}f(s,x')\Big]\,ds\Big|\\ &= \Big|\int_{t_0}^T\int_0^1 D^2_{x_1}\tilde{P}^{\tau,\xi}_{s,t}f(s,x+\lambda(x'-x))(x'-x)_1\,d\lambda ds\Big|.\end{aligned}$$

Then, Fubini theorem and the control on the frozen semigroup (Equation (3.7)) allow us to write that

$$\begin{aligned}\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x')\Big| &\leq C\|f\|_{L^\infty(C^\beta_{b,d})}|(x-x')_1|\int_{t_0}^T(s-t)^{\frac{\beta-2}{\alpha}}\,ds\\ &\leq C\|f\|_{L^\infty(C^\beta_{b,d})}|(x-x')_1|\Big[\frac{\alpha}{\alpha+\beta-2}(s-t)^{\frac{\alpha+\beta-2}{\alpha}}\Big]_{t_0}^T.\end{aligned}$$

Since by hypothesis [**P**], it holds that $\alpha/(\alpha+\beta-2)<0$, it follows that

$$\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x')\Big| \leq C\|f\|_{L^\infty(C^\beta_{b,d})}|(x-x')_1|(t_0-t)^{\frac{\alpha+\beta-2}{\alpha}}.$$

Using that $|(x-x')_1| = \mathbf{d}(x,x')$ and remembering our choice of t_0 in (4.16), we can then conclude that

$$\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x')\Big| \leq Cc_0^{\frac{\alpha+\beta-2}{\alpha}}\|f\|_{L^\infty(C^\beta_{b,d})}\mathbf{d}^{\alpha+\beta-1}(x,x').$$

Proof of Lemma 4.6 *Controls on frozen semigroup.* Using the change of variables $z = \tilde{m}^{\tau,\xi}_{T,t}(x) - y$, we can rewrite $\tilde{P}^{\tau,\xi}_{T,t} u_T(x)$ as

$$\begin{aligned}\tilde{P}^{\tau,\xi}_{T,t} u_T(x) &= \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t,T,x,y) u_T(y)\, dy \\ &= \int_{\mathbb{R}^{nd}} \frac{1}{\det\Big(\mathbb{M}_{T-t}\Big)} p_S(T-t, \mathbb{M}^{-1}_{T-t}\Big(\tilde{m}^{\tau,\xi}_{T,t}(x) - y\Big) u_T(y)\, dy \\ &= \int_{\mathbb{R}^{nd}} \frac{1}{\det\Big(\mathbb{M}_{T-t}\Big)} p_S(T-t, \mathbb{M}^{-1}_{T-t} z\Big) u_T(\tilde{m}^{\tau,\xi}_{T,t}(x) - z)\, dz.\end{aligned}$$

It then follows that

$$\begin{aligned}&\Big|\tilde{P}^{\tau,\xi}_{T,t} u_T(x) - \tilde{P}^{\tau,\xi}_{T,t} u_T(x')\Big| \\ &\quad = \Big|\int_{\mathbb{R}^{nd}} \frac{1}{\det\Big(\mathbb{M}_{T-t}\Big)} p_S\Big(T-t, \mathbb{M}^{-1}_{T-t} z\Big)\Big[u_T(\tilde{m}^{\tau,\xi}_{T,t}(x) - z) - u_T(\tilde{m}^{\tau,\xi}_{T,t}(x') - z)\Big]\, dz\Big|.\end{aligned}$$

We now observe that the function $x \to \tilde{m}^{\tau,\xi}_{T,t}(x)$ is affine (cf. Equation (3.4)) and thus, that

$$\Big(\tilde{m}^{\tau,\xi}_{T,t}(x) - z\Big)_1 = \Big(\tilde{m}^{\tau,\xi}_{T,t}(x') - z\Big)_1,$$

since $x_1 = x'_1$. It then holds that

$$\begin{aligned}\Big|u_T(\tilde{m}^{\tau,\xi}_{T,t}(x) - z) - u_T(\tilde{m}^{\tau,\xi}_{T,t}(x') - z)\Big| &\le C\|u_T\|_{C^{\alpha+\beta}_{b,d}} \mathbf{d}^{\alpha+\beta}\Big(\tilde{m}^{\tau,\xi}_{T,t}(x), \tilde{m}^{\tau,\xi}_{T,t}(x')\Big) \\ &\le C\|u_T\|_{C^{\alpha+\beta}_{b,d}} \mathbf{d}^{\alpha+\beta}(x,x').\end{aligned}$$

Hence, we can conclude using it to write

$$\begin{aligned}\Big|\tilde{P}^{\tau,\xi}_{T,t} u_T(x) - \tilde{P}^{\tau,\xi}_{T,t} u_T(x')\Big| &\le C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\, \mathbf{d}^{\alpha+\beta}(x,x') \int_{\mathbb{R}^{nd}} \frac{p_S\left(T-t, \mathbb{M}^{-1}_{T-t} z\right)}{\det \mathbb{M}_{T-t}}\, dz \\ &\le C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\, \mathbf{d}^{\alpha+\beta}(x,x').\end{aligned}$$

Controls on frozen Green kernel. We will assume the same notations appeared in the previous lemma for the frozen Green kernel. In particular, we decompose it as

$$\tilde{G}^{\tau,\xi}_{T,t} f(t,x) = \tilde{G}^{\tau,\xi}_{t_0,t} f(t,x) + \tilde{G}^{\tau,\xi}_{T,t_0} f(t,x)$$

with t_0 defined in Equation (4.16).
We start rewriting the off-diagonal regime contribution as

$$\begin{aligned}\Big|\tilde{G}^{\tau,\xi}_{t_0,t} f(t,x) - \tilde{G}^{\tau,\xi}_{t_0,t} f(t,x')\Big| &= \Big|\int_t^{t_0} \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t,s,x,y)\Big[f(s,y) \pm f(s,\tilde{m}^{\tau,\xi}_{s,t}(x))\Big] \\ &\qquad - \tilde{p}^{\tau,\xi}(t,s,x',y)\Big[f(s,y) \pm f(s,\tilde{m}^{\tau,\xi}_{s,t}(x'))\Big]\, dyds\Big| \\ &\le \Big|\int_t^{t_0} \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t,s,x,y)\Big[f(s,y) - f(s,\tilde{m}^{\tau,\xi}_{s,t}(x))\Big] \\ &\qquad - \tilde{p}^{\tau,\xi}(t,s,x',y)\Big[f(s,y) - f(s,\tilde{m}^{\tau,\xi}_{s,t}(x'))\Big]\, dyds\Big| \\ &\qquad + \Big|\int_t^{t_0} f(s,\tilde{m}^{\tau,\xi}_{s,t}(x)) - f(s,\tilde{m}^{\tau,\xi}_{s,t}(x'))\, ds\Big|.\end{aligned}$$

We can then use the smoothing effect for $\tilde{p}^{\tau,\xi}$ (Equation (3.6)) to show that

$$\begin{aligned}&\left|\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x)-\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x')\right|\\&\qquad\leq C\|f\|_{L^\infty(C^\beta_{b,d})}\int_t^{t_0}\left[(s-t)^{\beta/\alpha}+\mathbf{d}^\beta(\tilde{m}^{\tau,\xi}_{s,t}(x),\tilde{m}^{\tau,\xi}_{s,t}(x'))\right]ds.\end{aligned}\tag{4.25}$$

Recalling from Equation (3.4) that $x\to\tilde{m}^{\tau,\xi}_{s,t}(x)$ is affine, it follows that

$$\begin{aligned}\left|\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x)-\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x')\right|&\leq C\|f\|_{L^\infty(C^\beta_{b,d})}\int_t^{t_0}\left[(s-t)^{\beta/\alpha}+\mathbf{d}^\beta(x,x')\right]ds\\&\leq C\|f\|_{L^\infty(C^\beta_{b,d})}\left[(t_0-t)\mathbf{d}^\beta(x,x')+(t_0-t)^{\frac{\beta+\alpha}{\alpha}}\right].\end{aligned}$$

Using that $t_0-t\leq c_0\mathbf{d}^\alpha(x,x')$ for some $c_0\leq 1$, we can finally conclude that

$$\left|\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x)-\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x')\right|\leq C\|f\|_{L^\infty(C^\beta_{b,d})}\mathbf{d}^{\alpha+\beta}(x,x').$$

Now, we can focus our analysis to the diagonal regime contribution:

$$\left|\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x)-\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x')\right|.$$

We start applying a Taylor expansion on the frozen semigroup $\tilde{P}^{\tau,\xi}_{s,t}f$ with respect to the i-th variable x_i, which is, by hypothesis, the only one for which the entries of x and x' differ. Namely,

$$\begin{aligned}\left|\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x)-\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x')\right|&=\left|\int_{t_0}^T\tilde{P}^{\tau,\xi}_{s,t}f(s,x)-\tilde{P}^{\tau,\xi}_{s,t}f(s,x')\,ds\right|\\&=\left|\int_{t_0}^T\int_0^1 D_{x_i}\tilde{P}^{\tau,\xi}_{s,t}f(s,x+\lambda(x'-x))\cdot(x'-x)_i\,d\lambda ds\right|.\end{aligned}$$

The control on the frozen semigroup (Equation (3.7)) then implies that

$$\left|\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x)-\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x')\right|\leq C\|f\|_{L^\infty(C^\beta_{b,d})}|(x-x')_i|\int_{t_0}^T(s-t)^{\frac{\beta}{\alpha}-\frac{1}{\alpha_i}}\,ds.\tag{4.26}$$

Noticing from assumption [**P**] that $\beta+\alpha-1-\alpha(i-1)<0$ for $i\geq 2$, it holds that

$$\begin{aligned}\int_{t_0}^T(s-t)^{\frac{\beta}{\alpha}-\frac{1}{\alpha_i}}\,ds&=\int_{t_0}^T(s-t)^{\frac{\beta-[1+\alpha(i-1)]}{\alpha}}\,ds\\&\leq C\left[-(s-t)^{\frac{\beta+\alpha-1-\alpha(i-1)}{\alpha}}\right]_{t_0}^T\\&\leq C(t_0-t)^{\frac{\beta-1-\alpha(i-2)}{\alpha}}.\end{aligned}$$

Using that $|(x-x')_i|=\mathbf{d}^{1+\alpha(i-1)}(x,x')$ and our choice of t_0 (cf. Equation (4.16)), we can then conclude from Equation (4.26) that

$$\begin{aligned}&\left|\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x)-\tilde{G}^{\tau,\xi}_{T,t_0}f(t,x')\right|\\&\qquad\leq Cc_0^{\frac{\beta-1-\alpha(i-2)}{\alpha}}\|f\|_{L^\infty(C^\beta_{b,d})}\mathbf{d}^{\alpha+\beta}(x,x')\leq Cc_0^{\frac{\beta-\gamma_i}{\alpha}}\|f\|_{L^\infty(C^\beta_{b,d})}\mathbf{d}^{\alpha+\beta}(x,x'),\end{aligned}$$

remembering the definition of γ_i given in (2.21).

5 A priori estimates

Since the aim of this section is to prove Proposition 3.6, we will assume tacitly from this point further that assumption [**A'**] holds. Moreover, we recall here that throughout this section, we are considering the regularized framework of Section 3.2.

WARNING: For notational simplicity, we drop here the subscripts and the superscripts in m associated with the regularization. For any fixed (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$, we rewrite, with some abuse in notations, the Duhamel Expansion (3.16) as:

$$u(t,x) \;=\; \tilde{u}^{\tau,\xi}(t,x) + \int_t^T \tilde{P}^{\tau,\xi}_{s,t} R^{\tau,\xi}(s,x)\,ds, \tag{5.27}$$

where $\tilde{u}^{\tau,\xi}$ is defined through the Duhamel Representation (3.10) and

$$R^{\tau,\xi}(t,x) \;=\; \Big\langle F(t,x) - F(t,\theta_{t,\tau}(\xi)), D_x u(t,x)\Big\rangle, \quad (t,x)\;\in\;(0,T)\times\mathbb{R}^{nd}.$$

It is however important to keep in mind that f, u_T, F are now smooth and bounded functions so that all the terms above are clearly defined. We recall however that we aim at obtaining controls in the $L^\infty(C^{\alpha+\beta}_{b,d})$-norm, uniformly with respect to the regularization parameter.
From the expansion above, we know moreover that for any (t,ξ) in $[0,T]\times\mathbb{R}^{nd}$, it holds that

$$D_{x_1}u(t,x) \;=\; D_{x_1}\tilde{u}^{\tau,\xi}(t,x) + \int_t^T D_{x_1}\tilde{P}^{\tau,\xi}_{s,t} R^{\tau,\xi}(s,x)\,ds. \tag{5.28}$$

As seen in the previous section, these decompositions will allow us to obtain a control for u in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ analyzing separately the contributions from the Duhamel representation $\tilde{u}^{\tau,\xi}$ and those from the expansion error $R^{\tau,\xi}(t,x)$, for suitable choices of freezing parameters (τ,ξ).

5.1 Second Besov control

This sub-section focuses on the contribution associated with the remainder term $R^{m,\tau,\xi}$ appearing in the Duhamel-type Expansion (5.27). We recall that we aim at controlling it with the $L^\infty(C^{\alpha+\beta}_{b,d})$-norm of the coefficients, uniformly in the regularization parameter. Let us start decomposing it through

$$\Big|\int_t^T \tilde{P}^{\tau,\xi}_{s,t} R^{\tau,\xi}(s,x)\,ds\Big| \;=\; \Big|\sum_{j=1}^n \int_t^T \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\cdot D_{y_j}u(s,y)\,dyds\Big|,$$

where we have denoted for simplicity

$$\Delta^{\tau,\xi}F_j(s,y) \;:=\; F_j(s,y) - F_j(s,\theta_{s,\tau}(\xi)), \quad j\in[\![1,n]\!]. \tag{5.29}$$

We then notice that the non-degenerate contribution in the sum (corresponding to the index $j=1$) can be treated easily, remembering that u is differentiable with respect to

the first component with a bounded derivative. Indeed, using the smoothing effect for the frozen density $\tilde{p}^{\tau,\xi}$ (Equation (3.6)), it holds that

$$\begin{aligned}
&\Big|\int_t^T \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_1(s,y)\cdot D_{y_1}u(s,y)\,dyds\Big| \\
&\qquad \leq C\|D_{y_1}u(s,y)\|_{l^\infty(L^\infty)}\|F\|_H \int_t^T \int_{\mathbb{R}^{nd}} \tilde{p}^{\tau,\xi}(t,s,x,y)\mathbf{d}^{\alpha+\beta}\Big(y,\theta_{s,\tau}(\xi)\Big)\,dyds \\
&\qquad \leq C\|D_{y_1}u(s,y)\|_{l^\infty(L^\infty)}\|F\|_H \int_t^T (s-t)^{\frac{\beta}{\alpha}}\,ds \\
&\qquad \leq C\|D_{y_1}u(s,y)\|_{l^\infty(L^\infty)}\|F\|_H (T-t)^{\frac{\alpha+\beta}{\alpha}}.
\end{aligned}$$

In order to deal with the degenerate indexes, we will use, similarly to the previous subsection, a reasoning in Besov spaces. Since u is not differentiable with respect to y_j if $j>1$, we move the derivative to the other terms using integration by parts formula:

$$\Big|\int_t^T \int_{\mathbb{R}^{nd}} D_{y_j}\cdot\Big\{\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\Big\}u(s,y)\,dyds\Big|.$$

In order to rely again on the duality in Besov spaces (Equation (4.11)), we rewrite the above expression as

$$\begin{aligned}
&\Big|\int_t^T \int_{\mathbb{R}^{nd}} D_{y_j}\cdot\Big\{\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\Big\}u(s,y)\,dyds\Big| \\
&\qquad \leq \int_t^T \int_{\mathbb{R}^{(n-1)d}} \Big\|D_{y_j}\cdot\Big\{\tilde{p}^{\tau,\xi}(t,s,x,y_{\setminus j},\cdot)\Delta^{\tau,\xi}F_j(s,y_{\setminus j},\cdot)\Big\}\Big\|_{B_{1,1}^{-(\alpha_j+\beta_j)}} \\
&\qquad\qquad\qquad \times \|u(s,y_{\setminus j},\cdot)\|_{B_{\infty,\infty}^{\alpha_j+\beta_j}}\,dy_{\setminus j}ds.
\end{aligned}$$

Remembering the identification in Equation (4.10), it holds now that

$$\begin{aligned}
&\Big|\int_t^T \int_{\mathbb{R}^{nd}} D_{y_j}\cdot\Big\{\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\Big\}u(s,y)\,dyds\Big| \\
&\leq \|u\|_{L^\infty(C_{b,d}^{\alpha+\beta})}\int_t^T \int_{\mathbb{R}^{(n-1)d}} \Big\|D_{y_j}\cdot\Big\{\tilde{p}^{\tau,\xi}(t,s,x,y_{\setminus j},\cdot)\Delta^{\tau,\xi}F_j(s,y_{\setminus j},\cdot)\Big\}\Big\|_{B_{1,1}^{-(\alpha_j+\beta_j)}}\,dy_{\setminus j}ds.
\end{aligned}$$

It then remains to control the integral of the Besov norm above. To do that, we will need a refinement of the smoothing effect (Equation (3.6)) that involves only partial differences of variables. For a fixed i in $[\![2,n]\!]$, we start denoting by $d_{i:n}(\cdot,\cdot)$ the part of the anisotropic distance considering only the last $n-(i-1)$ variables. Namely,

$$d_{i:n}(x,x') := \sum_{j=i}^n |(x-x')_j|^{\frac{1}{1+\alpha(j-1)}}.$$

Lemma 5.1 (Partial Smoothing Effect)**.** *Let i be in $[\![2,n]\!]$, γ in $(0,1\wedge\alpha(1+\alpha(i-1)))$ and ϑ, ϱ two n-multi-indexes such that $|\vartheta+\varrho|\leq 3$. Then, there exists a constant $C:=C(\vartheta,\varrho,\gamma)$ such that for any $t<s$ in $[0,T]$, any x in $\mathbb{R}^{nd}$,*

$$\int_{\mathbb{R}^{nd}} |D_y^\varrho D_x^\vartheta \tilde{p}^{\tau,\xi}(t,s,x,y)|\mathbf{d}_{i:n}^\gamma\Big(y,\theta_{s,\tau}(\xi)\Big)\,dy \leq C(s-t)^{\frac{\gamma}{\alpha}-\sum_{i=k}^n \frac{\vartheta_k+\varrho_k}{\alpha_k}}, \tag{5.30}$$

taking $(\tau,\xi)=(t,x)$.

The above assumption on γ should not appear too strange. Indeed, in the partial distance $\mathbf{d}^{\gamma}_{i:n}(x,x')$, the stronger term to be integrated is at level i with intensity of order $\gamma/(1+\alpha(i-1))$. Since by the smoothing effect (Equation (3.6)) of the frozen density we know we can integrate against $\tilde{p}^{\tau,\xi}$ contributions of order up to α, the condition $\gamma < \alpha(1+\alpha(i-1))$ appears naturally.
A proof of this result can be obtained mimicking with slightly modifications the proof in Lemma 3.2.

As done above for the first Besov control (Lemma 4.3), we will however state the result considering a possibly additional derivative with respect to x_1. Namely, we would like to control the following:

$$D_{y_j}\cdot\Big\{\mathbf{d}^{\vartheta}_{x}\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\cdot)\otimes\Big[F_j(s,y_{\smallsetminus j},\cdot)-F_j(s,\theta_{s,\tau}(\xi))\Big]\Big\}$$

where we have denoted as in (4.7), $F_j(s,y_{\smallsetminus j},\cdot):=F_j(s,y_1,\dots,y_{j-1},\cdot,y_{j+1},\dots,y_n)$ and, with a slightly abuse of notation, by $D_{y_j}\cdot$ an extended form of the divergence over the j-th variable. In other words, this "enhanced" divergence form decreases by one the order of the input tensor.

Lemma 5.2 (Second Besov Control)**.** *Let j be in $[\![2,n]\!]$ and ϑ a multi-index in $\mathbb{N}^n$ such that $|\vartheta|\le 2$. Under (**A**'), there exists a constant $C:=C(j,\vartheta)$ such that for any x in $\mathbb{R}^{nd}$ and any $t<s$ in $[0,T]$*

$$\int_{\mathbb{R}^{(n-1)d}}\Big\|D_{y_j}\cdot\Big\{\mathbf{d}^{\vartheta}_{x}\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\cdot)\otimes\Big[F_j(s,y_{\smallsetminus j},\cdot)-F_j(s,\theta_{s,\tau}(\xi))\Big]\Big\}\Big\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}dy_{\smallsetminus j}$$
$$\le\ C\|F\|_H(s-t)^{\frac{\beta}{\alpha}-\sum_{k=1}^{n}\frac{\vartheta_k}{\alpha_k}},$$

taking $(\tau,\xi)=(t,x)$.

Proof. To control the Besov norm in $B^{-(\alpha_j+\beta_j)}_{1,1}(\mathbb{R}^d)$, we are going to use the Thermic Characterization (4.8) with $\tilde{\gamma}=-(\alpha_j+\beta_j)$. Since the first term can be controlled as in the first Besov control (Lemma 4.3), we will focus on the second one, i.e.

$$\int_0^1 v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}\partial_v p_h(v,z-y_j)D_{y_j}\cdot\Big\{D^{\vartheta}_{x}\tilde{p}^{\tau,\xi}(t,s,x,y)\otimes\Delta^{\tau,\xi}F_j(s,y)\Big\}dy_j\Big|dzdv,$$

where we exploited the same notations for $\Delta^{\tau,\xi}F_j$ given in (5.29).
We start applying integration by parts formula noticing that

$$D_{y_j}p_h(v,z-y_j)\ =\ -D_z p_h(v,z-y_j),$$

to write that

$$\int_0^1 v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d}\Big|\int_{\mathbb{R}^d}D_z\partial_v p_h(v,z-y_j)\cdot\Big\{D^{\vartheta}_{x}\tilde{p}^{\tau,\xi}(t,s,x,y)\otimes\Delta^{\tau,\xi}F_j(s,y)\Big\}dy_j\Big|dzdv.$$

Fixed a constant $\delta_j \geq 1$ to be chosen later, we then split the above integral with respect to v into two components:

$$\int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j+\beta_j}{\alpha}} \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} D_z \partial_v p_h(v, z-y_j) \cdot \Big\{ D_x^\vartheta \tilde{p}^{\tau,\xi}(t,s,x,y) \Delta^{\tau,\xi} F_j(s,y) \Big\} \, dy_j \Big| \, dz dv$$
$$+ \int_{(s-t)^{\delta_j}}^1 v^{\frac{\alpha_j+\beta_j}{\alpha}} \int_{\mathbb{R}^d} \Big| \int_{\mathbb{R}^d} D_z \partial_v p_h(v, z-y_j) \cdot \Big\{ D_x^\vartheta \tilde{p}^{\tau,\xi}(t,s,x,y) \Delta^{\tau,\xi} F_j(s,y) \Big\} \, dy_j \Big| \, dz dv$$
$$=: \Big(I_1 + I_2 \Big)(y_{\smallsetminus j}).$$

The second component I_2 has no time-singularity and it can be easily controlled using Fubini theorem in the following way

$$I_2(y_{\smallsetminus j}) \leq C \|F\|_H \int_{(s-t)^{\delta_j}}^1 v^{\frac{\alpha_j+\beta_j}{\alpha}} \int_{\mathbb{R}^d} \Big(\int_{\mathbb{R}^d} |D_z \partial_v p_h(v, z-y_j)| \, dz \Big) |D_x^\vartheta \tilde{p}^{\tau,\xi}(t,s,x,y)|$$
$$\times \mathbf{d}_{j:n}^{1+\alpha(j-2)+\beta}(y, \theta_{s,\tau}(\xi)) \, dy_j dv,$$

remembering that $F_j(t,\cdot)$ depends only on the last $(n-j)$ variables and it is in $C_{b,d}^{1+\alpha(j-2)+\beta}(\mathbb{R}^{nd})$ by assumption [**R**]. We can then use the smoothing effect of the heat-kernel p_h (Equation (4.9)) and Fubini theorem again, in order to write that

$$I_2(y_{\smallsetminus j}) \leq C \|F\|_H \int_{(s-t)^{\delta_j}}^1 \frac{v^{\frac{\alpha_j+\beta_j-1}{\alpha}}}{v} \int_{\mathbb{R}^d} |D_x^\vartheta \tilde{p}^{\tau,\xi}(t,s,x,y)| \mathbf{d}_{j:n}^{1+\alpha(j-2)+\beta}(y, \theta_{s,\tau}(\xi)) \, dy_j dv$$
$$\leq C \|F\|_H \Big(\int_{(s-t)^{\delta_j}}^1 \frac{v^{\frac{\alpha_j+\beta_j-1}{\alpha}}}{v} \, dv \Big) \Big(\int_{\mathbb{R}^d} |D_x^\vartheta \tilde{p}^{\tau,\xi}(t,s,x,y)| \mathbf{d}_{j:n}^{1+\alpha(j-2)+\beta}(y, \theta_{s,\tau}(\xi)) \, dy_j \Big).$$

Noticing from (2.27) that $\alpha_j + \beta_j - 1 < 0$, it holds now that

$$I_2(y_{\smallsetminus j}) \leq C \|F\|_H (s-t)^{\delta_j \frac{\alpha_j+\beta_j-1}{\alpha}} \int_{\mathbb{R}^d} |D_x^\vartheta \tilde{p}^{\tau,\xi}(t,s,x,y)| \mathbf{d}_{j:n}^{1+\alpha(j-2)+\beta}(y, \theta_{s,\tau}(\xi)) \, dy_j.$$

We can finally add the integral with respect to the other components $y_{\smallsetminus j}$. In order to use now the partial smoothing effect (Equation (5.30)), we take $\tau = t$ and $\xi = x$ and notice that by assumption [**P**],

$$1 + \alpha(j-2) + \beta < 1 + \alpha(j-1) - (1-\alpha)\Big(1 + \alpha(j-1)\Big) = \alpha(1 + \alpha(j-1)). \quad (5.31)$$

It then holds that

$$\int_{\mathbb{R}^{(n-1)d}} I_2(y_{\smallsetminus j}) \, dy_{\smallsetminus j}$$
$$\leq C \|F\|_H (s-t)^{\delta_j \frac{\alpha_j+\beta_j-1}{\alpha}} \int_{\mathbb{R}^{nd}} |D_x^\vartheta \tilde{p}^{\tau,\xi}(t,s,x,y)| \mathbf{d}_{j:n}^{1+\alpha(j-2)+\beta}(y, \theta_{s,\tau}(\xi)) \, dy$$
$$\leq C \|F\|_H (s-t)^{\delta_j \frac{\alpha_j+\beta_j-1}{\alpha} + \frac{1+\alpha(j-2)+\beta}{\alpha} - \sum_{k=1}^n \frac{\vartheta_k}{\alpha_k}}. \quad (5.32)$$

To control the other term I_1, we focus at first on the inner integral with respect to y_j:

$$\int_{\mathbb{R}^d} D_z \partial_v p_h(v, z-y_j) \cdot \Big\{ D_x^\vartheta \tilde{p}^{\tau,\xi}(t,s,x,y) \otimes \Delta^{\tau,\xi} F_j(s,y) \Big\} \, dy_j.$$

We start using a cancellation argument with respect to the density p_h to write that

$$\int_{\mathbb{R}^d} D_z\partial_v p_h(v, z-y_j)\cdot\Big\{D_x^\vartheta \tilde{p}^{\tau,\xi}(t,s,x,y)\otimes\Big[F_j(s,y)-F_j(s,\theta_{s,\tau}(\xi))\Big\}$$
$$-D_x^\vartheta \tilde{p}^{\tau,\xi}(t,s,x,y_{\setminus j},z)\otimes\Big[F_j(s,y_{\setminus j},z)-F_j(s,\theta_{s,\tau}(\xi))\Big]\Big\}\,dy_j$$

We can then divide the above integral into two components J_1+J_2 given by in

$$J_1(v,y_{\setminus j},z) := \int_{\mathbb{R}^d} D_z\partial_v p_h(v,z-y_j)\cdot\Big\{D_x^\vartheta \tilde{p}^{\tau,\xi}(t,s,x,y)$$
$$\otimes\Big[F_j(s,y)-F_j(s,y_{\setminus j},z)\Big]\Big\}\,dy_j;$$
$$J_2(v,y_{\setminus j},z) := \int_{\mathbb{R}^d} D_z\partial_v p_h(v,z-y_j)\cdot\Big\{\Big[D_x^\vartheta \tilde{p}^{\tau,\xi}(t,s,x,y)-D_x^\vartheta \tilde{p}^{\tau,\xi}(t,s,x,y_{\setminus j},z)\Big]$$
$$\otimes\Big[F_j(s,y_{\setminus j},z)-F_j(s,\theta_{s,\tau}(\xi))\Big]\Big\}\,dy_j.$$

Remembering the notation for $F_j(s,y_{\setminus j},z)$ in (4.7) and that F_j is $\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}$-Hölder continuous with respect to its j-th variable by assumption [**R**], the first component J_1 can be easily controlled using Fubini theorem by

$$\int_{\mathbb{R}^d}\Big|J_1(v,y_{\setminus j},z)\Big|\,dz$$
$$\le C\|F\|_H\int_{\mathbb{R}^d}\Big(\int_{\mathbb{R}^d}|z-y_j|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}}|D_z\partial_v p_h(v,z-y_j)|\,dz\Big)|D_x^\vartheta \tilde{p}^{\tau,\xi}(t,s,x,y)|\,dy_j$$
$$\le C\|F\|_H v^{\frac{1}{\alpha}\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}-\frac{1}{\alpha}-1}\int_{\mathbb{R}^d}|D_x^\vartheta \tilde{p}^{\tau,\xi}(t,s,x,y)|\,dy_j,$$

where in the last passage we used the smoothing effect of the heat-kernel p_h (Equation (4.9)), noticing that

$$\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)} = 1+\frac{\beta-\alpha}{1+\alpha(j-1)} < 1+\alpha,$$

since $\alpha>\beta$ by assumption [**P**]. Using now the identity

$$\frac{\alpha_j+\beta_j}{\alpha}+\frac{1}{\alpha}\Big(\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}-1\Big) = \frac{2\beta_j}{\alpha}, \tag{5.33}$$

we add the integral with respect to v and write that

$$\int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d}\Big|J_1(v,y_{\setminus j},z)\Big|\,dzdv \le C\|F\|_H\int_0^{(s-t)^{\delta_j}}\frac{v^{\frac{2\beta_j}{\alpha}}}{v}\int_{\mathbb{R}^d}|D_x^\vartheta \tilde{p}^{\tau,\xi}(t,s,x,y)|\,dy_j dv$$
$$\le C\|F\|_H(s-t)^{\delta_j\frac{2\beta_j}{\alpha}}\int_{\mathbb{R}^d}|D_x^\vartheta \tilde{p}^{\tau,\xi}(t,s,x,y)|\,dy_j.$$

Adding the integral with respect to the other components $y_{\setminus j}$, we can finally conclude that

$$\int_{\mathbb{R}^{(n-1)d}}\int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d}\Big|J_1(v,y_{\setminus j},z)\Big|\,dzdv\,dy_{\setminus j}$$
$$\le C\|F\|_H(s-t)^{\delta_j\frac{2\beta_j}{\alpha}}\int_{\mathbb{R}^{nd}}|D_x^\vartheta \tilde{p}^{\tau,\xi}(t,s,x,y)|\,dy \tag{5.34}$$
$$\le C\|F\|_H(s-t)^{\delta_j\frac{2\beta_j}{\alpha}-\sum_{k=1}^n\frac{\vartheta_k}{\alpha_k}}.$$

To control the second component J_2, we start applying a Taylor expansion on $\tilde{p}^{\tau,\xi}$ with respect to y_j:

$$J_2(v, y_{\setminus j}, z) = \int_{\mathbb{R}^d} D_z \partial_v p_h(v, z - y_j) \cdot \Big\{ \Delta^{\tau,\xi} F_j(s, y_{\setminus j}, z) \otimes \int_0^1 D_{y_j} D_x^{\vartheta} \tilde{p}^{\tau,\xi}(t, s, x, y_{\setminus j}, y_j + \lambda(z - y_j)) \cdot (z) \Big\} \, d\lambda dy_j. \quad (5.35)$$

We then notice that for any fixed λ in $[0, 1]$, it holds that

$$\begin{aligned}
|\Delta^{\tau,\xi} F_j(s, y_{\setminus j}, z)| &= |F_j(s, y_{\setminus j}, z) - F_j(s, \theta_{s,\tau}(\xi))| \\
&\leq C\|F\|_H \Big\{ \Big| \Big(z - \theta_{s,\tau}(\xi)\Big)_j \Big|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}} + \sum_{k=j+1}^{n} \Big| \Big(y - \theta_{s,\tau}(\xi)\Big)_k \Big|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(k-1)}} \Big\} \\
&\leq C\|F\|_H \Big\{ \Big|\lambda(y_j - z)\Big|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}} + \Big| \Big(y_j + \lambda(z - y_j) - \theta_{s,\tau}(\xi)\Big)_j \Big|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}} \\
&\qquad + \sum_{k=j+1}^{n} \Big| \Big(y - \theta_{s,\tau}(\xi)\Big)_k \Big|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(k-1)}} \Big\} \\
&\leq C\|F\|_H \Big\{ \Big|z - y_j\Big|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}} + \mathbf{d}_{j:n}^{1+\alpha(j-2)+\beta} \Big((y_{\setminus j}, y_j + \lambda(z - y_j)) \Big), \theta_{s,\tau}(\xi) \Big) \Big\},
\end{aligned}$$

where as in (4.7), we denoted

$$(y_{\setminus j}, y_j + \lambda(z - y_j)) := (y_1, \ldots, y_{j-1}, y_1, \ldots, y_{j-1}, y_j + \lambda(z - y_j), y_{j+1}, \ldots, y_n).$$

We can thus split J_2 as

$$|J_2(v, y_{\setminus j}, z)| \leq C\|F\|_H \int_0^1 \Big(J_{2,1} + J_{2,2} \Big)(v, y_{\setminus j}, z, \lambda) \, d\lambda, \quad (5.36)$$

where we denoted for simplicity:

$$\begin{aligned}
J_{2,1}(v, y_{\setminus j}, z, \lambda) &:= \int_{\mathbb{R}^d} |z - y_j|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}+1} |D_z \partial_v p_h(v, z - y_j)| \\
&\qquad \times |D_{y_j} D_x^{\vartheta} \tilde{p}^{\tau,\xi}(t, s, x, y_{\setminus j}, y_j + \lambda(z - y_j))| \, dy_j \\
J_{2,2}(v, y_{\setminus j}, z, \lambda) &:= \int_{\mathbb{R}^d} |z - y_j| \, |D_z \partial_v p_h(v, z - y_j)| \\
&\qquad \times |D_{y_j} D_x^{\vartheta} \tilde{p}^{\tau,\xi}(t, s, x, y_{\setminus j}, y_j + \lambda(z - y_j))| \\
&\qquad\quad \times \mathbf{d}_{j:n}^{1+\alpha(j-2)+\beta}((y_{\setminus j}, y_j + \lambda(z - y_j)), \theta_{s,\tau}(\xi)) \, dy_j
\end{aligned}$$

Adding now the integral with respect to z, the first term $J_{2,1}$ can be rewritten as

$$\begin{aligned}
\int_{\mathbb{R}^d} J_{2,1}(v, y_{\setminus j}, z, \lambda) \, dz &\leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |z - y_j|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}+1} |D_z \partial_v p_h(v, z - y_j)| \\
&\qquad \times |D_{y_j} D_x^{\vartheta} \tilde{p}^{\tau,\xi}(t, s, x, y_{\setminus j}, y_j + \lambda(z - y_j))| \, dy_j dz.
\end{aligned}$$

Fubini Theorem and the change of variables $\tilde{z} = z - y_j$ and $\tilde{y}_j = y_j + \lambda \tilde{z}$ allow then to split the integrals in the following way:

$$\begin{aligned}
&\int_{\mathbb{R}^d} J_{2,1}(v, y_{\setminus j}, z, \lambda) \, dz \\
&\quad \leq \Big(\int_{\mathbb{R}^d} |\tilde{z}|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}+1} |D_{\tilde{z}} \partial_v p_h(v, \tilde{z})| \, d\tilde{z} \Big) \Big(\int_{\mathbb{R}^d} |D_{\tilde{y}_j} D_x^{\vartheta} \tilde{p}^{\tau,\xi}(t, s, x, y_{\setminus j}, \tilde{y}_j)| \, d\tilde{y}_j \Big).
\end{aligned}$$

Noticing now from assumption [**P**] that

$$\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}+1 \;=\; 1-\frac{\beta-\alpha}{1+\alpha(j-1)}+1 \;<\; 2-(1-\alpha) \;=\; 1+\alpha,$$

we can use the smoothing effect of the heat-kernel p_h (Equation (4.9)) to show that

$$\int_{\mathbb{R}^d} J_{2,1}(v,y_{\setminus j},z,\lambda)\,dz \;\le\; \frac{v^{\frac{1+\alpha(j-2)+\beta}{\alpha(1+\alpha(j-1))}}}{v}\int_{\mathbb{R}^d}|D_{\tilde{y}_j}D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y_{\setminus j},\tilde{y}_j)|\,d\tilde{y}_j.$$

Remembering Equation (5.33), we can add the integral with respect to v and show that

$$\int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d} J_{2,1}(v,y_{\setminus j},z,\lambda)\,dz\,dv \\ \le\; (s-t)^{\delta_j\frac{2\beta_j+1}{\alpha}}\int_{\mathbb{R}^d}|D_{\tilde{y}_j}D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y_{\setminus j},\tilde{y}_j)|\,d\tilde{y}_j.$$

Adding the integral with respect to $y_{\setminus j}$, we can conclude with $J_{2,1}$ that

$$\int_{\mathbb{R}^{(n-1)d}}\int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d} J_{2,1}(v,y_{\setminus j},z,\lambda)\,dz\,dv\,dy_{\setminus j} \\ \le\; C(s-t)^{\delta_j\frac{2\beta_j+1}{\alpha}}\int_{\mathbb{R}^{nd}}|D_{y_j}D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)|\,dy \;\le\; C(s-t)^{\delta_j\frac{2\beta_j+1}{\alpha}-\frac{1}{\alpha_j}-\sum_{k=1}^n\frac{\vartheta_k}{\alpha_k}}, \quad (5.37)$$

where, for simplicity, we have changed back the variable $\tilde{y}_j$ with y_j.
To control instead the term $J_{2,2}$ (cf. Equation (5.36)), we can use again Fubini theorem and the changes of variables $\tilde{z}=z-y_j$, $\tilde{y}_j=y_j+\lambda\tilde{z}$ to split the integrals and show that

$$\int_{\mathbb{R}^d} J_{2,2}(v,y_{\setminus j},z,\lambda)\,dz \;\le\; \left(\int_{\mathbb{R}^d}|\tilde{z}|\,|D_{\tilde{z}}\partial_v p_h(v,\tilde{z})|\,d\tilde{z}\right) \\ \times\left(\int_{\mathbb{R}^d}|D_{\tilde{y}_j}D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y_{\setminus j},\tilde{y}_j)|\mathbf{d}_{j:n}^{1+\alpha(j-2)+\beta}((y_{\setminus j},\tilde{y}_j),\theta_{s,\tau}(\xi))d\tilde{y}_j\right) \\ \le\; \frac{1}{v}\int_{\mathbb{R}^d}|D_{y_j}D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)|\mathbf{d}_{j:n}^{1+\alpha(j-2)+\beta}(y,\theta_{s,\tau}(\xi))dy_j dv,$$

where in the second inequality we used the smoothing effect of the heat-kernel p_h (Equation (4.9)) and changed back the variable $\tilde{y}_j$ with y_j for simplicity. It then follows that

$$\int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d} J_{2,2}(v,z,y_{\setminus j})\,dzdv \\ \le\; (s-t)^{\delta_j\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d}|D_{y_j}D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)|\mathbf{d}_{j:n}^{1+\alpha(j-2)+\beta}(y,\theta_{s,\tau}(\xi))dy_j.$$

Taking now $\tau=t$ and $\xi=x$, we conclude with $J_{2,2}$ applying the partial smoothing effect (Equation (5.30)) of $\tilde{p}^{\tau,\xi}$ under the hypothesis $1+\alpha(j-2)+\beta\le\alpha(1+\alpha(j-1))$ (see Equation (5.31)) to write that

$$\int_{\mathbb{R}^{(n-1)d}}\int_0^{(s-t)^{\delta_j}} v^{\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^d} J_{2,2}(v,z,y_{\setminus j})\,dz\,dvdy_{\setminus j} \\ \le\; (s-t)^{\delta_j\frac{\alpha_j+\beta_j}{\alpha}}\int_{\mathbb{R}^{nd}}|D_{y_j}D_x^{\vartheta}\tilde{p}^{\tau,\xi}(t,s,x,y)|\mathbf{d}_{j:n}^{1+\alpha(j-2)+\beta}(y,\theta_{s,\tau}(\xi))dy \\ \le\; C(s-t)^{\delta_j\frac{\alpha_j+\beta_j}{\alpha}+\frac{1+\alpha(j-2)+\beta}{\alpha}-\frac{1}{\alpha_j}-\sum_{k=1}^n\frac{\vartheta_k}{\alpha_k}}. \quad (5.38)$$

Looking back to Equations (5.32), (5.34), (5.37) and (5.38), we can finally choose the right δ_j. Since $s-t \le T-t < 1$ by hypothesis [**ST**], it is enough to take δ_j such that the quantities

$$\delta_j \frac{\alpha_j+\beta_j-1}{\alpha} + \frac{1+\alpha(j-2)+\beta}{\alpha}, \qquad \delta_j \frac{2\beta_j}{\alpha}, \qquad \delta_j \frac{2\beta_j+1}{\alpha} - \frac{1}{\alpha_j}$$

and

$$\delta_j \frac{\alpha_j+\beta_j}{\alpha} + \frac{1+\alpha(j-2)+\beta}{\alpha} - \frac{1}{\alpha_j}$$

are bigger than β/α. This is true if, for example, we choose

$$\delta_j \;=\; \frac{[1+\alpha(j-2)][1+\alpha(j-1)]}{1+\alpha(j-2)-\beta}.$$

We have thus concluded th proof. □

5.2 Proof of Proposition 3.6

We have now all the tools necessary to prove the a priori estimates in Proposition 3.6. In Lemma 5.3 below, we will state the estimates for the supremum norms of the solution and its non-degenerate gradient while the controls of the Hölder moduli of the solution and its gradient with respect to the non-degenerate variable are given in Lemmas 5.7 and 5.8, respectively.
The Schauder estimates (Theorem 2.3) for a solution u in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ of IPDE (1.1) will then follows immediately.

Lemma 5.3 (Supremum Estimates)**.** *Let u be as in Equation* (5.27)*. Then, there exists a constant $C \ge 1$ such that for any t in $[0,T]$ and any x in $\mathbb{R}^{nd}$,*

$$|u(t,x)| + |D_{x_1}u(t,x)| \;\le\; C\Big[\|u_T\|_{C^{\alpha+\beta}_{b,d}} + \|f\|_{L^\infty(C^{\beta}_{b,d})} + \|F\|_H \|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\Big].$$

Proof. As indicated above, we can control the supremum norm of u and its gradient with respect to x_1 analyzing separately the contributions from the proxy $\tilde{u}^{\tau,\xi}$, that have already been handled in Lemma 4.4, and those from the perturbative term $R^{\tau,\xi}(s,x)$. To control the contribution $\int_t^T D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)\,ds$, we start splitting it up in the following way:

$$\begin{aligned}\int_t^T & D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)\,ds \\ &= \sum_{j=1}^n \int_t^T \int_{\mathbb{R}^{nd}} D_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y)\Big[F_j(s,y) - F_j(s,\theta_{s,\tau}(\xi))\Big]\cdot D_{y_j}u(s,y)\,dyds \\ &=: \sum_{j=1}^n I_j(t,x). \end{aligned} \tag{5.39}$$

Since by hypothesis u has a proper derivative with respect to the first variable x_1, it is possible to bound I_1 through

$$|I_1(t,x)| \;\le\; C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\int_t^T\int_{\mathbb{R}^{nd}}\Big|D_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y)\Big|\mathbf{d}^\beta(y,\theta_{s,\tau}(\xi))\,dyds.$$

We take now $(\tau,\xi)=(t,x)$ so that $\theta_{s,\tau}(\xi)=\tilde{m}^{\tau,\xi}_{s,t}$ (cf. Equation (3.5) in Lemma 3.1) and we then use the smoothing effect for the frozen density $\tilde{p}^{\tau,\xi}$ (Equation (3.6)) to show that

$$|I_1(t,x)| \leq C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}(T-t)^{\frac{\beta+\alpha-1}{\alpha}}. \tag{5.40}$$

Hence, it holds that $|I_1(t,x)| \leq C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}$, since $T\leq 1$ and $\alpha+\beta>1$ by assumptions [**ST**] and [**P**].
The control for the terms I_j with $j>1$ can be obtained easily from the second Besov control (Lemma 5.2). For this reason, we start applying integration by parts formula to show that

$$|I_j(t,x)| = \left|\int_t^T\int_{\mathbb{R}^{nd}} D_{y_j}\cdot\left\{D_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y)\Delta^{\tau,\xi}F_j(s,y)\right\}u(s,y)\,dyds\right|,$$

where we exploited the same notations for $\Delta^{\tau,\xi}F_j$ given in (5.29). We can then use identification (4.10) and duality in Besov spaces (4.11) to write that

$$|I_j(t,x)| \leq \\ \|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\int_{\mathbb{R}^{(n-1)d}}\left\|D_{y_j}\cdot\left\{D_{x_1}\tilde{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\cdot)\Delta^{\tau,\xi}F_j(s,y_{\smallsetminus j},\cdot)\right\}\right\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}dy_{\smallsetminus j}.$$

Taking now $(\tau,\xi)=(t,x)$, the second Besov control (Lemma 5.2) can be applied to show that

$$\begin{aligned}|I_j(t,x)| &\leq C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\int_t^T(s-t)^{\frac{\beta-1}{\alpha}}\,ds\\ &\leq C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}(T-t)^{\frac{\beta+\alpha-1}{\alpha}}.\end{aligned} \tag{5.41}$$

Since $T\leq 1$ by assumption [**ST**], we can conclude that $|I_j(t,x)|\leq C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}$.
The control on the pertubative term

$$\int_t^T \tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)\,ds$$

can be obtained in a similar way. Namely, Inequalities (5.40) and (5.41) hold again with $(T-t)^{\frac{\beta+\alpha-1}{\alpha}}$ replaced by $(T-t)^{\frac{\beta+\alpha}{\alpha}}$. □

As already specified in the previous sub-section, there is a big difference between the non-degenerate case $i=1$, where $\alpha+\beta$ is in $(1,2)$ and we have to deal with a proper derivative, and the other degenerate situations ($i>1$), where instead $(\alpha+\beta)/(1+\alpha(i-1))<1$ and the norm is calculated directly on the function. Again, we are going to analyze the two cases separately. Lemma 4.5 will focus on the non-degenerate setting ($i=1$) while lemma 4.6 will concerns the degenerate one ($i>1$).

Moreover, we will need to divide the proofs in two cases, depending on which regime we are considering. Since the global off-diagonal regime, i.e. when $T-t\leq c_0\mathbf{d}^\alpha(x,x')$, will work essentially as the already shown Schauder estimates (Proposition 3.3) for the proxy, the proof will be quite shorter.

Instead, in the global diagonal case, such that $T-t \geq c_0\mathbf{d}^\alpha(x,x')$, when a time integration is involved (for example in the control of the frozen Green kernel or the perturbative term), two different situations appear. There are again a local off-diagonal regime if $s-t \leq c_0\mathbf{d}^\alpha(x,x')$ and a local diagonal regime when $s-t \geq c_0\mathbf{d}^\alpha(x,x')$. In order to handle these terms properly, the key tool is to be able to change the freezing points depending on which regime we are. It seems reasonable that, when the spatial points are in a local diagonal regime, the auxiliary frozen densities are considered for the same freezing parameter and conversely that, in the local off-diagonal regime, the densities are frozen along their own spatial argument. For this reason, we have postponed the relative proofs in two specific sub-sections.

Before presenting the main results of this section, we are going to state three auxiliary estimates we will need below. We refer to the Section A.2 for a precise proof of these results.
The first one concerns the sensitivity of the Hölder flow $\theta_{s,t}$ with respect to the initial point. Indeed,

Lemma 5.4 (Controls on the Flows)**.** *Let $t<s$ be two points in $[0,T]$ and x,x' two points in $\mathbb{R}^{nd}$. Then, there exists a constant $C\geq 1$ such that*

$$\mathbf{d}(\theta_{s,t}(x),\theta_{s,t}(x')) \leq C\|F\|_H\Big[\mathbf{d}(x,x')+(s-t)^{1/\alpha}\Big].$$

The second result is the following:

Lemma 5.5. *Let $t<s$ be two points in $[0,T]$ and x,x' two points in $\mathbb{R}^{nd}$ and y,y' two points in $\mathbb{R}^{nd}$ such that $y_1=y_1'$. Then, there exists a constant $C\geq 1$ such that*

$$\Big|(\tilde{m}^{t,x}_{s,t}(y)-\tilde{m}^{t,x'}_{s,t}(y'))_1\Big| \leq C\|F\|_H\Big[(s-t)\mathbf{d}^\beta(x,x')+(s-t)^{\frac{\alpha+\beta}{\alpha}}\Big].$$

Finally, the impact of the freezing point in the linearization procedure is the argument of this last Lemma. Namely,

Lemma 5.6. *Let t be in $[0,T]$ and x,x' two points in $\mathbb{R}^{nd}$. Then, there exists a constant $C\geq 1$ such that*

$$\mathbf{d}(\tilde{m}^{t,x}_{t_0,t}(x'),\tilde{m}^{t,x'}_{t_0,t}(x')) \leq Cc_0^{\frac{1}{1+\alpha(n-1)}}\|F\|_H\mathbf{d}(x,x')$$

where t_0 is the change of regime time defined in (4.16)*.*

Thanks to the above controls, we will eventually prove the following results:

Lemma 5.7 (Controls on Hölder Moduli: Non-Degenerate)**.** *Let x,x' be in $\mathbb{R}^{nd}$ such that $x_j=x_j'$ for any $j\neq 1$ and u as in Equation* (5.27)*. Then, there exists a constant $C\geq 1$ such that for any t in $[0,T]$,*

$$\begin{aligned}\Big|D_{x_1}u(t,x)-D_{x_1}u(t,x')\Big| \leq C\Big\{&c_0^{\frac{\alpha+\beta-2}{\alpha}}\Big(\|u_T\|_{C^{\alpha+\beta}}+\|f\|_{L^\infty(C^\beta)}\Big)\\ &+\Big(c_0^{\frac{\alpha+\beta-1}{1+\alpha(n-1)}}+c_0^{\frac{\alpha+\beta-2}{\alpha}}\|F\|_H\Big)\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\Big\}\mathbf{d}^{\alpha+\beta-1}(x,x').\end{aligned}$$

We can point out now the analogous result in the degenerate setting.

Lemma 5.8 (Controls on Hölder Moduli: Degenerate)**.** *Let i be in $[\![1,n]\!]$ and x, x' in $\mathbb{R}^{nd}$ such that $x_j = x'_j$ for any $j \neq i$ and u as in Equation* (5.27)*. Then, there exists a constant $C \geq 1$ such that for any t in $[0,T]$,*

$$\Big|u(t,x) - u(t,x')\Big| \leq C\Big\{c_0^{\frac{\beta-\gamma_i}{\alpha}}\Big(\|u_T\|_{C^{\alpha+\beta}} + \|f\|_{L^\infty(C^\beta)}\Big) + \Big(c_0^{\frac{\alpha+\beta}{1+\alpha(n-1)}} + c_0^{\frac{\beta-\gamma_i}{\alpha}}\|F\|_H\Big)\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\Big\}\mathbf{d}^{\alpha+\beta}(x,x').$$

Off-Diagonal Regime

We focus here on the proof of the controls on the Hölder moduli, either in the non-degenerate setting (Lemma 5.7) and in the degenerate one (Lemma 5.8), when a off-diagonal regime is assumed. For this reason, all the statements presented in this subsection will tacitly assume that $T-t \leq c_0\mathbf{d}^\alpha(x,x')$ for some given (t,x,x') in $[0,T]\times\mathbb{R}^{2nd}$. To show these two controls, we will need to adapt the auxiliary estimates above to the off-diagonal regime case we consider here. Namely,

$$\mathbf{d}(\tilde{m}^{t,x}_{T,t}(x), \tilde{m}^{t,x'}_{T,t}(x')) = \mathbf{d}(\theta_{T,t}(x), \theta_{T,t}(x')) \leq C\|F\|_H\mathbf{d}(x,x'); \tag{5.42}$$

$$\text{if } x_1 = x'_1, \quad \Big|\Big(\tilde{m}^{t,x}_{T,t}(x) - \tilde{m}^{t,x'}_{T,t}(x')\Big)_1\Big| \leq C\|F\|_H\mathbf{d}^{\alpha+\beta}(x,x') \tag{5.43}$$

They can be obtained easily from Equation (3.5) in Lemma 3.1 and the sensitivity controls (Lemmas 5.4 and 5.5, respectively), taking $s = T$ and $(y,y') = (x,x')$.

Proof of Lemma 5.7 in the Off-Diagonal Regime. From the Duhamel-type Expansion (5.28), we can represent a mild solution u of IPDE (1.1) as

$$\begin{aligned}&|D_{x_1}u(t,x) - D_{x_1}u(t,x')| \\ &\quad\leq \Big|D_{x_1}\tilde{P}^{\tau,\xi}_{T,t}u_T(x) - D_{x_1}\tilde{P}^{\tau',\xi'}_{T,t}u_T(x')\Big| + \Big|D_{x_1}\tilde{G}^{\tau,\xi}_{T,t}f(t,x) - D_{x_1}\tilde{G}^{\tau',\xi'}_{T,t}f(t,x')\Big| \\ &\qquad + \Big|\int_t^T D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x) - D_{x_1}\tilde{P}^{\tau',\xi'}_{s,t}R^{\tau',\xi'}(s,x')\,ds\Big|,\end{aligned}$$

for any fixed $(\tau,\xi),(\tau',\xi')$ in $[0,T]\times\mathbb{R}^{nd}$ After possible differentiations, we will choose $\tau = \tau' = t$, $\xi = x$ and $\xi' = x'$ in order to exploit the sensitivity Controls (5.43) and (5.42).

Control on the frozen semigroup. It can be handled following the analogous part in the proof of the Hölder control for the proxy (Lemma 4.5). The only difference is that we cannot control

$$\mathbf{d}(\tilde{m}^{\tau,\xi}_{T,t}(x), \tilde{m}^{\tau',\xi'}_{T,t}(x'))$$

in Equation (4.18) using the affinity of the mapping $x \to \tilde{m}^{\tau,\xi}_{T,t}(x)$, since the two freezing point are now different. Instead, we can take $\tau = \tau' = t$, $\xi = x$ and $\xi' = x'$ and apply the sensitivity control (5.42) to write that

$$\mathbf{d}(\tilde{m}^{\tau,\xi}_{T,t}(x), \tilde{m}^{\tau',\xi'}_{T,t}(x')) = \mathbf{d}(\theta_{T,t}(x), \theta_{T,t}(x')) \leq C\|F\|_H\mathbf{d}(x,x').$$

Control on the Green kernel. It follows immediately from the proof of the Hölder control (Lemma 4.5) for the proxy, noticing that $t_0 = T$, since we are in the off-diagonal regime.

Control on the perturbative error. Since we do not exploit the difference of the spatial points (x, x') in the off-diagonal regime but instead we control the two contributions separately, we can rely on the controls on the supremum norms we have already shown in Lemma 5.3. Namely, we start writing that

$$\begin{aligned}\Big|\int_t^T D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x) - D_{x_1}\tilde{P}^{\tau',\xi'}_{s,t}R^{\tau',\xi'}(s,x')\,ds\Big| \\ \leq \Big|\int_t^T D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)\,ds\Big| + \Big|\int_t^T D_{x_1}\tilde{P}^{\tau',\xi'}_{s,t}R^{\tau',\xi'}(s,x')\,ds\Big|. \quad (5.44)\end{aligned}$$

Then, we can follow the same reasonings of Lemma 5.3 concerning the remainder term (cf. Equations (5.39), (5.40) and (5.41)) to show that

$$\Big|\int_t^T D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)\,ds\Big| \leq C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}(T-t)^{\frac{\alpha+\beta-1}{\alpha}}. \quad (5.45)$$

Using it in the above Equation (5.44), we can finally conclude that

$$\Big|\int_t^T D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x) - D_{x_1}\tilde{P}^{\tau',\xi'}_{s,t}R^{\tau',\xi'}(s,x')\,ds\Big| \leq C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\mathbf{d}^{\alpha+\beta-1}(x,x'), \quad (5.46)$$

remembering that we assumed to be in the off-diagonal regime, i.e. $T - t \leq c_0\mathbf{d}^\alpha(x,x')$ for some $c_0 \leq 1$.

Proof of Lemma 5.8 in the Off-Diagonal Regime. As done before, we are going to analyze separately the single terms appearing from the Duhamel-type Representation (5.27) of a solution u:

$$\begin{aligned}|u(t,x) - u(t,x')| \leq \Big|\tilde{P}^{\tau,\xi}_{T,t}u_T(x) - \tilde{P}^{\tau',\xi'}_{T,t}u_T(x')\Big| + \Big|\tilde{G}^{\tau,\xi}_{T,t}f(t,x) - \tilde{G}^{\tau',\xi'}_{T,t}f(t,x')\Big| \\ + \Big|\int_t^T \tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x) - \tilde{P}^{\tau',\xi'}_{s,t}R^{\tau',\xi'}(s,x')\,ds\Big|,\end{aligned}$$

for some $(\tau,\xi),(\tau',\xi')$ in $[0,T]\times\mathbb{R}^{nd}$ fixed but to be chosen later as $\tau = \tau' = t$, $\xi = x$ and $\xi' = x'$.

Control on the frozen semigroup. We can essentially follow the proof of the Hölder control (Lemma 4.6) for the proxy. However, this time we cannot exploit the affinity of the mapping $x \to \tilde{m}^{\tau,\xi}_{T,t}(x)$ to control the difference

$$\Big|u_T(\tilde{m}^{\tau,\xi}_{T,t}(x) - z) - u_T(\tilde{m}^{\tau,\xi}_{T,t}(x') - z)\Big|.$$

Instead, we notice now that we can bound it as

$$\begin{aligned}\Big|u_T(\tilde{m}^{\tau,\xi}_{T,t}(x) - z) - u_T(\tilde{m}^{\tau,\xi}_{T,t}(x') - z)\Big| \\ \leq C\|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big[\mathbf{d}^{\alpha+\beta}\Big(\tilde{m}^{\tau,\xi}_{T,t}(x), \tilde{m}^{\tau,\xi}_{T,t}(x')\Big) + \Big|\Big(\tilde{m}^{\tau,\xi}_{T,t}(x) - \tilde{m}^{\tau,\xi}_{T,t}(x')\Big)_1\Big|\Big],\end{aligned}$$

since u_T is differentiable and thus Lipschitz continuous, in the first non-degenerate variable.
Taking now $\tau = \tau' = t$, $\xi = x$ and $\xi' = x'$, we can use the sensitivity Controls (5.42)-(5.43) (noticing that by assumption, $x_1 = x_1'$) to write that

$$\left|u_T(\tilde{m}_{T,t}^{\tau,\xi}(x) - z) - u_T(\tilde{m}_{T,t}^{\tau,\xi}(x') - z)\right| \leq C\|F\|_H \|u_T\|_{C_{b,d}^{\alpha+\beta}} \mathbf{d}^{\alpha+\beta}(x, x').$$

Control on the Green kernel. It can be obtained following the analogous part in the proof of the Hölder control (Lemma 4.6) for the proxy. Similarly to the paragraph "Control on the frozen semigroup" in the previous proof, we need to take $(\tau, \xi) = (t, x)$, $(\tau, \xi') = (t, x)$ and apply the sensitivity Control (5.42) to bound the term

$$\mathbf{d}(\tilde{m}_{T,t}^{\tau,\xi}(x), \tilde{m}_{T,t}^{\tau',\xi'}(x'))$$

appearing in Equation (4.25).

Control on the perturbative error. The proof of this estimate essentially matches the previous, analogous one in the non-degenerate setting. Namely, Equations (5.44), (5.45) and (5.46) hold again with $(T-t)^{\frac{\beta+\alpha}{\alpha}}$ instead of $(T-t)^{\frac{\beta+\alpha-1}{\alpha}}$.

Diagonal Regime

Since the aim of this section is to prove Lemmas 5.7 and 5.8 when a diagonal regime is assumed, we will assume from this point further that $T - t \geq c_0 \mathbf{d}^\alpha(x, x')$ for some given (t, x, x') in $[0, T] \times \mathbb{R}^{2nd}$.
As preannounced in the introduction of this section, we need here a modification of the Duhamel-type Representation (3.16) that allows to change the freezing points along the time integration variable. Remembering the previous notations for $\tilde{G}_{r,v}^{\tau,\xi}$ and $R^{\tau,\xi}$ in (4.2) and (3.15) respectively, it holds that

Lemma 5.9 (Change of Frozen Point)**.** *Let (τ, ξ) be a freezing couple in $[0, T] \times \mathbb{R}^{nd}$ and $\tilde{\xi}$ another freezing point in $\mathbb{R}^{nd}$. Then, any classical solution u in $L^\infty(0, T; C_{b,d}^{\alpha+\beta}(\mathbb{R}^{nd}))$ of IPDE (1.1) can be represented for any (t, x) in $[0, T] \times \mathbb{R}^{nd}$ as*

$$\begin{aligned} u(t,x) \;=\; & \tilde{P}_{T,t}^{\tau,\tilde{\xi}} u_T(x) + \tilde{G}_{t_0,t}^{\tau,\xi} f(t,x) + \tilde{G}_{T,t_0}^{\tau,\tilde{\xi}} f(t,x) \\ & + \int_t^{t_0} \tilde{P}_{s,t}^{\tau,\xi} R^{\tau,\xi}(s,x)\, ds + \int_{t_0}^T \tilde{P}_{s,t}^{\tau,\tilde{\xi}} R^{\tau,\tilde{\xi}}(s,x)\, ds + \tilde{P}_{t_0,t}^{\tau,\xi} u(t_0,x) - \tilde{P}_{t_0,t}^{\tau,\tilde{\xi}} u(t_0,x), \end{aligned} \tag{5.47}$$

where t_0 is the change of regime time defined in (4.16).

Proof. Fixed t in $(0, T)$, we start considering another point r in (t, T). On $(0, r)$, it is clear that u is again a mild solution of IPDE (1.1) but with terminal condition $u(r, x)$. Then, Duhamel Expansion (3.16) can be applied with respect to the frozen couple (τ, ξ), allowing us to write that

$$u(t,x) \;=\; \tilde{P}_{r,t}^{\tau,\xi} u_T(x) + \int_t^r \tilde{P}_{s,t}^{\tau,\xi} f(s,x)\, ds + \int_t^r \tilde{P}_{s,t}^{\tau,\xi} R^{\tau,\xi} u(s,x)\, ds.$$

Noticing that u is independent from r, it is possible now to differentiate the above equation with respect to r in (t,T) in order to show that

$$0 \,=\, \partial_r\Big[\tilde{P}^{\tau,\xi}_{r,t}u(r,x)\Big] + \tilde{P}^{\tau,\xi}_{r,t}f(r,x) + \tilde{P}^{\tau,\xi}_{r,t}R^{\tau,\xi}(r,x). \tag{5.48}$$

We highlight now that the above expression holds for any chosen frozen couple (τ,ξ) and any fixed time r. Thus, it is possible to integrate it with respect to r for a fixed ξ between t and t_0 and for another frozen point $\tilde{\xi}$ between t_0 and T, leading to

$$\begin{aligned} 0 \,=\, & \tilde{P}^{\tau,\xi}_{t_0,t}u(t_0,x) - \tilde{P}^{\tau,\xi}_{t,t}u(t,x) + \int_t^{t_0} \tilde{P}^{\tau,\xi}_{r,t}f(r,x)\,dr + \int_t^{t_0} \tilde{P}^{\tau,\xi}_{r,t}R^{\tau,\xi}(r,x)\,dr \\ & + \tilde{P}^{\tau,\tilde{\xi}}_{T,t}u(T,x) - \tilde{P}^{\tau,\tilde{\xi}}_{t_0,t}u(t_0,x) + \int_{t_0}^{T} \tilde{P}^{\tau,\tilde{\xi}}_{r,t}f(r,x)\,dr + \int_{t_0}^{T} \tilde{P}^{\tau,\tilde{\xi}}_{r,t}R^{\tau,\tilde{\xi}}(r,x)\,dr. \end{aligned}$$

With our previous notations, the above expression can be finally rewritten as

$$\begin{aligned} 0 \,=\, & \tilde{P}^{\tau,\xi}_{t_0,t}u(t_0,x) - u(t,x) + \tilde{G}^{\tau,\xi}_{t_0,t}f(t,x) + \int_t^{t_0} \tilde{P}^{\tau,\xi}_{r,t}R^{\tau,\xi}(r,x)\,dr \\ & + \tilde{P}^{\tau,\tilde{\xi}}_{T,t}u_T(x) - \tilde{P}^{\tau,\tilde{\xi}}_{t_0,t}u(t_0,x) + \tilde{G}^{\tau,\tilde{\xi}}_{T,t_0}f(t,x) + \int_{t_0}^{T} \tilde{P}^{\tau,\tilde{\xi}}_{r,t}R^{\tau,\tilde{\xi}}(r,x)\,dr \end{aligned}$$

and we have concluded. □

Similarly to the off-diagonal case, we are going to apply the auxiliary estimates associated with the proxy (Lemmas 5.5 and 5.6) in the current diagonal regime. Namely, taking $s = t_0$ and $(y,y') = (x,x)$ in Lemma 5.5, we know that there exists a constant $C \geq 1$ such that for any t in $[0,T]$ and any x, x' in $\mathbb{R}^{nd}$,

$$\text{if } x_1 \,=\, x'_1, \quad \Big|(\tilde{m}^{t,x}_{t_0,t}(x) - \tilde{m}^{t,x'}_{t_0,t}(x))_1\Big| \,\leq\, C\|F\|_H \mathbf{d}^{\alpha+\beta}(x,x'). \tag{5.49}$$

Moreover, in order to control the perturbative term when a local diagonal regime appears, i.e. when the time integration variable s is in $[t_0,T]$, we will quite often use a Taylor expansion on the frozen density. To be able to exploit the already proven controls, such that the smoothing effect for the frozen density (Equation (3.6)) or the second Besov control (Lemma 5.2), we will need the following:

$$\text{if } s-t \geq c_0\mathbf{d}^{\alpha}(x,x'), \quad \Big|D^{\vartheta}_x\tilde{p}^{\tau,\xi'}(t,s,x+\lambda(x'-x),y)\Big| \,\leq\, C\Big|D^{\vartheta}_x\tilde{p}^{\tau,\xi'}(t,s,x,y)\Big|, \tag{5.50}$$

for any multi-index ϑ in $\mathbb{N}^d$ such that $|\vartheta| \leq 2$ and any λ in $[0,1]$. The proof of these results can be found in Section A.2.

We are now ready to prove Lemmas 5.7 and 5.8 when a global diagonal regime is considered.

Proof of Lemma 5.7 in the Diagonal Regime. We start recalling that in Lemma 5.7 we assumed fixed a time t in $[0,T]$ and two spatial points x, x' in $\mathbb{R}^{nd}$ such that $x_j = x'_j$ if $j \neq 1$.

From the above Representation (5.47) and the Duhamel-type Formula (3.16), we know that

$$\begin{aligned}
D_{x_1}u(t,x) - D_{x_1}u(t,x') \;=\; & \Big(D_{x_1}\tilde{P}^{\tau,\tilde{\xi}}_{T,t}u_T(x) - D_{x_1}\tilde{P}^{\tau',\xi'}_{T,t}u_T(x')\Big) \\
& + \Big(D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x) + D_{x_1}\tilde{G}^{\tau,\tilde{\xi}}_{T,t_0}f(t,x) - D_{x_1}\tilde{G}^{\tau',\xi'}_{T,t}f(t,x')\Big) \\
+ \Big(\int_t^{t_0} D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)\,ds + \int_{t_0}^{T} D_{x_1}\tilde{P}^{\tau,\tilde{\xi}}_{s,t}R^{\tau,\tilde{\xi}}(s,x)\,ds - \int_t^T D_{x_1}\tilde{P}^{\tau',\xi'}_{s,t}R^{\tau',\xi'}(s,x')\,ds\Big) \\
& + \Big(D_{x_1}\tilde{P}^{\tau,\xi}_{t_0,t}u(t_0,x) - D_{x_1}\tilde{P}^{\tau,\tilde{\xi}}_{t_0,t}u(t_0,x)\Big),
\end{aligned}$$

for some freezing couples $(\tau,\xi),(\tau,\tilde{\xi}),(\tau',\xi')$ in $[0,T]\times\mathbb{R}^{nd}$ fixed but to be chosen later. To help the readability of the following, we assume from this point further $\tau=\tau'$ and $\tilde{\xi}=\xi'$.

Control on frozen semigroup. We start focusing on the control of the frozen semigroup, i.e.

$$\Big|D_{x_1}\tilde{P}^{\tau,\xi'}_{T,t}u_T(x) - D_{x_1}\tilde{P}^{\tau,\xi'}_{T,t}u_T(x')\Big|.$$

Since the freezing couples coincide, the control on the frozen semigroup can be obtained following the proof of the Hölder control (Lemma 4.5) for the proxy.

Control on the Green kernel. As done before, we split the analysis with respect to the change of regime time t_0. Namely, we write

$$\begin{aligned}
&\Big|D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x) + D_{x_1}\tilde{G}^{\tau,\tilde{\xi}}_{T,t_0}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi'}_{T,t}f(t,x')\Big| \\
&\qquad \le \Big|D_{x_1}\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi'}_{t_0,t}f(t,x')\Big| + \Big|D_{x_1}\tilde{G}^{\tau,\tilde{\xi}}_{T,t_0}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi'}_{T,t_0}f(t,x')\Big|.
\end{aligned}$$

While in the local off-diagonal regime, the first term in the r.h.s. of the above expression can be handled as in the global off-diagonal regime, the local diagonal regime contribution represented by

$$\Big|D_{x_1}\tilde{G}^{\tau,\tilde{\xi}}_{T,t_0}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi'}_{T,t_0}f(t,x')\Big| \;=\; \Big|D_{x_1}\tilde{G}^{\tau,\xi'}_{T,t_0}f(t,x) - D_{x_1}\tilde{G}^{\tau,\xi'}_{T,t_0}f(t,x')\Big|$$

since $\tilde{\xi}=\xi'$, can be controlled following again the proof of the Hölder control (Lemma 4.5) for the proxy.

Control on the discontinuity term. We can now focus on the contribution

$$\Big|D_{x_1}\tilde{P}^{\tau,\xi}_{t_0,t}u(t_0,x) - D_{x_1}\tilde{P}^{\tau,\tilde{\xi}}_{t_0,t}u(t_0,x)\Big|,$$

arising from the change of freezing point in the Representation (5.47).
Since at fixed time t_0, the function u shows the same spatial regularity of u_T, this control can be handled following the paragraph in the proof of the Hölder control for the proxy (Lemma 4.5) concerning the frozen semigroup in the off-diagonal regime. The only main difference is in Equation (4.18) where, this time, we need to take $(\tau,\xi,\xi')=(t,x,x')$ and exploit the sensitivity estimate (Lemma 5.6) to control the quantity

$$\mathbf{d}(\tilde{m}^{\tau,\xi}_{t_0,t}(x),\tilde{m}^{\tau,\xi'}_{t_0,t}(x)).$$

In the end, it is possible to show again (cf. Equation (4.21)) that

$$\left|D_{x_1}\tilde{P}^{\tau,\xi}_{t_0,t}u(t_0,x) - D_{x_1}\tilde{P}^{\tau,\tilde{\xi}}_{t_0,t}u(t_0,x)\right| \leq C\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}c_0^{\frac{\alpha+\beta-1}{\alpha}}\mathbf{d}^{\alpha+\beta-1}(x,x').$$

Control on the perturbative term. We start splitting the analysis into two cases with respect to the critical time t_0 giving the change of regime. Namely, we write

$$\begin{aligned}\left|\int_t^{t_0} D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)\,ds + \int_{t_0}^T D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x)\,ds - \int_t^T D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x')\,ds\right| \\ \leq \left|\int_t^{t_0} D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x) - D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x')\,ds\right| \\ + \left|\int_{t_0}^T D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x) - D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x')\,ds\right|.\end{aligned}$$

We then notice that the local off-diagonal regime represented by

$$\left|\int_t^{t_0} D_{x_1}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x) - D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x')\,ds\right|$$

can be handled following the proof in the global off-diagonal regime of Lemma 5.7. We can then focus our attention on the local diagonal regime, i.e.

$$\left|\int_{t_0}^T D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x) - D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x')\,ds\right|.$$

Since the freezing couples coincide, we can use a Taylor expansion with respect to the first variable x_1 in order to write that

$$\begin{aligned}&\left|\int_{t_0}^T D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x) - D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x')\,ds\right| \\ &\qquad = \left|\int_{t_0}^T\int_{\mathbb{R}^{nd}}\int_0^1 D^2_{x_1}\tilde{p}^{\tau,\xi'}(t,s,x+\lambda(x'-x),y)(x'-x)_1R^{\tau,\xi'}(s,y)\,dydsd\lambda\right|.\end{aligned}$$

Noticing that we are integrating from t_0 to T, Equation (5.50) can be rewritten as

$$\begin{aligned}&\left|\int_{t_0}^T D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x) - D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x')\,ds\right| \\ &\qquad \leq |(x'-x)_1|\sum_{j=1}^n\int_0^1\int_{t_0}^T\Big|\int_{\mathbb{R}^{nd}} D^2_{x_1}\tilde{p}^{\tau,\xi'}(t,s,x,y) \\ &\qquad\qquad \times\Big\{\big[F_j(s,y) - F_j(s,\theta_{s,t}(\xi'))\big]\cdot D_{y_j}u(s,y)\Big\}\,dy\Big|dsd\lambda \\ &\qquad =: |(x-x')_1|\sum_{j=1}^n\int_{t_0}^T I^d_j(s)ds. \qquad (5.51)\end{aligned}$$

As done before, we are going to treat separately the cases $j=1$ and $j>1$. In the first case, the term I^d_1 can be easily controlled by

$$\begin{aligned}I^d_1(s) \leq \|D_{y_1}u\|_{L^\infty(L^\infty)}\int_{\mathbb{R}^{nd}}\left|D^2_{x_1}\tilde{p}^{\tau,\xi'}(t,s,x,y)\right|\left|F_1(s,y) - F_1(s,\theta_{s,t}(\xi'))\right|dy \\ \leq C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}(s-t)^{\frac{\beta-2}{\alpha}}, \quad (5.52)\end{aligned}$$

where in the last passage we used the smoothing effect for the frozen density $\tilde{p}^{\tau,\xi}$ (Equation (3.6)).
On the other side, the case $j > 1$ can be exploited using the second Besov control (Lemma 5.2). For this reason, we start using integration by parts formula to show that

$$I_j^d(s) \;=\; \Big|\int_{\mathbb{R}^{nd}} D_{y_j} \cdot \Big\{ D^2_{x_1}\tilde{p}^{\tau,\xi'}(t,s,x,y) \otimes \Big[F_j(s,y) - F_j(s,\theta_{s,t}(\xi'))\Big]\Big\} u(s,y)\,dy\Big|.$$

Through Duality (4.11) in Besov spaces and the identification in Equation (4.10), we then write that

$$\begin{aligned} I_j^d(s) \;\le\; C\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \int_{\mathbb{R}^{(n-1)d}} \|D_{y_j} \cdot \Big\{ & D^2_{x_1}\tilde{p}^{\tau,\xi'}(t,s,x,y_{\smallsetminus j},\cdot) \\ & \otimes \Big[F_j(s,y_{\smallsetminus j},\cdot) - F_j(s,\theta_{s,t}(\xi'))\Big]\Big\}\|_{B^{-(\alpha_j+\beta_j)}_{1,1}}\,dy_{\smallsetminus j}. \end{aligned}$$

We can now apply the second Besov control (Lemma 5.2) to show that

$$I_j^d(s) \;\le\; C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}(s-t)^{\frac{\beta-2}{\alpha}}. \tag{5.53}$$

Going back at Equations (5.51),(5.52) and (5.53), we then notice that

$$\begin{aligned} &\Big|\int_{t_0}^T D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x) - D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x')\,ds\Big| \qquad (5.54)\\ &\qquad\le\; C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}|(x-x')_1|\int_{t_0}^T (s-t)^{\frac{\beta-2}{\alpha}}ds \\ &\qquad\le\; C\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}|(x-x')_1|(t_0-t)^{\frac{\alpha+\beta-2}{\alpha}}, \end{aligned}$$

where in the last passage we used that $\frac{\alpha+\beta-2}{\alpha} < 0$ to pick the starting point t_0 in the integral.
Using that $t_0 - t = c_0\mathbf{d}^\alpha(x,x')$, we can finally write that

$$\begin{aligned} &\Big|\int_{t_0}^T D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x) - D_{x_1}\tilde{P}^{\tau,\xi'}_{s,t}R^{\tau,\xi'}(s,x')\,ds\Big| \\ &\qquad\le\; Cc_0^{\frac{\alpha+\beta-2}{\alpha}}\|F\|_H\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\mathbf{d}^{\alpha+\beta-1}(x,x'). \end{aligned}$$

Proof of Lemma 5.8 in Diagonal Regime. We conclude this section showing the Hölder control in the degenerate setting when a diagonal regime is assumed. We start recalling that in Lemma 5.8, we assumed fixed a time t in $[0,T]$ and two spatial points x, x' in $\mathbb{R}^{nd}$ such that $x_j = x'_j$ if $j \neq i$ for some i in $[\![2,n]\!]$.

Representation (5.47) and Duhamel-type Expansion (3.16) allows to control the Holder modulus of a solution u analyzing separately the different terms:

$$\begin{aligned} &u(t,x) - u(t,x') \\ &\quad= \Big(\tilde{P}^{\tau,\tilde{\xi}}_{T,t}u_T(x) - \tilde{P}^{\tau',\xi'}_{T,t}u_T(x')\Big) + \Big(\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x) + \tilde{G}^{\tau,\tilde{\xi}}_{T,t_0}f(t,x) - \tilde{G}^{\tau',\xi'}_{T,t}f(t,x')\Big) \\ &\qquad + \Big(\int_t^{t_0}\tilde{P}^{\tau,\xi}_{s,t}R^{\tau,\xi}(s,x)\,ds + \int_{t_0}^T \tilde{P}^{\tau,\tilde{\xi}}_{s,t}R^{\tau,\tilde{\xi}}(s,x)\,ds - \int_t^T \tilde{P}^{\tau',\xi'}_{s,t}R^{\tau',\xi'}(s,x')\,ds\Big) \\ &\qquad\quad + \Big(\tilde{P}^{\tau,\xi}_{t_0,t}u(t_0,x) - \tilde{P}^{\tau,\tilde{\xi}}_{t_0,t}u(t_0,x)\Big), \end{aligned}$$

for some freezing couples $(\tau,\xi),(\tau,\tilde{\xi}),(\tau,\xi')$ fixed but to be chosen later. As done before, we assume however from this point further that $\tau=\tau'$ and $\tilde{\xi}=\xi'$.

Control on the frozen semigroup. Noticing that we have taken the same freezing couples since $\tilde{\xi}=\xi'$, the control on the frozen semigroup $\left|\tilde{P}^{\tau,\xi'}_{T,t}u_T(x)-\tilde{P}^{\tau,\xi'}_{T,t}u_T(x')\right|$ can be obtained exploiting the same argument used in the proof of the Hölder control (Lemma 4.6) for the proxy.

Control on the Green kernel. The proof of this estimate essentially matches the previous, analogous one in the non-degenerate setting. Namely, we follow the proof in the global off-diagonal regime of Lemma 5.8 to control the local off-diagonal regime contribution $\left|\tilde{G}^{\tau,\xi}_{t_0,t}f(t,x)-\tilde{G}^{\tau,\xi'}_{t_0,t}f(t,x')\right|$ while in the locally diagonal regime term

$$\left|\tilde{G}^{\tau,\xi'}_{T,t_0}f(t,x)-\tilde{G}^{\tau,\xi'}_{T,t_0}f(t,x')\right|,$$

the freezing couples coincide and we can thus exploit the same argument used in the proof of the Hölder control (Lemma 4.6) for the proxy.

Control on the discontinuity term. The proof of this result will follow essentially the one about the off-diagonal regime of the frozen semigroup with respect to the degenerate variables. It holds that

$$\begin{aligned}\tilde{P}^{\tau,\xi}_{t_0,t}u(t_0,x) &= \int_{\mathbb{R}^{nd}}\tilde{p}^{\tau,\xi}(t,t_0,x,y)u(t_0,y)\,dy\\ &= \int_{\mathbb{R}^{nd}}\frac{1}{\det\Big(\mathbb{M}_{t_0-t}\Big)}p_S\Big(t_0-t,\mathbb{M}^{-1}_{t_0-t}(\tilde{m}^{\tau,\xi}_{t_0,t}(x)-y)\Big)u(t_0,y)\,dy\\ &= \int_{\mathbb{R}^{nd}}\frac{1}{\det\mathbb{M}_{t_0-t}}p_S(t_0-t,\mathbb{M}^{-1}_{t_0-t}z)u(t_0,\tilde{m}^{\tau,\xi}_{t_0,t}(x)-z)\,dz,\end{aligned}$$

where in the last passage we used the change of variable $z=\tilde{m}^{\tau,\xi}_{t_0,t}(x)-y$. Since a similar argument works also for $\tilde{P}^{\xi'}_{t_0,t}u(t_0,x)$, it then follows that

$$\begin{aligned}&\left|\tilde{P}^{\tau,\xi}_{t_0,t}u(t_0,x)-\tilde{P}^{\tau,\xi'}_{t_0,t}u(t_0,x)\right|\\ &\quad=\left|\int_{\mathbb{R}^{nd}}\frac{1}{\det\mathbb{M}_{t_0-t}}p_S\Big(t_0-t,\mathbb{M}^{-1}_{t_0-t}z\Big)\Big[u(t_0,\tilde{m}^{\tau,\xi}_{t_0,t}(x)-z)-u(t_0,\tilde{m}^{\tau,\xi'}_{t_0,t}(x)-z)\Big]\,dz\right|.\end{aligned}$$

Remembering that $u(t_0,\cdot)$ is Lipschitz with respect to the first non-degenerate variable, we can write now that

$$\begin{aligned}\left|\tilde{P}^{\tau,\xi}_{t_0,t}u(t_0,x)-\tilde{P}^{\tau,\xi'}_{t_0,t}u(t_0,x)\right| &\le C\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\left(\int_{\mathbb{R}^{nd}}p_S\Big(t_0-t,\mathbb{M}^{-1}_{t_0-t}z\Big)\frac{dz}{\det\mathbb{M}_{t_0-t}}\right)\\ &\qquad\times\Big[\mathbf{d}^{\alpha+\beta}(\tilde{m}^{\tau,\xi}_{t_0,t}(x),\tilde{m}^{\tau,\xi'}_{t_0,t}(x))+\left|(\tilde{m}^{\tau,\xi}_{t_0,t}(x)-\tilde{m}^{\tau,\xi'}_{t_0,t}(x))_1\right|\Big]\\ &\le C\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\Big[\mathbf{d}^{\alpha+\beta}(\tilde{m}^{\tau,\xi}_{t_0,t}(x),\tilde{m}^{\tau,\xi'}_{t_0,t}(x))+\left|(\tilde{m}^{\tau,\xi}_{t_0,t}(x)-\tilde{m}^{\tau,\xi'}_{t_0,t}(x))_1\right|\Big].\end{aligned}$$

Taking $\xi=\xi'=x$, we can then use the sensitivity controls (Lemma 5.6 and Equation (5.49)) to show that

$$\left|\tilde{P}^{\tau,\xi}_{t_0,t}u(t_0,x)-\tilde{P}^{\tau,\xi'}_{t_0,t}u(t_0,x)\right| \le C\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\|F\|_H c_0^{\frac{\alpha+\beta}{1+\alpha(n-1)}}\mathbf{d}^{\alpha+\beta}(x,x).$$

Control on the perturbative term. The proof of this estimate essentially matches the previous, analogous one in the non-degenerate setting. Namely, Inequalities (5.52), (5.53) and (5.54) hold again with $(s-t)^{\frac{\beta-2}{\alpha}}$ replaced by $(s-t)^{\frac{\beta}{\alpha}-\frac{1}{\alpha_i}}$.

Mollifying Procedure

We now make the mollifying parameter m appear again using the notations introduced in Section 3.2 (see Equation (3.16)). Then, Lemmas 5.3, 5.7 and 5.8 rewrite together in the following way. There exists a constant $C>0$ such that for any m in $\mathbb{N}$,

$$\begin{aligned}\|u_m\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \leq\, & C c_0^{\frac{\beta-\gamma_n}{\alpha}}\Big[\|u_{T,m}\|_{C^{\alpha+\beta}_{b,d}} + \|f_m\|_{L^\infty(C^{\beta}_{b,d})}\Big] \\ & + C\Big(c_0^{\frac{\beta-\gamma_n}{\alpha}}\|F_m\|_H + c_0^{\frac{\alpha+\beta-1}{1+\alpha(n-1)}}\Big)\|u_m\|_{L^\infty(C^{\alpha+\beta}_{b,d})}, \quad (5.1)\end{aligned}$$

where c_0 is assumed to be fixed but chosen later. Importantly, c_0 and C does not depends on the regularizing parameter m. Thus, letting m go to ∞ and remembering Definition 2.2 of a mild solution u, the above expression immediately implies the a priori estimates (Proposition 3.6).

6 Existence result

The aim of this section is to show the well-posedness in a mild sense of the original IPDE (1.1). Recalling Definition 2.2 for a mild solution of IPDE (1.1), let us consider three sequences $\{f_m\}_{m\in\mathbb{N}}$, $\{u_{T,m}\}_{m\in\mathbb{N}}$ and $\{F_m\}_{m\in\mathbb{N}}$ of "regularized" coefficients such that

- $\{f_m\}_{m\in\mathbb{N}}$ is in $C^\infty_b((0,T)\times\mathbb{R}^{nd})$ and f_m converges to f in $L^\infty\Big(0,T;C^\beta_{b,d}(\mathbb{R}^{nd})\Big)$;
- $\{u_{T,m}\}_{m\in\mathbb{N}}$ is in $C^\infty_b(\mathbb{R}^{nd})$ and $u_{T,m}$ converges to u_T in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})$;
- $\{F_m\}_{m\in\mathbb{N}}$ is in $C^\infty_b((0,T)\times\mathbb{R}^{nd};\mathbb{R}^{nd})$ and $\|F_m-F\|_H$ converges to 0.

