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A massive class of $\mathcal{N} = 2$ AdS$_4$ IIA solutions

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Abstract: We initiate a classification of $\mathcal{N} = 2$ supersymmetric AdS$_4$ solutions of (massive) type IIA supergravity. The internal space is locally equipped with either an SU(2) or an identity structure. We focus on the SU(2) structure and determine the conditions it satisfies, dictated by supersymmetry. Imposing as an Ansatz that the internal space is complex, we reduce the problem of finding solutions to a Riccati ODE, which we solve analytically. We obtain in this fashion a large number of new families of solutions, both regular as well as with localized O8-planes and conical Calabi-Yau singularities. We also recover many solutions already discussed in the literature.

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Contents

1 Introduction

The study of four-dimensional anti-de Sitter solutions of string/M-theory is of considerable interest both in the context of flux compactifications and of the AdS/CFT correspondence. The prototypical class of such solutions is the Freund-Rubin class [1] in M-theory, where the flux is along the anti-de Sitter spacetime and the internal manifold is Einstein. This class allows for various amounts of supersymmetry, which further constrain the geometry of the internal manifold M_7. Maximal supersymmetry imposes $M_7 \simeq S^7$, while with $\mathcal{N} = 2$ supersymmetry, which is the focus of this paper, M_7 is Sasaki-Einstein. Outside of this class very few solutions are known in M-theory [2–4].

From a holographic perspective, AdS_4 solutions are dual to Chern-Simons-matter field theories in three dimensions. A good control of the correspondence typically requires extended supersymmetry, and $\mathcal{N} = 2$ provides a nice balance between control and variety.

When in M-theory (and its type IIA reduction), the sum of the Chern-Simons levels of the gauge groups that characterize the field theory is zero. A non-zero sum corresponds to a non-zero Romans mass in type IIA string theory [5, 6]. So far, almost all known $\mathcal{N} = 2$ solutions of massive type IIA supergravity are numerical [7–10], with one notable exception being the Guarino-Jafferis-Varela solution [11]. In this paper, we overturn this status, finding a vast number of analytic solutions.

We will analyze the constraints imposed by supersymmetry by employing the "pure spinor" method, originally devised for $\mathcal{N} = 1$ solutions [12, 13]. Adapting this method to $\mathcal{N} = 2$ is not entirely straightforward. Until recently, in most of the literature one has resorted to imposing, on top of $\mathcal{N} = 1$ supersymmetry, the presence of a vector field that leaves all fields but the supersymmetry parameters invariant, thus representing the R-symmetry action. This has proven useful, but has to be supplemented by an inspired Ansatz, and so far has resulted in the aforementioned numerical solutions.

A different approach has been put forward in [14], based on the work of [15]. In the latter reference, the conditions for supersymmetry were expressed in terms of differential forms in the spirit of generalized geometry, without further assumptions on the form of the solution. In [14], these were adapted to the specific case of $\mathcal{N} = 2$ $\mathrm{AdS}_4 \times M_6$ solutions of type IIB supergravity, obtaining a set of $\mathcal{N} = 2$ pure spinor equations. After some work this resulted in a system of partial differential equations which characterize all possible solutions. In this paper we apply the same idea to type IIA supergravity.

The set of $\mathcal{N} = 2$ pure spinor equations we obtain superficially resembles that of [14] in IIB; as is the case for $\mathcal{N} = 1$, it is obtained by exchanging odd with even pure spinors. However, the geometric constraints that follow differ early on in the analysis. While in IIB the structure group on M_6 is exclusively the identity, in IIA it can be either the identity or SU(2).[1] In this paper we will focus on the SU(2) structure case, leaving the identity structure for future work.[2] The set of constraints we obtain from supersymmetry on the SU(2) structure also imply the Bianchi identities for the form fields, and all equations of motion.

Within the SU(2) structure case, we find two classes which we call "class K" and "class HK", because the internal manifold M_6 contains a four-dimensional subspace M_4 equipped with either a Kähler or a hyper-Kähler metric. We work out the supersymmetry constraints in full detail for both classes. Class HK leads to one local metric. On the other hand, class K leads to a very rich structure of solutions.

In particular, a simple and natural Ansatz (inspired by [18]) is that M_6 admits a complex structure. After imposing this, the problem of finding solutions reduces to a single ordinary differential equation (ODE) of Riccati type. The analysis is further subdivided according to whether M_4 is Kähler-Einstein, or is a product of two Riemann surfaces. In both cases we find the most general solution to the ODE analytically.

This results in two new families of analytic solutions. The fully regular ones are identified with the numerical solutions we mentioned above [7–10]. In the massless limit, they result in the IIA reduction of various previously-known Sasaki-Einstein manifolds:

[1] This parallels what happens for $\mathcal{N} = 1$: IIA solutions can have SU(3) or SU(2) structure, while in IIB only the latter is allowed [16].

[2] $\mathcal{N} = 2$ solutions with SU(2) structure have been previously studied in [17].

the so-called $Y^{p,k}$ [19], a generalization thereof called $A^{p,q,r}$ [10, 20, 21], and the older $M^{3,2}$ [22, 23], $Q^{1,1,1}$ [24], as well as the Fubini-Study metric on $\mathbb{CP}^3$. As already suggested by numerical evidence in [9], a certain "maximal" massive deformation exists, which results (when $M_4 = \mathbb{CP}^2$) in the Guarino-Jafferis-Varela solution [11].

Beyond this maximal deformation, the solutions still exist, but develop singularities that have a physical interpretation as corresponding to the presence of various orientifold planes. In most cases these are smeared in some directions and localized in others; some limits of the parameters however produce solutions with fully localized O8-planes.

The rest of the paper is structured as follows. In section 2 we specialize the system of ten-dimensional equations obtained in [15] to $\mathcal{N} = 2$ AdS$_4$ IIA solutions. As we mentioned, the analysis is similar to the one in IIB [14, section 2,3], but some differences begin to emerge already here, and in particular we see that the SU(2) structure case is admissible, which we then focus on. In section 3 we analyze the system for this case by eliminating redundancies and obtaining the geometrical consequences of the system; the two classes K and HK are analyzed in turn. Being class HK rather limited, we devote section 4 to class K, under the assumption that M_6 admits a complex structure. As anticipated, we obtain two main families of analytic solutions, depending on several parameters. We summarize our findings in section 4.3.

2 Reduction of the ten-dimensional supersymmetry equations

In this section we will specialize the system of equations obtained in [15] as a set of necessary and sufficient conditions for any ten-dimensional solution of type II supergravity to preserve superymmetry, to the case of an AdS$_4$ background of type IIA supergravity preserving $\mathcal{N} = 2$ supersymmetry. The process is similar to the one followed for type IIB supergravity in [14] and we refer the reader there for more details, especially on conventions.

We begin by reviewing the system of equations of [15], which are summarized in section 3.1 of that paper, focusing on the following subset of equations:

$$d_H(e^{-\phi}\Phi) = -(\widetilde{K} \wedge + \iota_K) F_{(10d)}\,, \tag{2.1a}$$

$$d\widetilde{K} = \iota_K H\,, \qquad \nabla_{(M} K_{N)} = 0\,. \tag{2.1b}$$

Here ϕ is the dilaton, H is the NS-NS three-form field strength, $d_H \equiv d - H\wedge$, and $F_{(10d)}$ is an even form obtained as a formal sum of all the R-R field strengths. $\lambda(F_p) \equiv (-1)^{\lfloor p/2 \rfloor} F_p$, for F_p a p-form.

Moreover, Φ in (2.1) is an even form that corresponds via the Clifford map $\gamma^{M_1 \ldots M_k} \mapsto dx^{M_1} \wedge \ldots \wedge dx^{M_k}$ to the bispinor $\epsilon_1 \otimes \overline{\epsilon_2}$, where $\epsilon_{1,2}$ are the parameters of the supersymmetry transformations (which we take to be two Majorana-Weyl spinors of positive and negative chirality respectively). The vector K and the one-form $\widetilde{K}$ are defined by

$$K \equiv \frac{1}{64}(\overline{\epsilon_1}\Gamma^M \epsilon_1 + \overline{\epsilon_2}\Gamma^M \epsilon_2)\partial_M\,, \qquad \widetilde{K} \equiv \frac{1}{64}(\overline{\epsilon_1}\Gamma_M \epsilon_1 - \overline{\epsilon_2}\Gamma_M \epsilon_2) dx^M\,. \tag{2.2}$$

As we see in the second equation of (2.1b), K is a Killing vector and is in fact a symmetry of all fields in a solution.

We will apply (2.1) to AdS_4 solutions, meaning that we will take all fields to preserve its $\mathrm{SO}(3,2)$ isometry group. In particular we will take the metric to be of the warped product form:

$$ds^2_{10} = e^{2A} ds^2_{\mathrm{AdS}_4} + ds^2_{M_6} \,. \tag{2.3}$$

The symmetry requirement also implies that the H field will only be a form on M_6, while the R-R field strengths will be decomposed as

$$F_{(10d)} = e^{4A} \mathrm{vol}_4 \wedge \star_6 \lambda(F) + F \,, \qquad F = F_0 + F_2 + F_4 + F_6 \,. \tag{2.4}$$

The supersymmetry parameters $\epsilon_{1,2}$ will also be sums of tensor products of spinors on AdS_4 and M_6. For $\mathcal{N}=2$ supersymmetry, the decomposition reads

$$\epsilon_1 = \sum_{I=1}^{2} \chi^I_+ \otimes \eta^I_{1+} + \sum_{J=1}^{2} \chi^J_- \otimes \eta^J_{1-} \,, \tag{2.5a}$$

$$\epsilon_2 = \sum_{I=1}^{2} \chi^I_+ \otimes \eta^I_{2-} + \sum_{J=1}^{2} \chi^J_- \otimes \eta^J_{2+} \,. \tag{2.5b}$$

The χ's are a basis of AdS_4 Killing spinors:

$$\nabla_\mu \chi^I_\pm = \frac{1}{2}\gamma_\mu \chi^I_\mp \,, \qquad \nabla_\mu \overline{\chi^I_\pm} = -\frac{1}{2}\overline{\chi^I_\mp}\gamma_\mu \,, \tag{2.6}$$

where $\mu = 0, \ldots, 3$. We will assume χ^1_+ and χ^2_+ to be linearly independent, since otherwise we would only have $\mathcal{N}=1$ supersymmetry. The gamma matrices decompose according to

$$\Gamma_\mu = e^A \gamma_\mu \otimes \mathbb{I} \,, \qquad \Gamma_{m+3} = \gamma_5 \otimes \gamma_m \,, \qquad \Gamma_{11} \equiv \Gamma^0 \ldots \Gamma^9 = \gamma_5 \otimes \gamma_7 \,, \tag{2.7}$$

with $m = 1, 2, \ldots 6$. γ_5 and γ_7 are the external and internal chirality operators.

Let us now reduce (2.1) for this case of $\mathcal{N}=2$ AdS_4 solutions. We start with (2.1b): $\widetilde{K}$, K decompose as

$$\widetilde{K}_\mu = \frac{e^A}{32} \sum_{I,J=1}^{2} \overline{\chi^I_+}\gamma_\mu \chi^J_+ (\overline{\eta^I_{1+}}\eta^J_{1+} - \overline{\eta^I_{2-}}\eta^J_{2-}) \,, \qquad \widetilde{K}_m = -\frac{1}{16}\mathrm{Re}(\overline{\chi^1_+}\chi^2_- \tilde{\xi}_m) \,, \tag{2.8a}$$

$$K^\mu = \frac{e^{-A}}{32} \sum_{I,J=1}^{2} \overline{\chi^I_+}\gamma^\mu \chi^J_+ (\overline{\eta^I_{1+}}\eta^J_{1+} + \overline{\eta^I_{2-}}\eta^J_{2-}) \,, \qquad K^m = -\frac{1}{16}\mathrm{Re}(\overline{\chi^1_+}\chi^2_- \xi^m) \,, \tag{2.8b}$$

where

$$\tilde{\xi}_m \equiv \overline{\eta^1_{1+}}\gamma_m \eta^2_{1-} - \overline{\eta^1_{2-}}\gamma_m \eta^2_{2+} \,, \qquad \xi^m \equiv \overline{\eta^1_{1+}}\gamma^m \eta^2_{1-} + \overline{\eta^1_{2-}}\gamma^m \eta^2_{2+} \,. \tag{2.9}$$

We thus find that (2.1b) gives

$$\overline{\eta^{(I}_{1+}}\eta^{J)}_{1+} = \overline{\eta^{(I}_{2+}}\eta^{J)}_{2+} \equiv \frac{1}{2}c^{IJ}e^A \,, \qquad -i\overline{\eta^{[I}_{1+}}\eta^{J]}_{1+} = -i\overline{\eta^{[I}_{2+}}\eta^{J]}_{2+} \equiv \epsilon^{IJ} f \,, \tag{2.10}$$

where c^{IJ} are constants, and

$$d(e^A f) = -\frac{1}{2}\mathrm{Im}\tilde{\xi} \,, \qquad d\tilde{\xi} = i_\xi H \,, \qquad \mathrm{Im}(\xi) = 0 \,, \qquad \nabla_{(n}\xi_{m)} = 0 \,. \tag{2.11}$$

Hence ξ is a Killing vector and in fact realizes the R-symmetry, acting on the I index in (2.5). Notice that there are subtle sign differences in these formulas with respect to similar ones in IIB [14].

To reduce (2.1a), we write Φ as a wedge product of external and internal forms:

$$\Phi = \sum_{IJ} \left(-\chi_+^I \overline{\chi_+^J} \wedge \eta_{1+}^I \overline{\eta_{2-}^J} + \chi_+^I \overline{\chi_-^J} \wedge \eta_{1+}^I \overline{\eta_{2+}^J} + \chi_-^I \overline{\chi_+^J} \wedge \eta_{1-}^I \overline{\eta_{2-}^J} + \chi_-^I \overline{\chi_-^J} \wedge \eta_{1-}^I \overline{\eta_{2+}^J} \right) . \tag{2.12}$$

Again we use bispinors to denote the corresponding polyform under the Clifford map. Using (2.6) we can derive the exterior differential of the four-dimensional spacetime polyforms:

$$d(\chi_\pm^I \overline{\chi_\pm^J}) = 2 \sum_k \left(1 - \frac{1}{4}(-1)^k (4-2k) \right) \mathrm{Re}(\chi_\mp^I \overline{\chi_\pm^J})_k \, , \tag{2.13a}$$

$$d(\chi_\pm^I \overline{\chi_\mp^J}) = 2i \sum_k \left(1 + \frac{1}{4}(-1)^k (4-2k) \right) \mathrm{Im}(\chi_\mp^I \overline{\chi_\mp^J})_k \, , \tag{2.13b}$$

where k is the form degree. Using this in (2.1a) and factoring spacetime forms, we get purely internal equations:

$$d_H \left(e^{2A-\phi} \phi_+^{(IJ)} \right) - 2 e^{A-\phi} \mathrm{Re} \phi_-^{(IJ)} = 0 \, , \tag{2.14a}$$

$$d_H \left(e^{3A-\phi} \mathrm{Re} \phi_-^{[IJ]} \right) = 0 \, , \tag{2.14b}$$

$$d_H \left(e^{A-\phi} \mathrm{Im} \phi_-^{[IJ]} \right) - e^{-\phi} \mathrm{Im} \phi_+^{[IJ]} = \frac{1}{8} e^A f F \epsilon^{IJ} \, ; \tag{2.14c}$$

and

$$d_H \left(e^{3A-\phi} \mathrm{Im} \phi_-^{(IJ)} \right) - 3 e^{2A-\phi} \mathrm{Im} \phi_+^{(IJ)} = \frac{1}{16} c^{IJ} e^{4A} \star \lambda(F) \, , \tag{2.14d}$$

$$d_H \left(e^{-\phi} \phi_+^{[IJ]} \right) = -\frac{1}{16} (\bar{\tilde{\xi}} \wedge + \iota_\xi) F \epsilon^{IJ} \, , \tag{2.14e}$$

$$d_H \left(e^{4A-\phi} \phi_+^{[IJ]} \right) - 4 e^{3A-\phi} \mathrm{Re} \phi_-^{[IJ]} = -\frac{i}{16} (\bar{\tilde{\xi}} \wedge + \iota_\xi) e^{4A} \star \lambda(F) \epsilon^{IJ} \, , \tag{2.14f}$$

where

$$\phi_+^{IJ} \equiv \eta_{1+}^I \overline{\eta_{2+}^J} \, , \qquad \phi_-^{IJ} \equiv \eta_{1+}^I \overline{\eta_{2-}^J} \, , \tag{2.15}$$

and $\bar{\tilde{\xi}}$ is the complex conjugate of $\tilde{\xi}$.

As was the case for the corresponding system of equations in type IIB supergravity [14], the system (2.14), although alarmingly large, has a high degree of redundancy. For instance, we will see soon that c^{IJ} can be set proportional to the identity; after that one can see that the equations that involve the R-R fields are redundant, except for (2.14c). Also, the $I \neq J$ components are redundant, since the $I = J$ ones furnish two copies of the pure spinor equations [12, 13] for $\mathcal{N} = 1$ AdS$_4$ solutions. Finally, the remaining "pairing equations" [15, (3.1c,d)] are redundant as for [14].

In spite of this redundancy, (2.14) will be more convenient for our analysis than a repeated application of the $\mathcal{N} = 1$ equations [13].

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2.1 Parametrization of the pure spinors

In this section we will parametrize the pure spinors $\phi^{IJ}_{\pm}$ in terms of a set of differential forms.

Before introducing the parametrization, we will fix the constants c^{IJ} of (2.10) as

$$c^{IJ} = 2\delta^{IJ}\,, \tag{2.16}$$

where δ^{IJ} is the Kronecker delta. This is permitted as the decomposition Ansatz (2.5) sets the internal spinors only up to a $\mathrm{GL}(2,\mathbb{R})$ transformation that leaves invariant the norms $\|\eta^I_{i+}\|$ (which are equal to e^A, by (2.10)). The details of this transformation can be found in [14]. Since $c^{12} = \overline{\eta^{(1}_{i+}}\eta^{2)}_{i+} = 0$, from $\overline{\eta^{[1}_{i+}}\eta^{2]}_{i+} = \overline{\eta^1_{i+}}\eta^2_{i+}$ and $|\overline{\eta^1_{i+}}\eta^2_{i+}| \leq \sqrt{\|\eta^1_{i+}\|\|\eta^2_{i+}\|}$ it follows that

$$|f| \leq e^A\,. \tag{2.17}$$

Furthermore, instead of η^I_{i+} we will work with

$$\eta^{\pm}_{i+} = \frac{1}{\sqrt{2}}(\eta^1_{i+} \pm i\eta^2_{i+}) \tag{2.18}$$

which have charge ± 1 under the $\mathrm{U}(1) \simeq \mathrm{SO}(2)$ R-symmetry. From (2.10) and (2.16) we then have

$$\overline{\eta^{\pm}_{i+}}\eta^{\mp}_{i+} = 0\,, \qquad \overline{\eta^{\pm}_{i+}}\eta^{\pm}_{i+} = f_{\mp} \equiv e^A \mp f\,. \tag{2.19}$$

The internal spinors $\eta^{\pm}_{i+}$ can be parametrized in terms of a chiral spinor η_+ of positive chirality (and its complex conjugate $\eta_- \equiv (\eta_+)^c$) as follows:

$$\eta^+_{1+} = \sqrt{f_-}\eta_+\,, \qquad \eta^+_{2+} = \sqrt{f_-}\left(a\eta_+ + \frac{1}{2}bw_3\eta_-\right)\,, \tag{2.20a}$$

$$\eta^-_{1+} = \sqrt{f_+}\frac{1}{2}w_1\eta_-\,, \qquad \eta^-_{2+} = \sqrt{f_+}\frac{1}{2}cw_2\left(a^*\eta_- - \frac{1}{2}b\overline{w_3}\eta_+\right)\,, \tag{2.20b}$$

where the w_i are one-forms, a is a function taking value in $\mathbb{C}$ and b, c are real functions. The latter satisfy

$$|a|^2 + b^2 = 1\,, \qquad c^{-1} = \left(|z_1|^2 b^2 + |a|^2\right)^{1/2}\,, \qquad z_1 \equiv \frac{1}{2}\overline{w_2}\cdot w_3\,, \tag{2.21}$$

where $\cdot$ denotes the inner product.

The chiral spinor η_+ defines an $\mathrm{SU}(3)$ structure, characterized by a real two-form J and a holomorphic three-form Ω, as

$$\eta_+\overline{\eta_+} = \frac{1}{8}e^{-iJ}\,, \qquad \eta_+\overline{\eta_-} = -\frac{1}{8}\Omega\,, \tag{2.22}$$

with J, Ω satisfying $J\wedge\Omega = 0$ and $J\wedge J\wedge J = \frac{3}{4}i\Omega\wedge\overline{\Omega}$.

When they are not all linearly dependent, the one-forms w_i parametrize an identity structure and are holomorphic with respect to the almost complex structure J defined by η_+. We will leave this generic case to future work; in this paper, we will limit ourselves to analyzing the case of an $\mathrm{SU}(2)$ structure, for which the w_i are all linearly dependent. Such a case is not guaranteed to be compatible with the supersymmetry equations a priori, and

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indeed it is not allowed in type IIB supergravity [14]. However, as we will see, in type IIA supergravity solutions with $\mathcal{N}=2$ supersymmetry and an SU(2) structure do exist.

We will thus take

$$w_2 = z_3 w_1\,, \qquad w_3 = z_2^* w_1\,, \qquad \{z_2\,, z_3 \in C(M_6, \mathbb{C}) : |z_2| = |z_3| = 1\}\,, \tag{2.23}$$

and set $w_1 \equiv w$, with normalized norm $||w||^2 = 2$. Note that $z_1 \equiv \frac{1}{2}\overline{w_2}\cdot w_3 = z_2^* z_3^*$, hence $|z_1|^2 = 1$ and $c = 1$. The SU(2) structure is defined by the one-form w, a real two-form j and a holomorphic two-form ω, with

$$J = j + \frac{i}{2} w \wedge \overline{w}\,, \qquad \Omega = w \wedge \omega\,, \tag{2.24}$$

and w, j and ω satisfying $\iota_w j = 0 = \iota_w \omega$, $j \wedge \omega = 0$ and $j \wedge j = \frac{1}{2}\omega \wedge \overline{\omega}$.

We can now express the pure spinors

$$\phi_+^{\pm\pm} \equiv \eta_{1+}^{\pm} \overline{\eta_{2+}^{\pm}}\,, \qquad \phi_-^{\pm\pm} \equiv \eta_{1+}^{\pm} \overline{\eta_{2-}^{\pm}}\,, \tag{2.25}$$

in terms of forms:

$$\phi_+^{++} = \frac{1}{8} f_- \left[a^* \left(e^{-ij} + \frac{1}{2} w \wedge \overline{w} \wedge e^{-ij} \right) + \frac{1}{2} b z_2 (\overline{w} \wedge w \wedge \omega - 2\omega) \right]\,, \tag{2.26a}$$

$$\phi_-^{++} = \frac{1}{8} f_- \left[-a w \wedge \omega - b z_2^* w \wedge e^{-ij} \right]\,, \tag{2.26b}$$

$$\phi_+^{+-} = \frac{1}{8} \sqrt{f_+ f_-} \left[\frac{1}{2} a z_3^* (\overline{w} \wedge w \wedge \omega - 2\omega) - b z_3^* z_2^* \left(e^{-ij} + \frac{1}{2} w \wedge \overline{w} \wedge e^{-ij} \right) \right]\,, \tag{2.26c}$$

$$\phi_-^{+-} = \frac{1}{8} \sqrt{f_+ f_-} \left[-a^* z_3 w \wedge e^{-ij} + b z_3 z_2 w \wedge \omega \right]\,, \tag{2.26d}$$

$$\phi_+^{-+} = \frac{1}{8} \sqrt{f_+ f_-} \left[\frac{1}{2} a^* (w \wedge \overline{w} \wedge \overline{\omega} + 2\overline{\omega}) + b z_2 \left(e^{ij} + \frac{1}{2} w \wedge \overline{w} \wedge e^{ij} \right) \right]\,, \tag{2.26e}$$

$$\phi_-^{-+} = \frac{1}{8} \sqrt{f_+ f_-} \left[a w \wedge e^{ij} - b z_2^* w \wedge \overline{\omega}) \right]\,, \tag{2.26f}$$

$$\phi_+^{--} = \frac{1}{8} f_+ \left[a z_3^* \left(e^{ij} + \frac{1}{2} w \wedge \overline{w} \wedge e^{ij} \right) - \frac{1}{2} b z_3^* z_2^* (w \wedge \overline{w} \wedge \overline{\omega} + 2\overline{\omega}) \right]\,, \tag{2.26g}$$

$$\phi_-^{--} = \frac{1}{8} f_+ \left[-a^* z_3 w \wedge \overline{w} \wedge \overline{\omega} - b z_3 z_2 w \wedge e^{ij} \right] \tag{2.26h}$$

We also have

$$(\xi)^\flat = i\sqrt{f_+ f_-}\,(\overline{w} + z_3 w)\,, \tag{2.27a}$$

$$\tilde{\xi} = i\sqrt{f_+ f_-}\,(\overline{w} - z_3 w)\,, \tag{2.27b}$$

where $(\xi)^\flat$ is the one-form dual to the vector ξ.