It can be derived through stochastic flows techniques (see e.g. [Kun04]) that there exists a solution u_m in $C^\infty_b\Big((0,T)\times\mathbb{R}^{nd}\Big)$ of the "regularized" IPDE:

$$\begin{cases}\partial_t u_m(t,x) + \mathcal{L}_\alpha u_m(t,x) + \langle Ax + F_m(t,x), D_x u_m(t,x)\rangle \;=\; -f_m(t,x) & \text{on } (0,T)\times\mathbb{R}^{nd};\\ u_m(T,x) \;=\; u_{T,m}(x) & \text{on } \mathbb{R}^{nd}.\end{cases}$$

In order to pass to the limit in m, we notice now that the arguments used above for the proof of the Schauder estimates (Equation (2.26)) can be applied to the above dynamics, too. Namely, there exists a constant $C>0$ such that

$$\|u_m\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \leq C\Big[\|f_m\|_{L^\infty(C^{\beta}_{b,d})} + \|u_{T,m}\|_{C^{\alpha+\beta}_{b,d}}\Big] \leq C\Big[\|f\|_{L^\infty(C^{\beta}_{b,d})} + \|u_T\|_{C^{\alpha+\beta}_{b,d}}\Big].$$

Importantly, the above estimates is uniformly in m and thus, the sequence $\{u_m\}_{m\in\mathbb{N}}$ is bounded in the space $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$. From Arzelà-Ascoli Theorem, we deduce

now that there exists u in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ and a sequence $\{u_{m_k}\}_{k\in\mathbb{N}}$ of smooth and bounded functions converging to u in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ and such that u_{m_k} is solution of the "regularized" IPDE (2.25). It is then clear that u is a mild solution of the original IPDE (1.1).

From Mild to Weak Solutions We conclude showing that any mild solution u of the IPDE (1.1) is indeed a weak solution. The proof of this result will be essentially an application of the arguments presented before, especially the second Besov control (Lemma 5.2). Let u be a mild solution of the IPDE (1.1) in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})\Big)$. Recalling the definition of weak solution in (2.23), we start fixing a test function ϕ in $C^\infty_0\Big((0,T]\times\mathbb{R}^{nd}\Big)$ and passing to the "regularized" setting (see Definition 2.2), we then notice that it holds that

$$\int_0^T\int_{\mathbb{R}^{nd}}\phi(t,y)\Big(\partial_t+L^m_t\Big)u_m(t,y)\,dy\;=\;-\int_0^T\int_{\mathbb{R}^{nd}}\phi(t,y)f_m(t,y)\,dy,$$

where L^m_t is the "complete" operator defined in (2.22) but with respect to the regularized coefficients. Integration by parts formula allows now to move the operators to the test function. Indeed, remembering that $u_m(T,\cdot)=u_{T,m}(\cdot)$, it holds that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^{nd}}\Big(-\partial_t+(L^m_t)^*\Big)\phi(t,y)u_m(t,y)\,dydt+\int_{\mathbb{R}^{nd}}\phi(T,y)u_{T,m}(y)\,dy\\ =\;-\int_0^T\int_{\mathbb{R}^{nd}}\phi(t,y)f_m(t,y)\,dydt,\end{aligned}\tag{6.2}$$

where $\mathcal{L}^*_{m,\alpha}$ denotes the formal adjoint of L^m_t. We would like now to go back to the solution u, letting m go to ∞. We start rewriting the right-hand side term in the following way:

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^{nd}}\phi(t,y)f_m(t,y)\,dydt\\ &\qquad=\int_0^T\int_{\mathbb{R}^{nd}}\phi(t,y)f(t,y)\,dydt+\int_0^T\int_{\mathbb{R}^{nd}}\phi(t,y)\Big[f_m-f\Big](t,y)\,dydt.\end{aligned}$$

Exploiting that f_m converges to f in $L^\infty\Big(0,T;C^{\beta}_{b,d}(\mathbb{R}^{nd})\Big)$ by assumption, it is easy to see that the second contribution above goes to 0 if we let m go to ∞. A similar argument can be used to show that

$$\int_{\mathbb{R}^{nd}}\phi(T,y)u_{T,m}(y)\,dy\;\overset{m}{\to}\;\int_{\mathbb{R}^{nd}}\phi(T,y)u_T(y)\,dy.$$

On the other hand, we can decompose the first term in the left-hand side of Equation (6.2) as

$$\begin{aligned}&\int_0^T\int_{\mathbb{R}^{nd}}\Big(-\partial_t+(L^m_t)^*\Big)\phi(t,y)u_m(t,y)\,dydt\\ &\qquad=\int_0^T\int_{\mathbb{R}^{nd}}\Big(-\partial_t+L^*_t\Big)\phi(t,y)u(t,y)\,dydt+R^1_m+R^2_m,\end{aligned}\tag{6.3}$$

where we have denoted

$$\begin{aligned} R^1_m &= \int_0^T \int_{\mathbb{R}^{nd}} \Big[\mathcal{L}^*_\alpha - (L^m_t)^*\Big]\phi(t,y)u_m(t,y)\,dydt; \\ R^2_m &= \int_0^T \int_{\mathbb{R}^{nd}} \Big(-\partial_t + \mathcal{L}^*_\alpha\Big)\phi(t,y)\Big[u_m(t,y) - u(t,y)\Big]\,dydt, \end{aligned}$$

with $\mathcal{L}^*_\alpha$ as the formal adjoint of the complete operator L_t. Noticing that

$$\Big[\mathcal{L}^*_\alpha - (L^m_t)^*\Big]\phi(t,y) \;=\; D_y \cdot \Big\{\phi(t,y)[F(t,y) - F_m(t,y)]\Big\},$$

it is clear that the remainder contribution R^1_m can be essentially handled as in the introduction of Section 5.1, exploiting that $\|F - F_m\|_H \to 0$.
To control instead the second contribution R^2_m, we start decomposing it as

$$\begin{aligned} R^2_m \;&=\; -\int_0^T \int_{\mathbb{R}^{nd}} \partial_t\phi(t,y)\Big[u_m(t,y) - u(t,y)\Big]\,dydt \\ &\qquad\qquad + \sum_{j=1}^n \int_0^T \int_{\mathbb{R}^{nd}} D_{y_j}\Big[\phi F_j\Big](t,y)\Big[u_m(t,y) - u(t,y)\Big]\,dydt \\ &=:\; R^2_{0,m} + \sum_{j=1}^n R^2_{j,m}. \end{aligned}$$

We firstly observe that $|R^2_{0,m}|$ goes to 0 if we let m go to ∞, since $\|u - u_m\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \overset{m}{\to} 0$.
On the other hand, integration by parts formula allows to show that

$$|R^2_{1,m}| \;=\; \Big|\int_0^T \int_{\mathbb{R}^{nd}} \Big[\phi F\Big](t,y) D_{y_j}\Big[u_m - u\Big](t,y)\,dydt\Big|$$

which again tends to 0 when m goes to ∞. To control instead the contributions $R^m_{j,m}$ for $j > 1$, the point is to use the Besov duality argument again. Namely, from Equations (4.11), (4.10) and with the notations in (4.7), it holds that

$$\begin{aligned} |R^2_{j,m}| \;&\le\; \int_0^T \int_{\mathbb{R}^{d(n-1)}} \Big\|D_{y_j}\Big[\phi F\Big](t,y_{\smallsetminus j},\cdot)\Big\|_{B^{-(\alpha_j+\beta_j)}_{1,1}} \Big\|\Big[u_m - u\Big](t,y_{\smallsetminus j},\cdot)\Big\|_{B^{\alpha_j+\beta_j}_{\infty,\infty}}\,dy_{\smallsetminus j}dt \\ &\le\; \int_0^T \int_{\mathbb{R}^{d(n-1)}} \Big\|D_{y_j}\Big[\phi F\Big](t,y_{\smallsetminus j},\cdot)\Big\|_{B^{-(\alpha_j+\beta_j)}_{1,1}} \Big\|\Big[u_m - u\Big](t,y_{\smallsetminus j},\cdot)\Big\|_{C^{\alpha_j+\beta_j}_b}\,dy_{\smallsetminus j}dt. \end{aligned}$$

Following the same arguments used in the proof of the second Besov control (Lemma 5.2), we know that there exists a constant C such that $\Big\|D_{y_j}\Big[\phi F\Big](t,y_{\smallsetminus j},\cdot)\Big\|_{B^{\alpha_j+\beta_j}_{1,1}} \le C\psi_j(t,y_{\smallsetminus j})$, where ψ_j has compact support on $\mathbb{R}^{d(n-1)}$.
Since moreover $\|u_m - u\|$ goes to zero with m, we easily deduce that $R^2_{m,j} \overset{m}{\to} 0$ for any j in $[\![2,n]\!]$. From the above controls, we can deduce now that $R^1_m + R^2_m \overset{m}{\to} 0$. From Equation (6.3), it then follows that

$$\int_0^T \int_{\mathbb{R}^{nd}} \Big(-\partial_t + (L^m_t)^*\Big)\phi(t,y)u_m(t,y)\,dydt \;\overset{m}{\to}\; \int_0^T \int_{\mathbb{R}^{nd}} \Big(-\partial_t + L^*_t\Big)\phi(t,y)u(t,y)\,dydt$$

and the proof is concluded.

7 Extensions

As already said in the introduction, our assumption of (global) Hölder regularity on the drift $\bar{F}$, as well as the choice of considering a perturbed Ornstein-Uhlenbeck operator instead of a more general non-linear dynamics, was done to preserve, as possible, the clarity and understandability of the article. In this conclusive section, we would like to explain briefly how it possible to naturally extend it.

7.1 General drift

Here, we illustrate how the perturbative method explained above can be easily adapted to work in a more general setting. In particular, the same results (well-posedness of the IPDE (1.1) and associated Schauder estimates) can be proven to hold also for an equation of the form:

$$\begin{cases} \partial_t u(t,x) + \mathcal{L}_\alpha u(t,x) + \langle \bar{F}(t,x), D_x u(t,x)\rangle \;=\; -f(t,x), & \text{on } (0,T)\times\mathbb{R}^{nd}; \\ u(T,x) \;=\; u_T(x) & \text{on } \mathbb{R}^{nd}, \end{cases} \tag{7.1}$$

where $\bar{F}(t,x) = \big(\bar{F}_1(t,x),\dots,\bar{F}_n(t,x)\big)$ has the following structure

$$\bar{F}_i(t, x_{(i-1)\vee 1},\dots,x_n).$$

We remark in particular that if for any i in $[\![2,n]\!]$, $\bar{F}_i$ is linear with respect to x_{i-1} and independent from time, the previous analysis works since we can rewrite $\bar{F}(t,x) = Ax + F(t,x)$.

In order to deal with this more general dynamics addressed in the diffusive setting in [CdRHM18a], we will need however to add some additional constraints and to modify slightly the ones presented in assumption [**A**]. First of all, the non-degeneracy assumption [**H**] does not make sense in this new framework and it will be replaced by the following one:

[H'] the matrix $D_{x_{i-1}}\bar{F}_i(t,x)$ has full rank d for any i in $[\![2,n]\!]$ and any (t,x) in $[0,T]\times\mathbb{R}^{nd}$.

In particular, we will say that assumption [$\bar{\mathbf{A}}$] is in force when

[S'] assumption [**ND**] and [**H'**] are satisfied and the drift $\bar{F} = (\bar{F}_1,\dots,\bar{F}_n)$ is such that for any i in $[\![2,n]\!]$, $\bar{F}_i$ depends only on time and on the last $n-(i-2)\vee 0$ components, i.e. $\bar{F}_i(t,x_{i-1},\dots,x_n)$;

[P'] α is a number in $(0,2)$, β is in $(0,1)$ and it holds that

$$\beta<\alpha, \quad \alpha+\beta\in(1,2) \;\text{ and }\; \beta<(\alpha-1)(1+\alpha(n-1));$$

[R'] Recalling the notations in (2.19)-(2.20), the source f is in $L^\infty(0,T;C^\beta_{b,d}(\mathbb{R}^{nd}))$, the terminal condition u_T is in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^{nd})$ and for any i in $[\![1,n]\!]$, the drift $\bar{F}_i$ belongs to $L^\infty(0,T;C^{\gamma_i+\beta}_d(\mathbb{R}^{nd}))$ where γ_i was defined in (2.21).

To prove Schauder-type estimates for a solution of IPDE (7.1), our idea is to adapt the perturbative approach to this new dynamics. In particular, we can exploit the differentiability of $\bar{F}_i$ with respect to x_{i-1} to "linearize" it along a flow that takes into account the perturbation (cf. Section 3.1). Namely, we are interested in the following equation:

$$\partial_t \bar{u}^{\tau,\xi}(t,x) + \mathcal{L}_\alpha \bar{u}^{\tau,\xi}(t,x) + \Big\langle \bar{A}_t^{\tau,\xi}\Big(x - \bar{\theta}_{t,\tau}(\xi)\Big) + \bar{F}(t,\bar{\theta}_{t,\tau}(\xi)), D_x \bar{u}^{\tau,\xi}(t,x)\Big\rangle = -f(t,x); \quad (7.2)$$

with initial condition $\bar{u}^{\tau,\xi}(T,x) = u_T(x)$, where the time-dependent matrix $\bar{A}_t^{\tau,\xi}$ is defined through

$$\Big[\bar{A}_t^{\tau,\xi}\Big]_{i,j} = \begin{cases} D_{x_{i-1}}\bar{F}_i(t,\theta_{t,\tau}(\xi)), & \text{if } j = i-1; \\ 0_{d\times d}, & \text{otherwise} \end{cases}$$

and $\bar{\theta}_{t,\tau}(\xi))$ is a fixed flow satisfying the dynamics

$$\bar{\theta}_{t,\tau}(\xi) = \xi + \int_\tau^t \bar{F}(v,\bar{\theta}_{v,\tau}(\xi))\,dv. \quad (7.3)$$

A first significant difference with respect to the previous approach consists in handling a time-dependent matrix $\bar{A}_t^{\tau,\xi}$. Indeed, it is possible to modify slightly the presentation in [PZ09] (allowing time-dependency on A) in order to show that under assumption [**S'**], the two parameters semigroup $\{\bar{P}_{s,t}^{\tau,\xi}\}_{t\le s}$ associated with the proxy operator

$$\mathcal{L}_\alpha + \langle \bar{A}_t^{\tau,\xi}\Big(x - \bar{\theta}_{t,\tau}(\xi)\Big) + \bar{F}(t,\bar{\theta}_{t,\tau}(\xi)), D_x\rangle$$

admits a density $\bar{p}^{\tau,\xi}$ and that it can be written as

$$\bar{p}^{\tau,\xi}(t,s,x,y) = \frac{1}{\det \mathbb{M}_{s-t}} p_S\Big(s-t, \mathbb{M}_{s-t}^{-1}(y - \bar{m}_{s,t}^{\tau,\xi}(x))\Big).$$

Here, the notations for p_S and $\mathbb{M}_t$ remain the same of above while this time the shift $\bar{m}_{s,t}^{\tau,\xi}$ is defined through

$$\bar{m}_{s,t}^{\tau,\xi}(x) = \mathcal{R}_{s,t}^{\tau,\xi}x + \int_t^s \mathcal{R}_{s,v}^{\tau,\xi}\Big[\bar{F}(v,\bar{\theta}_{v,\tau}(\xi)) - \bar{A}_v^{\tau,\xi}\bar{\theta}_{v,\tau}(\xi)\Big]\,dv,$$

where $\mathcal{R}_{s,t}^{\tau,\xi}$ is the time-ordered resolvent of $\bar{A}_s^{\tau,\xi}$ starting at time t, i.e.

$$\begin{cases} d\mathcal{R}_{s,t}^{\tau,\xi} = \bar{A}_s^{\tau,\xi}\mathcal{R}_{s,t}^{\tau,\xi}ds, & \text{on } [t,T]; \\ \mathcal{R}_{t,t}^{\tau,\xi} = I. \end{cases}$$

We can as well refer to [HM16] for related issues (see Proposition 3.2 and Section C about the linearization, therein).

Following the same reasonings of Propositions 3.4 and 3.5, it is then possible to state a Duhamel type formula suitable for IPDE (7.1):

$$u(t,x) = \bar{P}_{T,t}^{\tau,\xi}u_T(x) + \int_t^T \bar{P}_{s,t}^{\tau,\xi}\Big[f(s,\cdot) + \bar{R}^{\tau,\xi}(s,\cdot)\Big](x)\,ds, \quad (7.4)$$

where the remainder term is given now by

$$\bar{R}^{\tau,\xi}(t,x) \;=\; \langle F(t,x) - F(t,\bar{\theta}_{t,\tau}(\xi)) - \bar{A}_t^{\tau,\xi}\Big(x - \bar{\theta}_{t,\tau}(\xi)\Big), D_x u(t,x)\rangle.$$

Looking back at the first part of the article, it is important to notice that the main steps of proof (cf. Equation (3.6), Propositions 3.3 and 3.6 and Section 3.3) does not rely on the explicit formulas for $\bar{m}_{s,t}^{\tau,\xi}(x)$ and $\bar{R}^{\tau,\xi}$ but instead, they exploit only the Besov controls for the remainder $\bar{R}^{\tau,\xi}$ (cf. Section 5.1) and the controls on the shift $\bar{m}_{s,t}^{\tau,\xi}(x)$ (Section A.2). Hence, once we have proven the suitable controls, the proofs of the analogous results for the new IPDE (7.1) can be obtained easily modifying slightly the notations and following the same reasonings above.
For example, exploiting that

$$\bar{m}_{s,t}^{\tau,\xi}(x) \;=\; x + \int_t^s \mathcal{R}_{v,t}^{\tau,\xi}\Big(\bar{m}_{v,t}^{\tau,\xi}(x) - \theta_{v,\tau}(\xi)\Big) + F(v,\theta_{v,\tau}(\xi))\,dv,$$

we can follow the same method of proof in the above lemma 3.1 to show again that

$$\bar{m}_{s,t}^{\tau,\xi}(x) \;=\; \bar{\theta}_{s,\tau}(\xi),$$

taking $\tau = t$ and $\xi = x$.

Letting the interested reader look in the appendix for the suggestions on how to extend the controls on the shift $\bar{m}_{s,t}^{\tau,\xi}(x)$ in this more general setting, we will focus now on proving the Besov controls. First of all, we notice immediately that the proof of the first Besov control (Lemma 4.3) relies essentially only on the smoothing effect (Equation (3.6)) and thus, it can be obtained following the same reasoning above. The proof of the second Besov control (Lemma 5.2) in this framework is a bit more involved and we are going to explain it below more in details.
We start noticing that Lemma 5.2 can be reformulated for the new dynamics in the following way:

$$\int_{\mathbb{R}^{(n-1)d}} \Big\| D_{y_j} \cdot \Big\{ \mathbf{d}_x^{\vartheta} \bar{p}^{\tau,\xi}(t,s,x,y_{\smallsetminus j},\cdot) \otimes \bar{\Delta}_j^{\tau,\xi}(s,y_{\smallsetminus j},\cdot) \Big\} \Big\|_{B_{1,1}^{-(\alpha_j+\beta_j)}} dy_{\smallsetminus j}$$
$$\leq\; C\|\bar{F}\|_H (s-t)^{\frac{\beta}{\alpha} - \sum_{k=1}^n \frac{\vartheta_k}{\alpha_k}},$$

taking $(\tau,\xi) = (t,x)$, where we have denoted for simplicity

$$\bar{\Delta}_j^{\tau,\xi}(s,y) \;:=\; \bar{F}_j(s,y) - \bar{F}_j(s,\theta_{s,\tau}(\xi)) - D_{x_{j-1}}\bar{F}_j(s,\theta_{s,\tau}(\xi))\Big(y - \theta_{s,\tau}(\xi)\Big)_{j-1},$$

for any j in $[\![2,n]\!]$. The above control can be obtained mimicking the proof in the second Besov control (Lemma 5.2), exploiting this time that

$$|\bar{\Delta}_j^{\tau,\xi}(s,y)| \leq C\|\bar{F}\|_H \mathbf{d}_{j-1:n}^{1+\alpha(j-2)+\beta}\Big(y, \bar{\theta}_{s,\tau}(\xi)\Big)$$

and the additional assumption [**P'**] in order to make the partial smoothing effect (Equation (5.30)) work in this framework, too.

The main difference in the proof is related to the control of the component $J_2(v, y_{\setminus j}, z)$ appearing in Equation (5.35). Namely,

$$\int_{\mathbb{R}^d} D_z \partial_v p_h(v, z - y_j) \cdot \Big\{ \bar{\Delta}_j^{\tau,\xi}(s, y_{\setminus j}, z) \\ \otimes \int_0^1 D_{y_j} D_x^{\vartheta} \bar{p}^{\tau,\xi}(t, s, x, y_{\setminus j}, z + \lambda(y_j - z)) \cdot (y_j - z) \Big\} d\lambda dy_j$$

with our new notations. Indeed, the dependence of $\bar{F}$ on x_{j-1} pushes us to add a new term in the difference $|\bar{F}_j(s, y_{\setminus j}, z) - \bar{F}_j(s, \theta_{s,\tau}(\xi))|$ (now, $|\bar{\Delta}_j^{\tau,\xi}(s, y_{\setminus j}, z)|$) before splitting it up. In particular, it holds that

$$\begin{aligned} |\bar{\Delta}_j^{\tau,\xi}(s, y_{\setminus j}, z)| \;=\; & \Big| \bar{F}_j(s, y_{\setminus j}, z) - \bar{F}_j(s, \theta_{s,\tau}(\xi)) - D_{x_{j-1}} \bar{F}_j(s, \bar{\theta}_{s,\tau}(\xi)) \Big(y - \bar{\theta}_{s,\tau}(\xi) \Big)_{j-1} \\ & \pm \bar{F}_j(s, y_{1:j-1}, \Big(\bar{\theta}_{s,\tau}(\xi) \Big)_{j:n}) \Big| \\ \leq\; & C \|\bar{F}\|_H \Big(|z - \Big(\bar{\theta}_{s,\tau}(\xi) \Big)_j|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}} + \sum_{k=j+1}^{n} |\Big(y - \bar{\theta}_{s,\tau}(\xi) \Big)_k|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(k-1)}} \\ & + |\Big(y - \bar{\theta}_{s,\tau}(\xi) \Big)_{j-1}|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-2)}} \Big) \\ \leq\; & C \|\bar{F}\|_H \Big(|\lambda(z - y_j)|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}} + |z + \lambda(y_j - z) - \theta_{s,\tau}(\xi)_j|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}} \\ & + \sum_{k=j+1}^{n} |y - \bar{\theta}_{s,\tau}(\xi)_k|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(k-1)}} + |\Big(y - \bar{\theta}_{s,\tau}(\xi) \Big)_{j-1}|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-2)}} \Big) \\ \leq\; & C \|\bar{F}\|_H \Big(|z - y_j|^{\frac{1+\alpha(j-2)+\beta}{1+\alpha(j-1)}} + \mathbf{d}_{j+1:n}^{1+\alpha(j-2)+\beta}((y_{\setminus j}, z + \lambda(y_j - z)), \bar{\theta}_{s,\tau}(\xi)) \Big). \end{aligned}$$

The remaining part of the proof exactly matches the original method in Lemma 5.2.

Even in this more general framework, it is thus possible to obtain the following:

Theorem 7.1 (Well-posedness)**.** *Under* $[\bar{\boldsymbol{A}}]$*, there exists a unique mild solution* u *of IPDE* (7.1)*. Moreover, there exists a constant* $C := C(T)$ *such that*

$$\|u\|_{L^\infty(C_d^{\alpha+\beta})} \;\leq\; C\Big[\|f\|_{L^\infty(C_{b,d}^{\beta})} + \|u_T\|_{C_{b,d}^{\alpha+\beta}} \Big].$$

7.2 Locally Hölder drift

This part is designed to give a brief explanation on how it is possible to deal with the general IPDE (7.1) when the drift $\bar{F}$ is only locally Hölder continuous in space. Namely, we assume with the notations in (2.21) that

[LR'] there exists a constant $K_0 > 0$ such that for any i in $[\![1, n]\!]$

$$\mathbf{d}(\bar{F}(t, x), \bar{F}(t, x')) \;\leq\; K_0 \mathbf{d}^{\beta+\gamma_i}(x, x'), \quad t \in [0, T],\, x, x' \in \mathbb{R}^{nd} \text{ s.t. } \mathbf{d}(x, x') < 1.$$

In other words, it is required that $\bar{F}_i$ is in $L^\infty(0, T; C^{\beta+\gamma_i}(B(x_0, 1/2)))$, uniformly in $x_0 \in \mathbb{R}^{nd}$.

Under assumption [$\bar{\mathbf{A}}$] (with condition [**R'**] replaced by [**LR'**]), it is possible to recover the Schauder-type estimates (Theorem 2.3), following the approach developed successfully in [CdRMP20a] for the non-degenerate, super-critical stable setting. Roughly speaking, in order to handle the local assumption, as well as the potentially unboundedness of the drift $\bar{F}$, we need to introduce a "localized" version of the Duhamel formulation (cf. Equation (3.16)). The key point here is to multiply a solution u by a suitable bump function $\bar{\eta}^{\tau,\xi}$ that "localizes" in space along the deterministic flow $\bar{\theta}_{t,\tau}(\xi)$ that characterizes the proxy. Namely, we fix a smooth function ρ that is equal to 1 on $B(0,1/2)$ and vanishes outside $B(0,1)$) and then define for any (τ,ξ) in $[0,T]\times\mathbb{R}^{nd}$,

$$\bar{\eta}^{\tau,\xi}(t,x) \;:=\; \rho(x-\bar{\theta}_{t,\tau}(\xi)).$$

We mention however that in the setting of [CdRMP20a], the "localization" with the cut-off function $\bar{\eta}^{\tau,\xi}$ is not simply motivated by the local Hölder continuity condition but it is also needed to give a proper meaning to the Duhamel formulation for a solution (cf. Proposition 3.5) when $\alpha<1/2$, because of the low integrability properties of the underlying stable density. Such a problem does not however appear here since condition [**P**] forces us to consider only the case $\alpha>1/2$.

Given a mild solution u of IPDE (7.1) and assuming $\bar{F}$ to be only locally Hölder continuous as in [**LR'**], it is possible to show, at least formally, that the function $\bar{v}^{\tau,\xi}:=u\bar{\eta}^{\tau,\xi}$ solves the following equation on $(0,T)\times\mathbb{R}^{nd}$:

$$\begin{cases}\partial_t\bar{v}^{\tau,\xi}(t,x)+\langle\bar{F}(t,x),D_x\bar{v}^{\tau,\xi}(t,x)\rangle+\mathcal{L}_\alpha\bar{v}^{\tau,\xi}(t,x) \;=\; -\Big[\bar{\eta}^{\tau,\xi}f+\bar{\mathcal{S}}^{\tau,\xi}\Big](t,x);\\ \bar{v}^{\tau,\xi}(T,x) \;=\; \bar{\eta}^{\tau,\xi}(T,x)u_T(x),\end{cases} \tag{7.5}$$

where we have denoted

$$\begin{aligned}\bar{\mathcal{S}}^{\tau,\xi}(t,x) \;:=\; \int_{\mathbb{R}^d}\Big[u(t,x+By)-u(t,x)\Big]\Big[\bar{\eta}^{\tau,\xi}(t,t,x+By)-\bar{\eta}^{\tau,\xi}(t,x)\Big]\,\nu_\alpha(dy)\\ -\,u(t,x)\langle\bar{F}(t,x)-\bar{F}(t,\bar{\theta}_{t,\tau}(\xi)),D\rho(x-\bar{\theta}_{t,\tau}(\xi))\rangle.\end{aligned}$$

Equations as (7.5) can be essentially seen as a "local" version of the original one (7.1), depending on the freezing parameter (τ,ξ). In particular, it is important to notice that the difference

$$\bar{F}(t,x)-\bar{F}(t,\bar{\theta}_{t,\tau}(\xi))$$

appearing in the "localizing" error $\bar{\mathcal{S}}^{\tau,\xi}$ can be controlled exactly because it is multiplied by the derivative of the bump function ρ in the right point $x-\bar{\theta}_{t,\tau}(\xi)$, allowing us to exploit the *local* Hölder regularity. On the other hand, the first integral term in the r.h.s. can be seen as a commutator which involves only the non-degenerate variables and thus, that can be handled with interpolation techniques as in [CdRMP20a].

Even with the additional difficulty in controlling the remainder term, the perturbative approach explained in Section 3 can be applied, leading to show Schauder-type estimates as in Theorem 2.3 and the well-posedness of the IPDE (7.1) when assuming $\bar{F}$ to be only locally Hölder continuous.

Our procedure could be also used in order to establish Schauder-type estimates for the full Ornstein-Uhlenbeck operator as done, for example, in [Lun97] for the diffusive case. Indeed, a general operator of the form $\langle \bar{A}x, D_x\rangle + \mathcal{L}_\alpha$ can be treated, decomposing the matrix as $\bar{A} = A + U$ where A is, as before, the sub-diagonal matrix that makes the Ornstein-Ulhenbeck operator invariant by the dilation operator associated with the distance d, while U is an upper triangular matrix that could be seen as an additional *locally* Hölder term.

7.3 Diffusion coefficient

We conclude the article showing briefly how an additional diffusion term $\sigma\colon [0,T]\times \mathbb{R}^{nd} \to \mathbb{R}^d \otimes \mathbb{R}^d$ can be handled in the IPDE (7.1) with an operator $\mathcal{L}_{\alpha,t}$ of the form:

$$\mathcal{L}_{\alpha,t}\phi(t,x) \;:=\; \text{p.v.}\int_{\mathbb{R}^d}\Big[\phi(t,x+B\sigma(t,x)y)-\phi(t,x)\Big]\nu_\alpha(dy).$$

In this framework, it is quite standard (cf. [HWZ20] and [ZZ18]) to assume the Lévy measure ν_α to be absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ i.e. $\nu_\alpha(dy) = f(y)dy$, for some Lipschitz function $f\colon \mathbb{R}^d \to \mathbb{R}$. In particular, since ν_α is a symmetric, α-stable, Lévy measure, it holds passing to polar coordinates $y = \rho s$ where $(\rho, s) \in [0,\infty)\times \mathbb{S}^{d-1}$ that

$$f(y) \;=\; \frac{g(s)}{\rho^{d+\alpha}}$$

for an even, Lipschitz function g on $\mathbb{S}^{d-1}$ (see also Equation (2.6)). Moreover, σ is considered uniformly elliptic and in $L^\infty(0,T;C^\beta(\mathbb{R}^n,\mathbb{R}))$.
Introducing now the "frozen" operator

$$\bar{\mathcal{L}}^{\tau,\xi}_{\alpha,t}\phi(t,x) \;=\; \text{p.v.}\int_{\mathbb{R}^d}\Big[\phi(t,x+B\sigma(t,\bar{\theta}_{t,\tau}(\xi))y)-\phi(t,x)\Big]\nu_\alpha(dy),$$

this would lead to consider for the IPDE an additional term in the Duhamel formula (cf. Equation (7.4)) that would write:

$$u(t,x) \;=\; \breve{P}^{\tau,\xi}_{T,t}u_T(x) + \int_t^T \breve{P}^{\tau,\xi}_{s,t}f(s,x) + \breve{P}^{\tau,\xi}_{s,t}\bar{R}^{\tau,\xi}(s,x) + \breve{P}^{\tau,\xi}_{s,t}\Big[\Big(\mathcal{L}_{\alpha,t}-\bar{\mathcal{L}}^{\tau,\xi}_{\alpha,t}\Big)u(s,\cdot)\Big](x)\,ds. \tag{7.6}$$

Here, $\{\breve{P}^{\tau,\xi}_{s,t}\}_{t\le s}$ denotes the two parameter semigroup associated with the proxy operator

$$\bar{L}^{\tau,\xi}_\alpha + \langle \bar{A}^{\tau,\xi}_t\Big(x-\bar{\theta}_{t,\tau}(\xi)\Big) + \bar{F}(t,\bar{\theta}_{t,\tau}(\xi)), D_x\rangle.$$

Let us focus on the last term in the integral of Equation (7.6). Looking back at the proof of the a priori estimates (Proposition 3.6), we notice in particular that we aim to establish the following control:

$$\Big|\Big(\mathcal{L}_{\alpha,t}-\bar{\mathcal{L}}^{\tau,\xi}_{\alpha,t}\Big)u(t,x)\Big| \;\le\; C\|\sigma\|_{L^\infty(C^\beta_{b,d})}\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\mathbf{d}^\beta(x,\bar{\theta}_{t,\tau}(\xi)) \tag{7.7}$$

in order to apply the same reasoning above in this new framework. To this end, we write that

$$\begin{aligned}\Big(\mathcal{L}_{\alpha,t}-\bar{\mathcal{L}}^{\tau,\xi}_{\alpha,t}\Big)u(t,x) &= \text{p.v.}\int_{\mathbb{R}^d}\Big\{u(t,x+B\sigma(t,x)y)-u(t,x)\Big\}\,\nu_\alpha(dy)\\ &\qquad -\text{p.v.}\int_{\mathbb{R}^d}\Big\{u(t,x+B\sigma(t,\bar\theta_{t,\tau}(\xi))y)-u(t,x)\Big\}\,\nu_\alpha(dy)\\ &= \text{p.v.}\int_{\mathbb{R}^d}\Big\{u(t,x+Bz)-u(t,x)\Big\}\frac{f\Big(\sigma^{-1}(t,x)z\Big)}{\det\sigma(t,x)}\,dz\\ &\qquad -\int_{\mathbb{R}^d}\Big\{u(t,x+Bz)-u(t,x)\Big\}\frac{f\Big(\sigma^{-1}(t,\bar\theta_{t,\tau}(\xi))z\Big)}{\det\sigma(t,\bar\theta_{t,\tau}(\xi))}\,dz\\ &= \text{p.v.}\int_0^\infty\frac{1}{\rho^{1+\alpha}}\int_{\mathbb{S}^{d-1}}\Big\{u(t,x+B\rho s)-u(t,x)\Big\}\bar D^{\tau,\xi}(t,x,s)\,dsd\rho\end{aligned}$$

where we have denoted, for notational convenience

$$\bar D^{\tau,\xi}(t,x,s)\;:=\;\Big\{\frac{g\Big(\frac{\sigma^{-1}(t,x)s}{|\sigma^{-1}(t,x)s|}\Big)}{|\sigma^{-1}(t,x)s|^{d+\alpha}\det\sigma(t,x)}-\frac{g\Big(\frac{\sigma^{-1}(t,\bar\theta_{t,\tau}(\xi))s}{|\sigma^{-1}(t,\bar\theta_{t,\tau}(\xi))s|}\Big)}{|\sigma^{-1}(t,\bar\theta_{t,\tau}(\xi))s|^{d+\alpha}\det\sigma(t,\bar\theta_{t,\tau}(\xi))}\Big\}.$$

Using now that g is Lipschitz and the assumptions on σ, we can show that

$$|\bar D^{\tau,\xi}(t,x,s)|\;\le\;C|\sigma(t,x)-\sigma(t,\bar\theta_{t,\tau}(\xi))|\;\le\;C\|\sigma\|_{L^\infty(C^\beta_{b,d})}\mathbf{d}^\beta(x,\bar\theta_{t,\tau}(\xi)). \tag{7.8}$$

Finally, Equation (7.7) follows from the previous controls using Taylor expansions and the symmetry condition on ν_α. Namely, considering the case $\alpha\ge 1$, which is the most delicate one for this part and precisely requires the symmetry of u_T, we write that

$$\begin{aligned}\Big|\Big(\mathcal{L}_{\alpha,t}-\bar{\mathcal{L}}^{\tau,\xi}_{\alpha,t}\Big)u(t,x)\Big| &= \Big|\text{p.v.}\int_0^\infty\frac{1}{\rho^{1+\alpha}}\int_{\mathbb{S}^{d-1}}\Big\{u(t,x+B\rho s)-u(t,x)\Big\}\bar D^{\tau,\xi}(t,x,s)\,dsd\rho\Big|\\ &\le \Big|\text{p.v.}\int_{(0,1)}\frac{1}{\rho^{1+\alpha}}\int_{\mathbb{S}^{d-1}}\Big\{u(t,x+B\rho s)-u(t,x)\Big\}\bar D^{\tau,\xi}(t,x,s)\,dsd\rho\Big|\\ &\qquad+\int_{(1,\infty)}\frac{1}{\rho^{1+\alpha}}\int_{\mathbb{S}^{d-1}}\Big|u(t,x+B\rho s)-u(t,x)\Big|\,|\bar D^{\tau,\xi}(t,x,s)|\,dsd\rho\\ &=:\Big[\bar I^{\tau,\xi}_s+\bar I^{\tau,\xi}_l\Big](t,x).\end{aligned} \tag{7.9}$$

The *large jump* contribution $\bar I^{\tau,\xi}_l$ is easily handled from Equation (7.8). We get that

$$\begin{aligned}\bar I^{\tau,\xi}_l(t,x)\;&\le\;2C\|\sigma\|_{L^\infty(C^\beta_{b,d})}\|u\|_{L^\infty(L^\infty)}\mathbf{d}^\beta(x,\bar\theta_{t,\tau}(\xi))\\ &\le\;2C\|\sigma\|_{L^\infty(C^\beta_{b,d})}\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\mathbf{d}^\beta(x,\bar\theta_{t,\tau}(\xi)).\end{aligned} \tag{7.10}$$

On the other hand, from the symmetry assumption on ν_α, which transfers to u_T, we can control the *small jump* contribution $\bar I^{\tau,\xi}_s$ through Taylor expansion and a centering

argument. Indeed,

$$
\begin{aligned}
&\bar{I}_s^{\tau,\xi}(t,x)\\
&= \Big|\text{p.v.}\int_{(0,1)}\frac{1}{\rho^{1+\alpha}}\int_{\mathbb{S}^{d-1}}\int_0^1\Big[D_{x_1}u(t,x+\lambda B\rho s)-D_{x_1}u(t,x)\Big]\rho s\bar{D}^{\tau,\xi}(t,x,s)\,d\lambda ds d\rho\Big|\\
&\leq C\|\sigma\|_{L^\infty(C^\beta_{b,d})}\mathbf{d}^\beta(x,\bar{\theta}_{t,\tau}(\xi))\int_{(0,1)}\frac{1}{\rho^\alpha}\int_{\mathbb{S}^{d-1}}\int_0^1\Big|D_{x_1}u(t,x+\lambda B\rho s)-D_{x_1}u(t,x)\Big|\,d\lambda ds d\rho\\
&\leq C\|\sigma\|_{L^\infty(C^\beta_{b,d})}\|D_{x_1}u\|_{L^\infty(C^{\alpha+\beta-1}_{b,d})}\mathbf{d}^\beta(x,\bar{\theta}_{t,\tau}(\xi))\int_{(0,1)}\frac{1}{\rho^\alpha}\rho^{\alpha+\beta-1}\,d\rho\\
&\leq C\|\sigma\|_{L^\infty(C^\beta_{b,d})}\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})}\mathbf{d}^\beta(x,\bar{\theta}_{t,\tau}(\xi)). \qquad (7.11)
\end{aligned}
$$

Using Controls (7.10) and (7.11) in Equation (7.9), we obtain the expected bound (Equation (7.7)). We remark that the case $\alpha < 1$ could be handled similarly for the contribution $\bar{I}_l^{\tau,\xi}$ and even more directly for $\bar{I}_s^{\tau,\xi}$. Indeed, in that case, the centering argument is not needed since the Taylor expansion already yields an integrable singularity.

8 Appendix: Proofs of complementary results

8.1 Smoothing effects for Ornstein-Ulhenbeck operator

We state and prove here some of the key properties of the Ornstein-Uhlenbeck operator. Namely, we will prove the representation (2.11) and the associated α-smoothing effect (2.13). We highlight however that these results are only a slight modification to our purpose of those in [HMP19].

The two lemma below presents a deep connection with stochastic analysis and their proofs relies on tools that are more familiar in the probabilistic realm. For this reason, we are going to consider the stochastic counterpart of the Ornstein-Ulhenbeck operator L^{ou}. Namely, for a given starting point x in $\mathbb{R}^{nd}$, we are interested in the following dynamics

$$
\begin{cases} dX_t \;=\; AX_t dt + BdZ_t, & \text{on } [0,T] \\ X_0 \;=\; x \end{cases} \qquad (8.1)
$$

where $(Z_t)_{t\geq 0}$ is an α-stable, $\mathbb{R}^{nd}$-dimensional process with Lévy measure ν_α, defined on some complete probability space $(\Omega,\mathcal{F},\mathbb{P})$.

Lemma 8.1 (Representation)**.** *Under [**A**], the semigroup $\{P_t^{ou}\}_{t>0}$ generated by the Ornstein-Ulehnbeck operator L^{ou} (defined in* (2.9)*) admits for any fixed $t>0$, a density $p^{ou}(t,\cdot)$ which writes for any $t>0$ and any x,y in $\mathbb{R}^{nd}$*

$$
p^{ou}(t,x,y) \;=\; \frac{1}{\det \mathbb{M}_t}p_S(t,\mathbb{M}_t^{-1}\big(e^{At}x-y)\big)
$$

where $\mathbb{M}_t$ is the matrix defined in (2.12) *and p_S is the smooth density of an $\mathbb{R}^{nd}$-valued, symmetric and α-stable process S whose Lévy measure μ_S satisfies the non-degeneracy assumption [**ND**] on $\mathbb{R}^{nd}$.*

Proof. We start noticing that the above dynamics (8.1) can be explicitly integrated and gives

$$X_t \;=\; e^{tA}x + \int_0^t e^{(t-s)A} B\, dZ_s.$$

It is then readily derived from [PZ09] that, for any $t > 0$, the random variable X_t has a density $p_X(t, x, \cdot)$ with respect to the Lebesgue measure on $\mathbb{R}^{nd}$ and it is moreover well known (see for example [Dyn65]) that p_X coincides with the density p^{ou} of the Ornstein-Ulhenbeck operator L^{ou} .
For this reason, we fix $t \geq 0$ and consider, for a given N in $\mathbb{N}$, a uniform partition $\{t_i\}_{i\in[\![0,N]\!]}$ of $[0, t]$. Then, it holds for any p in $\mathbb{R}^{nd}$,

$$\mathbb{E}\Big[\exp\Big(i\langle p, \sum_{i=1}^N e^{(t-t_{i-1})A} B\big(Z_{t_i} - Z_{t_{i-1}}\big)\Big)\rangle\Big] \\ = \exp\Big(-\frac{1}{N}\sum_{i=1}^N \int_{\mathbb{S}^{d-1}} |\langle B^* e^{(t-t_{i-1})A^*} p, s\rangle|^\alpha\, \mu(ds)\Big)$$

where μ is the spherical measure associated with ν_α (see Equation (2.7)). By dominated convergence theorem, we let m goes to infinity and show that

$$\mathbb{E}\Big[\exp\Big(i\langle p, \int_0^t e^{(t-s)A} B\, dZ_s\Big)\Big] \;=\; \exp\Big(-\int_0^t \int_{\mathbb{S}^{d-1}} |\langle e^{uA^*} p, Bs\rangle\, \mu(ds)du\Big).$$

Thanks to the above equation, we can rewrite the characteristic function of X_t as:

$$\begin{aligned}\psi_{X_t}(p) \;&=\; \mathbb{E}\Big[\exp\Big(i\langle p, e^{tA}x + \int_0^t e^{(t-s)A} B\, dZ_s\Big)\Big] \\ &=\; \exp\Big(i\langle p, e^{tA}x\rangle - \int_0^t \int_{\mathbb{S}^{d-1}} |\langle e^{uA^*} p, Bs\rangle|^\alpha\, \mu(ds)du\Big) \\ &=\; \exp\Big(i\langle p, e^{tA}x\rangle - t\int_0^1 \int_{\mathbb{S}^{d-1}} |\langle e^{vtA^*} p, Bs\rangle|^\alpha\, \mu(ds)dv\Big)\end{aligned}$$

where in the last passage we used the change of variables $u = vt$. For the next step, we firstly notice that it holds

$$e^{tA} \;=\; \mathbb{M}_t e^A \mathbb{M}_t^{-1},$$

shown using the definition of matrix exponential and the trivial relation $\mathbb{M}_t A \mathbb{M}_t^{-1} = tA$. Exploiting the above identity, we then find that

$$\psi_{X_t}(p) \;=\; \exp\Big(i\langle p, e^{tA}x\rangle - t\int_0^1 \int_{\mathbb{S}^{d-1}} |\langle \mathbb{M}_t p, e^{vA}\mathbb{M}_t^{-1} Bs\rangle|^\alpha\, \mu(ds)dv\Big) \\ =\; \exp\Big(i\langle p, e^{tA}x\rangle - t\int_0^1 \int_{\mathbb{S}^{d-1}} |\langle \mathbb{M}_t p, e^{vA} Bs\rangle|^\alpha\, \mu(ds)dv\Big)$$

where in the last passage we used the straightforward identity $\mathbb{M}_t^1 By = By$. We focus now only on the double integral

$$\int_0^1 \int_{\mathbb{S}^{d-1}} |\langle \mathbb{M}_t p, e^{vA} Bs\rangle|^\alpha\, \mu(ds)dv.$$

If we consider the measure $m_\alpha(dv, ds) := |e^{vA}Bs|^\alpha \mu(ds)dv$ on $[0,1] \times \mathbb{S}^{d-1}$ and the normalized lift function $l\colon [0,1] \times \mathbb{S}^{d-1} \to \mathbb{S}^{nd-1}$ given by

$$l(v,s) := \frac{e^{vA}Bs}{|e^{vA}Bs|},$$

it then follows that

$$\begin{aligned}\int_0^1 \int_{\mathbb{S}^{d-1}} |\langle \mathbb{M}_t p, e^{vA}Bs\rangle|^\alpha \, \mu(ds)dv &= \int_0^1 \int_{\mathbb{S}^{d-1}} |\langle \mathbb{M}_t p, \frac{e^{vA}Bs}{|e^{vA}Bs|}\rangle|^\alpha \, m_\alpha(ds, dv) \\ &= \int_{\mathbb{S}^{nd-1}} |\langle \mathbb{M}_t p, \xi\rangle|^\alpha \, \mu_S(d\xi),\end{aligned}$$

where $\mu_S := \mathrm{Sym}(l_*(m_\alpha))$ is the symmetrized version of the measure m_α push-forwarded through l.
Noticing that μ_S is the Lévy measure of a symmetric α-stable process $\{S_t\}_{t\geq 0}$ satisfying assumption [**ND**] on $\mathbb{R}^{nd}$, we can finally write that

$$\psi_{X_t}(p) = \exp\Big(i\langle p, e^{tA}x\rangle - t\Phi_S(\mathbb{M}_t p)\Big)$$

where Φ_S is the Lévy symbol associated with S_t (cf. Equation (2.7)).
From Lemma A.1 in [HMP19], we know that under assumption [**ND**], the above calculations implies that

$$\int_0^1 \int_{\mathbb{S}^{d-1}} \Big|(\mathbb{M}_t p)\cdot(e^{Av}Bs)\Big|^\alpha \, \mu_S(ds)dv \geq C|\mathbb{M}_t p|^\alpha$$

for some constant $C > 0$. It follows in particular that the function $p \to \psi_{X_t}(p)$ is in $L^1(\mathbb{R}^{nd})$. Thus, by inverse fourier transform and a change of variables, we can prove that

$$\begin{aligned}\mathcal{F}^{-1}\Big[\psi_{X_t}\Big](y) &= \frac{1}{(2\pi)^{nd}} \int_{\mathbb{R}^{nd}} e^{-i\langle p,y\rangle} \exp\Big(i\langle p, e^{tA}x\rangle - t\Phi_S(\mathbb{M}_t p)\Big)\, dp \\ &= \frac{\det(\mathbb{M}_t^{-1})}{(2\pi)^{nd}} \int_{\mathbb{R}^{nd}} \exp\Big(-i\Big\langle \mathbb{M}_t^{-1}p, y - e^{tA}x\Big\rangle\Big) e^{-t\Phi(p)}\, dp \\ &= \frac{\det(\mathbb{M}_t^{-1})}{(2\pi)^{nd}} \int_{\mathbb{R}^{nd}} \exp\Big(-i\Big\langle p, \mathbb{M}_t^{-1}\Big(y - e^{tA}x\Big)\Big\rangle\Big) e^{-t\Phi(p)}\, dp \\ &= \frac{1}{\det \mathbb{M}_t} p_S(t, \mathbb{M}^{-1}(y - e^{At}x))\end{aligned}$$

and we have concluded since p_S is symmetric. □

We can now point out the smoothing effect (Equation (2.13)) associated with the Ornstein-Uhlenbeck density p^{ou}.

Lemma 8.2 (Smoothing Effect)**.** *Under* [$\boldsymbol{A}$]*, there exists a family* $\{q(t,\cdot)\colon t \in [0,T]\}$ *of densities on* $\mathbb{R}^{nd}$ *such that*

- *for any* l *in* $[\![0,3]\!]$*, there exists a constant* $C := C(l, nd)$ *such that* $|D_y^l p_S(t,y)| \leq Cq(t,y)t^{-l/\alpha}$ *for any* t *in* $[0,T]$ *and any* y *in* $\mathbb{R}^{nd}$*;*

- *(stable scaling property)* $q(t,y) = t^{-nd/\alpha} q(1, t^{-1/\alpha} y)$ *for any* t *in* $[0,T]$ *and any* y *in* $\mathbb{R}^{nd}$*;*
- *(stable smoothing effect) for any* γ *in* $[0,\alpha)$*, there exists a constant* $c := c(\gamma, nd)$ *such that*
$$\int_{\mathbb{R}^{nd}} q(t,y)\,|y|^{\gamma}\,dy \;\leq\; ct^{\gamma/\alpha} \text{ for any } t > 0. \tag{8.2}$$

Proof. Fixed a time $t > 0$, we start applying the Ito-Lévy decomposition to S at the associated characteristic stable time scale, i.e. we choose to truncate at threshold $t^{1/\alpha}$, so that we can write $S_t = M_t + N_t$ for some M_t, N_t independent random variables corresponding to the small jumps part and the large jumps part, respectively. Namely, we denote for any $s > 0$

$$N_s := \int_0^s \int_{|x|>t^{1/\alpha}} xP(du,dx) \;\text{ and }\; M_s := S_s - N_s$$

where P is the Poisson random measure associated with the process S. We can thus rewrite the density p_S in the following way

$$p_S(t,x) \;=\; \int_{\mathbb{R}^{nd}} p_M(t, x-y) P_{N_t}(dy)$$

where $p_M(t,\cdot)$ corresponds to the density of M_t and P_{N_t} is the law of N_t.
It is important now to notice that it is precisely our choice of the cutting threshold $t^{1/\alpha}$ that gives M and N the α-similarity property (for any fixed t)

$$N_t \overset{law}{=} t^{1/\alpha} N_1 \;\text{ and }\; M_t \overset{law}{=} t^{1/\alpha} M_1$$

we will need below. Indeed, to show the assertion for N, we can start from the Lévy-Khintchine formula for the characteristic function of N:

$$\mathbb{E}\Big[e^{i\langle p, N_t\rangle}\Big] \;=\; \exp\Big[t \int_{\mathbb{S}^{nd-1}} \int_{t^{1/\alpha}}^{\infty} \Big(\cos(\langle p, r\xi\rangle) - 1\Big) \frac{dr}{r^{1+\alpha}} \overline{\mu}_S(d\xi)\Big]$$

for any p in $\mathbb{R}^{nd}$. We then use the change of variable $rt^{-1/\alpha} = s$ to get that

$$\mathbb{E}\Big[e^{i\langle p, N_t\rangle}\Big] \;=\; \mathbb{E}\Big[e^{i\langle p, t^{1/\alpha} N_1\rangle}\Big].$$

This implies in particular our assertion on N. In a similar way, it is possible to get the analogous assertion on M.
From lemma A.2 in [HMP19] with $m = 3$, we know that there exist a family $\{p_{\overline{M}}(t,\cdot)\}_{t>0}$ of densities on $\mathbb{R}^{nd}$ and a constant $C := C(m,\alpha)$ such that

$$|D_y^l p_M(t,y)| \;\leq\; C p_{\overline{M}}(t,y) t^{-l/\alpha}$$

for any $t > 0$, any x in $\mathbb{R}^{nd}$ and any $l \in \{0,1,2\}$.
Moreover, denoting $\overline{M}_t$ the random variable with density $p_{\overline{M}}(t,\cdot)$ and independent from N_t, we can easily check from $p_{\overline{M}}(t,y) = t^{-nd/\alpha} p_{\overline{M}}(1, t^{-1/\alpha} x)$ that $\overline{M}$ is α-selfsimilar

$$\overline{M}_t \overset{law}{=} t^{1/\alpha} \overline{M}_1.$$

We can finally define the family $\{q(t,\cdot)\}_{t>0}$ of densities as

$$q(t,x) \;:=\; \int_{\mathbb{R}^{nd}} p_{\overline{M}}(t,x-y)P_{N_t}(dy)$$

corresponding to the density of the random variable

$$\overline{S}_t \,;=\, \overline{M}_t + N_t$$

for any fixed $t>0$. Using Fourier transform and the already proven α-selfsimilarity of $\overline{M}$ and N, we can show now that

$$\overline{S}_t \overset{law}{=} t^{1/\alpha}\overline{S}_1$$

or equivalently, that

$$q(t,y) = t^{-nd/\alpha}q(1,t^{-1/\alpha}y)$$

for any t in $[0,T]$ and any y in $\mathbb{R}^{nd}$. Moreover,

$$\mathbb{E}[|\overline{S}_t|^\gamma] \;=\; \mathbb{E}[|\overline{M}_t + N_t|^\gamma] \;=\; Ct^{\gamma/\alpha}\Big(\mathbb{E}[|\overline{M}_1|^\gamma] + \mathbb{E}[|N_t|^\gamma]\Big) \;\leq\; Ct^{\gamma/\alpha}.$$

This shows in particular that equation (8.2) holds. □

We conclude this sub-section showing Control (5.50) appearing in the proof of Proposition 3.6 for the diagonal regime. First of all, we will need the following lemma:

Lemma 8.3. *Let t in $[0,T]$, x,b in $\mathbb{R}^{nd}$ such that $|b|\leq ct^{1/\alpha}$ for some constant $c>0$. Under [**A**], there exists a constant $C:=C(c)$ such that*

$$|D^l_x p_S\Big(t,x+b)| \;\leq\; \tilde{C}|D^l_x p_S\Big(t,x)|$$

Proof. Looking back at the proof of the previous lemma 8.2, we know that

$$D^l_x p_S(t,x+b) \;=\; \int_{\mathbb{R}^{nd}} D^l_x p_M(t,x+b-y)P_{N_t}(dy)$$

where $p_M(t,\cdot)$ is the density of M_t and P_{N_t} is the law of N_t, corresponding to the small and big jumps in the Ito-Lévy decomposition.
From lemma $A.2$ in [HMP19] we know moreover that

$$|D^l_x p_M(t,x+b-y)| \;\leq\; \frac{C}{t^{\frac{l}{\alpha}}}p_{\overline{M}}(t,x+b-y) \;\text{ where }\; p_{\overline{M}}(t,z) \;=\; \frac{C}{t^{\frac{nd}{\alpha}}}\frac{1}{\Big(1+\frac{|z|}{t^{\frac{1}{\alpha}}}\Big)^3}.$$

It is then enough to show that

$$\begin{aligned} p_{\overline{M}}(t,z+b) \;&=\; \frac{C}{t^{\frac{nd}{\alpha}}}\frac{1}{\Big(1+\frac{|z+b|}{t^{\frac{1}{\alpha}}}\Big)^3} \;\leq\; \frac{\tilde{C}}{t^{\frac{nd}{\alpha}}}\frac{1}{\Big(1+c+\frac{|z+b|}{t^{\frac{1}{\alpha}}}\Big)^3} \\ &\leq\; \frac{C}{t^{\frac{nd}{\alpha}}}\frac{1}{\Big(1+c\frac{|z|}{t^{\frac{1}{\alpha}}}-\frac{|b|}{t^{\frac{1}{\alpha}}}\Big)^3} \;\leq\; \frac{C}{t^{\frac{nd}{\alpha}}}\frac{1}{\Big(1+\frac{|z|}{t^{\frac{1}{\alpha}}}\Big)^3} \\ &\leq\; Cp_{\overline{M}}(t,z). \end{aligned}$$

to conclude the proof. □

Proof of Equation (5.50). We start looking back to the proof of Lemma 3.2 to find that

$$\begin{aligned}&\Big|D_x^{\vartheta}\tilde{p}^{\tau,\xi'}(t,s,x+\lambda(x'-x),y)\Big|\\&\qquad = C(s-t)^{-\sum_{k=1}^{n}\frac{\vartheta_k}{\alpha_k}}\frac{1}{\det\mathbb{M}_{s-t}}\Big|\mathbf{d}_z^{|\vartheta|}p_S\Big(s-t,\cdot)(\mathbb{M}_{s-t}^{-1}(\tilde{m}_{s,t}^{\tau,\xi}(x)-y)\Big)\Big|\end{aligned}$$

Moreover, we notice that

$$\mathbb{M}_{s-t}^{-1}\Big(\tilde{m}_{s,t}^{\tau,\xi}(x+\lambda(x-x'))-y\Big) = \mathbb{M}_{s-t}^{-1}\Big(\tilde{m}_{s,t}^{\tau,\xi}(x)-y\Big)+\lambda\mathbb{M}_{s-t}^{-1}e^{A(s-t)}(x-x').$$

Then, Control (5.50) follows immediately from the previous lemma once we have shown that

$$\Big|\lambda\mathbb{M}_{s-t}^{-1}e^{A(s-t)}(x-x')\Big| \le C(s-t)^{1/\alpha}$$

for some constant $C := C(A)$. Indeed, fixed i in $[\![1,n]\!]$, we can exploit the structure of A and $\mathbb{M}_{s-t}$ (cf. Equation (2.18) in Scaling Lemma 2.1) to write that

$$\begin{aligned}\Big[\mathbb{M}_{s-t}^{-1}e^{A(s-t)}(x-x')\Big]_i &= \sum_{j=1}^{n}\sum_{k=1}^{n}\Big[\mathbb{M}_{s-t}^{-1}\Big]_{i,k}\Big[e^{A(s-t)}\Big]_{k,j}(x-x')_j\\&= \sum_{j=i}^{n}(s-t)^{-(i-1)}C_j(s-t)^{i-j}(x-x')_j.\end{aligned}$$

Since moreover we assumed to be in a local diagonal regime, i.e. $\mathbf{d}^{\alpha}(x,x')\le(s-t)^{1/\alpha}$, we have that

$$\begin{aligned}\Big|\Big[\mathbb{M}_{s-t}^{-1}e^{A(s-t)}(x-x')\Big]_i\Big| &\le C\sum_{j=i}^{n}(s-t)^{-(j-1)}|(x-x')_j|\\&\le C\sum_{j=i}^{n}(s-t)^{-(j-1)}(s-t)^{\frac{1+\alpha(j-1)}{\alpha}}\\&= C(s-t)^{1/\alpha}.\end{aligned}$$

The proof is thus concluded. □

8.2 Technical tools

In this section, we present the proof of some technical results already used in the article, for the sake of completeness.
We recall moreover that the results below can be proven also for the flow $\bar{\theta}_{s,\tau}(\xi)$ driven by a more general perturbation F under assumption [$\bar{\mathbf{A}}$] (cf. Section 7.1), exploiting that $\bar{F}_i$ is Lipschitz continuous in the x_{i-1} variable for any i in $[\![2,n]\!]$.

We begin proving Lemma 5.4 about the sensitivity of the Hölder flows, appearing in the proof of the a priori estimates (3.18) of Proposition 3.6. For this reason, we will assume from this point further to be under assumption [**A'**].

Proof of Lemma 5.4. We start noticing that our result follows immediately using Young inequality, once we have shown that it holds

$$\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_i\Big| \le C\Big[(s-t)^{\frac{1+\alpha(i-1)}{\alpha}}+\mathbf{d}^{1+\alpha(i-1)}(x,x')\Big] \quad \text{for any } i \text{ in } [\![1,n]\!]. \tag{8.3}$$

Our proof will rely essentially in iterative applications of the Grönwall lemma. We notice however that under [**A**], the perturbation F_i is only Hölder continuous with respect to its i-th variable. To overcome this problem, we are going to mollify (but only with respect to the variable of interest) the function F in the following way: fixed a mollifier ρ on $\mathbb{R}^d$, i.e. a compactly supported, non-negative, smooth function such that $\|\rho\|_{L^1}=1$ and a family δ_i of positive constants to be chosen later, the mollified version of the perturbation is given by $F^\delta=(F_1,F_2^{\delta_2},\dots,F_n^{\delta_n})$ where

$$F_i^{\delta_i}(t,z_{i:n}) \;:=\; F_i *_i \rho_{\delta_i}(t,z_{i:n}) \;=\; \int_{\mathbb{R}^d} F_i(t,z_i-\omega,z_{i+1},\dots,z_n)\frac{1}{\delta_i^d}\rho(\frac{\omega}{\delta_i})\,d\omega.$$

We remark in particular that we do not need to mollify the first component F_1 since it is regular enough, say β-Hölder continuous in the first d-dimensional variable x_1, by assumption [**R**].
Then, standard results on mollifier theory and our current assumptions on F show us that the following controls hold

$$|F_i(u,z)-F_i^\delta(u,z)| \le \|F_i\|_{L^\infty(C_d^{\gamma+\beta})}\delta_i^{\frac{\gamma_i+\beta}{1+\alpha(i-1)}}, \tag{8.4}$$

$$|F_i^\delta(u,z)-F_i^\delta(u,z')| \le C\|F_i\|_{L^\infty(C_d^{\gamma+\beta})}\Big[\delta_i^{\frac{\gamma_i+\beta}{1+\alpha(i-1)}-1}|(z-z')_i|+\sum_{j=i+1}^n|(z-z')_j|^{\frac{\gamma_i+\beta}{1+\alpha(j-1)}}\Big]. \tag{8.5}$$

We choose now δ_i for any i in $[\![2,n]\!]$ in order to have any contribution associated with the mollification appearing in (8.4) at a good current scale time. Namely, we would like δ_i to satisfy

$$\Big|\Big((s-t)^{\frac{1}{\alpha}}\mathbb{M}_{s-t}\Big)^{-1}\Big(F(u,z)-F^\delta(u,z)\Big)\Big| \le C(s-t)^{-1}$$

for any u in $[t,s]$ and any z in $\mathbb{R}^{nd}$. Using the mollifier controls (8.4), it is enough to ask for

$$\sum_{i=2}^n(s-t)^{-\frac{1}{\alpha_i}}\delta_i^{\frac{\gamma_i+\beta}{1+\alpha(i-1)}} \le C(s-t)^{-1}.$$

Recalling that $\gamma_i:=1+\alpha(i-2)$ by assumption [**R**], this is true if we fix for example,

$$\delta_i \;=\; (s-t)^{\frac{\gamma_i}{\alpha}\frac{1+\alpha(i-1)}{\gamma_i+\beta}} \quad \text{for } i \text{ in } [\![2,n]\!]. \tag{8.6}$$

After this introductive part, we start controlling the last component of the flow. By construction of $\theta_{s,t}$, we can write that

$$\begin{aligned}
&\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_n\Big| \qquad (8.7)\\
&= \Big|(x-x')_n+\int_t^s\Big\{\Big[A(\theta_{v,t}(x)-\theta_{v,t}(x'))\Big]_n+F_n(v,\theta_{v,t}(x))-F_n(v,\theta_{v,t}(x'))\Big\}\,dv\Big|\\
&\le |(x-x')_n|\\
&\qquad+\int_t^s\Big\{A_{n,n-1}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-1}|+\Big|F_n(v,\theta_{v,t}(x))-F_n(v,\theta_{v,t}(x'))\Big|\Big\}\,dv
\end{aligned}$$

where in the last passage we have exploited the sub-diagonal structure of A (cf. Equation (1.2)). If we focus only on the last term involving the difference of the drifts, It holds now that

$$\begin{aligned}\Big|F_n(v,\theta_{v,t}(x)) - F_n(v,\theta_{v,t}(x'))\Big| \le \Big|F_n(v,\theta_{v,t}(x)) - F_n^\delta(v,\theta_{v,t}(x))\Big| \\ + \Big|F_n(v,\theta_{v,t}(x')) - F_n^\delta(v,\theta_{v,t}(x'))\Big| + \Big|F_n^\delta(v,\theta_{v,t}(x)) - F_n^\delta(v,\theta_{v,t}(x'))\Big|.\end{aligned}$$

Using the controls (8.4), (8.5) on the mollified drifts, we then write from (8.7) and the previous equation that

$$\begin{aligned}\Big|(\theta_{s,t}(x) - \theta_{s,t}(x'))_n\Big| \le |(x-x')_n| + 2(s-t)\delta_n^{\frac{\gamma_n+\beta}{1+\alpha(n-1)}} \\ + C\int_t^s \Big\{\Big|(\theta_{v,t}(x) - \theta_{v,t}(x'))_{n-1}\Big| + \delta_n^{\frac{\gamma_n+\beta}{1+\alpha(n-1)}-1}\Big|(\theta_{v,t}(x) - \theta_{v,t}(x'))_n\Big|\Big\}\,dv.\end{aligned}$$

We now apply the Grönwall lemma to show that

$$\Big|(\theta_{s,t}(x) - \theta_{s,t}(x'))_n\Big| \le C\Big[|(x-x')_n| + (s-t)\delta_n^{\frac{\gamma_n+\beta}{1+\alpha(n-1)}} + \int_t^s \Big|(\theta_{v,t}(x) - \theta_{v,t}(x'))_{n-1}\Big|\,dv\Big].$$

From our previous choice for δ_n (cf. Equation (8.6)), we know that

$$(s-t)^{-\frac{1}{\alpha_n}}\delta_n^{\frac{\gamma_n+\beta}{1+\alpha(n-1)}} \le C(s-t)^{-1}$$

and thus, we can rewrite the last inequality as

$$\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_n\Big| \le C\Big[\Big|(x-x')_n\Big|+(s-t)^{\frac{1+\alpha(n-1)}{\alpha}}+\int_t^s\Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-1}\Big|\,dv\Big]. \quad (8.8)$$

We would like now to obtain a similar control on the $(n-1)$-th term. As already done at the beginning of the proof, we can write that

$$\begin{aligned}\Big|(\theta_{s,t}(x) - \theta_{s,t}(x'))_{n-1}\Big| \le \Big|(x-x')_{n-1}\Big| + C\delta_{n-1}^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-2)}}(s-t) + \int_t^s \Big|(\theta_{v,t}(x) - \theta_{v,t}(x'))_{n-2}\Big| \\ + \delta_{n-1}^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-2)}-1}\Big|(\theta_{v,t}(x) - \theta_{v,t}(x'))_{n-1}\Big| + \Big|(\theta_{v,t}(x) - \theta_{v,t}(x'))_n\Big|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}}\,dv\end{aligned}$$

We then apply the Grönwall lemma to find that

$$\begin{aligned}\Big|(\theta_{s,t}(x) - \theta_{s,t}(x'))_{n-1}\Big| \le C\Big[\Big|(x-x')_{n-1}\Big| + \delta_{n-1}^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-2)}}(s-t) \\ + \int_t^s \Big\{\Big|(\theta_{v,t}(x) - \theta_{v,t}(x'))_{n-2}\Big| + \Big|(\theta_{v,t}(x) - \theta_{v,t}(x'))_n\Big|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}}\Big\}\,dv\Big].\end{aligned}$$

Remembering our previous choice of δ_{n-1}, it holds now that

$$\begin{aligned}\Big|(\theta_{s,t}(x) - \theta_{s,t}(x'))_{n-1}\Big| \le C\Big[|(x-x')_{n-1}| + (s-t)^{\frac{1+\alpha(n-2)}{\alpha}} + \int_t^s \Big|(\theta_{v,t}(x) - \theta_{v,t}(x'))_{n-2}\Big| \\ + \Big|(\theta_{v,t}(x) - \theta_{v,t}(x'))_n\Big|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}}\,dv\Big]. \quad (8.9)\end{aligned}$$

We then use equation (8.8) and the Jensen inequality to write

$$\begin{aligned}
&\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_{n-1}\Big| \\
&\le C\Big[|(x-x')_{n-1}|+(s-t)^{\frac{1+\alpha(n-2)}{\alpha}}+\int_t^s\Big\{\Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-2}\Big|+\Big|(x-x')_n\Big|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} \\
&\qquad\qquad +(v-t)^{\frac{\gamma_{n-1}+\beta}{\alpha}}+\Big(\int_t^v\Big|(\theta_{\omega,t}(x)-\theta_{\omega,t}(x'))_{n-1}\Big|\,d\omega\Big)^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}}\Big\}\,dv\Big]. \qquad (8.10)
\end{aligned}$$

The idea now is to use Grönwall lemma again. To do so, we firstly move the exponent from the last integral term involving the $(n-1)$-th term using the Young inequality:

$$\begin{aligned}
&\Big(\int_t^v\Big|(\theta_{\omega,t}(x)-\theta_{\omega,t}(x'))_{n-1}\Big|\,d\omega\Big)^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} \\
&\qquad\qquad \le B^{-\frac{1+\alpha(n-1)}{\gamma_{n-1}+\beta}}\int_t^v\Big|(\theta_{\omega,t}(x)-\theta_{\omega,t}(x'))_{n-1}\Big|\,d\omega+B^{\frac{1+\alpha(n-1)}{2\alpha-\beta}}
\end{aligned}$$

for a quantity B to be fixed later.
Since we need homogeneity with respect to time in equation (8.9), we choose B such that

$$B^{\frac{1+\alpha(n-1)}{2\alpha-\beta}} = (v-t)^{\frac{\gamma_{n-1}+\beta}{\alpha}} \Leftrightarrow B=(v-t)^{\frac{\gamma_{n-1}+\beta}{\alpha}\frac{2\alpha-\beta}{1+\alpha(n-1)}}.$$

Plugging it into the general expression in (8.10), we find that

$$\begin{aligned}
\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_{n-1}\Big| &\le C\Big[|(x-x')_{n-1}|+(s-t)^{\frac{1+\alpha(n-2)}{\alpha}} \\
&\quad+\int_t^s\Big\{\Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-2}\Big|+\Big|(x-x')_n\Big|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}}+(v-t)^{\frac{\gamma_{n-1}+\beta}{\alpha}} \\
&\qquad+(v-t)^{\frac{\beta}{\alpha}-2}\int_t^v\Big|(\theta_{\omega,t}(x)-\theta_{\omega,t}(x'))_{n-1}\Big|\,d\omega\Big\}\,dv\Big] \\
&\le C\Big[|(x-x')_{n-1}|+(s-t)^{\frac{1+\alpha(n-1)}{\alpha}}+(s-t)\Big|(x-x')_n\Big|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}}+(s-t)^{\frac{\gamma_{n-1}+\beta+\alpha}{\alpha}} \\
&\quad+\int_t^s\Big\{\Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-2}\Big|+(v-t)^{\frac{\beta}{\alpha}-1}\sup_{\omega\in[t,v]}\Big|(\theta_{\omega,t}(x)-\theta_{\omega,t}(x'))_{n-1}\Big|\Big\}\,dv\Big].
\end{aligned}$$

Since the previous inequality is also true for any $\overline{s}$ in $[t,s]$, it follows that

$$\begin{aligned}
&\sup_{\overline{s}\in[0,s]}\Big|(\theta_{\overline{s},t}(x)-\theta_{\overline{s},t}(x'))_{n-1}\Big| \\
&\quad\le C\Big[|(x-x')_{n-1}|+(s-t)^{\frac{1+\alpha(n-2)}{\alpha}}+(s-t)\Big|(x-x')_n\Big|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}}+(s-t)^{\frac{\gamma_{n-1}+\beta+\alpha}{\alpha}} \\
&\qquad+\int_t^s\Big\{\Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-2}\Big|+(v-t)^{\frac{\beta}{\alpha}-1}\sup_{\omega\in[t,v]}\Big|(\theta_{\omega,t}(x)-\theta_{\omega,t}(x'))_{n-1}\Big|\Big\}\,dv\Big].
\end{aligned}$$

We can finally apply the Grönwall lemma to show that for any s in $[t,T]$, there exists a constant C such that

$$\begin{aligned}
\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_{n-1}\Big| &\le C\Big[|(x-x')_{n-1}|+(s-t)^{\frac{1+\alpha(n-2)}{\alpha}}+(s-t)|(x-x')_n|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} \\
&\qquad+\int_t^s\Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-2}\Big|\,dv\Big].
\end{aligned}$$

Moreover, thanks to the Young inequality we know that

$$(s-t)\Big|(x-x')_n\Big|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}} \leq C\Big\{(s-t)^{\frac{1+\alpha(n-2)}{\alpha}} + |(x-x')_n|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}\frac{1+\alpha(n-2)}{1+\alpha(n-3)}}\Big\}$$

and remembering that $\mathbf{d}(x,x') \leq 1$ by hypothesis,

$$|(x-x')_n|^{\frac{\gamma_{n-1}+\beta}{1+\alpha(n-1)}\frac{1+\alpha(n-2)}{1+\alpha(n-3)}} \leq |(x-x')_n|^{\frac{\gamma_{n-1}+\beta}{\gamma_{n-1}}\frac{1+\alpha(n-2)}{1+\alpha(n-1)}} \leq |(x-x')_n|^{\frac{1+\alpha(n-2)}{1+\alpha(n-1)}}.$$

We then use it to write for any v in $[t,T]$,

$$\Big|(\theta_{v,t}(x)-\theta_{v,t}(x'))_{n-1}\Big| \leq C\Big[|(x-x')_{n-1}| + (v-t)^{\frac{1+\alpha(n-2)}{\alpha}} + |(x-x')_n|^{\frac{1+\alpha(n-2)}{1+\alpha(n-1)}} + \int_t^v \Big|(\theta_{\omega,t}(x)-\theta_{\omega,t}(x'))_{n-2}\Big|\,d\omega\Big].$$

Going back to equation (8.8), we plug in the last bound to find that

$$\begin{aligned}\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_n\Big| &\leq C\Big[|(x-x')_n| + (s-t)^{\frac{1+\alpha(n-1)}{\alpha}} + (s-t)|(x-x')_{n-1}| \\ &\quad + (s-t)|(x-x')_n|^{\frac{1+\alpha(n-2)}{1+\alpha(n-1)}} + \int_t^s\int_t^v \Big|(\theta_{t,\omega}(x)-\theta_{\omega,t}(x'))_{n-2}\Big|\,d\omega dv\Big] \\ &\leq C\Big[|(x-x')_n| + (s-t)^{\frac{1+\alpha(n-1)}{\alpha}} + |(x-x')_{n-1}|^{\frac{1+\alpha(n-1)}{1+\alpha(n-2)}} \\ &\quad + \int_t^s\int_t^v \Big|(\theta_{\omega,t}(x)-\theta_{\omega,t}(x'))_{n-2}\Big|\,d\omega dv\Big]\end{aligned}$$

where in the last passage we used again the Young inequality to show that

$$(s-t)|(x-x')_{n-1}| \leq C(s-t)^{\frac{1+\alpha(n-1)}{\alpha}} + |(x-x')_{n-1}|^{\frac{1+\alpha(n-1)}{1+\alpha(n-2)}}$$

and

$$(s-t)|(x-x')_n|^{\frac{1+\alpha(n-2)}{1+\alpha(n-1)}} \leq C(s-t)^{\frac{1+\alpha(n-1)}{\alpha}} + |(x-x')_n|.$$

This approach may be naturally iterated up to the first term of the chain, so that

$$\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_n\Big| \leq C\Big[\sum_{j=2}^n |(x-x')_j|^{\frac{1+\alpha(n-1)}{1+\alpha(j-1)}} + (s-t)^{\frac{1+\alpha(n-1)}{\alpha}} + \int_t^{v_n=s} dv_{n-1}\cdots\int_t^{v=2} dv_1\Big|(\theta_{v_1,t}(x)-\theta_{v_1,t}(x'))_1\Big|\Big].$$

In a similar manner, we can show for any i in $[\![2,n]\!]$,

$$\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_i\Big| \leq C\Big[\sum_{j=2}^n |(x-x')_j|^{\frac{1+\alpha(i-1)}{1+\alpha(j-1)}} + (s-t)^{\frac{1+\alpha(i-1)}{\alpha}} + \int_t^{v_i=s} dv_{i-1}\cdots\int_t^{v=2} dv_1\Big|(\theta_{v_1,t}(x)-\theta_{v_1,t}(x'))_1\Big|\Big]. \quad (8.11)$$

Since all the non-integral terms in (8.11) are compatible with the statement of the lemma, it remains to find the proper bound for the first component of the flow. As before, let us consider $\overline{s}$ in $[t,s]$. We can write

$$|(\theta_{\overline{s},t}(x)-\theta_{\overline{s},t}(x'))_1| \le |(x-x')_1| + C\sum_{j=1}^{n}\int_t^{\overline{s}} |(\theta_{v,t}(x)-\theta_{v,t}(x'))_j|^{\frac{\beta}{1+\alpha(j-1)}}\,dv$$

or, passing to the supremum on both sides,

$$\begin{aligned}\sup_{\overline{s}\in[t,s]}|(\theta_{\overline{s},t}(x)-\theta_{\overline{s},t}(x'))_1| \le |(x-x')_1| + C\Big\{&(s-t)\Big(\sup_{v\in[t,s]}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_1|\Big)^{\beta}\\ &+\sum_{j=2}^{n}\int_t^{s}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_j|^{\frac{\beta}{1+\alpha(j-1)}}\,dv\Big\}.\end{aligned}$$

Using equation (8.11), it holds now that

$$\begin{aligned}\sup_{\overline{s}\in[t,s]}(|\theta_{\overline{s},t}(x)-\theta_{\overline{s},t}(x'))_1| \le |(x-x')_1| + C\Big\{&(s-t)\Big(\sup_{v\in[t,s]}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_1|\Big)^{\beta}\\ &+\sum_{j=2}^{n}\Big[(s-t)\Big((s-t)^{\frac{1+\alpha(j-1)}{\alpha}}+\sum_{k=2}^{n}|(x-x')_k|^{\frac{1+\alpha(j-1)}{1+\alpha(k-1)}}\\ &+(s-t)^{j-1}\sup_{v\in[t,s]}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_1|\Big)^{\frac{\beta}{1+\alpha(j-1)}}\Big]\Big\}.\end{aligned} \tag{8.12}$$

We then apply the Jensen inequality to show that

$$\begin{aligned}\sup_{\overline{s}\in[t,s]}(|\theta_{\overline{s},t}(x)-\theta_{\overline{s},t}(x'))_1| &\le |(x-x')_1| + C\Big\{(s-t)\Big[\sup_{v\in[t,s]}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_1|\Big]^{\beta}\\ &\quad+\sum_{j=2}^{n}C(s-t)\Big[(s-t)^{\frac{\beta}{\alpha}}+\sum_{k=2}^{n}|(x-x')_k|^{\frac{\beta}{1+\alpha(k-1)}}\\ &\quad+(s-t)^{\frac{(j-1)\beta}{1+\alpha(j-1)}}\sup_{v\in[t,s]}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_1|^{\frac{\beta}{1+\alpha(j-1)}}\Big]\Big\}\\ &\le C\Big\{|(x-x')_1|+(s-t)^{\frac{\alpha+\beta}{\alpha}}+(s-t)\sum_{k=2}^{n}|(x-x')_k|^{\frac{\beta}{1+\alpha(k-1)}}\\ &\quad+\sum_{j=1}^{n}(s-t)^{1+\frac{(j-1)\beta}{1+\alpha(j-1)}}\sup_{v\in[t,s]}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_1|^{\frac{\beta}{1+\alpha(j-1)}}\Big\}.\end{aligned} \tag{8.13}$$

From Young inequality, we can deduce now that

$$(s-t)|(x-x')_k|^{\frac{\beta}{1+\alpha(k-1)}} \le C\Big((s-t)^{\frac{1}{1-\beta}}+|(x-x')_k|^{\frac{1}{1+\alpha(k-1)}}\Big)$$

and

$$\begin{aligned}&(s-t)^{1+\frac{(j-1)\beta}{1+\alpha(j-1)}}\sup_{v\in[t,s]}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_1|^{\frac{\beta}{1+\alpha(j-1)}}\\ &\qquad\le C\Big\{(s-t)^{\frac{1+(\alpha+\beta)(j-1)}{1+\alpha(j-1)-\beta}}+\sup_{v\in[t,s]}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_1|\Big\}\end{aligned}$$

Plugging these inequalities into the main one (8.13), we find that

$$\begin{aligned}\sup_{\overline{s}\in[t,s]}(|\theta_{\overline{s},t}(x)-\theta_{\overline{s},t}(x'))_1| \le C\Big\{&|(x-x')_1|+(s-t)^{\frac{\alpha+\beta}{\alpha}}+\sum_{k=2}^{n}|(x-x')_k|^{\frac{1}{1+\alpha(k-1)}}\\ &+\sum_{j=1}^{n}(s-t)^{\frac{1+(\alpha+\beta)(j-1)}{1+\alpha(j-1)-\beta}}+\sup_{v\in[t,s]}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_1|\Big\}\\ \le C\Big\{&(s-t)^{\frac{\alpha+\beta}{\alpha}}+(s-t)^{\frac{1}{1-\beta}}+d(x,x')\\ &+\sum_{j=1}^{n}(s-t)^{\frac{1+(\alpha+\beta)(j-1)}{1+\alpha(j-1)-\beta}}+\sup_{v\in[t,s]}|(\theta_{v,t}(x)-\theta_{v,t}(x'))_1|\Big\}\end{aligned}$$

Remembering that $s-t\le T-t\le 1$, it finally holds that

$$|\theta_{s,t}(x)-\theta_{s,t}(x'))_1| \le C\Big((s-t)^{1/\alpha}+\mathbf{d}(x,x')\Big)$$

since by assumption [**P**],

$$\frac{\alpha+\beta}{\alpha} > \frac{1}{1-\beta} > \frac{1}{\alpha}$$

and

$$\frac{1+(\alpha+\beta)(j-1)}{1+\alpha(j-1)-\beta} = 1+\frac{\beta j}{1+\alpha j-(\alpha+\beta)} > 1+\frac{\beta j}{\alpha j} > 1+\Big(\frac{1-\alpha}{\alpha}\Big) = \frac{1}{\alpha}.$$

Plugging this control in equation (8.11), we then conclude since

$$\begin{aligned}&\Big|(\theta_{s,t}(x)-\theta_{s,t}(x'))_i\Big|\\ &\quad\le C\Big(\mathbf{d}^{1+\alpha(i-1)}(x,x')+(s-t)^{\frac{1+\alpha(i-1)}{\alpha}}+(s-t)^{i-1}\sup_{\overline{s}\in[t,s]}(|\theta_{\overline{s},t}(x)-\theta_{\overline{s},t}(x'))_1|\Big)\\ &\quad\le C\Big(\mathbf{d}^{1+\alpha(i-1)}(x,x')+(s-t)^{\frac{1+\alpha(i-1)}{\alpha}}+(s-t)^{i-1}\Big((s-t)^{1/\alpha}+\mathbf{d}(x,x')\Big)\Big)\\ &\quad\le C\Big((s-t)^{\frac{1+\alpha(i-1)}{\alpha}}+\mathbf{d}^{1+\alpha(i-1)}(x,x')\Big),\end{aligned}$$

using again the Young inequality in the last passage. The proof is complete.

We can now prove the two results (Lemmas 5.5 and Lemma 5.6) concerning the sensitivity of the frozen shift $\tilde{m}^{\tau,\xi}_{s,t}$.

Proof of Lemma 5.5. From the integral representation of $\tilde{m}^{t,x}_{s,t}(y)$ (cf. Equation (3.4)), we can write that

$$\begin{aligned}\Big|\Big(\tilde{m}^{t,x}_{s,t}(y)-\tilde{m}^{t,x'}_{s,t}(y')\Big)_1\Big| &\le \int_t^s\Big|F_1(v,\theta_{v,t}(x))-F_1(v,\theta_{v,t}(x'))\Big|\,dv\\ &\le C\|F\|_H\int_t^s\mathbf{d}^{\beta}\Big(\theta_{v,t}(x),\theta_{v,t}(x')\Big)\,dv\end{aligned}$$

where in the second passage we used that F_1 is in $C^{\beta}_{b,d}(\mathbb{R}^{nd})$. Thanks to the Control on the flows (Lemma 5.4), it then holds that

$$\Big|\Big(\tilde{m}^{t,x}_{s,t}(y)-\tilde{m}^{t,x'}_{s,t}(y')\Big)_1\Big| \le C\|F\|_H(s-t)\Big[\mathbf{d}^{\beta}(x,x')+(s-t)^{\frac{\beta}{\alpha}}\Big]$$

and we have concluded.