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2.2 System of equations

In terms of the pure spinors $\phi_{\pm}^{\pm\pm}$ introduced in (2.25), the system of supersymmetry equations (2.14) reads

$$d_H\left[e^{2A-\phi}\phi_+^{+-}\right] - e^{A-\phi}(\phi_-^{++} + \overline{\phi_-^{--}}) = 0\,, \tag{2.28a}$$

$$d_H\left[e^{2A-\phi}(\phi_+^{++} + \phi_+^{--})\right] - 2e^{A-\phi}\mathrm{Re}(\phi_-^{+-} + \phi_-^{-+}) = 0\,, \tag{2.28b}$$

$$d_H\left[e^{2A-\phi}\phi_+^{-+}\right] - e^{A-\phi}(\overline{\phi_-^{++}} + \phi_-^{--}) = 0\,, \tag{2.28c}$$

$$d_H\left[e^{3A-\phi}\mathrm{Im}(\phi_-^{+-} - \phi_-^{-+})\right] = 0\,, \tag{2.28d}$$

$$d_H\left[e^{3A-\phi}(\phi_-^{++} - \overline{\phi_-^{--}})\right] - 3e^{2A-\phi}(\phi_+^{+-} - \overline{\phi_+^{-+}}) = 0\,, \tag{2.28e}$$

$$d_H\left[e^{A-\phi}\mathrm{Re}(\phi_-^{+-} - \phi_-^{-+})\right] - e^{-\phi}\mathrm{Re}(\phi_+^{++} - \phi_+^{--}) = \frac{1}{4}e^{A} f F\,, \tag{2.28f}$$

and

$$d_H\left[e^{3A-\phi}\mathrm{Im}(\phi_-^{+-} + \phi_-^{-+})\right] - 3e^{2A-\phi}\mathrm{Im}(\phi_+^{++} + \phi_+^{--}) = \frac{1}{4}e^{4A} \star\lambda(F)\,, \tag{2.29a}$$

$$d_H\left[e^{-\phi}(\phi_+^{++} - \phi_+^{--})\right] = \frac{i}{8}(\bar{\tilde{\xi}}\wedge + \iota_\xi)F\,, \tag{2.29b}$$

$$d_H\left[e^{4A-\phi}(\phi_+^{++} - \phi_+^{--})\right] - 4ie^{3A-\phi}\mathrm{Im}(\phi_-^{+-} - \phi_-^{-+}) = -\frac{1}{8}(\bar{\tilde{\xi}}\wedge + \iota_\xi)e^{4A}\star\lambda(F)\,. \tag{2.29c}$$

We also have

$$\mathrm{Im}(\xi) = 0\,, \tag{2.30a}$$

$$d(e^A f) + \frac{1}{2}\mathrm{Im}(\tilde{\xi}) = 0\,, \tag{2.30b}$$

$$d\tilde{\xi} - i_\xi H = 0\,, \tag{2.30c}$$

$$\nabla_{(n}\xi_{m)} = 0\,, \tag{2.30d}$$

which were obtained from (2.1b) and the condition that the ten-dimensional vector K is Killing.

We are showing (2.29) separately because they are in fact implied by (2.28). Even if they are redundant, (2.29a) and (2.29b) are useful to show that the equations of motion and the Bianchi identities of the R-R fields are automatically satisfied; see also our comment after (2.15).

Acting with d_H on (2.29a), and using the imaginary part of (2.28b) it follows that

$$d_H(e^{4A}\star\lambda(F)) = 0\,, \tag{2.31}$$

which are the equations of motion. Acting with d_H on (2.28f), using (2.30b), and subtracting the real part of (2.29b), it follows that

$$d_H F = 0\,, \tag{2.32}$$

which are the Bianchi identities of the R-R fields. This holds under the assumption that the Bianchi identity for H, $dH = 0$, is satisfied. Although it is not immediately obvious, we shall see that the NS-NS Bianchi identity is in fact implied by the supersymmetry equations.

3 Analysis of the supersymmetry equations

In this section we analyze the supersymmetry equations obtained in section 2. As we anticipated, not all the equations are independent, and we will be able to reduce them to a significantly smaller set which characterizes the SU(2) structure on the internal manifold.

We will distinguish two cases. This is because certain equations, such as the zero-form component of (2.28c), have an overall factor of b, and can thus be solved either by setting $b = 0$ or by keeping $b \neq 0$ and setting to zero the remaining factor. It turns out that these two cases are qualitatively different, and we will consider them in separate subsections.

We will refer to the first case as "Class K" and the second one as "Class HK", because in these two cases M_6 will turn out to contain respectively a Kähler and a hyper-Kähler four-dimensional submanifold.

3.1 Class K

In this section we look at the case $b = 0$. The condition (2.30a), $\mathrm{Im}(\xi) = 0$, fixes $z_3 = -1$, while (2.30b) gives

$$d(e^A f) = -\sqrt{f_+ f_-}\mathrm{Re}w\,. \tag{3.1}$$

We define $y \equiv e^A f$, which we will use as a coordinate, so that

$$\mathrm{Re}w = -\frac{1}{\sqrt{f_+ f_-}}dy\,, \tag{3.2}$$

where now $f_+ f_- = e^{2A} - f^2 = e^{2A} - e^{-2A}y^2$. We will also introduce a coordinate ψ, adapted to the Killing vector as

$$\xi = 4\partial_\psi\,. \tag{3.3}$$

From (2.27a) it follows that

$$\mathrm{Im}w = \frac{1}{2}\sqrt{f_+ f_-}(d\psi + \rho)\,, \tag{3.4}$$

where ρ is a one-form on the four-dimensional subspace orthogonal to w.

The zero-form component of (2.28f) yields

$$\mathrm{Re}a = -e^{-A+\phi}yF_0\,, \tag{3.5}$$

while the one-form part of (2.28a)–(2.28e) give

$$d(e^{3A-\phi}\mathrm{Im}a) = 0\,. \tag{3.6}$$

We can thus write

$$a = e^{-A+\phi}(-yF_0 + ie^{-2A}\ell)\,, \qquad \ell = \text{constant}\,. \tag{3.7}$$

Note that from the above expression and (2.21), which for $b = 0$ yields $|a|^2 = 1$, it follows that F_0 and ℓ cannot be simultaneously zero.

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Given the above we find that the two-form part of (2.28a)–(2.28e) is automatically satisfied, while the three-form part yields:

$$F_0 d(e^{2A} y j - y^2 \text{Re}w \wedge \text{Im}w) + \ell H = 0\,, \tag{3.8a}$$
$$\ell d(e^{-2A} y^{-1} j - y^{-2} \text{Re}w \wedge \text{Im}w) - F_0 H = 0\,, \tag{3.8b}$$
$$d(e^{2A-\phi}\sqrt{f_+ f_-}\, a\omega) + 2e^{-\phi}(-y\text{Re}w + ie^{2A}\text{Im}w) \wedge a\omega = 0\,. \tag{3.8c}$$

We can combine the first two of the above equations so as to obtain one which does not involve the NS-NS field strength H:

$$d\left[(F_0^2 e^{2A} y + \ell^2 e^{-2A} y^{-1}) j - (F_0^2 y^2 + \ell^2 y^{-2})\text{Re}w \wedge \text{Im}w\right] = 0\,. \tag{3.9}$$

As pointed out earlier, F_0 and ℓ cannot be simultaneously zero, and hence when either of the two is, it follows from (3.8a) and (3.8b) that $H = 0$.

We will proceed by making a $2+4$ split of the internal manifold, with coordinates $\{y,\, \psi\}$ on the two-dimensional subspace. The differential operator is decomposed as

$$d = dy \wedge \partial_y + d\psi \wedge \partial_\psi + d_4\,, \tag{3.10}$$

and the metric takes the form

$$ds^2_{M_6} = \frac{1}{e^{2A} - e^{-2A} y^2} dy^2 + \frac{1}{4}(e^{2A} - e^{-2A} y^2)(d\psi + \rho)^2 + g^{(4)}_{ij}(y, x^i) dx^i dx^j\,, \tag{3.11}$$

with x^i, $i = 1, 2, 3, 4$ coordinates on the four-dimensional subspace, M_4.

We can now decompose the three-form equations (3.9) and (3.8c):

$$\begin{aligned}
&\partial_\psi j = 0\,, && \partial_y(y^{-1}|\hat{a}|^2 j) = \frac{1}{2}(F_0^2 y^2 + \ell^2 y^{-2}) d_4\rho\,, && d_4(y^{-1}|\hat{a}|^2 j) = 0\,,\\
&\partial_\psi \omega = -i\omega\,, && \partial_y(\sqrt{f_+ f_-}\,\hat{a}\omega) = -2\frac{e^{-2A} y}{\sqrt{f_+ f_-}}\hat{a}\omega\,, && d_4(\sqrt{f_+ f_-}\,\hat{a}\omega) = -i\rho \wedge \sqrt{f_+ f_-}\,\hat{a}\omega\,,
\end{aligned} \tag{3.12}$$

where we have introduced

$$\hat{a} \equiv e^{2A-\phi} a = -F_0 e^A y + ie^{-A}\ell\,. \tag{3.13}$$

To further analyze the above equations it is convenient to rescale the data of the four-dimensional base as follows:

$$j = y|\hat{a}|^{-2}\hat{j}\,, \qquad \omega = y|\hat{a}|^{-2} e^{-i(\psi+\theta)}\hat{\omega}\,, \qquad g^{(4)}_{ij} = y|\hat{a}|^{-2}\hat{g}^{(4)}_{ij}\,, \tag{3.14}$$

where $\theta \equiv \arg(\hat{a})$. Then (3.12) becomes

$$\begin{aligned}
&\partial_\psi \hat{j} = 0\,, && \partial_y \hat{j} = \frac{1}{2}(F_0^2 y^2 + \ell^2 y^{-2}) d_4\rho\,, && d_4 \hat{j} = 0\,,\\
&\partial_\psi \hat{\omega} = 0\,, && \partial_y \hat{\omega} = -\frac{1}{2}(F_0^2 y^2 + \ell^2 y^{-2}) T\hat{\omega}\,, && d_4\hat{\omega} = i\hat{P} \wedge \hat{\omega}\,,
\end{aligned} \tag{3.15}$$

where

$$\hat{P} \equiv -\rho + i\frac{2e^{4A}(\ell^2 + F_0^2 y^4)}{(e^{4A} - y^2)(\ell^2 + F_0^2 e^{4A} y^2)} d_4 A\,, \tag{3.16a}$$

$$T \equiv \frac{\partial_y (e^{4A} y^2)}{(e^{4A} - y^2)(\ell^2 + F_0^2 e^{4A} y^2)}\,. \tag{3.16b}$$

The last condition, $d_4\hat{\omega} = i\hat{P} \wedge \hat{\omega}$, suggests that the almost complex structure defined by $\hat{\omega}$ is independent of ψ and y and integrable on the four-dimensional subspace M_4, i.e. the latter is a complex manifold. In addition, $d_4\hat{\jmath} = 0$, and thus $\hat{g}^{(4)}$ is a family of Kähler metrics parametrized by y. Furthermore, $\hat{P}$ is the canonical Ricci form connection defined by the Kähler metric with the Ricci form $\hat{\mathfrak{R}} = d_4\hat{P}$.

It is worth noting the similarity of the SU(2) structure we are studying here with the one that characterizes $\mathcal{N} = 1$ supersymmetric AdS_5 solutions of M-theory, studied in [18].[3] This close resemblance allows us to draw upon certain results of the latter reference.

There are certain identities and conditions that derive from the system (3.15), to which we now turn. The equation for $\partial_y\hat{\omega}$ determines the dependence of the volume of M_4 on y:

$$\partial_y \log\sqrt{\hat{g}^{(4)}} = (F_0^2 y^2 + \ell^2 y^{-2})T\,. \tag{3.17}$$

Given that the complex structure is independent of y the following identity holds:

$$(\partial_y \hat{\jmath})^+ = \frac{1}{2}\partial_y \log\sqrt{\hat{g}^{(4)}}\,\hat{\jmath}\,, \tag{3.18}$$

where a plus superscript denotes the self-dual part of a two-form on M_4. Combining with the second equation of (3.15) we arrive at

$$(d_4\rho)^+ = -T\hat{\jmath}\,. \tag{3.19}$$

Finally, the restrictions below hold as consequences of (3.15):

$$d_4\rho \wedge \hat{\omega} = 0\,, \qquad \left[\frac{8y}{(e^{2A} - e^{-2A}y^2)^2} d_4 A + i\partial_y\rho\right] \wedge \hat{\omega} = 0\,. \tag{3.20}$$

The four- five- and six-form parts of (2.28a)–(2.28e) are automatically satisfied given the conditions we have derived so far.