Proof of Lemma 5.6. We know from Lemma 3.1 that $\tilde{m}^{t,x'}_{t_0,t}(x') = \theta_{t_0,t}(x')$. Fixed i in $[\![1,n]\!]$, we can then write that

$$\begin{aligned}\Big(\tilde{m}^{t,x}_{t_0,t}(x') - \tilde{m}^{t,x'}_{t_0,t}(x')\Big)_i &= \Big(\tilde{m}^{t,x}_{t_0,t}(x') - \theta_{t_0,t}(x')\Big)_i \\ &= \Big(\tilde{m}^{t,x}_{t_0,t}(x') - \theta_{t_0,t}(x)\Big)_i + \Big(\theta_{t_0,t}(x) - \theta_{t_0,t}(x')\Big)_i.\end{aligned}$$

We start focusing on the first term of the above expression. From the integral representation of $\tilde{m}^{t,x}_{t_0,t}(x')$ and $\theta_{t_0,t}(x)$, it holds that

$$\tilde{m}^{t,x}_{t_0,t}(x') - \theta_{t_0,t}(x) = x' - x + \int_t^{t_0} A\Big[\tilde{m}^{t,x}_{v,t}(x') - \theta_{v,t}(x)\Big]\,dv. \tag{8.14}$$

Remembering from (1.2) that A is sub-diagonal, it follows that

$$\Big(\tilde{m}^{t,x}_{t_0,t}(x') - \theta_{t_0,t}(x)\Big)_i = (x'-x)_i + A_{i,i-1}\int_t^{t_0}\Big(\tilde{m}^{t,x}_{v,t}(x') - \theta_{v,t}(x)\Big)_{i-1}\,dv \tag{8.15}$$

for any i in $[\![2,n]\!]$ and

$$\Big(\tilde{m}^{t,x}_{t_0,t}(x') - \theta_{t_0,t}(x)\Big)_1 = (x'-x)_1.$$

Iterating the process, we can find that

$$\Big|\Big(\tilde{m}^{t,x}_{t_0,t}(x') - \theta_{t_0,t}(x)\Big)_i\Big| \le C\sum_{k=1}^{i}\Big|(x'-x)_k\Big|(t_0-t)^{i-k}.$$

On the other side, the integral representation of $\theta_{s,\tau}(\xi)$ (Equation (3.1)) allows us to write that

$$\begin{aligned}\Big(\theta_{t_0,t}(x) - \theta_{t_0,t}(x')\Big)_i = (x-x')_i + A_{i,i-1}\int_t^{t_0}\Big\{&\Big(\theta_{t_0,t}(x) - \theta_{t_0,t}(x')\Big)_{i-1} \\ &+ F_i(v,\theta_{v,t}(x)) - F_i(v,\theta_{v,t}(x'))\Big\}\,dv\end{aligned} \tag{8.16}$$

for any i in $[\![2,n]\!]$ and

$$\Big(\theta_{t_0,t}(x) - \theta_{t_0,t}(x')\Big)_1 = (x-x')_1 + \int_t^{t_0}\Big\{F_1(v,\theta_{v,t}(x)) - F_1(v,\theta_{v,t}(x'))\Big\}\,dv. \tag{8.17}$$

Fixed i in $[\![2,n]\!]$, it then follows from (8.14) and (8.16) that

$$\begin{aligned}\Big|\Big(\tilde{m}^{t,x}_{t_0,t}(x') - \tilde{m}^{t,x'}_{t_0,t}(x')\Big)_i\Big| \le C\|F\|_H\Big(&\sum_{k=1}^{i-1}|(x'-x)_k|(t_0-t)^{i-k} \\ &+ \int_t^{t_0}\Big\{\Big|\Big(\theta_{v,t}(x) - \theta_{v,t}(x')\Big)_{i-1}\Big| + \sum_{j=i}^{n}\Big|\Big(\theta_{v,t}(x) - \theta_{v,t}(x')\Big)_j\Big|^{\frac{\gamma_i+\beta}{1+\alpha(j-1)}}\Big\}\,dv\Big).\end{aligned}$$

Also, from (8.15) and (8.17), it holds that

$$\Big|\Big(\tilde{m}^{t,x}_{t_0,t}(x') - \tilde{m}^{t,x'}_{t_0,t}(x')\Big)_1\Big| \le C\|F\|_H\int_t^{t_0}\sum_{j=1}^{n}\Big|\Big(\theta_{v,t}(x) - \theta_{v,t}(x')\Big)_j\Big|^{\frac{\beta}{1+\alpha(j-1)}}\,dv.$$

Using now Lemma 5.4, we can show that

$$\Big|\Big(\tilde{m}^{t,x}_{t_0,t}(x') - \tilde{m}^{t,x'}_{t_0,t}(x')\Big)_i\Big| \le C\|F\|_H\Big(\sum_{k=1}^{i-1}|(x'-x)_k|(t_0-t)^{i-k} + (t_0-t)^{\frac{1+\alpha(i-2)}{\alpha}+1}$$
$$+ (t_0-t)\mathbf{d}^{1+\alpha(i-2)}(x,x') + (t_0-t)^{\frac{1+\alpha(i-2)+\beta}{\alpha}+1} + (t_0-t)\mathbf{d}^{1+\alpha(i-2)+\beta}(x,x')\Big)$$

for any i in $[\![2,n]\!]$ and

$$\Big|\Big(\tilde{m}^{t,x}_{t_0,t}(x') - \tilde{m}^{t,x'}_{t_0,t}(x')\Big)_1\Big| \le C\|F\|_H(t_0-t)^{\frac{\beta+\alpha}{\alpha}} + (t_0-t)\mathbf{d}^{\beta}(x,x').$$

Since $t_0 - t = c_0\mathbf{d}^{\alpha}(x,x')$ by Equation (4.16), we can conclude that

$$\begin{aligned}\Big|\Big(\tilde{m}^{t,x}_{t_0,t}(x') - \tilde{m}^{t,x'}_{t_0,t}(x')\Big)_i\Big| &\le C\|F\|_H\Big\{\sum_{k=1}^{i-1}\mathbf{d}^{1+\alpha(k-1)}(x',x)c_0^{i-k}\mathbf{d}^{\alpha(i-k)}(x,x') \\ &\quad + c_0^{\frac{1+\alpha(i-1)}{\alpha}}\mathbf{d}^{1+\alpha(i-1)}(x,x') + c_0\mathbf{d}^{1+\alpha(i-1)}(x,x') \\ &\quad + c_0^{\frac{1+\alpha(i-2)+\beta}{\alpha}+1}\mathbf{d}^{1+\alpha(i-1)+\beta}(x,x') + c_0\mathbf{d}^{1+\alpha(i-1)+\beta}(x,x')\Big\} \\ &\le C\|F\|_H\Big\{\Big(c_0 + c_0^{\frac{1+\alpha(i-1)}{\alpha}}\Big)\mathbf{d}^{1+\alpha(i-1)}(x,x') \\ &\quad + \Big(c_0 + c_0^{\frac{1+\alpha(i-1)+\beta}{\alpha}}\Big)\mathbf{d}^{1+\alpha(i-1)+\beta}(x,x')\Big\} \\ &\le Cc_0\|F\|_H\mathbf{d}^{1+\alpha(i-1)}(x,x')\end{aligned}$$

for any i in $[\![2,n]\!]$ and

$$\Big|\Big(\tilde{m}^{t,x}_{t_0,t}(x') - \tilde{m}^{t,x'}_{t_0,t}(x')\Big)_1\Big| \le C\|F\|_H\Big(c_0^{\frac{\beta+\alpha}{\alpha}} + c_0\Big)\mathbf{d}^{\alpha+\beta}(x,x') \le Cc_0\|F\|_H\mathbf{d}^{\alpha+\beta}(x,x'),$$

where in the last passage we used that $c_0 \le 1$ and $\mathbf{d}(x,x') \le 1$. After summing all the terms together at the right scale, we finally show that

$$\mathbf{d}(\tilde{m}^{t,x}_{t_0,t}(x'), \tilde{m}^{t,x'}_{t_0,t}(x')) \le Cc_0^{\frac{1}{1+\alpha(n-1)}}\|F\|_H\mathbf{d}(x,x')$$

thanks to convexity inequalities and $c_0 \le 1$.

We conclude this section showing the reverse Taylor formula which was used in the proof of Lemma 5.7 in the diagonal regime to handle the discontinuity term:

Lemma 8.4 (Reverse Taylor Expansion)**.** *Let γ be in $(1,2)$, ϕ a function in $C^{\gamma}_{b,d}(\mathbb{R}^{nd})$ and x, x' two points in $\mathbb{R}^{nd}$. Then, there exists a constant $C := C(\gamma)$ such that*

$$|D_{x_1}\phi(x) - D_{x_1}\phi(x')| \le C\|\phi\|_{C^{\gamma}_{b,d}}\mathbf{d}^{\gamma-1}(x,x').$$

Proof. We start decomposing the left-hand side $D_{x_1}\phi(x) - D_{x_1}\phi(x')$ into $I_1 + I_2 + I_3$ where we denoted

$$\begin{aligned}I_1 &:= \Big(\int_0^1 D_{x_1}\phi(x) - D_{x_1}\phi(x_1 + \lambda\mathbf{d}(x,x'), (x)_{2:n})\,d\lambda\Big) \\ I_2 &:= -\Big(\int_0^1 D_{x_1}\phi(x') - D_{x_1}\phi(x_1 + \lambda\mathbf{d}(x,x'), (x')_{2:n})\,d\lambda\Big) \\ I_3 &:= -\Big(\int_0^1 D_{x_1}\phi(x_1 + \lambda\mathbf{d}(x,x'), (x')_{2:n}) - D_{x_1}\phi(x_1 + \lambda\mathbf{d}(x,x'), (x)_{2:n})\,d\lambda\Big).\end{aligned}$$

The first two components can be treated directly using that $D_{x_1}\phi$ is in $C^{\gamma-1}(\mathbb{R}^d)$ with respect to the first non-degenerate variable. Indeed,

$$\begin{aligned}|I_1| \leq \int_0^1 |D_{x_1}\phi(x) - D_{x_1}\phi(x_1 + \lambda \mathbf{d}(x,x'), (x)_{2:n})|\, d\lambda \\ \leq C\|\phi\|_{C^\gamma} \int_0^1 |\lambda \mathbf{d}(x,x')|^{\gamma-1}\, d\lambda \leq C\|\phi\|_{C^\gamma} \mathbf{d}^{\gamma-1}(x,x')\end{aligned}$$

and

$$\begin{aligned}|I_2| &\leq \int_0^1 |D_{x_1}\phi(x') - D_{x_1}\phi(x_1 + \lambda \mathbf{d}(x,x'), (x')_{2:n})|\, d\lambda \\ &\leq C\|\phi\|_{C^\gamma} \int_0^1 |(x'-x)_1 + \lambda \mathbf{d}(x,x')|^{\gamma-1}\, d\lambda \\ &\leq C\|\phi\|_{C^\gamma} \mathbf{d}^{\gamma-1}(x,x')\end{aligned}$$

where in the last expression we used Young inequality.
To control the last term, we assume for the sake of brevity to be in the scalar case, i.e. $d = 1$. In the general setting, the proof below can be reproduced component-wise. The idea is to use a reverse Taylor expansion to pass from the derivative to the function itself. Namely,

$$\begin{aligned}|I_3| &= \frac{1}{\mathbf{d}(x,x')}\Big|\int_0^1 \Big[\partial_\lambda \phi(x_1 + \lambda \mathbf{d}(x,x'), (x')_{2:n}) - \partial_\lambda \phi(x_1 + \lambda \mathbf{d}(x,x'), (x)_{2:n})\Big]\, d\lambda\Big| \\ &\leq \frac{1}{\mathbf{d}(x,x')}\Big|\phi(x_1 + \mathbf{d}(x,x'), (x')_{2:n}) - \phi(x_1, (x')_{2:n}) + \phi(x_1 + \mathbf{d}(x,x'), (x)_{2:n}) - \phi(x)\Big| \\ &\leq C\|\phi\|_{C^\gamma} \mathbf{d}^{\gamma-1}(x,x').\end{aligned}$$

We have thus concluded the proof. □

Chapter 3

Schauder estimates for degenerate Lévy Ornstein-Uhlenbeck operators

Abstract: We establish global Schauder estimates for integro-partial differential equations (IPDE) driven by a possibly degenerate Lévy Ornstein-Uhlenbeck operator, both in the elliptic and parabolic setting, using some suitable anisotropic Hölder spaces. The class of operators we consider is composed by a linear drift plus a Lévy operator that is comparable, in a suitable sense, with a possibly truncated stable operator. It includes for example, the relativistic, the tempered, the layered or the Lamperti stable operators. Our method does not assume neither the symmetry of the Lévy operator nor the invariance for dilations of the linear part of the operator. Thanks to our estimates, we prove in addition the well-posedness of the considered IPDE in suitable functional spaces. In the final section, we extend some of these results to more general operators involving non-linear, space-time dependent drifts.

1 Introduction

Fixed an integer N in $\mathbb{N}$, we consider the following integro-partial differential operator of Ornstein-Uhlenbeck type:

$$L^{\mathrm{ou}} := \mathcal{L} + \langle Ax, D_x \rangle \quad \text{on } \mathbb{R}^N, \tag{1.1}$$

where $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product on $\mathbb{R}^N$, A is a matrix in $\mathbb{R}^N \otimes \mathbb{R}^N$ and $\mathcal{L}$ is a possibly degenerate, Lévy operator acting non-degenerately only on a subspace of $\mathbb{R}^N$. We are interested in showing the *well-posedness* and the associated *Schauder estimates* for elliptic and parabolic equations involving the operator L^{ou} and with coefficients in a generalized family of Hölder spaces.

We only assume that A satisfies a natural controllability assumption, the so-called Kalman rank condition (condition [**K**] below), and that the operator $\mathcal{L}$ is comparable, in a suitable sense, to a non-degenerate, truncated α-stable operator on the same subspace of $\mathbb{R}^N$, for some $\alpha < 2$ (condition [**SD**] below).

The topic of Schauder estimates for Ornstein-Uhlenbeck operators has been widely studied in the last decades, especially in the diffusive, local setting, i.e. when $\mathcal{L} = \frac{1}{2}\mathrm{Tr}\Big(QD_x^2\Big)$

for some suitable matrix Q, and it is now quite well-understood. See e.g. [GT01].
On the other hand, a literature on the topic for the pure jump, non-local framework has been developed only in the recent years ([Bas09], [DK13], [BK15], [ROS16]), [FRRO17], [IJS18], [CdRMP20a], [Küh19], but mainly in the non-degenerate, α-stable setting, i.e. when $\mathcal{L} = \Delta_x^{\alpha/2}$ is the fractional Laplacian on $\mathbb{R}^N$ or similar. To the best of our knowledge, the only two articles dealing with the degenerate, non-local framework (if $\mathcal{L} = \Delta_x^{\alpha/2}$ acts non-degenerately only on a sub-space of $\mathbb{R}^N$) are [HPZ19], that takes into account the kinetics dynamics ($N = 2d$), and [Mar20], for the general chain. In order to use [HPZ19] or [Mar20] for our operator (1.1), we would need to impose the additional strong assumption of invariance for dilations of the matrix A.
The analysis of Ornstein-Uhlenbeck operators has been mainly developed following two different approaches. On the one hand, Da Prato and Lunardi in [DPL95] have been the first to use a priori estimates for the corresponding semi-group between suitable function spaces (See also [Lun97, Lor05, CdRHM18a, Pri18]). Such a semi-group approach only adresses the regularity in space and indeed, the associated anisotropic Hölder spaces and Schauder estimates reflect this fact. In particular, the parabolic Schauder estimates do not present a bootstrap effect with respect to the initial condition.
The second approach, introduced by Manfredini in [Man97], exploits instead the general analysis on Lie groups to construct intrinsic Hölder spaces (see [PPP16] for a definition) that takes into account the joint space-time regularity of the involved functions. For a more thorough explanation along this direction, we suggest the interested reader to see, for example, [Pas03], [DFP06] or the recent paper [IM21].

Even if the Ornstein-Uhlenbeck operator is usually exploited as a "toy model" for more general operators with space-time dependent, non-linear coefficients, we highlight that they appear naturally in various scientific contexts: for example in physics, for the analysis of anomalous diffusions phenomena or for Hamiltonian models in a turbulent regime (see e.g. [BBM01], [CPKM05] and the references therein) or in mathematical finance and econometrics (see e.g. [Bro01], [BNS01]). The interest in Schauder estimates involving this type of operator also follows from the natural application which consists in establishing the well-posedness of stochastic differential equations (SDE) driven by Lévy processes and the associated stochastic control theory. See e.g. [FM82], [CdRM20b], [HWZ20].

Under our assumptions, we have been able to consider more general Lévy operators not usually included in the literature, such as the relativistic stable process, the layered stable process or the Lamperti one (see Paragraph "Main Operators Considered" below for details). Moreover, we do not require the operator $\mathcal{L}$ to be symmetric. Here, we only mention one important example that satisfies our hypothesis, the Ornstein-Uhlenbeck operator on $\mathbb{R}^2$ driven by the relativistic fractional Laplacian $\Delta_{\text{rel}}^{\alpha/2}$ and acting only on the first component:

$$x_1(D_{x_1}\phi(x) + D_{x_2}\phi(x)) + \text{p.v.} \int_{\mathbb{R}} \Big[\phi(\begin{pmatrix} x_1 + z \\ x_2 \end{pmatrix}) - \phi(\begin{pmatrix} x_1 \\ x_2 \end{pmatrix})\Big] \frac{1 + |z|^{\frac{d+\alpha-1}{2}}}{|z|^{d+\alpha}} e^{-|z|}\, dz$$
$$= \langle Ax, D_x\phi(x)\rangle + \mathcal{L}\phi(x) \quad (1.2)$$

where $x = (x_1, x_2)$ in $\mathbb{R}^2$. Such an example is included in the framework of Equation (1.1)

considering $A = \begin{pmatrix} 1 & 0 \\ 1 & 0 \end{pmatrix}$. This operator appears naturally as a fractional generalization of the relativistic Schrödinger operator (See [Ryz02] for more details).
We remark that example (1.2) cannot be considered in [HPZ19] or in our previous work [Mar20]. Indeed, the matrix A_0 is not "dilation-invariant" (see example 2.1 below) and thus, it cannot be rewritten in the form used in [Mar20] (see also [LP94] Proposition 2.2 for a more thorough explanation). Furthermore, operators like the relativistic fractional Laplacian cannot be treated in [HPZ19] or [Mar20] that indeed have taken into account only stable-like operators on $\mathbb{R}^N$. Another useful advantage of our technique is that we do not need anymore the symmetry of the Lévy measure ν which was, again, a key assumption in [Mar20].

More in details, given an integer $d \leq N$ and a matrix B in $\mathbb{R}^N \otimes \mathbb{R}^d$ such that $\mathrm{rank}(B) = d$, we consider a family of operators $\mathcal{L}$ that can be represented for any sufficiently regular function $\phi\colon \mathbb{R}^N \to \mathbb{R}$ as

$$\begin{aligned} \mathcal{L}\phi(x) \;:=\; & \frac{1}{2}\mathrm{Tr}\Big(BQB^* D_x^2\phi(x)\Big) + \langle Bb, D_x\phi(x)\rangle \\ & + \int_{\mathbb{R}_0^d} \Big[\phi(x+Bz) - \phi(x) - \langle D_x\phi(x), Bz\rangle \mathbb{1}_{B(0,1)}(z)\Big]\, \nu(dz), \end{aligned} \tag{1.3}$$

where b is a vector in $\mathbb{R}^d$, Q is a symmetric, non-negative definite matrix in $\mathbb{R}^d \otimes \mathbb{R}^d$ and ν is a Lévy measure on $\mathbb{R}_0^d := \mathbb{R}^d \setminus \{0\}$, i.e. a σ-finite measure on $\mathcal{B}(\mathbb{R}_0^d)$, the Borel σ-algebra on $\mathbb{R}_0^d$, such that $\int (1 \wedge |z|^2)\, \nu(dz)$ is finite. We then suppose ν to satisfy the following *stable domination* condition:

[SD] there exists $r_0 > 0$, α in $(0,2)$ and a finite, non-degenerate measure μ on the unit sphere $\mathbb{S}^{d-1}$ such that

$$\nu(\mathcal{A}) \;\geq\; \int_0^{r_0} \int_{\mathbb{S}^{d-1}} \mathbb{1}_{\mathcal{A}}(r\theta)\, \mu(d\theta) \frac{dr}{r^{1+\alpha}}, \quad \mathcal{A} \in \mathcal{B}(\mathbb{R}_0^d).$$

We recall that a measure μ on $\mathbb{R}^d$ is non-degenerate if there exists a constant $\eta \geq 1$ such that

$$\eta^{-1}|p|^\alpha \;\leq\; \int_{\mathbb{S}^{d-1}} |p \cdot s|^\alpha\, \mu(ds) \;\leq\; \eta|p|^\alpha, \quad p \in \mathbb{R}^d, \tag{1.4}$$

where " $\cdot$ " stands for the inner product on the smaller space $\mathbb{R}^d$. Since any α-stable Lévy measure ν_α can be decomposed into a spherical part μ on $\mathbb{S}^{d-1}$ and a radial part $r^{-(1+\alpha)}dr$ (see e.g. Theorem 14.3 in [Sat13]), assumption [**SD**] roughly states that the Lévy measure of the integro-differential part of $\mathcal{L}$ is bounded from below by the Lévy measure of a possibly truncated, α-stable operator on $\mathbb{R}^d$.
It is assumed moreover that the matrices A, B satisfy the following *Kalman condition*:

[K] It holds that $N \;=\; \mathrm{rank}\Big[B, AB, \ldots, A^{N-1}B\Big]$,

where $\Big[B, AB, \ldots, A^{N-1}B\Big]$ is the matrix in $\mathbb{R}^N \otimes \mathbb{R}^{dN}$ whose columns are given by $B, AB, \ldots, A^{N-1}B$.
Such an assumption is equivalent, in the linear framework, to the Hörmander condition (see [Hör67]) on the commutators, ensuring the hypoellipticity of the operator $\partial_t - L^{\mathrm{ou}}$. Moreover, condition [**K**] is well-known in control theory (see e.g. [Zab92], [PZ09]).

Mathematical Outline. In the present paper, we aim at establishing global Schauder estimates for equations involving the operator L^{ou} on $\mathbb{R}^N$, both in the elliptic and parabolic settings. Namely, we consider for a fixed $\lambda > 0$ the following elliptic equation:

$$\lambda u(x) - L^{\mathrm{ou}}u(x) \;=\; g(x), \quad x \in \mathbb{R}^N, \tag{1.5}$$

and, for a fixed time horizon $T > 0$, the following parabolic Cauchy problem:

$$\begin{cases} \partial_t u(t,x) \;=\; L^{\mathrm{ou}}u(t,x) + f(t,x), & (t,x) \in (0,T) \times \mathbb{R}^N; \\ u(0,x) \;=\; u_0(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.6}$$

where f, g, u_0 are given functions. Since our aim is to show optimal regularity results in Hölder spaces, we will assume for the elliptic case (Equation (1.5)) that the source g belongs to a suitable *anisotropic* Hölder space $C^{\beta}_{b,d}(\mathbb{R}^N)$ for some β in $(0,1)$, where the Hölder exponent depends on the "direction" considered. The space $C^{\beta}_{b,d}(\mathbb{R}^N)$ can be understood as composed by the bounded functions on $\mathbb{R}^N$ that are Hölder continuous with respect to a distance $\mathbf{d}$ somehow induced by the operator L^{ou}. We refer to Section 2 for a detailed exposition of such an argument but we highlight already that the above mentioned distance $\mathbf{d}$ can be seen as a generalization of the classical parabolic distance, adapted to our degenerate, non-local framework. It is precisely assumption [**K**], or equivalently the hypoellipticity of $\partial_t + L^{\mathrm{ou}}$, that ensures the existence of such a distance $\mathbf{d}$ and gives it its anisotropic nature. Roughly speaking, it allows the smoothing effect of the Lévy operator $\mathcal{L}$ acting non-degenerately only on some components, say $B\mathbb{R}^N$, to spread in the whole space $\mathbb{R}^N$, even if with lower regularizing properties.
Concerning the parabolic problem (1.6), we assume similarly that u_0 is in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)$ and that $f(t,\cdot)$ is in $C^{\beta}_{b,d}(\mathbb{R}^N)$, uniformly in $t \in (0,T)$. The typical estimates we want to prove can be stated in the parabolic setting in the following way: there exists a constant C, depending only on the parameters of the model, such that any distributional solution u of the Cauchy problem (1.6) satisfies

$$\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \;\le\; C\Big[\|u_0\|_{C^{\alpha+\beta}_{b,d}} + \|f\|_{L^\infty(C^{\beta}_{b,d})}\Big]. \tag{$\mathscr{S}$}$$

As a by-product of the Schauder Estimates ($\mathscr{S}$), we will obtain the well-posedness of the Cauchy problem (1.6) in the space $L^\infty\Big(0,T; C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)\Big)$, once the existence of a solution is established. The additional regularity for the solution u with respect to the source f reflects the appearance of a smoothing effect associated with L^{ou} of order α, as it is expected by condition [**SD**]. It can be seen as a generalization of the "standard" parabolic bootstrap to our degenerate, non-local setting. We highlight that the parabolic bootstrap in ($\mathscr{S}$) is precisely derived from the non-degenerate stable-like part in $\mathcal{L}$ (lowest regularizing effect in the operator).
To show our result, we will follow the semi-group approach as firstly introduced in [DPL95], which became afterwards a very robust tool to study Schauder estimates in a wide variety of frameworks ([Lun97], [Lor05], [Sai07], [Pri09], [Pri12], [DK13], [KK15], [CdRHM18a], [Küh19]). The main idea is to consider the Markov transition semi-group P_t associated with L^{ou} and then, in the elliptic case, to use the Laplace transform formula in order to represent the unique distributional solution u of Equation (1.5) as:

$$u(x) \;=\; \int_0^\infty e^{-\lambda t}\Big[P_t g\Big](x)\,dt \;=:\; \int_0^\infty e^{-\lambda t} P_t g(x)\,dt.$$

In the parabolic setting, we exploit instead the variation of constants (or Duhamel) formula in order to show a similar representation for the weak solution of the Cauchy problem (1.6):

$$u(t,x) \;=\; P_t u_0(x) + \int_0^t \Big[P_{t-s} f(s,\cdot)\Big](x)\, ds \;=:\; P_t u_0(x) + \int_0^t P_{t-s} f(s,x)\, ds.$$

In order to prove global regularity estimates for the solutions, the crucial point is to understand the action of the operator P_t on the anisotropic Hölder spaces. In particular, we will show in Corollary 4.4 the continuity of P_t as an operator from $C^{\beta}_{b,d}(\mathbb{R}^N)$ to $C^{\gamma}_{b,d}(\mathbb{R}^N)$ for $\beta < \gamma$ and, more precisely, that it holds:

$$\|P_t\phi\|_{C^{\gamma}_{b,d}} \;\leq\; C\|\phi\|_{C^{\beta}_{b,d}}\Big(1 + t^{-\frac{\gamma-\beta}{\alpha}}\Big), \quad t > 0. \tag{1.7}$$

The above estimate can be obtained through interpolation techniques (see Equation (4.9)), once sharp controls in supremum norm (Theorem 4.3 below) are established for the spatial derivatives of $P_t\phi$ when $\phi \in C^{\beta}_{b,d}(\mathbb{R}^N)$. We think that such an estimate (1.7) and the controls in Theorem 4.3 can be of independent interest and used also beyond our scope in other contexts.
We face here two main difficulties to overcome. While in the gaussian setting, L^∞-estimates of this type have been established exploiting, for example, explicit formulas for the density of the semi-group P_t ([Lun97]), a priori controls of Bernstein type combined with interpolation methods ([Lor05] and [Sai07], when $n = 2$ in (2.2) below) or probabilistic representations of the semi-group P_t, allowing Malliavin calculus ([Pri09]), we cannot rely on these techniques in our non-local framework, mainly due to the lower integrability properties for P_t. Instead, we are going to use a *perturbative approach* which consists in considering the Lévy operator $\mathcal{L}$ as a perturbation, in a suitable sense, of an α-stable operator, at least for the associated small jumps. Indeed, we can "decompose" the operator $\mathcal{L}$ in a smoother part, $\mathcal{L}^\alpha$, whose Lévy measure is given by

$$\mu(d\theta)\frac{\mathbb{1}_{(0,r_0]}(r)}{r^{1+\alpha}}dr$$

and a remainder part. It is precisely condition [**SD**] that allows such a decomposition, since it ensures the positivity of the Lévy measure

$$d\nu - d\mu\frac{\mathbb{1}_{[0,r_0]}}{r^{1+\alpha}}dr$$

associated with the remainder term. The main difference with the previous techniques in the diffusive setting is that we will work mainly on the truncated α-stable contribution $\mathcal{L}^\alpha$, being the remainder term only bounded.
Following [SSW12], we will establish that the Hartman-Winter condition holds, ensuring the existence of a smooth density for the semi-group associated with $\mathcal{L}^\alpha$ and then, the required gradient estimates. Indeed, assumption [**SD**] roughly states that the small jump contributions of ν, the ones responsible for the creation of a density, are controlled from below by an α-stable measure, whose absolute continuity is well-known in our framework.

On the other hand, we will have to deal with the degeneracy of the operator $\mathcal{L}$, that acts non-degenerately, through the embedding matrix B, only on a subspace of dimension d. It will be managed by adapting the reasonings firstly appeared in [HM16]. Namely, we will show that the semi-group associated with the Ornstein-Uhlenbeck operator L^{ou} coincides with a non-degenerate one but "multiplied" by a time-dependent matrix that precisely takes into account the original degeneracy of the operator (see definition of matrix $\mathbb{M}_t$ in Section 2.1).

Main Operators Considered. We conclude this introduction showing that assumption [**SD**] applies to a large class of Lévy operators on $\mathbb{R}^d$. As already pointed out in [SSW12], it is satisfied by any Lévy measure ν that can be decomposed in polar coordinates as

$$\nu(\mathcal{A}) \;=\; \int_0^\infty \int_{\mathbb{S}^{d-1}} \mathbb{1}_{\mathcal{A}}(r\theta) Q(r,\theta)\, \mu(d\theta) \frac{dr}{r^{1+\alpha}}, \quad \mathcal{A} \in \mathcal{B}(R_0^d),$$

for a finite, non-degenerate (in the sense of Equation (1.4)), measure μ on $\mathbb{S}^{d-1}$ and a Borel function $Q\colon (0,\infty) \times \mathbb{S}^{d-1} \to \mathbb{R}$ such that there exists $r_0 > 0$ so that

$$Q(r,\theta) \;\geq\; c > 0, \quad \text{a.e. in } [0,r_0] \times \mathbb{S}^{d-1}.$$

In particular, assumption [**SD**] holds for the following families of "stable-like" examples with $\alpha \in (0,2)$:

1. Stable operator [Sat13]:
$$Q(r,\theta) \;=\; 1;$$

2. Truncated stable operator with $r_0 > 0$ [KS08]:
$$Q(r,\theta) \;=\; \mathbb{1}_{(0,r_0]}(r);$$

3. Layered stable operator with β in $(0,2)$ and $r_0 > 0$ [HK07]:
$$Q(r,\theta) \;=\; \mathbb{1}_{(0,r_0]}(r) + \mathbb{1}_{(r_0,\infty)}(r) r^{\alpha-\beta};$$

4. Tempered stable operator [Ros07]:
$$Q(\cdot,\theta) \text{ completely monotone, } Q(0,\theta) > 0 \text{ and } Q(\infty,\theta) = 0 \text{ a.e. in } S^{d-1};$$

5. Relativistic stable operator [CMS90], [BMR09]:
$$Q(r,\theta) \;=\; (1+r)^{(d+\alpha-1)/2} e^{-r};$$

6. Lamperti stable operator with $f\colon S^{d-1} \to \mathbb{R}$ such that $\sup f(\theta) < 1+\alpha$ [CPP10]:
$$Q(r,\theta) \;=\; e^{rf(\theta)} \Big(\frac{r}{e^r - 1}\Big)^{1+\alpha}.$$

Organization of Paper. The article is organized as follows. Section 2 introduces some useful notations and then, the anisotropic distance $\mathbf{d}$ induced by the dynamics as well as Zygmund-Hölder spaces associated with such a distance. In Section 3, we are going to show some analytical properties of the semi-group P_t generated by L^{ou}, such as the existence of a smooth density and, at least for small times, some controls for its derivatives. Section 4 is then dedicated to different estimates in the L^∞-norm for $P_t f$ and its spatial derivatives, involving the supremum or the Hölder norm of the function f. In particular, we show here the continuity of P_t as an operator between anisotropic Zygmund-Hölder spaces. In Section 5, we use the controls established in the previous parts in order to prove the elliptic Schauder estimates and show that Equation (1.5) has a unique solution. Similarly, we establish the well-posedness of the Cauchy problem (1.6) as well as the associated parabolic Schauder estimates. In the final section of the article, we briefly explain some possible extensions of the previous results to non-linear, space-time dependent operators.

2 Geometry of the dynamics

In this section, we are going to choose the right functional space "in which" to state our Schauder estimates. The idea is to construct an Hölder space $C^\beta_{b,d}(\mathbb{R}^N)$ with respect to a distance $\mathbf{d}$ that it is homogeneous to the dynamics, i.e. such that for any f in $C^\beta_{b,d}(\mathbb{R}^N)$, any distributional solution u of

$$L^{\mathrm{ou}}u(x) \,=\, \mathcal{L}u(x) + \langle Ax, Du(x)\rangle \,=\, f(x), \quad x \in \mathbb{R}^N \tag{2.1}$$

is in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)$, the expected parabolic bootstrap associated to this kind of operator. We recall in particular that the Kalman rank condition [**K**] is equivalent to the hypoellipticity (in the sense of Hörmander [Hör67]) of the operator L^{ou} that ensures the existence and smoothness of a distributional solution of Equation (2.1) for sufficiently regular f. See e.g. [Ish16] or [HPZ19] for more details.

2.1 The distance associated with the dynamics

To construct the suitable distance $\mathbf{d}$, we start noticing that the Kalman rank condition [**K**] allows us to denote

$$n \,:=\, \min\{r \in \mathbb{N} \colon N \,=\, \mathrm{rank}\Big[B, AB, \dots, A^{r-1}B\Big]\}. \tag{2.2}$$

Clearly, n is in $[\![1, N]\!]$, where $[\![\cdot,\cdot]\!]$ denotes the set of all the integers in the interval, and $n = 1$ if and only if $d = N$, i.e. if the dynamics is non-degenerate.
As done in [Lun97], the space $\mathbb{R}^N$ will be decomposed with respect to the family of linear operators $B, AB, \dots, A^{n-1}B$. We start defining the family $\{V_h \colon h \in [\![1, n]\!]\}$ of subspaces of $\mathbb{R}^N$ through

$$V_h \,:=\, \begin{cases} \mathrm{Im}(B), & \text{if } h = 1, \\ \bigoplus_{k=1}^h \mathrm{Im}(A^{k-1}B), & \text{otherwise.} \end{cases}$$

It is easy to notice that $V_h \neq V_k$ if $k \neq h$ and $V_1 \subset V_2 \subset \dots V_n = \mathbb{R}^N$. We can then construct iteratively the family $\{E_h \colon h \in [\![1,n]\!]\}$ of orthogonal projections from $\mathbb{R}^N$ as

$$E_h := \begin{cases} \text{projection on } V_1, & \text{if } h = 1; \\ \text{projection on } (V_{h-1})^\perp \cap V_h, & \text{otherwise.} \end{cases}$$

With a small abuse of notation, we will identify the projection operators E_h with the corresponding matrices in $\mathbb{R}^N \otimes \mathbb{R}^N$. It is clear that $\dim E_1(\mathbb{R}^N) = d$. Let us then denote $d_1 := d$ and

$$d_h := \dim E_h(\mathbb{R}^N), \qquad \text{for } h > 1.$$

We can define now the distance $\mathbf{d}$ through the decomposition $\mathbb{R}^N = \bigoplus_{h=1}^n E_h(\mathbb{R}^N)$ as

$$\mathbf{d}(x, x') := \sum_{h=1}^n |E_h(x - x')|^{\frac{1}{1+\alpha(h-1)}}.$$

The above distance can be seen as a generalization of the usual Euclidean distance when $n = 1$ (non-degenerate dynamics) as well as an extension of the standard parabolic distance for $\alpha = 2$. It is important to highlight that it does not induce a norm since it lacks of linear homogeneity.
The anisotropic distance $\mathbf{d}$ can be understood direction-wise: we firstly fix a "direction" h in $[\![1,n]\!]$ and then calculate the standard Euclidean distance on the associated subspace $E_h(\mathbb{R}^N)$, but scaled according to the dilation of the system in that direction. We conclude summing the contributions associated with each component. The choice of such a dilation will be discussed thoroughly in the example at the end of this section.

As emphasized by the result from Lanconelli and Polidoro recalled below (cf. [LP94], Proposition 2.1), the decomposition of $\mathbb{R}^N$ with respect to the projections $\{E_h \colon h \in [\![1,n]\!]\}$ determines a particular structure of the matrices A and B. It will be often exploited in the following.

Theorem 2.1 ([LP94])**.** *Let $\{e_i \colon i \in [\![1,N]\!]\}$ be an orthonormal basis consisting of generators of $\{E_h(\mathbb{R}^N) \colon h \in [\![1,n]\!]\}$. Then, the matrices A and B have the following form:*

$$B = \begin{pmatrix} B_0 \\ 0 \\ \vdots \\ 0 \end{pmatrix} \quad \textit{and} \quad A = \begin{pmatrix} * & * & \dots & \dots & * \\ A_2 & * & \ddots & \ddots & \vdots \\ 0 & A_3 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ 0 & \dots & 0 & A_n & * \end{pmatrix} \tag{2.3}$$

where B_0 is a non-degenerate matrix in $\mathbb{R}^{d_1} \otimes \mathbb{R}^{d_1}$ and A_h are matrices in $\mathbb{R}^{d_h} \otimes \mathbb{R}^{d_{h-1}}$ with rank$(A_h) = d_h$ for any h in $[\![2,n]\!]$. Moreover, $d_1 \geq d_2 \geq \dots \geq d_n \geq 1$.

Applying a change of variables if necessary, we will assume from this point further to have fixed such a canonical basis $\{e_i \colon i \in [\![1,N]\!]\}$. For notational simplicity, we denote by I_h, $h \in [\![1,n]\!]$, the family of indexes i in $[\![1,N]\!]$ such that $\{e_i \colon i \in I_h\}$ spans $E_h(\mathbb{R}^N)$.

The particular structure of A and B given by Theorem 2.1 allows us to decompose accurately the exponential e^{tA} of the matrix A in order to make the intrinsic scale of

the system appear. Further on, we will consider fixed a time-dependent matrix $\mathbb{M}_t$ on $\mathbb{R}^N \otimes \mathbb{R}^N$ given by

$$\mathbb{M}_t := \mathrm{diag}(I_{d_1\times d_1}, tI_{d_2\times d_2}, \dots, t^{n-1}I_{d_n\times d_n}), \quad t \geq 0.$$

Lemma 2.2. *There exists a time-dependent matrix $\{R_t \colon t \in [0,1]\}$ in $\mathbb{R}^N \otimes \mathbb{R}^N$ such that*

$$e^{tA}\mathbb{M}_t = \mathbb{M}_t R_t, \quad t \in [0,1]. \tag{2.4}$$

Moreover, there exists a constant $C > 0$ such that for any t in $[0,1]$,

- *any l, h in $[\![1,n]\!]$ and any θ in $\mathbb{S}^{N-1}$, it holds that*

$$\left|E_l e^{tA} E_h \theta\right| \leq \begin{cases} Ct^{l-h}, & \text{if } l \geq h \\ Ct, & \text{if } l < h. \end{cases}$$

- *any θ in $\mathbb{S}^{d-1}$, it holds that*

$$\left|R_t B\theta\right| \geq C^{-1}.$$

Proof. By definition of the matrix exponential, we know that

$$E_l e^{tA} E_h = \sum_{k=0}^{\infty} \frac{t^k}{k!} E_l A^k E_h. \tag{2.5}$$

Using now the representation of A given by Theorem 2.1, it is easy to check that $E_l A^k E_h = 0$ for $k < l - h$ (when $l - h$ is non-negative). Thus, for $l \geq h$, it holds that

$$\left|E_l e^{tA} E_h \theta\right| = \Big|\sum_{k=l-h}^{\infty} \frac{t^k}{k!} E_l A^k E_h \theta\Big| \leq Ct^{l-h},$$

where we exploited that t is in $[0,1]$ and $|\theta| = 1$. Assuming instead that $l < h$, it is clear that $E_l I_{N\times N} E_h$ vanishes. We can then write that

$$\left|E_l e^{tA} E_h \theta\right| = \Big|\sum_{k=l}^{\infty} \frac{t^k}{k!} E_l A^k E_h \theta\Big| \leq Ct,$$

using again that t is in $[0,1]$ and $|\theta| = 1$.
To show the other control, we highlight that the matrix $\mathbb{M}_t$ is not invertible in $t = 0$ and for this reason, we define the time-dependent matrix R_t as

$$R_t := \begin{cases} I_{N\times N}, & \text{if } t = 0; \\ \mathbb{M}_t^{-1} e^{tA} \mathbb{M}_t, & \text{if } t \in (0,1]. \end{cases}$$

We could have also defined $R_t := \left(\tilde{R}_s^t\right)_{|s=1}$ where $\tilde{R}_s^t$ solves the following ODE:

$$\begin{cases} \partial_s \tilde{R}_s^t = \mathbb{M}_t^{-1} tA \mathbb{M}_t \tilde{R}_s^t, & \text{on } (0,1], \\ \tilde{R}_0^t = I_{N\times N}. \end{cases}$$

Equivalently, $\tilde{R}^t_s$ is the resolvent matrix associated with $\mathbb{M}_t^{-1} t A \mathbb{M}_t$, whose sub-diagonal entries are "macroscopic" from the structure of A and $\mathbb{M}_t$.
It follows immediately that Equation (2.4) holds. Moreover, we notice that

$$\left|R_t B\theta\right| \geq \left|E_1 R_t B\theta\right| = \left|E_1 e^{tA} E_1 B\theta\right|.$$

Remembering the definition of matrix exponential (Equation (2.5) with $l = h = 1$), we use now that

$$E_1 A^k E_1 = (E_1 A E_1)^k = (A_{1,1})^k E_1,$$

where in the last expression the multiplication is meant block-wise, in order to conclude that

$$\left|R_t B\theta\right| \geq \left|e^{tA_{1,1}} B_0 \theta\right|.$$

Using that $e^{tA_{1,1}} B_0$ is non-degenerate and continuous in time and that θ is in $\mathbb{S}^{d-1}$, it is easy to conclude. □

We conclude this sub-section with a simpler example taken from [HMP19]. We hope that it will help the reader to understand the introduction of the anisotropic distance $\mathbf{d}$.

Example 2.1. Fixed $N = 2d$, $n = 2$ and $d = d_1 = d_2$, we consider the following operator:

$$L^{\text{ou}}_\alpha = \Delta^{\frac{\alpha}{2}}_{x_1} + x_1 \cdot \nabla_{x_2} \quad \text{on } \mathbb{R}^{2d},$$

where $(x_1, x_2) \in \mathbb{R}^{2d}$ and $\Delta^{\frac{\alpha}{2}}_{x_1}$ is the fractional Laplacian with respect to x_1. In our framework, it is associated with the matrices

$$A := \begin{pmatrix} 0 & 0 \\ I_{d\times d} & 0 \end{pmatrix} \quad \text{and} \quad B := \begin{pmatrix} I_{d\times d} \\ 0 \end{pmatrix}.$$

The operator L^{ou}_α can be seen as a generalization of the classical Kolmogorov example (see e.g. [Kol34]) to our non-local setting.
In order to understand how the system typically behaves, we search for a dilation

$$\delta_\lambda \colon [0,\infty) \times \mathbb{R}^{2d} \to [0,\infty) \times \mathbb{R}^{2d}$$

which is invariant for the considered dynamics, i.e. a dilation that transforms solutions of the equation

$$\partial_t u(t,x) - L^{\text{ou}}_\alpha u(t,x) = 0 \quad \text{on } (0,\infty) \times \mathbb{R}^{2d}$$

into other solutions of the same equation.
Due to the structure of A and the α-stability of $\Delta^{\frac{\alpha}{2}}$, we can consider for any fixed $\lambda > 0$, the following

$$\delta_\lambda(t, x_1, x_2) := (\lambda^\alpha t, \lambda x_1, \lambda^{1+\alpha} x_2).$$

It then holds that

$$\left(\partial_t - L^{\text{ou}}_\alpha\right) u = 0 \implies \left(\partial_t - L^{\text{ou}}_\alpha\right)(u \circ \delta_\lambda) = 0.$$

Introducing now the complete time-space distance $\mathbf{d}_P$ on $[0,\infty)\times\mathbb{R}^{2d}$ given by

$$\mathbf{d}_P\Big((t,x),(s,x')\Big) := |s-t|^{\frac{1}{\alpha}} + \mathbf{d}(x,x') = |s-t|^{\frac{1}{\alpha}} + |x_1-x_1'| + |x_2-x_2'|^{\frac{1}{1+\alpha}}, \quad (2.6)$$

we notice that it is homogeneous with respect to the dilation δ_λ, so that

$$\mathbf{d}_P\Big(\delta_\lambda(t,x);\delta_\lambda(s,x')\Big) = \lambda\mathbf{d}_P\Big((t,x);(s,x')\Big).$$

Precisely, the exponents appearing in Equation (2.6) are those which make each space-component homogeneous to the characteristic time scale $t^{1/\alpha}$. From a more probabilistic point of view, the exponents in Equation (2.6), can be related to the characteristic time scales of the iterated integrals of an α-stable process. It can be easily seen from the example, noticing that the operator L_α^{ou} corresponds to the generator of an isotropic α-stable process and its time integral.
Going back to the general setting, the appearance of this kind of phenomena is due essentially to the particular structure of the matrix A (cf. Theorem 2.1) that allows the smoothing effect of the operator $\mathcal{L}$, acting only on the first "component" given by B_0, to propagate into the system.

2.2 Anisotropic Zygmund-Hölder spaces

We are now ready to define the Zygmund-Hölder spaces $C^\gamma_{b,d}(\mathbb{R}^N)$ with respect to the distance $\mathbf{d}$. We start recalling some useful notations we will need below.
Given a function $f\colon\mathbb{R}^N\to\mathbb{R}$, we denote by $Df(x)$, $D^2f(x)$ and $D^3f(x)$ the first, second and third Fréchet derivative of f at a point x in $\mathbb{R}^N$ respectively, when they exist. For simplicity, we will identify $D^3f(x)$ as a 3-tensor so that $[D^3f(x)](u,v)$ is a vector in R^N for any u,v in $\mathbb{R}^N$. Moreover, fixed h in $[\![1,n]\!]$, we will denote by $D_{E_h}f(x)$ the gradient of f at x along the direction $E_h(\mathbb{R}^N)$. Namely,

$$D_{E_h}f(x) := E_hDf(x).$$

A similar notation will be used for the higher derivatives, too.
Given X,Y two real Banach spaces, $\mathcal{L}(X,Y)$ will represent the family of linear continuous operators between X and Y.
In the following, c or C denote generic *positive* constants whose precise value is unimportant. They may change from line to line and they will depend only on the parameters given by the model and assumptions [**SD**], [**K**]. Namely, d,N,A,B,α,ν,r_0 and μ. Other dependencies that may occur will be explicitly specified.

Let us introduce now some function spaces we are going to use. We denote by $B_b(\mathbb{R}^N)$ the family of Borel measurable and bounded functions $f\colon\mathbb{R}^N\to\mathbb{R}$. It is a Banach space endowed with the supremum norm $\|\cdot\|_\infty$. We will consider also its closed subspace $C_b(\mathbb{R}^N)$ consisting of all the uniformly continuous functions.
Fixed some k in $\mathbb{N}_0:=\mathbb{N}\cup\{0\}$ and β in $(0,1]$, we follow Lunardi [Lun97] denoting the Zygmund-Hölder semi-norm for a function $\phi\colon\mathbb{R}^N\to\mathbb{R}$ as

$$[\phi]_{C^{k+\beta}} := \begin{cases} \sup_{|\vartheta|=k}\sup_{x\neq y}\frac{|D^\vartheta\phi(x)-D^\vartheta\phi(y)|}{|x-y|^\beta}, & \text{if } \beta\neq 1;\\ \sup_{|\vartheta|=k}\sup_{x\neq y}\frac{\left|D^\vartheta\phi(x)+D^\vartheta\phi(y)-2D^\vartheta\phi(\frac{x+y}{2})\right|}{|x-y|}, & \text{if } \beta=1.\end{cases}$$

Consequently, The Zygmund-Hölder space $C_b^{k+\beta}(\mathbb{R}^N)$ is the family of functions $\phi\colon \mathbb{R}^N \to \mathbb{R}$ such that ϕ and its derivatives up to order k are continuous and the norm

$$\|\phi\|_{C_b^{k+\beta}} := \sum_{i=1}^{k} \sup_{|\vartheta|=i} \|D^\vartheta \phi\|_{L^\infty} + [\phi]_{C_b^{k+\beta}} \text{ is finite.}$$

We can define now the anisotropic Zygmund-Hölder spaces associated with the distance $\mathbf{d}$. Fixed $\gamma > 0$, the space $C_{b,d}^\gamma(\mathbb{R}^N)$ is the family of functions $\phi\colon \mathbb{R}^N \to \mathbb{R}$ such that for any h in $[\![1,n]\!]$ and any x_0 in $\mathbb{R}^N$, the function

$$z \in E_h(\mathbb{R}^N) \to \phi(x_0+z) \in \mathbb{R} \text{ belongs to } C_b^{\gamma/(1+\alpha(h-1))}\Big(E_h(\mathbb{R}^N)\Big),$$

with a norm bounded by a constant independent from x_0. It is endowed with the norm

$$\|\phi\|_{C_{b,d}^\gamma} := \sum_{h=1}^{n} \sup_{x_0\in\mathbb{R}^N} \|\phi(x_0+\cdot)\|_{C_b^{\gamma/(1+\alpha(h-1))}}. \tag{2.7}$$

We highlight that it is possible to recover the expected joint regularity for the partial derivatives, when they exist, as in the standard Hölder spaces. In such a case, they actually turn out to be Hölder continuous with respect to the distance $\mathbf{d}$ with order one less than the function (See Lemma 2.1 in [Lun97] for more details).

It will be convenient in the following to consider an equivalent norm in the "standard" Hölder-Zygmund spaces $C_b^\gamma(E_h(\mathbb{R}^N))$ that does not take into account the derivatives with respect to the different directions. We suggest the interested reader to see [Lun97], Equation (2.2) or [Pri09] Lemma 2.1 for further details.

Lemma 2.3. *Fixed γ in $(0,3)$ and h in $[\![1,n]\!]$ and ϕ in $C_b(E_h(\mathbb{R}^N))$, let us introduce*

$$\Delta_{x_0}^3\phi(z) := \phi(x_0+3z)-3\phi(x_0+2z)+3\phi(x_0+z)-\phi(x_0), \quad x_0 \in \mathbb{R}^N;\ z \in E_h(\mathbb{R}^N). \tag{2.8}$$

Then, ϕ is in $C_b^\gamma(E_h(\mathbb{R}^N))$ if and only if

$$\sup_{x_0\in\mathbb{R}^N} \sup_{z\in E_h(\mathbb{R}^N);z\neq 0} \frac{\left|\Delta_{x_0}^3\phi(z)\right|}{|z|^\gamma} < \infty.$$

We conclude this subsection with a result concerning the interpolation between the anisotropic Zygmund-Hölder spaces $C_{b,d}^\gamma(\mathbb{R}^N)$. We refer to Theorem 2.2 and Corollary 2.3 in [Lun97] for details.

Theorem 2.4. *Let r be in $(0,1)$ and β, γ in $[0,\infty)$ such that $\beta \le \gamma$. Then, it holds that*

$$\Big(C_{b,d}^\beta(\mathbb{R}^N), C_{b,d}^\gamma(\mathbb{R}^N)\Big)_{r,\infty} = C_{b,d}^{r\gamma+(1-r)\beta}(\mathbb{R}^N)$$

with equivalent norms, where we have denoted for simplicity: $C_{b,d}^0(\mathbb{R}^N) := C_b(\mathbb{R}^N)$.

3 Smoothing Effect for Truncated Density

We present here some analytical properties of the semi-group generated by the operator L^{ou}. Following [SSW12] and [SW12], we will show the existence of a smooth density for such a semi-group and its anisotropic smoothing effect, at least for small times.

Throughout this section, we consider fixed a stochastic base $\Big(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P}\Big)$ satisfying the usual assumptions (see [App09], page 72). Let us then consider the (unique in law) Lévy process $\{Z_t\}_{t\geq 0}$ on $\mathbb{R}^d$ characterized by the Lévy symbol

$$\Phi(p) \;=\; -ib\cdot p + \frac{1}{2}p\cdot Qp + \int_{\mathbb{R}^d_0}\Big(1 - e^{ip\cdot z} + ip\cdot z\mathbb{1}_{B(0,1)}(z)\Big)\,\nu(dz), \quad p\in\mathbb{R}^d.$$

It is well-known by the Lévy-Kitchine formula (see [Jac01]), that the infinitesimal generator of the process $\{BZ_t\}_{t\geq 0}$ is then given by $\mathcal{L}$ on $\mathbb{R}^N$.
Fixed x in $\mathbb{R}^N$, we denote by $\{X_t\}_{t\geq 0}$ the N-dimensional Ornstein-Uhlenbeck process driven by BZ_t, i.e. the unique (strong) solution of the following stochastic differential equation:

$$X_t \;=\; x + \int_0^t AX_s\,ds + BZ_t, \quad t\geq 0,\ \mathbb{P}\text{-almost surely.}$$

By the variation of constants method, it is easy to check that

$$X_t \;=\; e^{tA}x + \int_0^t e^{(t-s)A}B\,dZ_s, \qquad t\geq 0,\ \mathbb{P}\text{-almost surely.} \tag{3.1}$$

The *transition semi-group* associated with L^{ou} is then defined as the family $\{P_t\colon t\geq 0\}$ of linear contractions on $B_b(\mathbb{R}^N)$ given by

$$P_t\phi(x) \;=\; \mathbb{E}\Big[\phi(X_t)\Big], \quad x\in\mathbb{R}^N,\ \phi\in B_b(\mathbb{R}^N). \tag{3.2}$$

We recall that P_t is generated by L^{ou} in the sense that its infinitesimal generator $\mathcal{A}$ coincides with L^{ou} on $C_c^\infty(\mathbb{R}^N)$, the family of smooth functions with compact support.

The next result shows that the random part of X_t (see Equation (3.3)) satisfies again the non-degeneracy assumption [**SD**], even if re-scaled with respect to the anisotropic structure of the dynamics.

Proposition 3.1 (Decomposition)**.** *For any t in $(0,1]$, there exists a Lévy process $\{S_u^t\}_{u\geq 0}$ such that*

$$X_t \overset{law}{=} e^{tA}x + \mathbb{M}_t S_t^t.$$

*Moreover, $\{S_u^t\}_{u\geq 0}$ satisfies assumption [**SD**] with same α as before.*

Proof. For simplicity, we start denoting

$$\Lambda_t \;:=\; \int_0^t e^{(t-s)A}B\,dZ_s, \quad t>0, \tag{3.3}$$

so that $X_t = e^{tA}x + \Lambda_t$. To conclude, we need to construct a Lévy process $\{S_u^t\}_{u\geq 0}$ on $\mathbb{R}^N$ satisfying assumption [**SD**] and

$$\Lambda_t \overset{law}{=} \mathbb{M}_t S_t^t. \tag{3.4}$$

To show the identity in law, we are going to reason in terms of the characteristic functions. By Lemma 2.2 in [SW12], we know that Λ_t is an infinitely divisible random variable with associated Lévy symbol

$$\Phi_{\Lambda_t}(\xi) := \int_0^t \Phi\Big((e^{sA}B)^*\xi\Big)\,ds, \quad \xi \in \mathbb{R}^N.$$

Remembering the decomposition $e^{sA}B = e^{sA}\mathbb{M}_s B = \mathbb{M}_s R_s B$ from Lemma 2.2, we can now rewrite Φ_{Λ_t} as

$$\Phi_{\Lambda_t}(\xi) = t\int_0^1 \Phi\Big((e^{stA}B)^*\xi\Big)\,ds = t\int_0^1 \Phi\Big((R_{st}B)^*\mathbb{M}_s\mathbb{M}_t\xi\Big)\,ds.$$

The above equality suggests us to define, for any fixed t in $(0,1]$, the (unique in law) Lévy process $\{S_u^t\}_{u\geq 0}$ associated with the Lévy symbol

$$\tilde{\Phi}^t(\xi) := \int_0^1 \Phi\Big((R_{st}B)^*\mathbb{M}_s\xi\Big)\,ds, \quad \xi \in \mathbb{R}^N.$$

It is not difficult to check that $\tilde{\Phi}^t$ is indeed a Lévy symbol associated with the Lévy triplet $(\tilde{Q}^t, \tilde{b}^t, \tilde{\nu}^t)$ given by

$$\tilde{Q}^t = \int_0^1 \mathbb{M}_s R_{st} B Q (\mathbb{M}_s R_{st} B)^*\,ds; \tag{3.5}$$

$$\tilde{b}^t = \int_0^1 \mathbb{M}_s R_{st} B b\,ds + \int_0^1\int_{\mathbb{R}^d} \mathbb{M}_s R_{st} B z\Big[\mathbb{1}_{B(0,1)}(\mathbb{M}_s R_{st} B z) - \mathbb{1}_{B(0,1)}(z)\Big]\,\nu(dz)ds; \tag{3.6}$$

$$\tilde{\nu}^t(\mathcal{A}) = \int_0^1 \nu\Big((\mathbb{M}_s R_{st} B)^{-1}\mathcal{A}\Big)\,ds, \quad \mathcal{A} \in \mathcal{B}(\mathbb{R}_0^d). \tag{3.7}$$

Since we have that

$$\mathbb{E}\Big[e^{i\langle \xi, \Lambda_t\rangle}\Big] = e^{-\Phi_{\Lambda_t}(\xi)} = e^{-t\tilde{\Phi}^t(\mathbb{M}_t\xi)} = \mathbb{E}\Big[e^{i\langle \xi, \mathbb{M}_t S_t^t\rangle}\Big],$$

it follows immediately that the identity (3.4) holds.
It remains to show that the family of Lévy measure $\{\tilde{\nu}^t \colon t \in (0,1]\}$ satisfies the assumption [**SD**]. Recalling that condition [**SD**] is assumed to hold for ν, we know that

$$\tilde{\nu}^t(\mathcal{A}) = \int_0^1 \nu\Big((\mathbb{M}_s R_{st} B)^{-1}\mathcal{A}\Big)\,ds \geq \int_0^1\int_0^{r_0}\int_{\mathbb{S}^{d-1}} \mathbb{1}_{\mathcal{A}}(r\mathbb{M}_s R_{st} B\theta)\mu(d\theta)\frac{dr}{r^{1+\alpha}}ds, \tag{3.8}$$

for any $\mathcal{A}$ in $\mathcal{B}(\mathbb{R}_0^d)$. Furthermore, it holds from Lemma 2.2 that

$$\inf_{s\in(0,1),\, t\in(0,1],\, \theta\in S^{d-1}} \Big|\mathbb{M}_s R_{st} B\theta\Big| =: R_0 > 0. \tag{3.9}$$

It allows us to define two functions $l^t \colon [0,1]\times S^{d-1} \to S^{N-1}$, $m^t \colon [0,1]\times S^{d-1} \to \mathbb{R}$, given by

$$l^t(s,\theta) := \frac{\mathbb{M}_s R_{st} B\theta}{|\mathbb{M}_s R_{st} B\theta|} \quad \text{and} \quad m^t(s,\theta) := |\mathbb{M}_s R_{st} B\theta|.$$

Using the Fubini theorem, we can now rewrite Equation (3.8) as

$$\begin{aligned}\tilde{\nu}^t(\mathcal{A}) &\geq \int_0^1 \int_{\mathbb{S}^{d-1}} \int_0^{r_0} \mathbb{1}_{\mathcal{A}}(l^t(s,\theta)m^t(s,\theta)r)\frac{dr}{r^{1+\alpha}}\mu(d\theta)ds \\ &= \int_0^1 \int_{\mathbb{S}^{d-1}} \int_0^{m^t(s,\theta)r_0} \mathbb{1}_{\mathcal{A}}(l^t(s,\theta)r)\frac{dr}{r^{1+\alpha}}[m^t(s,\theta)]^\alpha\mu(d\theta)ds.\end{aligned}$$

Exploiting again Control (3.9), we can conclude that

$$\tilde{\nu}^t(C) \geq \int_0^1 \int_{\mathbb{S}^{d-1}} \int_0^{R_0} \mathbb{1}_C(l^t(s,\theta)r)\frac{dr}{r^{1+\alpha}}\tilde{m}^t(ds,d\theta) = \int_0^{R_0} \int_{\mathbb{S}^{N-1}} \mathbb{1}_C(\tilde{\theta}r)\tilde{\mu}^t(d\tilde{\theta})\frac{dr}{r^{1+\alpha}}, \tag{3.10}$$

where $\tilde{m}^t(ds,d\theta)$ is a measure on $[0,1]\times S^{d-1}$ given by

$$\tilde{m}^t(ds,d\theta) := [m^t(s,\theta)]^\alpha\mu(d\theta)ds$$

and $\tilde{\mu}^t := (l^t)_*\tilde{m}^t$ is the measure $\tilde{m}^t$ push-forwarded through l^t on S^{N-1}. It is easy to check that the measure $\tilde{\mu}^t$ is finite and non-degenerate in the sense of (1.4), replacing therein d by N. □

An immediate application of the above result is a first representation formula for the transition semi-group $\{P_t\colon t \geq 0\}$ associated with the Ornstein-Uhlenbeck process $\{X_t\}_{t\geq 0}$, at least for small times. Indeed, denoting by $\mathbb{P}_X$ the law of a random variable X, Equation (3.4) implies that for any ϕ in $B_b(\mathbb{R}^N)$, it holds that

$$P_t\phi(x) = \int_{\mathbb{R}^N} \phi(e^{tA}x+y)\,\mathbb{P}_{\Lambda_t}(dy) = \int_{\mathbb{R}^N} \phi(e^{tA}x+\mathbb{M}_t y)\,\mathbb{P}_{S_t^t}(dy), \quad x\in\mathbb{R}^N,\ t\in(0,1]. \tag{3.11}$$

Moreover, condition [**SD**] for $\{S_u^t\}_{u\geq 0}$ allows us to decompose it into two components: a truncated, α-stable part and a remainder one. Indeed, if we denote by ν_α^t the measure serving as lower bound to the Lévy measure $\tilde{\nu}^t$ in (3.10), i.e.

$$\nu_\alpha^t(\mathcal{A}) := \int_0^{R_0} \int_{\mathbb{S}^{N-1}} \mathbb{1}_{\mathcal{A}}(\theta r)\tilde{\mu}^t(d\theta)\frac{dr}{r^{1+\alpha}}, \quad C\in\mathcal{B}(\mathbb{R}_0^N), \tag{3.12}$$

we can consider $\{Y_u^t\}_{u\geq 0}$, the Lévy process on $\mathbb{R}^N$ associated with the Lévy triplet $(0,0,\nu_\alpha^t)$. We recall now a useful fact involving the Lévy symbol Φ_α^t of the process Y^t. The non-degeneracy of the measure $\tilde{\mu}^t$ is equivalent to the existence of a constant $C>0$ such that

$$\Phi_\alpha^t(\xi) \geq C|\xi|^\alpha, \quad \xi\in\mathbb{R}^N. \tag{3.13}$$

A proof of this result can be found, for example, in [Pri12] p.424.
In order to apply the results in [SSW12], we are going to truncate the above process at the typical time scale for an α-stable process. This is $t^{1/\alpha}$ when considering the process at time t (cf. Example 2.1). Namely, we consider the family $\{\mathbb{P}_t^{\mathrm{tr}}\}_{t\geq 0}$ of infinitely divisible probabilities whose characteristic function has the form $\widehat{\mathbb{P}_t^{\mathrm{tr}}}(\xi) := \exp[-\Phi_t^{\mathrm{tr}}(\xi)]$, where

$$\Phi_t^{\mathrm{tr}}(\xi) := \int_{|z|\leq t^{\frac{1}{\alpha}}} \left[1-e^{i\langle\xi,z\rangle}+i\langle\xi,z\rangle\right]\nu_\alpha^t(dz).$$

On the other hand, since the measure $\tilde{\nu}^t$ satisfies assumption [**SD**], we know that the remainder $\tilde{\nu}^t - \mathbb{1}_{B(0,t^{1/\alpha})}\nu_\alpha^t$ is again a Lévy measure on $\mathbb{R}^N$. Let $\{\pi_t\}_{t\geq 0}$ be the family of infinitely divisible probability associated with the following Lévy triplet:

$$(\tilde{Q}^t, \tilde{b}^t, \tilde{\nu}^t - \mathbb{1}_{B(0,t^{1/\alpha})}\nu_\alpha^t).$$

It follows immediately that $\mathbb{P}_{S_t^t} = \mathbb{P}_t^{\mathrm{tr}} * \pi_t$ for any $t > 0$. We can now disintegrate the measure $\mathbb{P}_{S_t^t}$ in Equation (3.11) in order to obtain

$$P_t\phi(x) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi\Big(e^{tA}x + \mathbb{M}_t(y_1+y_2)\Big)\,\mathbb{P}_t^{\mathrm{tr}}(dy_1)\pi_t(dy_2). \tag{3.14}$$

The next step is to use Proposition 2.3 in [SSW12] to show a smoothing effect for the family of truncated stable measures $\{\mathbb{P}_t^{\mathrm{tr}} : t \geq 0\}$, at least for small times. Namely,

Proposition 3.2. *Fixed m in $\mathbb{N}_0$, there exists $T_0 := T_0(m) > 0$ such that for any t in $(0, T_0]$, the probability $\mathbb{P}_t^{tr}$ has a density $p^{tr}(t,\cdot)$ that is m-times continuously differentiable on $\mathbb{R}^N$.*
Moreover, for any ϑ in $\mathbb{N}^N$ such that $|\vartheta| \leq m$, there exists a constant $C := C(m, |\vartheta|)$ such that

$$|D^\vartheta p^{tr}(t,y)| \leq Ct^{-\frac{N+|\vartheta|}{\alpha}}\left(1 + \frac{|y|}{t^{1/\alpha}}\right)^{|\vartheta|-m}, \quad t \in (0,T_0],\ y \in \mathbb{R}^N.$$

Proof. The result follows immediately applying Proposition 2.3 in [SSW12]. To do so, we need to show that the Lévy symbol Φ_α^t of the process $\{Y_u^t\}_{u\geq 0}$ satisfies the following assumptions:

- *Hartman-Wintner condition.* There exists $T > 0$ such that

$$\liminf_{|\xi|\to\infty} \frac{\mathrm{Re}\Phi_\alpha^t(\xi)}{\ln(1+|\xi|)} = \infty, \quad t \in (0,T];$$

- *Controllability condition.* There exist $T > 0$ and $c > 0$ such that

$$\int_{\mathbb{R}^N} e^{-t\mathrm{Re}\Phi_\alpha^t(\xi)}|\xi|^m \leq ct^{-\frac{m+N}{\alpha}}, \quad t \in (0,T].$$

In order to show that the above conditions hold, we fix $T \leq 1$ and we recall that the Lévy symbol Φ_α^t of Y_t, the truncated α-stable process with Lévy measure introduced in (3.12), can be written through the Lévy-Kitchine formula as

$$\Phi_\alpha^t(\xi) = \int_{\mathbb{R}_0^N}\Big(1 - e^{i\langle\xi,z\rangle} + i\langle\xi,z\rangle\Big)\nu_\alpha^t(dz) = \int_0^{R_0}\int_{\mathbb{S}^{N-1}}\Big(1-\cos(\langle\xi,r\theta\rangle)\Big)\,\tilde{\mu}^t(d\theta)\frac{dr}{r^{1+\alpha}}.$$

We have seen in Equation (3.13) that the non-degeneracy of $\tilde{\mu}^t$ implies that $\Phi_\alpha^t(\xi) \geq C|\xi|^\alpha$. The Hartman-Wintner condition then follows immediately since

$$\liminf_{|\xi|\to\infty} \frac{\mathrm{Re}\Phi_\alpha^t(\xi)}{\ln(1+|\xi|)} \geq \liminf_{|\xi|\to\infty} \frac{c|\xi|^\alpha}{\ln(1+|\xi|)} = \infty.$$

To show instead the controllability assumption, let us firstly notice that

$$e^{-t\mathrm{Re}\Phi_\alpha^t(\xi)} \leq \begin{cases} 1, & \text{if } |\xi| \leq R; \\ e^{-ct|\xi|^\alpha}, & \text{if } |\xi| > R, \end{cases}$$

for some $R > 0$. It then follows that

$$\begin{aligned} \int_{\mathbb{R}^N} e^{-t\mathrm{Re}\Phi_\alpha^t(\xi)}|\xi|^m \, d\xi &= \int_{|\xi|\leq R} |\xi|^m \, d\xi + \int_{|\xi|>R} e^{-ct|\xi|^\alpha}|\xi|^m \, d\xi \\ &\leq C + t^{-\frac{m+N}{\alpha}} \int_{|\xi|>t^{1/\alpha}R} e^{-c|\xi|^\alpha}|\xi|^m \, d\xi \\ &\leq C + t^{-\frac{m+N}{\alpha}} \int_{\mathbb{R}^N} e^{-c|\xi|^\alpha}|\xi|^m \, d\xi \\ &\leq Ct^{-\frac{m+N}{\alpha}}, \end{aligned}$$

where in the last step we used that $1 \leq t^{-\frac{m+N}{\alpha}}$. □

4 Estimates for Transition semi-group

The results in the previous section (Proposition 3.2 and Equation (3.14)) allow us to represent the semi-group P_t of the Ornstein-Uhlenbeck process $\{X_t\}_{t\geq 0}$ as

$$P_t\phi(x) = \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi(\mathbb{M}_t(y_1+y_2) + e^{tA}x)p^{\mathrm{tr}}(t,y_1)\, dy_1\pi_t(dy_2), \quad x \in \mathbb{R}^N, \tag{4.1}$$

at least for small time intervals.
Here, we will focus on estimates in $\|\cdot\|_\infty$-norm of the transition semi-group $\{P_t \colon t \geq 0\}$ given in Equation (3.2) and its derivatives. The main result in this section is Corollary 4.4 that shows the continuity of P_t between anisotropic Zygmund-Hölder spaces. These controls will be fundamental in the next section to prove Schauder Estimates in the elliptic and parabolic settings.
As we will see in the following result, the derivatives of the semi-group P_t with respect to a component i in I_h induces an additional time singularity of order $\frac{1+\alpha(h-1)}{\alpha}$, corresponding to the intrinsic time scale of the considered component.

Proposition 4.1. *Let h, h', h'' be in $[\![1,n]\!]$ and ϕ in $B_b(\mathbb{R}^N)$. Then, there exists a constant $C > 0$ such that for any i in I_h, any j in $I_{h'}$ and any k in $I_{h''}$, it holds that*

$$\|D_iP_t\phi\|_\infty \leq C\|\phi\|_\infty\Big(1 + t^{-\frac{1+\alpha(h-1)}{\alpha}}\Big), \quad t > 0; \tag{4.2}$$

$$\|D^2_{i,j}P_t\phi\|_\infty \leq C\|\phi\|_\infty\Big(1 + t^{-\frac{2+\alpha(h+h'-2)}{\alpha}}\Big), \quad t > 0; \tag{4.3}$$

$$\|D^3_{i,j,k}P_t\phi\|_\infty \leq C\|\phi\|_\infty\Big(1 + t^{-\frac{3+\alpha(h+h'+h''-3)}{\alpha}}\Big), \quad t > 0. \tag{4.4}$$

Proof. We start fixing a time horizon $T := 1 \wedge T_0(N+4) > 0$, where $T_0(m)$ was defined in Proposition 3.2. Our choice of $N+4$ is motivated by the fact that we consider derivatives up to order 3.
On the interval $(0,T]$, the representation formula (4.1) holds and $P_t\phi$ is three times

differentiable for any ϕ in $B_b(\mathbb{R}^N)$. We are going to show only Estimate (4.2) since the controls for the higher derivatives can be obtained similarly.
Fixed $t \leq T$, let us consider i in I_h for some h in $[\![1,n]\!]$. When $t \leq T$, we recall from Equation (4.1) that, up to a change of variables, it holds that

$$\Big|D_i P_t\phi(x)\Big| \;=\; \Big|D_i \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi(\mathbb{M}_t(y_1+y_2))p^{\mathrm{tr}}(t, y_1 - \mathbb{M}_t^{-1}e^{tA}x)\, dy_1\pi_t(dy_2)\Big|.$$

We can then move the derivative inside the integral and write that

$$\begin{aligned}\Big|D_i P_t\phi(x)\Big| \;&=\; \Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi(\mathbb{M}_t(y_1+y_2))\langle\nabla p^{\mathrm{tr}}(t, y_1 - \mathbb{M}_t^{-1}e^{tA}x), \mathbb{M}_t^{-1}e^{tA}e_i\rangle\, dy_1\pi_t(dy_2)\Big| \\ &\leq\; |\mathbb{M}_t^{-1}e^{tA}e_i| \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\phi(\mathbb{M}_t(y_1+y_2))|\,|\nabla p^{\mathrm{tr}}(t, y_1 - \mathbb{M}_t^{-1}e^{tA}x))|\, dy_1\pi_t(dy_2) \\ &\leq\; Ct^{-(h-1)}\|\phi\|_\infty \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} |\nabla p^{\mathrm{tr}}(t, y_1)|\, dy_1\pi_t(dy_2), \end{aligned} \tag{4.5}$$

where in the last step we exploited Lemma 2.2 to control

$$|\mathbb{M}_t^{-1}e^{tA}e_i| \;\leq\; \sum_{k=1}^n |\mathbb{M}_t^{-1}E_k e^{tA}E_h e_i| \;\leq\; C\Big[\sum_{k=1}^{h-1} t^{k-h}t + \sum_{k=h}^n t^{-(k-1)}t^{-(h-1)}\Big] \;\leq\; Ct^{-(h-1)}, \tag{4.6}$$

remembering that $t \leq 1$. We conclude the case $t \leq T$ using the control on p^{tr} (Proposition 3.2 with $m = N+2$) to write that

$$\begin{aligned}\Big|D_i P_t\phi(x)\Big| \;&\leq\; C\|\phi\|_\infty \pi_t(\mathbb{R}^N)t^{-(h-1)} \int_{\mathbb{R}^N} t^{-\frac{N+1}{\alpha}}\Big(1 + \frac{|y_1|}{t^{1/\alpha}}\Big)^{-(N+1)}\, dy_1 \\ &\leq\; C\|\phi\|_\infty t^{-\frac{1+\alpha(h-1)}{\alpha}} \int_{\mathbb{R}^N} (1+|z|)^{-(N+1)}\, dz \\ &\leq\; C\|\phi\|_\infty t^{-\frac{1+\alpha(h-1)}{\alpha}}. \end{aligned} \tag{4.7}$$

Above, we used the change of variables $z = t^{-1/\alpha}y_1$. When $t > T$, we can exploit the already proven controls for small times, the semi-group and the contraction properties of $\{P_t \colon t \geq 0\}$ on $B_b(\mathbb{R}^N)$ to write that

$$\|D_i P_t\phi\|_\infty \;=\; \|D_i P_T\Big(P_{t-T}\phi\Big)\|_\infty \;\leq\; C_T\|P_{t-T}\phi\|_\infty \;\leq\; C\|\phi\|_\infty. \tag{4.8}$$

We have thus shown Control (4.2) for any $t > 0$. □

The following interpolation inequality (see e.g. [Tri92])

$$\|\phi\|_{C_b^{r\delta_1+(1-r)\delta_2}} \;\leq\; C\|\phi\|_{C_b^{\delta_1}}^r \|\phi\|_{C_b^{\delta_2}}^{1-r} \tag{4.9}$$

valid for $0 \leq \delta_1 < \delta_2$, r in $(0,1)$ and ϕ in $C^{\delta_2}(\mathbb{R}^N)$, allows us to extend easily the above result.

Corollary 4.2. *Let γ be in $[0, 1+\alpha)$. Then, there exists a constant $C > 0$ such that*

$$\|P_t\|_{\mathcal{L}(C_b, C_{b,d}^\gamma)} \;\leq\; C\Big(1 + t^{-\frac{\gamma}{\alpha}}\Big), \quad t > 0. \tag{4.10}$$

Proof. Let us firstly assume that γ is in $(0,1]$. Remembering the definition of $C^{\gamma}_{b,d}$-norm in (2.7), we start fixing a point x_0 in $\mathbb{R}^N$ and h in $[\![2,n]\!]$. Then, the contraction property of the semi-group implies that

$$\|P_t\phi(x_0+\cdot)_{|E_h(\mathbb{R}^N)}\|_\infty \le C\|\phi\|_\infty.$$

Moreover, Control (4.2) in Proposition 4.1 ensures that

$$\|D_iP_t\phi(x_0+\cdot)_{|E_h(\mathbb{R}^N)}\|_\infty \le C\|\phi\|_\infty\Big(1+t^{-\frac{1+\alpha(h-1)}{\alpha}}\Big).$$

It follows immediately that

$$\|P_t\phi(x_0+\cdot)_{|E_h(\mathbb{R}^N)}\|_{C^1_b} \le C\|\phi\|_\infty(1+t^{-\frac{1+\alpha(h-1)}{\alpha}}).$$

We can now apply the interpolation inequality (4.9) with $\delta_1=0$, $\delta_2=1$ and $r=\gamma/(1+\alpha(h-1))$ in order to obtain that

$$\begin{aligned}\|P_t\phi(x_0+\cdot)_{|E_h(\mathbb{R}^N)}\|_{C^r_b} &\le C\|P_t\phi(x_0+\cdot)_{|E_h(\mathbb{R}^N)}\|^r_{C^1_b}\|P_t\phi(x_0+\cdot)_{|E_h(\mathbb{R}^N)}\|^{1-r}_\infty\\ &\le C\|\phi\|_\infty\Big(1+t^{-\frac{\gamma}{\alpha}}\Big).\end{aligned}$$

The argument is analogous for γ in $(1,3)$, considering only the case $h=0$. □

The next result allows us to extend the controls in Proposition 4.1 to functions in the anisotropic Zygmund-Hölder spaces. Roughly speaking, it states that the anisotropic γ-Hölder regularity induces a "homogeneous" gain in time of order γ/α that can be used to weaken, at least partially, the time singularities associated with the derivatives. The general argument of proof will mimic the one of Proposition 4.1 even if, this time, we will need to make the Hölder modulus of ϕ appear. It will be managed exploiting some "partial" cancellation arguments (cf. (4.17)).

Theorem 4.3. *Let h,h',h'' be in $[\![1,n]\!]$ and ϕ in $C^{\gamma}_{b,d}(\mathbb{R}^N)$ for some γ in $[0,1+\alpha)$. Then, there exists a constant $C>0$ such that for any i in I_h, any j in $I_{h'}$ and any k in $I_{h''}$, it holds that*

$$\|D_iP_t\phi\|_\infty \le C\|\phi\|_{C^{\gamma}_{b,d}}\Big(1+t^{\frac{\gamma-(1+\alpha(h-1))}{\alpha}}\Big),\quad t>0; \tag{4.11}$$

$$\|D^2_{i,j}P_t\phi\|_\infty \le C\|\phi\|_{C^{\gamma}_{b,d}}\Big(1+t^{\frac{\gamma-(2+\alpha(h+h'-2))}{\alpha}}\Big),\quad t>0; \tag{4.12}$$

$$\|D^3_{i,j,k}P_t\phi\|_\infty \le C\|\phi\|_{C^{\gamma}_{b,d}}\Big(1+t^{\frac{\gamma-(3+\alpha(h+h'+h''-3))}{\alpha}}\Big),\quad t>0. \tag{4.13}$$

Proof. Similarly to Proposition 4.1, we start fixing a time horizon

$$T := 1\wedge T_0(N+6)>0. \tag{4.14}$$

Then, Corollary 4.2 implies the continuity of P_t on $C^{\gamma}_{b,d}(\mathbb{R}^N)$, for any $t\ge T/2$. Indeed,

$$\|P_t\phi\|_{C^{\gamma}_{b,d}} \le C\|\phi\|_\infty\Big(1+t^{-\frac{\gamma}{\alpha}}\Big) \le C_T\|\phi\|_{C^{\gamma}_{b,d}}. \tag{4.15}$$

The same argument shown in Equation (4.8) can now be applied to prove Control (4.11) for $t > T$. Namely,

$$\|D_i P_t \phi\|_\infty \;=\; \|D_i P_{T/2}\Big(P_{t-T/2}\phi\Big)\|_\infty \;\le\; C_T \|P_{t-T/2}\phi\|_\infty \;\le\; C\|\phi\|_\infty.$$

The same reasoning can be used for the higher derivatives, too.
When $t \le T$, let us assume $\alpha > 1$, so that $1+\alpha > 2$. The case $\alpha \le 1$ can be handled similarly taking into account one less derivative. Moreover, we notice that we need to prove Controls (4.11)-(4.13) only for γ in $(2, 1+\alpha)$ thanks to interpolation techniques. Indeed, if we want, for example, to prove Estimates (4.11) for some γ' in $(0,2]$, we can use Theorem 2.4 to show that

$$\|D_i P_t\|_{\mathcal{L}(C^{\gamma'}_{b,d};B_b)} \;\le\; \Big(\|D_i P_t\|_{\mathcal{L}(B_b)}\Big)^{1-\gamma'/\gamma}\Big(\|D_i P_t\|_{\mathcal{L}(C^{\gamma}_{b,d},B_b)}\Big)^{\gamma'/\gamma} \;\le\; C\Big(1 + t^{\frac{\gamma'-(1+\alpha(h-1))}{\alpha}}\Big),$$

once we have proven Estimate (4.11) for $\gamma > 2$.
We are only going to show Control (4.11) for $t \le T$ and γ in $(2, 1+\alpha)$. The estimates (4.12) and (4.13) involving the higher derivatives can be obtained in an analogous way. Fixed i in I_h for some h in $[\![1,n]\!]$, we start noticing from Equation (4.1) that, up to the change of variables $\tilde{y}_1 = y_1 + \mathbb{M}_t^{-1}e^{tA}x$, it holds that

$$\begin{aligned} D_i P_t \phi(x) \;&=\; D_i \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi(\mathbb{M}_t(\tilde{y}_1 + y_2)) p^{\mathrm{tr}}(t, \tilde{y}_1 - \mathbb{M}_t^{-1}e^{tA}x)\, d\tilde{y}_1 \pi_t(dy_2) \\ &=\; \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi(\mathbb{M}_t(\tilde{y}_1 + y_2)) D_i\left[p^{\mathrm{tr}}(t, \tilde{y}_1 - \mathbb{M}_t^{-1}e^{tA}x)\right] d\tilde{y}_1 \pi_t(dy_2). \end{aligned} \tag{4.16}$$

Recalling that here, D_i stands for the derivative with respect to the variable x_i, we then notice that

$$\int_{\mathbb{R}^N} D_i\left[p^{\mathrm{tr}}(t, \tilde{y}_1 - \mathbb{M}_t^{-1}e^{tA}x)\right] d\tilde{y}_1 \;=\; D_i \int_{\mathbb{R}^N}\left[p^{\mathrm{tr}}(t, \tilde{y}_1 - \mathbb{M}_t^{-1}e^{tA}x)\right] d\tilde{y}_1 \;=\; 0.$$

In particular, it immediately follows that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi(\mathbb{M}_t y_2 + e^{tA}x) D_i\left[p^{\mathrm{tr}}(t, \tilde{y}_1 - \mathbb{M}_t^{-1}e^{tA}x)\right] dy_1 \pi_t(dy_2) \;=\; 0.$$

This property will allow to use a cancellation argument in Equation (4.17) below, once we split the small jumps in the non-degenerate contributions and the other ones. We thus get from (4.16) that

$$\begin{aligned} &D_i P_t \phi(x) \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Big((\phi(\mathbb{M}_t(\tilde{y}_1 + y_2))) - \phi(\mathbb{M}_t y_2 + e^{tA}x)\Big) D_i\left[p^{\mathrm{tr}}(t, \tilde{y}_1 - \mathbb{M}_t^{-1}e^{tA}x)\right] dy_1 \pi_t(dy_2) \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \Delta\phi(t, y_1, y_2, x) \langle \nabla p^{\mathrm{tr}}(t, y_1), \mathbb{M}_t^{-1}e^{tA}e_i\rangle\, dy_1 \pi_t(dy_2), \end{aligned} \tag{4.17}$$

where, after the backward change of variables, we have also denoted:

$$\Delta\phi(t, y_1, y_2, x) \;:=\; \phi(\mathbb{M}_t(y_1 + y_2) + e^{tA}x) - \phi(\mathbb{M}_t y_2 + e^{tA}x).$$

We can then decompose the difference $\Delta\phi$ in the following way:

$$\Delta\phi(t, y_1, y_2, x) \;=\; \Lambda_0(t, y_1, y_2, x) + \Lambda_1(t, y_1, y_2, x) \tag{4.18}$$

where we denoted

$$\begin{aligned}\Lambda_0(t,y_1,y_2,x) &:= \phi(E_1y_1+\mathbb{M}_ty_2+e^{tA}x)-\phi(\mathbb{M}_ty_2+e^{tA}x);\\ \Lambda_1(t,y_1,y_2,x) &:= \phi(\mathbb{M}_ty_1+\mathbb{M}_ty_2+e^{tA}x)-\phi(E_1y_1+\mathbb{M}_ty_2+e^{tA}x).\end{aligned}$$

Noticing now that the first contribution can be easily controlled

$$|\Lambda_1(t,y_1,y_2,x)| \le \|\phi\|_{C^\gamma_{b,d}}\sum_{k=2}^n |E_k\mathbb{M}_ty_1|^{\frac{\gamma}{1+\alpha(k-1)}}, \tag{4.19}$$

we can then write that

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Lambda_1(t,y_1,y_2,x)\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2)\Big|\\ &\qquad\le \|\phi\|_{C^\gamma_{b,d}}t^{-(h-1)}\sum_{k=2}^n\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|\nabla p^{\mathrm{tr}}(t,y_1)||E_k\mathbb{M}_ty_1|^{\frac{\gamma}{1+\alpha(k-1)}}\,dy_1\pi(dy_2),\end{aligned}$$

where we also exploited the control in (4.6).
The above expression allows us to conclude the control for Λ_1 as in (4.7), using Proposition 3.2 with $m=N+4$ and $|\vartheta|=1$. Namely,

$$\begin{aligned}&\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Lambda_1(t,y_1,y_2,x)\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2)\Big|\\ &\qquad\le C\|\phi\|_{C^\gamma_{b,d}}t^{-(h-1)}\sum_{k=2}^n\int_{\mathbb{R}^N}t^{-\frac{N+1}{\alpha}}\Big(1+\frac{|y_1|}{t^{\frac{1}{\alpha}}}\Big)^{-(N+3)}|E_k\mathbb{M}_ty_1|^{\frac{\gamma}{1+\alpha(k-1)}}\,dy_1\\ &\qquad\le C\|\phi\|_{C^\gamma_{b,d}}t^{\frac{\gamma-(1+\alpha(h-1))}{\alpha}}\sum_{k=2}^n\int_{\mathbb{R}^N}\Big(1+|z|\Big)^{-(N+3)}|z|^{\frac{\gamma}{1+\alpha(k-1)}}\,dz\\ &\qquad\le C\|\phi\|_{C^\gamma_{b,d}}t^{\frac{\gamma-(1+\alpha(h-1))}{\alpha}},\end{aligned}\tag{4.20}$$

where in the second step we used again the change of variable $z=y_1t^{-1/\alpha}$.
For the term Λ_0, we start instead applying a Taylor expansion of second order along E_1y_1:

$$\begin{aligned}\Lambda_0(t,x,y_1,y_2) &= \phi(E_1y_1+\mathbb{M}_ty_2+e^{tA}x)-\phi(\mathbb{M}_ty_2+e^{tA}x)\\ &= \langle\nabla\phi(\mathbb{M}_ty_2+e^{tA}x),E_1y_1\rangle+\int_0^1\langle D^2\phi(\mathbb{M}_ty_2+e^{tA}x+\lambda E_1y_1)R_1y_1,E_1y_1\rangle\,d\lambda\\ &=: \Lambda_2(t,x,y_1,y_2)+\Lambda_3(t,x,y_1,y_2).\end{aligned}\tag{4.21}$$

Now, we want to control the component involving the second term Λ_2:

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Lambda_2(t,y_1,y_2,x)\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2).$$

To do it, we exploit an integration by part formula with respect to the derivative of p^{tr}. Indeed, reasoning component-wise if necessary, it is not difficult to check that

$$\begin{aligned}&\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Lambda_2(t,y_1,y_2,x)\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2)\\ &\qquad=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\langle\nabla\phi(\mathbb{M}_ty_2+e^{tA}x),E_1y_1\rangle\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2)\\ &\qquad=\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\langle\nabla\phi(\mathbb{M}_ty_2+e^{tA}x),E_1\mathbb{M}_t^{-1}e^{tA}e_i\rangle p^{\mathrm{tr}}(t,y_1)\,dy_1\pi_t(dy_2).\end{aligned}\tag{4.22}$$

It then follows immediately that

$$\begin{aligned}\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Lambda_2(t,y_1,y_2,x)\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2)\Big| \\ \leq C\|\phi\|_{C^\gamma_{b,d}}|E_1e^{tA}e_i|\int_{\mathbb{R}^N}|p^{\mathrm{tr}}(t,y_1)|\,dy_1 \\ \leq C\|\phi\|_{C^\gamma_{b,d}},\end{aligned}\tag{4.23}$$

where in the last passage we also used Lemma 2.2.
To control the contribution involving the third term Λ_3, we will need an additional cancellation argument. Let us assume for the moment that the family of truncated probabilities $\{\mathbb{P}^{\mathrm{tr}}_t\}_{t\geq 0}$ has zero mean value, so that it holds that:

$$\int_{\mathbb{R}^N}E_1y_1p^{\mathrm{tr}}(t,y_1)\,dy_1 \;=\; 0_{\mathbb{R}^d}.$$

Under this additional hypothesis, it is possible to show that

$$\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\langle D^2\phi(\mathbb{M}_ty_2+e^{tA}x)E_1y_1,E_1y_1\rangle\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2) \;=\; 0.\tag{4.24}$$

Indeed, applying again an integration by parts formula with respect to the derivative on p^{tr}, we notice, reasoning as well component-wise as in (4.22), that:

$$\begin{aligned}\int_{\mathbb{R}^N}\langle D^2\phi(\mathbb{M}_ty_2+e^{tA}x)E_1y_1,&E_1y_1\rangle\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1 \\ &= \int_{\mathbb{R}^N}\langle D^2\phi(\mathbb{M}_ty_2+e^{tA}x)E_1y_1,E_1\mathbb{M}_t^{-1}e^{tA}e_i\rangle p^{\mathrm{tr}}(t,y_1)\,dy_1 \\ &= \left\langle D^2\phi(\mathbb{M}_ty_2+e^{tA}x)\left[\int_{\mathbb{R}^N}E_1y_1p^{\mathrm{tr}}(t,y_1)\,dy_1\right],E_1e^{tA}e_i\right\rangle \\ &= 0.\end{aligned}$$

The cancellation argument in (4.24) allows now to write that

$$\begin{aligned}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}&\Lambda_3(t,y_1,y_2,x)\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2) \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\int_0^1\Big\langle\Big[D^2\phi(\mathbb{M}_ty_2+e^{tA}x+\lambda E_1y_1)-D^2\phi(\mathbb{M}_ty_2+e^{tA}x)\Big]E_1y_1,E_1y_1\Big\rangle \\ &\qquad\times\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2).\end{aligned}$$

The same arguments presented in Control (4.20) can be also applied here to show that

$$\begin{aligned}\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Lambda_3(t,y_1,y_2,x)\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2)\Big| \\ \leq C\|\phi\|_{C^\gamma_{b,d}}t^{-(h-1)}\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}|E_1y_1|^{\gamma-2}|E_1y_1|^2|\nabla p^{\mathrm{tr}}(t,y_1)|\,dy_1\pi_t(dy_2) \\ \leq C\|\phi\|_{C^\gamma_{b,d}}t^{\frac{\gamma}{\alpha}-\frac{1+\alpha(h-1)}{\alpha}}.\end{aligned}\tag{4.25}$$

Going back to Expression (4.17), we notice that the decompositions in (4.18) and (4.21) implies immediately that

$$|D_iP_t\phi(x)| \;\leq\; \sum_{i=1}^3\Big|\int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Lambda_i(t,y_1,y_2,x)\langle\nabla p^{\mathrm{tr}}(t,y_1),\mathbb{M}_t^{-1}e^{tA}e_i\rangle\,dy_1\pi_t(dy_2)\Big|.$$

Then, we can finally use the estimates in (4.20), (4.23) and (4.25) to conclude that

$$\left|D_i P_t \phi(x)\right| \leq C\|\phi\|_{C^\gamma_{b,d}} \left(1 + t^{\frac{\gamma}{\alpha} - \frac{1+\alpha(i-1)}{\alpha}}\right).$$

In the general case, when the family of probabilities $\{\mathbb{P}^{\mathrm{tr}}_t\}_{t\geq 0}$ possibly has non-zero mean value, it is then enough to follow the same reasoning above, taking into account, in the first cancellation argument, the additional error given by the mean.
Namely, we will need to consider $\phi(\mathbb{M}_t y_2 + e^{tA}x - m_t)$, where m_t is the mean value associated with $\mathbb{P}_t$, in Equation (4.17). □

Next, we are going to use the controls in Theorem 4.3 to show the main result of this section. It states the continuity of the semi-group P_t between anisotropic Zygmund-Hölder spaces at a cost of additional time singularities.

Corollary 4.4. *Let β, γ be in $[0, 1+\alpha)$ such that $\beta \leq \gamma$. Then, there exists a constant $C > 0$ such that*

$$\|P_t\|_{\mathcal{L}(C^\beta_{b,d}, C^\gamma_{b,d})} \leq C\Big(1 + t^{\frac{\beta-\gamma}{\alpha}}\Big), \quad t > 0. \tag{4.26}$$

Proof. It is enough to show the result only for $\gamma = \beta$ non-integer, thanks to interpolation techniques. Indeed, fixed $\beta < \gamma$, we can use Theorem 2.4 to show that

$$\|P_t\|_{\mathcal{L}(C^\beta_{b,d}(\mathbb{R}^N), C^\gamma_{b,d}(\mathbb{R}^N))} \leq \left(\|P_t\|_{\mathcal{L}(C_b(\mathbb{R}^N), C^\gamma_{b,d}(\mathbb{R}^N))}\right)^{1-\frac{\beta}{\gamma}} \left(\|P_t\|_{\mathcal{L}(C^\gamma_{b,d}(\mathbb{R}^N))}\right)^{\frac{\beta}{\gamma}}.$$

On the other hand, if we fix γ integer, we can take γ' in $(\gamma, 1+\alpha)$ non-integer such that Theorem 2.4 implies:

$$\|P_t\|_{\mathcal{L}(C^\gamma_{b,d}(\mathbb{R}^N))} \leq \left(\|P_t\|_{\mathcal{L}(C_b(\mathbb{R}^N)}\right)^{1-\frac{\gamma}{\gamma'}} \left(\|P_t\|_{\mathcal{L}(C^{\gamma'}_{b,d}(\mathbb{R}^N))}\right)^{\frac{\gamma}{\gamma'}}.$$

The general result will then follows from the two above controls and Equation (4.10), once we have shown Estimate (4.26) for $\gamma = \beta$ non-integer.
Fixed again the time horizon T given in (4.14), we start noticing that Control (4.26) for $t \geq T$ has already been shown in Equation (4.15) .
To prove it when $t \leq T$, we are going to exploit the equivalent norm defined in (2.8) of Lemma 2.3. For this reason, we fix h in $[\![1, n]\!]$, a point x_0 in $\mathbb{R}^N$ and $z \neq 0$ in $E_h(\mathbb{R}^N)$ and we would like to show that

$$\left|\Delta^3_{x_0}\Big(P_t\phi\Big)(z)\right| \leq C\|\phi\|_{C^\gamma_{b,d}} |z|^{\frac{\gamma}{1+\alpha(h-1)}}, \tag{4.27}$$

for some constant $C > 0$ independent from x_0. Before starting with the calculations, we highlight the presence of three different "regimes" appearing below. On the one hand, we will firstly consider a *macroscopic regime* appearing for $|z| \geq 1$. On the other hand, we will say that the *off-diagonal regime* holds if $t^{\frac{1+\alpha(h-1)}{\alpha}} \leq |z| \leq 1$. It will mean in particular that the spatial distance is larger than the characteristic time-scale. Finally, a *diagonal regime* will be in force when $t^{\frac{1+\alpha(h-1)}{\alpha}} \geq |z|$ and the spatial point will be instead smaller than the typical time-scale magnitude. While for the two first regimes,

we are going to use the contraction property of the semi-group, the third regime will require to exploit the controls in Hölder norms given by Theorem 4.3.
As said above, Estimate (4.27) in the macroscopic regime (i.e. $|z| \geq 1$) follows immediately from the contraction property of P_t on $B_b(\mathbb{R}^N)$. Indeed,

$$\Big|\Delta^3_{x_0}\Big(P_t\phi\Big)(z)\Big| \leq C\|P_t\phi\|_\infty \leq C\|\phi\|_{C^\gamma_{b,d}}|z|^{\frac{\gamma}{1+\alpha(h-1)}}. \tag{4.28}$$

For $t^{\frac{1+\alpha(h-1)}{\alpha}} \leq |z| \leq 1$ and l in $[\![0,3]\!]$, we start noticing from Equation (4.1) that

$$\begin{aligned} P_t\phi(x_0+lz) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi\Big(\mathbb{M}_t(y_1+y_2)+e^{tA}(x_0+lz)\Big)p^{\mathrm{tr}}(t,y_1)\,dy_1\pi_t(dy_2) \\ &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \phi\Big(\xi_0+le^{tA}z)\Big)p^{\mathrm{tr}}(t,y_1)\,dy_1\pi_t(dy_2), \end{aligned}$$

where we have denoted for simplicity $\xi_0 = \mathbb{M}_t(y_1+y_2)+e^{tA}x_0$. We can then exploit Lemma 2.2 to write that

$$\begin{aligned} \Big|\Delta^3_{x_0}\Big(P_t\phi\Big)(z)\Big| &\leq \int_{\mathbb{R}^N}\int_{\mathbb{R}^N}\Big|\Delta^3_{\xi_0}\phi(e^{tA}z)\Big)\Big|p^{\mathrm{tr}}(t,y_1)\,dy_1\pi_t(dy_2) \\ &\leq \pi_t(\mathbb{R}^N)\|\phi\|_{C^\gamma_{b,d}}\sum_{k=1}^n |E_ke^{tA}z|^{\frac{\gamma}{1+\alpha(k-1)}} \\ &\leq C\|\phi\|_{C^\gamma_{b,d}}\Bigg[\sum_{k=1}^{h-1}\Big(t|z|\Big)^{\frac{\gamma}{1+\alpha(k-1)}} + \sum_{k=h}^n\Big(t^{k-h}|z|\Big)^{\frac{\gamma}{1+\alpha(k-1)}}\Bigg] \\ &\leq C\|\phi\|_{C^\gamma_{b,d}}|z|^{\frac{\gamma}{1+\alpha(h-1)}}. \end{aligned} \tag{4.29}$$

For $|z| \leq t^{\frac{1+\alpha(h-1)}{\alpha}}$, we are going to apply Taylor expansion three times in order to make $D^3_{I_h}$ appear. Namely,

$$\begin{aligned} &\Big|\Delta^3_{x_0}\Big(P_t\phi\Big)(z)\Big| \\ &\quad= \Big|\int_0^1 \langle D_{I_h}P_t\phi(x_0+\lambda z) - 2D_{I_h}P_t\phi(x_0+z+\lambda z) + D_{I_h}P_t\phi(x_0+2z+\lambda z), z\rangle\,d\lambda\Big| \\ &\quad\leq \Big|\int_0^1\int_0^1 \langle\Big[D^2_{I_h}P_t\phi(x_0+(\lambda+\mu)z) - D^2_{I_h}P_t\phi(x_0+z+(\lambda+\mu)z)\Big]z, z\rangle\,d\lambda d\mu\Big| \\ &\quad\leq \Big|\int_0^1\int_0^1\int_0^1 \langle\Big[D^3_{I_h}P_t\phi(x_0+(\lambda+\mu+\nu)z)\Big](z,z), z\rangle\,d\lambda d\mu d\nu\Big| \\ &\quad\leq C\|D^3_{I_h}P_t\phi\|_\infty|z|^3 \\ &\quad\leq C\|\phi\|_{C^\gamma_{b,d}}\Big(1+t^{\frac{\gamma-3(1+\alpha(h-1))}{\alpha}}\Big)|z|^3, \end{aligned} \tag{4.30}$$

where in the last step we used Control (4.13) with $h = h' = h''$. Since $|z| \leq t^{\frac{1+\alpha(h-1)}{\alpha}}$ and noticing that $\gamma - 3(1+\alpha(h-1)) < 0$, it holds that

$$\Big(1+t^{\frac{\gamma-3(1+\alpha(h-1))}{\alpha}}\Big)|z|^3 \leq |z|^{\frac{\gamma-3(1+\alpha(h-1))}{1+\alpha(h-1)}}|z|^3 = |z|^{\frac{\gamma}{1+\alpha(h-1)}}.$$

We can then conclude that

$$\Big|\Delta^3_{x_0}\Big(P_t\phi\Big)(z)\Big| \leq C\|\phi\|_{C^\gamma_{b,d}}|z|^{\frac{\gamma}{1+\alpha(h-1)}}. \tag{4.31}$$

Going back to Controls (4.28), (4.29) and (4.31), we have thus proven Estimate (4.27) for any non-integer $\gamma = \beta$. □

5 Elliptic and parabolic Schauder estimates

In this section, we use the controls shown before to prove Schauder Estimates both for the elliptic and the parabolic equation driven by the Ornstein-Ulhenbeck operator L^{ou}.

Fixed $\lambda > 0$ and g in $C_b(\mathbb{R}^N)$, we say that a function $u\colon \mathbb{R}^N \to \mathbb{R}^N$ is a *distributional solution* of Elliptic Equation (1.5) if u is in $C_b(\mathbb{R}^N)$ and for any ϕ in $C_c^\infty(\mathbb{R}^N)$ (i.e. smooth functions with compact support), it holds that

$$\int_{\mathbb{R}^N} u(x)\Big[\lambda\phi(x) - \Big(L^{\mathrm{ou}}\Big)^*\phi(x)\Big]\,dx \;=\; \int_{\mathbb{R}^N} \phi(x)g(x)\,dx, \tag{5.1}$$

where $\Big(L^{\mathrm{ou}}\Big)^*$ denotes the formal adjoint of L^{ou} on $L^2(\mathbb{R}^N)$, i.e.

$$\Big(L^{\mathrm{ou}}\Big)^*\phi(x) \;=\; \mathcal{L}^*\phi(x) - \langle Ax, D_x\phi(x)\rangle - \mathrm{Tr}(A)\phi(x), \quad (t,x) \in [0,T]\times\mathbb{R}^N, \tag{5.2}$$

and $\mathcal{L}^*$ is the adjoint of the operator $\mathcal{L}$ on $L^2(\mathbb{R}^N)$. It is well-known (see e.g. Section 4.2 in [App19]) that it can be represented for any ϕ in $C_c^\infty(\mathbb{R}^N)$ as

$$\begin{aligned}\mathcal{L}^*\phi(x) \;=\; &\frac{1}{2}\mathrm{Tr}\Big(BQB^*D^2\phi(x)\Big) - \langle Bb, D\phi(x)\rangle \\ &+ \int_{\mathbb{R}_0^d}\Big[\phi(x-Bz) - \phi(x) + \langle D\phi(x), Bz\rangle \mathbb{1}_{B(0,1)}(z)\Big]\,\nu(dz).\end{aligned}$$

We state now the main result for the elliptic case, ensuring the well-posedness (in a distributional sense) for Equation (1.5).

Theorem 5.1. *Fixed $\lambda > 0$, let g be in $C_b(\mathbb{R}^N)$. Then, the function $u\colon \mathbb{R}^N \to \mathbb{R}$ given by*

$$u(x) \;:=\; \int_0^\infty e^{-\lambda t}P_t g(x)\,dt, \quad x \in \mathbb{R}^N, \tag{5.3}$$

is the unique distributional solution of Equation (1.5).

Proof. Existence. We are going to show that the function u given in Equation (5.3) is indeed a distributional solution of the elliptic problem (1.5). It is straightforward to notice that u is in $C_b(\mathbb{R}^N)$, thanks to the contraction property of P_t on $C_b(\mathbb{R}^N)$. Fixed ϕ in $C_c^\infty(\mathbb{R}^N)$, we then use Fubini Theorem to write that

$$\begin{aligned}\int_{\mathbb{R}^N} u(x)\Big(L^{\mathrm{ou}}\Big)^*\phi(x)\,dx \;&=\; \lim_{\epsilon\to 0^+}\int_\epsilon^\infty\int_{\mathbb{R}^N} e^{-\lambda t}P_t g(x)\Big(L^{\mathrm{ou}}\Big)^*\phi(x)\,dxdt \\ &=\; \lim_{\epsilon\to 0^+}\int_\epsilon^\infty\int_{\mathbb{R}^N} e^{-\lambda t}L^{\mathrm{ou}}P_t g(x)\phi(x)\,dxdt,\end{aligned}$$

where, in the last step, we exploited that $P_t g$ is differentiable and bounded for $t > 0$ (Proposition 4.1). Since L^{ou} is the infinitesimal generator of the semi-group $\{P_t\colon t \geq 0\}$, we know that $\partial_t(P_t g)$ exists for any $t > 0$ and $\partial_t(P_t g)(x) = L^{\mathrm{ou}}P_t g(x)$ for any x in $\mathbb{R}^N$. Integration by parts formula allows then to conclude that

$$\begin{aligned}\int_{\mathbb{R}^N} u(x)\Big(L^{\mathrm{ou}}\Big)^*\phi(x)\,dx \;&=\; \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N}\phi(x)\int_\epsilon^\infty e^{-\lambda t}\partial_t P_t g(x)\,dtdx \\ &=\; \lim_{\epsilon\to 0^+}\int_{\mathbb{R}^N}\Big(-e^{-\lambda\epsilon}P_\epsilon g(x) + \lambda\int_\epsilon^\infty e^{-\lambda t}P_t g(x)\,dt\Big)\,dx \\ &=\; \int_{\mathbb{R}^N} -g(x)\phi(x)\,dx + \int_{\mathbb{R}^N}\lambda u(x)\phi(x)\,dx.\end{aligned}$$

Uniqueness. It is enough to show that any distributional solution u of Equation (1.5) for $g = 0$ coincides with the zero function, i.e. $u = 0$. To do so, we fix a function ρ in $C_c^\infty(\mathbb{R}^N)$ such that $\|\rho\|_{L^1} = 1$, $0 \leq |\rho| \leq 1$ and we then define the *mollifier* $\rho_m := m^N \rho(mx)$ for any m in $\mathbb{N}$. Denoting now, for simplicity, $u_m := u * \rho_m$, we define the function

$$g_m(x) \;:=\; \lambda u_m(x) - L^{\mathrm{ou}} u_m(x). \tag{5.4}$$

Using that u is in $C_b(\mathbb{R}^N)$, it is easy to notice that g_m is also in $C_b(\mathbb{R}^N)$ for any fixed m in $\mathbb{N}$. Truncating the functions if necessary, we can assume that u_m and g_m are integrable with integrable Fourier transform so that we can apply the Fourier transform in Equation (5.4):

$$\lambda \widehat{u}_m(\xi) - \mathcal{F}_x\Big(L^{\mathrm{ou}} u_m\Big)(\xi) \;=\; \widehat{g}_m(\xi). \tag{5.5}$$

We remember in particular that the above operator L^{ou} has an associated Lévy symbol $\Phi^{\mathrm{ou}}(\xi)$ and, following Section 3.3.2 in [App09], it holds that

$$\mathcal{F}_x\Big(L^{\mathrm{ou}} u_m\Big)(\xi) \;=\; \Phi^{\mathrm{ou}}(\xi)\widehat{u}_m(\xi). \tag{5.6}$$

We can then use it to show that $\widehat{u}_m$ is a classical solution of the following equation:

$$\Big[\lambda - \Phi^{\mathrm{ou}}(\xi)\Big]\widehat{u}_m(\xi) \;=\; \widehat{g}_m(\xi).$$

The above equation can be easily solved by direct calculation as

$$\widehat{u}_m(\xi) \;=\; \int_0^\infty e^{-\lambda t} e^{t\Phi^{\mathrm{ou}}(\xi)} \widehat{g}_m(\xi)\, ds.$$

In order to go back to u_m, we apply now the inverse Fourier transform to write that

$$u_m(x) \;=\; \int_0^\infty e^{-\lambda t} P_t g_m(x)\, dt.$$

The contraction property of the semi-group P_t then implies that $\|u_m\|_\infty \leq C\|g_m\|_\infty$. In order to conclude, we need to show that

$$\lim_{m\to\infty} \|g_m\|_\infty \;=\; 0. \tag{5.7}$$

We start noticing that, since u is a solution of Equation (1.5) with $g = 0$, it holds that

$$\begin{aligned} g_m(x) \;&=\; \int_{\mathbb{R}^N} u(y)\Big\{\lambda\rho_m(x-y) - \mathcal{L}[\rho_m(\cdot - y)](x) - \langle Ax, D_x\rho_m(x-y)\rangle\Big\}\, dy \\ &=\; \int_{\mathbb{R}^N} u(y)\Big\{\mathcal{L}^*[\rho_m(x-\cdot)](y) - \mathcal{L}[\rho_m(\cdot - y)](x) + \langle A(x-y), D_x\rho_m(x-y)\rangle \\ &\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad + \mathrm{Tr}(A)\rho_m(x-y)\Big\}\, dy \\ &=:\; R_m^1(x) + R_m^2(x) + R_m^3(x), \end{aligned}$$

where we have denoted

$$\begin{aligned} R_m^1(x) \;&:=\; \int_{\mathbb{R}^N} u(y)\Big[\mathcal{L}^*[\rho_m(x-\cdot)](y) - \mathcal{L}[\rho_m(\cdot - y)](x)\Big]\, dy; \\ R_m^2(x) \;&:=\; \int_{\mathbb{R}^N} u(y)\langle A(x-y), D_x\rho_m(x-y)\rangle\, dy; \\ R_m^3(x) \;&:=\; \int_{\mathbb{R}^N} u(y)\mathrm{Tr}(A)\rho_m(x-y)\, dy. \end{aligned}$$

On the one hand, it is easy to notice that $R^1_m = 0$, since $\mathcal{L}^*[\rho_m(x-\cdot)](y) = \mathcal{L}[\rho_m(\cdot-y)](x)$ for any m in $\mathbb{N}$ and any y in $\mathbb{R}^N$. Indeed, it holds that

$$\frac{1}{2}\mathrm{Tr}\Big(BQB^* D^2_y[\rho_m(x-\cdot)](y)\Big) - \langle Bb, D_y[\rho_m(x-\cdot)](y)\rangle \\ = \frac{1}{2}\mathrm{Tr}\Big(BQB^* D^2_x\rho_m(x-y)\Big) + \langle Bb, D_x\rho_m(x-y)\rangle$$

and

$$\int_{\mathbb{R}^d_0}\Big[\rho_m(x-y+Bz) - \rho_m(x-y) + \langle D_y[\rho_m(x-\cdot)](y), Bz\rangle \mathbb{1}_{B(0,1)}(z)\Big]\,\nu(dz) \\ = \int_{\mathbb{R}^d_0}\Big[\rho_m((x+Bz)-y) - \rho_m(x-y) - \langle D_x\rho_m(x-y), Bz\rangle \mathbb{1}_{B(0,1)}(z)\Big]\,\nu(dz).$$

On the other hand, it can be checked (see e.g. [Pri09]) that $\|R^2_m + R^3_m\|_\infty \to 0$ if m goes to infinity. Indeed, we firstly notice that R^3_m converges, when m goes to infinity, to the function $u\mathrm{Tr}(A)$, uniformly in x. On the other hand, applying the change of variables $y = x - z/m$ in R^2_m, we can obtain that

$$R^2_m(x) \;=\; m\int_{\mathbb{R}^N} u(x-z/m)\langle A(z/m), D_x\rho(z)\rangle\,dy.$$

Letting m goes to infinity above, we can then conclude that R^2_m converges to the function $-u\mathrm{Tr}(A)$, uniformly in x. □

Let us deal now with the parabolic setting. Since we are working with evolution equations, the functions we consider will often depend on time, too. We denote for any $\gamma > 0$ the space $L^\infty(0,T,C^\gamma_{b,d}(\mathbb{R}^N))$ as the family of functions ϕ in $B_b\Big([0,T]\times\mathbb{R}^N\Big)$ such that $\phi(t,\cdot)$ is in $C^\gamma_{b,d}(\mathbb{R}^N)$ at any fixed t and the norm

$$\|\phi\|_{L^\infty(C^\gamma_{b,d})} \;:=\; \sup_{t\in[0,T]}\|\phi(t,\cdot)\|_{C^\gamma_{b,d}} \text{ is finite.}$$

We define now the notion of solution we are going to consider. Fixed $T > 0$, u_0 in $C_b(\mathbb{R}^N)$ and f in $L^\infty\Big(0,T;C_b(\mathbb{R}^N)\Big)$, we say that a function $u\colon [0,T]\times\mathbb{R}^N \to \mathbb{R}^N$ is a *weak solution* of the Cauchy problem (1.6) if u is in $L^\infty\Big(0,T;C_b(\mathbb{R}^N)\Big)$ and for any ϕ in $C^\infty_c([0,T)\times\mathbb{R}^N)$, it holds that

$$\int_0^T\int_{\mathbb{R}^N} u(t,x)\Big[\partial_t\phi(t,x) + \Big(L^{\mathrm{ou}}\Big)^*\phi(t,x)\Big] + f(t,x)\phi(t,x)\,dxdt \\ + \int_{\mathbb{R}^N} u_0(x)\phi(0,x)\,dx \;=\; 0, \quad (5.8)$$

where $\Big(L^{\mathrm{ou}}\Big)^*$ denotes the formal adjoint of L^{ou} on $L^2(\mathbb{R}^N)$ given in Equation (5.2).

Similarly to the elliptic setting, we show firstly the weak well-posedness of the Cauchy problem (1.6).

Theorem 5.2. *Fixed $T > 0$, let u_0 be a function in $C_b(\mathbb{R}^N)$ and f in $L^\infty\Big(0,T;C_b(\mathbb{R}^N)\Big)$. Then, the function $u\colon [0,T]\times\mathbb{R}^N \to \mathbb{R}$ given by*

$$u(t,x) \;:=\; P_t u_0(x) + \int_0^t P_{t-s} f(s,x)\,ds, \quad (t,x)\in[0,T]\times\mathbb{R}^N, \tag{5.9}$$

is the unique weak solution of the Cauchy problem (1.6).

Proof. Existence. We start considering a "regularized" version of the coefficients appearing in Equation (1.6). Namely, we consider a family $\{u_{0,m}\}_{m\in\mathbb{N}}$ in $C_b^\infty(\mathbb{R}^N)$ such that $u_{0,m}\to u_0$ uniformly in x and a family $\{f_m\}_{m\in\mathbb{N}}$ in $L^\infty\Big(0,T;C_b^\infty(\mathbb{R}^N)\Big)$ such that $f_m\to f$ uniformly in t and x. They can be obtained through standard mollification methods in space.
Fixed m in $\mathbb{N}$, we denote now by $u_m\colon [0,T]\times\mathbb{R}^n\to\mathbb{R}$ the function given by

$$u_m(t,x) \;:=\; P_t u_{0,m}(x) + \int_0^t P_{t-s} f_m(s,x)\,ds, \quad t\in[0,T], x\in\mathbb{R}^N.$$

On the one hand, we use again that $\partial_t(P_t u_m)(t,x) = L^{\mathrm{ou}} P_t u_m(t,x)$ for any (t,x) in $[0,T]\times\mathbb{R}^N$ to check that u_m is indeed a *classical* solution of the "regularized" Cauchy Problem:

$$\begin{cases} \partial_t u_m(t,x) \;=\; L^{\mathrm{ou}} u_m(t,x) + f_m(t,x), & (t,x)\in(0,T)\times\mathbb{R}^N;\\ u_m(0,x) \;=\; u_{0,m}(x), & x\in\mathbb{R}^N. \end{cases}$$

On the other hand, we exploit the linearity and the continuity of the semi-group P_t on $C_b(\mathbb{R}^N)$ to show that

$$u_m \;=\; P_t u_{0,m}(x) + \int_0^t P_{t-s} f_m(s,x)\,ds \;\overset{m}{\to}\; P_t u_0(x) + \int_0^t P_{t-s} f(s,x)\,ds \;=\; u,$$

uniformly in t and x, where u is the function given in (5.9).
We fix now a test function ϕ in $C_0^\infty\Big([0,T)\times\mathbb{R}^N\Big)$ and we then notice that

$$\int_0^T\int_{\mathbb{R}^N} \phi(t,y)\Big(\partial_t - L^{\mathrm{ou}}\Big) u_m(t,y)\,dydt \;=\; \int_0^T\int_{\mathbb{R}^N} \phi(t,y) f_m(t,y)\,dydt.$$

An integration by parts allows now to move the operator to the test function, being careful to remember that $u_m(0,\cdot) = u_{0,m}(\cdot)$. Indeed, it holds that

$$\begin{aligned} -\int_0^T\int_{\mathbb{R}^N} &\Big(\partial_t + \big(L^{\mathrm{ou}}\big)^*\Big)\phi(t,y) u_m(t,y)\,dydt \\ &= \int_{\mathbb{R}^N} \phi(0,y) u_{0,m}(y)\,dy + \int_0^T\int_{\mathbb{R}^N} \phi(t,y) f_m(t,y)\,dydt, \end{aligned} \tag{5.10}$$

where $\big(L^{\mathrm{ou}}\big)^*$ denotes the formal adjoint of L^{ou} on $L^2(\mathbb{R}^N)$.
We would like now to go back to the solution u, letting m go to infinity. We start rewriting the right-hand side term of (5.10) as $R_m^1 + R_m^2$, where

$$\begin{aligned} R_m^1 \;&:=\; \int_{\mathbb{R}^N} \phi(0,y) u_{0,m}(y)\,dy;\\ R_m^2 \;&:=\; \int_0^T\int_{\mathbb{R}^N} \phi(t,y) f_m(t,y)\,dydt. \end{aligned}$$

We can rewrite R^2_m as

$$R^2_m = \int_0^T \int_{\mathbb{R}^N} \phi(t,y) f(t,y)\, dydt + \int_0^T \int_{\mathbb{R}^N} \phi(t,y)\Big[f_m - f\Big](t,y)\, dydt.$$

Exploiting that, by assumption, f_m converges to f uniformly in t and x, it is easy to see that the second contribution above converges to 0. A similar argument can be used to show that

$$\int_{\mathbb{R}^N} \phi(0,y) u_{0,m}(y)\, dy \overset{m}{\to} \int_{\mathbb{R}^N} \phi(0,y) u_0(y)\, dy.$$

On the other hand, we can rewrite the left-hand side of Equation (5.10) as

$$\begin{aligned} &-\int_0^T \int_{\mathbb{R}^N} \Big(\partial_t + \big(L^{\mathrm{ou}}\big)^*\Big)\phi(t,y) u_m(t,y)\, dydt \\ &\qquad = -\int_0^T \int_{\mathbb{R}^N} \Big(\partial_t + \big(L^{\mathrm{ou}}\big)^*\Big)\phi(t,y) u(t,y)\, dydt + L^1_m + L^2_m + L^3_m, \end{aligned}$$

where we have denoted

$$\begin{aligned} L^1_m &:= \int_0^T \int_{\mathbb{R}^N} \Big[\frac{1}{2}\mathrm{Tr}\Big(BQB^* D^2_y \phi(t,y)\Big) + \langle Ay + Bb, D_y\phi(t,y)\rangle + \mathrm{Tr}(A)\phi(t,y)\Big] \\ &\qquad\qquad \times [u_m - u](t,y)\, dydt; \\ L^2_m &:= \int_0^T \int_{\mathbb{R}^N} \partial_t \phi(t,y)[u - u_m](t,y)\, dydt; \qquad (5.11) \\ L^3_m &:= \int_0^T \int_{\mathbb{R}^N} \Big[\int_{\mathbb{R}^d_0} \phi(t,y-Bz) - \phi(t,y) + \langle D_y\phi(t,y), Bz\rangle \mathbb{1}_{B(0,1)}(z)\, \nu(dz)\Big] \\ &\qquad\qquad \times [u - u_m](t,y) dydt. \end{aligned}$$

To conclude, we need to show that the remainder $L^1_m + L^2_m + L^3_m$ is negligible, if m goes to infinity. Exploiting that ϕ has a compact support and that $\|u_m - u\|_\infty \overset{m}{\to} 0$, it is easy to show that $|L^1_m + L^2_m| \overset{m}{\to} 0$.
In order to control L^3_m, we need firstly to decompose it as $L^{3,1}_m + L^{3,2}_m$, where

$$\begin{aligned} L^{3,1}_m &:= \int_0^T \int_{\mathbb{R}^N} \Big[u(t,y) - u_m(t,y)\Big] \\ &\qquad \times \Big[\int_{0<|z|<1} \phi(t,y-Bz) - \phi(t,y) + \langle D_y\phi(t,y), Bz\rangle\, \nu(dz)\Big] dydt; \\ L^{3,2}_m &:= \int_0^T \int_{\mathbb{R}^N} \Big[u(t,y) - u_m(t,y)\Big]\Big[\int_{|z|>1} \phi(t,y-Bz) - \phi(t,y)\, \nu(dz)\Big] dydt. \end{aligned}$$

The second term $L^{3,2}_m$ can be controlled easily using the Fubini Theorem. Indeed, denoting by K the support of ϕ and by λ the Lebesgue measure on $\mathbb{R}^N$, we notice that

$$\begin{aligned} |L^{3,2}_m| &\le \|u - u_m\|_\infty \int_0^T \int_{|z|>1} \int_{\mathbb{R}^N} |\phi(t,y-Bz) - \phi(t,y)|\, dy\nu(dz)dt \\ &\le CT2\lambda(K)\nu\Big(B^c(0,1)\Big)\|u - u_m\|_\infty. \end{aligned}$$

Exploiting that $\nu\Big(B^c(0,1)\Big)$ is finite since ν is a Lévy measure, we can then conclude that $|L^{3,2}_m|$ tends to zero if m goes to infinity.

The argument for $L_m^{3,1}$ is similar but we need firstly to apply a Taylor expansion twice to make a term $|z|^2$ appear in the integral and exploit that $|z|^2\nu(dz)$ is finite on $B(0,1)$.

Uniqueness. This proof will follow essentially the same arguments as for Theorem 5.1. Let u be any weak solution of Cauchy problem (1.6) with $u_0 = f = 0$. We are going to show that $u = 0$.
We start considering a mollyfing sequence $\{\rho_m\}_{m\in\mathbb{N}}$ in $C_c^\infty((0,T)\times\mathbb{R}^N)$. Denoting for simplicity $u_m(t,x) = u * \rho_m(t,x)$, we then notice that u_m is continuously differentiable in time and that $u_m(0,x) = 0$. It makes sense to define now the function

$$f_m(t,x) \;:=\; \partial_t u_m(t,x) - L^{\mathrm{ou}} u_m(t,x). \tag{5.12}$$

Moreover, we can truncate f_m and u_m if necessary, so that they are integrable with integrable Fourier transform. Then, the same reasoning in Equations (5.5), (5.6) allows us to write that

$$\begin{cases} \partial_t \widehat{u}_m(t,\xi) - \Phi^{\mathrm{ou}}(\xi)\widehat{u}_m(t,\xi) \;=\; \widehat{f}_m(t,\xi), \\ \widehat{u}_m(0,\xi) \;=\; 0. \end{cases}$$

The above equation can be easily solved integrating in time, giving the following representation:

$$\widehat{u}_m(t,\xi) \;=\; \int_0^t e^{(t-s)\Phi^{\mathrm{ou}}(\xi)} \widehat{f}_m(s,\xi)\, ds.$$

In order to go back to u_m, we apply now the inverse Fourier transform to write that

$$u_m(t,x) \;=\; \int_0^t P_{t-s} f_m(s,x)\, ds.$$

The contraction property of P_t allows us to conclude that $\|u_m\|_\infty \le C\|f_m\|_\infty$. Letting m goes to zero, we obtain the desired result. Indeed, it is possible to show that

$$\lim_{m\to\infty} \|f_m\|_\infty \;=\; 0,$$

relying on the same reasonings used in the analogous elliptic case (Theorem 5.1). □

The next two conclusive theorems provide the Schauder estimates both in the elliptic and in the parabolic setting.

Theorem 5.3 (Elliptic Schauder Estimates)**.** *Fixed $\lambda > 0$ and β in $(0,1)$, let g be in $C_{b,d}^{\alpha+\beta}(\mathbb{R}^N)$. Then, the distributional solution u of Equation (1.5) is in $C_{b,d}^{\beta}(\mathbb{R}^N)$ and there exists a positive constant C such that*

$$\|u\|_{C_{b,d}^{\alpha+\beta}} \;\le\; C\Big(1 + \frac{1}{\lambda}\Big)\|g\|_{C_{b,d}^{\beta}}. \tag{5.13}$$

Proof. Thanks to Theorem 5.1, we know that the unique solution u of the elliptic equation (1.5) is given in (5.3). In order to show that such a function u satisfies Schauder estimates (5.13), we exploit again the equivalent norm defined in (2.8) of Lemma 2.3. Namely, we fix h in $[\![1,n]\!]$ and x_0 in $\mathbb{R}^N$ and we show that

$$|\Delta_{x_0}^3 u(z)| \;=\; \Big|\int_0^\infty e^{-\lambda t}\Delta_{x_0}^3\Big(P_t g\Big)(z)\, dt\Big| \;\le\; C\|g\|_{C_{b,d}^{\beta}}|z|^{\frac{\alpha+\beta}{1+\alpha(h-1)}}, \quad z\in E_h(\mathbb{R}^N),$$

for some constant $C > 0$ independent from x_0. For $|z| \geq 1$, it can be easily obtained from the contraction property of P_t on $B_b(\mathbb{R}^N)$:

$$\Big|\int_0^\infty e^{-\lambda t}\Delta^3_{x_0}\Big(P_t g\Big)(z)\,dt\Big| \;\leq\; 3\int_0^\infty e^{-\lambda t\|P_t g\|_\infty\,dt} \;\leq\; \frac{3}{\lambda}\|g\|_\infty |z|^{\frac{\alpha+\beta}{1+\alpha(h-1)}}. \tag{5.14}$$

When $|z| \leq 1$, we start fixing a *transition time* t_0 given by

$$t_0 \;=\; |z|^{\frac{\alpha}{1+\alpha(h-1)}}. \tag{5.15}$$

Notably, t_0 represents the transition time between the diagonal and the off-diagonal regime, accordingly to the intrinsic time scales of the system. We then decompose $\Delta^3_{x_0}u(z)$ as $R_1(z) + R_2(z)$, where

$$\begin{aligned} R_1(z) \;&:=\; \int_0^{t_0} e^{-\lambda t}\Delta^3_{x_0}\Big(P_t g\Big)(z)\,dt;\\ R_2(z) \;&:=\; \int_{t_0}^\infty e^{-\lambda t}\Delta^3_{x_0}\Big(P_t g\Big)(z)\,dt. \end{aligned}$$

The first component R_1 is controlled easily using Corollary 4.4 for $\beta = \gamma$. Indeed,

$$|R_1(z)| \;\leq\; \int_0^{t_0} |\Delta^3_{x_0}\Big(P_t g\Big)(z)|\,dt \;\leq\; |z|^{\frac{\beta}{1+\alpha(h-1)}}\int_0^{t_0}\|P_t g\|_{C^\beta_{b,d}}\,dt \;\leq\; C\|g\|_{C^\beta_{b,d}}|z|^{\frac{\alpha+\beta}{1+\alpha(h-1)}}. \tag{5.16}$$

On the other hand, the control for R_2 can be obtained following Equation (4.30) in order to write that

$$\begin{aligned} |R_2(z)| \;&\leq\; C\|g\|_{C^\beta_{b,d}}|z|^3\int_{t_0}^\infty e^{-\lambda t}\Big(1 + t^{\frac{\beta-3(1+\alpha(h-1))}{\alpha}}\Big)\,dt\\ &\leq C\|g\|_{C^\beta_{b,d}}|z|^3\Big(\lambda^{-1} + |z|^{\frac{\alpha+\beta-3(1+\alpha(h-1))}{1+\alpha(h-1)}}\Big)\\ &\leq C\Big(1+\frac{1}{\lambda}\Big)\|g\|_{C^\beta_{b,d}}|z|^{\frac{\alpha+\beta}{1+\alpha(h-1)}}, \end{aligned} \tag{5.17}$$

where, in the last step, we exploited that $|z| \leq 1$. □

Theorem 5.4 (Parabolic Schauder Estimates)**.** *Fixed $T > 0$ and β in $(0,1)$, let u_0 be in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)$ and f in $L^\infty\Big(0,T;C^\beta_{b,d}(\mathbb{R}^N)\Big)$. Then, the weak solution u of Cauchy Problem* (1.6) *is in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)\Big)$ and there exists a constant $C := C(T) > 0$ such that*

$$\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \;\leq\; C\Big[\|u_0\|_{C^{\alpha+\beta}_{b,d}} + \|f\|_{L^\infty(C^\beta_{b,d})}\Big]. \tag{5.18}$$

Proof. We are going to show that any function u given by Equation (5.9) satisfies the Schauder Estimates (5.18). We start splitting the function u in $u_1 + u_2$, where

$$u_1(t,x) := P_t u_0(x); \tag{5.19}$$

$$u_2(t,x) := \int_0^t P_s f(t-s,x)\,ds. \tag{5.20}$$

Corollary 4.4 allows then to control u_1 in the following way:

$$\|u_1\|_{L^\infty(C^{\alpha+\beta}_{b,d})} = \sup_{t\in[0,T]} \|P_t u_0\|_{C^{\alpha+\beta}_{b,d}} \leq C\|u_0\|_{C^{\alpha+\beta}_{b,d}}.$$

In order to deal with the contribution u_2, we will follow essentially the same reasoning for the Schauder Estimates in the elliptic setting. Namely, we use again the equivalent norm defined in (2.8) of Lemma 2.3 in order to estimate

$$\|u_2\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \leq C\|f\|_{L^\infty(C^{\beta}_{b,d})}.$$

Fixed h in $[\![1,n]\!]$ and x_0 in $\mathbb{R}^N$, our aim is to show that

$$|\Delta^3_{x_0} u_2(z)| = \left|\int_0^t \Delta^3_{x_0}\Big(P_{t-s}f\Big)(s,z)\,ds\right| \leq C\|f\|_{L^\infty(C^{\beta}_{b,d}\beta)}|z|^{\frac{\alpha+\beta}{1+\alpha(h-1)}}, \quad z\in E_h(\mathbb{R}^N),$$

for some constant $C>0$ independent from x_0. When $|z|\geq 1$, it can be obtained easily from the contraction property of P_t on $C_b(\mathbb{R}^N)$ as in (5.14). For $|z|\leq 1$, we fix again the transition time t_0 given in (5.15) and we then decompose $\Delta^3_{x_0}u_2(t,z)$ as $\tilde{R}_1(t,z)+\tilde{R}_2(t,z)$, where

$$\begin{aligned}\tilde{R}_1(t,z) &:= \int_0^{t\wedge t_0} \Delta^3_{x_0}\Big(P_s f\Big)(t-s,z)\,ds:\\ \tilde{R}_2(t,z) &:= \int_{t\wedge t_0}^{t} \Delta^3_{x_0}\Big(P_s f\Big)(t-s,z)\,ds.\end{aligned}$$

The first component R_1 can be controlled easily as in (5.16):

$$\begin{aligned}|\tilde{R}_1(t,z)| &\leq \int_0^{t\wedge t_0} |\Delta^3_{x_0}\Big(P_s f\Big)(t-s,z)|\,ds\\ &\leq |z|^{\frac{\beta}{1+\alpha(h-1)}}\int_0^{t\wedge t_0}\|P_s f(t-s,\cdot)\|_{C^{\beta}_{b,d}}\,ds\\ &\leq C\|f\|_{L^\infty(C^{\beta}_{b,d})}|z|^{\frac{\alpha+\beta}{1+\alpha(h-1)}}.\end{aligned}$$

On the other hand, the control for R_2 is obtained following the same steps used in Equation (5.17). Namely,

$$\begin{aligned}|\tilde{R}_2(t,z)| &\leq C\|f\|_{L^\infty(C^{\beta}_{b,d})}|z|^3\int_{t\wedge t_0}^{\infty}\Big(1+s^{\frac{\beta-3(1+\alpha(h-1))}{\alpha}}\Big)\,ds\\ &\leq C\|f\|_{L^\infty(C^{\beta}_{b,d})}|z|^{\frac{\alpha+\beta}{1+\alpha(h-1)}}.\end{aligned}$$

□

6 Extensions to time dependent operators

In this final section, we would like to show some possible extensions of our method in order to include more general operators with non-linear, space-time dependent coefficients. Even in this framework, we will prove the well-posedness of the parabolic

Cauchy problem and show the associated Schauder estimates.
Following [KP10], our first step is to consider a time-dependent Ornstein-Uhlenbeck operator of the following form:

$$L_t^{\mathrm{ou}}\phi(t,x) := \frac{1}{2}\mathrm{Tr}\Big(B_tQB_t^*D^2\phi(x)\Big) + \langle A_tx, D\phi(x)\rangle + \int_{\mathbb{R}_0^d}\Big[\phi(x+B_tz)-\phi(x)-\langle D_x\phi(x), B_tz\rangle \mathbb{1}_{B(0,1)}(z)\Big]\,\nu(dz),$$

where $B_t := B\sigma_0(t)$ and A_t, $\sigma_0(t)$ are two time-dependent matrices in $\mathbb{R}^N\otimes\mathbb{R}^N$ and $\mathbb{R}^d\otimes\mathbb{R}^d$, respectively. From this point further, we assume that the matrices A_t, $\sigma_0(t)$ are measurable in time and that they satisfy the following conditions:

[tK] for any fixed t in $[0,T]$, it holds that $N = \mathrm{rank}\Big[B, A_tB, \dots, A_t^{N-1}B\Big]$;

[B] the matrix A_t is bounded in time, i.e. there exists a constant $\eta > 0$ such that

$$|A_t\xi| \le \eta|\xi|, \quad \xi\in\mathbb{R}^N;$$

[UE] the matrix σ_0 is uniformly elliptic, i.e. it holds that

$$\eta^{-1}|\xi|^2 \le \langle\sigma_0(t)\xi,\xi\rangle \le \eta|\xi|^2, \quad (t,\xi)\in[0,T]\times\mathbb{R}^d.$$

It is important to highlight already that this new "time-dependent" version [**tK**] of the Kalman rank condition [**K**] allows us to reproduce the same reasonings of Section 2. In particular, the anisotropic distance **d** and the Zygmund-Hölder spaces $C^\beta_{b,d}(\mathbb{R}^N)$ can be constructed under these assumptions, even if only at any *fixed* time t. A priori, the number of sub-divisions of the space $\mathbb{R}^N$ may change for different times, leading to consider a time-dependent $n(t)$ in Equation (2.2) and, consequently, time-dependent anisotropic distances and Hölder spaces. We will however drop the subscript in t below since it does not add any difficulty in the arguments but it may damage the readability of the article.

Proposition 6.1. *Let u_0 be in $C_b(\mathbb{R}^N)$ and f in $L^\infty\Big(0,T;C_b(\mathbb{R}^N)\Big)$. Then, there exists a unique solution $u\colon [0,T]\times\mathbb{R}^N\to\mathbb{R}$ of the following Cauchy problem:*

$$\begin{cases}\partial_t u(t,x) = L_t^{ou}u(t,x) + f(t,x), & (t,x)\in(0,T)\times\mathbb{R}^N;\\ u(0,x) = u_0(x), & x\in\mathbb{R}^N.\end{cases} \tag{6.1}$$

Furthermore, if u_0 is in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)$ and f in $L^\infty\Big(0,T;C^\beta_{b,d}(\mathbb{R}^N)\Big)$, then the solution u is in $L^\infty\Big(0,T;C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)\Big)$ and there exists a constant $C := C(T,\eta) > 0$ such that

$$\|u\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \le C\Big[\|u_0\|_{C^{\alpha+\beta}_{b,d}} + \|f\|_{L^\infty(C^\beta_{b,d})}\Big]. \tag{6.2}$$

Proof. The proof of this result can be obtained mimicking the arguments already presented in the first part of the article with some slight modifications. The main difference is the introduction of the resolvent $\mathcal{R}_{s,t}$ associated with the matrix A_t in place of the

matrix exponential e^{tA}. Namely, $\mathcal{R}_{t,u}$ is a time-dependent matrix in $\mathbb{R}^N \otimes \mathbb{R}^N$ that is solution of the following ODE:

$$\begin{cases} \partial_t \mathcal{R}_{t,u} \;=\; A_t \mathcal{R}_{t,u}, \quad t \in [u,T]; \\ \mathcal{R}_{u,u} \;=\; \mathrm{Id}_{N\times N}\,. \end{cases} \tag{6.3}$$

As said before, Section 2 follows exactly in the same manner as above except for Lemma 2.2 (structure of the resolvent), whose proof can be found in [HM16], Lemmas 5.1 and 5.2. The arguments in Section 3 and 4 can be applied again, even if the formulation of some objects presented there changes slightly. For example in Equation (3.1), the N-dimensional Ornstein-Uhlenbeck process $\{X_t\}_{t\geq 0}$ driven by $B_t Z_t$ should be now represented by

$$X_t \;=\; \mathcal{R}_{t,0}x + \int_0^t \mathcal{R}_{t,u} B_u \, dZ_u, \quad t \geq 0, x \in \mathbb{R}^N.$$

Finally in Section 5, the uniform ellipticity [**UE**] of $\sigma_0(t)$ and the boundedness [**B**] of A_t allow us to control the remainder terms appearing in Equation (5.11) as done above and thus, to conclude as in Theorems 5.2 and 5.4. □

Once we have shown our results for the time-dependent Ornstein-Uhlenbeck operator L_t^{ou}, we add now a non-linearity to the problem, even if only dependent in time. Namely, we are interested in operators of the following form:

$$L_t\phi(t,x) \;:=\; L_t^{\mathrm{ou}}\phi(t,x) + \langle F_0(t), D_x\phi(x)\rangle - c_0(t)\phi(x), \quad (t,x) \in [0,T]\times\mathbb{R}^N, \tag{6.4}$$

where $c_0\colon [0,T] \to \mathbb{R}$ and $F_0\colon [0,T] \to \mathbb{R}^N$ are two functions. For any sufficiently regular function $\phi\colon [0,T] \to \mathbb{R}$, we are going to denote

$$\mathcal{T}\phi(t,x) \;:=\; e^{-\int_0^t c_0(s)\,ds}\phi\Big(t, x + \int_0^t F_0(s)\,ds\Big), \quad (t,x) \in [0,T]\times\mathbb{R}^N. \tag{6.5}$$

We will see in the next result that the "operator" $\mathcal{T}$ transforms solutions of the Cauchy problem associated with $\mathcal{L}_t^{\mathrm{ou}}$ to solutions of the Cauchy problem driven by L_t, even if for a modified drift $\mathcal{T}f$.

Lemma 6.2. *Fixed $T > 0$, let u_0 be in $C_b(\mathbb{R}^N)$, f in $L^\infty\big(0,T;C_b(\mathbb{R}^N)\big)$ and c_0, F_0 in $C_b([0,T])$. Then, a function $u\colon [0,T]\times\mathbb{R}^N \to \mathbb{R}$ is a weak solution of Cauchy Problem* (6.1) *if and only if the function $v\colon [0,T]\times\mathbb{R}^N \to \mathbb{R}$ given by $v(t,x) = \mathcal{T}u(t,x)$ is a weak solution of the following Cauchy problem:*

$$\begin{cases} \partial_t u(t,x) \;=\; L_t u(t,x) + \mathcal{T}f(t,x), \quad (t,x) \in (0,T)\times\mathbb{R}^N; \\ u(0,x) \;=\; u_0(x), \quad x \in \mathbb{R}^N. \end{cases} \tag{6.6}$$

In particular, there exists a unique weak solution of Cauchy Problem (6.6).

Proof. Given a weak solution u of Cauchy problem (6.1), we are going to show that the function v given in (6.5) is indeed a weak solution of Cauchy Problem (6.6). The inverse implication can be obtained in a similar manner and we will not prove it here. By mollification if necessary, we can take two sequences $\{c_m\}_{m\in\mathbb{N}}$ and $\{F_m\}_{m\in\mathbb{N}}$ in

$C_b^\infty([0,T])$ such that $c_m \to c_0$ and $F_m \to F_0$ uniformly in t. Furthermore, we denote for simplicity

$$\tilde{c}_m(t) := \int_0^t c_m(s)\,ds; \quad \tilde{F}_m(t) := \int_0^t F_m(s)\,ds.$$

Given a test function ϕ in $C_c^\infty\Big([0,T)\times\mathbb{R}^N\Big)$, let us consider for any m in $\mathbb{N}$, the following function

$$\psi_m(t,x) := e^{-\tilde{c}_m(t)}\phi(t, x-\tilde{F}_m(t)) \quad (t,x)\in[0,T]\times\mathbb{R}^N.$$

Since $\tilde{c}_m$ and $\tilde{F}_m$ are smooth and bounded, it is easy to check that ψ_m is in $C_c^\infty\Big([0,T)\times\mathbb{R}^N\Big)$. We can then use ψ_m in Equation (5.8) (with time-dependent A_t and B_t) to show that

$$\int_0^T\int_{\mathbb{R}^N}\Big[\partial_t + \Big(L_t^{\mathrm{ou}}\Big)^*\Big]\psi_m(t,y)u(t,y) + f(t,y)\,dydt + \int_{\mathbb{R}^N}\psi_m(0,y)u_0(y)\,dy = 0.$$

A direct calculation then show that $\psi_m(0,y)=\phi(0,y)$ and

$$\begin{aligned}\Big(L_t^{\mathrm{ou}}\Big)^*\psi_m(t,y) &= e^{-\tilde{c}_m(t)}\Big(L_t^{\mathrm{ou}}\Big)^*\phi(t,y-\tilde{F}_m(t));\\ \partial_t\psi_m(t,y) &= e^{-\tilde{c}_m(t)}\Big[\partial_t\phi(t,y-\tilde{F}_m(t)) - \langle F_m(t), D_y\phi(t,y-\tilde{F}_m(t))\rangle\\ &\qquad - c_m(t)\phi(t,y-\tilde{F}_m(t))\Big].\end{aligned}$$

The above calculations and a change of variable then imply that

$$\begin{aligned}\int_0^T\int_{\mathbb{R}^N}\Big[\Big(\partial_t + \Big(L_t^{\mathrm{ou}}\Big)^*\Big)\phi(t,y) - \langle F_m(t), D_y\phi(t,y)\rangle - c_m(t)\phi(t,y)\Big]\mathcal{T}_m u(t,y)\\ + \phi(t,y)\mathcal{T}_m f(t,y)\,dydt + \int_{\mathbb{R}^N}u_0(y)\phi(0,y)\,dy = 0,\end{aligned}$$

where, analogously to (6.5), we have denoted for any function $\varphi\colon [0,T]\times\mathbb{R}^N\to\mathbb{R}$,

$$\mathcal{T}_m\varphi(t,y) := e^{-\tilde{c}_m(t)}\varphi(t,y+\tilde{F}_m(t)).$$

Following similar arguments exploited in the "existence" part in the proof of Theorem 5.2, i.e. exploiting the compact support of ϕ and the uniform convergence of the coefficients, it is possible to show that the above expression converges, when m goes to infinity, to

$$\int_0^T\int_{\mathbb{R}^N}\Big[\partial_t + \Big(L_t\Big)^*\Big]\phi(t,y)v(t,y) + \mathcal{T}f(t,y)\,dxdt + \int_{\mathbb{R}^N}\phi(0,x)u_0(x)\,dx] = 0$$

and thus, that v is a weak solution of Cauchy problem (6.6). □

Thanks to the previous lemma, we are now able to show the Schauder estimates for the solution v of the Cauchy problem (6.6) and, more importantly, without changing the constant C appearing in Equation (6.2).

Proposition 6.3. *Fixed $T > 0$, $\beta \in (0,1)$, let u_0 in $C^{\alpha+\beta}_{b,d}(\mathbb{R}^N)$, f in $L^\infty\Big(0,T;C^\beta_b(\mathbb{R}^N)\Big)$ and c_0, F_0 in $B_b([0,T])$. Then, the unique solution v of Cauchy Problem (6.6) is in $L^\infty\Big(0,T;C^{\alpha+\beta}_b(\mathbb{R}^N)\Big)$ and it holds that*

$$\|v\|_{L^\infty(C^{\alpha+\beta}_{b,d})} \le C\Big[\|u_0\|_{C^{\alpha+\beta}_{b,d}} + \|f\|_{L^\infty(C^\beta_{b,d})}\Big], \tag{6.7}$$

where $C := C(T,\eta) > 0$ is the same constant appearing in Theorem 5.4.

Proof. We start denoting for simplicity

$$\tilde{c}_0(t) := \int_0^t c_0(s)\,ds \quad \text{and} \quad \tilde{F}_0(t) := \int_0^t F_0(s)\,ds.$$

By Lemma 6.2, we know that if v is a weak solution of Cauchy problem (6.6), then the function

$$u(t,x) := e^{\tilde{c}_0(t)} v(t, x - \tilde{F}_0(t))$$

is the weak solution of Cauchy problem (6.1) with $\tilde{f}$ instead of f, where

$$\tilde{f}(t,x) := e^{\tilde{c}_0(t)} f(t, x - \tilde{F}_0(t)), \quad (t,x) \in (0,T) \times \mathbb{R}^N.$$

Moreover, we have that $\tilde{f}$ is in $L^\infty\Big(0,T;C^\beta_b(\mathbb{R}^N)\Big)$. Considering, if necessary, a smaller time interval $[0,t]$ for some $t \le T$, it is not difficult to check from Proposition 6.1 that

$$\|e^{\tilde{c}_0(t)} v(t, \cdot - \tilde{F}_0(t))\|_{C^{\alpha+\beta}_{b,d}} \le C\Big[\|u_0\|_{C^{\alpha+\beta}_{b,d}} + \sup_{s\in[0,t]} \|e^{\tilde{c}_0(s)} f(s, \cdot - \tilde{F}_0(t))\Big].$$

Using now the invariance of the Hölder norm under translations, we can show that

$$\begin{aligned}\|v(t,\cdot)\|_{C^{\alpha+\beta}_{b,d}} &\le C\Big[e^{-\tilde{c}_0(t)}\|u_0\|_{C^{\alpha+\beta}_{b,d}} + e^{-\tilde{c}_0(t)} \sup_{s\in[0,t]} \|e^{\tilde{c}_0(s)} f(s,\cdot)\Big] \\ &\le C\Big[\|u_0\|_{C^{\alpha+\beta}_{b,d}} + \sup_{s\in[0,t]} \|f(s,\cdot)\Big],\end{aligned}$$

where in the last step we exploited that $\tilde{c}_0(t)$ is non-decreasing. Taking the supremum with respect to t on both sides of the above inequality, we obtain our result. □

Remark 6.1 (About space-time dependent coefficients)**.** We briefly explain here how to extend the Schauder estimates (5.18) to a class of non-linear, space-time dependent operators, whose coefficients are only locally Hölder continuous in space and may be unbounded. Namely, we are interested in operators of the following form:

$$\begin{aligned}L_t\phi(t,x) &:= \langle F(t,x), D_x\phi(x)\rangle \\ &\quad + \int_{\mathbb{R}^d_0} \Big[\phi(x + B\sigma(t,x)z) - \phi(x) - \langle D_x\phi(x), B\sigma(t,x)z\rangle \mathbb{1}_{B(0,1)}(z)\Big]\,\nu(dz),\end{aligned}$$

where B is as in (2.3) and $\sigma\colon [0,T]\times\mathbb{R}^N \to \mathbb{R}^d\otimes\mathbb{R}^d$, $F\colon [0,T]\times\mathbb{R}^N \to \mathbb{R}^N$ are two measurable functions such that $F(t,0)$ is locally bounded in time and σ satisfies assumption [**UE**] at any fixed (t,x) in $[0,T]\times\mathbb{R}^N$.
We would like now the operator L_t to present a similar "dynamical" behaviour as above, i.e. the transmission of the smoothing effect of the Lévy operator to the degenerate components of the system (cf. Example 2.1). For this reason, we suppose the following:

- the drift $F = (F_1, \dots, F_n)$ is such that for any i in $[\![1, n]\!]$, F_i depends only on time and on the last $n - (i-2)$ components, i.e. $F_i(t, x_{i-1}, \dots, x_n)$;
- the matrices $D_{x_{i-1}} F_i(t, x)$ have full rank d_i at any fixed (t, x) in $[0, T] \times \mathbb{R}^N$.

As said before, the functions F and σ are assumed to be only locally Hölder in space, uniformly in time. Namely, there exists a positive constant K_0 such that

$$\mathbf{d}\Big(\sigma(t,x), \sigma(t,y)\Big) \le K_0 \mathbf{d}^{\beta}(x,y); \qquad \mathbf{d}\Big(F_i(t,x), F_i(t,y)\Big) \le K_0 \mathbf{d}^{\beta+\gamma_i}(x,y) \tag{6.8}$$

for any i in $[\![1, n]\!]$, any t in $[0, T]$ and any x, y in $\mathbb{R}^N$ such that $\mathbf{d}(x, y) \le 1$, where

$$\gamma_i := \begin{cases} 1 + \alpha(i-2), & \text{if } i > 1; \\ 0, & \text{if } i = 1. \end{cases} \tag{6.9}$$

We remark in particular that the function F may be unbounded in space.
In order to recover Schauder-type estimates even in this framework, we can follow a perturbative method firstly introduced in [KP10] that allows to exploit the already proven results for time-dependent operators. Let us assume for the moment that σ and F are *globally* Hölder continuous in space, i.e. they satisfy (6.8) for any x, y in $\mathbb{R}^N$. Informally speaking, the method links the operator L_t with the space independent operator L_t defined in (6.4), by "freezing" the coefficients of L_t along a *reference path* $\theta \colon [0, T] \to \mathbb{R}^N$ given by

$$\theta_t := x_0 + \int_{t_0}^{t} F(s, \theta_s)\, ds,$$

for some (t_0, x_0) in $[0, T] \times \mathbb{R}^N$. It is important to highlight that, since F is only Hölder continuous, we need to fix one of the possible paths satisfying the above dynamics. We point out that the deterministic flow θ_t associated with the drift F is introduced precisely to handle the possible unboundedness of F. We could then consider a proxy operator L_t whose coefficients are given by $\sigma_0(t) := \sigma(t, \theta_t)$, $F_0(t) := F(t, \theta_t)$ and

$$\Big[A_t\Big]_{i,j} = \begin{cases} D_{x_{i-1}} F_i(t, \theta_t), & \text{if } j = i-1; \\ 0, & \text{otherwise} \end{cases}$$

In particular, Theorem 6.3 assures the well-posedness and the Schauder estimates for the Cauchy problem associated with L_t.
The final step of the proof would be to expand a solution u of the Cauchy problem associated with L_t around the proxy L_t through a Duhamel-like formula and finally show that the expansion error only brings a negligible contribution so that the Schauder estimates still hold for the original problem.
The a priori estimates for the expansion error are however quite involved (and they are the main reason why we have decided to not show here the complete proof), since they rely on some non-trivial controls in appropriate Besov norms.
In order to deal with coefficients that are only locally Hölder in space, we need in addition to introduce a "localized" version of the above reasoning. It would be necessary to multiply a solution u by a suitable bump function δ that localizes in space along the deterministic flow θ_t that characterizes the proxy. Namely, to fix a smooth function ρ

that is equal to 1 on $B(0,1/2)$ and vanishes outside $B(0,1)$ and define $\delta(t,x) := \rho(x-\theta_t)$. We would then follow the above method but with respect to the "localized" solution

$$v(t,x) \;:=\; \delta(t,x)u(t,x), \quad (t,x) \in [0,T] \times \mathbb{R}^N.$$

We suggest the interested reader to see [CdRHM18a] for a detailed treatise of the argument in the degenerate diffusive setting, [CdRMP20a] in the non-degenerate stable framework or [Mar20] for the precise assumptions on the coefficients.

Chapter 4

Weak well-posedness for degenerate SDEs driven by Lévy processes

Abstract: We study the effects of the propagation of a non-degenerate Lévy noise through a chain of deterministic differential equations whose coefficients are Hölder continuous and satisfy a weak Hörmander-like condition. In particular, we assume some non-degeneracy with respect to the components which transmit the noise. Moreover, we characterize, for some specific dynamics, through suitable counter-examples, the almost sharp regularity exponents that ensure the weak well-posedness for the associated SDE. As a by-product of our approach, we also derive some Krylov-ype estimates for the density of the weak solutions of the considered SDE.

1 Introduction

We investigate the effects of the propagation of a d-dimensional Lévy noise through a chain of $n \geq 2$ differential equations. Namely, we are interested in a degenerate, Lévy-driven stochastic differential equation (SDE in short) of the following form:

$$\begin{cases} dX_t^1 &= [[A_t]_{1,1}X_t^1 + \cdots + [A_t]_{1,n}X_t^n + F_1(t, X_t^1, \ldots, X_t^n)]\, dt + \sigma(t, X_{t-}^1, \ldots, X_{t-}^n) dZ_t, \\ dX_t^2 &= [[A_t]_{2,1}X_t^1 + \cdots + [A_t]_{2,n}X_t^n + F_2(t, X_t^2, \ldots, X_t^n)]\, dt, \\ dX_t^3 &= [[A_t]_{3,2}X_t^2 + \cdots + [A_t]_{2,n}X_t^n + F_3(t, X_t^3, \ldots, X_t^n)]\, dt, \\ \vdots \\ dX_t^n &= \Big[[A_t]_{n-1,n}X_t^{n-1} + [A_t]_{n,n}X_t^n + F_n(t, X_t^n)\Big]\, dt, \end{cases} \tag{1.1}$$

where for $i \in [\![1, n]\!]$ ($[\![\cdot, \cdot]\!]$ denotes the set of all the integers in the interval), X_t^i is $\mathbb{R}^{d_i}$ valued, with $d_1 = d$ and $d_i \geq 1$, $i \in [\![2, n]\!]$. Set $N = \sum_{i=1}^n d_i$. We suppose that the $F_i \colon [0, +\infty) \times \mathbb{R}^{\sum_{j=i}^n d_j} \to \mathbb{R}^d$, $\sigma \colon [0, +\infty) \times \mathbb{R}^N \to \mathbb{R}^d \otimes \mathbb{R}^d$ are Borel and respectively locally bounded and uniformly elliptic and bounded.

We also assume the entries $([A_t]_{ij})_{1 \leq i \leq n,\ i-1 \leq j \leq n}$ are Borel bounded and such that the blocks $[A_t]_{i,i-1}$ in $\mathbb{R}^{d_i} \otimes \mathbb{R}^{d_{i-1}}$, $2 \leq i \leq n$ have rank d_i, uniformly in time. This is a kind

of non-degeneracy assumption which can be viewed as weak Hörmander-like condition. It actually precisely allows the noise to propagate into the system.

Eventually, the noise $\{Z_t\}_{t\geq 0}$ belongs to a class of d-dimensional, symmetric, Lévy processes with suitable properties. In particular, to handle non trivial diffusion coefficients, we will assume that the Lévy measure of $\{Z_t\}_{t\geq 0}$ is absolutely continuous with respect to the Lévy measure of a rotationally invariant α-stable process (with α in $(1,2]$) and its Radon-Nikodym derivative enjoys some natural properties. The class of processes $\{Z_t\}_{t\geq 0}$ we can consider, includes for example, the tempered, the layered or the relativistic α-stable processes. In the case of an additive noise, cylindrical stable processes could be handled as well.

Here, the major issue is linked with the specific degenerate framework we consider. Indeed, the noise only acts on the first component of the dynamics (1.1) and it implies, in particular, that the random perturbation on the i-th line of SDE (1.1) only comes from the previous $(i-1)$-th one, through the non-degeneracy of the matrixes $[A_t]_{i,i-1}$. Hence, the smoothing effect associated with the Lévy noise decreases along the chain, making thus more and more difficult to regularize by noise the furthest lines of Equation (1.1).
We nevertheless prove the weak well-posedness, i.e. the existence and uniqueness in law, for the above SDE (1.1) when the drift $F = (F_1, \dots, F_n)$ and σ lie in a suitable anisotropic Hölder space with multi-indices of regularity. We assume that F_1 and σ have spatial Hölder regularity $\beta^1 > 0$ with respect to the j-th variable. We highlight already that we could have considered different regularity indexes β^1_j for the regularity of F_1 with respect to the j-th variable. We keep only one common index for notational simplicity. We also suppose that for fixed $j \in [\![2, n]\!]$, $(F_2, \cdots, F_j)$ has Hölder regularity β^j with respect to the j-th variable, where:

$$\beta^j \in \Big(\frac{1+\alpha(j-2)}{1+\alpha(j-1)}; 1\Big].$$

We indeed recall that from the dynamics (1.1) the variable x_j does not appear in the chain after level j.

Furthermore, we will show through suitable counter-examples that the above threshold for the regularity exponents β^j is "almost" sharp for a perturbation of the jth level of the chain. Such counter-examples are based on Peano-type dynamics adapted to our degenerate, fractional framework.

Models of the form (1.1) naturally appear in various scientific contexts: in physics, for the analysis of anomalous diffusions phenomena or for Hamiltonian models in turbulent regimes (see e.g. [BBM01], [CPKM05], [EPRB99]); in mathematical finance and econometrics, for example in the pricing of Asian options (see e.g. [JYC09], [Bro01], [BNS01]). In particular, models that consider Lévy noises, such as SDE (1.1), seem more natural and realistic for many applications since they allow the presence of jumps.

Weak well posedness for non-degenerate stable SDEs. The topic of weak well-posedness for non-degenerate (i.e. $d = N$) SDEs of the form:

$$X_t = x + \int_0^t F(X_s)ds + Z_t, \quad t \geq 0, \tag{1.2}$$

where $\{Z_t\}_{t\geq 0}$ is a symmetric α-stable process on $\mathbb{R}^N$, has been widely studied in the last decades, especially in the diffusive, local setting, i.e. when $\alpha = 2$ and $\{Z_t\}_{t\geq 0}$ is a Brownian Motion, and it is now quite well-understood. We can first refer to the seminal work [SV79] where the Authors considered additionally a multiplicative noise with bounded drift and non-degenerate, continuous in space diffusion coefficient. We recall moreover that in the framework of (1.2) with bounded drift, strong uniqueness also holds (cf. [Ver81]).
SDEs like (1.2) with a proper α-stable process ($\alpha < 2$) were firstly investigated in [TTW74] where the weak well-posedness was obtained for the one-dimensional case when the drift F is bounded, continuous and the Lévy exponent Φ of $\{Z_t\}_{t\geq 0}$ satisfies $\Re\Phi(\xi)^{-1} = 0(1/|\xi|)$ if $|\xi| \to \infty$. The multidimensional case ($d > 1$) can be similarly obtained following [Kom83] if the drift is bounded, continuous and the law of $\{Z_t\}_{t\geq 0}$ admits a density with respect to the Lebesgue measure on $\mathbb{R}^d$. Equations as (1.2) with drift in some suitable L^p-spaces and non-degenerate noise were also considered in [Jin18] (see also the references therein). We can eventually quote the recent work by Krylov who obtained even strong uniqueness for Brownian SDEs with drifts in critical L^p-spaces, see [Kry21].

In recent years, SDEs driven by singular (distributional) drift have gained a lot of interest, especially in the Brownian setting, where they arise as a model for diffusions in random media (see e.g. [FRW03],[FRW04],[FIR17], [DD16], [CC18]).
In the non-local α-stable framework, a first work to appear was [ABM20] where the authors considered the one-dimensional case with a time-homogeneous drift of (negative) Hölder regularity strictly greater than $(1-\alpha)/2$. We remark that in the one-dimensional framework, the regularity thresholds on the drift are the same for the strong and the weak well-posedness, since it is possible to exploit local time arguments (see also [BC01] in the diffusive setting). On the same side, the almost simultaneous works [LZ19] and [CdRM20a] take into account time-homogeneous and time-inhomogeneous, respectively, distributional drift in general Besov spaces with suitable conditions on the parameters. These results rely on Young integrals in order to give a meaningful sense to the dynamics. Beyond the Young regime, we instead refer to [KP20] where techniques such as paracontrolled products (which have also been popular in the recent developments in the SPDE theory) are exploited to analyze the martingale problem associated with a time-inhomogeneous drift of regularity index strictly greater than $(2 - 2\alpha)/3$.

Moreover, we would like to remark that the above works concerned the so-called *sub-critical* case, i.e. when $\alpha > 1$. Indeed, SDEs like (1.2) are much more difficult to handle if $\alpha \leq 1$ since in this case, the noise does not dominate the system for small time scales. Two recent works along this path are [Zha19] and [CdRMP20b] where the authors consider $\alpha < 1$, $(1-\alpha)$-Hölder drift F and $\alpha = 1$, continuous drift, respectively. We also mention that for Hölder drifts, the well-posedness of the associated martingale

problem can be obtained following [MP14] if F is bounded or through the Schauder estimates given in [CdRMP20a] when F is unbounded.

Regularization by noise in a degenerate setting. All the above results present a common phenomenon that, following the terminology in [Fla11], is usually called *regularization by noise*. This occurs when a deterministic ODE is ill-posed (for example if the drift is less than Lipschitz) but its stochastic counterpart (SDE) is well posed in a strong or a weak sense.
To obtain such phenomenon, the noise plays a fundamental role. A usual assumption is that the noise should act on every line of the dynamics, regularizing the coefficients. It is then clear that in our degenerate framework, when the noise acts only on the first component of the chain (1.1), the situation is even more delicate. In order to obtain some kind of regularization effect in this case, we need that the noise propagates through the system, reaching all the lines of Equation (1.1). A typical assumption ensuring such type of behaviour is the so-called Hörmander condition for hypoellipticity (cf. [Hör67]). From the structure of the equation (1.1) at hand, we will consider a *weak* type Hörmander condition, i.e. up to some regularization of the diffusion coefficient, the drift is needed to span the space through Lie bracketing.

In the Hamiltonian setting $n = 2$, when $\{Z_t\}_{t\geq 0}$ is a Brownian Motion and for a more general, non-linear, drift than in (1.1) which still satisfies a weak Hörmander type condition, Chaudru de Raynal showed in [CdR18] that the associated SDE is weakly well-posed as soon as the drift is Hölder continuous in the degenerate variable with regularity index strictly greater than 1/3. It was also established through an appropriate counter-example, that the 1/3-threshold is (almost) sharp for the second component of the drift. Such a result has been then extended in [CdRM20b] in order to consider the more general case of n oscillators. Therein, the regularity thresholds that ensure weak uniqueness also depend on the variable and the level of the chain. This seems intuitively clear, the further the variable in the oscillator chain, the larger its typical time scale, the weaker the regularity needed to regularize components which are above that variable in the chain. Also, some corresponding Krylov type estimates, giving existence and integrability properties of the density of the SDE are derived. We can mention as well the recent work by Gerencsér [Ger20] who obtain similar regularization properties for the iterated time integrals of a fractional Brownian motion.

In the jump case, the situation is much more delicate. Within the proper regularization by noise framework (when the coefficients are less than Lipschitz continuous), we cite [HM16] where the Authors showed the weak well-posedness for (1.1) with $F = 0$ and a Hölder continuous diffusion coefficient, under some constraints on the dimensions d,n. In that framework, the Authors obtained as well same point-wise density estimates. The driving noises considered were stable and tempered stable processes.
Finally, we mention that it is possible to derive the weak well-posedness of dynamics (1.1) via the martingale formulation, exploiting the Schauder estimates given in [HWZ20] for the kinetic model ($n = 2$). In that work, the Authors actually characterized conditions for strong uniqueness, using Littlewood-Paley decomposition techniques.

We will here proceed through a perturbative approach. Namely, we will expand the

formal generator associated with (1.1) around the one of a well understood process, with possibly time inhomogeneous coefficients which are anyhow frozen in space. We will call such a process a *proxy*. The most natural candidate to be a *proxy* for (1.1) is a degenerate Ornstein-Uhlenbeck process. In the case of time homogeneous coefficients, Priola and Zabczyk established in [PZ09] existence of the density for such processes under the same previously indicated non-degeneracy conditions on the matrix A (which turn out to be equivalent in that setting to the well known Kalman condition).

Intrinsic difficulties associated with large jumps. When Z is a strict stable process, the density of the corresponding degenerate Ornstein-Uhlenbeck process can somehow be related to the one of a multi-scale stable process which has however a very singular associated *spectral measure* (spherical part of the α-stable Lévy measure) on $\mathbb{S}^{N-1}$, see e.g. [HM16], [HMP19] and Proposition 2.10 below. From Watanabe [Wat07], it is known that the tails of stable densities are highly related to the nature of this spectral measure. Specifically, the concentration properties worsen when the measure becomes singular. This renders delicate the characterization of the smoothing properties for the proxy, especially when it depends on parameters and that one would like to obtain estimates which are uniform w.r.t. those parameters (see Proposition 2.11 and Section 2.2 below).

Even for smooth coefficients, the stable like jump setting is much more delicate to establish the existence of the density for (degenerate) SDEs. For multiplicative noises, we cannot indeed rely on the flow techniques considered in [BGJ87] or [Kun19] in the non-degenerate case, and Léandre in the degenerate one, see [Léa85],[Léa88]. Still for smooth coefficients, we can refer to the work of Zhang [Zha14] who obtained existence and smoothness results for the density of equations of type (1.1) in arbitrary dimension, for a possibly more general non linear drift, still satisfying a weak Hörmander type condition when the driving process is a rotationally invariant stable process. The strategy therein is based on the *subordinated* Malliavin calculus, which consists in applying the *usual* Malliavin calculus techniques on a Brownian motion observed along the path of an independent α-stable subordinator. In whole generality a *complete* version of the Hörmander theorem in the jump case seems to lack. We can refer to the work by Cass [Cas09] who gets smoothness of the density in the weak Hörmander framework under technical restrictions.

1.1 Complete model and assumptions

Let us now specify the assumptions on equation (1.1) that we rewrite in the shortened form:

$$dX_t \,=\, G(t, X_t)dt + B\sigma(t, X_{t-})dZ_t, \quad t \geq 0, \tag{1.3}$$

where B is the embedding from $\mathbb{R}^d$ to $\mathbb{R}^N$ given in matricial form as

$$B := \Big(I_{d\times d}, \quad 0_{d\times(N-d)},\Big)^t$$

and $G(t,x) = A_t x + F(t,x)$ with:

$$A_t := \begin{pmatrix} [A_t]_{1,1} & \dots & \dots & \dots & [A_t]_{1,n} \\ [A_t]_{2,1} & [A_t]_{2,2} & \dots & \dots & [A_t]_{2,n} \\ 0 & [A_t]_{3,2} & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & [A_t]_{n,n-1} & [A_t]_{n,n} \end{pmatrix}. \tag{1.4}$$

A classical assumption in this degenerate framework (cf. [SV79], [Kry04], [CdRM20b]) is the *uniform ellipticity* of the underlying non-degenerate component of the diffusion matrix at any fixed space-time point. Namely,

[UE] There exists a constant $\eta > 1$ such that for any $t \geq 0$ and any x in $\mathbb{R}^N$, it holds that

$$\eta^{-1}|\xi|^2 \leq \sigma(t,x)\xi \cdot \xi \leq \eta|\xi|^2, \quad \xi \in \mathbb{R}^d,$$

where "$\cdot$" stands for the inner product on the smaller space $\mathbb{R}^d$.
We will suppose that the drift $G(t,x) = A_t x + F(t,x)$ has a particular "upper diagonal" structure and its sub-diagonal elements are linear and non-degenerate, i.e.

[H]
- $F = (F_1, \dots, F_n)\colon [0,\infty) \times \mathbb{R}^N \to \mathbb{R}^N$ is such that F_i depends only on time and on the last $n-(i-1)$ components, i.e. $F_i(t, x_i, \dots, x_n)$, for any i in $[\![1,n]\!]$;
- $A\colon [0,\infty) \to \mathbb{R}^N \otimes \mathbb{R}^N$ is bounded and the blocks $[A_t]_{i,j} \in \mathbb{R}^{d_i} \otimes \mathbb{R}^{d_j}$ in (1.4) are such that

$$[A_t]_{i,j} = \begin{cases} \text{is non-singular (i.e. it has rank } d_i\text{) uniformly in } t, \text{ if } j = i-1; \\ 0, \text{ if } j < i-1. \end{cases}$$

Clearly, n is in $[\![1,N]\!]$ and $n = 1$ if and only if $d = N$, i.e. if the dynamics is non-degenerate.
In the linear framework ($F = 0$) and for constant diffusion coefficients ($\sigma(t,x) = \sigma$), this last assumption can be seen as a Hörmander-type condition, ensuring the hypoellipticity of the infinitesimal generator associated with the process $\{X_t\}_{t\geq 0}$, which is in this setting equivalent to the Kalman condition, see e.g. [PZ09]. We highlight however that in our framework, the "classic" Hörmander assumption (cf. [Hör67]) cannot be considered, due to the low regularity of the coefficients we will consider in (1.3) (see Theorem 2.6). This prevents us from explicitly calculating the commutators.
In Equation (1.3) above, $\{Z_t\}_{t\geq 0}$ is a d-dimensional, symmetric and adapted Lévy process with respect to some stochastic basis $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq 0}, \mathbb{P})$. We recall that a d-valued Lévy process is a stochastically continuous process on $\mathbb{R}^d$ starting from zero and such that its increments are independent and stationary. Moreover, it is well-known (see e.g. [Sat13]) that any Lévy process admits a càdlàg modification, i.e. a right continuous modification having left limits $\mathbb{P}$-almost surely. We will always assume to have chosen such a version.

A fundamental tool in the analysis of Lévy processes is given by the Lévy-Kitchine formula (see for instance [Jac01]) that allows us to represent the Lévy symbol $\Phi(\xi)$ of $\{Z_t\}_{t\geq 0}$, given by

$$\mathbb{E}[e^{i\xi\cdot Z_t}] \,=\, e^{t\Phi(\xi)}, \quad \xi\in\mathbb{R}^d$$

in terms of the generating triplet (b,Σ,ν) as:

$$\Phi(\xi) \,=\, ib\cdot\xi - \frac{1}{2}\Sigma\xi\cdot\xi + \int_{\mathbb{R}^d_0}\Big(e^{i\xi\cdot z} - 1 - i\xi\cdot z\mathbb{1}_{B(0,1)}(z)\Big)\,\nu(dy), \quad \xi\in\mathbb{R}^d,$$

where b is a vector in $\mathbb{R}^d$, Σ is a symmetric, non-negative definite matrix in $\mathbb{R}^d\otimes\mathbb{R}^d$ and ν is a Lévy measure on $\mathbb{R}^d_0 := \mathbb{R}^d\smallsetminus\{0\}$, i.e. a σ-finite measure on $\mathcal{B}(\mathbb{R}^d_0)$, the Borel σ-algebra on $\mathbb{R}^d_0$, such that $\int(1\wedge|z|^2)\,\nu(dz)$ is finite. In particular, any Lévy process is completely determined by its generating triplet (b,Σ,ν).
Importantly, we point out already that a change on the truncation set $B(0,1)$ for the Lévy-Kitchine formula does not affect the formulation of the Lévy symbol Φ, since we assumed ν to be symmetric. Namely, given a threshold $c>0$, the Lévy symbol $\Phi(\xi)$ of $\{Z_t\}_{t\geq 0}$ could be also represented as

$$\Phi(\xi) \,=\, ib\cdot\xi - \frac{1}{2}\Sigma\xi\cdot\xi + \int_{\mathbb{R}^d_0}\Big(e^{i\xi\cdot z} - 1 - i\xi\cdot z\mathbb{1}_{B(0,c)}(z)\Big)\,\nu(dz), \quad \xi\in\mathbb{R}^d, \tag{1.5}$$

where b, Σ and ν are as above. Here, we only consider pure jump processes, i.e. $\Sigma=0$. Indeed, the more general case, where a Gaussian component is considered, can be obtained from already existing results (cf. [CdRM20b]).
We will suppose moreover that, additionally to the symmetry, the Lévy measure ν of $\{Z_t\}_{t\geq 0}$ satisfies the following *non-degeneracy condition*:

[ND] there exist a Borel function $Q\colon\mathbb{R}^d\to[0,\infty)$ such that

- ess-sup$\{Q(z)\colon z\in\mathbb{R}^d\}<+\infty$;
- there exist $r_0>0$ and $c>0$ such that $Q(z)\geq c$ and Lipschitz continuous in $B(0,r_0)$;
- there exists $\alpha\in(1,2)$ and a finite, non-degenerate measure μ on $\mathbb{S}^{d-1}$ such that
$$\nu(\mathcal{A}) \,=\, \int_0^\infty\int_{\mathbb{S}^{d-1}}\mathbb{1}_{\mathcal{A}}(rs)Q(rs)\,\mu(ds)\frac{dr}{r^{1+\alpha}}, \quad \mathcal{A}\in\mathcal{B}(\mathbb{R}^d_0),$$

where $\mathcal{B}(\mathbb{R}^d_0)$ stands for the Borelian σ-field on $\mathbb{R}^d_0$. We recall moreover that a spherical measure μ on $\mathbb{S}^{d-1}$ is non-degenerate (in the sense of Kolokoltsov [Kol00]) if there exists a constant $\tilde\eta\geq 1$ such that

$$\tilde\eta^{-1}|\xi|^\alpha \,\leq\, \int_{\mathbb{S}^{d-1}}|\xi\cdot s|^\alpha\,\mu(ds) \,\leq\, \tilde\eta|\xi|^\alpha, \quad \xi\in\mathbb{R}^d. \tag{1.6}$$

Since any α-stable Lévy measure can be decomposed into a spherical part μ on $\mathbb{S}^{d-1}$ and a radial part $r^{-(1+\alpha)}dr$ (see e.g. Theorem 14.3 in [Sat13]), assumption [**ND**] roughly states that the Lévy measure of $\{Z_t\}_{t\geq 0}$ is absolutely continuous with respect to the non-degenerate (in the sense of (1.6)), Lévy measure of a α-stable process and that

their Radon-Nikodym derivative is given by the function Q. From this point further, we will denote such a Lévy measure by $\nu_\alpha(dz) := \mu(ds)r^{-(1+\alpha)}dr$ with $z = rs$.
In order to deal with a possibly multiplicative noise, i.e. in the presence of a space-inhomogeneous diffusion coefficient σ in Equation (1.3), we will need the following:

[AC] If $x \to \sigma(t,x)$ is non-constant for some $t \geq 0$, then the measure ν_α is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^N$ with Lipschitz Radon-Nykodim derivative.

Assumptions [**ND**] and [**AC**] together imply in particular that in the multiplicative case, the Lévy measure ν of $\{Z_t\}_{t\geq 0}$ can be decomposed as

$$\nu(dz) \;=\; Q(z)\frac{g(\frac{z}{|z|})}{|z|^{d+\alpha}}dz, \tag{1.7}$$

for some Lipschitz function $g\colon \mathbb{S}^{d-1} \to \mathbb{R}$.
At this point, we would like to remark that no regularity is assumed for the Lévy measure ν of $\{Z_t\}_{t\geq 0}$ in the additive framework (or more generally, for a space-independent σ). In particular, the measure μ in condition [**ND**] may not be absolutely continuous with respect to the Lebesgue measure on $\mathbb{S}^{d-1}$. Indeed, our model can also include very singular (with respect to the Lebesgue measure) examples such as the cylindrical α-stable process associated with $\mu = \sum_{i=1}^d \frac{1}{2}(\delta_{e_i} + \delta_{-e_i})$. See e.g. [BC06] for more details.
From this point further, we always assume that the above hypotheses on the coefficients are satisfied.