Let us now look at the rest of the fields. The dilaton is determined by (3.7) and the condition $|a|^2 = 1$ descending from (2.21):

$$e^{2\phi} = \frac{e^{2A}}{y^2 F_0^2 + e^{-4A}\ell^2}\,. \tag{3.21}$$

The NS-NS field strength H is given by either (3.8a) or (3.8b), and its Bianchi identity $dH = 0$ is manifestly satisfied. The R-R fields are determined by (2.28f) and are given by

[3] A similar resemblance occurs in the study of AdS_3 solutions [25].

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the expressions

$$F_2 = \ell\left(\alpha_2 + e^{-2A}y^{-1}j\right)\,, \tag{3.22a}$$

$$F_4 = -F_0\left(e^{2A}yj + \frac{1}{2}y^2 dy\wedge D\psi\right)\wedge\alpha_2 - \frac{1}{2}F_0 j\wedge j\,, \tag{3.22b}$$

$$F_6 = -3\ell e^{-4A}\mathrm{vol}_6\,, \tag{3.22c}$$

where we have introduced the auxiliary two-form

$$\alpha_2 \equiv -\frac{1}{2}d\left(e^{-4A}y(d\psi+\rho)\right) + \frac{1}{2}y^{-1}d\rho \tag{3.23}$$

and $\mathrm{vol}_6 = \mathrm{Re}w\wedge\mathrm{Im}w\wedge\frac{1}{2}j\wedge j$.

For future reference, let us also note the following B-twisted fluxes $\tilde{F}\equiv e^{-B}F$, which will play a role when we examine flux quantization. This necessitates differentiating between the cases with the constants ℓ, F_0 either generic or vanishing. We will consider the twisted fluxes only for the generic case with $F_0\neq 0$, $\ell\neq 0$. Local expressions for the NS-NS potential B are easily read off from (3.8a) or (3.8b),

$$B_1 = -\frac{F_0}{\ell}\left(e^{2A}yj + \frac{1}{2}y^2 dy\wedge D\psi\right)\,, \tag{3.24a}$$

$$B_2 = \frac{\ell}{F_0}\left(e^{-2A}y^{-1}j + \frac{1}{2}y^{-2}dy\wedge D\psi\right)\,. \tag{3.24b}$$

While B_2 leads to shorter expressions for the remaining potentials, it has a singularity at $y=0$; we will thus work with B_1. We thus find the following expressions

$$\tilde{F}_2 = \frac{\ell}{2}d\left(\left(y^{-1}-e^{-4A}y\right)D\psi\right) - F_0(B_1-B_2)\,, \tag{3.25a}$$

$$\tilde{F}_4 = \frac{1}{2}\frac{F_0}{\ell^2}\left(\frac{e^{4A}y^2}{F_0^2e^{4A}y^2+\ell^2}\hat{\jmath}\wedge\hat{\jmath} + y^2 dy\wedge D\psi\wedge\hat{\jmath}\right)\,, \tag{3.25b}$$

$$\tilde{F}_6 = \frac{y^2}{4\ell^3}\frac{F_0^2e^{4A}y^2+3\ell^2}{F_0^2e^{4A}y^2+\ell^2}dy\wedge D\psi\wedge\hat{\jmath}\wedge\hat{\jmath}\,, \tag{3.25c}$$

where explicitly $\tilde{F}_4\equiv F_4 - B_1\wedge F_2 + \frac{1}{2}F_0B_1\wedge B_1$ and $\tilde{F}_6\equiv F_6 - B_1\wedge F_4 + \frac{1}{2}B_1^2\wedge F_2 - \frac{1}{6}F_0B_1^3$.

We will come back to these expressions in section 4.

3.2 Class HK

In this section we look at the case $b\neq 0$. We find that in contrast to Class K this class of solutions is rather restricted and determined up to constant parameters.

The condition (2.30a), $\mathrm{Im}(\xi) = 0$, fixes $z_3 = -1$, while (2.30b) gives $d(e^Af) = -\sqrt{f_+f_-}\mathrm{Re}w$. Similar to section 3.1, we define the coordinate $y\equiv e^Af$ by

$$\mathrm{Re}w = -\frac{1}{\sqrt{f_+f_-}}dy\,, \tag{3.26}$$

where once again $f_+f_- = e^{2A} - f^2 = e^{2A} - e^{-2A}y^2$, and a coordinate ψ such that $\xi = 4\partial_\psi$. From (2.27a) it follows that

$$\mathrm{Im}w = \frac{1}{2}\sqrt{f_+f_-}(d\psi + \rho)\,, \tag{3.27}$$

where ρ is a one-form on the four-dimensional subspace orthogonal to w.

The zero-form component of (2.28f) yields $\mathrm{Re}a = -e^{-A+\phi}yF_0$ while the one-form component of (2.28b) gives $d(e^{3A-\phi}\mathrm{Im}a) = 0$. We can thus write

$$a = e^{-A+\phi}(-yF_0 + ie^{-2A}\ell), \qquad \ell = \text{constant}\,. \tag{3.28}$$

So far things are akin to section 3.1.

From now on, however, analysis of the rest of the equations (2.28) puts strong constraints on the SU(2) structure and the functions that determine the solution. We find:[4]

$$\rho = d\varphi\,, \qquad dj = 0\,, \qquad d(e^{i(\psi+\varphi)}\omega) = 0\,, \tag{3.29}$$

where φ is defined via $z_2 = e^{i(\varphi+\psi)}$ and satisfies $\partial_y\varphi = 0 = \partial_\psi\varphi$. We thus conclude that the four-dimensional base of the internal manifold is hyper-Kähler[5] and its metric is independent of y. Also, the connection of the fibration of the U(1) isometry generated by ξ over the base is flat. Furthermore, for the warp factor and the dilaton we find

$$e^A = Ly^{-1/2}\,, \qquad e^\phi = g_s y^{-3/2}\,, \tag{3.30}$$

where L and g_s are constants. b is also constant and is fixed by the relation $|a|^2 + b^2 = 1$ which becomes $L^{-2}g_s^2(F_0^2 + L^{-4}\ell^2) + b^2 = 1$.

The metric on the internal manifold, after a coordinate transformation $y = L\cos^{1/2}(\alpha)$, reads

$$ds^2_{M_6} = \frac{1}{4}e^{2A}(d\alpha^2 + \sin^2(\alpha)D\psi^2) + ds^2_{\mathrm{HK}}(x), \qquad e^{2A} = \frac{L}{\cos^{1/2}(\alpha)}\,. \tag{3.31}$$

Here $D\psi \equiv d\psi + \rho$ and $ds^2_{\mathrm{HK}}(x)$ is the line element on the hyper-Kähler base, with x denoting its coordinates.

Turning to the rest of the fields, the NS-NS field strength H is zero, while the R-R fields can be read from (2.28f). Their expressions are:[6]

$$F_2 = -\frac{1}{2}\frac{\ell}{L}d\left(\cos^{3/2}(\alpha)\right)\wedge D\psi + \frac{\ell}{L^2}j + Le^{-\phi_0}b\mathrm{Re}\omega\,, \tag{3.32a}$$

$$F_4 = \frac{1}{2}Ld\left(\cos^{3/2}(\alpha)\right)\wedge D\psi\wedge\left(F_0 j - Le^{-\phi_0}b\mathrm{Im}\omega\right) - \frac{1}{2}F_0 j\wedge j\,, \tag{3.32b}$$

$$F_6 = -3\frac{\ell}{L^2}\cos\alpha\mathrm{vol}_6\,, \tag{3.32c}$$

where $\mathrm{vol}_6 = \mathrm{Re}w\wedge\mathrm{Im}w\wedge\frac{1}{2}j\wedge j$.

[4]In particular, these constraints are implied by the remaining one-form constraint and the three-form constraints. The two-, four-, five- and six-form constraints are trivially satisfied.

[5]The phase $e^{i(\psi+\varphi)}$ can be absorbed in ω.

[6]The ω appearing here is the one following the redefinition $e^{i(\psi+\varphi)}\omega \to \omega$.

4 Class K: complex Ansatz

In this section we explore an Ansatz for the Class K of solutions consisting of

$$d_4 A = 0\,, \qquad \partial_y \rho = 0\,. \tag{4.1}$$

This Ansatz is equivalent to requiring that the holomorphic three-form $\Omega = w \wedge \omega$ that characterizes the SU(3) structure on the internal space M_6 satisfies $d\Omega = V \wedge \Omega$ for a one-form V, which in turn is equivalent to requiring that M_6 is a complex manifold.

From (4.1) and the definition (3.16a) it follows that $\rho = -\hat{P}$. The Ricci form $\hat{\mathfrak{R}} = d_4 \hat{P}$ then reads $\hat{\mathfrak{R}} = -d_4\rho$, and the second equation of (3.15) and (3.19) can be rewritten as

$$\hat{\mathfrak{R}} = -\frac{2}{F_0^2 y^2 + \ell^2 y^{-2}} \partial_y \hat{\jmath}\,, \tag{4.2a}$$

$$\hat{\mathfrak{R}}^+ = T\hat{\jmath}\,. \tag{4.2b}$$

From the above it can be inferred [18] that the Ricci tensor on M_4, at fixed y, has two pairs of constant eigenvalues. For compact M_4, which is the case of interest, we can invoke [26] stating (under the assumption that the Goldberg conjecture is true) that a compact Kähler four-manifold whose Ricci tensor has two distinct pairs of constant eigenvalues is locally the product of two Riemann surfaces of constant curvature. If the two pairs of eigenvalues are the same, then by definition the manifold is Kähler-Einstein. There are thus two classes to consider: either M_4 is Kähler-Einstein or is the product of two Riemann surfaces.

4.1 Kähler-Einstein base

In this class

$$\hat{\mathfrak{R}} = \frac{\kappa}{Q(y)}\hat{\jmath}\,. \tag{4.3}$$

with $\kappa = 0$ or $\kappa = \pm 1$. The case $\kappa = 0$, corresponds to M_4 being hyper-Kähler and turns out to be the $b = 0$ limit of the Class HK of solutions we examined in the previous section. We will thus restrict to $\kappa = \pm 1$. The dependence of the metric of M_4 on y is given by

$$\hat{g}^{(4)}(y, x^i) = Q(y) g_{\mathrm{KE}_4}(x^i)\,, \tag{4.4}$$

where g_{KE_4} is a Kähler–Einstein metric of constant curvature $R = 4\kappa$.

When combined with (4.2a), and the fact that $\partial_y \hat{\mathfrak{R}} = -d_4 \partial_y \rho = 0$ which is part of the Ansatz (4.1), the condition (4.3) fixes

$$\frac{Q}{\kappa} = \frac{1}{6}(3\ell^2 y^{-1} - F_0^2 y^3) + \nu\,, \qquad \nu = \text{constant}\,. \tag{4.5}$$

In combination with (4.2b), (4.3) gives the ordinary differential equation (ODE) $T = \kappa/Q$, which determines the warp factor A. Given the expression (3.16b) for T and defining

$$p(y) \equiv e^{4A} y^2\,, \tag{4.6}$$

this becomes a Riccati:

$$y^2 \frac{Q}{\kappa}\frac{dp}{dy} = F_0^2 p^2 + (\ell^2 - F_0^2 y^4)p - \ell^2 y^4\,. \tag{4.7}$$

We were able to solve this Riccati equation analytically:

$$p = \ell^2 y^2 \frac{3\ell^2\mu - 9\ell^2 y^2 - 12\nu y^3 + F_0^2 y^6}{9\ell^4 + 36\ell^2\nu y + (36\nu^2 - 3\ell^2 F_0^2\mu)y^2 + 3\ell^2 F_0^2 y^4}\,, \tag{4.8}$$

where μ is a constant parameter. Note that in this parametrization the limit $\ell \to 0$ is not well-defined since the solution becomes trivial. The $\ell \to 0$ limit is well-defined after shifting $\mu \to 12\nu^2/(F_0^2\ell^2) + \mu$.

4.1.1 Regularity and boundary conditions

We now turn to the analysis of the geometry of the solutions, which we will carry out in terms of a rescaled coordinate $x \propto y$. We will specify the constant rescaling factor later on, for the cases (i) $F_0 \neq 0$ and $\ell \neq 0$ (generic), (ii) $\ell = 0$, and (iii) $F_0 = 0$, separately.