We would like to conclude the introduction with some comments concerning our assumptions with particular reference with our previous works.
In [Mar21], the Schauder estimates, an important analytical first step for proving the well-posedness of SDEs, has been showed for degenerate Ornstein-Uhlenbeck operators driven by a more general class of Lévy noises. It also includes, for example, the asymmetric version of the stable-like noises we consider in this work. We start highlighting that a similar family of noises could not have been introduced here, as in [Mar20], due to the non-linear structure of our problem and, especially, our technique of proof through a perturbative approach. Indeed, it requires more delicate regularizing properties for the involved operators and, in particular, a compatibility between some proxy and the original operator, seen as a perturbation of the first one.
Here, we have followed a backward perturbative approach as firstly introduced by McKean-Singer in [MS67]. This terminology comes from the fact that the underlying proxy process will be associated with a backward in time flow. This method appears more natural for proving weak uniqueness in a degenerate $L^p - L^q$ framework (cf. [CdRM20b] in the diffusive case). Roughly speaking, it only requires controls on the gradients (in a weak sense) for the solutions of the associated PDE in order to apply the inversion technique on the infinitesimal generator. However, we are confident that the Schauder estimates presented in [Mar20] could be extended to the class of noises we consider here. Relying on them, we could have then proven the uniqueness in law for dynamics such as (1.3). This method appears really involved and long since it structurally requires to establish pointwise estimates for the first order derivatives with respect to the degenerate components of the dynamics. Another useful advantage of

the backward perturbative approach is that it allows us to show Krylov-type estimates on the solution process X_t of SDE (1.3). These kind of controls seems of independent interest and new for random dynamics involving degenerate stable-like noises.

The drawback of our approach is that it leads to a specific structure in Equation (1.3), given by assumption **[H]**. Namely, we cannot consider drift of the form $F_i(x) = F_i(x_{i-1}, \cdots, x_n)$ with non-linear dependence w.r.t. x_{i-1}, variable which transmits the noise. This case is often investigated for Brownian noises (see e.g. [DM10], [CdRM20b]). This feature is specifically linked to the structure of the joint law of a stable process and its iterated integrals which generate a multi-scale stable process with highly singular associated spectral measure, see e.g. Proposition 2.10 and Remark 2.2 below or [HM16]. Similar issues constrain us to assume in the multiplicative noise case that the driving process has an absolutely continuous spectral measure with respect to the Lebesgue measure on $\mathbb{S}^{d-1}$. This precisely allows us to get estimates which will be uniform with respect to the parameters for the considered class of proxys.

Main Driving Processes Considered. Here, we highlight that assumption **[ND]** applies to a large class of Lévy processes on $\mathbb{R}^d$. As already pointed out in [SSW12], it holds for the following families of stable-like examples with $\alpha \in (0, 2)$:

1. Stable process [Sat13]:
$$Q(z) \;=\; 1;$$

2. Truncated stable process with $r_0 > 0$ [KS08]:
$$Q(z) \;=\; \mathbb{1}_{(0,r_0]}(|z|);$$

3. Layered stable process with $\beta > \alpha$ and $r_0 > 0$ [HK07]:
$$Q(z) \;=\; \mathbb{1}_{(0,r_0]}(|z|) + \mathbb{1}_{(r_0,\infty)}(|z|)|z|^{\alpha-\beta};$$

4. Tempered stable process [Ros07] with $Q(z) = Q(rs)$ such that for all s in $\mathbb{S}^{d-1}$,
$$r \to Q(rs) \text{ is completely monotone, } Q(0) > 0 \text{ and } \lim_{r\to+\infty} Q(rs) = 0.$$

5. Relativistic stable process [CMS90], [BMR09]:
$$Q(z) \;=\; (1 + |z|)^{(d+\alpha-1)/2} e^{-|z|};$$

6. Lamperti process with $f\colon \mathbb{S}^{d-1} \to \mathbb{R}$ even such that $\sup f(s) < 1 + \alpha$ [CPP10]:
$$Q(z) \;=\; \exp\Big(|z| f(\frac{z}{|z|})\Big)\Big(\frac{|z|}{e^{|z|} - 1}\Big)^{1+\alpha}, \quad z \in \mathbb{R}^d_0.$$

Organization of Paper. The article is organized as follows. In Section 2, we introduce some useful notations and we present the associated martingale problem. In particular, we state there our main results. Section 3 contains all the associated analytical tools that allow to derive our results. Namely, we follow there a perturbative approach, considering a suitable linearization of our dynamics (1.3) around a Cauchy-Peano flow which takes into account the deterministic part of our model (corresponding to (1.3) with $\sigma = 0$). Section 4 is then dedicated to prove the well-posedness of the associated martingale problem, exploiting the analytical results given in Section 3. In Section 5, we finally construct an "ad hoc" Peano counter-example to the uniqueness in law for SDE (1.3).

2 Basic notations and main results

We start recalling some useful notations we will need below. In the following, C will denote a generic *positive* constant. It may change from line to line and it will depend only on the parameters appearing in the previously stated assumptions, as for instance: $d, N, \alpha, \eta, b, g, r_0, \mu$. We will explicitly specify any other dependence that may occur.
Given a function $f\colon \mathbb{R}^N \to \mathbb{R}$, we denote by $Df(x)$, and $D^2f(x)$ the first and second Fréchet derivative of f at a point x in $\mathbb{R}^N$ respectively, when they exist. We denote by $B_b(\mathbb{R}^N)$ the family of all the Borel and bounded functions $f\colon \mathbb{R}^N \to \mathbb{R}$. It is a Banach space endowed with the supremum norm $\|\cdot\|_\infty$. We also consider its closed subspace $C_b(\mathbb{R}^N)$ consisting of all the continuous functions. Moreover, $C_c^\infty(\mathbb{R}^N) \subseteq C_b(\mathbb{R}^N)$ denotes the space of smooth functions with compact support.

We now recall two correlated definitions of solution associated with SDE (1.3). Let us consider fixed μ in $\mathcal{P}(\mathbb{R}^N)$, the family of the probability measures on $\mathbb{R}^N$ and an initial time $t \geq 0$.

Definition 2.1. A weak solution of SDE (1.3) with starting condition (t, μ) is a N-dimensional, càdlàg, adapted process $\{X_s\}_{s\geq 0}$ on some stocastic base $(\Omega, \mathcal{F}, \{\mathcal{F}_s\}_{s\geq 0}, \mathbb{P})$ such that

- the law of X_t is μ;
- there exists a d-dimensional, adapted Lévy process $\{Z_s\}_{s\geq t}$ satisfying [**ND**] and [**AC**] such that

$$X_s \;=\; X_t + \int_t^s G(u, X_u)\, du + \int_t^s \sigma(u, X_{u-}) B\, dZ_u, \quad s \geq t,\ \mathbb{P}\text{-a.s.} \tag{2.1}$$

To state our second definition, we need to consider the infinitesimal generator $\partial_s + L_s$ (formally) associated with the solutions of SDE (1.3). Noticing that the term involving the constant drift b can be absorbed in the expression for G without loss of generality, the operator L_s can be represented for any ϕ in $C_c^\infty(\mathbb{R}^N)$ as

$$\begin{aligned} L_s\phi(s,x) \;&:=\; \langle G(s,x), D_x\phi(x)\rangle + \mathcal{L}_s\phi(s,x) \\ &:=\; \langle G(s,x), D_x\phi(x)\rangle + \int_{\mathbb{R}_0^d} \Big[\phi(x + B(s,x)z) - \phi(x)\Big]\, \nu(dz), \end{aligned} \tag{2.2}$$

where $\langle \cdot, \cdot \rangle$ denotes the inner product on the bigger space $\mathbb{R}^N$ and, for brevity, $B(s,x) := B\sigma(s,x)$. As done in [Pri15b], we introduce the following definition:

Definition 2.2. A solution of the martingale problem for $\partial_s + L_s$ with initial condition (t,μ) is an N-dimensional, càdlàg process $\{X_s\}_{s\geq t}$ on some probability space $(\Omega, \mathcal{F}, \mathbb{P})$ such that

- the law of X_t is μ;
- for any ϕ in $\mathrm{dom}\Big(\partial_s + L_s\Big)$, the process

$$\left\{\phi(s,X_s) - \phi(t,X_t) - \int_t^s \Big(\partial_u + L_u\Big)\phi(u,X_u)\,du\right\}_{s\geq t}$$

  is a $\mathbb{P}$-martingale with respect to the natural filtration $\{\mathcal{F}_s^X\}_{s\geq 0}$ of the process $\{X_s\}_{s\geq 0}$.

We can now recall some known results that enlighten the link between the two definitions presented above. For a more thorough analysis on the topic of martingale problems in a rather abstract and general framework, we refer to Chapter 4 in [EK86].
Given a solution $\{X_s\}_{s\geq 0}$ of SDE (1.3), an application of the Itô formula immediately shows that the process $\{X_s\}_{s\geq 0}$ is a solution of the martingale problem for $\partial_s + L_s$ with initial condition (t,μ), too.
On the other hand, if there exists a solution $\{X_s\}_{s\geq 0}$ of the martingale problem for $\partial_t + L_t$ with initial condition (t,μ), it is possible to construct an "enhanced" filtered probability space $(\tilde{\Omega}, \tilde{\mathcal{F}}, \{\tilde{\mathcal{F}}\}_{s\geq 0}, \tilde{\mathbb{P}})$ on which there exists a solution $\{\tilde{X}_s\}_{s\geq 0}$ of the SDE (1.3). Moreover, the two processes $\{X_s\}_{s\geq t}$ and $\{\tilde{X}_s\}_{s\geq t}$ have the same law (See, for more details, [Kur11]). Thus, it holds that:

Proposition 2.3. *Let μ be in $\mathcal{P}(\mathbb{R}^N)$ and $t \geq 0$. The existence of a weak solution for SDE* (1.3) *with initial condition (t,μ) is equivalent to the existence of a solution to the martingale problem for $\partial_s + L_s$ with initial condition (t,μ).*

We can now move on the notion of uniqueness associated with our problem.

Definition 2.4. We say that weak uniqueness holds for the SDE (1.3) with initial condition (t,μ) if any two solutions $\{X_s\}_{s\geq 0}$, $\{Y_s\}_{s\geq 0}$ of SDE (1.3) with initial condition (t,μ) have same finite dimensional distributions. In particular, we say that SDE (1.3) is weakly well-posed if for any μ in $\mathcal{P}(\mathbb{R}^N)$ and any $t \geq 0$, there exists a unique weak solution of SDE (1.3) with initial condition (t,μ).

Since the definition above takes into account only the law of the solutions $\{X_s\}_{s\geq t}$, $\{Y_s\}_{s\geq t}$, they may, in general, have been defined on different stochastic bases or with respect to two different underlying Lévy processes. The definition of uniqueness for a solution of the martingale problem for $\partial_s + L_s$ can be stated similarly.

It is not difficult to check that the uniqueness of the martingale problem for $\partial_s + L_s$ with initial condition (t,μ) implies the weak uniqueness of the SDE (1.3). Furthermore, it has been shown in [Kur11], Corollary 2.5 that the converse is also true.

Proposition 2.5. *Let μ be in $\mathcal{P}(\mathbb{R}^N)$ and $t \geq 0$. Then, weak uniqueness holds for SDE* (1.3) *with initial condition (t, μ) if and only if uniqueness holds for the martingale problem for $\partial_s + L_s$ with initial condition (t, μ).*

Thanks to Propositions 2.3 and 2.5, we can conclude that the two approaches, i.e. the martingale formulation and the dynamics given in (1.3), are equivalent in specifying a Lévy diffusion process on $\mathbb{R}^N$. We recall however that a third, yet equivalent, method is given by the forward Fokker-Plank equation governing the law of the process. We will not explicitly define it since we will not exploit it afterwards (see, for more details, e.g. [Fig08], [LBL08]).
From now on, we write x in $\mathbb{R}^N$ as $x = (x_1, \dots, x_n)$ where $x_i = (x_i^1, \dots, x_i^{d_i})$ is in $\mathbb{R}^{d_i}$ for any i in $[\![1, n]\!]$.
We can now state our main theorem.

Theorem 2.6. *For any j in $[\![1, n]\!]$, let β^j be an index in $(0, 1]$ such that*

- *$x_j \to \sigma(t, x_1, \dots, x_j, \dots, x_n)$ is β^1-Hölder continuous, uniformly in t and in x_i for $i \neq j$;*
- *$x_j \to F_1(t, x_1, \dots, x_j, \dots, x_n)$ is β^1-Hölder continuous, uniformly in t and in x_i for $i \neq j$;*
- *$x_j \to F_i(t, x_i, \dots, x_j, \dots, x_n)$ is β^j-Hölder continuous, uniformly in t and in x_k, for $k \neq j$ and $2 \leq i \leq j$.*

Additionally, we suppose that there exists $K \geq 1$ such that $|F_i(t, 0)| \leq K$ for any i in $[\![1, n]\!]$ and any $t \geq 0$. Then, the SDE (1.3) *is weakly well-posed if*

$$\beta^j > \frac{1 + \alpha(j-2)}{1 + \alpha(j-1)}, \ j \geq 2. \tag{2.3}$$

Theorem 2.6 above will follow from Propositions 2.3 and 2.5, once we have shown that under the same assumptions, there exists a unique weak solution to the martingale problem for $(\partial_s + L_s, \delta_x)$ at any x in $\mathbb{R}^N$.
As a by-product of our method of proof, we have been able to show a Krylov-type estimates for the solutions of SDE (1.3). For notational convenience, we will say that two real numbers $p > 1$, $q > 1$ satisfy Condition $(\mathscr{C})$ when the following inequality holds:

$$\Big(\frac{1-\alpha}{\alpha} N + \sum_{i=1}^{n} i d_i\Big) \frac{1}{q} + \frac{1}{p} < 1. \tag{$\mathscr{C}$}$$

Roughly speaking, such a threshold guarantees the necessary integrability in time with respect to the associated intrinsic scale of the system when considering the $L_t^p - L_x^q$ theory (see Equation (2.43) for more details). Furthermore, when considering the homogeneous case, i.e. when all the components of the system has the same dimension ($d_i = d$ and $N = nd$), condition $(\mathscr{C})$ can be rewritten in the following, clearer, way:

$$\left(\frac{2 + \alpha(n-1)}{\alpha}\right) \frac{nd}{q} + \frac{2}{p} < 2.$$

In particular, taking $\alpha = 2$ above, we find the same threshold appearing in [CdRM20b] for the diffusive setting. We highlight moreover that our thresholds can be seen as a natural extension of the ones appearing in [KR05] in the non-degenerate, Brownian setting.

Corollary 2.7. *Under the same assumptions of Theorem 2.6, let $T > 0$ and $p > 1$, $q > 1$ such that Condition $(\mathscr{C})$ holds. Then, there exists a constant $C := C(T, p, q)$ such that for any f in $L^p\Big(0, T; L^q(\mathbb{R}^N)\Big)$, it holds*

$$\left|\mathbb{E}^{\mathbb{P}_{t,x}}\Big[\int_t^T f(s, X_s)\, ds\Big]\right| \leq C\|f\|_{L^p_t L^q_x}, \quad (t, x) \in [0, T] \times \mathbb{R}^N, \tag{2.4}$$

where $\{X_s\}_{s\geq 0}$ is the canonical process associated with $\mathbb{P}_{t,x}[\cdot] := \mathbb{E}[\cdot|X_t = x]$ which is also the unique weak solution of SDE (1.3) with initial condition (t, x). In particular, the random variable X_s admits a density $p(t, s, x, \cdot)$ for any $t < s$ and any x in $\mathbb{R}^N$.

Additionally, we have been able to show the following non uniqueness result.

Theorem 2.8. *Let us consider SDE (1.3) with $\sigma = 1$ and assume that*

- *$x_j \to F_i(t, x_i, \ldots, x_j, \ldots, x_n)$ is β_i^j-Hölder continuous, uniformly in t and in x_k, for $k \neq j$.*

Then, for given i, j in $[\![2, n]\!]$ with $j \geq i$ there exists $F_i(t, x_i, \ldots, x_j, \ldots, x_n) = F_i(t, x_j)$ with

$$\beta_i^j < \frac{1 + \alpha(i-2)}{1 + \alpha(j-1)},$$

for which weak uniqueness fails for the SDE (1.3).

The above result will be proven in Section 5, showing a suitable, explicit Peano-type counter-example.

Remark 2.1. As opposed to the Gaussian driven case, we did not succeed to obtain regularity indexes which are *sharp* at any level of the chain (cf. [CdRM20b]). However, we point out that for diagonal systems of the form:

$$\begin{cases} dX_t^1 &= F_1(t, X_t^1, \ldots, X_t^n)dt + \sigma(t, X_{t-}^1, \ldots, X_{t-}^n)dZ_t, \\ dX_t^2 &= [A_t^2 X_t^1 + F_2(t, X_t^2)]\, dt, \\ dX_t^3 &= [A_t^3 X_t^2 + F_3(t, X_t^3)]\, dt, \\ \vdots & \\ dX_t^n &= \Big[A_t^n X_t^{n-1} + F_n(t, X_t^n)\Big]\, dt, \end{cases} \tag{2.5}$$

i.e. the degenerate components are perturbed by a function which only depends of the current level on the chain, we have that the previous thresholds are *almost* sharp. Indeed, in this case, we are led to consider $\beta^j > \frac{1+\alpha(j-2)}{1+\alpha(j-1)}$ which gives the well-posedness from the conditions in Theorem 2.6 while Theorem 2.8 shows that uniqueness fails as soon as $\beta_j^j < \frac{1+\alpha(j-2)}{1+\alpha(j-1)}$. For this diagonal system, Theorems 2.6 and 2.8 together then provide an "almost" complete understanding of the weak well-posedness for degenerate SDEs of type (2.5) with Hölder coefficients. Indeed, the problem for the critical

exponents

$$\overline{\beta}_j^j = \frac{1+\alpha(j-2)}{1+\alpha(j-1)}, \quad j \in [\![1,n]\!],$$

remains to be investigated and, up to our best knowledge, there are no general available results even in the diffusive case. We can only mention [Zha18] in the kinetic case.

We present in this section the analytical tools we will need to show the well-posedness of the associated martingale problem. In particular, they will be fundamental in the derivation of our main Theorem 2.6, thanks to Propositions 2.3 and 2.5. For this reason, we will assume in this section to be under the same conditions of Theorem 2.6. Moreover, we will suppose that the final time horizon T is small enough for our scopes. Indeed, we could always exploit the Markov property of the involved processes and standard chaining in time arguments to extend the results to arbitrary (but finite) time intervals.

2.1 The "frozen" dynamics

The crucial element in our approach consists in choosing wisely a suitable proxy operator with well-known properties and controls, along which we can expand the infinitesimal generator L_s, with an additional negligible error.
In order to deal with potentially unbounded perturbations F, it is natural to use a proxy involving a non-zero first order term associated with a flow associated with $G(t,x) := Ax + F(t,x)$, the transport part of SDE (1.3) (see e.g. [KP10] or [CdRMP20a]).
Remembering that we assume F to be Hölder continuous, we know from the classical Peano-Lipschitz Theorem that there exists a solution of

$$\begin{cases} d\theta_{t,\tau}(\xi) = \Big[A_t\theta_{t,\tau}(\xi) + F(t,\theta_{t,\tau}(\xi))\Big]\,dt & \text{on } [0,\tau]; \\ \theta_{\tau,\tau}(\xi) = \xi, \end{cases} \tag{2.1}$$

even if it may be not unique. For this reason, we are going to choose one particular flow, denoted by $\theta_{t,\tau}(\xi)$, and consider it fixed throughout the work. As it will be shown below in Lemma 2.13, it is always possible to take a measurable version of such a flow. More precisely, given a freezing couple (τ,ξ) in $(0,T]\times\mathbb{R}^N$, the backward flow will be defined on $[0,\tau]$ as

$$\theta_{t,\tau}(\xi) = \xi - \int_t^\tau \Big[A_u\theta_{u,\tau}(\xi) + F(u,\theta_{u,\tau}(\xi))\Big]\,du.$$

Fixed the reference flow, the next step is to consider the stochastic dynamics linearized along the backward flow $\theta_{t,\tau}(\xi)$. Namely, for any fixed starting point (t,x) in $[0,\tau]\times\mathbb{R}^N$, we consider $\{\tilde{X}_s^{\tau,\xi,t,x}\}_{s\in[t,T]}$ solving the following SDE:

$$\begin{cases} d\tilde{X}_u^{\tau,\xi,t,x} = \Big[A_u\tilde{X}_u^{\tau,\xi,t,x} + \tilde{F}_u^{\tau,\xi}\Big]\,du + B\tilde{\sigma}_u^{\tau,\xi}\,dZ_u, \quad u\in[t,T], \\ \tilde{X}_t^{\tau,\xi,t,x} = x, \end{cases} \tag{2.2}$$

where $\tilde{\sigma}_s^{\tau,\xi} := \sigma(s,\theta_{s,\tau}(\xi))$ and $\tilde{F}_s^{\tau,\xi} := F(s,\theta_{s,\tau}(\xi))$.
In order to obtain an integral representation of the process $\{\tilde{X}_s^{\tau,\xi,t,x}\}_{s\in[t,T]}$, we now

introduce the time-ordered resolvent $\mathcal{R}_{s,t}$ of the matrix A_s starting at time t. Namely, $\mathcal{R}_{s,t}$ is a time-dependent matrix in $\mathbb{R}^N \otimes \mathbb{R}^N$ that is solution of the following ODE:

$$\begin{cases} \partial_s \mathcal{R}_{s,t} \;=\; A_s \mathcal{R}_{s,t}, \quad s \in [0,T]; \\ \mathcal{R}_{t,t} \;=\; \mathrm{Id}_{N\times N}\,. \end{cases}$$

By the variation of constants method, it is now easy to check that the solution $\tilde{X}_s^{\tau,\xi,t,x}$ of SDE (2.2) satisfies that

$$\tilde{X}_s^{\tau,\xi,t,x} \;=\; \tilde{m}_{s,t}^{\tau,\xi}(x) + \int_t^s \mathcal{R}_{s,u} B \tilde{\sigma}_u^{\tau,\xi} dZ_u, \tag{2.3}$$

where the "frozen shift" $\tilde{m}_{s,t}^{\tau,\xi}(x)$ is given by:

$$\tilde{m}_{s,t}^{\tau,\xi}(x) \;=\; \mathcal{R}_{s,t} x + \int_t^s \mathcal{R}_{s,u} \tilde{F}_u^{\tau,\xi}\, du. \tag{2.4}$$

We point out already two important properties of the shift $\tilde{m}_{s,t}^{\tau,\xi}(x)$.

Lemma 2.9. *Let s in $[0,T]$ and x, y two points in $\mathbb{R}^N$. Then, for any $t < s$, it holds that*

$$\tilde{m}_{s,t}^{t,x}(x) \;=\; \theta_{s,t}(x) \tag{2.5}$$

$$y - \tilde{m}_{s,t}^{s,y}(x) \;=\; \theta_{t,s}(y) - x \tag{2.6}$$

Proof. We start noticing that by construction in (2.4), $\tilde{m}_{s,t}^{\tau,\xi}(x)$ satisfies

$$\partial_s \tilde{m}_{s,t}^{\tau,\xi}(x) \;=\; A_s \tilde{m}_{s,t}^{\tau,\xi}(x) + F(s, \theta_{s,\tau}(\xi)), \tag{2.7}$$

for any freezing parameters (τ, ξ). Choosing $\tau = t$, $\xi = x$ above, it then holds that

$$\partial_s \left[\tilde{m}_{s,t}^{s,x}(x) - \theta_{s,t}(x)\right] \;=\; A_s \left[\tilde{m}_{s,t}^{t,x}(x) - \theta_{s,t}(x)\right].$$

Since, $\tilde{m}_{t,t}^{t,x}(x) = \theta_{t,t}(x) = x$, Equation (2.5) then follows immediately applying the Grönwall lemma.
The second identity in (2.6) follows in a similar manner. □

We are now interested in investigating the analytical properties of the "frozen" solution process $\tilde{X}_s^{\tau,\xi,t,x}$. In particular, we will show in the next results the existence of a density for such a process and its anisotropic regularizing effect, at least for small times. Further on, we will consider fixed a time-dependent matrix $\mathbb{M}_t$ on $\mathbb{R}^N \otimes \mathbb{R}^N$ given by

$$\mathbb{M}_t := \mathrm{diag}(I_{d_1\times d_1}, tI_{d_2\times d_2}, \ldots, t^{n-1} I_{d_n\times d_n}), \quad t \geq 0, \tag{2.8}$$

which reflects the multi-scale nature of the underlying dynamics in (2.2).

Proposition 2.10 (Decomposition)**.** *Let the freezing couple (τ,ξ) be in $[0,T]\times\mathbb{R}^N$, $t<s$ in $[0,T]$ and x in $\mathbb{R}^N$. Then, there exists a Lévy process $\{\tilde{S}_u^{\tau,\xi,t,s}\}_{u\geq 0}$ such that*

$$\tilde{X}_s^{\tau,\xi,t,x} \overset{(law)}{=} \tilde{m}_{s,t}^{\tau,\xi}(x)+\mathbb{M}_{s-t}\tilde{S}_{s-t}^{\tau,\xi,t,s}. \tag{2.9}$$

In particular, the random variable $\tilde{X}_s^{\tau,\xi,t,x}$ admits a continuous density $\tilde{p}^{\tau,\xi}(t,s,x,\cdot)$ given by

$$\begin{aligned}\tilde{p}^{\tau,\xi}(t,s,x,y) &= \frac{1}{\det\mathbb{M}_{s-t}}p_{\tilde{S}^{\tau,\xi,t,s}}\Big(t-s,\mathbb{M}_{s-t}^{-1}(y-\tilde{m}_{s,t}^{\tau,\xi}(x))\Big) \qquad (2.10)\\ &:= \frac{\det\mathbb{M}_{s-t}^{-1}}{(2\pi)^N}\int_{\mathbb{R}^N}e^{-i\langle\mathbb{M}_{s-t}^{-1}(y-\tilde{m}_{s,t}^{\tau,\xi}(x)),z\rangle}\exp\Big((s-t)\int_{\mathbb{R}^N}[\cos(\langle z,p\rangle)-1]\,\nu_{\tilde{S}^{\tau,\xi,t,s}}(dp)\Big)\,dz,\end{aligned}$$

where $\nu_{\tilde{S}^{\tau,\xi,t,s}}$ and $p_{\tilde{S}^{\tau,\xi,t,s}}(u,\cdot)$ are the Lévy measure and the density associated with the process $\{\tilde{S}_u^{\tau,\xi,t,s}\}_{u\geq 0}$, respectively.

Proof. For simplicity, we start denoting

$$\tilde{\Lambda}^{\tau,\xi,t,s} := \int_t^s \mathcal{R}_{s,u}B\tilde{\sigma}_u^{\tau,\xi}dZ_u, \quad s\geq t,$$

so that we have from Equation (2.3) that $\tilde{X}_s^{\tau,\xi,t,x}=\tilde{m}_{s,t}^{\tau,\xi}(x)+\tilde{\Lambda}^{\tau,\xi,t,s}$. To conclude, we need to construct a Lévy process $\{\tilde{S}_u^{\tau,\xi,t,s}\}_{u\geq 0}$ on $\mathbb{R}^N$ such that

$$\tilde{\Lambda}^{\tau,\xi,t,s} \overset{(\text{law})}{=} \mathbb{M}_{s-t}\tilde{S}_{s-t}^{\tau,\xi,t,s}. \tag{2.11}$$

To show the identity in law, we are going to reason in terms of the characteristic functions. We start recalling that the Lévy process $\{Z_t\}_{t\geq 0}$ on $\mathbb{R}^d$ is characterized by the Lévy symbol

$$\Phi(p) = \int_{\mathbb{R}_0^d}[\cos(p\cdot q)-1]\,Q(q)\,\nu_\alpha(dq), \quad p\in\mathbb{R}^d,$$

where $\nu_\alpha(dq)=\mu(d\theta)\frac{dr}{r^{1+\alpha}}$ is the Lévy measure of an α-stable process. It is well-known (see e.g. Lemma 2.2 in [SW12]) that at any fixed $t\leq s$ in $[0,1]$, $\tilde{\Lambda}^{\tau,\xi,t,s}$ is an infinitely divisible random variable with associated Lévy symbol

$$\Phi_{\tilde{\Lambda}^{\tau,\xi,t,s}}(z) := \int_t^s \Phi\Big((\mathcal{R}_{s,u}B\tilde{\sigma}_u^{\tau,\xi})^*z\Big)\,du, \quad z\in\mathbb{R}^N,$$

where, we recall, we have denoted $\tilde{\sigma}_u^{\tau,\xi}=\sigma(u,\theta_{u,\tau}(\xi))$.
Setting $v:=(u-t)/(s-t)$ and noticing that $u=u(v):=t+v(s-t)$, we can now rewrite the Lévy symbol of $\tilde{\Lambda}^{\tau,\xi,t,s}$ as

$$\Phi_{\tilde{\Lambda}^{\tau,\xi,t,s}}(z) := (s-t)\int_0^1 \Phi\Big((\mathcal{R}_{s,u(v)}B\tilde{\sigma}_{u(v)}^{\tau,\xi})^*z\Big)\,dv. \tag{2.12}$$

From the analysis performed in [HM16], Lemmas 5.1 and 5.2 (see also [DM10] Proposition 3.7), we then know that we can decompose the first column of the resolvent $\mathcal{R}_{s,u(v)}$ in the following way:

$$\mathcal{R}_{s,u(v)}B = \mathbb{M}_{s-t}\widehat{\mathcal{R}}_vB,$$

where $\{\widehat{\mathcal{R}}_v \colon v \in [0,T]\}$ are non-degenerate and bounded matrixes in $\mathbb{R}^N \otimes \mathbb{R}^N$ and the multi-scale matrix $\mathbb{M}_t$ is given in (2.8). We can now rewrite the Lévy symbol of $\tilde{\Lambda}^{\tau,\xi,t,s}$ as

$$\Phi_{\tilde{\Lambda}^{\tau,\xi,t,s}}(z) \;=\; (s-t)\int_0^1 \Phi\Big((\widehat{\mathcal{R}}_v B\tilde{\sigma}^{\tau,\xi}_{u(v)})^*\mathbb{M}_{s-t}z\Big)\,dv, \quad z \in \mathbb{R}^N.$$

The above equality suggests us to define, for any fixed $t \le s$ in $(0,1]$, the (unique in law) Lévy process $\{\tilde{S}^{\tau,\xi,t,s}_u\}_{u\ge 0}$ associated with the Lévy symbol

$$\begin{aligned}\Phi_{\tilde{S}^{\tau,\xi,t,s}}(z) \;&:=\; \int_0^1 \Phi\Big((\widehat{\mathcal{R}}_v B\tilde{\sigma}^{\tau,\xi}_{u(v)})^* z\Big)\,dv \\ &=\; \int_0^1\int_{\mathbb{R}^d}\Big[\cos\Big(\langle z, \widehat{\mathcal{R}}_v B\tilde{\sigma}^{\tau,\xi}_{u(v)}p\rangle\Big) - 1\Big]\;\nu(dp)dv.\end{aligned} \tag{2.13}$$

Since we have that

$$\mathbb{E}\Big[e^{i\langle z,\tilde{\Lambda}^{\tau,\xi,t,s}\rangle}\Big] \;=\; e^{\Phi_{\tilde{\Lambda}^{\tau,\xi,t,s}}(z)} \;=\; e^{(s-t)\Phi_{\tilde{S}^{\tau,\xi,t,s}}(\mathbb{M}_t z)} \;=\; \mathbb{E}\Big[e^{i\langle z,\mathbb{M}_t\tilde{S}^{\tau,\xi,t,s}_{s-t}\rangle}\Big], \tag{2.14}$$

it follows immediately that Equation (2.11) holds.
To show the existence of a density for $\tilde{X}^{\tau,\xi,t,x}_s$, we want to exploit the Fourier inversion formula in (2.14). To do it, we firstly need to prove that $\exp(\Phi_{\tilde{\Lambda}^{\tau,\xi,t,s}}(z))$ is integrable. From (2.13), we notice that

$$\begin{aligned}\Phi_{\tilde{S}^{\tau,\xi,t,s}}(z) \;&=\; \int_0^1\int_{\mathbb{R}^d}\Big[\cos\Big(\langle z, \widehat{\mathcal{R}}_v B\tilde{\sigma}^{\tau,\xi}_{u(v)}p\rangle\Big) - 1\Big]\;\nu(dp)dv \\ &=\; \int_0^1\int_{\mathbb{R}^d}\Big[\cos\Big(\langle z, \widehat{\mathcal{R}}_v B\tilde{\sigma}^{\tau,\xi}_{u(v)}p\rangle\Big) - 1\Big]\,Q(p)\,\nu_\alpha(dp)dv,\end{aligned}$$

where in the last step we used hypothesis [**ND**]. Exploiting now that the quantities above are non-positive and $Q(p) \ge c > 0$ for p in $B(0,r_0)$, we write that

$$\begin{aligned}\Phi_{\tilde{S}^{\tau,\xi,t,s}}(z) \;&\le C\int_0^1\int_{B(0,r_0)}\Big[\cos\Big(\langle z, \widehat{\mathcal{R}}_v B\tilde{\sigma}^{\tau,\xi}_{u(v)}p\rangle\Big) - 1\Big]\;\nu_\alpha(dp)dv \\ &=\; C\Big\{-\int_0^1\Big|(\widehat{\mathcal{R}}_v B\tilde{\sigma}^{\tau,\xi}_{u(v)})^* z\Big|^\alpha\,dv + \int_0^1\int_{B^c(0,r_0)}\Big[1-\cos\Big(\langle z, \widehat{\mathcal{R}}_v B\tilde{\sigma}^{\tau,\xi}_{u(v)}p\rangle\Big)\Big]\,\nu_\alpha(dp)dv\Big\} \\ &\le\; C\Big\{-\int_0^1\Big|(\widehat{\mathcal{R}}_v B\tilde{\sigma}^{\tau,\xi}_{u(v)})^* z\Big|^\alpha\,dv + 1\Big\}.\end{aligned}$$

To conclude, we recall that Lemma 5.4 in [HM16] states that

$$\int_0^1\Big|(\widehat{\mathcal{R}}_v B\tilde{\sigma}^{\tau,\xi}_{u(v)})^* z\Big|^\alpha\,dv \;\ge\; C|z|^\alpha,$$

for some positive constant C independent from t,s,τ,ξ. It then follows in particular that

$$\Phi_{\tilde{S}^{\tau,\xi,t,s}}(z) \;\le\; C\left[1-|z|^\alpha\right], \quad z \in \mathbb{R}^N. \tag{2.15}$$

Since $\exp(\Phi_{\tilde{\Lambda}^{\tau,\xi,t,s}}(z))$ is integrable, it implies that there exists a density $p_{\tilde{\Lambda}^{\tau,\xi,t,s}}(t,s,\cdot)$ of the random variable $\tilde{\Lambda}^{\tau,\xi,t,s}$. We can now apply the Fourier inversion formula in Equation (2.14) showing that $p_{\tilde{\Lambda}^{\tau,\xi,t,s}}(t,s,\cdot)$ is given by

$$p^{\tau,\xi}_{\tilde{\Lambda}}(t,s,y) \;:=\; \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{-i\langle y,z\rangle}\exp\left((s-t)\Phi_{\tilde{S}^{\tau,\xi,t,s}}(z)\right)\,dz. \tag{2.16}$$

From Decomposition (2.9) and Equation (2.16), the representation for $\tilde{p}^{\tau,\xi}(t,s,x,\cdot)$ follows immediately. □

Once we have proven the existence of a density $\tilde{p}^{\tau,\xi}(t,s,x,\cdot)$ for the "frozen" stochastic dynamics $\tilde{X}_s^{\tau,\xi,t,x}$, we move now on determining its associated smoothing effects. In particular, we show in the following proposition that the derivatives of the "frozen" density are controlled by another density at the price of an additional time singularity of the order corresponding to the intrinsic time scale of the considered component in the stable regime. Importantly, such a control holds uniformly in the freezing parameters (τ,ξ).
Let us introduce for simplicity the following time-dependent scale matrix:

$$\mathbb{T}_t := t^{\frac{1}{\alpha}}\mathbb{M}_t, \quad t \geq 0. \tag{2.17}$$

Proposition 2.11. *There exists a family $\{\overline{p}(u,\cdot)\colon u \geq 0\}$ of densities on $\mathbb{R}^N$ and a positive constant $C := C(N,\alpha)$ such that*

- *for any $u \geq 0$ and any z in $\mathbb{R}^N$, $\overline{p}(u,z) = u^{-N/\alpha}\overline{p}(1,u^{-1/\alpha}z)$; (stable scaling property)*
- *for any γ in $[0,\alpha)$,*

$$\int_{\mathbb{R}^N} \overline{p}(u,z)\,|z|^\gamma\,dz \leq Cu^{\gamma/\alpha}, \quad u > 0; \tag{2.18}$$

- *for any k in $[\![0,2]\!]$, any i in $[\![1,n]\!]$, any $t < s$ in $[0,T]$ and any x,y in $\mathbb{R}^N$,*

$$|D_{x_i}^k \tilde{p}^{\tau,\xi}(t,s,x,y)| \leq C\frac{(s-t)^{-k\frac{1+\alpha(i-1)}{\alpha}}}{\det \mathbb{T}_{s-t}}\overline{p}\left(1,\mathbb{T}_{s-t}^{-1}(y-\tilde{m}_{s,t}^{\tau,\xi}(x))\right). \tag{2.19}$$

  where we denoted, coherently with the notations introduced before Theorem 2.6, $D_{x_i} = \left(D_{x_i^1},\dots,D_{x_i^{d_i}}\right)$.

Remark 2.2 (About the freezing parameters)**.** We carefully point out that since we will later on choose as parameters $(\tau,\xi) = (s,y)$, it is particularly important that we manage to obtain an upper bound by a density which is independent from those parameters, since they will be as well the integration variables (see Section 2.2 below). This is precisely why we actually impose the specific semi-linear drift structure in SDE (1.3) (cf. assumption **[H]**), as opposed to the more general one that can be handled in the Gaussian case [CdRM20b]. This is a framework which naturally gives the independence of the large jumps of the *proxy* process $\tilde{X}_s^{\tau,\xi,t,x}$ as used in (2.24) below. The more general case for the first order dynamics considered in [CdRM20b] would actually lead to linearize around a matrix which would depend on the freezing parameters. For such models, we did not succeed in proving that the corresponding densities can be bounded independently of the parameters (see also the proof of Lemma 5.2 below for a similar issue regarding the diffusion coefficient).

Proof. Fixed the freezing parameters (τ,ξ) in $[0,T]\times\mathbb{R}^N$, and the times $t < s$ in $[0,T]$, we start applying the Itô-Lévy decomposition to the process $\{\tilde{S}_u^{\tau,\xi,t,s}\}_{u\geq 0}$ introduced in

Proposition 2.10 at the associated characteristic stable time, i.e. we choose to truncate at threshold $u^{1/\alpha}$. Thus, we can write

$$\tilde{S}_u^{\tau,\xi,t,s} \;=\; \tilde{M}_u^{\tau,\xi,t,s} + \tilde{N}_u^{\tau,\xi,t,s} \tag{2.20}$$

for some $\tilde{M}_u^{\tau,\xi,t,s}$, $\tilde{N}_u^{\tau,\xi,t,s}$ independent random variables corresponding to the small jumps part and the large jumps part, respectively. Namely, we denote for any $v > 0$,

$$\tilde{N}_v^{\tau,\xi,t,s} \;:=\; \int_0^v \int_{|z|>u^{1/\alpha}} z P_{\tilde{S}^{\tau,\xi,t,s}}(dr,dz) \;\text{ and }\; \tilde{M}_v^{\tau,\xi,t,s} := \tilde{S}_v^{\tau,\xi,t,s} - \tilde{N}_v^{\tau,\xi,t,s}, \tag{2.21}$$

where $P_{\tilde{S}^{\tau,\xi,t,s}}(dr,dz)$ is the Poisson random measure associated with the process $\tilde{S}^{\tau,\xi,t,s}$. It can be shown, similarly to Proposition 2.10, that the process $\{\tilde{M}_u^{\tau,\xi,t,s}\}_{u\geq 0}$ admits a density $p_{\tilde{M}^{\tau,\xi,t,s}}(u,\cdot)$. Indeed, it is well-known that the small jump part leads to a density which is in the Schwartz class $\mathcal{S}(\mathbb{R}^N)$ (see Lemma 5.1 below). We can then rewrite the density $p_{\tilde{S}^{\tau,\xi,t,s}}$ of $\tilde{S}^{\tau,\xi,t,s}$ in the following way:

$$p_{\tilde{S}^{\tau,\xi,t,s}}(u,z) \;=\; \int_{\mathbb{R}^N} p_{\tilde{M}^{\tau,\xi,t,s}}(u,z-y) P_{\tilde{N}_u^{\tau,\xi,t,s}}(dy) \tag{2.22}$$

where $P_{\tilde{N}_u^{\tau,\xi,t,s}}$ is the law of $\tilde{N}_u^{\tau,\xi,t,s}$.
We need now to control the modulus of the density $p_{\tilde{S}^{\tau,\xi,t,s}}$ with another density, independently from the parameters τ, ξ. From Lemma 5.1 in the Appendix (see also Lemma B.2 in [HM16]) with $m = N+1$, we know that there exists a positive constant C, independent from τ, ξ such that

$$\left|D_z^k p_{\tilde{M}^{\tau,\xi,t,s}}(u,z)\right| \;\leq\; C u^{-(N+k)/\alpha} \left(\frac{u^{1/\alpha}}{u^{1/\alpha}+|z|}\right)^{N+2} \;=:\; C u^{-k/\alpha} p_{\overline{M}}(u,z), \tag{2.23}$$

for any k in $[\![0,2]\!]$, any $u > 0$, and any z in $\mathbb{R}^N$.
Moreover, denoting by $\overline{M}_u$ the random variable with density $p_{\overline{M}}(u,\cdot)$ that is independent from $\tilde{N}_u^{\tau,\xi,t,s}$, we can easily check that $p_{\overline{M}}(u,z) = u^{-N/\alpha} p_{\overline{M}}(1, u^{-1/\alpha} z)$ and thus, that $\overline{M}$ is α-selfsimilar:

$$\overline{M}_u \;\overset{\text{law}}{=}\; u^{1/\alpha}\overline{M}_1.$$

On the other hand, Lemma 5.2 in the Appendix (see also Lemma A.2 in [FKM21]) ensures the existence of a family $\{\overline{P}_u\}_{u\geq 0}$ of probability measures such that

$$P_{\tilde{N}_u^{\tau,\xi,t,s}}(\mathcal{A}) \;\leq\; C\overline{P}_u(\mathcal{A}), \quad \mathcal{A} \in \mathcal{B}(\mathbb{R}^N), \tag{2.24}$$

for some positive constant C independent from the parameters τ, ξ, t, s.
For any fixed $u \geq 0$, let us now denote by $\overline{N}_u$ the random variable with law $\overline{P}_u$ that is independent from $\overline{M}_u$. Thanks to the representation of the measure $\overline{P}_u$ in (5.7), it is then immediate to check that

$$\overline{N}_u \;\overset{(\text{law})}{=}\; u^{1/\alpha}\overline{N}_1.$$

We can finally define the family $\{\overline{p}(u,\cdot)\}_{u\geq 0}$ of densities as

$$\overline{p}(u,z) \;:=\; \int_{\mathbb{R}^N} p_{\overline{M}}(u,z-w)\overline{P}_u(dw), \tag{2.25}$$

which corresponds to the density of the following random variable:

$$\overline{S}_u \,:=\, \overline{M}_u + \overline{N}_u$$

for any fixed $u \geq 0$. Using Fourier transform and the already proven α-selfsimilarity of $\overline{M}$ and $\overline{N}$, we now show that

$$\overline{S}_u \overset{(\text{law})}{=} u^{1/\alpha}\overline{S}_1,$$

or equivalently, that

$$\overline{p}(u,z) \,=\, u^{-N/\alpha}\overline{p}(1, u^{-1/\alpha}z)$$

for any $u \geq 0$ and any z in $\mathbb{R}^N$. Moreover,

$$\mathbb{E}[|\overline{S}_u|^\gamma] \,=\, \mathbb{E}[|\overline{M}_u + \overline{N}_u|^\gamma] \,=\, Cu^{\gamma/\alpha}\Big(\mathbb{E}[|\overline{M}_1|^\gamma] + \mathbb{E}[|\overline{N}_1|^\gamma]\Big) \,\leq\, Cu^{\gamma/\alpha},$$

for any $\gamma < \alpha$. In particular, Equation (2.18) holds. We emphasize that the integrability constraints precisely come from the Poisson measure $\overline{P}_u$ which behaves as the one associated with the large jumps of an α-stable density.
Equation (2.19) now follows easily from the previous arguments. From Equation (2.22), we start noticing that Controls (2.23), (2.24) and (2.25) imply that for any k in $[\![0,2]\!]$,

$$\Big|D_z^k p_{\tilde{S}^{\tau,\xi,t,s}}(u,z)\Big| \,\leq\, Ct^{-k/\alpha}\overline{p}(u,z), \quad u \geq 0,\, z \in \mathbb{R}^N,$$

for some constant $C > 0$, independent from the parameters τ, ξ, t, s. Recalling the decomposition in (2.9), Equation (2.19) for $k = 0$ already follows.
To show instead the case $k = 1$, we can write that

$$\begin{aligned}
\Big|D_{x_i}\tilde{p}^{\tau,\xi}(t,s,x,y)\Big| \,&=\, \left|\frac{1}{\det(\mathbb{M}_{s-t})}D_{x_i}\left[p_{\tilde{S}^{\tau,\xi,t,s}}\Big(s-t, \mathbb{M}_{s-t}^{-1}(y - \tilde{m}_{s,t}^{\tau,\xi}(x))\Big)\right]\right| \\
&=\, \left|\frac{1}{\det(\mathbb{M}_{s-t})}\langle D_z p_{\tilde{S}^{\tau,\xi,t,s}}\left(s-t,\cdot\right)\left(\mathbb{M}_{s-t}^{-1}(y - \tilde{m}_{s,t}^{\tau,\xi}(x))\right), D_{x_i}\mathbb{M}_{s-t}^{-1}\tilde{m}_{s,t}^{\tau,\xi}(x)\rangle\right| \\
&=\, \frac{(s-t)^{-1/\alpha}}{\det(\mathbb{T}_{s-t})}\overline{p}\left(1, \mathbb{T}_{s-t}^{-1}(y - \tilde{m}_{s,t}^{\tau,\xi}(x))\right)\left|D_{x_i}\mathbb{M}_{s-t}^{-1}\tilde{m}_{s,t}^{\tau,\xi}(x)\right|,
\end{aligned}$$

where in the last step we exploited the α-scaling property of $\overline{p}$. From Equation (2.7), we now notice that the function $x \to \tilde{m}_{s,t}^{\tau,\xi}(x)$ is affine, so that

$$\left|D_{x_i}\mathbb{M}_{s-t}^{-1}\tilde{m}_{s,t}^{\tau,\xi}(x)\right| \,\leq\, C(s-t)^{-(i-1)}.$$

Hence, it follows that

$$|D_{x_i}\tilde{p}^{\tau,\xi}(t,s,x,y)| \,\leq\, C\frac{(s-t)^{-\frac{1+\alpha(i-1)}{\alpha}}}{\det(\mathbb{T}_{s-t})}\overline{p}\left(1, \mathbb{T}_{s-t}^{-1}(y - \tilde{m}_{s,t}^{\tau,\xi}(x))\right).$$

The other case ($k = 2$) can be derived in an analogous way. □

We conclude this section with a useful control on the powers of the density $\overline{p}(u,z)$.

Corollary 2.12. *Let $q \geq 1$. Then, there exists a positive constant $C := C(q)$ such that*

$$[\overline{p}(u,z)]^q \leq u^{(1-q)\frac{N}{\alpha}} C\overline{p}(u,z), \quad (u,z) \in (0,T] \times \mathbb{R}^N. \tag{2.26}$$

Proof. We start noticing that we can assume without loss of generality that $u = 1$, thanks to the scaling property of $\overline{p}(u,z)$ in Proposition 2.11. Moreover, we know that there exists a constant K such that $\overline{p}(1,z) \leq 1$ for any z in $B^c(0,K)$, since $\overline{p}(1,\cdot)$ is a density. It then clearly follows that

$$[\overline{p}(1,z)]^q \leq \overline{p}(1,z), \quad z \in B^c(0,K).$$

On the other hand, we recall that $\overline{p}(1,\cdot)$ is continuous. For any z in $B(0,K)$, it then holds that

$$[\overline{p}(1,z)]^q = \overline{p}(1,z)[\overline{p}(1,z)]^{q-1} \leq C\overline{p}(1,z),$$

where C is the maximum of $[\overline{p}(1,\cdot)]^q$ on $B(0,K)$. □

2.2 Regularity of the density along the terminal condition

We briefly explain here how we want to prove the well-posedness of the martingale formulation associated with $\partial_s + L_s$ at some starting point (t,x). We will mainly focus on the problem of uniqueness since the existence of a solution can be easily handled from already known results. Indeed, we recall that under the assumptions we consider, the main part of the operator L_s is of order $\alpha > 1$ while the perturbation is sub-linear. Thus, the existence of a solution can be obtained, for example, from Theorem 2.2 in [Str75].

In particular, uniqueness for the martingale problem will follow once the Krylov-like estimates (2.4) have been shown.

Starting from a solution $\{X_s^{t,x}\}_{s\in[0,T]}$ of the martingale problem with starting point (t,x), the idea is to exploit the properties of the frozen dynamics $\{\tilde{X}_s^{\tau,\xi,t,x}\}_{s\in[0,T]}$ in (2.3). For this reason, let us denote by $\tilde{L}_s^{\tau,\xi}$ its infinitesimal generator and define for f in $C_c^{1,2}([0,T) \times \mathbb{R}^N)$ the associated Green kernel:

$$\tilde{G}^{\tau,\xi}(t,x) = \int_t^T ds \int_{\mathbb{R}^N} \tilde{p}^{\tau,\xi}(t,s,x,y) f(s,y) dy.$$

Standard results now give that

$$\left(\partial_t + \tilde{L}_t^{\tau,\xi}\right) \tilde{G}^{\tau,\xi} f(t,x) = -f(t,x), \quad (t,x) \in [0,T) \times \mathbb{R}^N, \tag{2.27}$$

for any (fixed) freezing parameters (τ,ξ).

The first step of our method then consists in applying the Itô formula on the function $\tilde{G}^{\tau,\xi} f$, which is indeed smooth enough, and the solution process $\{X_s^{t,x}\}_{s\in[0,T]}$:

$$\tilde{G}^{\tau,\xi} f(t,x) + \mathbb{E}\left[\int_t^T (\partial_s + L_s)\tilde{G}^{\tau,\xi} f(s, X_s^{t,x})\, ds\right] = 0.$$

Exploiting (2.27), we can then write

$$\tilde{G}^{\tau,\xi} f(t,x) - \mathbb{E}\Big[\int_t^T f(s, X_s^{t,x})\, ds\Big] + \mathbb{E}\Big[\int_t^T \Big(L_s - \tilde{L}_s^{\tau,\xi}\Big)\tilde{G}^{\tau,\xi} f(s, X_s^{t,x})\, ds\Big] = 0$$

or, equivalently,

$$\mathbb{E}\Big[\int_t^T f(s, X_s^{t,x})\, ds\Big] \;=\; \tilde{G}^{\tau,\xi} f(t,x) + \mathbb{E}\Big[\int_t^T \Big(L_s - \tilde{L}_s^{\tau,\xi}\Big)\tilde{G}^{\tau,\xi} f(s, X_s^{t,x})\, ds\Big].$$

While an estimate of the frozen Green kernel $\tilde{G}^{\tau,\xi} f$ can be obtained from Proposition 2.11, the main difficulty of our approach will be to control, uniformly in (t,x), the following quantity:

$$\int_t^T \int_{\mathbb{R}^N} \Big(L_s - \tilde{L}_s^{\tau,\xi}\Big)\tilde{G}^{\tau,\xi} f(s,x)\, ds.$$

Focusing for example only on the component associated with the deterministic drift F, i.e.

$$\int_t^T \int_{\mathbb{R}^N} \langle F(t,x) - F(t,\theta_{t,\tau}(\xi)), D_x \tilde{p}^{\tau,\xi}(t,s,x,y)\rangle f(s,y)\, dyds,$$

it is clear that we need some kind of compatibility between the arguments of the drift F and those of the frozen density $\tilde{p}^{\tau,\xi}(t,s,x,\cdot)$, in order to exploit the associated smoothing effect (Proposition 2.11). Namely, we need to compare the quantities $(x - \theta_{t,\tau}(\xi))$ and $(y - \tilde{m}_{s,t}^{\tau,\xi}(x))$.
Noticing that for $\tau = s$ and $\xi = y$, $(y - \tilde{m}_{s,t}^{\tau,\xi}(x)) = \theta_{t,s}(y) - x$, it follows from Proposition 2.11 that this choice of freezing parameters gives the natural compatibility between the difference of the generators and the upper-bounds of the derivatives of the corresponding proxy.

The above reasoning requires however a more thorough analysis on the properties of the "density" $\tilde{p}^{s,y}(t,s,x,\cdot)$ frozen along the terminal condition (τ,ξ). Indeed, the freezing parameter y appears also as the integration variable. In other words, with this approach, the freezing parameter cannot be fixed once for all. The present section is precisely dedicated to the handling of such a choice. This will lead us to introduce a *pseudo* Green kernel, see (2.41) below, from which will then derive uniqueness to the martingale problem following the Stroock and Varadhan approach (see Chapter 7 in [SV79]), through appropriate inversion in $L_t^q - L_x^p$ spaces, proving that the remainder has a small corresponding norm.

We start with a lemma showing the existence of at least one version of the flow $\theta_{t,s}(y)$ which is measurable in s and y. This result will be fundamental to make licit any integration of this flow along the terminal condition y.

Lemma 2.13. *There exists a measurable mapping $\theta\colon [0,T]^2 \times \mathbb{R}^N \to \mathbb{R}^N$ such that*

$$\theta(t,s,z) \;:=\; \theta_{t,s}(z) \;=\; z + \int_t^s \Big[A_u \theta_{u,s}(z) + F(u, \theta_{u,s}(z))\Big]\, du. \tag{2.28}$$

Proof. The result can be obtained from [Zub12] and a standard compactness argument. □

From this point further, we assume without loss of generality to have chosen such a measurable version $\theta_{t,s}(x)$ of the reference flow.

The next Lemma 2.14 (Approximate Lipschitz condition of the flows) will be a key technical tool for our method. Roughly speaking, it says that a kind of equivalence

between the rescaled forward and backward flows appears even in our framework (where the drift F is not regular enough), up to an additional constant contribution, for any two measurable flows satisfying Equation (2.1). We only remark that similar results has been thoroughly exploited in [DM10, Men11, Men18] when considering Lipschitz drifts or [CdRM20b] in the degenerate diffusive setting with Hölder coefficients.
This *approximated* Lipschitz property will be fundamental later on in the proof of Lemma 2.18 (Dirac Convergence of frozen density) below. It will be proved in Appendix $A.1$, adapting the lines of [CdRM20b].

Lemma 2.14. *Let $\theta\colon [0,T]^2\times\mathbb{R}^N \to \mathbb{R}^N$, $\check{\theta}\colon [0,T]^2\times\mathbb{R}^N \to \mathbb{R}^N$ be two measurable flows satisfying Equation* (2.28)*. Then, there exist two positive constants $(C,C') := (C,C')(T)$ such that for any $t<s$ in $[0,T]$ and any x,y in $\mathbb{R}^N$,*

$$C^{-1}|\mathbb{T}^{-1}_{s-t}(\check{\theta}_{s,t}(x)-y)| - C' \le |\mathbb{T}^{-1}_{s-t}(x-\theta_{t,s}(y))| \le C\left[|\mathbb{T}^{-1}_{s-t}(\check{\theta}_{s,t}(x)-y)|+1\right]. \quad (2.29)$$

From the above lemma, we also derive the following important estimate for the rescaled difference between the forward flow $\theta_{s,t}(x)$ and the linearized forward dynamics $\tilde{m}^{s,y}_{s,t}(x)$ (defined in (2.4)) where the linearization is considered along any backward flow.

Corollary 2.15. *Let $\theta\colon [0,T]^2\times\mathbb{R}^N \to \mathbb{R}^N$ be a measurable flow satisfying Equation* (2.28)*. Then, there exist a positive constant $C := C(T)$ and ζ in $(0,1)$ such that for any $t<s$ in $[0,T]$ and any x,y in $\mathbb{R}^N$,*

$$|\mathbb{T}^{-1}_{s-t}(\theta_{s,t}(x)-\tilde{m}^{s,y}_{s,t}(x))| \le C(s-t)^{\frac{1}{\alpha}\wedge\zeta}\left(1+|\mathbb{T}^{-1}_{s-t}(\theta_{s,t}(x)-y)|\right). \quad (2.30)$$

Proof. We start exploiting the differential dynamics given in Equation (2.7) to write that

$$\begin{aligned}\mathbb{T}^{-1}_{s-t}\left(\theta_{s,t}(x)-\tilde{m}^{s,y}_{s,t}(x)\right) &= \mathbb{T}^{-1}_{s-t}\int_t^s\Big\{\left[F(u,\theta_{u,t}(x))-F(u,\theta_{u,s}(y))\right]\\ &\qquad+\left[A_u(\theta_{u,t}(x)-\tilde{m}^{s,y}_{u,t}(x))\right]\Big\}\,du \\ &:= (\mathcal{I}^1_{s,t}+\mathcal{I}^2_{s,t})(x,y).\end{aligned} \quad (2.31)$$

We start dealing with $\mathcal{I}^1_{s,t}(x,y)$. The key idea is to use the sub-linearity of F and the appropriate Hölder exponents. Namely, using the Young inequality, we derive that

$$\begin{aligned}|\mathcal{I}^1_{s,t}(x,y)| &\le C\sum_{i=1}^n(s-t)^{-\frac{1+\alpha(i-1)}{\alpha}}\sum_{j=i}^n\int_t^s|(\theta_{u,t}(x)-\theta_{u,s}(y))_j|^{\beta^j}\,du\\ &\le C\Big\{(s-t)^{-\frac{1}{\alpha}}\int_t^s\left[|\theta_{u,t}(x)-\theta_{u,s}(y)|+1\right]du\\ &\quad+\sum_{i=2}^n(s-t)^{-\frac{1+\alpha(i-1)}{\alpha}}\sum_{j=i}^n\int_t^s\left[(s-t)^{-\gamma^j}|((\theta_{u,t}(x)-\theta_{u,s}(y))_j)|+(s-t)^{\gamma^j\frac{\beta^j}{1-\beta^j}}\right]du\Big\},\end{aligned}$$

for some parameters $\gamma^j>0$ to be specified below. Denoting now for simplicity,

$$\Gamma_j := -\frac{1+\alpha(i-1)}{\alpha}+\gamma^j\frac{\beta^j}{1-\beta^j},$$

we get that

$$
\begin{aligned}
|\mathcal{I}^1_{s,t}(x,y)| \;\leq\; & C\Big\{(s-t)^{\frac{\alpha-1}{\alpha}} + \int_t^s |\mathbb{T}^{-1}_{s-t}(\theta_{u,t}(x)-\theta_{u,s}(y))|\,du \\
& + \sum_{i=2}^n \sum_{j=i}^n \int_t^s \Big[(s-t)^{-i+j-\gamma^j}\Big(\frac{|((\theta_{u,t}(x)-\theta_{u,s}(y))_j)|}{(s-t)^{\frac{1+\alpha(j-1)}{\alpha}}}\Big) + (s-t)^{-\Gamma_j}\Big]\,du\Big\} \\
\leq\; & C\Big\{(s-t)^{\frac{\alpha-1}{\alpha}} + \int_t^s |\mathbb{T}_(s-t)^{-1}(\theta_{u,t}(x)-\theta_{u,s}(y))|\,du \\
& + \sum_{i=2}^n \sum_{j=i}^n \int_t^s \Big[(s-t)^{-i+j-\gamma^j}|\mathbb{T}^{-1}_{s-t}(\theta_{u,t}(x)-\theta_{u,s}(y))| + (s-t)^{\Gamma_j}\Big]\,du\Big\}.
\end{aligned}
$$

We now use Lemma 2.14 (Approximate Lipschitz condition of the flows) to derive that

$$
|\mathbb{T}^{-1}_{s-t}(\theta_{u,t}(x)-\theta_{u,s}(y))| \;\leq\; C(|\mathbb{T}^{-1}_{s-t}(\theta_{s,t}(x)-y)|+1).
$$

We emphasize here that in our current framework we should *a priori* write $\theta_{s,u}(\theta_{u,t}(x))$ in the above equation since we do not have the flow property. Anyhow, since Lemma 2.14 (Approximate Lipschitz condition of the flows) is valid for any flow starting from $\theta_{u,t}(x)$ at time u associated with the ODE (see Equation (2.29)) we can proceed along the previous one, i.e. $(\theta_{v,t}(x))_{v\in[u,s]}$. The previous reasoning yields that

$$
\begin{aligned}
|\mathcal{I}^1_{s,t}(x,y)| \;\leq\; C\Big\{ & (s-t)^{\frac{\alpha-1}{\alpha}} + (s-t)\Big[|\mathbb{T}^{-1}_{s-t}(\theta_{s,t}(x)-y)|+1\Big] \\
& \times \Big[1+\sum_{i=2}^n\sum_{j=i}^n\Big((s-t)^{-i+j-\gamma^j} + (s-t)^{-\frac{1+\alpha(i-1)}{\alpha}+\gamma^j\frac{\beta^j}{1-\beta^j}}\Big)\Big]\Big\}.
\end{aligned}
\tag{2.32}
$$

We now choose for j in $[\![i,n]\!]$,

$$
-i+j-\gamma^j \;=\; -\frac{1+\alpha(i-1)}{\alpha} + \gamma^j\frac{\beta^j}{1-\beta^j} \;\Leftrightarrow\; \gamma^j \;=\; \Big(j-\frac{\alpha-1}{\alpha}\Big)\Big(1-\beta^j\Big),
$$

to balance the two previous contributions associated with the indexes i,j.
To obtain a global smoothing effect with respect to $s-t$ in (2.32) we need to impose:

$$
-i+j-\gamma^j \;>\; -1 \;\Leftrightarrow\; \beta^j \;>\; \frac{1+\alpha(i-2)}{1+\alpha(j-1)}, \quad \forall i\leq j. \tag{2.33}
$$

Hence, under our assumptions, we have that there exists ζ in $(0,1)$ depending on β^j for any $j\in[\![i,n]\!]$ such that

$$
|\mathcal{I}^1_{s,t}(x,y)| \;\leq\; C(s-t)^{\zeta}\Big[1+|\mathbb{T}^{-1}_{s-t}(\theta_{s,t}(x)-y)|\Big]. \tag{2.34}
$$

Recalling from the structure of A that

$$
|\mathbb{T}^{-1}_{s-t}A_u\mathbb{T}_{s-t}| \leq C(s-t)^{-1},
$$

Control (2.30) now follows from (2.31), (2.34) and the Grönwall lemma. □

Thanks to the Approximate Lipschitz property of the flow presented in Lemma 2.14 above and Corollary 2.15, we can now adapt the controls on the derivatives of the frozen density (Proposition 2.11) to the "density" $\tilde{p}^{s,y}(t,s,x,y)$. Indeed, we recall again that the function $\tilde{p}^{s,y}(t,s,x,y)$ is not a proper density in y since the integration variable y stands also as freezing parameter. This is one of the main difficulties of the approach.

The following result is the key to our analysis since it precisely quantifies the smoothing effect in time of the proxy we chose.

Corollary 2.16. *There exists a positive constant $C := C(N,\alpha)$ such that for any γ in $[0,\alpha)$, any $t < s$ in $[0,T]$ and any x, y in $\mathbb{R}^N$,*

$$\int_{\mathbb{R}^N} \frac{|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)|^\gamma}{\det \mathbb{T}_{s-t}} \overline{p}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x))\, dy \;\leq\; C. \tag{2.35}$$

Moreover, if $K > 0$ is large enough, it holds that

$$\int_{\mathbb{R}^N} \mathbb{1}_{|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)|\geq K} \frac{1}{\det \mathbb{T}_{s-t}} \overline{p}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x))\, dy \\ \leq\; C \int_{\mathbb{R}^N} \mathbb{1}_{|z|\geq \frac{K}{2}} \check{p}(1,z) dz, \tag{2.36}$$

where $\check{p}$ enjoys the same integrability properties as $\overline{p}$ (stated in Proposition 2.11).

The strengthened assumptions concerning the integrability thresholds in Theorem 2.6 with respect to the natural ones appearing in (2.33) might seem awkward at first sight. It is actually the specific current framework, which involves as a proxy a stochastic integral with respect to a stable-like jump process and its associated iterated integrals that leads to additional constraints on the regularity indexes needed for our method to work.

The natural approach to get rid of the flow involving the integration variable in (2.35) would have been to use the approximate Lipschitz property of the flow established in Lemma 2.14. This indeed readily yields that:

$$|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)|^\gamma \leq C(1 + |\mathbb{T}_{s-t}^{-1}(y-\theta_{s,t}(x))|^\gamma).$$

The main difficulty is that we do not actually succeed in establishing in whole generality that:

$$\overline{p}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)) \leq C\check{p}(1, \mathbb{T}_{s-t}^{-1}(y-\theta_{s,t}(x)), \tag{2.37}$$

for a density $\check{p}$ which shares the same integrability properties as $\bar{p}$.

Equation (2.37) is absolutely direct in the diffusive setting from the explicit form of the Gaussian density and it has been thoroughly used in [CdRM20b] to derive sharp thresholds for weak uniqueness. It is clear that the above control has to be considered point-wise and one of the huge difficulties with stable type processes consists in describing precisely their tail behavior which is actually very much related to the geometry of their corresponding spectral measure on the sphere. We refer to the seminal work of Watanabe [Wat07] for a precise description of the tails in terms of the dimension of the

support of the spectral measure, in the stable case, and to the extension by Sztonyk [Szt10a] for the tempered stable case. The delicate point comes of course from the behavior of the Poisson measure (large jumps) as illustrated in the following computation. From (2.23) and (2.25), we write that

$$\begin{aligned}\overline{p}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)) &= \int_{\mathbb{R}^N} p_{\bar{M}}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)-w)\,\overline{P}_1(dw)\\ &\le C\int_{\mathbb{R}^N} \frac{1}{(C+|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)-w|)^M}\,\overline{P}_1(dw).\end{aligned}$$

- Let us first emphasize that, when $|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)| \le K$ (*diagonal type regime*) for some fixed K, then Control (2.37) holds. Indeed, since from Corollary 2.15,

$$|\mathbb{T}_{s-t}^{-1}(\tilde{m}_{s,t}^{t,y}(x)-\theta_{s,t}(x))| \le \tilde{C}(s-t)^{\frac{1}{\alpha}\wedge\zeta}\left(1+|\mathbb{T}_{s-t}^{-1}(\theta_{s,t}(x)-y)|\right),$$

we would get, recalling from Lemma 2.9, Equation (2.6) that $\theta_{t,s}(y)-x = y-\tilde{m}_{s,t}^{s,y}(x)$, that

$$\begin{aligned}\overline{p}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)) &\le C\int_{\mathbb{R}^N} \frac{1}{(C+|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)-w|)^M}\,\overline{P}_1(dw)\\ &\le C\int_{\mathbb{R}^N} \frac{1}{([C+|\mathbb{T}_{s-t}^{-1}(y-\theta_{s,t}(x))-w|-(s-t)^{\frac{1}{\alpha}\wedge\zeta}|\mathbb{T}_{s-t}^{-1}(\theta_{s,t}(x)-y)|]\vee 1)^M}\,\overline{P}_1(dw)\\ &\le C\int_{\mathbb{R}^N} \frac{1}{([\check{C}+|\mathbb{T}_{s-t}^{-1}(y-\theta_{s,t}(x))-w|)^M}\,\overline{P}_1(dw)\\ &=: \check{p}(1, \mathbb{T}_{s-t}^{-1}(y-\theta_{s,t}(x))),\end{aligned}$$

and $\check{p}$ plainly satisfies the required integrability conditions. These computations actually emphasize that (2.37) holds, up to a modification of $\check{C}$ above, up to the threshold

$$|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)| \le c_0(s-t)^{-(\frac{1}{\alpha}\wedge\zeta)},$$

for some $c_0>0$ small enough with respect to C. It would therefore remain to investigate the complementary *very off-diagonal regime*.

- Let us now concentrate on the *off-diagonal regime* $|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)| > K$. In that case, we write:

$$\begin{aligned}\overline{p}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)) &\le C\int_{\mathbb{R}^N} \frac{1}{(C+|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)-w|)^M}\overline{P}_1(dw)\\ &\le C\int_0^1 \overline{P}_1(\{w\in\mathbb{R}^N\colon (1+|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)-w|)^{-M}|>u\})du\\ &\le C\int_0^1 \overline{P}_1(B(\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x), u^{-1/M})du. \qquad (2.38)\end{aligned}$$

It now follows from the proof of Proposition 2.10 that the support of the spectral measure on $\mathbb{S}^{N-1}$ associated with $\{\tilde{S}_u^{\tau,\xi,t,s}\}_{u\ge 0}$ has dimension d. The related concentration properties also transmit to $\bar{N}_1$ (see the proofs of Proposition 2.11 and Lemma 5.2).

Thus, we get from [Wat07], [Szt10a] (see respectively Lemma 3.1 and Corollary 6 in those references) that there exists a constant $C > 0$ such that for all z $\mathbb{R}^N$ and $r > 0$:

$$\overline{P}_1(B(z,r)) \leq Cr^{d+1}(1+r^\alpha)|z|^{-(d+1+\alpha)}. \tag{2.39}$$

In other words, the global bound is given by the worst decay deriving from the dimension of the support of the spectral measure. In the current case $|z| \geq K$, this bound is clearly of interest for *large* values of z. Hence, from (2.38) and (2.39), it holds that

$$\begin{aligned}\overline{p}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)) &\leq C\int_0^1 u^{-(d+1)/M}(1+u^{-\alpha/M})\,du|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)|^{-(d+1+\alpha)} \\ &\leq C(1+|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)|)^{-(d+1+\alpha)}\int_0^1[u^{-(d+1)/M}+u^{-(d+1+\alpha)/M}du],\end{aligned}$$

Choosing $M > d+1+\alpha$ then gives that there exists $C \geq 1$ such that

$$\overline{p}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)) \leq C(1+|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)|)^{-(d+1+\alpha)}.$$

We thus get from Lemma 2.14, up to a modification of C, that:

$$\overline{p}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)) \leq C(1+|\mathbb{T}_{s-t}^{-1}(y-\theta_{s,t}(x))|)^{-(d+1+\alpha)}. \tag{2.40}$$

This actually leads to strong dimension constraints for this bound to be integrable. This phenomenon already appeared e.g. in [HM16] and induced therein to consider $d = 1, n = 3$ at most to address the well posedness of the martingale problem associated with a linear drift and a multiplicative isotropic stable noise. Those thresholds and dimension constraints remain with this approach.
Actually, from the threshold appearing in (2.3), we would like to consider the left-hand side of (2.35) with $\gamma > \frac{1+\alpha}{1+2\alpha}$ corresponding to $j = 3 = n$ therein. From Control (2.40), this would require $-\frac{1+\alpha}{1+2\alpha} + (d+1+\alpha) > 3, d = 1 \iff \alpha^2 - \alpha - 1 > 0$, which in our framework imposes that $\alpha \in (\frac{1+\sqrt{5}}{2}, 2)$.

Another possibility would have been, in the tempered case, to keep track of the tempering function, instead of bounding $\tilde{p}^{\tau,\xi}$ by a self-similar density $\bar{p}$, in order to benefit from the tempering at infinity to compensate the bad concentration rate in (2.40). However, see [HM16] and [Szt10a], we would have obtained bounds of the form

$$\tilde{p}^{\tau,\xi}(t,s,x,y) \leq C(1+|\mathbb{T}_{s-t}^{-1}(y-\theta_{s,t}(x))|)^{-(d+1+\alpha)}Q\left(|\mathbb{M}_{s-t}^{-1}(y-\theta_{s,t}(x))|\right).$$

Such a bound will give space integrability but deteriorates as well the time-integrability. This difficulty would occur even in the truncated case, thoroughly studied in the non-degenerate case by Chen *et al.* [CKK08]. Thus, we will develop here another approach.

Namely, we would like to change variable to $\bar{y} := \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)$ in the left-hand side of Equation (2.35). Of course, this is not bluntly possible since the coefficients at hand are not smooth enough. The point is then to introduce a flow $\theta_{t,s}^\delta(y)$ associated with mollified coefficients (for which the difference with respect to the initial flow will be controlled similarly to what is done to establish the approximate Lipschitz property of the flows in Lemma 2.14) and then, to control $\det(\nabla\theta_{t,s}^\delta(y))$ (see Lemma 5.3 below).

Since we do not have here a summation with respect to the single rescaled components as in the previous Lemma 2.14 above or as in Corollary 2.15, this will conduct to reinforce our assumptions and suppose that $(F_i)_{i\in[\![2,n]\!]}$ has the same regularity with respect to the variable x_j, $j \in [\![2,n]\!]$, whatever the level of the chain. This is precisely what leads to consider the condition

- $x_j \to F_i(t, x_i, \dots, x_j, \dots, x_n)$ is β^j-Hölder continuous, uniformly in t and in x_k for $k \neq j$, with
$$\beta^j \;>\; \frac{1+\alpha(j-2)}{1+\alpha(j-1)}.$$

For the sake of clarity the proof of Corollary 2.16 is postponed to the Appendix.

Let us introduce now some useful tools for the study of the martingale problem for $\partial_s + L_s$. The first step is to consider a suitable Green-type kernel associated with the frozen density $\tilde{p}^{s,y}$ and establish which Cauchy-like problem it solves. Namely, we define for any function $f\colon [0,T]\times\mathbb{R}^N \to \mathbb{R}$ regular enough, the *pseudo* Green kernel $\tilde{G}_\epsilon$ given by:

$$\tilde{G}_\epsilon f(t,x) \;:=\; \int_{(t+\epsilon)\wedge T}^{T}\int_{\mathbb{R}^N} \tilde{p}^{s,y}(t,s,x,y) f(s,y)\, dyds, \quad (t,x)\in[0,T)\times\mathbb{R}^N, \tag{2.41}$$

where ϵ is meant to be small.
We only remark that the above Green kernel $\tilde{G}_\epsilon$ is well-defined, since the frozen density $\tilde{p}^{s,y}(t,s,x,y)$ is measurable in (s,y) thanks to Lemma 2.13 (measurability of the flow in these parameters).

Proposition 2.17. *Let p, q in $(1,+\infty)$ such that the integrability Condition $(\mathscr{C})$ holds. Then, there exists a positive constant $C := C(T,p,q)$ such that for any f in $L^p\Big(0,T;L^q(\mathbb{R}^N)\Big)$,*
$$\|\tilde{G}_\epsilon f\|_\infty \;\le\; C\|f\|_{L^p_tL^q_x}.$$
Moreover, it holds that $\lim_{T\to 0} C(T,p,q) = 0$.

Proof. We start using the Hölder inequality in order to split the component with f and the part with the density $\tilde{p}(t,s,x,y)$:

$$\begin{aligned}|\tilde{G}_\epsilon f(t,x)| \;&\le\; C\|f\|_{L^p_tL^q_x}\left(\int_{(t+\epsilon)\wedge T}^{T}\left(\int_{\mathbb{R}^N}|\tilde{p}^{s,y}(t,s,x,y)|^{q'}\,dy\right)^{\frac{p'}{q'}}ds\right)^{\frac{1}{p'}}\\ &=:\; C\|f\|_{L^p_tL^q_x}|I_\epsilon(t,x)|,\end{aligned}$$

where we have denoted by p', q' the conjugate of p and q, respectively.
In order to control the remainder term $I_\epsilon(t,x)$, we now apply (2.19) from Proposition 2.11 with $k=0$ and $(\tau,\xi)=(s,y)$ to write that

$$|I_\epsilon(t,x)|^{p'} \;\le\; C\int_{(t+\epsilon)\wedge T}^{T}\left(\int_{\mathbb{R}^N}\left(\frac{1}{\det\mathbb{T}_{s-t}}\overline{p}\left(1,\mathbb{T}_{s-t}^{-1}(y-\tilde{m}^{s,y}_{s,t}(x))\right)\right)^{q'}dy\right)^{\frac{p'}{q'}}ds,$$

where we recall that $\mathbb{T}_t = t^{1/\alpha}\mathbb{M}_t$ (see (2.17) and (2.8)).
From Corollaries 2.12 and 2.16, we then write that

$$\begin{aligned}|I_\epsilon(t,x)|^{p'} &\le C\int_{(t+\epsilon)\wedge T}^T \left(\int_{\mathbb{R}^N} \frac{1}{(\det \mathbb{T}_{s-t})^{q'}}\overline{p}\left(1, \mathbb{T}_{s-t}^{-1}(y-\tilde{m}_{s,t}^{s,y}(x))\right)\,dy\right)^{\frac{p'}{q'}} ds \\ &\le C\int_{(t+\epsilon)\wedge T}^T (\det \mathbb{T}_{s-t})^{\frac{p'}{q'}-p'} ds \;=\; C\int_{(t+\epsilon)\wedge T}^T \frac{1}{(\det \mathbb{T}_{s-t})^{\frac{p'}{q}}} ds.\end{aligned}$$

Since by definition of matrix $\mathbb{T}_t$, it holds that

$$\det \mathbb{T}_{s-t} \;=\; (s-t)^{\sum_{i=1}^n d_i \frac{1+\alpha(i-1)}{\alpha}}, \tag{2.42}$$

we can conclude that under the integrability assumption $(\mathscr{C})$, we have that

$$\Big(\sum_{i=1}^n d_i \frac{1+\alpha(i-1)}{\alpha}\Big)\frac{p'}{q} \;<\; 1 \;\Leftrightarrow\; \Big(\sum_{i=1}^n d_i \frac{1+\alpha(i-1)}{\alpha}\Big)\frac{1}{q} + \frac{1}{p} \;<\; 1. \tag{2.43}$$

The proof is complete. □

Now, we understand which Cauchy-like problem is solved by the "density" $\tilde{p}^{s,y}(t,s,x,y)$ frozen at the terminal point (s,y). We start denoting by $\tilde{L}_t^{s,y}$ the infinitesimal generator of the proxy process $\{\tilde{X}_s^{s,y,t,x}\}_{s\in[t,T]}$. For any smooth function $\phi\colon \mathbb{R}^N \to \mathbb{R}$, it writes:

$$\begin{aligned}\tilde{L}_t^{s,y}\phi(x) &:= \langle A_t x + \tilde{F}_t^{s,y}, D_x\phi(x)\rangle + \tilde{\mathcal{L}}_t^{s,y} \\ &:= \langle A_t x + \tilde{F}_t^{s,y}, D_x\phi(x)\rangle + \int_{\mathbb{R}_0^d}\Big[\phi(x + B\tilde{\sigma}_t^{s,y}w) - \phi(x)\Big]\,\nu(dw),\end{aligned} \tag{2.44}$$

where, we recall, $\tilde{F}_t^{s,y} := F(t,\theta_{t,s}(y))$ and $\tilde{\sigma}_t^{s,y} := \sigma(t,\theta_{t,s}(y))$.
By direct calculation, it is not difficult to check now that for any (s,x,y) in $[0,T]\times\mathbb{R}^{2N}$ it holds that

$$\Big(\partial_t + \tilde{L}_t^{s,y}\Big)\,\tilde{p}^{s,y}(t,s,x,z) \;=\; 0, \quad (t,z)\in[0,s)\times\mathbb{R}^N. \tag{2.45}$$

However, we carefully point out that some attention is requested to establish the following lemma, which is crucial to derive which Cauchy-type problem the function $\tilde{G}f := \lim_{\epsilon\to 0}\tilde{G}_\epsilon f$ actually solves. In particular, it is important to highlight that Lemma 2.18 (Dirac Convergence of frozen density) below cannot be obtained directly from the convergence in law of the frozen process $\tilde{X}_s^{s,y,t,x}$ towards the Dirac mass (cf. Equation (2.45)). Indeed, the integration variable y also appears as a freezing parameter which makes the argument more complicated.
The proofs of the following two lemmas is quite involved and technical. For this reason, we decided to postpone them to the Appendix, Section 5.2.

Lemma 2.18. *Let (t,x) be in $[0,T)\times\mathbb{R}^N$ and $f\colon\mathbb{R}^N\to\mathbb{R}$ a bounded continuous function. Then,*

$$\lim_{\epsilon\to 0}\left|\int_{\mathbb{R}^N} f(y)\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)\,dy - f(x)\right| \;=\; 0.$$

Moreover, the above limit is uniform with respect to t in $[0,T]$.

A similar result involving the $L^p_t L^q_x$-norm can also be obtained. For notational simplicity, let us set

$$I_\epsilon f(t,x) \;:=\; \int_{\mathbb{R}^N} f(t+\epsilon,y)\mathbb{1}_{[0,T-\epsilon]}(t)\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)\,dy \tag{2.46}$$

for any sufficiently regular function $f\colon [0,T]\times\mathbb{R}^N \to \mathbb{R}$.

Lemma 2.19. *Let $p>1$, $q>1$ and f in $C^{1,2}_c([0,T)\times\mathbb{R}^N)$. Then,*

$$\lim_{\epsilon\to 0} \|I_\epsilon f - f\|_{L^p_t L^q_x} \;=\; 0.$$

We want now to understand which Cauchy-like problem is solved by our frozen Green kernel $\tilde{G}_\epsilon f(t,x)$. For this reason, we introduce for any function f in $C^{1,2}_0([0,T)\times\mathbb{R}^N,\mathbb{R})$ the following quantity:

$$\tilde{M}_\epsilon f(t,x) \;:=\; \int_{t+\epsilon}^T \int_{\mathbb{R}^N} \tilde{L}^{s,y}_t \tilde{p}^{s,y}(t,s,x,y) f(s,y)\,dyds, \quad (t,x)\in[0,T)\times\mathbb{R}^N, \tag{2.47}$$

for some fixed $\epsilon>0$ that is assumed to be small enough. Then, we can derive from Equation (2.45) and Proposition 2.11 that the following equality holds:

$$\partial_t \tilde{G}_\epsilon f(t,x) + \tilde{M}_\epsilon f(t,x) \;=\; -I_\epsilon f(t,x), \quad (t,x)\in[0,T)\times\mathbb{R}^N, \tag{2.48}$$

where we used the same notation in (2.46) for $I_\epsilon f$. We point out that the localization with respect to ϵ is precisely needed to exploit directly (2.45) and thus, to derive (2.48) for any fixed $\epsilon>0$, by usual dominated convergence arguments. In particular, we point out that in the limit case ($\epsilon\to 0$), the smoothness on f is not a sufficient condition to derive the smoothness of $\tilde{G}f$. This is again due to the dependence of the proxy upon the integration variable.

3 Well-Posedness of the martingale problem

This section is devoted to the proof of the well-posedness of the martingale problem for $\partial_s + L_s$ with initial condition (t,x), under the assumptions of Theorem 2.6.

Since by definition the paths of any solution $\{X_t\}_{t\geq 0}$ of the martingale problem for $\partial_s + L_s$ are càdlàg, it will be convenient afterwards to give an alternative definition. We denote by $\mathcal{D}[0,\infty)$ the family of all the càdlàg paths from $[0,\infty)$ to $\mathbb{R}^N$, equipped with the "standard" Skorokhod topology. For further details, we suggest the interested reader to see [Bas11], [EK86] or [JS03].
Fixed a starting point (t,x) in $[0,\infty)\times\mathbb{R}^N$, we will say that a a probability measure $\mathbb{P}$ on $\mathcal{D}[0,\infty)$ is a solution of the martingale problem for $\partial_t + L_t$ starting at (t,x) if the coordinate process $\{y_t\}_{t\geq 0}$ on $\mathcal{D}[0,\infty)$, defined by

$$y_t(\omega) \;=\; \omega(t), \quad \omega\in\mathcal{D}[0,\infty)$$

is a solution (in the previous sense) of the martingale problem for $\partial_s + L_s$.
Similarly, we will say that uniqueness holds for the martingale problem for $\partial_s + L_s$ with starting point (t,x) if

$$\mathbb{P}\circ y^{-1} \;=\; \tilde{\mathbb{P}}\circ y^{-1},$$

for any two solutions $\mathbb{P}$, $\tilde{\mathbb{P}}$ of the martingale problem for $\partial_s + L_s$ starting at (t, x).

The existence of a solution $\mathbb{P}$ of the martingale problem for $\partial_s + L_s$ can be obtained adapting the proof of Theorem 2.2 in [Str75] exploiting the sublinear structure of the drift F and localization arguments in order to deal with possibly unbounded coefficients.

Proposition 3.1 (existence)**.** *Under the assumptions of Theorem 2.6, let (t, x) be in $[0, \infty) \times \mathbb{R}^N$. Then, there exists a solution $\mathbb{P}$ of the martingale problem for $\partial_s + L_s$ starting at (t, x).*

We move to the question of uniqueness for the martingale problem associated with $\partial_s + L_s$. As shown already in the introduction of Section 3, the analytical properties on the frozen process $(\tilde{X}_u^{s,y,t,x})_{u\in[t,T]}$ we presented there will be the crucial tools for the reasoning in the following section.
We will start proving directly that the Krylov-type estimates (2.4) holds but first for p, q big enough (but finite). It will imply in particular the existence of a density for the canonical process associated with any solution of the martingale problem. As a consequence, the weak well-posedness of SDE (1.3) under our assumptions can be shown to hold.
Only in a second moment, we will then show that the Krylov estimates holds for *any* p, q satisfying condition $(\mathscr{C})$ through a regularization technique. Namely, we regularize the driving noise Z_t by introducing an additional isotropic α-stable process depending from a regularizing parameter. Following the previous arguments for the regularized dynamics, we will then prove that the solution process satisfies again the Krylov-type estimates for any p, q in the considered range, *uniformly* with respect to the regularizing parameter.
Letting the regularizing parameter go to zero, we will then conclude the proof of Corollary 2.7.

3.1 Uniqueness of martingale problem

The first step in proving the uniqueness of the Martingale problem for $\partial_s + L_s$ is to show that any solution to the martingale problem satisfies the Krylov-like estimates in Equation (2.4). To do so, we prove that the difference operator between the genuine generator L_t and a suitable associated perturbation (associated with the frozen generator $\tilde{L}_t^{s,y}$ given in (2.44)) has small $L_t^p L_x^q$-norm when considering a sufficiently small final horizon T. Namely, we introduce the following remainder:

$$\tilde{R}_\epsilon f(t,x) \;:=\; (L_t \tilde{G}_\epsilon f - \tilde{M}_\epsilon f)(t,x) \;=\; \int_{t+\epsilon}^T \int_{\mathbb{R}^N} (L_t - \tilde{L}_t^{s,y}) \tilde{p}^{s,y}(t,s,x,y) f(s,y)\, dyds, \quad (3.1)$$

for some ϵ to be small enough. We recall that $\tilde{G}_\epsilon f, \tilde{M}_\epsilon f$ and $\tilde{p}^{s,y}(t,s,x,y)$ were defined in (2.41), (2.47) and (2.10), respectively.
We firstly present a point-wise control for the remainder term $\tilde{R}_\epsilon f$. Importantly, the constant C below does not depend on ϵ, allowing to pass to the limit in Equation (3.1). This will be discussed at the end of the present section.

Proposition 3.2. *There exist $q_0 > 1$, $p_0 > 1$ and $C := C(T, p_0, q_0)$ such that for any*

$q \geq q_0$, $p \geq p_0$ and any f in $L^p([0,T]; L^q(\mathbb{R}^N))$, it holds that

$$\|\tilde{R}_\epsilon f\|_\infty \leq C\|f\|_{L^p_t L^q_x}. \tag{3.2}$$

Proof. We start recalling from (2.2)-(2.44) (exploiting also the change of truncation in (1.5))) that we can decompose $\tilde{R}_\epsilon f$ in the following way:

$$\begin{aligned}\tilde{R}_\epsilon f(t,x) &= \int_{t+\epsilon}^T \int_{\mathbb{R}^N} (\mathcal{L}_t - \tilde{\mathcal{L}}_t^{s,y})\tilde{p}^{s,y}(t,s,x,y) f(s,y)\, dyds \\ &\quad + \int_{t+\epsilon}^T \int_{\mathbb{R}^N} \langle F(t,x) - \tilde{F}_t^{s,y}, D_x \tilde{p}^{s,y}(t,s,x,y)\rangle f(s,y)\, dyds \\ &=: \tilde{R}_\epsilon^0 f(t,x) + \tilde{R}_\epsilon^1 f(t,x)\end{aligned} \tag{3.3}$$

where the operators $\mathcal{L}_t$ and $\tilde{\mathcal{L}_t^{s,y}}$ have been defined in (2.2) and (2.44), respectively. Since by assumptions, $x_j \to F_i(t,x)$ is β^j-Hölder continuous, we can control the second term $\tilde{R}_\epsilon^1 f$, associated with the difference of the drifts, using Proposition 2.11 with $(\tau,\xi) = (s,y)$:

$$\begin{aligned}\left|\langle F(t,x) - \tilde{F}_t^{s,y}, D_x \tilde{p}^{s,y}(t,s,x,y)\rangle\right| &\leq \sum_{i=1}^n |F_i(t,x) - F_i(t,\theta_{t,s}(y))|\, |D_{x_i}\tilde{p}^{s,y}(t,s,x,y)| \\ &\leq C\sum_{i=1}^n (s-t)^{-\frac{1+\alpha(i-1)}{\alpha}} \frac{\overline{p}(1, \mathbb{T}_{s-t}^{-1}(y - \tilde{m}_{s,t}^{s,y}(x)))}{\det \mathbb{T}_{s-t}} \sum_{j=i}^n |(x-\theta_{t,s}(y))_j|^{\beta^j} \\ &\leq C\sum_{i=1}^n\sum_{j=i}^n (s-t)^{\zeta_i^j} \left|\mathbb{T}_{s-t}^{-1}(x-\theta_{t,s}(y))\right|^{\beta^j} \frac{\overline{p}(1, \mathbb{T}_{s-t}^{-1}(y - \tilde{m}_{s,t}^{s,y}(x)))}{\det \mathbb{T}_{s-t}},\end{aligned}$$

with the following notation at hand:

$$\zeta_i^j := -\frac{1+\alpha(i-1)}{\alpha} + \beta^j \frac{1+\alpha(j-1)}{\alpha}.$$

Then, we write with the notations of (3.3) that

$$\begin{aligned}\left|\tilde{R}_\epsilon^1 f(t,x)\right| &\leq C\sum_{i=1}^n\sum_{j=i}^n \int_t^T \int_{\mathbb{R}^N} |f(s,y)| \frac{\overline{p}(1, \mathbb{T}_{s-t}^{-1}(y - \tilde{m}_{s,t}^{s,y}(x)))}{\det \mathbb{T}_{s-t}} \frac{\left|\mathbb{T}_{s-t}^{-1}(x-\theta_{t,s}(y))\right|^{\beta^j}}{(s-t)^{-\zeta_i^j}}\, dyds \\ &=: C\sum_{i=1}^n\sum_{j=i}^n \int_t^T \int_{\mathbb{R}^N} |f(s,y)| \mathcal{I}_{ij}(t,s,x,y)\, dyds.\end{aligned} \tag{3.4}$$

Then, from the Hölder inequality,

$$\left|\tilde{R}_\epsilon^1 f(t,x)\right| \leq C\|f\|_{L^p_t L^q_x} \sum_{i=1}^n\sum_{j=i}^n \left(\int_t^T \left(\int_{\mathbb{R}^N} [\mathcal{I}_{ij}(t,s,x,y)]^{q'}\, dy\right)^{\frac{p'}{q'}} ds\right)^{\frac{1}{p'}}, \tag{3.5}$$

where q' and p' are the conjugate exponents of q and p, respectively.
Now, the integrals with respect to y can be easily controlled by Corollary 2.12. Indeed,

$$\begin{aligned}&\int_{\mathbb{R}^N} \left[\mathcal{I}_{ij}(t,s,x,y)\right]^{q'} dy \\ &\qquad \leq C\left(\frac{(s-t)^{\zeta_i^j}}{\det \mathbb{T}_{s-t}}\right)^{q'} \int_{\mathbb{R}^N} \left|\mathbb{T}_{s-t}^{-1}(x-\theta_{t,s}(y))\right|^{\beta^j q'} \overline{p}(1, \mathbb{T}_{s-t}^{-1}(y - \tilde{m}_{s,t}^{s,y}(x)))\, dy.\end{aligned} \tag{3.6}$$

Choosing $q_0 > 1$ big enough so that $\beta^j q' < \alpha$ for any j in $[\![1, n]\!]$ and any $q \geq q_0$, we can use Corollary 2.16 to show that

$$\int_{\mathbb{R}^N} [\mathcal{I}_{ij}(t, s, x, y)]^{q'} \, dy \;\leq\; C(s-t)^{\zeta_i^j q'} \left(\det \mathbb{T}_{s-t}\right)^{1-q'} . \tag{3.7}$$

Going back to Equation (3.5), we can thus write that

$$\left|\tilde{R}^1_\epsilon f(t,x)\right| \;\leq\; C\|f\|_{L^p_t L^q_x} \sum_{i=1}^n \sum_{j=i}^n \left(\int_t^T (s-t)^{\zeta_i^j p'} \left(\det \mathbb{T}_{s-t}\right)^{\frac{p'}{q'}-p'} ds \right)^{\frac{1}{p'}} .$$

Noticing now that for any $i \leq j$ in $[\![1, n]\!]$

$$\zeta_i^j > -1 \;\Leftrightarrow\; -\frac{1+\alpha(i-1)}{\alpha} + \beta^j \frac{1+\alpha(j-1)}{\alpha} > -1 \;\Leftrightarrow\; \beta^j > \frac{1+\alpha(i-2)}{1+\alpha(j-1)}, \tag{3.8}$$

we can choose $q_0 > 1$, $p_0 > 1$ large enough so that p', q' are sufficiently close to 1 in order to conclude that

$$\left|\tilde{R}^1_\epsilon f(t,x)\right| \;\leq\; C\|f\|_{L^p_t L^q_x}. \tag{3.9}$$

We can now focus on the control for the first remainder term $\tilde{R}^0_\epsilon f$. Since clearly $\tilde{R}^0_\epsilon f = 0$ if $\sigma(t,x)$ is constant in space, we can assume without loss of generality that ν is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^d$ (cf. assumption [**AC**]). In particular, we know that it can be decomposed as in (1.7):

$$\nu(dz) \;=\; Q(z) \frac{g(\frac{z}{|z|})}{|z|^{d+\alpha}} dz.$$

Given now a smooth enough function $\phi\colon \mathbb{R}^N \to \mathbb{R}$, we start noticing that

$$\begin{aligned}
\mathcal{L}_t \phi(x) \;&=\; \int_{\mathbb{R}^d_0} \left[\phi(x + B\sigma(t,x)z) - \phi(x)\right] \nu(dz) \\
&=\; \int_{\mathbb{R}^d_0} \left[\phi(x + B\sigma(t,x)z) - \phi(x)\right] Q(z) g\left(\frac{z}{|z|}\right) \frac{dz}{|z|^{d+\alpha}} \\
&=\; \int_{\mathbb{R}^d_0} \left[\phi(x + B\tilde{z}) - \phi(x)\right] Q(\sigma^{-1}(t,x)\tilde{z}) \frac{g\left(\frac{\sigma^{-1}(t,x)\tilde{z}}{|\sigma^{-1}(t,x)\tilde{z}|}\right)}{\det \sigma(t,x)} \frac{d\tilde{z}}{|\sigma^{-1}(t,x)\tilde{z}|^{d+\alpha}},
\end{aligned}$$

where we assumed, without loss of generality from [**UE**], that $\det \sigma(t,x) > 0$. A similar representation holds for $\tilde{\mathcal{L}}^{s,y}_t \phi(x)$, too. Now, let us introduce for any z in $\mathbb{R}^d$, the following quantity:

$$\begin{aligned}
&\tilde{H}^{s,y}_{t,x}(z) \\
&\quad:=\; Q(\sigma^{-1}(t,x)z) \frac{g\left(\frac{\sigma^{-1}(t,x)z}{|\sigma^{-1}(t,x)z|}\right)}{\det \sigma(t,x) |\sigma^{-1}(t,x)\frac{z}{|z|}|^{d+\alpha}} - Q((\tilde{\sigma}^{s,y}_t)^{-1}z) \frac{g\left(\frac{(\tilde{\sigma}^{s,y}_t)^{-1}z}{|(\tilde{\sigma}^{s,y}_t)^{-1}z|}\right)}{\det \tilde{\sigma}^{s,y}_t |(\tilde{\sigma}^{s,y}_t)^{-1}\frac{z}{|z|}|^{d+\alpha}},
\end{aligned}$$

where we have normalized z above in order to make the usual isotropic stable Lévy measure appear.

Fixed $\eta > 0$, local to this section, meant to be small and to be chosen later (and not to be confused with the ellipticity constant in assumption [**UE**]), we then define

$$\alpha_\eta = \alpha/(1-\eta), \tag{3.10}$$

and we decompose the integral in the difference of the generators in the following way:

$$\begin{aligned}
\left(\mathcal{L}_t - \tilde{\mathcal{L}}_t^{s,y}\right)\phi(x) &= \int_{\mathbb{R}_0^d} \left[\phi(x+Bz) - \phi(x)\right] \tilde{H}_{t,x}^{s,y}(z)\, s\frac{dz}{|z|^{d+\alpha}} \\
&= \int_{\Delta_\eta} \left[\phi(x+Bz) - \phi(x) - \langle D_x\phi(x), Bz\rangle\right] \tilde{H}_{t,x}^{s,y}(z)\, \frac{dz}{|z|^{d+\alpha}} \\
&\qquad + \int_{\Delta_\eta^c} \left[\phi(x+Bz) - \phi(x)\right] \tilde{H}_{t,x}^{s,y}(z)\, \frac{dz}{|z|^{d+\alpha}} \\
&=: \sum_{i=1}^{2} \left[\Delta_i \phi(t,s,\cdot,y)\right](x),
\end{aligned}$$

where we have denoted, for simplicity,

$$\begin{aligned}
\Delta_\eta &:= B(0,(s-t)^{\frac{1}{\alpha\eta}}); \\
\Delta_\eta^c &:= B^c(0,(s-t)^{\frac{1}{\alpha\eta}}).
\end{aligned}$$

We highlight in particular that it is precisely the symmetry of ν that ensures that the function $\tilde{H}_{t,x}^{s,y}$ is even and that allow us to introduce the odd first order term $\langle D_x\phi(x), Bz\rangle$ in the first integral above on the simmetric space Δ_η.
Noticing from Proposition 2.10 that the frozen "density" $\tilde{p}^{s,y}$ is regular enough in x, we can now replace ϕ in the above decomposition with $\tilde{p}^{s,y}(t,s,\cdot,y)$. Going back to $\tilde{R}_\epsilon^0 f$ given in (3.3), we start rewriting it as

$$\begin{aligned}
|\tilde{R}_\epsilon^0 f(t,x)| &\le C\sum_{i=1}^{2}\int_t^T\int_{\mathbb{R}^N} |f(s,y)|\, \left|\left[\Delta_i \tilde{p}^{s,y}(t,s,\cdot,y)\right](x)\right| dyds \\
&=: \sum_{i=1}^{2}\int_t^T\int_{\mathbb{R}^N} |f(s,y)| \mathcal{I}_{0i}(t,s,x,y)\, dyds.
\end{aligned} \tag{3.11}$$

As before, we can then apply Hölder inequality to show that

$$\left|\tilde{R}_\epsilon^0 f(t,x)\right| \le C\|f\|_{L_t^p L_x^q} \sum_{i=1}^{2}\left(\int_t^T \left(\int_{\mathbb{R}^N} \left[\mathcal{I}_{0i}(t,s,x,y)\right]^{q'} dy\right)^{\frac{p'}{q'}} ds\right)^{\frac{1}{p'}}, \tag{3.12}$$

where q' and p' are again the conjugate exponents of q and p, respectively.
To control the second term involving $\mathcal{I}_{02}$, we start noticing that

$$|\tilde{H}_{t,x}^{s,y}(z)| \le C \tag{3.13}$$

for some constant C independent from the parameters, thanks to assumption [**UE**] for σ and the boundedness of g and Q.

Then, we can use Control (3.13), Corollary 2.12 and the Hölder inequality to write that

$$\begin{aligned}
|\mathcal{I}_{02}(t,s,x,y)|^{q'} &\leq C\Big[\int_{\Delta_\eta^c} |\tilde{p}^{s,y}(t,s,x+Bz,y) - \tilde{p}^{s,y}(t,s,x,y)| \frac{dz}{|z|^{d+\alpha}}\Big]^{q'} \\
&\leq C\left(\int_{\Delta_\eta^c} \frac{dz}{|z|^{d+\alpha}}\right)^{\frac{q'}{q}} \int_{\Delta_\eta^c} |\tilde{p}^{s,y}(t,s,x+Bz,y) - \tilde{p}^{s,y}(t,s,x,y)|^{q'} \frac{dz}{|z|^{d+\alpha}} \\
&\leq C\frac{(s-t)^{(\eta-1)\frac{q'}{q}}}{(\det \mathbb{T}_{s-t})^{q'}} \int_{\Delta_\eta^c} \Big[\overline{p}(1,\mathbb{T}_{s-t}^{-1}(y-\theta_{s,t}(x+Bz))) + \overline{p}(1,\mathbb{T}_{s-t}^{-1}(y-\theta_{s,t}(x)))\Big] \frac{dz}{|z|^{d+\alpha}},
\end{aligned}$$

recalling from (3.10) that $\alpha_\eta = \alpha/(1-\eta)$ for the last inequality. The Fubini theorem and the change of variables $\tilde{y} = y - \theta_{s,t}(x+Bz)$ now show that

$$\begin{aligned}
\int_{\mathbb{R}^N} |\mathcal{I}_{02}(t,s,x,y)|^{q'}\, dy &\leq 2C\frac{(s-t)^{(\eta-1)\frac{q'}{q}}}{(\det \mathbb{T}_{s-t})^{q'-1}} \int_{B^c(0,(s-t)^{\frac{1}{\alpha\eta}})} \int_{\mathbb{R}^N} \overline{p}(1,\tilde{y})\, d\tilde{y} \frac{dz}{|z|^{d+\alpha}} \\
&\leq C(\det \mathbb{T}_{s-t})^{1-q'}(s-t)^{(\eta-1)\frac{q'}{q}} \int_{\Delta_\eta^c} \frac{dz}{|z|^{d+\alpha}} \\
&\leq C(\det \mathbb{T}_{s-t})^{1-q'}(s-t)^{q'(\eta-1)}.
\end{aligned} \tag{3.14}$$

Going back to Equation (3.12), we can then conclude that

$$\begin{aligned}
\int_t^T \left(\int_{\mathbb{R}^N} |\mathcal{I}_{02}(t,s,x,y)|^{q'}\, dy\right)^{\frac{p'}{q'}} ds &\leq C\int_t^T (\det \mathbb{T}_{s-t})^{-\frac{p'}{q}}(s-t)^{p'(\eta-1)}\, ds \\
&\leq C\int_t^T (s-t)^{-p'(1-\eta+\frac{1}{q}\sum_{i=1}^n d_i \frac{1+\alpha(i-1)}{\alpha})}\, ds,
\end{aligned}$$

where in the last step we also exploited (2.42).
Assuming now that $\eta < 1$ and p, q are big enough so that

$$p'(1-\eta+\frac{1}{q}\sum_{i=1}^n d_i \frac{1+\alpha(i-1)}{\alpha}) < 1,$$

we immediately obtain that

$$\left(\int_t^T \left(\int_{\mathbb{R}^N} |\mathcal{I}_{02}(t,s,x,y)|^{q'}\, dy\right)^{\frac{p'}{q'}} ds\right)^{\frac{1}{p'}} \leq C_T. \tag{3.15}$$

We can now focus on the integral with respect to y of the first term $\mathcal{I}_{01}$ in Equation (3.12). Using the Lipschitz continuity of Q in a neighborhood of zero and the Hölder regularity of the diffusion matrix σ, it is not difficult to check that

$$|\tilde{H}_{t,x}^{s,y}(z)| \leq C\sum_{j=1}^n |(x-\theta_{t,s}(y))_j|^{\beta^1}.$$

Thanks to the above estimate, we exploit a Taylor expansion on the density $\tilde{p}^{s,y}$ and Proposition 2.11 with $k=2$ and $(\tau,\xi)=(s,y)$ to show that

$$\begin{aligned}
\Theta(t,s,x,y,z) &:= \left|\left[\tilde{p}^{s,y}(t,s,x+Bz,y)-\tilde{p}^{s,y}(t,s,x,y)-\langle D_x\tilde{p}^{s,y}(t,s,x,y),Bz\rangle\right]\tilde{H}^{s,y}_{t,x}(z)\right| \\
&\leq \sum_{j=1}^n\int_0^1 |(x-\theta_{t,s}(y))_j|^{\beta^1}\ |D^2_{x_1}\tilde{p}^{s,y}(t,s,x+\lambda Bz,y)||z|^2\,d\lambda \\
&\leq \frac{C}{\det\mathbb{T}_{s-t}}\int_0^1 |z|^2\frac{\overline{p}(1,\mathbb{T}^{-1}_{s-t}(y-\tilde{m}^{s,y}_{s,t}(x+\lambda Bz)))}{(s-t)^{\frac{2}{\alpha}}} \\
&\qquad\qquad\times\left[\sum_{j=1}^n|(x+\lambda Bz-\theta_{t,s}(y))_j|^{\beta^1}+|\lambda Bz|^{\beta^1}\right]d\lambda \\
&\leq \frac{C}{\det\mathbb{T}_{s-t}}\int_0^1 |z|^2\overline{p}(1,\mathbb{T}^{-1}_{s-t}(y-\tilde{m}^{s,y}_{s,t}(x+\lambda Bz))) \qquad (3.16)\\
&\qquad\qquad\times\left[\sum_{j=1}^n\frac{\left|\mathbb{T}^{-1}_{s-t}(x+\lambda Bz-\theta_{t,s}(y))\right|^{\beta^1}}{(s-t)^{\zeta_0^j}}+\frac{|z_1|^{\beta^1}}{(s-t)^{\frac{2}{\alpha}}}\right]d\lambda,
\end{aligned}$$

where, similarly to above, we have denoted:

$$\zeta_0^j := \frac{2}{\alpha}-\beta^1\frac{1+\alpha(j-1)}{\alpha}. \qquad (3.17)$$

It then follows from the Hölder inequality and Corollary 2.12 that

$$\begin{aligned}
|\mathcal{I}_{01}(t,s,x,y)|^{q'} &\leq \left[\int_{\Delta_\eta}\Theta(t,s,x,y,z)\,\frac{dz}{|z|^{d+\alpha}}\right]^{q'} \\
&\leq \frac{C}{(\det\mathbb{T}_{s-t})^{q'}}\left(\int_{\Delta_\eta}1\,dz\right)^{\frac{q'}{q}}\int_0^1\int_{\Delta_\eta}\overline{p}(1,\mathbb{T}^{-1}_{s-t}(y-\tilde{m}^{s,y}_{s,t}(x+\lambda Bz))) \\
&\qquad\times\left[\sum_{j=1}^n\frac{\left|\mathbb{T}^{-1}_{s-t}(x+\lambda Bz-\theta_{t,s}(y))\right|^{\beta^1}}{(s-t)^{\zeta_0^j}}+\frac{|z_1|^{\beta^1}}{(s-t)^{\frac{2}{\alpha}}}\right]^{q'}\frac{dz}{|z|^{q'(d+\alpha-2)}}d\lambda.
\end{aligned}$$

If we now add the integral with respect to y, Fubini Theorem readily implies that

$$\begin{aligned}
&\int_{\mathbb{R}^N}|\mathcal{I}_{01}(t,s,x,y)|^{q'}\,dy \\
&\quad\leq C\frac{(s-t)^{\frac{d}{\alpha\eta}(q'-1)}}{(\det\mathbb{T}_{s-t})^{q'}}\int_0^1\int_{\Delta_\eta}\int_{\mathbb{R}^N}\overline{p}(1,\mathbb{T}^{-1}_{s-t}(y-\tilde{m}^{s,y}_{s,t}(x+\lambda Bz))) \\
&\qquad\times\left[\sum_{j=1}^n\frac{\left|\mathbb{T}^{-1}_{s-t}(x+\lambda Bz-\theta_{t,s}(y))\right|^{\beta^1q'}}{(s-t)^{\zeta_0^jq'}}+\frac{|z_1|^{\beta^1q'}}{(s-t)^{\frac{2}{\alpha}q'}}\right]dy\frac{dz}{|z|^{q'(d+\alpha-2)}}d\lambda.
\end{aligned}$$

If we assume to have taken q' close enough to 1 so that $\beta^1q'<\alpha$, we can use Corollary

2.16 to show that

$$\begin{aligned}
&\int_{\mathbb{R}^N} |\mathcal{I}_{01}(t,s,x,y)|^{q'}\,dy \\
&\qquad \le C\frac{(s-t)^{\frac{d}{\alpha_\eta}(q'-1)}}{(\det \mathbb{T}_{s-t})^{q'-1}} \int_{B(0,(s-t)^{\frac{1}{\alpha_\eta}})} \left[\sum_{j=1}^n \frac{1}{(s-t)^{q'\zeta_0^j}} + \frac{|z_1|^{q'\beta^1}}{(s-t)^{q'\frac{2}{\alpha}}}\right] \frac{dz}{|z|^{q'(d+\alpha-2)}} \\
&\qquad \le C\frac{(s-t)^{\frac{d}{\alpha_\eta}(q'-1)}}{(\det \mathbb{T}_{s-t})^{q'-1}} \int_0^{(s-t)^{\frac{1}{\alpha_\eta}}} \left[\sum_{j=1}^n \frac{r^{d-1-(d+\alpha-2)q'}}{(s-t)^{q'\zeta_0^j}} + \frac{r^{d-1-(d+\alpha-2-\beta^1)q'}}{(s-t)^{\frac{2}{\alpha}q'}}\right] dr.
\end{aligned}$$

Similarly, if q is big enough (so that q' is close to 1), it holds that

$$d-1-q'(d+\alpha-2) > -1 \;\Leftrightarrow\; q' < \frac{d}{d+\alpha-2}$$

and we can integrate with respect to r:

$$\begin{aligned}
\int_{\mathbb{R}^N} |\mathcal{I}_{01}(t,s,x,y)|^{q'}\,dy \;&\le\; C\frac{(s-t)^{\frac{d}{\alpha_\eta}(q'-1)}}{(\det \mathbb{T}_{s-t})^{q'-1}} \left[\sum_{j=1}^n \frac{r^{d-q'(d+\alpha-2)}}{(s-t)^{q'\zeta_0^j}} + \frac{r^{d-q'(d+\alpha-2-\beta^1)}}{(s-t)^{q'\frac{2}{\alpha}}}\right]\Bigg|_0^{(s-t)^{\frac{1}{\alpha_\eta}}} \\
&\le\; C(\det \mathbb{T}_{s-t})^{1-q'}(s-t)^{\frac{q'}{\alpha_\eta}(2-\alpha)} \left[\sum_{j=1}^n (s-t)^{-q'\zeta_0^j} + (s-t)^{q'(\frac{\beta^1}{\alpha_\eta}-\frac{2}{\alpha})}\right]. \qquad (3.18)
\end{aligned}$$

Hence, it follows from Equation (2.42) that

$$\begin{aligned}
&\int_t^T \left(\int_{\mathbb{R}^N} |\mathcal{I}_{01}(t,s,x,y)|^{q'}\,dy\right)^{\frac{p'}{q'}} ds \\
&\qquad \le\; C\int_t^T (\det \mathbb{T}_{s-t})^{-\frac{p'}{q}}(s-t)^{\frac{p'}{\alpha_\eta}(2-\alpha)} \left[\sum_{j=1}^n (s-t)^{-p'\zeta_0^j} + (s-t)^{p'(\frac{\beta^1}{\alpha_\eta}-\frac{2}{\alpha})}\right] ds \\
&\qquad \le\; C\int_t^T (s-t)^{p'\left(\frac{(2-\alpha)}{\alpha_\eta}-\frac{1}{q}\sum_{i=1}^n d_i\frac{1+\alpha(i-1)}{\alpha}\right)} \left[\sum_{j=1}^n (s-t)^{-p'\zeta_0^j} + (s-t)^{p'(\frac{\beta^1}{\alpha_\eta}-\frac{2}{\alpha})}\right] ds
\end{aligned}$$

To conclude, we need to show that the two terms above are integrable with respect to s. Namely,

$$\begin{aligned}
p'\left(\frac{(2-\alpha)}{\alpha_\eta} - \frac{1}{q}\sum_{i=1}^n d_i\frac{1+\alpha(i-1)}{\alpha} - \zeta_0^j\right) \;&>\; -1, \quad \forall\, j \in [\![1,n]\!]; \\
p'\left(\frac{(2-\alpha)}{\alpha_\eta} - \frac{1}{q}\sum_{i=1}^n d_i\frac{1+\alpha(i-1)}{\alpha} + \frac{\beta^1}{\alpha_\eta} - \frac{2}{\alpha}\right) \;&>\; -1.
\end{aligned}$$

Recalling again that we can choose p, q big enough as we want, so that Equation (2.43) holds, it is now sufficient to take η in $(0,1)$ in order to have:

$$\frac{(2-\alpha)}{\alpha_\eta} - \zeta_0^j \;=\; \frac{(2-\alpha)}{\alpha_\eta} - \frac{2}{\alpha} + \beta^1\frac{1+\alpha(j-1)}{\alpha} \;>\; -1, \quad \forall\, j \in [\![1,n]\!]; \qquad (3.19)$$

$$\frac{(2-\alpha)}{\alpha_\eta} + \frac{\beta^1}{\alpha_\eta} - \frac{2}{\alpha} \;>\; -1. \qquad (3.20)$$

By direct calculations, recalling from (3.10) that $\alpha_\eta = \alpha/(1-\eta)$, we now notice that Conditions (3.19)-(3.20) can be rewritten as follows

$$\begin{aligned} \eta &< \frac{\beta^1(1+\alpha(j-1)}{2-\alpha}, \quad \forall\, j \in [\![1,n]\!]; \\ \eta &< \frac{\beta^1}{2+\beta^1-\alpha}. \end{aligned}$$

Choosing $\epsilon > 0$ so that the above conditions holds, we have that

$$\left(\int_t^T \left(\int_{\mathbb{R}^N} |\mathcal{I}_{01}(t,s,x,y)|^{q'}\, dy\right)^{\frac{p'}{q'}} ds\right)^{\frac{1}{p'}} \leq C_T. \tag{3.21}$$

Going back to Equation (3.12), we use Controls (3.15)-(3.21) to write that

$$\left|\tilde{R}^0_\epsilon f(t,x)\right| \leq C\|f\|_{L^p_t L^q_x}. \tag{3.22}$$

Exploiting Controls (3.9) and (3.22) in Equation (3.3), we have concluded our proof. □

A similar control in $L^p_t L^q_x$-norms can be obtained. In particular, we point out that Equation (3.23) below implies that the operator $I - \tilde{R}_\epsilon$ is invertible in $L^p\left(0,T;L^q(\mathbb{R}^N)\right)$, provided T is small enough. From Lemma 2.19, the same holds for $I_\epsilon - \tilde{R}_\epsilon$.

Proposition 3.3. *Let $q > 1$, $p > 1$ be such that Condition ($\mathscr{C}$) holds. Then, there exists $C := C(T,p,q) > 0$ such that for any f in $L^p\left(0,T;L^q(\mathbb{R}^N)\right)$,*

$$\|\tilde{R}_\epsilon f\|_{L^p_t L^q_x} \leq C\|f\|_{L^p_t L^q_x}. \tag{3.23}$$

In particular, it holds that $\lim_{T\to 0} C(T,p,q) = 0$.

Proof. We are going to keep the same notations used in the previous proof. In particular, we recall the following decomposition

$$\tilde{R}_\epsilon f(t,x) = \tilde{R}^0_\epsilon f(t,x) + \tilde{R}^1_\epsilon f(t,x),$$

given in Equation (3.3).
In order to control the second term $\tilde{R}^1_\epsilon f$ in $L^p_t L^q_x$-norm, we start from Equation (3.4) to write that

$$\|\tilde{R}^1_\epsilon f(t,\cdot)\|_{L^q_x} \leq C\sum_{i=1}^n \sum_{j=i}^n \int_t^T \left\|\int_{\mathbb{R}^N} |f(s,y)|\mathcal{I}_{ij}(t,s,\cdot,y)\, dy\right\|_{L^q_x} ds.$$

The Young inequality now implies that

$$\begin{aligned} \left\|\int_{\mathbb{R}^N} |f(s,y)|\mathcal{I}_{ij}(t,s,\cdot,y)\, dy\right\|^q_{L^q_x} &= \int_{\mathbb{R}^N} \left|\int_{\mathbb{R}^N} |f(s,y)|\mathcal{I}_{ij}(t,s,x,y)\, dy\right|^q dx \\ &\leq \int_{\mathbb{R}^n} \left(\int_{\mathbb{R}^N} |f(s,y)|^q \mathcal{I}_{ij}(t,s,x,y)\, dy\right) \|\mathcal{I}_{ij}(t,s,x,\cdot)\|^{q/q'}_{L^1} dx \\ &\leq C(s-t)^{\zeta^j_i q/q'} \int_{\mathbb{R}^N} |f(s,y)|^q \left(\int_{\mathbb{R}^N} \mathcal{I}_{ij}(t,s,x,y)\, dx\right) dy \end{aligned}$$

using Control (3.7) and the Fubini Theorem for the last inequality. From (3.4), (3.6) and the correspondence (2.6) which gives $y - \tilde{m}^{s,y}_{s,t}(x) = \theta_{t,s}(y) - x$ it is plain to derive that:

$$\int_{\mathbb{R}^N} dx \mathcal{J}_{ij}(t,s,x,y) \leq C(s-t)^{\zeta_i^j}.$$

Thus,

$$\|\tilde{R}^1_\epsilon f(t,\cdot)\|_{L^q_x} \leq C \sum_{i=1}^n \sum_{j=i}^n \int_t^T (s-t)^{\zeta_i^j} \|f(t,\cdot)\|_{L^q_x} ds,$$

where, in the last step, we exploited Equation (3.7) with $q' = 1$, recalling that $\beta^j < 1 < \alpha$.
We can then use the above control to write that

$$\begin{aligned}
\|\tilde{R}^1_\epsilon f\|^p_{L^p_t L^q_x} &\leq \sum_{i=1}^n \sum_{j=i}^n \int_0^T \|\tilde{R}^1_\epsilon f(t,\cdot)\|^p_{L^q_x} dt \\
&\leq C \sum_{i=1}^n \sum_{j=i}^n \int_0^T \|f(t,\cdot)\|^p_{L^q_x} \left(\int_t^T (s-t)^{\zeta_i^j} ds \right)^p dt \\
&\leq C_T \sum_{i=1}^n \sum_{j=i}^n \int_0^T \|f(t,\cdot)\|^p_{L^q_x} dt \leq C_T \|f\|^p_{L^p_t L^q_x},
\end{aligned}$$

where $C_T := C(p', q', T)$ denotes a positive constant that tends to zero if T goes to zero (recall indeed from (3.8) that $\zeta_i^j > -1$) .
The control for $\tilde{R}^0_\epsilon f$ can be obtained following the same arguments above, exploiting Equation (3.11) instead of (3.4) and Equations (3.14)-(3.18) with $q' = 1$ for the controls of $\|\mathcal{J}_{0j}(t,s,x,\cdot)\|_{L^1_x}$. □

Let us fix now a function f in $C^{1,2}_c([0,T) \times \mathbb{R}^N)$. The first step of our method consists in applying the Itô formula on the Green kernel $\tilde{G}_\epsilon f$ and the process $\{X^{t,x}_s\}_{s \in [t,T]}$, solution of the martingale problem with starting point (t,x):

$$\mathbb{E}\left[\tilde{G}_\epsilon f(t,x) + \int_t^T (\partial_s + L_s) \tilde{G}_\epsilon f(s, X^{t,x}_s) ds \right] = 0.$$

We then exploit Equation (2.48) to write that

$$\tilde{G}_\epsilon f(t,x) - \mathbb{E}\left[\int_t^T I_\epsilon f(s, X^{t,x}_s)\, ds \right] + \mathbb{E}\left[\int_t^T \left[L_s \tilde{G}_\epsilon f - \tilde{M}_\epsilon f \right](s, X^{t,x}_s)\, ds \right] = 0.$$

Thus, it holds that

$$\mathbb{E}\Big[\int_t^T I_\epsilon f(s, X^{t,x}_s)\, ds \Big] = \tilde{G}_\epsilon f(t,x) + \mathbb{E}\Big[\int_t^T \tilde{R}_\epsilon f(s, X^{t,x}_s)\, ds \Big]. \tag{3.24}$$

Thanks to Proposition 2.17, we know that there exists $C(T) := C(T) \underset{T \to 0}{\longrightarrow} 0$ such that

$$\|\tilde{G}_\epsilon f\|_\infty \leq C \|f\|_{L^p_t L^q_x}. \tag{3.25}$$

Let us assume for now that p, q are large enough so that the control (3.2) of Lemma 3.2 (pointwise control of the remainder) holds. From Equations (3.24), (3.25) and (3.2), we readily get that

$$\left|\mathbb{E}\left[\int_t^T I_\epsilon f(X_s^{t,x})\,ds\right]\right| \leq C\|f\|_{L_t^p L_x^q}.$$

Letting ϵ go to zero, we thus derive that any solution $\{X_s^{t,x}\}_{s\in[t,T]}$ of the martingale problem for $\partial_s + L_s$ with initial condition (t,x) satisfies

$$\left|\mathbb{E}\left[\int_t^T f(s, X_s^{t,x})\,ds\right]\right| \leq C\|f\|_{L_t^p L_x^q},$$

for any f in f in $C_c^{1,2}([0,T)\times\mathbb{R}^N)$. Above, we have exploited Lemma 2.18 for the integral in space and the bounded convergence Theorem for that in time.
To show the result for a general f in $L^p\left(0,T;L^q(\mathbb{R}^N)\right)$, we now use a density argument and the Fatou Lemma. Indeed, let $\{f_n\}_{n\in\mathbb{N}}$ a sequence of functions in $C_c^{1,2}([0,T)\times\mathbb{R}^N)$ such that $\|f_n - f\|_{L_t^p L_x^q} \to 0$. We then have that:

$$\begin{aligned}\left|\mathbb{E}\left[\int_t^T f(s, X_s^{t,x})\,ds\right]\right| &\leq \left|\mathbb{E}\left[\int_t^T \liminf_n f_n(s, X_s^{t,x})\,ds\right]\right| \\ &\leq \liminf_n \left|\mathbb{E}\left[\int_t^T f_n(s, X_s^{t,x})\,ds\right]\right| \\ &\leq C\liminf_n \|f_n\|_{L_t^p L_x^q} \\ &= C\|f\|_{L_t^p L_x^q}.\end{aligned} \tag{3.26}$$

This is precisely the Estimate (2.4) in Corollary 2.7, provided that p, q are large enough.

Thanks to Estimates (3.26), we then know that the process $\{X_s^{t,x}\}_{s\in[t,T]}$ has a density we will denote by $p(t,s,x,y)$. From Equation (3.24) it now follows that

$$\begin{aligned}\tilde{G}_\epsilon f(t,x) &= \mathbb{E}\left[\int_t^T I_\epsilon f(s, X_s^{t,x})\,ds\right] - \mathbb{E}\left[\int_t^T \tilde{R}_\epsilon f(s, X_s^{t,x})\,ds\right] \\ &= \int_t^T \int_{\mathbb{R}^N} I_\epsilon f(s,y) p(t,s,x,y)\,dyds - \int_t^T \int_{\mathbb{R}^N} \tilde{R}_\epsilon f(s,y) p(t,s,x,y)\,dyds \\ &= \int_t^T \int_{\mathbb{R}^N} (I_\epsilon - \tilde{R}_\epsilon) f(s,y) p(t,s,x,y)\,dyds.\end{aligned} \tag{3.27}$$

Then, Proposition 2.17, Lemma 2.19 (with an additional approximation argument) and Control (3.23) imply that both sides of the above control are bounded in the $L_t^p L_x^q$-norm, uniformly in $\epsilon > 0$. Thus, we can conclude that Equation (3.27) holds for any f in $L^p\left(0,T;L^q(\mathbb{R}^N)\right)$. We then conclude from Lemma 3.2 (pointwise control of the remainder) that letting ϵ go to zero, it holds that

$$\mathbb{E}\left[\int_t^T f(s, X_s^{t,x})\,ds\right] = \tilde{G}\circ(I-\tilde{R})^{-1} f(t,x),$$

which gives uniqueness if the final time T is small enough. Global well-posedness is again derived from a chaining argument in time.

To complete the proof of Corollary 2.7, it remains to derive the Krylov estimates (2.4) under Condition ($\mathscr{C}$) and not only for p, q large enough.
Fixed a parameter $\delta > 0$ meant to be small, we consider a "mollified" version of the solution process $X_s^{t,x}$, given by

$$\overline{X}_s^{t,x,\delta} \;:=\; X_s^{t,x} + \delta \mathbb{M}_{s-t} \overline{Z}_{s-t}, \tag{3.28}$$

where $\{\overline{Z}_s\}_{s\geq 0}$ is an isotropic α-stable process on $\mathbb{R}^N$.
Let us denote now by $p^\delta(t,s,x,\cdot)$ the density associated with the random variable $\overline{X}_s^{t,s,\delta}$. We notice that Equation (3.28) implies in particular that

$$p^\delta(t,s,x,y) \;=\; \Big[p(t,s,x,\cdot) * q^\delta(s-t,\cdot)\Big](y),$$

where $q^\delta(t,\cdot)$ is the density of the process $\delta \mathbb{M}_t \overline{Z}_t$ and thus, under the integrability condition ($\mathscr{C}$) and thanks to the Young inequality, the quantity $\|p^\delta\|_{L_t^{p'} L_x^{q'}}$, where p', q' are the conjugate exponents of p, q, respectively, is finite (possibly explosive with δ). The point is now to reproduce the previous perturbative analysis in order to prove that the controls on $\|p^\delta\|_{L_t^{p'} L_x^{q'}}$ actually do no depend on δ.

For this reason, we introduce the mollified "frozen" process $\tilde{X}_s^{s,y,t,x,\delta}$ along the flow $\theta_{t,s}(y)$ as

$$\tilde{X}_s^{s,y,t,x,\delta} \;:=\; \tilde{X}_s^{s,y,t,x} + \delta \mathbb{M}_{s-t} \overline{Z}_{s-t}. \tag{3.29}$$

Following the same arguments presented in Propositions 2.10 and 2.11, it is now possible to show that the process $\tilde{X}_s^{s,y,t,x,\delta}$ admits a density $\tilde{p}^{s,y,\delta}(t,s,x,y)$ and that it enjoys a multi-scale bound similar to (2.35). Namely,

Proposition 3.4. *There exists a positive constant $C := C(N,\alpha)$ such that for any k in $[\![0,2]\!]$, any i in $[\![1,n]\!]$, any $t < s$ in $[0,T]$ and any x, y in $\mathbb{R}^N$,*

$$|D_{x_i}^k \tilde{p}^{s,y,\delta}(t,s,x,y)| \;\leq\; C \frac{\big((s-t)(1+\delta)\big)^{-k\frac{1+\alpha(i-1)}{\alpha}}}{\det \mathbb{T}_{(s-t)(1+\delta)}} \overline{p}\left(1, \mathbb{T}^{-1}_{(s-t)(1+\delta)}(y - \theta_{s,t}(x))\right). \tag{3.30}$$

A sketch of proof for the above Proposition has been briefly presented in the Appendix section. Importantly, we highlight that the constant C appearing in (3.30) is independent from the "smoothing" parameter δ.
Then, the same arguments leading to (3.24) can be applied here to show that

$$\mathbb{E}\Big[\int_t^T I_\epsilon f(s, \overline{X}_s^{t,x,\delta})\, ds\Big] \;=\; \tilde{G}_\epsilon^\delta f(t,x) + \mathbb{E}\Big[\int_t^T \int_{\mathbb{R}^N} \Big(L_t^\delta - \tilde{L}_t^{s,y,\delta}\Big) \tilde{G}_\epsilon^\delta f(s, \overline{X}_s^{t,x,\delta})\, ds\Big], \tag{3.31}$$

where $\tilde{G}_\epsilon^\delta$ and $\tilde{\mathcal{L}}^{s,y,\delta}$ are the frozen Green kernel and the frozen infinitesimal generator associated with the process $\tilde{X}_s^{s,y,t,x,\delta}$, respectively (cf. Equations (2.41) and (2.44)). In particular, we point out that the pointwise bound (3.25) on the Green kernel and the controls of Proposition 3.2 (pointwise control of the remainder) are uniform with respect to the additional parameter δ, thanks to Proposition 3.4.

From Equation (3.31) and Proposition 3.3 ($L^p_t L^q_x$ control of the remainder) we can then deduce that

$$\left|\int_t^T \int_{\mathbb{R}^N} I_\epsilon f(s,y) p^\delta(t,s,x,y)\, dyds\right| \leq C_T \left(1 + \|p^\delta\|_{L^{p'}_t L^{q'}_x}\right) \|f\|_{L^p_t L^q_x}.$$

From the Riesz representation theorem and the above inequality, we then deduce that $\|p^\delta\|_{L^{p'}_t L^{q'}_x} \leq C_T$, for T small enough and uniformly in δ. Hence,

$$\left|\int_t^T \int_{\mathbb{R}^N} I_\epsilon f(s,y) p^\delta(t,s,x,y)\, dyds\right| = \left|\int_t^T \mathbb{E}[I_\epsilon f(s, X^{t,x}_s + \delta \overline{Z}_s)]\, ds\right| \leq C_T \|I_\epsilon f\|_{L^p_t L^q_x}.$$

The Krylov-type estimate (2.4) can be then derived exploiting the dominated convergence theorem and Lemma 2.18 (Dirac Convergence of frozen density), letting firstly ϵ and then δ go to zero. We have thus concluded the proof of Corollary 2.7.

4 A counter-example to uniqueness

. In this section, we present a counter-example to the uniqueness in law for the equation (1.3) when the Hölder regularity in space of the coefficients is low enough. In particular, we show here the almost sharpness of the thresholds appearing in Theorem 2.6 for diagonal perturbations, proving also Theorem 2.8. In order to test the threshold associated with the critical Hölder exponent for the i-th component of the drift F with respect to the variables x_j, we adapt the *ad hoc* Peano example constructed in [CdRM20b] to our Lévy framework.
Let us briefly recall it. It is well-known that the following deterministic equation

$$\begin{cases} dy_t = \operatorname{sgn}(y_t)|y_t|^\beta dt, & t \geq 0, \\ y_0 = 0, \end{cases} \tag{4.1}$$

for some β in $(0,1)$, is ill-posed since it admits an infinite number of solutions of the form

$$y_t = \pm c(t-t_0)^{1/(1-\beta)} \mathbb{1}_{[t_0,\infty)}(t), \quad \text{for some } t_0 \text{ in } [0,+\infty).$$

Nevertheless, Bafico and Baldi in [BB81] proved that the associated SDE, obtained by adding a Brownian Motion $\{W_t\}_{t\geq 0}$ to the dynamics:

$$\begin{cases} dX_t = \operatorname{sgn}(X_t)|X_t|^\beta dt + \epsilon dW_t, & t \geq 0 \\ X_0 = 0, \end{cases}$$

is well-posed for any $\epsilon > 0$ in a strong (probabilistic) sense. Furthermore, they showed that, letting ϵ goes to zero, the limit law concentrates around the two extremal solutions $\pm c t^{1/(1-\beta)}$ of the deterministic equation (4.1), thus providing a selection "criterion" between the infinite deterministic solutions.
In a subsequent article [DF14], Delarue and Flandoli highlighted the hidden dynamical mechanism behind this counter-intuitive behaviour. Heuristically, this *regularization*

by noise happens since, at least in a small time interval, the mean fluctuations of the Brownian noise are stronger than the irregularity of the deterministic drift. Indeed, they showed that before some transition time t_ϵ, the dominating noise pushes the solution to leave the drift singularity at 0, while afterwards, the deterministic part of the system prevails, constraining the (stochastic) solution to fluctuate around one of the extremal deterministic solutions, given by $\pm ct^{1/(1-\beta)}$.
More quantitatively, we can compare the fluctuations of the noise, say of order $\gamma > 0$ with the fluctuations of the deterministic extremal solutions, giving that

$$t^\gamma \,>\, t^{1/(1-\beta)}.$$

Since it should happen in small times, we then obtain that

$$\beta \,>\, 1 - \frac{1}{\gamma},$$

should be the heuristic relation that guarantees the noise dominates in short time. Clearly, the above inequality holds for any β in $(0,1)$ in the Brownian case ($\gamma = 1/2$), which would actually give $\beta > -1$. We can refer to [DD16] which is the closest work to this threshold since the authors manage to reach $-2/3^+$.

In view the above arguments, we fix $n = N$, $d_i = d = 1$ and i, j in $[\![1, n]\!]$ such that $j \geq i$ and we consider the drift

$$Ax + e_i \mathrm{sgn}(x_j)|x_j|^\beta$$

where $\{e_i \colon i \in [\![1, N]\!]\}$ is the canonical orthonormal basis for $\mathbb{R}^N$, A is the matrix in $\mathbb{R}^N \otimes \mathbb{R}^N$ given by

$$A \,:=\, \begin{pmatrix} 0 & \dots & \dots & \dots & 0 \\ 1 & 0 & \dots & \dots & 0 \\ 0 & 1 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & \dots & 0 & 1 & 0 \end{pmatrix}.$$

We will assume moreover that β is in $(0,1)$ such that

$$\beta \,<\, \frac{1 + \alpha(i-2)}{1 + \alpha(j-1)},$$

so that we are clearly outside the framework given by condition $(\mathscr{C})$.
Our aim is to prove that uniqueness in law fails for the following equation:

$$\begin{cases} dX_t \,=\, \Big[Ax + e_i \mathrm{sgn}(X_t^j)|X_t^j|^\beta\Big]dt + BdZ_t, \quad t \geq 0, \\ X_0 \,=\, 0, \end{cases} \tag{4.2}$$

where $\{Z_t\}_{t\geq 0}$ is a symmetric, d-dimensional α-stable process such that $\mathbb{E}[|Z_1|]$ is finite. In particular, we are interested on the i-th component of the above Equation (4.2) that can be rewritten in integral form as:

$$X_t^j \,=\, \int_0^t \mathrm{sgn}\Big(I_t^{j-i}(X^j)\Big)\Big|I_t^{j-i}(X^j)\Big|^\beta dt + I_t^{i-1}(Z), \quad t \geq 0, \tag{4.3}$$

where we have denoted by $I_t^k(y)$ the k-th *iterated integral* of a càdlàg path $y\colon [0,\infty) \to \mathbb{R}$ at a time t. Namely,

$$I_t^k(y) \;:=\; \int_0^{t_k=t} \cdots \int_0^{t_2} y_{t_0}\, dt_0 \dots dt_{k-1}, \quad t \geq 0. \tag{4.4}$$

In order to improve the readability of the next part, we are going to present our reasoning in a slightly more general way. It is not difficult to check that Equation (4.3) satisfies the assumptions of the following proposition.

Proposition 4.1. *Let k be in $\mathbb{N}$, β in $(0,1)$, x in $\mathbb{R}$ and $\{\mathcal{Z}_t\}_{t\geq 0}$ a continuous process on $\mathbb{R}$ such that*

- $\mathbb{E}\Big[\sup_{s\in[0,1]} |\mathcal{Z}_s|\Big] < \infty$;
- *it is symmetric and γ-self-similar in law for some $\gamma > 0$. Namely,*

$$\Big(\mathcal{Z}_t\Big)_{t\geq 0} \overset{\text{(law)}}{=} \Big(-\mathcal{Z}_t\Big)_{t\geq 0} \text{ and } \forall \rho > 0,\ \Big(\mathcal{Z}_{\rho t}\Big)_{t\geq 0} \overset{\text{(law)}}{=} \Big(\mathcal{Z}_t \rho^\gamma\Big)_{t\geq 0}.$$

Then, uniqueness in law fails for the following SDE:

$$\begin{cases} dX_t \;=\; \operatorname{sgn}\Big(I_t^k(X)\Big)\Big|I_t^k(X)\Big|^\beta dt + d\mathcal{Z}_t, \quad t \geq 0 \\ X_0 \;=\; x, \end{cases} \tag{4.5}$$

if $x = 0$ and $\beta < \frac{\gamma-1}{\gamma+k}$.

Since we can clearly apply Proposition 4.1 to Equation (4.3) taking $\gamma = i - 1 + \frac{1}{\alpha}$, $k = j - i$, it implies that SDE (4.2) lacks of uniqueness in law if

$$\beta \;<\; \frac{\gamma-1}{\gamma+k} \;=\; \frac{1+\alpha(i-2)}{1+\alpha(j-1)}.$$

Hence, to complete the proof of Theorem 2.8, it suffices to establish Proposition 4.1.

Before proving Proposition 4.1, we need however an auxiliary result. It roughly states that any solution of SDE (4.5) starting outside zero cannot immediately reach the extremal solutions of the associated deterministic Peano example. Importantly, the constant ρ appearing below does not depend on the starting point x.

Lemma 4.2. *Fixed $x > 0$ and $\beta < \frac{\gamma-1}{\gamma+k}$, let $\{X_t\}_{t\geq 0}$ be a solution of Equation (4.5) starting from x. Then, there exist two positive constants $\rho := \rho(k,\beta,\gamma,\mathbb{E}[\sup_{s\in[0,1]} |\mathcal{Z}_s|])$ and $c_0 := c_0(k,\beta)$ such that*

$$\mathbb{P}\Big(\tau(X) \geq \rho\Big) \;\geq\; 3/4, \tag{4.6}$$

where $\tau(X)$ is the stopping time on Ω given by

$$\tau(X) \;=\; \inf\{t \geq 0\colon X_t \;\leq\; c_0 t^{\frac{k\beta+1}{1-\beta}}\}. \tag{4.7}$$

Proof. We start noticing that the process $\{X_t\}_{t\geq 0}$ is continuous in 0, since it is càdlàg. Fixed $c_0 > 0$ to be chosen later, it implies that $\tau(X) > 0$, almost surely. In particular, it makes sense to consider the random interval $(0, \tau(X)]$.
Fixed t in $(0, \tau(X)]$, it holds, by definition of $\tau(X)$, that $X_t > c_0 t^{\frac{k\beta+1}{1-\beta}}$. It follows then that

$$\int_0^t \left|I_s^k(X)\right|^\beta ds \;>\; \tilde{C} c_0^\beta t^{\frac{k\beta+1}{1-\beta}} \quad \text{where} \quad \tilde{C} \;:=\; \Big(\prod_{i=1}^k \frac{k\beta+1}{1-\beta} + (i-1)\Big)^{-\beta}.$$

Since $x > 0$ by assumption and $X > 0$ on $(0, \tau(X)]$, we can now show that

$$X_t \;=\; x + \int_0^t \operatorname{sgn}\Big(I_s^k(X)\Big)\Big|I_s^k(X)\Big|^\beta ds + \mathcal{Z}_t \;>\; \tilde{C} c_0^\beta t^{\frac{k\beta+1}{1-\beta}} + \mathcal{Z}_t.$$

The next step is to write $\tilde{C} c_0^\beta = c_0 + \hat{C}$ for some constant $\hat{C} > 0$. To do so, we need to choose carefully c_0. In particular, the condition above is equivalent to the following

$$\hat{C} = \tilde{C} c_0^\beta - c_0 \;>\; 0 \Leftrightarrow c_0 \;<\; \tilde{C}^{\frac{1}{1-\beta}}.$$

Fixed $c_0 = \tilde{C}^{\frac{1}{1-\beta}}/2$, it then holds that

$$X_t \;>\; c_0 t^{\frac{k\beta+1}{1-\beta}} + \hat{C} t^{\frac{k\beta+1}{1-\beta}} + \mathcal{Z}_t$$

for any t in $(0, \tau(X)]$. Fixed $\rho > 0$ to be chosen later, we can now define the event $\mathcal{A}$ in Ω as

$$\mathcal{A} \;:=\; \{\omega \in \Omega \colon \hat{C} t^{\frac{k\beta+1}{1-\beta}} + \mathcal{Z}_t > 0,\ \forall\, t \in (0, \rho]\}.$$

On $\mathcal{A}$ and for any t in $(0, \tau(X)]$, it then holds that

$$X_t \;>\; c_0 t^{\frac{k\beta+1}{1-\beta}}.$$

In particular, we have that $\tau(X) \geq \rho$ on $\mathcal{A}$ and thus, $\mathcal{A} \subseteq \{\tau(X) \geq \rho\}$ on Ω. It immediately implies that

$$\mathbb{P}\Big(\tau(X) \geq \rho\Big) \;\geq\; \mathbb{P}(\mathcal{A}).$$

It remains to choose $\rho > 0$ such that $\mathbb{P}(\mathcal{A}) \geq 3/4$. Write:

$$\begin{aligned}\mathbb{P}(\mathcal{A}) =& \mathbb{P}[\forall t \in (0,\rho],\ \hat{C} t^{\frac{k\beta+1}{1-\beta}} + \mathcal{Z}_t > 0] = \mathbb{P}[\forall t \in (0,1],\ \hat{C}(\rho t)^{\frac{k\beta+1}{1-\beta}} + \mathcal{Z}_{\rho t} > 0] \\ =& \mathbb{P}[\forall t \in (0,1],\ \hat{C}(\rho t)^{\frac{k\beta+1}{1-\beta}} + \rho^\gamma \mathcal{Z}_t > 0] = \mathbb{P}[\forall t \in (0,1],\ \hat{C}\rho^{\frac{k\beta+1}{1-\beta}-\gamma} + t^{-\frac{k\beta+1}{1-\beta}} \mathcal{Z}_t > 0],\end{aligned}$$

from the self-similarity assumption on $\mathcal{Z}$. Since by assumption $\beta < \frac{\gamma-1}{\gamma+k} \iff \frac{k\beta+1}{1-\beta} - \gamma < 0$, the statement will follow taking ρ small enough as soon as we prove the process $\mathcal{R}_t := t^{-\frac{k\beta+1}{1-\beta}} \mathcal{Z}_t,\ t \in (0,1]$, which is continuous on the open set $(0,1]$, can be extended by continuity in 0 with $\mathcal{R}_0 = 0$. Observe that $\mathbb{E}[|\mathcal{R}_t|] = t^{\gamma - \frac{k\beta+1}{1-\beta}} \mathbb{E}[|\mathcal{Z}_1|] \underset{t\to 0}{\longrightarrow} 0$. Setting $\delta := \gamma - \frac{k\beta+1}{1-\beta} > 0$ and introducing $t_n := n^{-1/\delta(1+\eta)}, \eta > 0$, we get that for all $\varepsilon > 0$,

$$\mathbb{P}[|\mathcal{R}_{t_n}| \geq \varepsilon] \leq \varepsilon^{-1}\mathbb{E}[|\mathcal{R}_{t_n}|] = \varepsilon^{-1} t_n^\delta \mathbb{E}[|\mathcal{Z}_1|] = \varepsilon^{-1} n^{-(1+\eta)} \mathbb{E}[|\mathcal{Z}_1|].$$

We thus get from the Borel-Cantelli lemma that $\mathcal{R}_{t_n} \underset{n,\ a.s.}{\longrightarrow} 0$. Namely, we have almost sure convergence along the subsequence t_n going to zero with n. It now remains to prove that the process $\mathcal{R}_t$ does not fluctuate much between two successive times t_n and t_{n+1}. Write for $t \in [t_{n+1}, t_n]$:

$$\begin{aligned}|R_t| := |t^{-\frac{k\beta+1}{1-\beta}}\mathcal{Z}_t| \leq & t_{n+1}^{-\frac{k\beta+1}{1-\beta}}\Big(|\mathcal{Z}_{t_{n+1}}| + \sup_{s\in[t_{n+1},t_n]}|\mathcal{Z}_s - \mathcal{Z}_{t_{n+1}}|\Big)\\ \leq & t_{n+1}^{-\frac{k\beta+1}{1-\beta}}\Big(2|\mathcal{Z}_{t_{n+1}}| + \sup_{s\in[0,t_n]}|\mathcal{Z}_s|\Big).\end{aligned} \tag{4.8}$$

The first term of the above left hand side tends almost surely to zero with n. Observe as well that, from the scaling properties of $\mathcal{Z}$, for any $\varepsilon > 0$:

$$\begin{aligned}\mathbb{P}[t_{n+1}^{-\frac{k\beta+1}{1-\beta}}\sup_{s\in[0,t_n]}|\mathcal{Z}_s| \geq \varepsilon] = \mathbb{P}[t_{n+1}^{-\frac{k\beta+1}{1-\beta}}t_n^{\gamma}\sup_{s\in[0,1]}|\mathcal{Z}_s| \geq \varepsilon] \leq \varepsilon^{-1}t_n^{\delta}(\frac{t_n}{t_{n+1}})^{\frac{k\beta+1}{1-\beta}}\mathbb{E}[\sup_{s\in[0,1]}|\mathcal{Z}_s|]\\ \leq C\varepsilon^{-1}n^{-(1+\eta)}\mathbb{E}[\sup_{s\in[0,1]}|\mathcal{Z}_s|],\end{aligned}$$

which again gives from the Borel-Cantelli lemma the a.s. convergence with n of the second term in the r.h.s of (4.8). We eventually derive that $\mathcal{R}_t \underset{t\to 0, a.s.}{\longrightarrow} 0$. Again, the key point is that we normalize the process $\mathcal{Z}$ at a rate, $t^{\frac{k\beta+1}{1-\beta}}$, which is lower than its own characteristic time scale, t^{γ}. This is precisely what leaves some margin to establish continuity.

□

Exploiting the lower bound for the random time $\tau(X)$ given in Lemma 4.2, we are now ready to show uniqueness in law fails for SDE (4.5) when $x = 0$ and $\beta < \frac{\gamma-1}{\gamma+k}$.

Proof of Proposition 4.1. By contradiction, we start assuming that uniqueness in law holds for SDE (4.5) starting at $x = 0$. Fixed any solution $\{X_t\}_{t\geq 0}$ of Equation (4.5) starting at zero, it follows by symmetry that $\{-X_t\}_{t\geq 0}$ is also a solution of the same dynamics. Since by hypothesis, $-\mathcal{Z}_t \overset{\text{(law)}}{=} \mathcal{Z}_t$, uniqueness in law for SDE (4.5) implies that the laws of X and $-X$ are identical.
Assuming for the moment that Lemma 4.2 is applicable for $x = 0$, we easily find a contradiction. Indeed, it follows from Lemma 4.2 that

$$\mathbb{P}\Big(\tau(X) \geq \rho\Big) \geq 3/4$$

but on the same time, thanks to the uniqueness in law, we have that

$$\mathbb{P}^0\Big(\tau(-X) \geq \rho\Big) \geq 3/4,$$

which is clearly impossible. To show the validity of Lemma 4.2 in $x = 0$, we consider a a sequence $\{\{X_t^n\}_{t\geq 0} \colon n \in \mathbb{N}\}$ of solutions of SDE (4.5) starting at $1/n$. It is then easy to check that such a sequence satisfies the Aldous criterion:

$$\mathbb{E}[|X_t^n - X_0^n|^p] \leq ct^{p\gamma}, \quad t \geq 0$$

for some $p > 0$ and $c > 0$ independent from t and n. It follows (Proposition 34.8 in [Bas11]) that the sequence $\{\mathbb{P}^n\}_{n\in\mathbb{N}}$ of the laws of $\{X_t^n\}_{t\geq 0}$ is tight. Prohorov Theorem (cf. Theorem 30.4 in [Bas11]) ensures now the existence of a converging sub-sequence $\{\mathbb{P}^{n_k}\}_{k\in\mathbb{N}}$. The uniqueness in law then implies that the sequence $\{\mathbb{P}^{n_k}\}_{k\in\mathbb{N}}$ converges, as expected, to $\mathbb{P}^0$ the law of the solution starting at 0. Noticing that inequality (4.6) holds for any solution $\{X_t^n\}_{t\geq 0}$ and moreover, the constant ρ is independent from the starting points $1/n$, we find that

$$\mathbb{P}\Big(\tau(X) \geq \rho\Big) \;\geq\; 3/4.$$

The proof of Proposition 4.1 is thus concluded.

5 Appendix: proofs of complementary results

5.1 Controls on the density of the proxy process

We present here two useful lemmas needed to complete the proof of Proposition 2.11. We will analyze the behavior of the laws of the independent random variables $\tilde{M}^{\tau,\xi,t,s}$ and $\tilde{N}^{\tau,\xi,t,s}$ obtained in (2.20) by truncation of the process $\tilde{S}^{\tau,\xi,t,s}$ at the associated stable time scale $u^{1/\alpha}$.

Lemma 5.1. *Let m be in $\mathbb{N}$. Then, there exists a positive constant $C := C(m,T)$ such that for any k in $[\![0,m]\!]$,*

$$\Big|D_z^k p_{\tilde{M}^{\tau,\xi,t,s}}(u,z)\Big| \;\leq\; Cu^{-(N+k)/\alpha}\left(1+\frac{|z|}{u^{1/\alpha}}\right)^{-m} \;=:\; Cu^{-k/\alpha}p_{\overline{M}}(u,z),$$

for any $u > 0$, any z in $\mathbb{R}^N$, any $t \leq s$ in $[0,T]$ and any (τ,ξ) in $[0,T]\times\mathbb{R}^N$.

Proof. Similarly to the proof of Proposition 2.10 (see in particular Equation (2.16)), we start writing

$$p_{\tilde{M}^{\tau,\xi,t,s}}(u,z) \;=\; \frac{1}{(2\pi)^N}\int_{\mathbb{R}^N} e^{-i\langle z,y\rangle}\exp\left(u\int_{|p|\leq u^{1/\alpha}}\left[\cos(\langle y,p\rangle)-1\right]\,\nu_{\tilde{S}^{\tau,\xi,t,s}}(dp)\right)dy,$$

where, we recall, $\nu_{\tilde{S}}^{\tau,\xi,t,s}$ is the Lévy measure associated with the process $\{\tilde{S}_u^{\tau,\xi,t,s}\}_{u\geq 0}$ in Proposition 2.10. Setting $u^{1/\alpha}y = \tilde{y}$ then yields

$$\begin{aligned} p_{\tilde{M}^{\tau,\xi,t,s}}(u,z) \;&=\; \frac{u^{-N/\alpha}}{(2\pi)^N}\int_{\mathbb{R}^N} e^{-i\langle z,\frac{\tilde{y}}{u^{1/\alpha}}\rangle}\exp\left(u\int_{|p|\leq u^{1/\alpha}}\left[\cos(\langle \tilde{y},\frac{p}{u^{1/\alpha}}\rangle)-1\right]\nu_{\tilde{S}^{\tau,\xi,t,s}}(dp)\right)d\tilde{y} \\ &=:\; \frac{u^{-N/\alpha}}{(2\pi)^N}\int_{\mathbb{R}^N} e^{-i\langle\frac{z}{u^{1/\alpha}},\tilde{y}\rangle}\,\hat{f}_u^{\tau,\xi,t,s}(\tilde{y})\,d\tilde{y} \qquad (5.1) \end{aligned}$$

Since the Lévy measure $\nu_{\tilde{S}}^{\tau,\xi,t,s}$ in the expression above has finite support, Theorem 3.7.13 in Jacob [Jac01] implies that $\hat{f}_u^{\tau,\xi,t,s}$ is infinitely differentiable in $\tilde{y}$. We can thus

calculate

$$|\partial_{\tilde{y}} \hat{f}_u^{\tau,\xi,t,s}(\tilde{y})| \leq u \int_{|p|\leq u^{1/\alpha}} \frac{|p|}{u^{1/\alpha}} \left|\sin\left(\left\langle \tilde{y}, \frac{p}{u^{1/\alpha}}\right\rangle\right)\right| \nu_{\tilde{S}^{\tau,\xi,t,s}}(dp)$$
$$\times \exp\left(u \int_{|p|\leq u^{1/\alpha}} \left[\cos\left(\left\langle \frac{\tilde{y}}{u^{1/\alpha}}, p\right\rangle\right) - 1\right] \nu_{\tilde{S}^{\tau,\xi,t,s}}(dp)\right).$$

Recalling that $\alpha > 1$, we can now write that

$$\begin{aligned} u \int_{|p|\leq u^{1/\alpha}} \frac{|p|}{u^{1/\alpha}} \left|\sin\left(\left\langle \tilde{y}, \frac{p}{u^{1/\alpha}}\right\rangle\right)\right| \nu_{\tilde{S}^{\tau,\xi,t,s}}(dp) &\leq Cu \int_{r\leq u^{1/\alpha}} \frac{r}{u^{1/\alpha}} \frac{|\tilde{y}|r}{u^{1/\alpha}} \frac{dr}{r^{1+\alpha}} \\ &\leq Cu \int_{r\leq u^{1/\alpha}} |\tilde{y}| \frac{r^{1-\alpha}}{u^{2/\alpha}}\, dr \\ &\leq C(1+|\tilde{y}|). \end{aligned}$$

It then follows that

$$\begin{aligned} &|\partial_{\tilde{y}} \hat{f}_u^{\tau,\xi,t,s}(\tilde{y})| \\ &\quad \leq C(1+|\tilde{y}|)\exp\left(u \int_{\mathbb{R}^N} \left[\cos\left(\left\langle \frac{\tilde{y}}{u^{1/\alpha}}, p\right\rangle\right) - 1\right] \nu_{\tilde{S}^{\tau,\xi,t,s}}(dp)\right) e^{2u\nu_{\tilde{S}^{\tau,\xi,t,s}}(B^c(0,u^{1/\alpha}))} \\ &\quad \leq C(1+|\tilde{y}|)\exp(-C^{-1}|\tilde{y}|^\alpha), \end{aligned}$$

where in second inequality we exploited Control (2.15) and

$$\nu_{\tilde{S}^{\tau,\xi,t,s}}(B^c(0,u^{1/\alpha})) \leq C/u. \tag{5.2}$$

Iterating the above reasoning, we can then show that for any l in $\mathbb{N}$,

$$|\partial_{\tilde{y}}^l \hat{f}_u^{\tau,\xi,t,s}(\tilde{y})| \leq C_l(1+|\tilde{y}|^l)\exp(-C^{-1}|\tilde{y}|^\alpha),$$

for some positive constant $C := C(l)$. It implies in particular that $\hat{f}_u^{\tau,\xi,t,s}(\tilde{y})$ is a Schwartz test function. Denoting by $f_u^{\tau,\xi,t,s}$ its inverse Fourier transform, we thus have that for any m in $\mathbb{N}$, there exists a positive constant $C := C(m)$ such that

$$|f_u^{\tau,\xi,t,s}(y)| \leq C_m(1+|y|)^{-m}, \quad y \in \mathbb{R}^N.$$

The result for $k = 0$ now follows immediately noticing that

$$p_{\overline{M}}(t-s,y) = (t-s)^{-\frac{d}{\alpha}} f_{s,t}(y/(t-s)^{\frac{1}{\alpha}}).$$

The controls on the derivatives can be derived analogously. □

We can now show a similar control on the law of the process $\tilde{N}^{\tau,\xi,t,s}$.

Lemma 5.2. *There exists a family $\{\overline{P}_u\}_{u\geq 0}$ of Poisson measures and a positive constant $C := C(T,N)$ such that for any $\mathcal{A}$ in $\mathcal{B}(\mathbb{R}^N)$ and $\tilde{N}^{\tau,\xi,t,s}$ as in (2.21),*

$$P_{\tilde{N}_u^{\tau,\xi,t,s}}(\mathcal{A}) \leq C\overline{P}_u(\mathcal{A}). \tag{5.3}$$

Proof. For notational simplicity, we start introducing the truncated Lévy measure associated with the big jumps of the process $\{\tilde{S}_u^{\tau,\xi,t,s}\}_{u\geq 0}$:

$$\nu_{\mathrm{tr}}^{\tau,\xi,t,s}(dp) \;=\; \mathbb{1}_{|p|\geq u^{1/\alpha}}(p)\nu_{\tilde{S}}^{\tau,\xi,t,s}(dp).$$

It follows immediately that $\nu_{\mathrm{tr}}^{\tau,\xi,t,s}$ is a finite measure (see (5.2) above). With this notation at hand, we can write:

$$\begin{aligned}\widehat{P_{\tilde{N}_u^{\tau,\xi,t,s}}}(y) \;&=\; \exp\left(u\int_{|p|>u^{\frac{1}{\alpha}}}\left[\cos(\langle y,p\rangle)-1\right]\,\nu_{\tilde{S}}^{\tau,\xi,t,s}(dp)\right)\\ &=\; \exp\left(u\widehat{\nu_{\mathrm{tr}}^{\tau,\xi,t,s}}(y)-u\nu_{\mathrm{tr}}^{\tau,\xi,t,s}(\mathbb{R}^N)\right),\end{aligned}$$

where $\widehat{\nu}$ denotes the Fourier-Stieltjes transform of the considered measure ν. Let us introduce then the following measure:

$$\zeta^{\tau,\xi,t,s} \;:=\; u\nu_{\mathrm{tr}}^{\tau,\xi,t,s}.$$

Expanding the previous exponential and by termwise Fourier inversion, we now find that

$$P_{\tilde{N}_u^{\tau,\xi,t,s}} \;=\; \exp\left(\zeta^{\tau,\xi,t,s}-u\nu_{\mathrm{tr}}^{\tau,\xi,t,s}(\mathbb{R}^N)\right) \;=\; \exp\left(-u\nu_{\mathrm{tr}}^{\tau,\xi,t,s}(\mathbb{R}^N)\right)\sum_{n\in\mathbb{N}}\frac{\left(\zeta^{\tau,\xi,t,s}\right)^{\star n}}{n!}, \tag{5.4}$$

where, for a finite measure ρ on $\mathbb{R}^N$, $(\rho)^{\star n} := \rho\star\cdots\star\rho$ denotes its nth fold convolution. For now, let us assume that $\sigma(t,x)$ is non-constant in space, so that

$$B\tilde{\sigma}_{u(v)}^{\tau,\xi} \;=\; B\sigma\left(u(v),\theta_{u(v),\tau}(\xi)\right)$$

appearing in the definition of $\nu_{\tilde{S}}^{\tau,\xi,t,s}$, truly depends on the parameters τ,ξ. Assumption [**AC**] then ensures the existence of a bounded function $g\colon \mathbb{S}^{d-1}\to\mathbb{R}$ such that

$$\nu(dp) \;=\; Q(p)\frac{g(\frac{p}{|p|})}{|p|^{d+\alpha}}dp.$$

From Equation (5.4), it is clear that we need to control the measure $\zeta^{\tau,\xi,t,s}$, uniformly in the parameters τ,ξ,t,s. Namely, for any $\mathcal{A}$ in $\mathcal{B}(\mathbb{R}^N)$, we write from (2.13) that

$$\begin{aligned}\zeta^{\tau,\xi,t,s}(\mathcal{A}) \;&=\; u\int_{|p|>u^{\frac{1}{\alpha}}}\mathbb{1}_{\mathcal{A}}(p)\,\nu_{\tilde{S}}^{\tau,\xi,t,s}(dp) \;=\; u\int_0^1\int_{|\widehat{\mathcal{R}}_vB\tilde{\sigma}_{u(v)}^{\tau,\xi}p|>u^{\frac{1}{\alpha}}}\mathbb{1}_{\mathcal{A}}(\widehat{\mathcal{R}}_vB\tilde{\sigma}_{u(v)}^{\tau,\xi}p)\,\nu(dp)dv\\ &=u\int_0^1\int_{|\widehat{\mathcal{R}}_vB\tilde{\sigma}_{u(v)}^{\tau,\xi}p|>u^{\frac{1}{\alpha}}}\mathbb{1}_{\mathcal{A}}(\widehat{\mathcal{R}}_vB\tilde{\sigma}_{u(v)}^{\tau,\xi}p)\frac{g(\frac{p}{|p|})}{|p|^{d+\alpha}}Q(p)\,dpdv\\ &\leq\; u\int_0^1\int_{|\widehat{\mathcal{R}}_vB\tilde{\sigma}_{u(v)}^{\tau,\xi}p|>u^{\frac{1}{\alpha}}}\mathbb{1}_{\mathcal{A}}(\widehat{\mathcal{R}}_vB\tilde{\sigma}_{u(v)}^{\tau,\xi}p)\frac{dp}{|p|^{d+\alpha}}dv\end{aligned}$$

We can then exploit assumption [**UE**] on σ to conclude that

$$\begin{aligned}\zeta^{\tau,\xi,t,s}(\mathcal{A}) \;&\leq\; u\int_0^1\int_{|\widehat{\mathcal{R}}_vBq|>u^{\frac{1}{\alpha}}}\mathbb{1}_{\mathcal{A}}(\widehat{\mathcal{R}}_vBq)\frac{1}{\det(\tilde{\sigma}_{u(v)}^{\tau,\xi})}\frac{dq}{|(\tilde{\sigma}_{u(v)}^{\tau,\xi})^{-1}q|^{d+\alpha}}dv\\ &\leq\; Cu\int_0^1\int_{|\widehat{\mathcal{R}}_vBq|>u^{\frac{1}{\alpha}}}\mathbb{1}_{\mathcal{A}}(\widehat{\mathcal{R}}_vBq)\frac{dq}{|q|^{d+\alpha}}dv.\end{aligned}$$

Denoting now by $\Lambda_{\rm tr} := c\mathbb{1}_{p>u^{1/\alpha}}\frac{dp}{p^{d+\alpha}}$ the truncated Lévy measure of the isotropic α-stable process and by $\overline{\nu}_{\rm tr}$ the following push-forward measure

$$\overline{\nu}_{\rm tr}(\mathcal{A}) \;:=\; \int_0^1 \Lambda_{\rm tr}\left((\widehat{\mathcal{R}}_v B)^{-1}\mathcal{A}\right) dv, \quad \mathcal{A}\in\mathcal{B}(\mathbb{R}^N)$$

we derive that there exists a constant C such that for any (τ,ξ) in $[0,T]\times\mathbb{R}^N$, $t\le s$ in $[0,T]$,

$$\zeta^{\tau,\xi,t,s}(\mathcal{A}) \;\le\; Cu\int_0^1 \Lambda_{\rm tr}\left((\widehat{\mathcal{R}}_v B)^{-1}\mathcal{A}\right) dv \;=\; u\overline{\nu}_{\rm tr}(\mathcal{A}) \;=:\; \overline{\zeta}(\mathcal{A}). \tag{5.5}$$

Equation (5.3) now follows from the above control, (5.2) and (5.4), denoting

$$\overline{P}_u \;:=\; \exp\left(-u\overline{\nu}_{\rm tr}(\mathbb{R}^N)\right)\sum_{n\in\mathbb{N}}\frac{(\overline{\zeta})^{\star n}}{n!},$$

up to a modification of the constant C in (5.5). Following backwards the same reasoning presented at the beginning of the proof, we then notice that

$$\begin{aligned}
\widehat{\overline{P}_u}(y) \;&=\; \exp\left(u\int_0^1\int_{\mathbb{R}^N}\left[\cos(\langle y,p\rangle)-1\right]\,\overline{\nu}_{\rm tr}(dp)dv\right)\\
&=\; \exp\left(u\int_0^1\int_{\mathbb{R}^d}\mathbb{1}_{\{|\widehat{\mathcal{R}}_v Bp|>u^{\frac{1}{\alpha}}\}}\left[\cos(\langle y,\widehat{\mathcal{R}}_v Bp\rangle)-1\right]\,\Lambda(dp)dv\right)\\
&=\; \exp\left(u\int_0^1\int_0^\infty\int_{\mathbb{S}^{d-1}}\mathbb{1}_{\{|\widehat{\mathcal{R}}_v B\theta r|>u^{\frac{1}{\alpha}}\}}\left[\cos(\langle y,\widehat{\mathcal{R}}_v B\theta r\rangle)-1\right]\,\mu_{\rm leb}(d\theta)\frac{dr}{r^{1+\alpha}}dv\right),
\end{aligned}$$

where we used the spherical decomposition for the Lévy measure Λ of an isotropic α-stable process:

$$\Lambda(dp) := \frac{dp}{p^{d+\alpha}} \;=\; C\mu_{\rm leb}(d\theta)\frac{dr}{r^{1+\alpha}}, \tag{5.6}$$

with $p=r\theta$ and $\mu_{\rm leb}$ Lebesgue measure on the sphere $\mathbb{S}^{d-1}$.
We exploit now the non-degeneracy of $\widehat{\mathcal{R}}_v$ to to define two functions $k\colon [0,1]\times\mathbb{S}^{d-1}\to\mathbb{R}$ and $l\colon [0,1]\times\mathbb{S}^{d-1}\to\mathbb{S}^{N-1}$, given by

$$k(v,\theta) \;:=\; |\widehat{\mathcal{R}}_v B\theta| \;\text{ and }\; l(v,\theta) \;:=\; \frac{\widehat{\mathcal{R}}_v B\theta}{|\widehat{\mathcal{R}}_v B\theta|}.$$

Using the Fubini theorem, we can now write that

$$\begin{aligned}
&\widehat{\overline{P}_u}(y)\\
&\;=\; \exp\left(u\int_0^1\int_0^\infty\int_{\mathbb{S}^{d-1}}\mathbb{1}_{\{|l(v,\theta)k(v,\theta)r|>u^{\frac{1}{\alpha}}\}}\left[\cos\left(\langle z,l(v,\theta)k(v,\theta)r\rangle\right)-1\right]\,\mu_{\rm leb}(d\theta)\frac{dr}{r^{1+\alpha}}dv\right)\\
&\;=\; \exp\left(u\int_0^1\int_0^\infty\int_{\mathbb{S}^{d-1}}\mathbb{1}_{\{|l(v,\theta)\tilde{r}|>u^{\frac{1}{\alpha}}\}}\left[\cos\left(\langle z,l(v,\theta)\tilde{r}\rangle\right)-1\right]\,[k(v,\theta)]^\alpha\mu_{\rm leb}(d\theta)\frac{d\tilde{r}}{\tilde{r}^{1+\alpha}}dv\right).
\end{aligned}$$

Denoting now by $\tilde{k}(dv,d\theta)$ the measure on $[0,1]\times\mathbb{S}^{d-1}$ given by

$$\tilde{k}(dv,d\theta) \;:=\; [k(v,\theta)]^\alpha\mu_{\rm leb}(d\theta)dv$$

and by $\tilde{\mu}_{\text{sym}} := \text{Sym}(l)_* \tilde{k}$ the symmetrization of the measure $\tilde{k}(dv, d\theta)$ push-forwarded through l on $\mathbb{S}^{N-1}$, we can finally conclude that

$$\begin{aligned}\widehat{\overline{P}_u}(y) &= \exp\left(u \int_0^\infty \int_{[0,1]\times\mathbb{S}^{d-1}} \mathbb{1}_{\{|l(v,\theta)\tilde{r}|>u^{\frac{1}{\alpha}}\}} \left[\cos\left(\langle z, l(v,\theta)\tilde{r}\rangle\right) - 1\right] \tilde{k}(dv, d\theta) \frac{d\tilde{r}}{\tilde{r}^{1+\alpha}}\right) \\ &= \exp\left(u \int_{|u|^{\frac{1}{\alpha}}}^\infty \int_{\mathbb{S}^{N-1}} \left[\cos\left(\langle z, \tilde{\theta}\tilde{r}\rangle\right) - 1\right] \tilde{\mu}_{\text{sym}}(d\tilde{\theta}) \frac{d\tilde{r}}{\tilde{r}^{1+\alpha}}\right). \end{aligned} \tag{5.7}$$

It is easy to check now that the measure $\tilde{\mu}_{\text{sym}}$ is finite and non-degenerate in the sense of (1.6). This concludes the proof of our result under the additional assumption that ν is absolutely continuous with respect to the Lebesgue measure.
If this is not the case, assumption [**AC**] implies immediately that $\sigma(t,x) =: \sigma_t$ does not depends on x. Thus, the "frozen" diffusion $\tilde{\sigma}_t^{\tau,\xi}$ does not depends on the parameters τ, ξ as well. The same arguments above then allow to conclude in a similar manner. □

Sketch of proof for Proposition 3.4 We briefly present here the proof of Proposition 3.4 concerning the existence and the associated controls for the density of the mollified frozen process $\tilde{X}_s^{\tau,\xi,t,x,\delta}$.
We start noticing that the reasoning in the proof of Proposition 2.10 can be similarly applied. Indeed, from the definition in (3.29), it follows immediately that

$$\tilde{X}_s^{\tau,\xi,t,x,\delta} = \tilde{m}_{s,t}^{\tau,\xi}(x) + \mathbb{M}_{s-t}\left(\tilde{S}_{s-t}^{\tau,\xi,t,s} + \delta \overline{Z}_{s-t}\right),$$

and thus, that there exists a density $\tilde{p}^{\tau,\xi,\delta}(t,s,x,y)$ associated with the frozen process $\tilde{X}_s^{\tau,\xi,t,x,\delta}$. Moreover, the representation in (2.10) holds again if we change there the Lévy measure $\nu_{\tilde{S}}^{\tau,\xi,t,s}$ with the one associated with the following Lévy symbol:

$$\Phi_{\tilde{S}^{\tau,\xi,t,s,\delta}}(z) := \Phi_{\tilde{S}^{\tau,\xi,t,s}}(z) + c_\alpha \delta |z|^\alpha = \int_0^1 \Phi\Big((\widehat{\mathcal{R}}_v B \tilde{\sigma}_{u(v)}^{\tau,\xi})^* z\Big)\, dv + c_\alpha \delta |z|^\alpha.$$

Namely, it holds that

$$\begin{aligned}\tilde{p}^{\tau,\xi,\delta}(t,s,x,y) = \frac{\det \mathbb{M}_{s-t}^{-1}}{(2\pi)^N} \int_{\mathbb{R}^N} & e^{-i\langle \mathbb{M}_{s-t}^{-1}(y - \tilde{m}_{s,t}^{\tau,\xi}(x)), z\rangle} \\ & \times \exp\left((s-t)\int_{\mathbb{R}^N} \left[\cos(\langle z, p\rangle) - 1\right] \nu_{\tilde{S}^{\tau,\xi,t,s,\delta}}(dp)\right) dz,\end{aligned}$$

where the Lévy measure $\nu_{\tilde{S}^{\tau,\xi,t,s,\delta}}$ is given by

$$\nu_{\tilde{S}^{\tau,\xi,t,s,\delta}}(\mathcal{A}) = \nu_{\tilde{S}^{\tau,\xi,t,s}}(A) + \delta^\alpha \nu_{\overline{Z}}(\mathcal{A}), \quad \mathcal{A} \in \mathcal{B}(\mathbb{R}^N), \tag{5.8}$$

with $\nu_{\overline{Z}}$ Lévy measure of the isotropic α-stable process $\overline{Z}_t$. In particular, the Lévy symbol $\Phi_{\tilde{S}^\delta}$ satisfies Control (2.15) for a constant C independent from δ.
We can now move to show the controls on the derivatives of the mollified frozen density. It is not difficult to check that the arguments presented in the proofs of Proposition 2.11, Lemmas 5.1 and 5.2 can be applied again if we substitute there the Lévy measure $\nu_{\tilde{S}^{\tau,\xi,t,s}}$ with the mollified one $\nu_{\tilde{S}^{\tau,\xi,t,s,\delta}}$. Indeed, taking into account the decomposition in (5.8), we notice that the Lévy measure $\nu_{\tilde{S}^{\tau,\xi,t,s,\delta}}$ only considers an additional term

$(\delta\nu_{\overline{Z}})$ that has the same α-scaling nature considered before (but is however much less singular).
To show instead that the estimates (3.30) are indeed uniform in the parameter δ, it is sufficient to notice from (5.8) that we have that

$$\nu_{\tilde{S}^{\tau,\xi,t,s,\delta}}(\mathcal{A}) \leq \nu_{\tilde{S}^{\tau,\xi,t,s}}(\mathcal{A}) + \nu_{\overline{Z}}(\mathcal{A}), \quad \mathcal{A} \in \mathcal{B}(\mathbb{R}^N).$$

To conclude the proof of Proposition 3.4, it is then enough to take $\xi = y$, $\tau = s$ and to follow the same arguments introduced in the proof of Corollary 2.16.