The metric (3.11) on the internal manifold takes the form:

$$e^{-2A} ds^2_{M_6} = -\frac{1}{4}\frac{q'}{xq}dx^2 - \frac{q}{xq' - 4q}D\psi^2 + \frac{\kappa q'}{3q' - xq''}ds^2_{\mathrm{KE}_4}\,, \tag{4.9a}$$

where $q = q(x)$ is a polynomial (of degree 6 if $F_0 \neq 0$), and a prime denotes differentiation. The warp factor is given by

$$L^{-2}e^{2A} = \sqrt{\frac{x^2 q' - 4xq}{q'}}\,. \tag{4.9b}$$

The dilaton is given by

$$g_s^{-2}e^{2\phi} = \frac{xq'}{(3q' - xq'')^2}\left(\frac{x^2q' - 4xq}{q'}\right)^{3/2}. \tag{4.9c}$$

L and g_s are two integration constants which we will specify in terms of the constants appearing in p later on.

Positivity of the metric and the dilaton requires

$$q < 0\,, \qquad xq' > 0\,, \qquad \kappa(3xq' - x^2q'') > 0\,. \tag{4.10}$$

These conditions will only be realized on an interval of x. What happens to q at an endpoint x_0 of this interval dictates the physical interpretation of the solution around that point. We summarize our conclusions in table 1. For example, we see from there that if q has a simple zero at a point $x_0 \neq 0$, the S^1 parametrized by ψ shrinks in such a way as to make the geometry regular, provided that the periodicity $\Delta\psi$ is chosen to be 2π. If this happens at both endpoints of the interval, the solution is fully regular.

Here are some details about each of these cases.

Simple zero: regular endpoint. Near a simple zero $x = x_0$ of q, the warp factor and dilaton go to constants, while the internal metric behaves as

$$\frac{1}{L^2|x_0|}ds^2_{M_6} \sim \frac{1}{x_0}\left(\frac{1}{4}\frac{dx^2}{x_0 - x} + (x_0 - x)D\psi^2\right) + \frac{\kappa q_0'}{3q_0' - x_0 q_0''}ds^2_{\mathrm{KE}_4}\,, \tag{4.11}$$

x_0	$q(x_0)$	$q'(x_0)$	$q''(x_0)$	interpretation
	0			regular
		0		O4
	0	0	0	conical CY
0		0	0	O8

Table 1. Various boundary conditions for the polynomial q at an endpoint x_0, and their interpretation. Empty entries are meant to be non-zero.

where $q_0' \equiv q'(x_0)$, $q_0'' \equiv q''(x_0)$ (x_0 will never be zero). Positivity of the metric requires that if $x_0 > 0 \Rightarrow x < x_0$, and if $x_0 < 0 \Rightarrow x > x_0$. Choosing for definiteness the first case, and introducing $r = \sqrt{x_0 - x}$ we see that the parenthesis in (4.11) becomes $dr^2 + r^2 D\psi^2$, which is the metric of $\mathbb{R}^2$ (fibred over the Kähler-Einstein base), with the condition that the periodicity of ψ is taken to be $\Delta\psi = 2\pi$.

Extremum: O4-plane. At a point $x_0 \neq 0$ where $q'(x_0) = 0$, the ten-dimensional metric and dilaton behave as

$$\frac{1}{2L^2}\sqrt{-\frac{q_0''}{x_0 q_0}}ds^2 \sim \frac{1}{\sqrt{x-x_0}}\left(ds^2_{\mathrm{AdS}_4}+\frac{1}{4}D\psi^2\right)+\sqrt{x-x_0}\left(-\frac{q_0'' dx^2}{4x_0 q_0}-\frac{\kappa}{x_0}ds^2_{\mathrm{KE}_4}\right), \tag{4.12}$$

$$g_s^{-2}e^{2\phi} \sim \frac{1}{x_0 q_0''}\left(\frac{-4x_0 q_0}{q_0''}\right)^{3/2}\frac{1}{\sqrt{x-x_0}}, \tag{4.13}$$

where $q_0 \equiv q(x_0)$. Positivity of the metric and the dilaton requires that if $x_0 q_0'' > 0 \Rightarrow x > x_0$, and if $q_0'' x_0 < 0 \Rightarrow x < x_0$ (in the above equation we have recorded the first case). It also requires $\kappa q_0'' < 0$. One recognizes the usual structure $H^{-1/2}ds^2_\parallel + H^{1/2}ds^2_\perp$ for extended objects, with $H \sim x - x_0$. Since there are five parallel directions, this signals the presence of a four-dimensional object; the fact that the function is linear matches with the behavior of an O4-plane near the point where its harmonic function goes to zero. The dilaton matches the behavior $e^{2\phi} \propto H^{(3-p)/2}$ of an Op-plane again for $p = 4$. Thus we conclude that this singularity corresponds to the presence of an O4-plane extended along AdS_4.

We should also point out, however, that the local structure of the singularity does not clarify if the orientifold is smeared over KE_4. Suppose one places a fully localized O4-plane at the tip of a cone $C(Y_4)$ of metric $dx^2 + x^2 ds^2_{Y_4}$. Near the tip, the backreacted metric is then of the form

$$ds^2 = H_{\mathrm{O4}}^{-1/2}ds^2_\parallel + H_{\mathrm{O4}}^{1/2}(dx^2 + x^2 ds^2_{Y_4}), \qquad H_{\mathrm{O4}} = 1 - \left(\frac{x_0}{x}\right)^3. \tag{4.14}$$

On the other hand, an O4-plane that is partially smeared along a four-dimensional manifold Y_4 would have a metric

$$ds^2 = H_{\mathrm{smO4}}^{-1/2}ds^2_\parallel + H_{\mathrm{smO4}}^{1/2}(dx_9^2 + ds^2_{Y_4}), \qquad H_{\mathrm{smO4}} = a + b|x_9|; \tag{4.15}$$

since H_{smO4} is now a harmonic function of one dimension only, it is piecewise linear.

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The metric (4.14) ceases to make sense at $x = x_0$. Expanding around this point, $H_{O4} \sim (x - x_0)$, and we would obtain (4.12) (up to constants that can be reabsorbed). On the other hand, (4.15) for $a = 0$ also gives (4.12), upon identifying $x - x_0 = |x_9|$. In this sense, it is not entirely clear if (4.12) should be considered as smeared over KE_4 or not. (Such considerations also apply to Op-planes for $p \neq 4$.) For $Y_4 = S^4$, (4.14) is the simplest interpretation; for Y_4 a Kähler-Einstein manifold, the singularity $C(Y_4)$ would be bad (in that for example it would not be Ricci-flat, as one would expect before placing an object on it), and (4.15) seems the simplest interpretation. We thus conclude (4.12) is an O4-plane that is smeared over KE_4.

Of course smeared orientifolds have rather limited physical validity; nevertheless, for completeness, we will include them in our analysis.

Triple zero: conical Calabi-Yau singularity. Near a triple zero $x = x_0$ of q, the warp factor and dilaton go to constants, while the internal metric behaves as

$$\frac{1}{L^2|x_0|} ds^2_{M_6} \sim \frac{3}{x_0} \left[\frac{1}{4} \frac{dx^2}{x_0 - x} + (x_0 - x) \left(\frac{1}{9} D\psi^2 + \frac{\kappa}{6} ds^2_{\mathrm{KE}_4} \right) \right] . \tag{4.16}$$

Positivity works as in the case of a simple zero, but it also requires $\kappa = 1$. Upon introducing $r = \sqrt{x_0 - x}$, (4.16) becomes proportional to

$$dr^2 + r^2 \left(\frac{1}{9} D\psi^2 + \frac{1}{6} ds^2_{\mathrm{KE}_4} \right) . \tag{4.17}$$

The metric in parenthesis is a regular Sasaki-Einstein metric, built as U(1) bundle over the Kähler-Einstein base (KE_4). Thus (4.17) represents a conical Calabi-Yau singularity. In the particular case that KE_4 is $\mathbb{CP}^2$, this is in fact an orbifold singularity.

We can be a little more precise. $d(D\psi)$ is the Ricci form of KE_4, which in de Rham cohomology represents the first Chern class c_1. The integral of the latter over the two-cycles $\mathcal{C}_i$ of KE_4 is $2\pi n_i$, $n_i \in \mathbb{Z}$. If the periodicity of the S^1 coordinate is $\Delta\psi = 2\pi$, the total space is the U(1) bundle associated to the canonical bundle over KE_4. In that case the conical singularity (4.17) is the complex cone over KE_4. If $\gcd(n_i) \geq 1$, there is also the possibility of taking the periodicity to be $2\pi \times \gcd(n_i)$.

For example, if $\mathrm{KE}_4 = \mathbb{CP}^2$, there is only one cycle $\mathcal{C}$ with $n = 3$; taking $\Delta\psi = 2\pi$ gives the orbifold singularity

$$\mathbb{C}^3/\mathbb{Z}_3 \,, \tag{4.18}$$

but one can also consider $\Delta\psi = 6\pi$, for which (4.17) in fact becomes the fully regular space $\mathbb{C}^3$. This possibility is not available if at the other endpoint one has a single zero, where $\Delta\psi$ is necessarily 2π. However, as we will see, in one case there is a triple zero is present at both endpoints, and in that case $\Delta\psi = 6\pi$ is possible. This will correspond to the Guarino-Jafferis-Varela (GJV) solution [11].

If $\mathrm{KE}_4 = \mathbb{CP}^1 \times \mathbb{CP}^1$, there are two $\mathcal{C}_i$ with $n_i = 2$. With $\Delta\psi = 2\pi$, (4.17) becomes a $\mathbb{Z}_2$ quotient of the conifold singularity; one also has the possibility of taking $\Delta\psi = 4\pi$, for which one obtains the original conifold singularity. Again this option is only available if a triple zero appears at both endpoints. We will see later, when considering the product base class in section 4.2, that this case has in fact a richer array of possibilities.

Inflection point at the origin: O8-plane. When $q'(0) = q''(0) = 0$, around $x = 0$, the metric and dilaton behave as

$$\frac{1}{L^2}\sqrt{-\frac{q_3}{8q_0}}ds^2_{10} \sim \frac{1}{\sqrt{x}}\left(ds^2_{\mathrm{AdS}_4} + \frac{1}{4}D\psi^2 + \kappa ds^2_{\mathrm{KE}_4}\right) - \frac{q_3}{8q_0}\sqrt{x}\,dx^2\,, \tag{4.19}$$

$$g_s^{-2}e^{2\phi} \sim \frac{2}{q_3}\left(-\frac{8q_0}{q_3}\right)^{3/2} x^{-5/2}\,. \tag{4.20}$$

where $q_0 \equiv q(0)$ and $q_3 \equiv q^{(3)}(0)$. This time we recognize an O8-plane localized at $x = 0$.