5.2 Proof of the technical lemmas

Proof of Lemma 2.14 (Approximate Lipschitz condition of the flows) We start considering two measurable flows $\theta, \check{\theta}$ satisfying dynamics (2.28). Recalling the decomposition $G(t,x) = A_t x + F(t,x)$, it follows immediately that:

$$\begin{aligned} \mathbb{T}_{s-t}^{-1}(x - \theta_{t,s}(y)) &= \mathbb{T}_{s-t}^{-1}\Big[\check{\theta}_{s,t}(x) - y - \int_t^s \Big(G(u,\check{\theta}_{u,t}(x)) - G(u,\theta_{u,s}(y))\Big)\,du\Big] \\ &= \mathbb{T}_{s-t}^{-1}\Big(\check{\theta}_{s,t}(x) - y\Big) + \mathcal{I}_{s,t}(x,y), \end{aligned} \tag{5.9}$$

where in the last step, we denoted

$$\mathcal{I}_{s,t}(x,y) = \mathbb{T}_{s-t}^{-1}\int_t^s \Big[A_u\Big(\theta_{u,s}(y) - \check{\theta}_{u,t}(x)\Big) + \Big(F(u,\theta_{u,s}(y)) - F(u,\check{\theta}_{u,t}(x))\Big)\Big]\,du.$$

To conclude, we need to show the following bound for $\mathcal{I}_{s,t}(x,y)$:

$$|\mathcal{I}_{s,t}(x,y)| \leq C\left[1 + (s-t)^{-1}\int_t^s |\mathbb{T}_{s-t}^{-1}(\check{\theta}_{u,t}(x) - \theta_{u,s}(y))|\,du\right]. \tag{5.10}$$

Indeed, Control (5.10) together with (5.9) and the Gronwall lemma imply the right-hand side of Control (2.29). The left-hand side one can be obtained analogously and we will not show it here.
We start decomposing $\mathcal{I}_{s,t}$ into $\mathcal{I}^1_{s,t} + \mathcal{I}^2_{s,t}$, where we denote

$$\begin{aligned} \mathcal{I}^1_{s,t}(x,y) &:= \mathbb{T}_{s-t}^{-1}\int_t^s A_u\Big(\theta_{u,s}(y) - \check{\theta}_{u,t}(x)\Big)\,du; \\ \mathcal{I}^2_{s,t}(x,y) &:= \mathbb{T}_{s-t}^{-1}\int_t^s \Big[F(u,\theta_{u,s}(y)) - F(u,\check{\theta}_{u,t}(x))\Big]\,du. \end{aligned}$$

The first remainder $\mathcal{I}^1_{s,t}$ can be controlled easily, exploiting the linearity of $z \to A_u z$. Indeed, for any z,z' in $\mathbb{R}^N$ and any u in $[s,t]$, we have that

$$\begin{aligned} \left|\mathbb{T}_{s-t}^{-1}A_u(z-z')\right| &\leq \sum_{i=1}^n \sum_{j=(i-1)\vee 1}^n (s-t)^{-\frac{1+\alpha(i-1)}{\alpha}}|A_u^{i,j}|\,|(z-z')_j| \\ &\leq C(s-t)^{-1}|\mathbb{T}_{s-t}^{-1}(z-z')|. \end{aligned} \tag{5.11}$$

To control instead the second term $\mathcal{I}^2_{s,t}$, we will need to thoroughly exploit an appropriate smoothing method, due to the low regularity in space of the drift F. To overcome this

problem, we are going to mollify the function F in the following way. We start fixing a family $\{\rho_i \colon i \in [\![1,n]\!]\}$ of mollifiers on $\mathbb{R}^{D_i}$ where $D_i = N - \sum_{j=1}^{i-1} d_j$, i.e. for any i in $[\![1,n]\!]$, ρ_i is a compactly supported, non-negative, smooth function on $\mathbb{R}^{D_i}$ such that $\|\rho_i\|_{L^1} = 1$, and a family $\{\delta_{ij} \colon i \leq j\}$ of positive constants to be chosen later. Then, the mollified version of the drift is defined by $F^\delta := (F_1, F_2^\delta, \dots, F_n^\delta)$ where

$$\begin{aligned} F_i^\delta(t,z) &:= F_i *_x \rho_i^\delta(t,z) \\ &:= \int_{\mathbb{R}^{D_i}} F_i(t, z_i - \omega_i, \dots, z_n - \omega_n) \frac{1}{\prod_{j=i}^n \delta_{ij}^{d_i}} \rho_i(\frac{\omega_i}{\delta_{ii}}, \dots, \frac{\omega_n}{\delta_{in}})\, d\omega. \end{aligned} \tag{5.12}$$

Roughly speaking, we have mollified any component F_i by convolution in space with a mollifier with multi-scaled dilations. Then, standard results on mollifier theory and our current assumptions on F show us that the following controls hold

$$|F_i(u,z) - F_i^\delta(u,z)| \leq C \sum_{j=i}^n \delta_{ij}^{\beta^j}, \tag{5.13}$$

$$|F_i^\delta(u,z) - F_i^\delta(u,z')| \leq C \sum_{j=i}^n \delta_{ij}^{\beta^j - 1} |(z-z')_j|. \tag{5.14}$$

We can now pick δ_{ij} for any $i \leq j$ in $[\![2,n]\!]$ in order to have any contribution associated with the mollification appearing in (5.13) at a good current scale time. Namely, we would like δ_{ij} to satisfy

$$\left| \mathbb{T}_{s-t}^{-1} \left(F(u,z) - F^\delta(u,z) \right) \right| \leq C(s-t)^{-1}, \tag{5.15}$$

for any u in $[t,s]$ and any z in $\mathbb{R}^N$. Using the mollifier controls (5.13), it is enough to ask for

$$\sum_{i=2}^n (s-t)^{-\frac{1+\alpha(i-1)}{\alpha}} \sum_{j=i}^n \delta_{ij}^{\beta^j} \leq C(s-t)^{-1}. \tag{5.16}$$

This is true if we fix for example,

$$\delta_{ij} = (s-t)^{\frac{1+\alpha(i-2)}{\alpha\beta^j}} \quad \text{for } i \leq j \text{ in } [\![2,n]\!]. \tag{5.17}$$

Next, we would like to show that, for our choice of the regularization parameter δ_{ij}, the mollified drift F^δ satisfies an *approximate* Lipschitz condition with a constant that, once the drift is integrated, does not yield any additional singularity. Namely, we want to derive the following control:

$$\left| \mathbb{T}_{s-t}^{-1} \left(F^\delta(u,z) - F^\delta(u,z') \right) \right| \leq C \left[(s-t)^{-\frac{1}{\alpha}} + (s-t)^{-1} |\mathbb{T}_{s-t}^{-1}(z-z')| \right]. \tag{5.18}$$

To show it, we start noticing that F_1 is Hölder continuous with Hölder index $\beta^1 > 0$. By Young inequality, it then yields that there exists a positive constant C possibly depending on β^1 such that $|z|^{\beta^1} \leq C(1+|z|)$ for any z in $\mathbb{R}^N$. It then follows from

Equation (5.14) that

$$
\begin{aligned}
&|\mathbb{T}_{s-t}^{-1}\Big(F^\delta(u,z)-F^\delta(u,z')\Big)| \\
&\qquad \leq C\Big[(s-t)^{-\frac{1}{\alpha}}(1+|(z-z')|)+\sum_{i=2}^{n}\sum_{j=i}^{n}(s-t)^{-\frac{1+\alpha(i-1)}{\alpha}}\delta_{ij}^{\beta^j-1}|(z-z')_j|\Big] \\
&\qquad \leq C\Big[(s-t)^{-\frac{1}{\alpha}}+|\mathbb{T}_{s-t}^{-1}(z-z')|\Big(1+\sum_{i=2}^{n}\sum_{j=i}^{n}\frac{(s-t)^{j-i}}{\delta_{ij}^{1-\beta^j}}\Big)\Big].
\end{aligned}
$$

Hence, Control (5.18) follows from the fact that, from our previous choice of δ_{ij}, one gets

$$
\frac{(s-t)^{j-i}}{\delta_{ij}^{1-\beta^j}} \;=\; (s-t)^{(j-i)-\frac{1+\alpha(i-2)}{\alpha\beta^j}(1-\beta^j)} \;\leq\; C(s-t)^{-1}, \tag{5.19}
$$

recalling that we assumed $s-t$ to be small enough and since from the assumption (2.3) on the indexes of Hölder continuity β^j for F:

$$
\beta^j > \frac{1+\alpha(i-2)}{1+\alpha(j-1)} \Leftrightarrow (j-i)-\frac{1+\alpha(i-2)}{\alpha\beta^j}(1-\beta^j) \;>\; -1.
$$

We recall that the above inequality should precisely give the natural threshold, namely an exponent β_i^j satisfying this condition. The current choice for β^j is sufficient to ensure this bound holds for any $i \leq j$ and is *sharp* for $i = j$. We can finally show the bound for the second remainder $\mathcal{I}_{s,t}^2(x,y)$ as given in (5.10). It holds that:

$$
\begin{aligned}
|\mathcal{I}_{s,t}^2(x,y)| \;&\leq\; \int_t^s \Big|\mathbb{T}_{s-t}^{-1}\Big(F(u,\theta_{u,s}(y))-F(u,\check{\theta}_{u,t}(x))\Big)\Big|\,du \\
&\leq \int_t^s \Big|\mathbb{T}_{s-t}^{-1}(F(u,\check{\theta}_{u,t}(x))-F^\delta(u,\check{\theta}_{u,t}(x)))\Big|\,du \\
&\qquad\qquad + \int_t^s \Big|\mathbb{T}_{s-t}^{-1}(F^\delta(u,\check{\theta}_{u,t}(x))-F^\delta(u,\theta_{u,s}(y)))\Big|\,du \\
&\qquad\qquad\qquad + \int_t^s \Big|\mathbb{T}_{s-t}^{-1}\Big(F^\delta(u,\theta_{u,s}(y))-F(u,\theta_{u,s}(y))\Big)\Big|\,du \\
&=:\; \mathcal{I}_{s,t}^{21}(x,y)+\mathcal{I}_{s,t}^{22}(x,y)+\mathcal{I}_{s,t}^{23}(x,y).
\end{aligned}
$$

From Control (5.13) with our choice of δ_{ij}, we easily obtain from Control (5.15) that there exists a positive constant $C := C(T)$ such that

$$
|\mathcal{I}_{s,t}^{21}(x,y)|+|\mathcal{I}_{s,t}^{23}(x,y)| \;\leq\; C, \tag{5.20}
$$

for any $t \leq s$ in $[0,T]$ and x, y in $\mathbb{R}^N$. On the other hand, we exploit (5.18) to derive that

$$
|\mathcal{I}_{s,t}^{22}(x,y)| \;\leq\; C\left[1+\int_t^s (s-t)^{-1}|\mathbb{T}_{s-t}^{-1}(\check{\theta}_{u,t}(x)-\theta_{u,s}(y))|\,du\right]
$$

for any $t \leq s$ in $[0,T]$ and x, y in $\mathbb{R}^N$. To conclude, we finally derive (5.10) from the last inequality together with Controls (5.11)-(5.20).

Proof of Lemma 2.18 (Dirac Convergence of frozen density). Fixed (t,x) in $[0,T]\times\mathbb{R}^N$ and a bounded, continuous function $f\colon \mathbb{R}^N\to\mathbb{R}$, we want to show that the following limit

$$\lim_{\epsilon\to 0}\left|\int_{\mathbb{R}^N} f(y)\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)\,dy - f(x)\right| \;=\; 0$$

holds, uniformly in $t\in[0,T]$.
We start rewriting the argument of the limit in the following way:

$$\begin{aligned}\int_{\mathbb{R}^N} f(y)\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)\,dy - f(x) \qquad (5.21)\\ = \int_{\mathbb{R}^N} f(y)\left[\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)-\tilde{p}^{t,x}(t,t+\epsilon,x,y)\right]dy\\ + \int_{\mathbb{R}^N} f(y)\tilde{p}^{t,x}(t,t+\epsilon,x,y)\,dy - f(x).\end{aligned}$$

By Proposition 2.10, we know that the second term in (5.21) tends to zero, uniformly in t in $[0,T]$ (scaling property of the upper bound for the density), when ϵ goes to zero. We can then focus on the first one. We start splitting the space $\mathbb{R}^N$ in the diagonal/off-diagonal regime associated with our anisotropic dynamics. Namely, we fix $\beta>0$ to be chosen later and we consider the following subsets:

$$\begin{aligned}D_1 &:= \{y\in\mathbb{R}^N\colon \left|\mathbb{T}_\epsilon^{-1}(y-\theta_{t+\epsilon,t}(x))\right| \le \epsilon^{-\beta}\};\\ D_2 &:= \{y\in\mathbb{R}^N\colon \left|\mathbb{T}_\epsilon^{-1}(y-\theta_{t+\epsilon,t}(x))\right| > \epsilon^{-\beta}\},\end{aligned}$$

where $\mathbb{T}_\epsilon$ was defined in (2.17). We can then decompose the first term in (5.21) in the following way:

$$\begin{aligned}\Big|\int_{\mathbb{R}^N} f(y)\big[\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)-\tilde{p}^{t,x}(t,t+\epsilon,x,y)\big]\,dy\Big|\\ \le \|f\|_\infty\int_{D_1}\left|\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)-\tilde{p}^{t,x}(t,t+\epsilon,x,y)\right|dy\\ + \|f\|_\infty\int_{D_2}\left|\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)-\tilde{p}^{t,x}(t,t+\epsilon,x,y)\right|dy\\ =: \|f\|_\infty\left(\mathcal{D}_1+\mathcal{D}_2\right)(t,t+\epsilon,x). \qquad (5.22)\end{aligned}$$

We will follow different approaches to control the two terms $\mathcal{D}_1$, $\mathcal{D}_2$. In the off-diagonal regime D_2, the idea is to exploit tail estimates of the single densities while in the diagonal one D_1, a more thorough sensibility analysis between the spectral measures and the Fourier transform is needed. Let us consider first the off-diagonal term $\mathcal{D}_2$. We can write that

$$\begin{aligned}\mathcal{D}_2(t,t+\epsilon,x,y) &\le \int_{D_2}\left|\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)\right| + \left|\tilde{p}^{t,x}(t,t+\epsilon,x,y)\right|\,dy\\ &\le \int_{D_2}\frac{1}{\det\mathbb{T}_\epsilon}\Big(\bar{p}(1,\mathbb{T}_\epsilon^{-1}(x-\theta_{t,t+\epsilon}(y))) + \bar{p}(1,\mathbb{T}_\epsilon^{-1}(\theta_{t+\epsilon,t}(x)-y))\Big)dy\end{aligned}$$

using Proposition 2.11 together with Lemma 2.9 for the last inequality. From Lemma 2.14 (to use the *approximate* Lipschitz property of the flows) and introducing

$$\bar{D}_2 := \{y\in\mathbb{R}^N\colon \left|\mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y)-x)\right| > \frac{1}{2}\epsilon^{-\beta}\},$$

we thus deduce that for ϵ small enough we get:

$$\mathcal{D}_2(t,t+\epsilon,x,y) \le \int_{\bar{D}_2} \frac{1}{\det \mathbb{T}_\epsilon} \bar{p}(1, \mathbb{T}_\epsilon^{-1}(x-\theta_{t,t+\epsilon}(y)))\, dy + \int_{D_2} \frac{1}{\det \mathbb{T}_\epsilon} \bar{p}(1, \mathbb{T}_\epsilon^{-1}(\theta_{t+\epsilon,t}(x)-y))\, dy.$$

Using now Equation (2.36) from Corollary 2.16 for the first integral and the direct change of variable $z = \mathbb{T}_\epsilon^{-1}(y-\theta_{t+\epsilon,t}(x))$ for the second, we can conclude that

$$\mathcal{D}_2(t,t+\epsilon,x) \le C \int_{\mathbb{R}^N} \mathbb{1}_{B^c(0,\frac{1}{2}\epsilon^{-\beta})}(z)(\check{p}+\overline{p})(1,z)\, dz,$$

where $\check{p}$ is a density enjoying the same integrability properties as $\overline{p}$.
By dominated convergence theorem, it is easy to notice that $\mathcal{D}_2(t,t+\epsilon,x)$ tends to zero if ϵ goes to zero, uniformly in the time variable t in $[0,T]$.
We can now focus on the diagonal term $\mathcal{D}_1$ appearing in (5.22). We start recalling from Equation (2.16) that the density $\tilde{p}^\omega(t,s,x,y)$ (for $\omega \in \{(t,x),(t+\epsilon,y)\}$) can be written as

$$\tilde{p}^\omega(t,t+\epsilon,x,y) = \frac{\det \mathbb{M}_\epsilon^{-1}}{(2\pi)^N} \int_{\mathbb{R}^N} e^{\mathcal{F}_{t,t+\epsilon}(z,\omega)} \exp\left(-i\langle \mathbb{M}_\epsilon^{-1}(y-\tilde{m}^\omega_{t+\epsilon,t}(x)), z\rangle\right) dz,$$

where we have denoted:

$$\mathcal{F}_\epsilon(t,z,\omega) := \epsilon \int_0^1 \int_{\mathbb{R}^d} \left[\cos\left(\langle z, \widehat{\mathcal{R}}_v B \tilde{\sigma}^\omega_{u(v)} p\rangle\right) - 1\right] \nu(dp) dv,$$

with $u(v) = t+\epsilon v$ (cf. notations in (2.12) of Proposition 2.10) and $\Phi(p)$ the Lévy symbol of the process $\{Z_t\}_{t\ge 0}$. We can now consider the two following terms

$$\mathcal{P}_1(t,t+\epsilon,x,y) := \frac{\det \mathbb{M}_\epsilon^{-1}}{(2\pi)^N} \int_{\mathbb{R}^N} \left[e^{\mathcal{F}_\epsilon(t,z,t,x)} - e^{\mathcal{F}_\epsilon(t,z,t+\epsilon,y)}\right] e^{-i\langle \mathbb{M}_\epsilon^{-1}(y-\tilde{m}^{t,x}_{t+\epsilon,t}(x)),z\rangle}\, dz$$

$$\mathcal{P}_2(t,t+\epsilon,x,y) := \frac{\det \mathbb{M}_\epsilon^{-1}}{(2\pi)^N} \int_{\mathbb{R}^N} e^{\mathcal{F}_\epsilon(t,z,t+\epsilon,y)} \left[e^{-i\langle \mathbb{M}_\epsilon^{-1}(y-\tilde{m}^{t,x}_{t+\epsilon,t}(x)),z\rangle} - e^{-i\langle \mathbb{M}_\epsilon^{-1}(y-\tilde{m}^{t+\epsilon,y}_{t+\epsilon,t}(x)),z\rangle}\right] dz$$

and decompose $\mathcal{D}_1$ as follows:

$$\mathcal{D}_1 = \int_{D_1} |\mathcal{P}_1(t,t+\epsilon,x,y)| + |\mathcal{P}_2(t,t+\epsilon,x,y)|\, dy.$$

To control the first term $\mathcal{P}_1$, we can exploit a Taylor expansion. Indeed,

$$|\mathcal{P}_1(t,t+\epsilon,x,y)| \le \frac{C}{\det \mathbb{M}_\epsilon} \int_{\mathbb{R}^N} \int_0^1 |\mathcal{F}_\epsilon(t,z,t+\epsilon,y) - \mathcal{F}_\epsilon(t,z,t,x)|\, e^{\lambda \mathcal{F}_\epsilon(t,z,t+\epsilon,y) + (1-\lambda)\mathcal{F}_\epsilon(t,z,t,x)}\, d\lambda dz.$$

We then notice from (2.15) that

$$\mathcal{F}_\epsilon(t,z,\omega) \le C\epsilon[1-|z|^\alpha],$$

and thus, we obtain that

$$e^{\lambda\mathcal{F}_\epsilon(t,z,t+\epsilon,y)+(1-\lambda)\mathcal{F}_\epsilon(t,z,t,x)} \leq e^{C\epsilon(1-|z|^\alpha)},$$

for some constant C independent from λ in $[0,1]$. From our non-degenerate structure, any linear combination of the symbols remains homogeneous to a non-degenerate symbol. Thus, we have that

$$|\mathcal{P}_1(t,t+\epsilon,x,y)| \leq \frac{C}{\det \mathbb{M}_\epsilon}\int_{\mathbb{R}^N} |\mathcal{F}_\epsilon(t,z,t+\epsilon,y)-\mathcal{F}_\epsilon(t,z,t,x)|\, e^{C\epsilon(1-|z|^\alpha)}\, dz. \tag{5.23}$$

On the other hand, we can decompose the difference in absolute value in the following way:

$$\begin{aligned}
&|\mathcal{F}_\epsilon(t,z,t+\epsilon,y)-\mathcal{F}_\epsilon(t,z,t,x)| \\
&\qquad \leq \epsilon\int_0^1\left|\int_{\mathbb{R}^d}\left[\cos\left(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t+\epsilon,y}_{u(v)}p\rangle\right)-\cos\left(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t,x}_{u(v)}p\rangle\right)\right]\nu(dp)\right|dv \\
&\qquad \leq \epsilon\int_0^1\left|\left(\Delta_s^{t,\epsilon,x,y}+\Delta_l^{t,\epsilon,x,y}\right)(v,z)\right|dv,
\end{aligned} \tag{5.24}$$

where we denoted

$$\begin{aligned}
\Delta_s^{t,\epsilon,x,y}(v,z) &= \int_{B(0,r_0)}\left[\cos\left(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t+\epsilon,y}_{u(v)}p\rangle\right)-\cos\left(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t,x}_{u(v)}p\rangle\right)\right]Q(p)\,\nu_\alpha(dy);\\
\Delta_l^{t,\epsilon,x,y}(v,z) &= \int_{B^c(0,r_0)}\left[\cos\left(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t+\epsilon,y}_{u(v)}p\rangle\right)-\cos\left(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t,x}_{u(v)}p\rangle\right)\right]Q(p)\,\nu_\alpha(dp),
\end{aligned}$$

with r_0 defined in assumption [**ND**]. The term $\Delta_l^{t,\epsilon,x,y}$ involving the large jumps can be easily controlled using that $\sup_{p\in\mathbb{R}^d} Q(p)<\infty$:

$$\begin{aligned}
|\Delta_l^{t,\epsilon,x,y}(v,z)| &\leq \int_{B^c(0,r_0)}\left|\cos\left(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t+\epsilon,y}_{u(v)}p\rangle\right)-\cos\left(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t,x}_{u(v)}p\rangle\right)\right|\nu_\alpha(dp)\\
&\leq C.
\end{aligned} \tag{5.25}$$

To bound the term $\Delta_s^{t,\epsilon,x,y}$ associated with the small jumps, we want to exploit instead that Q is Lipschitz continuous on $B(0,r_0)$. For this reason, we write that

$$\begin{aligned}
&|\Delta_s^{t,\epsilon,x,y}(v,z)| \\
&\quad\leq \left|\int_{B(0,r_0)}\left[\cos\left(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t+\epsilon,y}_{u(v)}p\rangle\right)-\cos\left(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t,x}_{u(v)}p\rangle\right)\right][Q(p)-Q(0)]\,\nu_\alpha(dp)\right| \\
&\qquad\qquad+\left|\int_{B(0,r_0)}\left[\cos\left(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t+\epsilon,y}_{u(v)}p\rangle\right)-\cos\left(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t,x}_{u(v)}p\rangle\right)\right]Q(0)\,\nu_\alpha(dp)\right| \\
&\quad=: \left(\Delta_{s,1}^{t,\epsilon,x,y}+\Delta_{s,2}^{t,\epsilon,x,y}\right)(v,z).
\end{aligned} \tag{5.26}$$

Since Q and the cosine function are Lipschitz continuous in a neighborhood of 0, we have that

$$\begin{aligned}
\Delta_{s,1}^{t,\epsilon,x,y}(v,z) &\leq C\int_{B(0,r_0)}|p||z|\left|\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t+\epsilon,y}_{u(v)}p-\widehat{\mathcal{R}}_v B\tilde{\sigma}^{t,x}_{u(v)}p\right|\nu_\alpha(dp)\\
&\leq C\int_{B(0,r_0)}|p||z|\left|\sigma(u(v),\theta_{u(v),t+\epsilon}(y))p-\sigma(u(v),\theta_{u(v),t}(x))p\right|\nu_\alpha(dp)\\
&\leq C|z|\int_{B(0,r_0)}|p|^2\,\nu_\alpha(dp) \leq C|z|,
\end{aligned} \tag{5.27}$$

where in the last step, we used that the diffusion coefficient σ is bounded (cf. assumption [**UE**]).
The control of the other term $\Delta_{s,2}^{t,\epsilon,x,y}$ now follows from the classical characterization of the Lévy symbol of a non-degenerate α-stable process (see e.g. [Sat13]). Indeed,

$$\begin{aligned}\Delta_{s,2}^{t,\epsilon,x,y}(v,z) &= \left|\int_{\mathbb{R}^d}\left[\cos\left(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}_{u(v)}^{t+\epsilon,y}p\rangle\right)-1\right]-\left[\cos\left(\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}_{u(v)}^{t,x}p\rangle\right)-1\right]\nu(dp)\right| \\ &\leq C\int_{\mathbb{S}^{d-1}}\left|\left|\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}_{u(v)}^{t+\epsilon,y}s\rangle\right|^\alpha-\left|\langle z,\widehat{\mathcal{R}}_v B\tilde{\sigma}_{u(v)}^{t,x}s\rangle\right|^\alpha\right|\mu(ds).\end{aligned}$$

We now exploit the β^1-Hölder regularity in space of the diffusion coefficient σ to show that

$$\begin{aligned}\Delta_{s,2}^{t,\epsilon,x,y}(v,z) &\leq C|z|^\alpha\left|\theta_{u(v),t+\epsilon}(y)-\theta_{u(v),t}(x)\right|^{\beta^1(\alpha\wedge 1)} \\ &\leq C|z|^\alpha\left[|y-\theta_{t+\epsilon,t}(x)|^{\beta^1}+\epsilon^{\beta^1}\right],\end{aligned} \tag{5.28}$$

where in the last step we used that $\alpha>1$ and the approximate Lipschitz property of the flow (cf. Lemma 2.14 up to a normalization, see also Lemma 1.1 in [MPZ21]).
We can now use Controls (5.27)-(5.28) in Equation (5.26) to show that

$$|\Delta_s^{t,\epsilon,x,y}(v,z)| \leq C\left(|z|+|z|^\alpha+|y-\theta_{t+\epsilon,t}(x)|^{\beta^1}|z|^\alpha\right). \tag{5.29}$$

Similarly, Controls (5.29)-(5.25) with Equation (5.24) allow us to conclude that

$$|\mathcal{F}_\epsilon(t,z,t+\epsilon,y)-\mathcal{F}_\epsilon(t,z,t,x)| \leq C\epsilon\left(1+|z|+\epsilon^{\beta^1}|z|^\alpha+|y-\theta_{t+\epsilon,t}(x)|^{\beta^1}|z|^\alpha\right). \tag{5.30}$$

We can now go back to Equation (5.23). Changing variable and integrating over z, we find that

$$\begin{aligned}|\mathcal{P}_1(t,t+\epsilon,x,y)| &\leq \frac{C\epsilon}{\det\mathbb{M}_\epsilon}\int_{\mathbb{R}^N}\left(1+|z|+\epsilon^{\beta^1}|z|^\alpha+|y-\theta_{t+\epsilon,t}(x)|^{\beta^1}|z|^\alpha\right)e^{C\epsilon(1-|z|^\alpha)}\,dz \\ &\leq \frac{C}{\det\mathbb{T}_\epsilon}\int_{\mathbb{R}^N}\left(\epsilon+\epsilon^{\frac{\alpha-1}{\alpha}}|\tilde{z}|+\epsilon^{\beta^1}|\tilde{z}|^\alpha+|y-\theta_{t+\epsilon,t}(x)|^{\beta^1}|\tilde{z}|^\alpha\right)e^{C(1-|\tilde{z}|^\alpha)}\,dz \\ &\leq \frac{C}{\det\mathbb{T}_\epsilon}\left(\epsilon^{(1-\frac{1}{\alpha})\wedge\beta^1}+|y-\theta_{t+\epsilon,t}(x)|^{\beta^1}\right)\end{aligned}$$

where we recall that $\mathbb{T}_t=t^{1/\alpha}\mathbb{M}_t$.
To conclude, we apply the change of variable $\tilde{y}=y-\theta_{t+\epsilon,t}(x)$:

$$\begin{aligned}\int_{D_1}|\mathcal{P}_1(t,t+\epsilon,x,y)|\,dy &\leq \frac{C}{\det\mathbb{T}_\epsilon}\int_{D_1}\left[|y-\theta_{t+\epsilon,t}(x)|^{\beta^1}+\epsilon^{(1-\frac{1}{\alpha})\wedge\beta^1}\right]dy \\ &= C\int_{|\tilde{y}|\leq\epsilon^{-\beta}}\left[|\mathbb{T}_\epsilon\tilde{y}|^{\beta^1}+\epsilon^{(1-\frac{1}{\alpha})\wedge\beta^1}\right]d\tilde{y} \\ &\leq C[\epsilon^{\beta^1/\alpha-\beta(N+\beta^1)}+\epsilon^{(1-\frac{1}{\alpha})\wedge\beta^1-\beta N}].\end{aligned}$$

The above control then tends to zero letting ϵ go to zero, if we choose β such that

$$0<\beta<\frac{\beta^1}{\alpha(N+\beta^1)}\wedge\frac{(1-\frac{1}{\alpha})\wedge\beta^1}{N}.$$

To control the second term $\mathcal{P}_2$, we use again Control (2.15) and a Taylor expansion to write, similarly to above, that

$$\begin{aligned}
&|\mathcal{P}_2(t,t+\epsilon,x,y)| \\
&\quad\leq \frac{C}{\det \mathbb{M}_\epsilon}\int_{\mathbb{R}^N} e^{C\epsilon(1-|z|^\alpha)}\left|\langle \mathbb{M}_\epsilon^{-1}(y-\tilde{m}^{t,x}_{t+\epsilon,t}(x)),z\rangle - \langle \mathbb{M}_\epsilon^{-1}(y-\tilde{m}^{t+\epsilon,y}_{t+\epsilon,t}(x)),z\rangle\right| dz \\
&\quad\leq \frac{C}{\det \mathbb{T}_\epsilon}\left|\mathbb{T}_\epsilon^{-1}\left(\theta_{t+\epsilon,t}(x)-\tilde{m}^{t+\epsilon,y}_{t+\epsilon,t}(x)\right)\right|, \qquad (5.31)
\end{aligned}$$

where in the last passage we used Lemma 2.9. To bound the above right-hand side, we now exploit Corollary 2.15 to show that

$$|\mathcal{P}_2(t,t+\epsilon,x,y)| \;\leq\; C\epsilon^{\frac{1}{\alpha}\wedge\zeta}\frac{1}{\det \mathbb{T}_\epsilon}\left(1+|\mathbb{T}_\epsilon^{-1}(\theta_{t+\epsilon,t}(x)-y)|\right).$$

Similarly to above, we can then apply a change of variables:

$$\int_{D_1}|\mathcal{P}_2(t,t+\epsilon,x,y)|\,dy \;\leq\; C\epsilon^{\frac{1}{\alpha}\wedge\zeta}\int_{|z|\leq\epsilon^{-\beta}}(1+|z|)\;dz.$$

We can then notice again that the above control tends to zero letting ϵ goes to zero, if we choose β small enough.

Proof of Lemma 2.19. As in the previous Lemma 2.18, we want to show the following limit:

$$\lim_{\epsilon\to 0}\|I_\epsilon f - f\|_{L^p_tL^q_x} \;=\; 0,$$

for some $p\in(1,+\infty)$, $q\in(1,+\infty)$ and f in $C^{1,2}_c([0,T)\times\mathbb{R}^N)$. We start writing that

$$\|I_\epsilon f - f\|^p_{L^p_tL^q_x} \;=\; \int_0^T \|I_\epsilon f(t,\cdot)-f(t,\cdot)\|^p_{L^q}\,dt.$$

We then notice that, up to a middle point-type argument, the indicator function in the definition (2.46) of $I_\epsilon f$ can be easily controlled. We can now write that

$$\begin{aligned}
\|I_\epsilon f(t,\cdot)-f(t,\cdot)\|^p_{L^p} \;&=\; \int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N} f(t+\epsilon,y)\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)\,dy - f(t,x)\right|^p dx \\
&\leq C\Bigg(\int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N} f(t+\epsilon,y)\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)\,dy - f(t+\epsilon,\theta_{t+\epsilon,t}(x))\right|^p dx \\
&\qquad\qquad + \int_{\mathbb{R}^N}|f(t+\epsilon,\theta_{t+\epsilon,t}(x))-f(t,x)|^p\,dx\Bigg) \\
&=:\; C\left(\mathcal{I}+\mathcal{I}'\right)(\epsilon,t). \qquad (5.32)
\end{aligned}$$

Since f is smooth and with compact support in time and space, it follows immediately that $\mathcal{I}'(\epsilon,t)$ tends to zero if ϵ goes to zero, thanks to the bounded convergence Theorem. We can then focus on the first term $\mathcal{I}(\epsilon,t)$. We start splitting it in the following way:

$$\begin{aligned}
\mathcal{I}(\epsilon,t) \;&\leq\; C\Bigg(\int_{\mathbb{R}^N}\left|\int_{\mathbb{R}^N}\left[f(t+\epsilon,y)-f(t+\epsilon,\theta_{t+\epsilon,t}(x))\right]\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)\,dy\right|^p dx \\
&\quad+\int_{\mathbb{R}^N}\left|f(t+\epsilon,\theta_{t+\epsilon,t}(x))\int_{\mathbb{R}^N}\left[\tilde{p}^{t+\epsilon,y}(t,t+\epsilon,x,y)-\tilde{p}^{t,x}(t,t+\epsilon,x,y)\right]dy\right|^p dx\Bigg) \\
&=:\; C\left(\mathcal{I}_1+\mathcal{I}_2\right)(\epsilon,t),
\end{aligned}$$

where we used that $\tilde{p}^{t,x}(t,s,x,y)$ is indeed a *true* density with respect to y. The second term $\mathcal{I}_2(\epsilon,t)$ already appeared in the proof of Lemma 2.18 (Dirac Convergence of frozen density) (cf. term $\mathcal{D}_2$ in (5.22)) and a similar analysis readily gives that $\mathcal{I}_2(\epsilon,t) \xrightarrow{\epsilon\to 0} 0$. To control instead the first term $\mathcal{I}_1(\epsilon,t)$, we decompose the whole space $\mathbb{R}^N$ into $\Delta_1 \cup \Delta_2$ given by

$$\begin{aligned}\Delta_1 &:= \{x \in \mathbb{R}^N \colon |\theta_{t+\epsilon,t}(x) - \mathrm{supp}[f(t+\epsilon,\cdot)]| \le 1\};\\ \Delta_2 &:= \{x \in \mathbb{R}^N \colon |\theta_{t+\epsilon,t}(x) - \mathrm{supp}[f(t+\epsilon,\cdot)]| > 1\}.\end{aligned}$$

Using Proposition 2.11 with $(\tau,\xi) = (t+\epsilon, y)$, we write that

$$\begin{aligned}\mathcal{I}_1(\epsilon,t) &\le \int_{\mathbb{R}^N}\left(\int_{\mathbb{R}^N} |f(t+\epsilon,y) - f(t+\epsilon,\theta_{t+\epsilon,t}(x))| \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(y - \tilde{m}^{t+\epsilon,y}_{t+\epsilon,t}(x))\right)}{\det \mathbb{T}_\epsilon}\, dy\right)^p dx\\ &\le \int_{\Delta_1}\left(\int_{\mathbb{R}^N} |f(t+\epsilon,y) - f(t+\epsilon,\theta_{t+\epsilon,t}(x))| \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(y - \tilde{m}^{t+\epsilon,y}_{t+\epsilon,t}(x))\right)}{\det \mathbb{T}_\epsilon}\, dy\right)^p dx\\ &\quad + \int_{\Delta_2}\left(\int_{\mathbb{R}^N} |f(t+\epsilon,y) - f(t+\epsilon,\theta_{t+\epsilon,t}(x))| \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(y - \tilde{m}^{t+\epsilon,y}_{t+\epsilon,t}(x))\right)}{\det \mathbb{T}_\epsilon}\, dy\right)^p dx\\ &=: \left(\mathcal{I}_{11} + \mathcal{I}_{12}\right)(\epsilon,t).\end{aligned}$$

To control $\mathcal{I}_{11}$, we start noticing that f is Hölder continuous with a Hölder exponent $\gamma < \alpha$ in $(0,1]$, since it has a compact support. Moreover, Δ_1 is a bounded set (uniformly in ϵ). Then, from Lemma 2.9 (cf. Equation (2.6)), Lemma 2.14 and Corollary 2.16,

$$\begin{aligned}\mathcal{I}_{11}(\epsilon,t) &\le C\int_{\Delta_1}\left(\int_{\mathbb{R}^N} |y - \theta_{t+\epsilon,t}(x)|^\gamma \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(y - \tilde{m}^{t+\epsilon,y}_{t+\epsilon,t}(x))\right)}{\det \mathbb{T}_\epsilon}\, dy\right)^p dx\\ &\le C\epsilon^{p\gamma/\alpha}\int_{\Delta_1}\left(\int_{\mathbb{R}^N} |\mathbb{T}_\epsilon^{-1}(y - \theta_{t+\epsilon,t}(x))|^\gamma \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y) - x)\right)}{\det \mathbb{T}_\epsilon}\, dy\right)^p dx\\ &\le C\epsilon^{p\gamma/\alpha}\int_{\Delta_1}\left(\int_{\mathbb{R}^N} \left[|\mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y) - x)|^\gamma + 1\right] \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y) - x)\right)}{\det \mathbb{T}_\epsilon}\, dy\right)^p dx\\ &\le C\epsilon^{p\gamma/\alpha}.\end{aligned}$$

To control instead $\mathcal{I}_{12}$ we firstly notice that if x is in Δ_2, then, $\theta_{t+\epsilon,t}(x)$ is not in the support of f. Thus,

$$\begin{aligned}\mathcal{I}_{12}(\epsilon,t) &= \int_{\Delta_2}\left(\int_{\mathbb{R}^N} |f(t+\epsilon,y) - f(t+\epsilon,\theta_{t+\epsilon,t}(x))| \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(y - \tilde{m}^{t+\epsilon,y}_{t+\epsilon,t}(x))\right)}{\det \mathbb{T}_\epsilon}\, dy\right)^p dx\\ &\le \int_{\Delta_2}\left(\int_{\mathrm{supp} f} |f(t+\epsilon,y)| \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y) - x)\right)}{\det \mathbb{T}_\epsilon}\, dy\right)^p dx\\ &\le \|f\|_\infty^p \int_{\Delta_2}\left(\int_{\mathrm{supp} f} \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y) - x)\right)}{\det \mathbb{T}_\epsilon}\, dy\right)^{p-1+1} dx\\ &\le C\int_{\mathrm{supp} f}\int_{\Delta_2} \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y) - x)\right)}{\det \mathbb{T}_\epsilon}\, dx dy,\end{aligned}$$

where in the last step we used that, from Corollary 2.16

$$\left(\int_{\text{supp}f} \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y)-x)\right)}{\det \mathbb{T}_\epsilon}\, dy\right)^{p-1} \leq \left(\int_{\mathbb{R}^N} \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y)-x)\right)}{\det \mathbb{T}_\epsilon}\, dy\right)^{p-1} \leq C_p.$$

We notice now that for any y in supp$f$ and any x in Δ_2, we have that $|y-\theta_{t+\epsilon,t}(x))| \geq 1$. Exploiting Corollary 2.16 and Lemma 2.14, we write that

$$\begin{aligned}
\mathcal{I}_{12}(\epsilon,t) &\leq \int_{\text{supp}f}\int_{\Delta_2} |y-\theta_{t+\epsilon,t}(x)| \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y)-x)\right)}{\det \mathbb{T}_\epsilon}\, dxdy \\
&\leq C\epsilon^{\frac{1}{\alpha}} \int_{\text{supp}f}\int_{\Delta_2} |\mathbb{T}_\epsilon^{-1}(y-\theta_{t+\epsilon,t}(x))| \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y)-x)\right)}{\det \mathbb{T}_\epsilon}\, dxdy \\
&\leq C\epsilon^{\frac{1}{\alpha}} \int_{\text{supp}f}\int_{\mathbb{R}^N} \left[|\mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y)-x)|+1\right] \frac{\overline{p}\left(1, \mathbb{T}_\epsilon^{-1}(\theta_{t,t+\epsilon}(y)-x)\right)}{\det \mathbb{T}_\epsilon}\, dxdy \\
&\leq C\epsilon^{\frac{1}{\alpha}} \int_{\text{supp}f}\int_{\mathbb{R}^N} \left[|z|+1\right] \overline{p}\left(1, z\right)\, dzdy \\
&\leq C\epsilon^{\frac{1}{\alpha}}.
\end{aligned}$$

Knowing the convergence of $\mathcal{I}(\epsilon,t)$ and $\mathcal{I}'(\epsilon,t)$ to zero, we can finally conclude the proof using the dominated convergence theorem in (5.32).

5.3 Controls associated with the change of variable

Proof of Corollary 2.16

We first concentrate on the proof of Control (2.35). We start exploiting the decomposition of $\overline{p}(t,z)$ in terms of small and large jumps, as in (2.25), to rewrite the left-hand side of Equation (2.35) in the following way:

$$\begin{aligned}
I(s,t,x) &:= \int_{\mathbb{R}^N} \frac{|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)|^\gamma}{\det \mathbb{T}_{s-t}} \bar{p}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x))\, dy \\
&= \int_{\mathbb{R}^N} \frac{|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)|^\gamma}{\det \mathbb{T}_{s-t}} \int_{\mathbb{R}^N} p_{\overline{M}}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)-w)\overline{P}_1(dw)dy,
\end{aligned}$$

Then, the Fubini Theorem and the definition of $p_{\overline{M}}$ in (2.23) immediately imply that

$$\begin{aligned}
I(s,t,x) &= \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)|^\gamma}{\det \mathbb{T}_{s-t}} p_{\overline{M}}(1, \mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)-w)\, dy\overline{P}_1(dw) \\
&\leq C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \det \mathbb{T}_{s-t}^{-1} \frac{\left[|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y))-x-w|^\gamma + |w|^\gamma\right]}{\left[1+|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)-w|\right]^{N+2}}\, dy\overline{P}_1(dw).
\end{aligned}$$

To conclude, it is now enough to show that for any $M > N+1$, there exists $C := C(M)$ such that

$$\int_{\mathbb{R}^N} \frac{\det \mathbb{T}_{s-t}^{-1}}{\left[1+|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)-w|\right]^M}\, dy \leq C. \tag{5.33}$$

Indeed, it would follow from Control (5.33) that

$$\begin{aligned} I(t,s,x) \;\le\; & C\int_{\mathbb{R}^N}\int_{\mathbb{R}^N} \frac{\det \mathbb{T}_{s-t}^{-1}}{\left[1+|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)-w|\right]^{N+2-\gamma}}\, dy\overline{P}_1(dw) \\ & + \int_{\mathbb{R}^N}\int_{\mathbb{R}^N} [1+|w|^\gamma]\, \frac{\det \mathbb{T}_{s-t}^{-1}}{\left[1+|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)-w|\right]^{N+2}}\, dy\overline{P}_1(dw) \\ \le\; & C\int_{\mathbb{R}^N} [1+|w|^\gamma]\, \overline{P}_1(dw) \;\le\; C. \end{aligned}$$

In order to show Control (5.33), we start noticing that it would be enough to apply the change of variable $\tilde{y} = \mathbb{T}_{s-t}^{-1}(x-\theta_{t,s}(y)) - w$ and then, to control the Jacobian matrix of the transformation. Unfortunately, our coefficients are not smooth enough in order to follow this kind of reasoning. Indeed, the drift F is only Hölder continuous.
As done already in in the proof of Lemma 2.14, we firstly need to regularize F through a multi-scale mollification procedure. Namely, we reintroduce the mollified drift $F^\delta := (F_1^\delta, \dots, F_n^\delta)$ similarly to what we did in Equation (5.12). However we modify a bit the mollifying parameters and set

$$\delta_{ij} \;=\; \bar{C}(s-t)^{\frac{1+\alpha(j-2)}{\alpha\beta j}} \quad \text{for } 2 \le i \le j \le n, \tag{5.34}$$

for a constant $\bar{C}$ meant to be large enough. We also mollify the first component F_1 at a macro scale, i.e. $\delta_{1j} = C_1$, with C_1 large enough as well.

In particular, this choice of parameters gives that the controls (5.11), (5.15) and (5.18) hold again.
We can now define the mollified flow $\theta_{t,s}^\delta(y)$ associated with the drift F^δ given by

$$\theta_{t,s}^\delta(y) \;=\; y - \int_t^s \left[A_u\theta_{u,s}^\delta + F^\delta(u,\theta_{u,s}^\delta(y))\right] du. \tag{5.35}$$

Denoting now, for brevity,

$$\Delta^\delta\theta_{u,s}(y) \;:=\; \theta_{u,s}(y) - \theta_{u,s}^\delta(y),$$

it is not difficult to check from the Grönwall Lemma and Controls (5.11), (5.15) and (5.18) that

$$\begin{aligned} |\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-\theta_{t,s}^\delta(y))| \;\le\; & \left|\int_t^s \mathbb{T}_{s-t}^{-1}\left[A_u(\Delta^\delta\theta_{u,s}(y)) + F(u,\theta_{u,s}(y)) - F^\delta(u,\theta_{u,s}^\delta(y))\right] du\right| \\ \le\; & \int_t^s \left|\mathbb{T}_{s-t}^{-1}A_u(\Delta^\delta\theta_{u,s}(y))\right| du + \int_t^s \left|\mathbb{T}_{s-t}^{-1}\Big(F(u,\theta_{u,s}(y)) - F^\delta(u,\theta_{u,s}(y))\Big)\right| du \\ & + \int_t^s \left|\mathbb{T}_{s-t}^{-1}\Big(F^\delta(u,\theta_{u,s}(y)) - F^\delta(u,\theta_{u,s}^\delta(y))\Big)\right| du \\ \le\; & C_0, \end{aligned} \tag{5.36}$$

for some positive constant C_0.
Exploiting now Control (5.36), we firstly notice that for $C \ge 2C_0$,

$$\begin{aligned} C + |\mathbb{T}_{s-t}^{-1}(x-\theta_{t,s}(y)) - w| \;\ge\; & C + |\mathbb{T}_{s-t}^{-1}(x-\theta_{t,s}^\delta(y)) - w| - |\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-\theta_{t,s}^\delta(y))| \\ \ge\; & C_0 + |\mathbb{T}_{s-t}^{-1}(x-\theta_{s,t}^\delta(y)) - w| \end{aligned}$$

and we then use it to write that

$$\int_{\mathbb{R}^N} \frac{\det \mathbb{T}_{s-t}^{-1}}{\left(1+|\mathbb{T}_{s-t}^{-1}(x-\theta_{t,s}(y))-w|\right)^M}\,dy \le C\int_{\mathbb{R}^N} \frac{\det \mathbb{T}_{s-t}^{-1}}{\left(1+|\mathbb{T}_{s-t}^{-1}(x-\theta_{t,s}^{\delta}(y))-w|\right)^M}\,dy$$
$$= C\int_{\mathbb{R}^N} \frac{1}{(1+|\tilde{y}|)^M}\frac{1}{\det J_{t,s}^{\delta}(\tilde{y})}\,dy \tag{5.37}$$

where in the last step we used the change of variables $\tilde{y} = \mathbb{T}_{s-t}^{-1}(x-\theta_{t,s}^{\delta}(y)) - w$ and denoted by $J_{t,s}^{\delta}(\tilde{y})$ the Jacobian matrix of $y \to \theta_{t,s}^{\delta}(y)$.
It is now clear that the last term in (5.37) is indeed controlled by a constant C, if we show the existence of a positive constant c, independent from y in $\mathbb{R}^N$, $t < s$ in $[0,T]$ and δ , such that

$$|\det J_{t,s}^{\delta}(y)| \ \ge\ c > 0. \tag{5.38}$$

This is precisely the result provided by Lemma 5.3 below. From the previous computations it is clear that (2.35) holds.

Let us now turn to the proof of Control (2.36). Following the previous approach, we can write

$$\int_{\{|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)|\ge K\}} \frac{1}{\det \mathbb{T}_{s-t}}\overline{p}(1,\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x))\,dy$$
$$\le\ C\int_{\{|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}^{\delta}(y)-x)|\ge K-|\mathbb{T}_{s-t}^{-1}(\Delta^{\delta}\theta_{u,s}(y))|\}}\int_{\mathbb{R}^N} \frac{\det \mathbb{T}_{s-t}^{-1}}{\left(1+|\mathbb{T}_{s-t}^{-1}(x-\theta_{s,t}^{\delta}(y))-w|\right)^M}\overline{P}_1(dw)dy$$
$$\le C\int_{\{|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}^{\delta}(y)-x)|\ge K-C_0\}}\int_{\mathbb{R}^N} \frac{\det \mathbb{T}_{s-t}^{-1}}{\left(1+|\mathbb{T}_{s-t}^{-1}(x-\theta_{s,t}^{\delta}(y))-w|\right)^M}\overline{P}_1(dw)dy,$$

exploiting also (5.36) for the last inequality. Using now the Fubini Theorem and the change of variables $z = \mathbb{T}_{s-t}^{-1}(x-\theta_{s,t}^{\delta}(y))$, we derive from (5.38) that

$$\int_{\{|\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x)|\ge K\}} \frac{1}{\det \mathbb{T}_{s-t}}\overline{p}(1,\mathbb{T}_{s-t}^{-1}(\theta_{t,s}(y)-x))\,dy$$
$$\le\ C\int_{\mathbb{R}^N}\int_{\{|z|\ge K-C_0\}} \frac{1}{(1+|z-w|)^M}\,dz\overline{P}_1(dw)$$
$$=:\ C\int_{\{|z|\ge \frac{K}{2}|\}} \check{p}(1,z)\,dz,$$

where $\check{p}$ is a density satisfying the same integrability properties as $\bar{p}$ assuming as well K large enough. Thus (2.36) holds and the proof of Corollary 2.16 is now complete.

Jacobian of the mollified system

This is a technical part dedicated to the proof of control (5.38) appearing in the proof of key Corollary 2.16 which precisely gives the expected smoothing effect of the frozen density where the freezing parameters also correspond to the integration variable.

Lemma 5.3 (Control of the determinant for the change of variable)**.** *Let $\theta^\delta_{t,s}(y)$ denote the mollified flow associated with the drift F^δ where the mollifying parameter δ has the form* (5.34)*. Its dynamics writes:*

$$\theta^\delta_{t,s}(y) \;=\; y - \int_t^s \left[A_u \theta^\delta_{u,s} + F^\delta(u, \theta^\delta_{u,s}(y))\right] du.$$

Then, there exists a constants $c_0 := c_0(T) > 0$ s.t., denoting for $0 \le t \le s \le T$ by $J^\delta_{t,s}(y)$ the Jacobian matrix associated with the mapping $y \mapsto \theta^\delta_{s,t}(y)$

$$\det(J^\delta_{t,s}(y)) \ge c_0.$$

Importantly, c_0 does not depend on δ.

Proof. Let us first mention that the even though the coefficients F^δ are smooth, the above control is not direct because there is a subtle balance between the mollifying, matrix valued, parameter δ and the length of the considered time interval $[t, s]$. We recall indeed that the entries δ_{ij} given in (5.34) do depend on $s - t$.

We also recall that, similarly to (5.14), it holds that

$$|D_{x_j} F^\delta_1(t, z)| \;\le\; C(\delta_{1j})^{\beta^1 - 1},\ \forall 2 \le i \le j \le n,\ |D_{x_j} F^\delta_i(t, z)| \;\le\; C(\delta_{ij})^{\beta^j - 1}. \tag{5.39}$$

To prove the statement, we have thus to justify, somehow similarly to the control for the flows of Lemma 2.14, that the explosive behavior of the Lipschitz moduli is indeed well balanced by the time-integration.

Let us now start from the dynamics of $J^\delta(y)$ which writes:

$$\begin{aligned} J^\delta_{t,s}(y) = \;\; D_y \theta^\delta_{t,s}(y) &= \mathbb{I} - \int_t^s \left[\left(A_u + D_z F^\delta(u, z)|_{z=\theta^\delta_{u,s}(y)}\right) D_y \theta^\delta_{u,s}(y)\right] du \\ &= \mathbb{I} - \int_t^s \left[\left(A_u + D_z F^\delta(u, z)|_{z=\theta^\delta_{u,s}(y)}\right) J^\delta_{u,s}(y)\right] du. \end{aligned}$$

The above equation can be partially integrated using the resolvent $(R_{u,s})_{u\in[t,s]}$ associated with A, i.e. the $\mathbb{R}^N \otimes \mathbb{R}^N$ valued function satisfying

$$\frac{d}{du} R_{u,s} = A_u R_{u,s},\ \ R_{s,s} = I_{nd\times nd}. \tag{5.40}$$

This yields:

$$J^\delta_{t,s}(y) = R_{t,s} - \int_t^s R_{t,u} D_z F^\delta(u, z)|_{z=\theta^\delta_{u,s}(y)} J^\delta_{u,s}(y) du. \tag{5.41}$$

We actually have the following important structure property of the resolvent $(R_{u,s})_{u\in[t,s]}$. There exists a non-degenerate family of matrices $(\hat{R}^{t,s}_{\frac{u-t}{s-t}})_{u\in[t,s]}$, which is bounded uniformly on $u \in [t, s]$ with constants depending on T s.t.

$$R_{u,s} = \mathbb{T}_{s-t} \hat{R}^{t,s}_{\frac{u-t}{s-t}} (\mathbb{T}_{s-t})^{-1}. \tag{5.42}$$

Indeed, setting for all $v \in [0,1]$, $\hat{R}^{t,s}_v := (\mathbb{T}_{s-t})^{-1} R_{t+v(s-t),s} \mathbb{T}_{s-t}$ and differentiating yields:

$$\begin{aligned}\partial_v \hat{R}^{t,s}_v &= (s-t)(\mathbb{T}_{s-t})^{-1} A_{t+v(s-t)} R_{t+v(s-t),s} \mathbb{T}_{s-t} \\ &= \Big((s-t)(\mathbb{T}_{s-t})^{-1} A_{t+v(s-t)} \mathbb{T}_{s-t}\Big) \hat{R}^{t,s}_v := A^{t,s}_v \hat{R}^{t,s}_v.\end{aligned}$$

The identity (5.42) then actually follows from the structure of the matrix A_t (see assumption **[H]** and (1.4)) which ensures that $(A^{t,s}_v)_{v\in[0,1]}$ has bounded entries.

As a by-product of (5.42), we derive that there exists $C \geq 1$ s.t. for all $(i,j) \in [\![1,n]\!]$,

$$|(R_{t,u})_{ij}| \leq C(\mathbb{1}_{j\geq i} + (s-t)^{i-j}\mathbb{1}_{i>j}). \tag{5.43}$$

From (5.41) we thus derive

$$\begin{aligned}|J^\delta_{t,s}(y)| \leq& C + \int_t^s \sum_{i,j,k=1}^n \Big|R_{t,u} D_z F^\delta(u,z)|_{z=\theta^\delta_{u,s}(y)}\Big|_{ik} |J^\delta_{u,s}(y)|_{kj} du \\ \leq& C + \int_t^s \sum_{i,j,k=1}^n \Big|R_{t,u} D_z F^\delta(u,z)|_{z=\theta^\delta_{u,s}(y)}\Big|_{ik} |J^\delta_{u,s}(y)|_{kj} du.\end{aligned} \tag{5.44}$$

Remember now that $D_z F^\delta(u,z)$ is upper triangular. Then for fixed $(j,k) \in [\![1,n]\!]^2$, using (5.43),

$$\begin{aligned}\Big|R_{t,u} D_z F^\delta(u,z)|_{z=\theta^\delta_{u,s}(y)}\Big|_{ik} \leq& \sum_{\ell=1}^k |R_{t,u}|_{i\ell} |DF^\delta_{\ell k}|_\infty \\ \leq& C \sum_{\ell=1}^k (\mathbb{1}_{\ell\geq i} + (t-s)^{i-l}\mathbb{1}_{\ell<i}) |D_k F^\delta_\ell|_\infty.\end{aligned} \tag{5.45}$$

It is now clearly seen that, for a fixed line index i and $\ell \geq i$, there is no time regularity, contrarily to what happened with the control of the renormalized flows. Recall that if we had chosen F^δ as in the proof of Lemma 2.14 then, for $\ell \geq 2$ (recall that we regularize at macro scale C_1 for F^δ_1) $|D_k F^\delta_\ell|_\infty \leq C(\delta_{\ell k})^{-1+\beta^k_\ell}$. It is then clear that the $\left(\delta_{lk}^{-1+\beta^k_\ell}\right)_{\ell\in[\![2,k]\!]}$ must have the same order, which precisely prevents from the choice in (5.17) which allows to consider minimal Hölder regularity exponents distinguishing the regularity with respect to the kth variable in function of the level ℓ of the chain. We are here led to consider $\beta^k_\ell = \beta^k_k = \beta^k$ (condition (5.34)), imposing the strongest integrability threshold, associated with the diagonal perturbation at level k all along the previous levels (up to the second one), which in principle lead to less singularity when the corresponding gradients are considered.

Such a phenomenon naturally appears when investigating the strong uniqueness of the SDE because of the Zvonkin approach, see e.g. [HWZ20] for the Kinetic case deriving from our framework or [CdR17] for the kinetic Brownian case. It was also the case, still for the Brownian kinetic case, in [CdR18] where the parametrix approach freezing the initial coefficients was considered. The author had to impose the same regularity for

the drift, in the degenerate variable, on the whole F. Hence, adapting the work [Mar20] to derive pointwise bound of the gradients, which could have been another approach would have led to the same constraints. Here, we have slightly more freedom since we manage to have arbitrary smoothness indexes for the non-degenerate component of the drift.

We thus derive from (5.44) and for $\bar{C}, C_1$ large enough there exists $c_0 > 0$ such that

$$\left[\sum_{k=2}^{n}\sum_{\ell=2}^{k}(\delta_{\ell k})^{-1+\beta^k} + \sum_{k=1}^{n}(\delta_{1k})^{-1+\beta_1^k}\right](s-t) \leq c_0$$

meant to be small that, under the current assumptions, there exists $C \geq 1$ s.t.

$$|J^{\delta}_{t,s}(y)| \leq C\exp(c_0),$$

and similarly, $\forall u \in [t,s]$,

$$|J^{\delta}_{u,s}(y)| \leq C\exp(c_0). \tag{5.46}$$

Rewriting:

$$J^{\delta}_{t,s}(y) = R_{t,s}\Big(I - \int_t^s R_{s,u}D_zF^{\delta}(u,z)|_{z=\theta^{\delta}_{u,s}(y)}J^{\delta}_{u,s}(y)du\Big),$$

we derive from (5.43), (5.46) that the matrix $\Big(I - \int_t^s R_{s,u}D_zF^{\delta}(u,z)|_{z=\theta^{\delta}_{u,s}(y)}J^{\delta}_{u,s}(y)du\Big)$ has diagonal dominant and this gives, from the non degeneracy of R, the statement concerning the determinant. □

Chapter 5

About the sharp constants in Sobolev and Schauder estimates for degenerate Kolmogorov operators

Abstract: We consider a possibly degenerate Kolmogorov-Ornstein-Uhlenbeck operator of the form $L = \mathrm{Tr}(BD^2) + \langle Az, D\rangle$, where A, B are $N \times N$ matrices, $N \geq 1$, which satisfy the Kalman condition which is equivalent to the hypoellipticity condition. We prove the following stability result: the Schauder and Sobolev estimates associated with the corresponding parabolic Cauchy problem remain valid, with the same constant, for the parabolic Cauchy problem associated with a second order perturbation of L, namely for $L + \mathrm{Tr}(S(t)D^2)$ where $S(t)$ is a *non-negative* $N \times N$ matrix depending continuously on $t \geq 0$. Our approach relies on the perturbative technique based on the Poisson process introduced in [KP17].

1 Introduction

Let us first consider the following parabolic Cauchy problem:

$$\begin{cases} \partial_t u(t,x,y) \;=\; \Delta_x u(t,x,y) + x \cdot \nabla_y u(t,x,y) + f(t,x,y), \\ u(0,x,y) \;=\; 0, \end{cases} \tag{1.1}$$

where (t,x,y) is in $(0,+\infty)\times\mathbb{R}^{d+d'}$ for two integers $d, d' \geq 1$. The underlying differential operator

$$L^{\mathrm{K}} \;=\; \Delta_x + x \cdot \nabla_y = \sum_{i=1}^{d} \partial^2_{x_i x_i} \;+\; \sum_{i=1}^{d}\sum_{j=1}^{d'} x_i\, \partial_{y_j}$$

is the so-called Kolmogorov operator whose fundamental solution was derived in the seminal paper [Kol34]. This particular operator was also mentioned by Hörmander as the starting point for his theory of hypoelliptic operators [Hör67].
Let us write $z = (x,y) \in \mathbb{R}^d \times \mathbb{R}^{d'} = \mathbb{R}^N$ and by ∂_{z_j} and $\partial^2_{z_i z_j}$ we denote respectively the first and the second partial derivatives with $i, j = 1, \ldots, N$.

We are interested in studying the influence of a second order perturbation on Equation (1.1). Precisely, for a time-dependent matrix $\{S(t)\colon t \geq 0\}$ in $\mathbb{R}^N \otimes \mathbb{R}^N$ such that $t \mapsto S(t)$ is continuous and $S(t)$ is *symmetric* and *non-negative* for any fixed t, we consider the *perturbed* Cauchy problem:

$$\begin{cases} \partial_t u_S(t,z) \;=\; L^{\mathrm{K}} u_S(t,z) + \sum_{i,j=1}^N S_{ij}(t)\, \partial^2_{z_i z_j} u_S(t,z) + f(t,z); \\ u_S(0,z) = 0, \end{cases} \tag{1.2}$$

In particular, we will show that Sobolev (and Schauder) estimates which hold for solutions u of the Cauchy Problem (1.1) are also true, with the same constants, for solutions u_S to (1.2). Clearly, the operator $L^{\mathrm{K},S}$ given by

$$L^{\mathrm{K},S}\, u_S(t,z) \;:=\; L^{\mathrm{K}} u_S(t,z) + \sum_{i,j=1}^N S_{ij}(t)\, \partial^2_{z_i z_j} u_S(t,z).$$

can be seen as a perturbation of L^{K} involving second order partial derivatives with continuous time-dependent coefficients.

For now, let us explain our main results for a special form of Equation (1.1) in the case of L^p-estimates (or Sobolev estimates). For a statement of our results in the whole generality, we instead refer to Section 2.
For a fixed final time $T > 0$ and a source f in $C_0^\infty((0,T) \times \mathbb{R}^N)$, it is known from the work of Bramanti *et al.* [BCM96], Theorem 3.1, that Equation (1.1) admits a unique classical bounded solution u which satisfies for p in $(1, +\infty)$ the following estimates:

$$\|\Delta_x u\|_{L^p((0,T)\times\mathbb{R}^N)} \leq C_p\, \|f\|_{L^p((0,T)\times\mathbb{R}^N)} = C_p \|\partial_t u - L^{\mathrm{K}} u\|_{L^p((0,T)\times\mathbb{R}^N)}. \tag{1.3}$$

Note that $C_p = C_p(T, d, d') > 0$. We will actually manage to prove that the unique classical bounded solution u_S to (1.2) satisfies the estimate

$$\|\Delta_x u_S\|_{L^p((0,T)\times\mathbb{R}^N)} \leq C_p\, \|f\|_{L^p((0,T)\times\mathbb{R}^N)} = C_p \|\partial_t u - L^{\mathrm{K},S} u\|_{L^p((0,T)\times\mathbb{R}^N)}, \tag{1.4}$$

with the **same** previous constant C_p as in (1.3). This result seems to be new even in dimension $N = 2$ and even if we only consider $S(t) = S$, $t \geq 0$, where S is a 2×2 *symmetric non-negative definite* matrix.

For a uniformly elliptic second order perturbation $S(t) = S$, $t \geq 0$, where S is *positive definite*, we could also have appealed to [BCLP10] to derive estimates like in (1.4). Dor related estimates in the uniformly elliptic case, see also Section 4 in Metafune *et al.* [MPRS02]. However, note that from [BCLP10] and [MPRS02] we could only deduce that the constant C_p depends on the ellipticity constant of the perturbation (this is the first eigenvalue λ_S of S if $0 < \lambda_S \leq 1$) and on the maximum eigenvalue of S (on this respect, see also [Kry02] and [Pri15a]).

The remarkable point in (1.4) is that the L^p-estimates are stable under second order perturbations, which can be possibly degenerate. Namely, the fact that $S(t)$ might be degenerate for some t in $(0,T)$, or even in some non-empty sub-intervals of $(0,T)$, does not affect the estimates in (1.4).

To prove (1.4), we combine the results of [BCM96] with a probabilistic *perturbative* approach based on the Poisson process inspired by [KP17]. There, it was established in particular that the L^p-estimates for non-degenerate parabolic heat equations with space homogeneous coefficients are valid with constants that are independent of the dimension.

Remark 1.1. Importantly, the approach of [KP17] turns out to be sufficiently robust to handle the estimates in the degenerate directions as well. We recall that the associated maximal L^p regularity was studied e.g. in [Bou02], [HMP19] or [CZ19]. Fixed p in $(1, +\infty)$, there exists $\tilde{C}_p > 0$ such that for f in $C_0^\infty((0,T) \times \mathbb{R}^N)$ the unique classical bounded solution u of (1.1) verifies

$$\|(\Delta_y)^{\frac{1}{3}} u\|_{L^p((0,T)\times\mathbb{R}^N} \leq \tilde{C}_p \, \|f\|_{L^p((0,T)\times\mathbb{R}^N)} = \tilde{C}_p \|\partial_t u - L^{\mathrm{K}} u\|_{L^p((0,T)\times\mathbb{R}^N)}, \tag{1.5}$$

where $(\Delta_y)^{\frac{1}{3}}$ denotes the fractional Laplacian with respect to the degenerate variables y in $\mathbb{R}^{d'}$. It turns out that this estimate is also stable for the previously described second order perturbation. Namely, for u_S solving (1.2),

$$\|(\Delta_y)^{\frac{1}{3}} u_S\|_{L^p((0,T)\times\mathbb{R}^d} \leq \tilde{C}_p \, \|f\|_{L^p((0,T)\times\mathbb{R}^d)} = \tilde{C}_p \|\partial_t u - L^{\mathrm{K},S} u\|_{L^p((0,T)\times\mathbb{R}^N)}, \tag{1.6}$$

where again $\tilde{C}_p$ is the same as in (1.5). □

Remark 1.2. The same type of stability results will also hold for the corresponding Schauder estimates, first established in the framework of anisotropic Hölder spaces for the solution of (1.1) by Lunardi [Lun97].

We point out that our results in Section 3 could be possibly obtained by using the general theorems of Section 4 in [KP17], too. This section in [KP17] introduces a more general probabilistic approach and provides quite unexpected regularity results. However checking in our case all the assumptions given in that section is quite involved. On the other hand, we provide self-contained proofs inspired by Sections 2 and 3 of [KP17].

It remains a challenging open problem to have a purely analytic proof of the above regularity results.

Lp estimates for degenerate Ornstein-Uhlenbeck operators. Let us now describe the more general framework we are going to consider here. Fixed $\mathbb{R}^N = \mathbb{R}^d \times \mathbb{R}^{d'}$ where d, d' are two *non-negative* integers such that $d + d' = N$ and $d \geq 1$. let us introduce the non-negative, symmetric matrix B in $\mathbb{R}^N \otimes \mathbb{R}^N$ given by

$$B = \begin{pmatrix} B_0 & 0 \\ 0 & 0 \end{pmatrix},$$

where B_0 is a symmetric, positive definite matrix in $\mathbb{R}^d \otimes \mathbb{R}^d$.

We will use, as underlying *proxy* operators, the family of degenerate Ornstein-Uhlenbeck generators of the form

$$L^{\mathrm{ou}} f(z) = \mathrm{Tr}(BD^2 f(z)) + \langle Az, Df(z) \rangle, \quad z \in \mathbb{R}^N, \tag{1.7}$$

for a matrix A in $\mathbb{R}^N \otimes \mathbb{R}^N$, where $\langle \cdot, \cdot \rangle$ denotes the usual inner product in $\mathbb{R}^N$.

Moreover, we assume the Kalman condition:

[K] There exists a non-negative integer n, such that

$$\mathrm{Rank}[B, AB, \cdots, A^{n-1}B] = N, \tag{1.8}$$

where $[B, AB, ..., A^{n-1}B]$ is the $\mathbb{R}^N \otimes \mathbb{R}^{Nn}$ matrix whose blocks are $B, AB, \cdots A^{n-1}B$.

From the non-degeneracy of B_0, the above condition amounts to say that the vectors

$$\{e_1, \ldots, e_d, Ae_1, \ldots, Ae_d, \ldots, A^{n-1}e_1, \ldots, A^{n-1}e_d\} \quad \text{generate} \quad \mathbb{R}^N, \tag{1.9}$$

where $\{e_i \colon i \in \{1, \cdots, d\}\}$ are the first d vectors of the canonical basis for $\mathbb{R}^N$.

In the linear framework, assumption **[K]** (which also often appears in control theory; see e.g. [Zab92]) is equivalent to the Hörmander condition on the commutators (c.f. [Hör67]) ensuring the hypoellipticity of the operator $\partial_t - L^{\mathrm{ou}}$. In particular, it implies the existence and the smoothness of a distributional solution for the following equation:

$$\begin{cases} \partial_t u(t,z) \;=\; L^{\mathrm{ou}} u(t,z) + f(t,z), & \text{on } (0,T) \times \mathbb{R}^N; \\ u(0,z) \;=\; 0, & \text{on } \mathbb{R}^N, \end{cases} \tag{1.10}$$

where f is a function in $C_0^\infty((0,T) \times \mathbb{R}^N)$.

Similarly to [KP17], we will prove below the existence and the uniqueness of bounded regular solutions to (1.10) assuming that the source f belongs to $B_b\left(0, T; C_0^\infty(\mathbb{R}^N)\right)$, which contains $C_0^\infty((0,T) \times \mathbb{R}^N)$, and that can be roughly described as the family of functions which are bounded measurable in time and compactly supported in space uniformly in time (see Section 1.2 for a precise definition). Equation (1.10) will be understood in an integral form.

By Theorem 3 in [BCLP10] and exploiting some explicit properties of the underlying heat kernel (see Section 2.3 below), it can be derived that for any fixed p in $(1, +\infty)$, there exists $C_p = C_p(B_0, d, d', T)$ such that

$$\|D_x^2 u\|_{L^p((0,T)\times\mathbb{R}^N)} \;\leq\; C_p \|u_t - L^{\mathrm{ou}} u\|_{L^p((0,T)\times\mathbb{R}^N)} \;=\; C_p \|f\|_{L^p((0,T)\times\mathbb{R}^N)}, \tag{1.11}$$

where for any z in $\mathbb{R}^N$, $D_x^2 u(z)$ stands for the Hessian matrix in $\mathbb{R}^d \otimes \mathbb{R}^d$ with respect to the variable x. We note, in particular, that (1.11) can be rewritten, up to a modification of C_p involving the ellipticity constant of B_0, in the following, equivalent way:

$$\begin{aligned} \|B^{1/2} D^2 u \, B^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N)} \;&=\; \|B_0^{1/2} D_x^2 u \, B_0^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N)} \\ &\leq\; C_p \|u_t - L^{\mathrm{ou}} u\|_{L^p((0,T)\times\mathbb{R}^N)} \\ &=\; C_p \|f\|_{L^p((0,T)\times\mathbb{R}^N)}, \end{aligned} \tag{1.12}$$

where $D^2 u = D_z^2 u$ represents instead the full Hessian matrix in $\mathbb{R}^N \otimes \mathbb{R}^N$ with respect to z.

Fixed a continuous mapping $t \mapsto S(t)$ such that $S(t)$ is a *symmetric* and *non-negative* matrix in $\mathbb{R}^N \otimes \mathbb{R}^N$, we consider again the following perturbation of L^{ou} :

$$\begin{aligned} L_t^{\mathrm{ou},S} f(z) \ &:= \ \mathrm{Tr}(BD^2 f(z)) + \mathrm{Tr}(S(t)D^2 f(z)) + \langle Az, Df(z)\rangle \\ &= \ L^{\mathrm{ou}} f(z) + \mathrm{Tr}(S(t)D^2 f(z)), \end{aligned} \tag{1.13}$$

where z is in $\mathbb{R}^N$. For the solution u_s of the related Cauchy problem

$$\begin{cases} \partial_t u_S(t,z) \ = \ L_t^{\mathrm{ou},S} u_S(t,z) + f(t,z), & \text{on } (0,T)\times\mathbb{R}^N; \\ u_S(0,z) \ = \ 0, & \text{on } \mathbb{R}^N, \end{cases} \tag{1.14}$$

we will prove the following main theorem:

Theorem 1.1. *Let us consider* (1.14) *with* $f \in B_b\Big(0,T; C_0^\infty(\mathbb{R}^N)\Big)$. *Then, there exists a unique solution* u_S *of Cauchy Problem* (1.14) *which verifies, with the same constant* C_p *as in* (1.12),

$$\|B^{1/2} D^2 u_S \, B^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N)} \ \leq \ C_p \|f\|_{L^p((0,T)\times\mathbb{R}^N)}. \tag{1.15}$$

For time-homogeneous non-negative matrices S, the corresponding elliptic estimate could also be derived from (1.15) following the approach of Corollary 3.5 in [KP17].

Independently from the constant preservation, we also emphasize that the L^p estimates in (1.15) for the perturbed operator seem, to the best of our knowledge, to be new and have some interest by their own. We can also refer to the recent work by Fornaro *et al.* [FMPS21] for a full description of the spectrum of degenerate OU operators in L^p spaces.

Let us eventually mention that such stability results could turn out to be useful to investigate the well posedness of some related stochastic differential equations through the corresponding martingale problem.

We could actually derive more general estimates, possibly involving the degenerate directions as well, dependingly on the structure of A. Some results in that direction are gathered in Section 4. Anyhow, to illustrate our approach we now briefly present the various steps to derive (1.15).

1.1 Strategy of the proof for Estimates (1.15)

Fixed a classical bounded solution u to Cauchy Problem (1.10), let us introduce ,for brevity, $v(t,z) := u(t, e^{-tA}z)$. This well-known (cf. [DPL95]) transformation precisely allows to get rid of the drift term in the PDE satisfied by v. Indeed, we have that $u(t,z) = v(t, e^{tA}z)$ and since u solves (1.10), it holds for any (t,z) in $(0,T)\times\mathbb{R}^N$, that:

$$\begin{aligned} f(t,z) \ &= \ \partial_t u(t,z) - L^{\mathrm{ou}} u(t,z) \\ &= \ v_t(t,e^{tA}z) + \langle Dv(t,e^{tA}z), Ae^{tA}z\rangle - \mathrm{Tr}\Big(e^{tA}Be^{tA^*}D^2 v(t,e^{tA}z)\Big) \\ &\qquad - \langle Dv(t,e^{tA}z), Ae^{tA}z\rangle \\ &= \ v_t(t,e^{tA}z) - \mathrm{Tr}\Big(e^{tA}Be^{tA^*}D^2 v(t,e^{tA}z)\Big). \end{aligned} \tag{1.16}$$

Denoting $\tilde{f}(t,z) := f(t, e^{-tA}z)$, It now follows that v satisfies the PDE:

$$\begin{cases} \partial_t v(t,z) \;=\; \mathrm{Tr}\Big(e^{tA}Be^{tA^*}D^2v(t,z)\Big) + \tilde{f}(t,z) & \text{on } (0,T)\times\mathbb{R}^N; \\ v(0,z) = 0 & \text{on } \mathbb{R}^N. \end{cases} \tag{1.17}$$

In terms of the function v, the estimates in (1.12) rewrites as:

$$\|B^{1/2}e^{-tA^*}D^2v(t,e^{tA}\cdot)\,e^{tA}B^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N)} \le C_p\|\tilde{f}(t,e^{tA}\cdot)\|_{L^p((0,T)\times\mathbb{R}^N)}, \tag{1.18}$$

where we used the notation $\|B^{1/2}e^{-tA^*}D^2v(t,e^{tA}\cdot)\,e^{tA}B^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N)}$ to stress the dependence on t instead of the more precise formulation

$$\|B^{1/2}e^{-\cdot A^*}D^2v(\cdot,e^{-\cdot A}\cdot)\,e^{-\cdot A}B^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N)}.$$

By changing variable in the integrals, Control (1.18) is equivalent to

$$\|B^{1/2}e^{tA^*}D^2v(t,\cdot)\,e^{tA}B^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N,m)} \le C_p\|\tilde{f}\|_{L^p((0,T)\times\mathbb{R}^N,m)}, \tag{1.19}$$

where $L^p((0,T)\times\mathbb{R}^N,m)$ denotes the L^p norms w.r.t. the measure

$$m(dt,dx) \;:=\; \det(e^{-At})dtdx.$$

Considering now the following, more general equation

$$\begin{cases} \partial_t w(t,z) + \mathrm{Tr}\Big(e^{tA}Be^{tA^*}D^2w(t,z)\Big) + \mathrm{Tr}\Big(e^{tA}S(t)e^{tA^*}D^2w(t,z)\Big) \;=\; \tilde{f}(t,z); \\ w(0,z) = 0, \end{cases} \tag{1.20}$$

we can establish the well-posedness of the Cauchy problem (1.20), exploting, for instance, probabilistic arguments and the underlying Gaussian process.

The crucial step consists in adapting some arguments from [KP17] to derive that the same L^p-estimates in (1.19) still hold for w, independently from the non-negative definite, symmetric matrix S. Precisely,

$$\|B^{1/2}e^{tA^*}D^2w(t,\cdot)\,e^{tA}B^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N,m)} \le C_p\|\tilde{f}(t,\cdot)\|_{L^p((0,T)\times\mathbb{R}^N,m)}, \tag{1.21}$$

with the *same* constant C_p appearing in (1.19).

The last step then consists in coming back to the Ornstein-Uhlenbeck operators framework. Namely, we introduce $\tilde{u}(t,z) := w(t,e^{tA}z)$ which solves, by definition, the following equation:

$$\begin{cases} \partial_t\tilde{u}(t,z) + L_t^{\mathrm{ou},S}\tilde{u}(t,z) = f(t,z),\ (t,z)\in(0,T)\times\mathbb{R}^N, \\ \tilde{u}(0,z) = 0,\ z\in\mathbb{R}^N. \end{cases}$$

Noticing that $D^2w(t,\cdot) = D^2[\tilde{u}(t,e^{-tA}\cdot)] = e^{-tA^*}D^2\tilde{u}(t,e^{-tA}\cdot)e^{-tA}$ we thus get from (1.21) that the following estimates hold:

$$\|B^{1/2}D^2\tilde{u}\,B^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N)} \;\le\; C_p\|f\|_{L^p((0,T)\times\mathbb{R}^N)}. \tag{1.22}$$

Through the previous steps we have then constructed a solution $\tilde{u}$ of Cauchy Problem (1.14) which indeed satisfies the estimates in (1.15) with the same C_p, associated with the *unperturbed* or *proxy* operator. The maximum principle will eventually provide uniqueness for the solution $\tilde{u}$.

Remark 1.3. i) We point out that we could also consider more general time-dependent Ornstein-Uhlenbeck operators like:

$$M = \mathrm{Tr}(B(t)D^2 f(z)) + \langle Az, Df(z)\rangle.$$

Arguing as before starting from L^p-estimates (or Schauder estimates) for M we can derive the same L^p-estimates (or Schauder estimates) for a perturbation of M like (1.13).

ii) Moreover, we could extend the L^p-estimates (or the Schauder estimates) related to L^{ou} to more general operators like

$$L_t^{\mathrm{ou},S} f(z) + \langle b(t), Df(z)\rangle$$

where $b : \mathbb{R}_+ \to \mathbb{R}^N$ is continuous and even add a further possibly degenerate non-local perturbation (cf. Section 7 of [KP17]). For the sake of simplicity in the sequel we will consider $b(t) = 0$ and we will not deal with non-local perturbations of $L_t^{\mathrm{ou},S}$.

Organization of Paper. The article is organized as follows. At the end of the current section, we first give some useful notations. In Section 2 we will then focus on driftless second order Cauchy problems associated with a non-negative, possibly degenerate, diffusion matrix. In particular, we will establish through the probabilistic perturbation approach of [KP17] that if some L^p-estimates hold for a particular diffusion matrix so does it, with the same associated constant, for a non-negative perturbation of the diffusion matrix as explained before (see Section 3). Finally by the argument of Section 1.1 we will obtain (1.22).

1.2 Definition of solution and useful notations

Let us consider the following Cauchy problem:

$$\begin{cases} \partial_t v(t,z) = \mathrm{tr}\,(Q(t)D^2 v(t,z)) + \langle b(t,z), Dv(t,z)\rangle + f(t,z), & \text{on } (0,T)\times\mathbb{R}^N; \\ v(0,z) = 0, & \text{on } \mathbb{R}^N; \end{cases} \tag{1.23}$$

where $Q\colon [0,T] \to \mathbb{R}^N \otimes \mathbb{R}^N$ is a *continuous time-dependent, symmetric non-negative* matrix and $b : [0,T]\times\mathbb{R}^N \to \mathbb{R}^N$ is a *continuous* function such that $|b(t,z)| \le K_T(1+|z|)$, $(t,z) \in [0,T]\times\mathbb{R}^N$, for some constant $K_T > 0$.

The function f belongs to $B_b\left(0,T; C_0^\infty(\mathbb{R}^N)\right)$, the space of all Borel bounded functions $\phi\colon [0,T]\times\mathbb{R}^N \to \mathbb{R}$ such that $\phi(t,\cdot)$ is smooth and compactly supported for any t in $[0,T]$, for any k in $\mathbb{N}$ the $C^k(\mathbb{R}^N)$-norms of $\phi(t,\cdot)$ are bounded in time and the supports of the functions $\phi(t,\cdot)$ are contained in the same ball. Moreover, we require that, for any $z \in \mathbb{R}^N$, the mapping:

$$t \mapsto \phi(t,z) \tag{1.24}$$

is a *piece-wise continuous* function on $[0,T]$, i.e. it is continuous except for a finite number of points.

Remark 1.4. Note that to perform the technique used in [KP17] and based on the Poisson process we need to consider equations like (1.23) with a source f which is possibly discontinuous in time (cf. the proof in Section 2 of [KP17] and Section 3.2 below).

We interpret Cauchy Problem (1.23) in an *integral* form:

$$v(t,z) = \int_0^t \Big(f(s,z) + \mathrm{Tr}(Q(s)D^2v(s,z)) + \langle b(s,z), Dv(s,z)\rangle \Big) ds. \tag{1.25}$$

In particular, we say that a continuous and bounded function $v : [0,T] \times \mathbb{R}^N \to \mathbb{R}$ is a solution to Equation (1.23) if $v(t,\cdot)$ is in $C^2(\mathbb{R}^N)$, for any $t \in [0,T]$, and (1.25) holds as well.

We finally note that, for any $z \in \mathbb{R}^N$, the function $t \mapsto v(t,z)$ is a C^1-piece-wise function on $[0,T]$.

By Theorem 4.1 in [KP10] we know that if a solution v exists then it is unique and the following maximum principle holds:

$$\sup_{(t,z)\in[0,T]\times\mathbb{R}^N} |v(t,z)| \le T \sup_{(t,z)\in[0,T]\times\mathbb{R}^N} |f(t,z)|. \tag{1.26}$$

2 Estimates for driftess second order operators and related perturbation

Throughout this section, we consider the following Cauchy problem:

$$\begin{cases} \partial_t v(t,z) \;=\; \mathrm{Tr}\left(Q(t)D^2v(t,z)\right) + f(t,z) & \text{on } (0,T)\times\mathbb{R}^N; \\ v(0,z) \;=\; 0 & \text{on } \mathbb{R}^N, \end{cases} \tag{2.27}$$

which can be seen as a special case of (1.23) when $b = 0$. Moreover, we assume that Q is not identically zero.

2.1 Well-posedness

Proposition 2.1 (Well-posedness in integral form for the driftless Cauchy problem)**.** *Let f be in $B_b\left(0,T;C_0^\infty(\mathbb{R}^N)\right)$. Then, there exists a unique solution v to Cauchy problem* (2.27) *in an integral sense, i.e. it solves for $(t,z) \in [0,T]\times\mathbb{R}^N$:*

$$v(t,z) = \int_0^t \Big(f(s,z) + \mathrm{Tr}(Q(s)D^2v(s,z) \Big) ds. \tag{2.28}$$

We will denote in short $v = PDE(Q,f)$.

Proof. By the maximum principle (cf. Equation (1.26)) uniqueness holds for Cauchy Problem (2.27). We can then focus on proving the existence of a solution.

Let us introduce now

$$v(t,z) \;:=\; \int_0^t \mathbb{E}[f(s,z+I_{s,t})]\, ds$$

with the following notation: $I_{s,u} := \sqrt{2}\int_s^u Q(v)^{1/2} dW_v$, where W is an N-dimensional Brownian motion on some probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ and $Q(v)^{1/2}$ stands for a square root of $Q(v)$, i.e. $Q(v) = Q(v)^{1/2}(Q(v)^{1/2})^*$.

Applying the Itô formula in space to $f(s, z + I_{s,u})_{u\in[s,t]}$, we get that

$$\mathbb{E}f(s, z + I_{s,t}) \;=\; f(s,z) + \mathbb{E}\left[\int_s^t \mathrm{Tr}(Q(u)D^2 f(s, z + I_{s,u}))\, du\right].$$

Hence,

$$v(t,z) \;=\; \int_0^t \left(f(s,z) + \mathbb{E}\left[\int_s^t \mathrm{Tr}(Q(u)D^2 f(s, z + I_{s,u}))\, du\right]\right) ds,$$

from which it readily follows that

$$\begin{aligned}
\partial_t v(t,z) \;&=\; f(t,z) + \int_0^t \mathbb{E}\left[\mathrm{Tr}(Q(t)D^2 f(s, z + I_{s,t}))\right] ds \\
&=\; f(t,z) + \mathrm{Tr}\Big(Q(t)D^2 \int_0^t \mathbb{E}\left[f(s, z + I_{s,t})\right] ds\Big) \\
&=\; f(t,z) + \mathrm{Tr}\Big(Q(t)D^2 v(t,z)\Big),
\end{aligned}$$

for almost every $t \in [0,T]$ and any $z \in \mathbb{R}^N$. □

2.2 Relation to the Ornstein-Uhlenbeck dynamics

If now in particular, $Q(t)$ has the particular form $Q(t) = e^{tA}Be^{tA^*}$ (cf. Equation (1.17)), we introduce

$$u(t,z) := v(t, e^{tA}z),$$

where v is the solution to (2.28) (see Proposition 2.1). Since we can differentiate with respect to t the function $u(\cdot, z)$ for a.e. $t \in [0,T]$, we can perform computations similar to (1.16) and get that $u(t,z)$ solves in integral form:

$$\begin{cases} \partial_t u(t,z) \;=\; L^{\mathrm{ou}}\, u(t,z) + \bar{f}(t,z), & \text{on } (0,T)\times\mathbb{R}^N; \\ u(0,z) \;=\; 0, & \text{on } \mathbb{R}^N; \end{cases} \tag{2.29}$$

with L^{ou} as in (1.7), $\bar{f}(t,z) = f(t, e^{tA}z)$. Precisely, for all $(t,z) \in [0,T]\times\mathbb{R}^N$,

$$u(t,z) = \int_0^t \Big(\bar{f}(s,z) + L^{\mathrm{ou}} v(s,z)\Big) ds. \tag{2.30}$$

Let us also point out that the well-posedness of (2.29) could also have been obtained directly from Gaussian type calculations, similar to those in the proof of Proposition 2.1, introducing $u^{\mathrm{ou}}(t,z) := \int_0^t \mathbb{E}[\bar{f}(s, e^{(t-s)A}z + I^{\mathrm{ou}}_{s,t})] ds$ where $I^{\mathrm{ou}}_{s,u} := \sqrt{2}\int_s^u e^{(u-v)A} B dW_v$.

2.3 About the Lp estimates in (1.11)

The aim of this section is to fully justify the estimates in (1.11). This is a consequence of the previous probabilistic representation and Theorem 3 in [BCLP10]. For u solving (1.10) it holds that for all $(t,z) \in [0,T] \times \mathbb{R}^N$,

$$u(t,z) \;=\; \int_0^t \mathbb{E}\left[f(s, e^{A(t-s)}z + I^{\mathrm{ou}}_{s,t})\right] ds \;=\; \int_0^t \int_{\mathbb{R}^N} f(s,z') p^{\mathrm{ou}}(t-s,z,z')\, dz' ds, \quad (2.31)$$

where for $v > 0$, $p^{\mathrm{ou}}(v,z,\cdot)$ stands for the density at time v of the stochastic process

$$X^{\mathrm{ou}}_u \;:=\; e^{Au}z + \sqrt{2}\int_0^u e^{A(u-w)} B dW_w \;=\; z + \int_0^u A X^{\mathrm{ou}}_w dw + BW_u,\; u \geq 0.$$

We recall from [LP94] that assumption **[K]** is equivalent to the fact that there exist $n \in \mathbb{N}$ and positive integers $\{d_i \colon i \in 1, \cdots, n\}$ such that $d = d_1$, $\sum_{i=1}^n d_i = N$ and for all $i \in \{2, \cdots, n\}$ the matrixes

$$\mathscr{A}^i \;:=\; (A_{j,\ell})_{(j,\ell) \in \{\sum_{m=1}^{i-1} d_m + 1, \cdots, \sum_{m=1}^{i} d_i\} \times \{\sum_{m=1}^{i-2} d_m + 1, \cdots, \sum_{m=1}^{i-1} d_m\}},$$

with the natural notation $\sum_{m=1}^0 = 0$, have rank d_i. The matrix A writes:

$$A \;=\; \begin{pmatrix} * & * & \cdots & \cdots & * \\ \mathscr{A}^2 & * & \ddots & \ddots & \vdots \\ 0_{d_3,d} & \mathscr{A}^3 & * & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ 0_{d_n,d} & \cdots & 0_{d_n,d_{n-2}} & \mathscr{A}^n & * \end{pmatrix}.$$

Following the proof of Lemma 5.5 in [DM10], where the case $d_i = d$ for any i in $[\![1,n]\!]$ is addressed, it can be derived that there exists $C \geq 1$ s.t. for all $(v,z,z') \in (0,T] \times (\mathbb{R}^N)^2$,

$$|D^2_x p^{\mathrm{ou}}(v,z,z')| \;\leq\; \frac{C}{v^{\sum_{i=1}^n d_i(i-\frac{1}{2})+1}} \exp\left(-C^{-1} v |\overline{\mathbb{M}}_v^{-1}(e^{Av}z - z')|^2\right), \quad (2.32)$$

where

$$\overline{\mathbb{M}}_v \;:=\; \mathrm{diag}(vI_{d\times d}, v^2 I_{d_2 \times d_2}, \ldots, v^n I_{d_n \times d_n}), \quad v \geq 0,$$

reflects the various scales of the system. For a given function $f \in B_b\left(0,T; C_0^\infty(\mathbb{R}^N)\right)$, it is then clear from (2.31) and (2.32) that for all $(t,z) \in (0,T] \times \mathbb{R}^N$:

$$D^2_x u(t,z) \;=\; \text{p.v.} \int_0^t \int_{\mathbb{R}^N} f(s,z') D^2_x p^{\mathrm{ou}}(t-s,z,z') dz' ds. \quad (2.33)$$

It indeed suffices to observe that:

$$\begin{aligned}
&\left|\text{p.v.} \int_0^t \int_{\mathbb{R}^N} f(s,z') D^2_x p^{\mathrm{ou}}(t-s,z,z')\, dz' ds\right| \\
&\quad = \left|\text{p.v.} \int_0^t \int_{\mathbb{R}^N} [f(s,z') - f(s, e^{A(t-s)}x)] D^2_x p^{\mathrm{ou}}(t-s,z,z')\, dz' ds\right| \\
&\quad \underset{(2.32)}{\leq} \sup_{s\in[0,T]} \|Df(s,\cdot)\|_\infty \\
&\qquad \times \int_0^t \int_{\mathbb{R}^N} \frac{C}{(t-s)^{\sum_{i=1}^n d_i(i-\frac{1}{2})+\frac{1}{2}}} \exp\left(-C^{-1}(t-s)|\mathbb{T}^{-1}_{t-s}(e^{A(t-s)}z - z')|^2\right) dz' ds \\
&\quad \leq C \sup_{s\in[0,T]} \|Df(s,\cdot)\|_\infty T^{\frac{1}{2}}.
\end{aligned}$$

The estimates in (1.11) now follows from the proof of Theorem 3 in [BCLP10], starting from (2.33) instead of (16) therein. The strategy is clear. It is necessary to introduce a cut-off function which separates the points (s, z') which do not induce any singularity in (2.33) for the derivatives of the density, namely such that $t-s \geq c_0$ or $|e^{A(t-s)}z-z'| \geq c_0$, for some fixed constant $c_0 > 0$, from those who are close to the singularity. For the non-singular part of the integral the expected L^p-control readily follows from (2.32) and the Young inequality (see also Proposition 5 in [BCLP10]), whereas the derivation of the bound for the singular part requires some involved harmonic analysis, see Section 4 on the same reference. We can also refer to Theorem 11 and its proof in [Pri15b] for similar issues linked with the corresponding L^p-estimates for degenerate Ornstein-Uhlenbeck operators in an elliptic setting.

2.4 The main result for Equation (2.27)

Let us fix p in $(1, +\infty)$ and assume that there exist $R(t) \in \mathbb{R}^N \otimes \mathbb{R}^N$ depending continuously on $t \geq 0$ and a constant $C_p > 0$, such that for any f in $B_b\left(0, T; C_0^\infty(\mathbb{R}^N)\right)$, the unique solution $v = PDE(Q, f)$ to equation (2.27) satisfies

$$\|R(t)^* D^2 v\, R(t)\,\|_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})} \leq C_p \|f\|_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})}, \tag{2.34}$$

for some absolutely continuous measure $\mathfrak{m}$ w.r.t. the Lebesgue measure on $[0, T] \times \mathbb{R}^N$ such that $\mathfrak{m}(dt, dx) = g(t)dtdx$ for some borel bounded function g (note that in (1.19) we have $R(t) = e^{tA}B^{1/2}$, $\mathfrak{m}(dt, dx) = g(t)dtdx = \det(e^{-At})dtdx$).

We would like to exhibit that a control like (2.34) also holds for the solution w to the following Cauchy Problem:

$$\begin{cases} \partial_t w(t,z) \;=\; \mathrm{tr}\,(Q(t)D^2 w(t,z)) + \mathrm{tr}\,(Q'(t)D^2 w(t,z)) + f(t,z), & \text{on } (0,T)\times\mathbb{R}^N; \\ w(0,z) \;=\; 0, & \text{on } \mathbb{R}^N, \end{cases} \tag{2.35}$$

Namely we have to prove the following result.

Theorem 2.2. *Let us consider equations* (2.27) *and* (2.35) *where $Q(t)$, $Q'(t)$ are two continuous in time, non-negative matrices in $\mathbb{R}^N \otimes \mathbb{R}^N$ and $f \in B_b\left(0, T; C_0^\infty(\mathbb{R}^N)\right)$. Assume that estimate* (2.34) *holds as explained above.*

Then the solution w to (2.35) *verifies*

$$\|R(t)^* D^2 w\, R(t)\,\|_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})} \leq C_p \|f\|_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})}, \tag{2.36}$$

$p \in (1, \infty)$ with the same constant C_p as in (2.34).

From Theorem 2.2 using the argument of Section 1.1 we can easily derive Theorem 1.1.

3 A perturbation argument for proving Theorem 2.2

We aim here at applying the probabilistic perturbative approach considered in [KP17]. The key idea in that work was, for a well-posed PDE which enjoys some quantitative

given estimates, to introduce a *small* random perturbation of the source through a suitable Poisson type process and to investigate the properties of the associated PDE.

When taking the expectation in the associated integral formulation, the contributions associated with the jumps then yield, for an appropriate intensity of the underlying Poisson process, a finite difference operator. For the PDE satisfied by the expectation, involving the finite difference operator, the initial estimates are preserved. Compactness arguments then allow to derive that, the initial estimates still hold at the limit with the finite difference operator replaced by the corresponding differential one of order two.

Below, we start recalling the basic properties on Poisson type processes, and their corresponding stochastic integrals, that are needed for our approach.

3.1 Poisson stochastic integrals

We briefly recall here the very definition of the stochastic integral driven by a Poisson process. We start reminding the construction of such processes.
Given a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ to be fixed from this point further, we start considering a sequence of independent real-valued random variables $\{\tau_m\}_{m\in\mathbb{N}}$ on Ω whose distribution is exponential of parameter $\lambda > 0$:

$$\mathbb{P}(\tau_m > r) \,=\, e^{-r\lambda}, \quad r \geq 0.$$

We can then define the partial sums sequence $\{\sigma_n\}_{n\in\mathbb{N}}$ as follows:

$$\sigma_0 \,=\, 0; \quad \sigma_m = \sum_{i=1}^{m} \tau_i, \quad n = 1, 2, \ldots$$

For any fixed $t \geq 0$, π_t now denotes the number of consecutive sums of τ_i which lie on $[0, t]$, i.e.

$$\pi_t = \sum_{n=0}^{\infty} \mathbb{1}_{\sigma_m \leq t}, \tag{3.37}$$

where $\mathbb{1}_{\sigma_m \leq t}$ represents the indicator function of the event $\{\sigma_m \leq t\}$. The process $\{\pi_t\}_{t\geq 0}$ we have just constructed is usually known in the literature as a *Poisson process* with intensity λ (see, for instance, [Pro05]).

Now, let $c : [0, T] \to \mathbb{R}^N$ be a continuous function. We can define the Poisson stochastic integral $\int_0^t c(s)d\pi_s = \int_{(0,t]} c(s)d\pi_s$, $t \in (0, T]$, as

$$b_t \,:=\, \int_0^t c(s)d\pi_s \,=\, \sum_{\sigma_k \leq t,\ k\geq 1} c(\sigma_k) = \sum_{0<s\leq t} c(s)(\pi_s - \pi_{s-}) \tag{3.38}$$

$b_0 = 0$ (as usual $\pi_{s-}(\omega)$ denotes the left limit at s, for any ω, $\mathbb{P}$-a.s.). We now recall the following formula for the expectation of the stochastic integral:

$$\mathbb{E}[\int_0^t c(s)d\pi_s] = \lambda \int_0^t c(s)ds. \tag{3.39}$$

(cf. Lemma 2.1 in [KP17] for a direct proof; see also Theorem 16 in [Pro05] and Theorem 5.3 in [KP17] for a more general formula involving stochastic integrals with predictable processes against the Poisson process). We also recall the following more general result.

Lemma 3.1. *Let $\{\pi_t\}_{t\geq 0}$ be a Poisson Process of intensity λ on a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Let us consider a stochastic process $(\xi_t)_{t\in[0,T]}$ with values in $\mathbb{R}$ which has cadlag paths ($\mathbb{P}$-a.s.) and is $\mathcal{F}_t$-adapted where $\mathcal{F}_t$ is the σ-algebra generated by the random variables π_s, $0 \leq s \leq t$. Suppose that $\sup_{\omega\in\Omega,\, s\in[0,T]} |\xi_s(\omega)| < \infty$. Then*

$$\mathbb{E}\int_0^t \xi_{s-}\, d\pi_s = \lambda \int_0^t \mathbb{E}\xi_s ds. \tag{3.40}$$

3.2 Proof of Theorem 2.2

According to the notations in Proposition 2.1, let $v = PDE(Q, f)$ and $w = PDE(Q + Q', f)$ be the unique solutions of equations (2.27) and (2.35), respectively.
The proof of Theorem 2.2 will be obtained adapting the method developed in [KP17] (see in particular Section 3, therein). Let e_1 be the first unit vector in $\mathbb{R}^N$. We define

$$X_t = \int_0^t \sqrt{Q'(r)}\, e_1 d\pi_r$$

where $\sqrt{Q'(t)}$ is the unique $N \times N$ symmetric non-negative square root of $Q'(t)$ and $\{\pi_t\}_{t\geq 0}$ is a Poisson Process of intensity λ. The parameter λ will be chosen appropriately later on.

Recall that the solution v to (2.27) is given by

$$v(t, z) = \int_0^t ds \int_{\mathbb{R}^N} [f(s, z + z')\mu_{s,t}(dz') \tag{3.41}$$

where $\mu_{s,t}$ is the Gaussian law of the stochastic integral $I_{s,t} := \sqrt{2}\int_s^t Q(v)^{1/2} dW_v$ (see the proof of Proposition 2.1).

Let us fix $\epsilon > 0$. We notice that the shifted source $f_\epsilon(t, z) := f(t, z - \epsilon X_t)$ (which also depends on ω; we have omitted to write such dependence on ω) is again in $B_b\left(0, T; C_0^\infty(\mathbb{R}^N)\right)$. This is the reason why we considered such a function space for the source. It precisely allows to take into account the time discontinuities coming from the jumps of the Poisson process.

For any fixed ω in Ω, Proposition 2.1 readily gives that there exists a unique solution $v_\epsilon = \mathrm{PDE}(Q, f(t, z - \epsilon X_t))$, depending also on ϵ and ω as parameters, such that

$$\sup_{(t,z)\in[0,T]\times\mathbb{R}^N} |v_\epsilon(t, z)| \leq T \sup_{(t,z)\in[0,T]\times\mathbb{R}^N} |f(t, z)|. \tag{3.42}$$

Moreover, thanks to the invariance for translations of the L^p-norms, it follows from (2.34) that

$$\|R(t)^* D^2 v_\epsilon\, R(t)\,\|_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})} \leq C_p \|f\|_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})}. \tag{3.43}$$

By Equation (3.41), we know that v_ϵ is given by

$$v_\epsilon(t, z) = \int_0^t \int_{\mathbb{R}^N} [f(s, z - \epsilon X_s + z')\, \mu_{s,t}(dz') ds.$$

For each $z \in \mathbb{R}^N$, the stochastic process $(v_\epsilon(t,z))_{t\in[0,T]}$ has continuous paths ($\mathbb{P}$-a.s.) and is $\mathcal{F}_t$-adapted where $\mathcal{F}_t$ is the σ-algebra generated by the random variables π_s, $0 \le s \le t$.

For fixed $z \in \mathbb{R}^N$, let us introduce the process $(v_\epsilon(t, z+\epsilon X_t))_{t\in[0,T]}$ which is given by

$$v_\epsilon(t, z+\epsilon X_t) = \int_0^t \int_{\mathbb{R}^N} [f(s, z+\epsilon X_t - \epsilon X_s + z')\, \mu_{s,t}(dz')ds.$$

It is not difficut to check that it is is $\mathcal{F}_t$-adapted and it has bounded and càdlàg paths.

Applying (2.28) on each interval $[\sigma_m, \sigma_{m+1}\wedge t), m \in \{0, \cdots, \pi_t]$ on which X_s is constant, one then derives that:

$$v_\epsilon(t, z+\epsilon X_t) \;=\; \int_0^t \Big(\mathrm{tr}(Q(s)D_z^2 v_\epsilon(s, z+\epsilon X_s)) + f(s,z)\Big)\, ds + \int_0^t g_\epsilon(s,z)\, d\pi_s, \quad (3.44)$$

where $g_\epsilon(s,z) = v_\epsilon(s, z+\epsilon\sqrt{Q'(s)}\, e_1 + \epsilon X_{s-}) - v_\epsilon(s, z+\epsilon X_{s-})$ is precisely the contribution associated with the jump times. It is clear that $g_\epsilon(s,z) \neq 0$ if and only if π_s has a jump at time s. We then have by Lemma 3.1:

$$\mathbb{E}\int_0^t g_\epsilon(s,z)\, d\pi_s \;=\; \lambda \int_0^t \Big(\bar{v}_\epsilon(s, z+\epsilon\sqrt{Q'(s)}\, e_1) - v_\epsilon(s,z)\Big)\, ds, \quad (3.45)$$

where $\bar{v}_\epsilon(s,z) = \mathbb{E}[v_\epsilon(s, z+\epsilon X_s)]$. Let us denote

$$l(t) := \sqrt{Q'(t)}\, e_1.$$

Taking the expectation on both sides of equation (3.44), we find out that $\bar{v}_\epsilon$ is an integral solution of the following PDE:

$$\partial_t \bar{v}_\epsilon(t,z) \;=\; \mathrm{tr}(Q(t)D_z^2 \bar{v}_\epsilon(t,z)) + \lambda\,(\bar{v}_\epsilon(t, z+\epsilon l(t)) - \bar{v}_\epsilon(t,z)) + f(t,z), \quad (3.46)$$

with zero initial condition. Moreover, we obtain from the Jensen inequality and the Fubini theorem that

$$\begin{aligned}
\|R(t)^* D^2\bar{v}_\epsilon\, R(t)\|^p_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})} &= \int_{(0,T)\times\mathbb{R}^N} |R(t)^* D^2\bar{v}_\epsilon(t,z)R(t)|^p \mathfrak{m}(dt,dz) \\
&= \int_0^T \int_{\mathbb{R}^N} |\mathbb{E}[R(t)^* D^2 v_\epsilon(t, z+\epsilon X_t)R(t)]\,|^p dz g(t)dt \\
&\le \int_0^T \int_{\mathbb{R}^N} \mathbb{E}[|R(t)^* D^2 v_\epsilon(t, z+\epsilon X_t)R(t)\,|^p] dz g(t)dt \\
&= \mathbb{E}\int_0^T \int_{\mathbb{R}^N} |R(t)^* D^2 v_\epsilon(t, z+\epsilon X_t)R(t)\,|^p dz g(t)dt \\
&= \mathbb{E}\int_0^T \int_{\mathbb{R}^N} |R(t)^* D^2 v_\epsilon(t, \bar{z})R(t)\,|^p d\bar{z} g(t)dt \\
&\le C_p^p \|f\|^p_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})},
\end{aligned}$$

using (3.43) for the last inequality (L^p estimate for the PDE with random source). Choosing $\lambda = \epsilon^{-2}$ we have from (3.46)

$$\partial_t \bar{v}_\epsilon(t,z) \;=\; \mathrm{tr}(Q(t)D_z^2 \bar{v}_\epsilon(t,z)) + \;\epsilon^{-2}\,(\bar{v}(t, z+\epsilon l(t)) - \bar{v}_\epsilon(t,z)) + f(t,z), \quad (3.47)$$

with zero initial condition and moreover

$$\|R(t)^* D^2 \bar{v}_\epsilon R(t)\|^p_{L^p((0,T)\times\mathbb{R}^N)} \le C_p^p \|f\|^p_{L^p((0,T)\times\mathbb{R}^N)}. \tag{3.48}$$

Now, the idea is to apply again the same reasoning above to the equation (3.47) with respect to $\bar{v}_\epsilon$ and $f(t, z + \epsilon X_t)$ again with $\lambda = \epsilon^{-2}$. We obtain first a solution p_ϵ to (3.47) corresponding to $f(t, z + \epsilon X_t)$ and then derive that

$$w_\epsilon(t, z) = \mathbb{E}[p_\epsilon(t, z - \epsilon X_t)]$$

is the unique bounded continuous (integral) solution w_ϵ of the following problem:

$$\begin{aligned} \partial_t w_\epsilon(t, z) \;=\; & \operatorname{tr}(Q(t) D^2 w_\epsilon(t, z)) \\ & + \epsilon^{-2} \left[w_\epsilon(t, z + \epsilon l(t)) - 2 w_\epsilon(t, z) + w_\epsilon(t, z - \epsilon l(t)) \right] + f(t, z), \end{aligned} \tag{3.49}$$

with initial condition $w_\epsilon(0, z) = 0$. Uniqueness follows by the maximum principle as in the proof of Lemma 2.2 in [KP17].

Moreover, the previous estimates still hold with w_ϵ instead of v_ϵ, i.e.,

$$\sup_{(t,z)\in[0,T]\times\mathbb{R}^N} |w_\epsilon(t, z)| \le T \sup_{(t,z)\in[0,T]\times\mathbb{R}^N} |f(t, z)|; \tag{3.50}$$

$$\|R(t)^* D^2 w_\epsilon \, R(t)\|_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})} \le C_p \|f\|_{L^p((0,T)\times\mathbb{R}^N,\mathfrak{m})}. \tag{3.51}$$

We would like now to let ϵ goes to zero, possibly passing to a subsequence $\epsilon_n \to 0$, and prove that the associated limit w solves

$$\begin{cases} \partial_t w(t, z) \;=\; \operatorname{tr}(Q(t) D^2 w(t, z)) + \langle D^2 w(t, z) \sqrt{Q'(t)} e_1, \sqrt{Q'(t)} e_1 \rangle + f(t, z), \\ w(0, z) \;=\; 0 \end{cases} \tag{3.52}$$

and estimates (3.50) and (3.51) hold with w_ϵ replaced by w.

To do so we will proceed by compactness. Namely, we are going to prove that the family of solutions w_ϵ solving (3.49), indexed by the parameter ϵ, is equi-Lipschitz on any compact subset of $[0, T] \times \mathbb{R}^N$ and the same holds for any derivative in space of w_ϵ. Indeed, one can apply the finite difference operators with respect to z at any order in (3.49). We recall that for a *smooth* function $\phi\colon \mathbb{R}^N \to \mathbb{R}$, the first finite difference $\delta_{h,i}\phi$, $i \in \{1, \cdots, N\}$ of step h in the direction e_i (i^{th} basis vector) is given by

$$\delta_{h,i}\phi(z) \;=\; \frac{\phi(z + h e_i) - \phi(z)}{h}, \quad z \in \mathbb{R}^N.$$

For a given multi-index $\gamma \in \mathbb{N}^N$, the γ-th order finite difference operator $\delta_{h,\gamma}$, is then defined, for any $h > 0$, through composition. Namely,

$$\delta_{h,\gamma}\phi(z) \;=\; \delta_{h,1}^{\gamma_1} \delta_{h,2}^{\gamma_2} \dots \delta_{h,N}^{\gamma_N} \phi(z),$$

where $\delta_{h,i}^{\gamma_i}$ denotes the γ_i-th times composition of $\delta_{h,i}$ with itself.
Using (3.50) and the fact that any spatial derivative of f belongs to $B_b\left(0, T; C_0^\infty(\mathbb{R}^N)\right)$,

we deduce first that any finite difference of any order of w_ϵ is bounded. Consequently, w_ϵ is infinitely differentiable in space with bounded derivatives on $[0,T]\times\mathbb{R}^N$. Equation (3.49), to be understood in its integral form similarly to (2.28), then gives that those derivatives are themselves Lipschitz continuous in time. This precisely gives the equi-Lipschitz on any compact subset of $[0,T]\times\mathbb{R}^N$ of the family w_ϵ and any spatial derivative.
We can now apply the Arzelà-Ascoli theorem to w_ϵ showing the existence of a subsequence $\{w_{\epsilon_n}\}_{n\in\mathbb{N}}$ which converges to a function $w\colon [0,T]\times\mathbb{R}^N\to\mathbb{R}$, uniformly on any compact set. Similarly, $\partial_t w_{\epsilon_n}$ and any derivative in space of w_{ϵ_n} tend to the respective derivatives of w, uniformly on the compact sets.
Passing to the limit for $n\to\infty$ along the sequence $(\epsilon_n)_n$ in Equation (3.49) (written in the integral form), we can then conclude that w solves (3.52).

Moreover, estimates (3.50) and (3.51) holds with w_ϵ replaced by w.

Iterating the previous argument in N steps we finally prove that the unique solution w to

$$\begin{cases}\partial_t w(t,z) \;=\; \mathrm{tr}(Q(t)D^2w(t,z)) + \sum_{k=1}^N \langle D^2w(t,z)\sqrt{Q'(t)}e_k, \sqrt{Q'(t)}e_k\rangle + f(t,z),\\ w(0,z) \;=\; 0\end{cases} \tag{3.53}$$

verifies estimates (3.50) and (3.51) with w_ϵ replaced by w. The proof is complete. □

4 Additional stability results

In this section we extend the previous approach to derive the stability with respect to a second order perturbation of the general Ornstein-Uhlenbeck operator in (1.7) for L^p estimates involving the degenerate components of the operator and some associated Schauder estimates, under the Kalman condition **[K]**.

4.1 Anisotropic Sobolev spaces and maximal Lp regularity.

With the notations of Section 2.3 we write $z\in\mathbb{R}^N$ as $z=(x,y_2,\cdots,y_n)$ with $x\in\mathbb{R}^d$, $y_i\in\mathbb{R}^{d_i},\ i\in\{2,\cdots,n\}$, recalling also that $\sum_{i=2}^n d_i = d'$.

Given β in $(0,1)$ and i in $[\![2,n]\!]$, we want to introduce the β-fractional Laplacian $\Delta_{y_i}^\beta$ along the *component* y_i. To do so, we follow [HMP19] by considering the orthogonal projection $p_i\colon\mathbb{R}^N\to\mathbb{R}^{d_i}$ such that $p_i(z)=p_i\big((x,y_2,\dots,y_n)\big)=y_i$ and denoting its adjoint by $E_i\colon\mathbb{R}^{d_i}\to\mathbb{R}^N$. We can now define the β-fractional Laplacian $\Delta_{y_i}^\beta$ as:

$$\Delta_{y_i}^\beta\phi(z) \;:=\; \text{p.v.}\int_{\mathbb{R}^{d_i}}[\phi(z+E_iw)-\phi(z)]\,\frac{dw}{|w|^{d_i+2\beta}},\quad z\in\mathbb{R}^N,$$

for any sufficiently regular function $\phi\colon\mathbb{R}^N\to\mathbb{R}$.
Fixed p in $(1,+\infty)$, we recall that we have denoted by $L^p((0,T)\times\mathbb{R}^N)$ the standard L^p-space with respect to the Lebesgue measure.

We can now define the appropriate anisotropic Sobolev space to state our results. For notational simplicity, let us denote

$$\alpha_i := \frac{1}{2i-1}. \tag{4.54}$$

Set now $\alpha := (\alpha_1, \cdots, \alpha_k) \in \mathbb{R}^k$. The *homogeneous* space $\dot{W}^{2,p}_\alpha([0,T]\times\mathbb{R}^N)$ is composed by all the functions $\varphi\colon [0,T]\times\mathbb{R}^N \to \mathbb{R}$ in $L^p([0,T]\times\mathbb{R}^N)$ such that $(t,z) \in [0,T]\times\mathbb{R}^N \mapsto \Delta_x\varphi(t,z) \in L^p([0,T]\times\mathbb{R}^N)$, where $\Delta_x\varphi$ is intended in distributional sense, and for any i in $[\![2,n]\!]$, $\Delta^{\alpha_i}_{y_i}\varphi(t,z)$ is well defined for almost every (t,z) and

$$\Delta^{\alpha_i}_{y_i}\varphi(t,z) := \Delta^{\alpha_i}_{y_i}\varphi(t,\cdot)(z) \text{ belongs to } L^p([0,T]\times\mathbb{R}^N).$$

It is endowed with the natural *semi*-norm $\|\varphi\|_{\dot{W}^{2,p}_\alpha}$ where

$$\|\varphi\|^p_{\dot{W}^{2,p}_\alpha} = \|\Delta_x\varphi\|^p_{L^p} + \sum_{i=2}^n \|\Delta^{\alpha_i}_{y_i}\varphi\|^p_{L^p}. \tag{4.55}$$

The thresholds in (4.54) might seem awkward at first sight. They actually correspond to the indexes needed to get stability of the harmonic functions associated with the principal part of (1.7), that is considering A_0 consisting in the subdiagonal part of A only, along an associated dilation operator. Namely, setting

$$L^{\mathrm{ou}}_0 f(z) = \mathrm{Tr}(BD^2 f(z)) + \langle A_0 z, Df(z)\rangle, \quad z \in \mathbb{R}^N, \tag{4.56}$$

so that A_0, B satisfy **[K]**, if it holds that

$$(\partial_t - L^{\mathrm{ou}}_0)u(t,z) = 0$$

then for all $\lambda > 0$, we have that

$$(\partial_t - L^{\mathrm{ou}}_0)u\Big(\delta_\lambda(t,z)\Big) = 0$$

where the dilation operator

$$\delta_\lambda(t,z) = (\lambda^{1/2}t, \lambda x, \lambda^{1/3}y_1, \cdots, y_k^{1/(2n-1)})$$

precisely exhibits the exponents in (4.54) for the degenerate components.

In [HMP19], see also [CZ19] and [Men18] where time inhomogeneous coefficients are considered as well, it has been proven that when the strictly upper diagonal elements of A in (1.7) are equal to zero then the following Sobolev estimates hold:

$$\|u\|_{\dot{W}^{2,p}_\alpha} \le C_p\|f\|_{L^p}, \tag{4.57}$$

where again u is the unique bounded solution to the corresponding Cauchy problem (1.10). In particular we get also the maximal smoothing effects w.r.t. the degenerate directions. The specific structure assumed on A is actually due to the fact that for such matrices there is an underlying homogeneous space structure which makes easier to establish maximal regularity estimates (see e.g. [CW71] in this general setting).

For a general A, having non zero strictly upper diagonal entries, such that A, B satisfy [**K**], we believe that the approach in [BCLP10] could extend to show that for (4.57) still hold in this general setting, but such estimates have not been, up to our best knowledge, proven yet.

Setting now, as in Section 1.1, $u(t,z) = v(t, e^{tA}z)$ and since u solves (1.10) we have that v in turn solves (1.17). From the previous computations and setting

$$B_I := \begin{pmatrix} I_{d,d} & 0_{d,d'} \\ 0_{d',d} & 0_{d',d'} \end{pmatrix},$$

we then derive that

$$\begin{aligned} \|D_x^2 u\|_{L^p((0,T)\times\mathbb{R}^N)} &= \|B_I^{1/2} e^{tA^*} D^2 v(t, e^{tA}\cdot) e^{tA} B_I^{1/2}\|_{L^p((0,T)\times\mathbb{R}^N)} \\ &\leq C_p \|\tilde{f}(t, e^{tA}\cdot)\|_{L^p((0,T)\times\mathbb{R}^N)}. \end{aligned}$$

On the other hand, for all $i \in \{2, \cdots, n\}$ and with α_i as in (4.54),

$$\begin{aligned} \|\Delta_{y_i}^{\alpha_i} u\|^p_{L^p((0,T)\times\mathbb{R}^N)} &= \int_0^T \int_{\mathbb{R}^N} \left| \text{p.v.} \int_{\mathbb{R}^{d_i}} [u(t, z + E_i w) - u(t,z)] \frac{dw}{|w|^{d_i + 2\alpha_i}} \right|^p dz dt \\ &= \int_0^T \int_{\mathbb{R}^N} \left| \text{p.v.} \int_{\mathbb{R}^{d_i}} [v(t, e^{tA}(z + E_i w)) - v(t, e^{tA} z)] \frac{dw}{|w|^{d_i + 2\alpha_i}} \right|^p dz dt \\ &=: \|\Delta^{\alpha_i, i, A} v\|^p_{L^p((0,T)\times\mathbb{R}^N)}. \end{aligned}$$

Hence, setting now

$$\|\Delta^{\alpha_0, 0, A} v\|^p_{L^p((0,T)\times\mathbb{R}^N)} := \|\mathrm{Tr}\Big(B_I^{1/2} e^{-tA^*} D^2 v(t, e^{tA}\cdot)\, e^{tA} B_I^{1/2}\Big)\|^p_{L^p((0,T)\times\mathbb{R}^N)},$$

we get from Definition (4.55) that the Estimates (4.57) rewrite in term of v as:

$$\|v\|^p_{\dot{W}^{2,p,A}_\alpha} := \sum_{i=1}^n \|\Delta^{\alpha_i, i, A} v\|^p_{L^p((0,T)\times\mathbb{R}^N)} \leq \tilde{C}_p \|f\|^p_{L^p((0,T)\times\mathbb{R}^N)} \tag{4.58}$$

with $\tilde{C}_p = C_p^p$. We now want to prove that for w solving (1.20), namely

$$\begin{cases} \partial_t w(t,z) + \mathrm{Tr}\Big(e^{tA} B e^{tA^*} D^2 w(t,z)\Big) + \mathrm{Tr}\Big(e^{tA} S(t) e^{tA^*} D^2 w(t,z)\Big) = \tilde{f}(t,z), \\ w(0,z) = 0, \end{cases}$$

it also holds that

$$\|w\|^p_{\dot{W}^{2,p,A}_\alpha} := \sum_{i=1}^n \|\Delta^{\alpha_i, i, A} w\|^p_{L^p((0,T)\times\mathbb{R}^N)} \leq \tilde{C}_p \|f\|^p_{L^p((0,T)\times\mathbb{R}^N)}, \tag{4.59}$$

with the same constants $\tilde{C}_p$ as in (4.58). This can be done through the previous perturbative approach of Section 3.2 employed to prove Theorem 2.2, which actually gives the expected control for the the second order derivatives contribution of the semi-norm $\|\cdot\|_{\dot{W}^{2,p,A}_\alpha}$.

For the other contributions, we would get, using the notations of Section 3.2 with $Q'(s) = e^{sA}S(s)e^{sA^*}$ and with m which is the Lebesgue measure on $[0,T]\times\mathbb{R}^N$, that

$$\begin{aligned}
\|\bar{v}_\epsilon\|^p_{\dot{W}^{2,p,A}_\alpha} &= \sum_{i=1}^n \|\Delta^{\alpha_i,i,A}\bar{v}_\epsilon\|^p_{L^p((0,T)\times\mathbb{R}^N)} \\
&= \sum_{i=1}^n \int_0^T\int_{\mathbb{R}^N} |\Delta^{\alpha_i,i,A}\bar{v}_\epsilon(t,z)|^p\,dzdt \\
&= \sum_{i=1}^n \int_0^T\int_{\mathbb{R}^N} \left|\mathbb{E}[\Delta^{\alpha_i,i,A}v_\epsilon(t,z+\epsilon X_t)]\right|^p\,dzdt \\
&\le \sum_{i=1}^n \int_0^T\int_{\mathbb{R}^N} \mathbb{E}\left[|\Delta^{\alpha_i,i,A}v_\epsilon(t,z+\epsilon X_t)|^p\right]\,dzdt \\
&= \sum_{i=1}^n \mathbb{E}\left[\int_0^T\int_{\mathbb{R}^N} |\Delta^{\alpha_i,i,A}v_\epsilon(t,z+\epsilon X_t)|^p\,dzdt\right] \\
&= \sum_{i=1}^n \mathbb{E}\left[\int_0^T\int_{\mathbb{R}^N} |\Delta^{\alpha_i,i,A}v_\epsilon(t,\bar{z})|^p\,d\bar{z}dt\right] \\
&\le \tilde{C}_p\|f\|^p_{L^p((0,T)\times\mathbb{R}^N)},
\end{aligned}$$

using for the last inequality that v_ϵ also satisfies (4.58) (similarly to what had been established in (3.43)). The same previous compactness argument then yields (4.59). Setting eventually $\tilde{u}(t,z) := w(t,e^{tA}z)$, which is the unique integral solution, smooth in space, of

$$\begin{cases}\partial_t\tilde{u}(t,z) + L^{\mathrm{ou},S}_t\tilde{u}(t,z) = f(t,z),\ (t,z)\in(0,T)\times\mathbb{R}^N,\\ \tilde{u}(0,z) = 0,\ z\in\mathbb{R}^N,\end{cases}$$

where $L^{\mathrm{ou},S}_t$ introduced in (1.13) is the Ornstein-Uhlenbeck operator perturbed at second order, we derive that

$$\|\tilde{u}\|_{\dot{W}^{2,p}_\alpha} \le C_p\|f\|_{L^p}, \tag{4.60}$$

with C_p as in (4.57). We have thus extended the results of Theorem 1.1 for the anisotropic Sobolev norm in (4.55). The estimate (4.57) is stable for a continuous, non-negative second order perturbation of the underlying degenerate Ornstein-Uhlenbeck operator.

4.2 Anisotropic Schauder estimates

Following Krylov [Kry96], for some fixed ℓ in $\mathbb{N}_0 := \mathbb{N}\cup\{0\}$ and β in $(0,1]$, we introduce for a function $\phi\colon\mathbb{R}^N\to\mathbb{R}$ the Zygmund-Hölder semi-norm as

$$[\phi]_{C^{\ell+\beta}} := \begin{cases}\sup_{|\vartheta|=\ell}\sup_{x\neq y}\frac{|D^\vartheta\phi(x)-D^\vartheta\phi(y)|}{|x-y|^\beta}, & \text{if } \beta\neq 1;\\ \sup_{|\vartheta|=\ell}\sup_{x\neq y}\frac{\left|D^\vartheta\phi(x)+D^\vartheta\phi(y)-2D^\vartheta\phi(\frac{x+y}{2})\right|}{|x-y|}, & \text{if } \beta = 1\end{cases}$$

(we are using usual multi-indices ϑ for the partial derivatives). Consequently, the Zygmund-Hölder space $C^{\ell+\beta}_b(\mathbb{R}^N)$ is the family of bounded functions $\phi\colon\mathbb{R}^N\to\mathbb{R}$ such

that ϕ and its derivatives up to order ℓ are continuous and the norm

$$\|\phi\|_{C_b^{\ell+\beta}} := \sum_{i=0}^{\ell} \sup_{|\vartheta|=i} \|D^{\vartheta}\phi\|_\infty + [\phi]_{C^{\ell+\beta}} \text{ is finite.}$$

We can define now the anisotropic Zygmund-Hölder spaces associated with the current setting and which again reflect the various scales already introduced in (4.54). Fixed $\gamma \in (0,3)$, the space $C^{\gamma}_{b,d}(\mathbb{R}^N)$ is the family of functions $\phi\colon \mathbb{R}^N \to \mathbb{R}$ such that for any i in $[\![1,n]\!]$ and any z_0 in $\mathbb{R}^N$, the function

$$w \in \mathbb{R}^{d_i} \to \phi(z_0 + E_i(w)) \in \mathbb{R} \text{ belongs to } C_b^{\gamma/(2i-1)}\left(\mathbb{R}^{d_i}\right),$$

with a norm bounded by a constant independent from z_0. In the above expression, we recall that the $\{E_i\colon i \in \{2,\cdots,n\}\}$ have been defined in the previous paragraph, $d_1 = d$ and E_1 is the embedding matrix from $\mathbb{R}^d$ into $\mathbb{R}^N$. It is endowed with the norm

$$\|\phi\|_{C^{\gamma}_{b,d}} := \sup_{z_0\in\mathbb{R}^N} \|\phi\Big(z_0 + E_0(\cdot)\Big)\|_{C_b^{\gamma}(\mathbb{R}^d)} + \sum_{i=1}^{k} \sup_{z_0\in\mathbb{R}^N} [\phi\Big(z_0 + E_i(\cdot)\Big)]_{C^{\gamma/(1+2i)}(\mathbb{R}^{d_i})}. \quad (4.61)$$

We denote by $C^{\gamma}_{b,d}$ this function space because the regularity exponents reflect again the multi-scale features of the system and the norm could equivalently be defined through the corresponding spatial parabolic distance d defined as follows. For all $z = (x,y)$, $z' = (x',y')$ in $\mathbb{R}^N = \mathbb{R}^d \times \mathbb{R}^{d'}$:

$$d(z,z') := |x - x'| + \sum_{i=2}^{n} |y_i - y_i'|^{\frac{1}{2i-1}},$$

where the exponents are again those who appeared in (4.54).
Let as before $f \in B_b\Big(0,T;C_0^{\infty}(\mathbb{R}^N)\Big)$. We recall that under **[K]**, Lunardi showed in [Lun97] that the following anisotropic Schauder estimates hold for the solution u of Cauchy Problem (1.10):

$$\|u\|_{L^\infty((0,T),C^{2+\beta}_{b,d})} \le C_\beta \|f\|_{L^\infty((0,T),C^{\beta}_{b,d})}, \quad (4.62)$$

for some constant C_β independent from f, i.e.,

$$\sup_{0\le t\le T} \|u(t,\cdot)\|_{C^{2+\beta}_{b,d}} \le C_\beta \sup_{0\le t\le T} \|f(t,\cdot)\|_{C^{\beta}_{b,d}},$$

We again set as in the previous paragraph $u(t,z) = v(t,e^{tA}z)$ and since u solves (1.10) we have that v in turn solves (1.17). Write:

$$\begin{aligned}\|u\|_{L^\infty((0,T),C^{2+\beta}_{b,d})} &= \|v(t,e^{tA}\cdot)\|_{L^\infty((0,T),C^{2+\beta}_{b,d})} =: \|v\|_{L^\infty((0,T),C^{2+\beta}_{b,d,A})} \\ &\le C_\beta \|f\|_{L^\infty((0,T),C^{\beta}_{b,d})} \le C_\beta \|\tilde{f}(t,e^{tA}\cdot)\|_{L^\infty((0,T),C^{\beta}_{b,d})} \\ &= C_\beta \|\tilde{f}\|_{L^\infty((0,T),C^{\beta}_{b,d,A})}.\end{aligned} \quad (4.63)$$

We again want to prove as in Section 1.1 that for w solving (1.20),

$$\|w\|_{L^\infty((0,T),C^{2+\beta}_{b,d,A})} \leq C_\beta \|\tilde{f}\|_{L^\infty((0,T),C^{\beta}_{b,d,A})} \tag{4.64}$$

with the same constant C_β as in (4.63). We proceed one more time through the previous perturbative approach of Section 3.2. With the notations employed therein, we deduce that there exists a unique solution $v_\epsilon = \mathrm{PDE}(Q, \tilde{f}(t, z - \epsilon X_t))$, depending also on ϵ and ω as parameters such that

$$\sup_{(t,z)\in[0,T]\times\mathbb{R}^N} |v_\epsilon(t,z)| \;\leq\; T \sup_{(t,z)\in[0,T]\times\mathbb{R}^N} |\tilde{f}(t,z)|. \tag{4.65}$$

From the translation invariance of the Hölder-norms, it is not difficult to prove that, for any ω, $\mathbb{P}$-a.s.,

$$\|\tilde{f}\|_{L^\infty((0,T),C^{\beta}_{b,d,A})} = \|\tilde{f}(\cdot,\cdot - \epsilon X_\cdot)\|_{L^\infty((0,T),C^{\beta}_{b,d,A})}. \tag{4.66}$$

Thus it also holds from (4.63)

$$\|v_\epsilon\|_{L^\infty((0,T),C^{2+\beta}_{b,d,A})} \leq C_\beta \|\tilde{f}\|_{L^\infty((0,T),C^{\beta}_{b,d,A})}. \tag{4.67}$$

Recalling now that $\bar{v}_\epsilon(s,z) = \mathbb{E}[v_\epsilon(s, z + \epsilon X_s)]$, is an integral solution of

$$\partial_t \bar{v}_\epsilon(t,z) \;=\; \mathrm{tr}(Q(t) D^2_z \bar{v}_\epsilon(t,z)) + \lambda\left(\bar{v}_\epsilon(t, z + \epsilon l(t)) - \bar{v}_\epsilon(t,z)\right) + \tilde{f}(t,z),$$

with zero initial condition, we write for any i in $\{2, \cdots, n\}$, w, w' in $\mathbb{R}^{d_i}$ e (t, z_0) in $[0,T]\times\mathbb{R}^N$, that

$$\begin{aligned}
&|\bar{v}_\epsilon(t, e^{At}(z_0 + E_i(w)) - \bar{v}_\epsilon(t, e^{At}(z_0 + E_i(w'))| \\
&\quad\leq \mathbb{E}\left[|v_\epsilon(t, e^{At}(z_0 + E_i(w)) + \epsilon e^{At} e^{-At} X_t) - v_\epsilon(t, e^{At}(z_0 + E_i(w')) + \epsilon e^{At} e^{-At} X_t)|\right] \\
&\quad\leq \mathbb{E}\left[[v_\epsilon(t, e^{At}(z_0 + E_i(\cdot)))]_{C^{\frac{2+\beta}{2i-1}}}\right] |w - w'|^{\frac{2+\beta}{2i-1}}.
\end{aligned}$$

Hence,

$$\left[\bar{v}_\epsilon(t, e^{-At}(z_0 + E_i(\cdot)))\right]_{C^{\frac{2+\beta}{2i-1}}} \;\leq\; \mathbb{E}\left[[v_\epsilon(t, e^{-At}(z_0 + E_i(\cdot)))]_{C^{\frac{2+\beta}{2i-1}}}\right].$$

We would get similarly, for $w, w' \in \mathbb{R}^d$,

$$\left[D^2_x \bar{v}_\epsilon(t, e^{At}(z_0 + E_0(\cdot)))\right]_{C^\beta} \;\leq\; \mathbb{E}\left[[D^2_x v_\epsilon(t, e^{At}(z_0 + E_0(\cdot)))]_{C^\beta}\right],$$

and for all $k \in \{0,1,2\}$,

$$\left|D^k_x \bar{v}_\epsilon(t, e^{At}(z_0 + E_0(\cdot)))\right|_\infty \;\leq\; \mathbb{E}\left[|D^k_x v_\epsilon(t, e^{At}(z_0 + E_0(\cdot)))|_\infty\right].$$

Summing all those contributions, we thus derive from (4.61), (4.63) that:

$$\|\bar{v}_\epsilon\|_{L^\infty((0,T),C^{2+\beta}_{b,d,A})} \;\leq\; \sup_{0\leq t\leq T} \mathbb{E}\left[\|v_\epsilon(t,\cdot)\|_{C^{2+\beta}_{b,d,A})}\right] \;\leq\; C_\beta \|\tilde{f}\|_{L^\infty((0,T),C^{\beta}_{b,d,A})}, \tag{4.68}$$

using (4.67) for the last inequality. The other key property we used above is the triangle inequality, namely we put the absolute value inside the expectation. Iterating the procedure, one would derive from the same previous compactness argument that (4.64) indeed holds.

Going eventually backwards setting $\tilde{u}(t,z) := w(t, e^{tA}z)$, which is the unique solution to (1.14) in the function class considered through the document, we derive as in the previous paragraph that

$$\|\tilde{u}\|_{L^\infty((0,T),C^{2+\beta}_{b,d})} \leq C_\beta \|f\|_{L^\infty((0,T),C^{\beta}_{b,d})}, \tag{4.69}$$

where C_β is the same constant as in (4.62). Equation (4.69) provides the extension of Theorem 1.1 for the anisotropic Schauder estimates.

Let us mention that for the perturbative analysis to work, few properties were actually needed on the underlying norm. Namely, we used the translation invariance and some kind of commutation between the norm (or a function of the norm in the L^p case) and the expectation. Hence, this approach could possibly be applied to a much wider class of estimates in other function spaces (like e.g. Besov spaces). This will concern further research.

Bibliography

[ABM20] S. Athreya, O. Butkovsky, and L. Mytnik. Strong existence and uniqueness for stable stochastic differential equations with distributional drift. *Ann. Probab.*, 48(1):178–210, 2020.

[AKQ14] T. Alberts, K. Khanin, and J. Quastel. The intermediate disorder regime for directed polymers in dimension 1+1. *Ann. Probab.*, 42(3):1212–1256, 2014.

[Ale12] R. Alexandre. Fractional order kinetic equations and hypoellipticity. *Anal. Appl. (Singap.)*, 10(3):237–247, 2012.

[Amb04] L. Ambrosio. Transport equation and Cauchy problem for BV vector fields. *Invent. Math.*, 158(2):227–260, 2004.

[AP77] S. Anulova and G. Pragarauskas. Weak Markov solutions of stochastic equations. *Litovsk. Mat. Sb.*, 17(2):5–26, 219, 1977.

[App09] D. Applebaum. *Lévy processes and stochastic calculus*, volume 116 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, second edition, 2009.

[App19] D. Applebaum. *Semigroups of linear operators*, volume 93 of *London Mathematical Society Student Texts*. Cambridge University Press, Cambridge, 2019.

[Aro59] D. G. Aronson. The fundamental solution of a linear parabolic equation containing a small parameter. *Illinois J. Math.*, 3:580–619, 1959.

[Aro67] D. G. Aronson. Bounds for the fundamental solution of a parabolic equation. *Bull. Amer. Math. Soc.*, 73:890–896, 1967.

[Bas09] R. F. Bass. Regularity results for stable-like operators. *J. Funct. Anal.*, 257(8):2693–2722, 2009.

[Bas11] R. F. Bass. *Stochastic Processes*, volume 33 of *Cambridge Series in Statistical and Probabilistic Mathematics*. Cambridge University Press, 2011.

[BB81] R. Bafico and P. Baldi. Small random perturbations of Peano phenomena. *Stochastics*, 6(3-4):279–292, 1981.

[BB99] K. Bogdan and T. Byczkowski. Potential theory for the α-stable Schrödinger operator on bounded Lipschitz domains. *Studia Math.*, 133(1):53–92, 1999.

[BBM01] B. Baeumer, D. A. Benson, and M. M. Meerschaert. Operator Lévy motion and multiscaling anomalous diffusion. *Phys. Rev. E*, 63:021112, 2001.

[BC01] R. F. Bass and Z.-Q. Chen. Stochastic differential equations for Dirichlet processes. *Probab. Theory Related Fields*, 121(3):422–446, 2001.

[BC03] R. F. Bass and Z.-Q. Chen. Brownian motion with singular drift. *Ann. Probab.*, 31(2):791–817, 2003.

[BC06] R. F. Bass and Z.-Q. Chen. Systems of equations driven by stable processes. *Probab. Theory Related Fields*, 134(2):175–214, 2006.

[BCLP10] M. Bramanti, G. Cupini, E. Lanconelli, and E. Priola. Global L^p estimates for degenerate Ornstein-Uhlenbeck operators. *Math. Z.*, 266(4):789–816, 2010.

[BCM96] M. Bramanti, M. C. Cerutti, and M. Manfredini. L^p estimates for some ultraparabolic operators with discontinuous coefficients. *J. Math. Anal. Appl.*, 200(2):332–354, 1996.

[BGJ87] K. Bichteler, J.-B. Gravereaux, and J. Jacod. *Malliavin calculus for processes with jumps*, volume 2 of *Stochastics Monographs*. Gordon and Breach Science Publishers, New York, 1987.

[BJ07] K. Bogdan and T. Jakubowski. Estimates of heat kernel of fractional Laplacian perturbed by gradient operators. *Comm. Math. Phys.*, 271(1):179–198, 2007.

[BK15] J. Bae and M. Kassmann. Schauder estimates in generalized Hölder spaces. *arXiv preprint: 1505.05498*, 2015.

[BKH10] V. Bally and A. Kohatsu-Higa. Lower bounds for densities of Asian type stochastic differential equations. *J. Funct. Anal.*, 258(9):3134–3164, 2010.

[BKRS15] V. I. Bogachev, N. V. Krylov, M. Röckner, and S. V. Shaposhnikov. *Fokker-Planck-Kolmogorov equations*, volume 207 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015.

[BMPT09] A. Bensoussan, L. Mertz, O. Pironneau, and J. Turi. An ultra weak finite element method as an alternative to a Monte Carlo method for an elasto-plastic problem with noise. *SIAM J. Numer. Anal.*, 47(5):3374–3396, 2009.

[BMR09] T. Byczkowski, J. Małecki, and M. Ryznar. Bessel potentials, hitting distributions and Green functions. *Trans. Amer. Math. Soc.*, 361(9):4871–4900, 2009.

[BNS01] O. E. Barndorff-Nielsen and N. Shephard. Non-Gaussian Ornstein-Uhlenbeck-based models and some of their uses in financial economics. *J. R. Stat. Soc. Ser. B Stat. Methodol.*, 63(2):167–241, 2001.

[Bou02] F. Bouchut. Hypoelliptic regularity in kinetic equations. *J. Math. Pures Appl. (9)*, 81(11):1135–1159, 2002.

[BPV01] E. Barucci, S. Polidoro, and V. Vespri. Some results on partial differential equations and Asian options. *Mathematical Models and Methods in Applied Sciences*, 11(03):475–497, 2001.

[Bro86] T. Brox. A one-dimensional diffusion process in a Wiener medium. *Ann. Probab.*, 14(4):1206–1218, 1986.

[Bro01] P. J. Brockwell. Lévy-driven CARMA processes. nonlinear non-gaussian models and related filtering methods. *Ann. Inst. Statist. Math.*, 53(1):113–124, 2001.

[BT08] A. Bensoussan and J. Turi. Degenerate Dirichlet problems related to the invariant measure of elasto-plastic oscillators. *Appl. Math. Optim.*, 58(1):1–27, 2008.

[Cas09] T. Cass. Smooth densities for solutions to stochastic differential equations with jumps. *Stochastic Process. Appl.*, 119(5):1416–1435, 2009.

[CC07] S. Cerrai and P. Clément. Schauder estimates for a degenerate second order elliptic operator on a cube. *J. Differential Equations*, 242(2):287–321, 2007.

[CC18] G. Cannizzaro and K. Chouk. Multidimensional SDEs with singular drift and universal construction of the polymer measure with white noise potential. *Ann. Probab.*, 46(3):1710–1763, 2018.

[CDL08] G. Crippa and C. De Lellis. Estimates and regularity results for the DiPerna-Lions flow. *J. Reine Angew. Math.*, 616:15–46, 2008.

[CdR17] P.-É. Chaudru de Raynal. Strong existence and uniqueness for degenerate SDE with Hölder drift. *Ann. Inst. Henri Poincaré Probab. Stat.*, 53(1):259–286, 2017.

[CdR18] P.-É. Chaudru de Raynal. Weak regularization by stochastic drift: result and counter example. *Discrete Contin. Dyn. Syst.*, 38(3):1269–1291, 2018.

[CdRHM18a] P.-É Chaudru de Raynal, I. Honoré, and S. Menozzi. Sharp Schauder estimates for some degenerate Kolmogorov equations. *To appear in Ann. Sc. N. Sup. Pisa*, 2018.

[CdRHM18b] P.-É Chaudru de Raynal, I. Honoré, and S. Menozzi. Strong regularization by brownian noise propagating through a weak hörmander structure. *arXiv preprint: 1810.12225*, 2018.

[CdRM20a] P.-E. Chaudru de Raynal and S. Menozzi. On Multidimensional stable-driven Stochastic Differential Equations with Besov drift. *arXiv preprint: 1907.12263*, 2020.

[CdRM20b] P.-É Chaudru de Raynal and S. Menozzi. Regularization effects of a noise propagating through a chain of differential equations: an almost sharp result. *Transactions of AMS*, 2020.

[CdRMP20a] P.-É. Chaudru de Raynal, S. Menozzi, and E. Priola. Schauder estimates for drifted fractional operators in the supercritical case. *J. Funct. Anal.*, 278(8):108425, 57, 2020.

[CdRMP20b] P.-É. Chaudru de Raynal, S. Menozzi, and E. Priola. Weak well-posedness of multidimensional stable driven SDEs in the critical case. *Stoch. Dyn.*, 20(6):2040004, 20, 2020.

[CKK08] Z.-Q. Chen, P. Kim, and T. Kumagai. Weighted Poincaré inequality and heat kernel estimates for finite range jump processes. *Math. Ann.*, 342(4):833–883, 2008.

[CMS90] R. Carmona, W. C. Masters, and B. Simon. Relativistic Schrödinger operators: asymptotic behavior of the eigenfunctions. *J. Funct. Anal.*, 91(1):117–142, 1990.

[CPKM05] J. H. Cushman, M. Park, N. Kleinfelter, and M. Moroni. Super-diffusion via Lévy lagrangian velocity processes. *Geophysical Research Letters*, 32(19), 2005.

[CPP10] M. E. Caballero, J. Pardo, and J. L. Pérez. On the Lamperti stable processes. *Probab. Math. Statist.*, 30:1–28, 03 2010.

[CW71] R. Coifman and G. Weiss. *Analyse harmonique non-commutative sur certains espaces homogènes.* Lecture Notes in Mathematics, Vol. 242. Springer-Verlag, Berlin-New York, 1971. Étude de certaines intégrales singulières.

[CZ18] Z.-Q. Chen and X. Zhang. L^p-maximal hypoelliptic regularity of nonlocal kinetic Fokker-Planck operators. *J. Math. Pures Appl. (9)*, 116:52–87, 2018.

[CZ19] Z.-Q. Chen and X. Zhang. Propagation of regularity in L^p-spaces for Kolmogorov-type hypoelliptic operators. *J. Evol. Equ.*, 19(4):1041–1069, 2019.

[dCN10] D. del Castillo-Negrete. Non-diffusive, non-local transport in fluids and plasmas. *Nonlinear Processes in Geophysics*, 17(6):795–807, 2010.

[DD16] F. Delarue and R. Diel. Rough paths and 1d SDE with a time dependent distributional drift: application to polymers. *Probab. Theory Related Fields*, 165(1-2):1–63, 2016.

[DF14] F. Delarue and F. Flandoli. The transition point in the zero noise limit for a 1D Peano example. *Discrete Contin. Dyn. Syst.*, 34(10):4071–4083, 2014.

[DFP05] M. Di Francesco and A. Pascucci. On a class of degenerate parabolic equations of Kolmogorov type. *AMRX Appl. Math. Res. Express*, 3:77–116, 2005.

[DFP06] M. Di Francesco and S. Polidoro. Schauder estimates, Harnack inequality and Gaussian lower bound for Kolmogorov-type operators in non-divergence form. *Adv. Differential Equations*, 11(11):1261–1320, 2006.

[DK13] H. Dong and D. Kim. Schauder estimates for a class of non-local elliptic equations. *Discrete Contin. Dyn. Syst.*, 33(6):2319–2347, 2013.

[DL89] R. J. DiPerna and P.-L. Lions. Ordinary differential equations, transport theory and Sobolev spaces. *Inventiones Mathematicae*, 98(3):511–547, 1989.

[DM10] F. Delarue and S. Menozzi. Density estimates for a random noise propagating through a chain of differential equations. *J. Funct. Anal.*, 259(6):1577–1630, 2010.

[DPL95] G. Da Prato and A. Lunardi. On the Ornstein-Uhlenbeck operator in spaces of continuous functions. *J. Funct. Anal.*, 131(1):94–114, 1995.

[Dyn65] E. B. Dynkin. *Markov Processes*, volume 1 of *Grundlehren der mathematischen Wissenschaften*. Springer-Verlag Berlin Heidelberg, 1965.

[EK86] S. Ethier and T. Kurtz. *Markov processes. Characterization and convergence*. Wiley Series in Probability and Mathematical Statistics: Probability and Mathematical Statistics. John Wiley & Sons, Inc., New York, 1986.

[EPRB99] J.-P. Eckmann, C.-A. Pillet, and L. Rey-Bellet. Non-equilibrium statistical mechanics of anharmonic chains coupled to two heat baths at different temperatures. *Comm. Math. Phys.*, 201(3):657–697, 1999.

[Fef86] C. Fefferman. The N-body problem in quantum mechanics. *Communications on Pure and Applied Mathematics*, 39(S1):S67–S109, 1986.

[FF11] E. Fedrizzi and F. Flandoli. Pathwise uniqueness and continuous dependence of SDEs with non-regular drift. *Stochastics. An International Journal of Probability and Stochastic Processes*, 83(3):241–257, 2011.

[FFPV17] E. Fedrizzi, F. Flandoli, E. Priola, and J. Vovelle. Regularity of stochastic kinetic equations. *Electronic Journal of Probability*, 22, 2017.

[FGP10] F. Flandoli, M. Gubinelli, and E. Priola. Well-posedness of the transport equation by stochastic perturbation. *Inventiones Mathematicae*, 180(1):1–53, 2010.

[FH14] P. K. Friz and M. Hairer. *A course on rough paths*. Universitext. Springer, Cham, 2014.

[Fig08] A. Figalli. Existence and uniqueness of martingale solutions for SDEs with rough or degenerate coefficients. *J. Funct. Anal.*, 254(1):109–153, 2008.

[FIR17] F. Flandoli, E. Issoglio, and F. Russo. Multidimensional stochastic differential equations with distributional drift. *Transactions of the American Mathematical Society*, 369:1665–1688, 2017.

[FKM21] N. Frikha, V. Konakov, and S. Menozzi. Well-posedness of some nonlinear stable driven SDEs. *Discrete Contin. Dyn. Syst.*, 41(2):849–898, 2021.

[Fla11] F. Flandoli. *Random perturbation of PDEs and fluid dynamic models*, volume 2015 of *Lecture Notes in Mathematics*. Springer, Heidelberg, 2011. Lectures from the 40th Probability Summer School held in Saint-Flour, 2010.

[FM82] W. H. Fleming and S. K. Mitter. Optimal control and nonlinear filtering for nondegenerate diffusion processes. *Stochastics*, 8(1):63–77, 1982.

[FMPS21] S. Fornaro, G. Metafune, D. Pallara, and R. Schnaubelt. L^p-spectrum of degenerate hypoelliptic Ornstein-Uhlenbeck operators. *J. Funct. Anal.*, 280(2):108807, 22, 2021.

[Fri64] A. Friedman. *Partial differential equations of parabolic type*. Prentice-Hall, Inc., Englewood Cliffs, N.J., 1964.

[FRRO17] X. Fernández-Real and X. Ros-Oton. Regularity theory for general stable operators: parabolic equations. *J. Funct. Anal.*, 272(10):4165–4221, 2017.

[FRW03] F. Flandoli, F. Russo, and J. Wolf. Some SDEs with distributional drift. I. General calculus. *Osaka J. Math.*, 40(2):493–542, 2003.

[FRW04] F. Flandoli, F. Russo, and J. Wolf. Some SDEs with distributional drift. II. Lyons-Zheng structure, Itô's formula and semimartingale characterization. *Random Oper. Stochastic Equations*, 12(2):145–184, 2004.

[Ger20] M. Gerencsér. Regularisation by regular noise. *arXiv preprint: 2009.08418*, 2020.

[GIP15] M. Gubinelli, P. Imkeller, and N. Perkowski. Paracontrolled distributions and singular PDEs. *Forum Math. Pi*, 3:e6, 75, 2015.

[GT01] D. Gilbarg and N. S. Trudinger. *Elliptic partial differential equations of second order*. Classics in Mathematics. Springer-Verlag, Berlin, 2001.

[Had32] J. Hadamard. *Le probleme de Cauchy et les équations aux dérivées partielles linéaires hyperboliques*, volume 220. Paris, 1932.

[Had64] J. Hadamard. *La théorie des équations aux derivées partielles*. Gauthier-Villars Éditeur, Paris, éditions scientifiques edition, 1964.

[HK07] C. Houdré and R. Kawai. On layered stable processes. *Bernoulli*, 13(1):252–278, 2007.

[HM16] L. Huang and S. Menozzi. A parametrix approach for some degenerate stable driven SDEs. *Ann. Inst. Henri Poincaré Probab. Stat.*, 52(4):1925–1975, 2016.

[HMP19] L. Huang, S. Menozzi, and E. Priola. L^p estimates for degenerate non-local Kolmogorov operators. *J. Math. Pures Appl. (9)*, 121:162–215, 2019.

[HN04] F. Hérau and F. Nier. Isotropic hypoellipticity and trend to equilibrium for the Fokker-Planck equation with a high-degree potential. *Arch. Ration. Mech. Anal.*, 171(2):151–218, 2004.

[Hör67] L. Hörmander. Hypoelliptic second order differential equations. *Acta Math.*, 119:147–171, 1967.

[HPZ19] Z. Hao, X. Peng, and X. Zhang. Hörmander's hypoelliptic theorem for nonlocal operators. *Journal of Theoretical Probability*, pages 1–47, 2019.

[HWZ20] Z. Hao, M. Wu, and X. Zhang. Schauder estimates for nonlocal kinetic equations and applications. *J. Math. Pures Appl. (9)*, 140:139–184, 2020.

[IJS18] C. Imbert, T. Jin, and R. Shvydkoy. Schauder estimates for an integro-differential equation with applications to a nonlocal Burgers equation. *Ann. Fac. Sci. Toulouse Math. (6)*, 27(4):667–677, 2018.

[IKO62] A. M. Il'in, A. S. Kalashnikov, and O. A. Oleinik. Second-order linear equations of parabolic type. *Uspekhi Mat. Nauk*, 17(3):3–146, 1962.

[IM21] C. Imbert and C. Mouhot. The Schauder estimate in kinetic theory with application to a toy nonlinear model. *Ann. H. Lebesgue*, 4:369–405, 2021.

[Ish16] Y. Ishikawa. *Stochastic calculus of variations*, volume 54 of *De Gruyter Studies in Mathematics*. De Gruyter, Berlin, second edition, 2016.

[Jac01] N. Jacob. *Pseudo differential operators and Markov processes. Vol. I.* Imperial College Press, London, 2001.

[Jin18] P. Jin. On weak solutions of SDEs with singular time-dependent drift and driven by stable processes. *Stoch. Dyn.*, 18(2):1850013, 23, 2018.

[JS03] J. Jacod and A. Shiryaev. *Limit theorems for stochastic processes*, volume 288 of *Fundamental Principles of Mathematical Sciences*. Springer-Verlag, Berlin, second edition, 2003.

[JYC09] M. Jeanblanc, M. Yor, and M. Chesney. *Mathematical methods for financial markets*. Springer Finance. Springer-Verlag London, Ltd., London, 2009.

[Kal60a] R. E. Kalman. Contributions to the theory of optimal control. *Bol. Soc. Mat. Mexicana (2)*, 5:102–119, 1960.

[Kal60b] R. E. Kalman. A new approach to linear filtering and prediction problems. *Trans. ASME Ser. D. J. Basic Engrg.*, 82(1):35–45, 1960.

[KHN63] R. E. Kalman, Y. C. Ho, and K. S. Narendra. Controllability of linear dynamical systems. *Contr. to Diff. Equ.*, 1:189–213, 1963.

[KK15] I. Kim and K.-H. Kim. A Hölder regularity theory for a class of non-local elliptic equations related to subordinate Brownian motions. *Potential Anal.*, 43(4):653–673, 2015.

[KMM10] V. Konakov, S. Menozzi, and S. Molchanov. Explicit parametrix and local limit theorems for some degenerate diffusion processes. *Ann. Inst. Henri Poincaré Probab. Stat.*, 46(4):908—923, 2010.

[Kol34] A. Kolmogoroff. Zufällige Bewegungen (zur Theorie der Brownschen Bewegung). *Ann. of Math. (2)*, 35(1):116–117, 1934.

[Kol00] V. Kolokoltsov. Symmetric stable laws and stable-like jump-diffusions. *Proc. London Math. Soc. (3)*, 80(3):725–768, 2000.

[Kol11] V. Kolokoltsov. *Markov processes, semigroups and generators*, volume 38 of *De Gruyter Studies in Mathematics*. Walter de Gruyter & Co., Berlin, 2011.

[Kom83] T. Komatsu. Markov processes associated with pseudodifferential operators. In *Probability theory and mathematical statistics (Tbilisi, 1982)*, volume 1021 of *Lecture Notes in Math.*, pages 289–298. Springer, Berlin, 1983.

[KP10] N. V. Krylov and E. Priola. Elliptic and parabolic second-order PDEs with growing coefficients. *Comm. Partial Differential Equations*, 35(1):1–22, 2010.

[KP17] N. V. Krylov and E. Priola. Poisson stochastic process and basic Schauder and Sobolev estimates in the theory of parabolic equations. *Arch. Ration. Mech. Anal.*, 225(3):1089–1126, 2017.

[KP20] H. Kremp and N. Perkowski. Multidimensional SDE with distributional drift and Lévy noise. *arXiv preprint: 2008.05222*, 2020.

[KR05] N. V. Krylov and M. Röckner. Strong solutions of stochastic equations with singular time dependent drift. *Probability Theory and Related Fields*, 131:154–196, 2005.

[Kry71] N. V. Krylov. An inequality in the theory of stochastic integrals. *Theory of Probability & Its Applications*, 16(3):438–448, 1971.

[Kry96] N. V. Krylov. *Lectures on elliptic and parabolic equations in Hölder spaces*, volume 12 of *Graduate Studies in Mathematics*. AMS, 1996.

[Kry02] N. Krylov. Parabolic equations in L_p-spaces with mixed norms. *Algebra i Analiz*, 14(4):91–106, 2002.

[Kry04] N. V. Krylov. On weak uniqueness for some diffusions with discontinuous coefficients. *Stochastic Process. Appl.*, 113(1):37–64, 2004.

[Kry21] N. V. Krylov. On diffusion processes with drift in L_d. *Probab. Theory Related Fields*, 179(1-2):165–199, 2021.

[KS84] S. Kusuoka and D. Stroock. Applications of the Malliavin calculus. I. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 32(2):271–306, 1984.

[KS85] S. Kusuoka and D. Stroock. Applications of the Malliavin calculus. II. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 32(1):1–76, 1985.

[KS87] S. Kusuoka and D. Stroock. Applications of the Malliavin calculus. III. *J. Fac. Sci. Univ. Tokyo Sect. IA Math.*, 34(2):391–442, 1987.

[KS08] P. Kim and R. Song. Boundary behavior of harmonic functions for truncated stable processes. *J. Theoret. Probab.*, 21(2):287–321, 2008.

[KT01] T. Komatsu and A. Takeuchi. On the smoothness of PDF of solutions to SDE of jump type. *Int. J. Differ. Equ. Appl.*, 2(2):141–197, 2001.

[Küh19] F. Kühn. Schauder estimates for equations associated with Lévy generators. *Integral Equations Operator Theory*, 91(2):Paper No. 10, 21, 2019.

[Kun04] H. Kunita. Stochastic differential equations based on Lévy processes and stochastic flows of diffeomorphisms. In *Real and stochastic analysis*, Trends Math., pages 305–373. Birkhäuser Boston, Boston, MA, 2004.

[Kun19] H. Kunita. *Stochastic flows and jump-diffusions*, volume 92 of *Probability Theory and Stochastic Modelling*. Springer, Singapore, 2019.

[Kur98] T. Kurtz. Martingale problems for conditional distributions of Markov processes. *Electron. J. Probab.*, 3:no. 9, 29, 1998.

[Kur08] V. Kurenok. A note on L_2-estimates for stable integrals with drift. *Trans. Amer. Math. Soc.*, 360(2):925–938, 2008.

[Kur11] T. Kurtz. Equivalence of stochastic equations and martingale problems. In *Stochastic analysis 2010*, pages 113–130. Springer, Heidelberg, 2011.

[LBL08] C. Le Bris and P.-L. Lions. Existence and uniqueness of solutions to Fokker-Planck type equations with irregular coefficients. *Comm. Partial Differential Equations*, 33(7-9):1272–1317, 2008.

[Léa85] R. Léandre. Régularité de processus de sauts dégénérés. *Ann. Inst. H. Poincaré Probab. Statist.*, 21(2):125–146, 1985.

[Léa88] R. Léandre. Régularité de processus de sauts dégénérés. II. *Ann. Inst. H. Poincaré Probab. Statist.*, 24(2):209–236, 1988.

[Lev07] E. E. Levi. Sulle equazioni lineari totalmente ellittiche alle derivate parziali. *Rendiconti del Circolo Matematico di Palermo*, 1:275–317, 1907.

[Lie90] E. Lieb. The stability of matter: from atoms to stars. *Bull. Amer. Math. Soc. (N.S.)*, 22(1):1–49, 1990.

[LM76] J.-P. Lepeltier and B. Marchal. Problème des martingales et équations différentielles stochastiques associées à un opérateur intégro-différentiel. *Ann. Inst. H. Poincaré Sect. B (N.S.)*, 12(1):43–103, 1976.

[Lor05] L. Lorenzi. Schauder estimates for degenerate elliptic and parabolic problems with unbounded coefficients in $\mathbb{R}^N$. *Differential Integral Equations*, 18(5):531–566, 2005.

[LP94] E. Lanconelli and S. Polidoro. On a class of hypoelliptic evolution operators. *Rend. Sem. Mat. Univ. Politec. Torino*, 52(1):29–63, 1994.

[LR02] P. G. Lemarié-Rieusset. *Recent developments in the Navier-Stokes problem*, volume 431 of *Chapman & Hall/CRC Research Notes in Mathematics*. Chapman & Hall/CRC, Boca Raton, FL, 2002.

[LR21] A. Lunardi and M. Röckner. Schauder theorems for a class of pseudo-differential operators on finite- and infinite-dimensional state spaces. *J. of the London Math. Soc.*, 2021.

[Lun97] A. Lunardi. Schauder estimates for a class of degenerate elliptic and parabolic operators with unbounded coefficients in $\mathbf{R}^n$. *Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4)*, 24(1):133–164, 1997.

[Lun18] A. Lunardi. *Interpolation theory*, volume 16 of *Appunti. Scuola Normale Superiore di Pisa (Nuova Serie)*. Edizioni della Normale, Pisa, 2018.

[LZ19] C. Ling and G. Zhao. Nonlocal elliptic equation in Hölder space and the martingale problem. *arXiv preprint: 1907.00588*, 2019.

[Man97] M. Manfredini. The Dirichlet problem for a class of ultraparabolic equations. *Adv. Differential Equations*, 2(5):831–866, 1997.

[Mar20] L. Marino. Schauder estimates for degenerate stable Kolmogorov equations. *Bull. Sci. Math.*, 162:102885, 98, 2020.

[Mar21] L. Marino. Schauder estimates for degenerate Lévy Ornstein-Uhlenbeck operators. *Journal of Mathematical Analysis and Applications*, 500(1):125168, 2021.

[MBP10] T. Meyer-Brandis and F. Proske. Construction of strong solutions of SDE's via Malliavin calculus. *J. Funct. Anal.*, 258(11):3922–3953, 2010.

[Mel83] A. Melnikov. Stochastic equations and Krylov's estimates for semimartingales. *Stochastics*, 10(2):81–102, 1983.

[Mel16] A. Mellet. Anomalous diffusion phenomena: a kinetic approach. In *Séminaire Laurent Schwartz—Équations aux dérivées partielles et applications. Année 2014–2015*, pages Exp. No. XII, 16. Ed. Éc. Polytech., Palaiseau, 2016.

[Men11] S. Menozzi. Parametrix techniques and martingale problems for some degenerate Kolmogorov equations. *Electron. Commun. Probab.*, 16:234–250, 2011.

[Men18] S. Menozzi. Martingale problems for some degenerate Kolmogorov equations. *Stochastic Process. Appl.*, 128(3):756–802, 2018.

[MM21] L. Marino and S. Menozzi. Weak well-posedness for degenerate SDEs driven by Lévy processes. *arXiv preprint: 2107.04325*, 2021.

[MMM11] A. Mellet, S. Mischler, and C. Mouhot. Fractional diffusion limit for collisional kinetic equations. *Arch. Ration. Mech. Anal.*, 199(2):493–525, 2011.

[MP14] R. Mikulevicius and H. Pragarauskas. On the Cauchy problem for integro-differential operators in Hölder classes and the uniqueness of the martingale problem. *Potential Analysis*, 40(4):539–563, 2014.

[MPMBN+13] O. Menoukeu-Pamen, T. Meyer-Brandis, T. Nilssen, F. Proske, and T. Zhang. A variational approach to the construction and Malliavin differentiability of strong solutions of SDE's. *Math. Ann.*, 357(2):761–799, 2013.

[MPP02] G. Metafune, D. Pallara, and E. Priola. Spectrum of Ornstein-Uhlenbeck operators in L^p spaces with respect to invariant measures. *J. Funct. Anal.*, 196(1):40–60, 2002.

[MPRS02] G. Metafune, J. Prüss, A. Rhandi, and R. Schnaubelt. The domain of the Ornstein-Uhlenbeck operator on an L^p-space with invariant measure. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 1(2):471–485, 2002.

[MPZ21] S. Menozzi, A. Pesce, and X. Zhang. Density and gradient estimates for non degenerate Brownian SDEs with unbounded measurable drift. *J. Differential Equations*, 272:330–369, 2021.

[MS67] H. P. McKean, Jr. and I. M. Singer. Curvature and the eigenvalues of the Laplacian. *Journal of Differential Geometry*, 1(1):43–69, 1967.

[Nam20] K. Nam. Stochastic differential equations with critical drifts. *Stochastic Process. Appl.*, 130(9):5366–5393, 2020.

[Pas03] A. Pascucci. Hölder regularity for a Kolmogorov equation. *Trans. Amer. Math. Soc.*, 355(3):901–924, 2003.

[Pic96] J. Picard. On the existence of smooth densities for jump processes. *Probab. Theory Related Fields*, 105(4):481–511, 1996.

[PPP16] S. Pagliarani, A. Pascucci, and M. Pignotti. Intrinsic Taylor formula for Kolmogorov-type homogeneous groups. *J. Math. Anal. Appl.*, 435(2):1054–1087, 2016.

[Pri09] E. Priola. Global Schauder estimates for a class of degenerate Kolmogorov equations. *Studia Math.*, 194(2):117–153, 2009.

[Pri12] E. Priola. Pathwise uniqueness for singular SDEs driven by stable processes. *Osaka J. Math.*, 49(2):421–447, 2012.

[Pri15a] E. Priola. L^p-parabolic regularity and non-degenerate Ornstein-Uhlenbeck type operators. In *Geometric methods in PDE's*, volume 13 of *Springer INdAM Ser.*, pages 121–139. Springer, Cham, 2015.

[Pri15b] E. Priola. On weak uniqueness for some degenerate SDEs by globalL^p estimates. *Potential Anal.*, 42(1):247–281, 2015.

[Pri18] E. Priola. Davie's type uniqueness for a class of SDEs with jumps. *Ann. Inst. Henri Poincaré Probab. Stat.*, 54(2):694–725, 2018.

[Pro05] P. E. Protter. *Stochastic integration and differential equations*, volume 21 of *Stochastic Modelling and Applied Probability.* Springer-Verlag, Berlin, 2005.

[PT69] W. E. Pruitt and S. J. Taylor. The potential kernel and hitting probabilities for the general stable process in $\mathbb{R}^N$. *Trans. Amer. Math. Soc.*, 146:299–321, 1969.

[PZ09] E. Priola and J. Zabczyk. Densities for Ornstein-Uhlenbeck processes with jumps. *Bull. Lond. Math. Soc.*, 41(1):41–50, 2009.

[RBT00] L. Rey-Bellet and L. Thomas. Asymptotic behavior of thermal nonequilibrium steady states for a driven chain of anharmonic oscillators. *Communications in Mathematical Physics*, 215(01), 2000.

[Ros07] J. Rosiński. Tempering stable processes. *Stochastic Process. Appl.*, 117(6):677–707, 2007.

[ROS16] X. Ros-Oton and J. Serra. Regularity theory for general stable operators. *J. Differential Equations*, 260(12):8675–8715, 2016.

[Ryz02] M. Ryznar. Estimates of Green function for relativistic α-stable process. *Potential Anal.*, 17(1):1–23, 2002.

[RZ21] M. Röckner and G. Zhao. SDEs with critical time dependent drifts: weak solutions. *arXiv preprint: 2012.04161*, 2021.

[Sai07] N. Saintier. Schauder estimates for degenerate elliptic and parabolic equations in $\mathbb{R}^N$ with Lipschitz drift. *Differential Integral Equations*, 20(4):397–428, 2007.

[Sat13] K. Sato. *Lévy processes and infinitely divisible distributions*, volume 68 of *Cambridge Studies in Advanced Mathematics*. Cambridge University Press, Cambridge, 2013.

[Sko56] A. V. Skorohod. Limit theorems for stochastic processes. *Teor. Veroyatnost. i Primenen.*, 1:289–319, 1956.

[SSW12] R. L. Schilling, P. Sztonyk, and J. Wang. Coupling property and gradient estimates of Lévy processes via the symbol. *Bernoulli*, 18(4):1128–1149, 2012.

[Ste93] E. M. Stein. *Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals*, volume 43 of *Princeton Mathematical Series*. Princeton University Press, Princeton, NJ, 1993.

[Str75] D. Stroock. Diffusion processes associated with Lévy generators. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete*, 32(3):209–244, 1975.

[SV79] D.W. Stroock and S.R.S. Varadhan. *Multidimensional diffusion processes*. Springer-Verlag Berlin Heidelberg New-York, 1979.

[SW12] R. L. Schilling and J. Wang. On the coupling property and the Liouville theorem for Ornstein-Uhlenbeck processes. *J. Evol. Equ.*, 12(1):119–140, 2012.

[Szt10a] P. Sztonyk. Estimates of tempered stable densities. *Journal of Theoretical Probability*, 23(1):127–147, 2010.

[Szt10b] P. Sztonyk. Regularity of harmonic functions for anisotropic fractional Laplacians. *Math. Nachr.*, 283(2):289–311, 2010.

[Tri92] H. Triebel. *Theory of function spaces. II*, volume 84 of *Monographs in Mathematics*. Birkhäuser Verlag, Basel, 1992.

[TTW74] H. Tanaka, M. Tsuchiya, and S. Watanabe. Perturbation of drift-type for Lévy processes. *J. Math. Kyoto Univ.*, 14:73–92, 1974.

[Ver81] A J Veretennikov. Strong solutions and explicit formulas for solutions of stochastic integral equations. *Mathematics of the USSR-Sbornik*, 39(3):387–403, 1981.

[Vil02] C. Villani. A review of mathematical topics in collisional kinetic theory. In *Handbook of mathematical fluid dynamics, Vol. I*, pages 71–305. North-Holland, Amsterdam, 2002.

[Wat07] T. Watanabe. Asymptotic estimates of multi-dimensional stable densities and their applications. *Trans. Amer. Math. Soc.*, 359(6):2851–2879, 2007.

[Woy01] W. A. Woyczyński. Lévy processes in the physical sciences. In *Lévy processes*, pages 241–266. Birkhäuser Boston, Boston, MA, 2001.

[XZ16] L. Xie and X. Zhang. Sobolev differentiable flows of SDEs with local Sobolev and super-linear growth coefficients. *Ann. Probab.*, 44(6):3661–3687, 2016.

[YW71] T. Yamada and S. Watanabe. On the uniqueness of solutions of stochastic differential equations. *J. Math. Kyoto Univ.*, 11:155–167, 1971.

[Zab92] J. Zabczyk. *Mathematical control theory: an introduction*. Systems & Control: Foundations & Applications. Birkhäuser Boston, Inc., Boston, MA, 1992.

[Zha13a] X. Zhang. Stochastic differential equations with Sobolev drifts and driven by α-stable processes. *Ann. Inst. Henri Poincaré Probab. Stat.*, 49(4):1057–1079, 2013.

[Zha13b] X. Zhang. Well-posedness and large deviation for degenerate SDEs with Sobolev coefficients. *Rev. Mat. Iberoam.*, 29(1):25–52, 2013.

[Zha14] X. Zhang. Densities for SDEs driven by degenerate α-stable processes. *Ann. Probab.*, 42(5):1885–1910, 2014.

[Zha15] X. Zhang. Stochastic differential equations with Sobolev diffusion and singular drift. *arXiv preprint: 1406.7446*, 2015.

[Zha18] X. Zhang. Stochastic Hamiltonian flows with singular coefficients. *Sci. China Math.*, 61(8):1353–1384, 2018.

[Zha19] G. Zhao. Weak uniqueness for SDEs driven by supercritical stable processes with Hölder drifts. *Proc. Amer. Math. Soc.*, 147(2):849–860, 2019.

[Zub12] O. Zubelevich. Several Notes on Existence Theorem of Peano. *Funkcialaj Ekvacioj*, 55(1):89–97, 2012.

[Zvo74] A. K. Zvonkin. A transformation of the phase space of a diffusion process that will remove the drift. *Mat. Sb. (N.S.)*, 93(135):129–149, 152, 1974.

[ZZ18] X. Zhang and G. Zhao. Dirichlet problem for supercritical non-local operators. *arXiv preprint: 1809.05712*, 2018.

université PARIS-SACLAY
École doctorale de mathématiques Hadamard (EDMH)

Titre: Régularisation faible par un bruit de Lévy dégénéré et applications

Mots clés: EDOs mal posées, Solutions faibles via régularisations, Équations dégénérées de Kolmogorov, Estimées de Schauder pour opérateurs intégro-différentiels, Méthodes de la parametrix, Processus de Lévy

Résumé: Après une introduction générale sur le phénomène de régularisation par le bruit dans le cadre dégénéré, la première partie de cette thèse est consacrée à l'obtention d'estimées de Schauder, un outil analytique utile pour établir le caractère bien posé des équations différentielles stochastiques (EDS), pour deux différents classes d'équations de Kolmogorov sous une condition de type Hörmander faible, dont les coefficients appartiennent à des espaces de Holder anisotropes appropriés avec des multi-indices de régularité. La première classe considère un système non linéaire dirigé par un opérateur α-stable symétrique agissant uniquement sur certaines composantes. Notre méthode de preuve repose sur une approche perturbative basée sur des développements parametrix progressifs par des formules de type Duhamel. En raison des faibles propriétés de régularisation données par le cadre dégénéré, nous exploitons également certains contrôles sur les normes de Besov, afin de traiter la perturbation non-linéaire. Dans le prolongement de la première, nous présentons également les estimées de Schauder pour un opérateur dégénéré d'Ornstein-Uhlenbeck associé à une classe plus large d'opérateurs de type α-stable, comme l'opérateur stable relativiste ou de Lamperti. La preuve de ce résultat repose plutôt sur une analyse précise du comportement du semi-groupe de Markov correspondante entre les espaces de Hölder anisotropes et quelques techniques d'interpolation. En exploitant une approche parametrix rétrograde, la deuxième partie de cette thèse cherche à établir le caractère bien-posé au sens faible d'une chaîne dégénérée de EDS dirigées par la même classe de processus de type α-stable, sous des hypothèses de régularité de Hölder minimale sur les coefficients. Comme corollaire de notre méthode, nous présentons également des estimations de type Krylov d'intérêt indépendant pour le processus canonique sous-jacent. Enfin, nous soulignons à travers des contre-exemples appropriés qu'il existe bien un seuil (presque) optimal sur les exposants de régularité assurant le caractère faiblement bien posé pour l'EDS. En lien avec quelques applications mécaniques pour des dynamiques cinétique avec frottement, nous concluons en étudiant la stabilité des perturbations du second ordre pour des opérateurs de Kolmogorov dégénérés en normes Lp et Hölder.

Title: Weak regularization by degenerate Lévy noise and its applications

Keywords: Ill posed ODEs, Weak solutions through noise regularization, Kolmogorov degenerate equations, Schauder estimates for integro-differential operators, Parametrix Methods, Lévy processes

Abstract: After a general introduction about the regularization by noise phenomenon in the degenerate setting, the first part of this thesis focuses at establishing the Schauder estimates, a useful analytical tool to prove also the well-posedness of stochastic differential equations (SDEs), for two different classes of Kolmogorov equations under a weak Hörmander-like condition, whose coefficients lie in suitable anisotropic Hölder spaces with multi-indices of regularity. The first class considers a nonlinear system controlled by a symmetric α-stable operator acting only on some components. Our method of proof relies on a perturbative approach based on forward parametrix expansions through Duhamel-type formulas. Due to the low regularizing properties given by the degenerate setting, we also exploit some controls on Besov norms, in order to deal with the non-linear perturbation. As an extension of the first one, we also present Schauder estimates associated with a degenerate Ornstein-Uhlenbeck operator driven by a larger class of α-stable-like operators, like the relativistic or the Lamperti stable one. The proof of this result relies instead on a precise analysis of the behaviour of the associated Markov semigroup between anisotropic Hölder spaces and some interpolation techniques. Exploiting a backward parametrix approach, the second part of this thesis aims at establishing the well-posedness in a weak sense of a degenerate chain of SDEs driven by the same class of α-stable-like processes, under the assumptions of the minimal Hölder regularity on the coefficients. As a by-product of our method, we also present Krylov-type estimates of independent interest for the associated canonical process. Finally, we emphasize through suitable counter-examples that there exists indeed an (almost) sharp threshold on the regularity exponents ensuring the weak well-posedness for the SDE. In connection with some mechanical applications for kinetic dynamics with friction, we conclude by investigating the stability of second-order perturbations for degenerate Kolmogorov operators in Lp and Hölder norms.

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