4.1.2 Generic case

We now turn to examining the parameter space of solutions for the generic case, by which we mean $F_0 \neq 0$ and $\ell \neq 0$. The rescaling of the coordinate y that we mentioned at the beginning of 4.1.1 is

$$x = \sqrt{\frac{F_0}{\sqrt{3}\ell}}y\,. \tag{4.21}$$

We will also rescale the constant parameters that appeared in (4.8) and introduce

$$\beta = \frac{F_0}{2\sqrt{3}\ell}\mu\,, \qquad \gamma = \frac{1}{\ell^2}\sqrt{\frac{\sqrt{3}\ell}{F_0}}\nu\,. \tag{4.22}$$

The polynomial q that determines the solution then reads

$$q = x^6 + 3(2\gamma^2 - \beta)x^4 + 8\gamma x^3 + 3x^2 - \beta\,, \tag{4.23}$$

while the constants L, g_s introduced in the previous section are:

$$L^4 = \frac{\sqrt{3}\ell}{F_0}\,, \qquad g_s^2 = \frac{3^{7/4}\cdot 8}{\sqrt{\ell F_0^3}}\,. \tag{4.24}$$

For later use, we note that the Riccati ODE (4.7) implies an ODE directly on the polynomial q:

$$\frac{1}{24}(3q' - xq'')^2 = (xq' - 4q)(1 + 3x^4) + 4q\,. \tag{4.25}$$

It is also useful to notice that in this case

$$3q' - xq'' = 12x(1 + 2\gamma x - x^4)\,. \tag{4.26}$$

We now have to study for which values of the parameters β, γ the positivity conditions (4.10) are satisfied. First of all, in order for $q < 0$ we need to require $\beta > 0$. We then have to subdivide this region according to the nature of the zeros of q, and the presence of maxima or minima. To identify these subregions, it is useful to look at the discriminant of q:

$$\Delta(q) = 2^{10}3^6\left(\beta^4 - 6\gamma^2\beta^3 - (2 - 12\gamma^4)\beta^2 - \gamma^2(10 + 8\gamma^4)\beta + \gamma^4 + 1\right)^2\beta\,. \tag{4.27}$$

For every β, $\Delta(q) = 0$ has two solutions $\beta = \beta_\pm$, $\beta_- \leq \beta_+$. (This can be seen by considering the discriminant of $\Delta(q)$ with respect to β, which is always negative.) Notice that

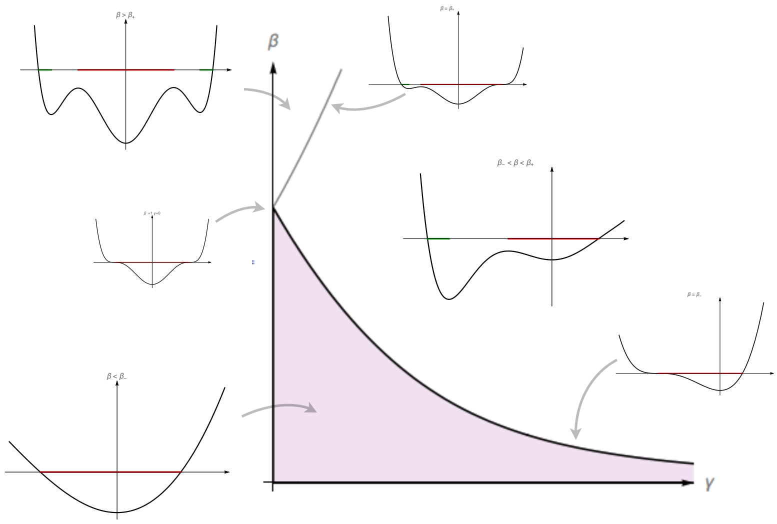

Figure 1. Plots of $q(x)$ corresponding to various regions of parameter space.

$\beta\Delta(q')^2 \propto \Delta(q)$, and that $\mathrm{res}(q, q'')$ divides $\Delta(q')$; this implies a double zero is in fact also a triple zero. This also follows from (4.25).

There are six different cases, which we discuss in turn and describe in figure 1. For definiteness, we will consider $\gamma \geq 0$; the discussion for $\gamma \leq 0$ is similar.

- For $\beta < \beta_-$, q has two simples zeros $x_\pm$; in the interval $[x_-, x_+]$, the conditions (4.10) are met with $\kappa = +1$. According to our discussion in section 4.1.1, at both simple zeros the S^1 circle shrinks in such a way that the solution is regular. Thus the internal space is smooth and the solution is fully regular.

  The solutions previously found numerically fall in this region. The first to be found were the ones in [7], which should correspond to $\gamma = 0$, $\beta \leq 1$, with $\mathrm{KE}_4 = \mathbb{CP}^2$. In [8] it was later suggested that the $\mathbb{CP}^2$ could be replaced by an arbitrary regular KE_4. This was worked out explicitly in [10] for $\mathrm{KE}_4 = \mathbb{CP}^1 \times \mathbb{CP}^1$; our regular solutions corresponds to the $q_1 = \tilde{q}_1$ slice in [10, figure 3]. (We will see later how the rest of that figure is reproduced.)

- At $\beta = \beta_-$, the discriminant $\Delta(q) = 0$, and as we remarked also $\Delta(q') = 0$; so the simple zero x_- becomes a triple zero. As we discussed in section 4.1.1, this means that one of the regular points becomes a conical Calabi-Yau singularity; if $\mathrm{KE}_4 = \mathbb{CP}^2$, this is a $\mathbb{Z}_3$ singularity.

- For $\beta_- < \beta < \beta_+$, the triple zero x_- splits into a single zero x_-, a local minimum x_1 and a local maximum x_2 ($x_- < x_1 < x_2$). Now the positivity conditions (4.10) are met for $\kappa = +1$ between the maximum and the zero: $x \in [x_2, x_+]$. However, a new possibility also appears: for $\kappa = -1$, the positivity conditions are met in the interval $x \in [x_-, x_1]$. Both of these correspond to solutions with a single O4-plane singularity.

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- At $\beta = \beta_+$, the other simple zero x_+ becomes a triple zero. Thus the solution with $\kappa = +1$ develops a Calabi-Yau singularity, besides the O4-plane singularity it already had; the solution with $\kappa = -1$ still has a single O4-plane singularity.

- For $\beta > \beta_+$, the new triple zero splits into a local maximum x_3, a local minimum x_4 and a simple zero x_+. Now the $\kappa = +1$ interval is $x \in [x_2, x_3]$, and corresponds to a solution with two O4-plane singularities. Moreover there are two intervals that are allowed for $\kappa = -1$: $x \in [x_-, x_1]$ and $x \in [x_4, x_+]$. Both correspond to a solution with a single O4-plane singularity.

- Finally, at $\beta = 1$, $\gamma = 0$ we have that $\beta_- = \beta_+$. In this case the two triple zeros appear together; the allowed interval is between them and works for $\kappa = +1$. The solution has two Calabi-Yau singularities. When $\mathrm{KE}_4 = \mathbb{CP}^2$, with $\Delta\psi = 2\pi$ these are two orbifold singularities (4.18). (This is the periodicity originally considered in [7].) As we discussed there, in this case one can also take the periodicity of the S^1 to be $\Delta\psi = 6\pi$, in which case the space becomes fully smooth again; this is the GJV solution [11].[7] For more general KE_4, this solution was discussed in [27].

4.1.3 Flux quantization for the generic case

Let us now discuss flux quantization for the generic solutions of section 4.1.2.

First we need to return to the B-twisted fluxes (3.25) and introduce potentials C_{k-1} such that $dC_{k-1} = \tilde{F}_k$. Explicitly,

$$C_1 = f_2 D\psi \,, \qquad C_3 = f_4 D\psi \wedge j_{\mathrm{KE}} \,, \qquad C_5 = f_6 D\psi \wedge j_{\mathrm{KE}} \wedge j_{\mathrm{KE}} \,, \tag{4.28}$$

where

$$f_2 = 2\ell \sqrt{\frac{F_0}{\sqrt{3}\ell}} \left(\frac{1}{x} \frac{q}{4q - xq'} - \frac{3q' - xq''}{48x^2} \right) , \tag{4.29a}$$

$$f_4 = \frac{\ell}{16\sqrt{3}} \kappa (4q - xq') \,, \tag{4.29b}$$

$$f_6 = \frac{\ell}{16} \sqrt{\frac{\sqrt{3}\ell}{F_0}} \left[\frac{q'}{24}(3 + x^4) - \frac{q}{6}x^3 + \frac{x}{9} \left(3 + x^4 + \frac{3q' - xq''}{8x} \right)^2 \right] . \tag{4.29c}$$

The fluxes $\tilde{F}_k$ are closed; they have been defined using a particular choice B_1 for the B field. In fact it is also possible to add to it a closed two-form b, so that $B = B_1 + b$; this defines new fluxes

$$\tilde{F}^b \equiv e^b \tilde{F} \,, \tag{4.30}$$

which are also closed. Explicitly, $\tilde{F}^b_2 \equiv \tilde{F}_2 + bF_0$, $\tilde{F}^b_4 \equiv \tilde{F}_4 + b\tilde{F}_2 + \frac{1}{2}b^2 F_0$, $\tilde{F}^b_6 \equiv \tilde{F}_6 + b\tilde{F}_4 + \frac{1}{2}b^2\tilde{F}_2 + \frac{1}{6}b^3 F_0$. Flux quantization now imposes that the periods of these should be

[7] The coordinate transformation that brings the solution to the form of [11] is $x = \cos\alpha$ and $D\psi = 3\eta$. The parameters e^{ϕ_0} and L of [11] are identified as $e^{\phi_0} = 3^{-3/8}2^{1/4}\ell^{-1/4}F_0^{-3/4}$ and $L^2 = 3^{-1/16}2^{-5/8}\ell^{5/8}F_0^{-1/8}$. Note also that the solution in [11] is in the Einstein frame, whereas we work in the string frame.

quantized, as well as that $2\pi F_0 \equiv n_0 \in \mathbb{Z}$ (working in string units $l_s = 1$). It constrains the parameters ℓ, β, γ of the solution, as well as the two-form b.

We will now work out more precisely what this implies for regular generic solutions. In particular, we will assume $\Delta\psi = 2\pi$ and $\beta < \beta_-$, in the language of section 4.1.2; topologically, M_6 is an S^2-bundle over KE_4.

The second homology of M_6 is given by the fiber $\mathcal{C}_0 \equiv S^2_{\psi,x}$ spanned by ψ and x, and by the two-cycles $\mathcal{C}_i$, $i = 1, \ldots, h_2(\mathrm{KE}_4)$. More precisely, a lift of these two-cycles is given by a section of the fibration obtained by setting x to one of the endpoints, say x_+. (x_- would also work, but a random value would not define a cycle in M_6, because of the topological non-triviality of the fibration of the ψ coordinate.) A basis for the cohomology H^2 can be taken to be ω^I, $I = 0, \ldots, h^2(\mathrm{KE}_4)$; $\omega^0 \equiv d(s(x)D\psi)$, where $s(x)$ is a function which at the two simple zeros $x_\pm$ has second-order expansion $s \sim \pm(1 + (x - x_\pm)^2) + \ldots$, and ω^i are the elements of a basis for $H^2(\mathrm{KE}_4)$. We will expand the closed two-form b in this basis: $b = b_I \omega^I$. Similarly, a basis of four-cycles can be obtained by the $\tilde{\mathcal{C}}^i \equiv S^2_{\psi,x} \times \mathcal{C}_i$ and by $\tilde{\mathcal{C}}^0 \equiv \mathrm{KE}_4$. Finally, the triple intersection form d^{IJK} of M_6 will have non-zero entries $d^{0JK} = c^{JK}$, the intersection form of KE_4.

We can now define the periods

$$n^b_{2I} \equiv \frac{1}{2\pi}\int_{\mathcal{C}_I} \tilde{F}^b_2\,, \qquad n^{I\,b}_4 \equiv \frac{1}{(2\pi)^3}\int_{\tilde{\mathcal{C}}^I} \tilde{F}^b_4\,, \qquad n^b_6 \equiv \frac{1}{(2\pi)^3}\int_{M_6} \tilde{F}^b_6\,. \tag{4.31}$$

The periods at $b = 0$, $n_{2I} \equiv n^{b=0}_{2I}$, $n^I_4 \equiv n^{I\,b=0}_4$, $n_6 \equiv n^{b=0}_6$, are computed more directly as integrals of the $\tilde{F}_k$. The two are related by the b-transform (4.30): this gives $n^b_{2I} = n_{2I} - b_I n_0$, $n^{I\,b}_4 = n^I_4 + d^{IJK} b_J \left(n_{2K} + \frac{1}{2} b_K n_0\right)$, $n^b_6 = n_6 + b_I n^I_4 + \frac{1}{2} d^{IJK} b_I b_J \left(n_{2K} + \frac{1}{3} b_K n_0\right)$. From (4.28), (4.29) we can now compute the relevant integrals:

$$\begin{aligned} &n_{20} = f_{2+} - f_{2-}\,, && n_{2i} = K_i f_{2+}\,, \\ &n^i_4 = \frac{K_i}{2\pi\kappa}(f_{4+} - f_{4-})\,, && n^0_4 = -\frac{K^2}{2\pi\kappa} f_{4+}\,, \qquad n_6 = \frac{K^2}{4\pi^2}(f_{6+} - f_{6-})\,, \end{aligned} \tag{4.32}$$

where $f_{k\pm} \equiv f_k(x_\pm)$ the K_i are the Chern class integers of the canonical bundle; $2\pi K_i$ are the integrals of the Ricci form over the two-cycles $\mathcal{C}_i$ of the KE_4. We also defined $K^2 \equiv K_i K_j c^{ij}$. (4.32) can be further evaluated using the expressions for the f_k in (4.29). In doing so, it is useful to note that (4.25) implies that at a single zero x_0 of q:

$$(3q'_0 - x_0 q''_0)^2 = 24 q'_0 x_0 (1 + 3x_0^4)\,. \tag{4.33}$$

So in particular

$$\begin{aligned} &f_2(x_0) = -\ell\sqrt{\frac{F_0}{\sqrt{3}\ell}}\sqrt{\frac{q'_0(1 + 3x_0^4)}{24x_0^3}}\,, \qquad f_4(x_0) = -\frac{\ell}{16\sqrt{3}}\kappa x_0 q'_0\,, \\ &f_6(x_0) = \frac{\ell}{16}\sqrt{\frac{\sqrt{3}\ell}{F_0}}(3 + x_0^4)\left[\frac{1}{6}q'_0 - \frac{1}{3\sqrt{6}}\sqrt{x_0 q'_0(1 + 3x_0^4)} + \frac{x_0}{9}(3 + x_0^4)\right]\,. \end{aligned} \tag{4.34}$$

The n_{2I} now determine $b_I = \frac{1}{n_0}(n^b_{2I} - n_{2I})$; one can then eliminate them from the remaining quantization conditions. A practical way of doing this is to introduce some combinations of the flux quanta that are invariant under b-transform $\tilde{F} \to \tilde{F}^b$, generalizing slightly results in [9]:

$$I_4^I \equiv d^{IJK} n_{2J} n_{2K} - 2n_0 n_4^I \,, \qquad I_6 \equiv d^{IJK} n_{2I} n_{2J} n_{2K} + 3n_0^2 n_6 - 3n_0 n_{2I} n_4^I \,. \tag{4.35}$$

(These come from the expansion in form basis of $\tilde{F}_2^2 - 2F_0\tilde{F}_4$ and $\tilde{F}_2^3 + 3F_0^2\tilde{F}_6 - 3F_0\tilde{F}_2\tilde{F}_4$.) Indeed one can check that the I_4^I and I_6 remain the same if one replaces $n_{2I} \to n_{2I}^b$, $n_4^I \to n_4^{I\,b}$, $n_6 \to n_6^b$. For us these invariants evaluate to

$$\begin{aligned} &I_4^0 = K^2 \left(f_{2+}^2 + \frac{n_0}{\pi k} f_{4+} \right) \,, \qquad I_4^i = 2K^i \left[-f_{2+}(f_{2+} - f_{2-}) - \frac{n_0}{2\pi\kappa}(f_{4+} - f_{4-}) \right] \\ &I_6 = 3K^2 \left[f_{2+}^2 (f_{2+} - f_{2-}) + \frac{n_0}{2\pi\kappa}(2f_{4+}f_{2+} - f_{4+}f_{2-} + f_{4-}f_{2+}) + \frac{3n_0}{4\pi^2}(f_{6+} - f_{6-}) \right] . \end{aligned} \tag{4.36}$$

Once a set of flux quanta is specified, solving these equations will specify the parameters ℓ, β, γ of the solution.

4.1.4 $\ell = 0$

We will now examine the solutions with $\ell = 0$. The rescaling of the coordinate y, appropriate for this case, is

$$x = \frac{y}{y_0} \,, \qquad y_0 \equiv \left(\frac{6\nu}{F_0^2} \right)^{1/3} \,, \tag{4.37}$$

and we will also introduce

$$\sigma = -\frac{3\mu}{y_0^2} \,. \tag{4.38}$$

The solution is then in the form (4.9) with

$$q = x^6 + \frac{\sigma}{2} x^4 + 4x^3 - \frac{1}{2} \,, \tag{4.39}$$

which gives

$$3q' - xq'' = -12x^2(x^3 - 1) \,. \tag{4.40}$$

The constants that appear in the warp factor and the dilaton are $L^2 = |y_0|$ and $g_s^2 = 72/(|y_0| F_0^2)$.

In this case, the analysis is easier, because there is only one parameter, σ, to vary. The discriminant of q is $2(\sigma^3 + 9^3)^2$; thus there always two simple zeros, except for $\sigma = -9$, when one of the two becomes a triple zero. q' always has a double zero at the origin, but the discriminant of q'/x^2 is $-192(\sigma^3 + 9^3)$ (so again we have $\Delta(q) \propto \Delta(q')^2$), and the sign shows that there is a single extremum x_1 for $\sigma > -9$, and three extrema x_1, x_2, x_3 for $\sigma < -9$. In addition, there is a inflection point at $x = 0$, which from section 4.1.1 we know to correspond to an O8-plane.

We divide the analysis in three cases:

- For $\sigma > -9$, between the two zeros $x_- < 0$ and $x_+ > 0$, q has a minimum at $x = x_1 < 0$ and the inflection point at $x = 0$. The intervals where (4.10) are realized are $x \in [x_-, x_1]$ with $\kappa = -1$, and $x \in [0, x_+]$ with $\kappa = +1$. The former corresponds to a solution with a single O4-plane singularity; the latter to a solution with a single O8-plane singularity.

- For $\sigma = -9$, the zero x_+ becomes triple; the allowed intervals remain the same as in the previous case, but the $\kappa = +1$ case now develops a Calabi-Yau conical singularity at $x = x_+$.

- For $\sigma < -9$, the zero x_+ splits in a maximum x_2, a minimum x_3 and a simple zero x_+ (all three greater than zero). There are now three allowed intervals: the old one $x \in [x_-, x_1]$, still for $\kappa = -1$, an interval $x \in [0, x_2]$ for $\kappa = +1$, and a new one $x \in [x_2, x_+]$ for $\kappa = -1$. These two new possibilites correspond to a solution with an O8-plane and an O4-plane singularity, and to a solution with a single O4-plane singularity, respectively.

4.1.5 $F_0 = 0$

In this limit, the rescaling of the coordinate y to the coordinate x is

$$y = \frac{\ell^2}{\nu} x \,. \tag{4.41}$$

Furthermore the constant parameter μ is rescaled to s as $\mu = 3(\ell^4/\nu^2)s$. From (4.22) we see

$$s = \frac{3}{2}\beta\gamma^2 \,. \tag{4.42}$$

The massless limit is now obtained by taking $\beta \to 0$ with s constant. We obtain (4.9) with

$$q = x^4 + \frac{4}{3}x^3 + \frac{1}{2}x^2 - \frac{s}{4} \,, \qquad L^2 = \frac{\ell^2}{\nu} \,, \qquad g_s^2 = \frac{4\ell^4}{\nu^3} \,. \tag{4.43}$$

These solutions uplift to M-theory $\mathrm{AdS}_4 \times Y^{p,k}$ solutions, where $Y^{p,k}$ are the well-known Sasaki-Einstein seven-manifolds of [19]. To see this, one has to perform the further change of coordinate

$$x = \rho^2 - \frac{1}{2} \,, \tag{4.44}$$

and set the constants in [19, section 2] as $\{\Lambda, \kappa, \lambda\} = \{8, 12s - 1, 2\ell/\nu\}$.

In the limit $s \to 0$, the interval of definition for x shrinks to zero. However, one can define $S \equiv s/\nu^2$ and take $S \to 0$; in this limit the solution remains well-defined. After introducing the coordinate θ by $\cos\theta \equiv \sqrt{2/s}\, x$, it becomes (for $\mathrm{KE}_4 = \mathbb{CP}^2$) the IIA reduction of $M^{3,2}$ [22, 23], as worked out in [7, (2.10)].

4.2 Product base

In this class the metric of M_4 splits as

$$\hat{g}^{(4)} = \hat{g}_1 + \hat{g}_2 \,. \tag{4.45}$$

Accordingly, $\hat{\jmath} = \hat{\jmath}_1 + \hat{\jmath}_2$ and $\hat{\mathfrak{R}} = \hat{\mathfrak{R}}_1 + \hat{\mathfrak{R}}_2$, with

$$\hat{\mathfrak{R}}_1 = \frac{\kappa_1}{Q_1(y)}\hat{\jmath}_1 \,, \qquad \hat{\mathfrak{R}}_2 = \frac{\kappa_2}{Q_2(y)}\hat{\jmath}_2 \,, \qquad \kappa_1, \kappa_2 = \pm 1, 0 \,. \tag{4.46}$$

The dependence of the metric of M_4 on y is given by

$$\hat{g}^{(4)}(y, x^i) = Q_1(y) g_{\Sigma_1} + Q_2(y) g_{\Sigma_2} \,, \tag{4.47}$$

where g_{Σ_1}, g_{Σ_2} are the metrics of the two Riemann surfaces Σ_1, Σ_2 of scalar curvature $R_1 = 2\kappa_1$ and $R_2 = 2\kappa_2$ respectively.

When combined with (4.2a), and the fact that $\partial_y \hat{\mathfrak{R}} = 0$ the condition (4.46) fixes

$$Q_1 = \frac{\kappa_1}{6}(3\ell^2 y^{-1} - F_0^2 y^3) + \nu_1 \,, \tag{4.48}$$

$$Q_2 = \frac{\kappa_2}{6}(3\ell^2 y^{-1} - F_0^2 y^3) + \nu_2 \,, \tag{4.49}$$

where ν_1, ν_2 are constants. In combination with (4.2b), (4.46) gives the ODE $2T = (\kappa_1 Q_2 + \kappa_2 Q_1)/(Q_1 Q_2)$ which, given the expression (3.16b) for T and defining again $p(y) = e^{4A} y^2$, becomes a Riccati:

$$y^2 \frac{2 Q_1 Q_2}{\kappa_1 Q_2 + \kappa_2 Q_1} \frac{dp}{dy} = F_0^2 p^2 + (\ell^2 - F_0^2 y^4) p - \ell^2 y^4 \,. \tag{4.50}$$

This is solved by:

$$p = \ell^2 y^2 \frac{3\ell^2 \mu - 9\kappa_1\kappa_2 \ell^2 y^2 - 6(\kappa_1\nu_2 + \kappa_2\nu_1) y^3 + \kappa_1\kappa_2 F_0^2 y^6}{9\kappa_1\kappa_2 \ell^4 + 18\ell^2(\kappa_1\nu_2 + \kappa_2\nu_1) y + (36\nu_1\nu_2 - 3\ell^2 F_0^2 \mu) y^2 + 3\kappa_1\kappa_2 \ell^2 F_0^2 y^4} \,, \tag{4.51}$$

where μ is a constant parameter. The $\ell \to 0$ limit is well-defined after shifting $\mu \to 12\nu_1\nu_2/(F_0^2 \ell^2) + \mu$.

4.2.1 Regularity and boundary conditions

We now turn to the analysis of the geometry of the solutions, in a manner similar to the one in the previous section.

The metric (3.11) on the internal manifold takes the form:

$$e^{-2A} ds^2_{M_6} = -\frac{1}{4}\frac{q'}{xq} dx^2 - \frac{q}{xq' - 4q} D\psi^2 + \frac{\kappa_1 q'}{x u_1} ds^2_{\Sigma_1} + \frac{\kappa_2 q'}{x u_2} ds^2_{\Sigma_2} \,, \tag{4.52a}$$

where q, u_1, u_2 are polynomials. The warp factor is given by

$$L^{-2} e^{2A} = \sqrt{\frac{x^2 q' - 4xq}{q'}} \,. \tag{4.52b}$$

The dilaton is given by

$$g_s^{-2} e^{2\phi} = \frac{q'}{x u_1 u_2} \left(\frac{x^2 q' - 4xq}{q'} \right)^{3/2} \,. \tag{4.52c}$$

The above is valid for κ_1, $\kappa_2 \neq 0$. When one of the two Riemann surfaces is flat, a slight modification is required and we will treat the case $\kappa_2 = 0$ separately.

Notice that

$$3q' - xq'' = \frac{x}{2}(u_1 + u_2) \,; \tag{4.53}$$

so we see that for $u_1 = u_2$ we recover (4.9).

Positivity of the metric and the dilaton now requires either

$$q < 0\,, \qquad xq' > 0\,, \qquad \kappa_a u_a > 0\,, \qquad u_1 u_2 > 0\,, \tag{4.54a}$$

or

$$q > 0\,, \qquad xq' < 0\,, \qquad \kappa_a u_a < 0\,, \qquad u_1 u_2 < 0\,, \tag{4.54b}$$

$a = 1, 2$ (no summation over repeated indices). (4.54a) generalizes (4.10), while (4.54b) is a new possibility.

Given the similarity between (4.52) and (4.9), most of the analysis leading to table 1 is the same. There is the additional possibility of the occurrence of a double zero. Moreover, the case of an inflection point now ramifies into three different branches. See table 2.

Double zero: orbifold singularity. This did not occur in section 4.1, because a multiple zero was always a triple zero. This is no longer the case in the present section: a double zero which is not also triple can occur, and we must analyze it separately. A crucial fact is that $\Delta(q) \propto \mathrm{res}(q, u_1)\mathrm{res}(q, u_2)$. Thus when a double zero occurs either u_1 or u_2 has a zero. Choosing the latter, one finds the local expression for the metric

$$\frac{1}{L^2|x_0|} ds^2_{M_6} \sim \frac{1}{2x_0}\left[\frac{dx^2}{x_0 - x} + (x_0 - x)\left(D\psi^2 - \frac{2\kappa_1 q_0''}{u_{10}} ds^2_{\Sigma_1}\right)\right] + \frac{\kappa_2 q_0''}{x_0 u_{20}'} ds^2_{\Sigma_2}\,, \tag{4.55}$$

while the warp factor and dilaton are constant. Here $u_{10} \equiv u_1(x_0)$ and $u_{20}' \equiv u_2'(x_0)$. From (4.53) we see that $-2q_0''/u_{10}' = 1$. Positivity of the metric requires that $x_0(x_0 - x) > 0$, and $\kappa_1 = 1$, which selects $\Sigma_1 = S^2$. With the choice $r = \sqrt{x_0 - x}$, the parenthesis becomes proportional to $dr^2 + \frac{1}{4}r^2(D\psi^2 + ds^2_{S^2})$. If the S^1 periodicity is $\Delta\psi = 2\pi$, this is the local metric for an $\mathbb{R}^4/\mathbb{Z}_2$ singularity; if $\Delta\psi = 4\pi$, it is $\mathbb{R}^4$, and we have a regular point.

Inflection point at the origin: O4-, O6- or O8-plane. In section 4.1, at an inflection point at the origin the denominator of the coefficient of $ds^2_{\mathrm{KE}_4}$ has a double zero, canceling the double zero of the numerator, so that the full coefficient remains constant. In (4.52a), however, the functions u_a in the denominator are independent of q. If neither of u_a vanishes, the local metric and dilaton read

$$\begin{aligned}\frac{1}{L^2}\sqrt{-\frac{q_3}{8q_0}} ds^2_{10} &\sim \frac{1}{\sqrt{x}}\left(ds^2_{\mathrm{AdS}_4} + \frac{1}{4}D\psi^2\right) - \frac{q_3}{8q_0}\sqrt{x}\left(dx^2 - \frac{4q_0\kappa_1}{u_{10}} ds^2_{\Sigma_1} - \frac{4q_0\kappa_2}{u_{20}} ds^2_{\Sigma_2}\right),\\ g_s^{-2} e^{2\phi} &\sim \frac{-4q_0}{u_{10}u_{20}}\sqrt{-\frac{8q_0}{xq_3}}\,,\end{aligned} \tag{4.56}$$

According to the discussion underneath (4.14), this locus describes an O4-plane smeared over $\Sigma_1 \times \Sigma_2$.

If only one of the u_a, say u_1, vanishes,[8] we have

$$\begin{aligned}L^{-2}\sqrt{-\frac{q_3}{8q_0}} ds^2_{10} &\sim \frac{1}{\sqrt{x}}\left(ds^2_{\mathrm{AdS}_4} + \frac{1}{4}D\psi^2 + \frac{q_3\kappa_1}{2u_{10}'} ds^2_{\Sigma_1}\right) - \frac{q_3}{8q_0}\sqrt{x}\left(dx^2 - \frac{4q_0\kappa_2}{u_{20}} ds^2_{\Sigma_2}\right),\\ g_s^{-2} e^{2\phi} &\sim \frac{q_3}{2u_{10}'u_{20}}\left(-\frac{8q_0}{xq_3}\right)^{3/2}.\end{aligned} \tag{4.57}$$

[8]Equation (4.53) appears to imply that at an inflection point either the u_a both vanish or both do not vanish. However, we will see in section 4.2.4 that for $\kappa_2 = 0$ this formula is modified, so that only u_1 needs to vanish.

x_0	$q(x_0)$	$q'(x_0)$	$q''(x_0)$	$q'''(x_0)$	$u_1(x_0)$	$u_2(x_0)$	interpretation
	0	0			0		$\mathbb{R}^4/\mathbb{Z}_2$
0		0	0				O4
0		0	0		0		O6
0		0	0		0	0	O8
0		0	0	0	0	0	O8/O4

Table 2. Additional singularities that occur in the product base case. The O8-plane one, which already appeared in table 1, is repeated for comparison.

This locus corresponds to an O6-plane singularity. Adapting our discussion below (4.14), we conclude that it is localized if $\Sigma_2 = S^2$ ($\kappa_2 = 1$), while it is smeared over Σ_2 if $\kappa_2 = -1$.

Finally, if both u_a vanish at the inflection point, there is an O8-plane singularity as in (4.19), with the KE_4 base replaced by $\Sigma_1 \times \Sigma_2$.

Quartic maximum at the origin: O4/O8-plane. When $q \sim q_0 + \frac{1}{4!} q_4 x^4$, then both u_a have a single zero. We have

$$\begin{aligned} L^{-2}\sqrt{-\frac{q_4}{24 q_0}}ds^2_{10} &\sim \frac{1}{x}\left(ds^2_{\text{AdS}_4} + \frac{1}{4}D\psi^2\right) + \frac{q_4}{6}\left(\frac{\kappa_1}{u'_{10}}ds^2_{\Sigma_1} + \frac{\kappa_2}{u'_{20}}ds^2_{\Sigma_2}\right) - \frac{q_4}{24 q_0}x dx^2 \,, \\ g_s^{-2}e^{2\phi} &\sim \frac{8\sqrt{6}}{u'_{10}u'_{20}}\sqrt{-\frac{q_0^3}{q_4}}x^{-3} \,. \end{aligned} \tag{4.58}$$

which is the appropriate behavior for an O4/O8-plane singularity. This case occurs only if κ_a have opposite signs, so the O4-plane is smeared over one of the Σ_a.

4.2.2 Generic case

Here we define new parameters

$$\gamma_a = \frac{1}{\kappa_a}\frac{1}{\ell^2}\sqrt{\frac{\sqrt{3}\ell}{F_0}}\nu_a \,, \qquad \beta = \frac{1}{\kappa_1\kappa_2}\frac{F_0}{2\sqrt{3}\ell}\mu \,, \tag{4.59}$$

$a = 1, 2$, and a new coordinate $x = \sqrt{\frac{F_0}{\sqrt{3}\ell}}y$. The solution is then cast in the form (4.52), with

$$\begin{aligned} q &= x^6 + 3(2\gamma_1\gamma_2 - \beta)x^4 + 4(\gamma_1 + \gamma_2)x^3 + 3x^2 - \beta \,, \\ u_1 &= 12(1 + 2\gamma_1 x - x^4) \,, \qquad u_2 = 12(1 + 2\gamma_2 x - x^4) \end{aligned} \tag{4.60}$$

and the constants L, g_s determined by (4.24).

We again note that the Riccati ODE (4.50) implies an ODE directly on q, u_1, u_2:

$$\frac{x}{12}(3q' - xq'')^2 = -\left(\frac{1}{u_1} + \frac{1}{u_2}\right)\left[(xq' - 4q)(1 + 3x^4) + 4q\right] \,, \tag{4.61}$$

similar to (4.25).

In this case we will not give an exhaustive discussion as in section 4.1.2 for the Kähler-Einstein base. There are many different possibilities, and a full discussion would be rather tedious. Let us instead give here a summary of the main features of the parameter space.

As in the Kähler-Einstein base case, the interpretation of the solution depends on the properties of the polynomial q, and less importantly on u_1, u_2. The most important features are the presence of extrema, and the presence of zeros. These can be decided again by looking at the discriminants of q and q'. Unlike in section 4.1, these are unrelated, and vanish on different loci of parameter space. It is also useful to notice that $2^{26}3^{10}\Delta(q) = \beta\,\mathrm{res}(q',u_1)\mathrm{res}(q',u_2) \propto \Delta(q)$; thus, the sign of u_1, u_2 at an extremum of q (which has to do with the sign of κ_1, κ_2) changes on the discriminant of q. The expressions of the resultants are

$$\begin{aligned}\mathrm{res}(q',u_1) &= -2^{18}3^9\left(\beta^4 - 6\gamma_1\gamma_2\beta^3 + (12\gamma_1^2\gamma_2^2 - 2)\beta^2\right.\\ &\qquad \left. - 2(2\gamma_1^2 + \gamma_1\gamma_2 + 2\gamma_2^2 + 4\gamma_1^3\gamma_2^3)\beta + 1 - \gamma_1^4 + 2\gamma_1^3\gamma_2\right),\\ \mathrm{res}(q',u_2) &= \mathrm{res}(q',u_1) + 2^{18}3^9(\gamma_1+\gamma_2)(\gamma_1-\gamma_2)^3\,. \end{aligned} \tag{4.62}$$

Depending on the signs of κ_a, there are three possibilities to consider: (i) For $\kappa_a = -1$ there are solutions with a regular endpoint and an endpoint with an O4-plane singularity. (ii) For $\kappa_a = +1$ there are regular solutions and ones with orbifold singularities which were found numerically in [10]. There are also solutions which for $\Delta\psi = 4\pi$ have $\mathbb{CP}^3$ topology and were found numerically in [9]. There are also solutions with one or two O4-plane singularities. (iii) For $\kappa_1 = +1$, $\kappa_2 = -1$ (or viceversa) there are solutions with one or two O4-plane singularities, as well as solutions with an orbifold singularity.

The flux potentials are again defined by (3.25), now with $\hat{\jmath} = Q_1 j_1 + Q_2 j_2$. Using $\rho = \rho_1 + \rho_2$, with $d_4\rho_a = -\kappa_a j_a$ (no summation), we find

$$C_1 = f_2 D\psi + \frac{1}{2}(\kappa_1\nu_1 - \kappa_2\nu_2)(\rho_1-\rho_2)\,, \quad C_3 = D\psi\wedge(g_1 j_1 + g_2 j_2)\,, \quad C_5 = f_6 D\psi\wedge j_1\wedge j_2\,, \tag{4.63}$$

where

$$f_2 = 2\ell\sqrt{\frac{F_0}{\sqrt{3}\ell}}\left(\frac{1}{x}\frac{q}{4q - xq'} - \frac{3q' - xq''}{48x^2}\right), \tag{4.64a}$$

$$g_a = \frac{\ell^3}{8\sqrt{3}F_0}\int x u_a dx\,, \tag{4.64b}$$

$$f_6 = \frac{\ell}{72}\sqrt{\frac{\sqrt{3}\ell}{F_0}}\left[q' - \frac{x}{768}(3u_1^2 - 10u_1u_2 + 3u_2^2) + x\left(3 + x^4 + \frac{u_1+u_2}{16}\right)^2\right]. \tag{4.64c}$$

The integration constant in the indefinite integral for the g_a is chosen such that

$$g_a(x=0) = -\frac{\ell^3}{8\sqrt{3}F_0}\frac{\beta}{3}\,. \tag{4.65}$$

4.2.3 $\ell = 0$

In this case we introduce

$$\sigma = -\frac{3\mu}{y_0^2}\,, \qquad t = \frac{\kappa_2\nu_1}{\kappa_1\nu_2}\,, \qquad x = \frac{y}{y_0}\,, \qquad y_0 \equiv \left(\frac{6\nu_2}{\kappa_2 F_0^2}\right)^{1/3}. \tag{4.66}$$

The solution is now (4.52) with

$$q = x^6 + \frac{\sigma}{2}x^4 + 2(1+t)x^3 - \frac{t}{2}\,, \qquad u_1 = 12x(1-x^3)\,, \qquad u_2 = 12x(t-x^3)\,, \tag{4.67}$$

and the constants $L^2 = |y_0|$, $g_s^2 = 72/(|y_0|F_0^2)$.

For $t = 1$, we recover the solutions in section 4.1.4, with $\mathrm{KE}_4 = \mathbb{CP}^1 \times \mathbb{CP}^1$. For other values of t, the analysis is similar.

Define

$$\sigma_1 \equiv -9 \cdot 4^{-1/3}|1+t|^{2/3}\,, \qquad \sigma_+ \equiv -3(2+t)\,, \qquad \sigma_- \equiv -3t^{-1/3}(1+2t)\,. \tag{4.68}$$

The discriminant $\Delta(q) = 2t^3(\sigma^3 - \sigma_+^3)(\sigma^3 - \sigma_-^3)$, while $\Delta\left(q'/x^2\right) = -192(\sigma^3 - \sigma_1^3)$. In terms of these, the following solutions are possible:

- $\kappa_a = +1$: With a single O4-plane singularity for $t < 0$, $\sigma < \sigma_-$, and also for $t > 0$ and any σ. With an O8-plane singularity for $t > 0$; for $\sigma > \sigma_-$ it also has an O4-plane singularity, for $\sigma = \sigma_-$ it has an orbifold singularity, for $\sigma < \sigma_-$ no other singularity.
- $\kappa_a = -1$: With a single O4-plane singularity for $\sigma < \sigma_+$ and any t.
- $\kappa_1 = +1$, $\kappa_2 = -1$: With two O4-plane singularities for $\sigma_+ < \sigma < \sigma_-$; with an O4-plane singularity and an orbifold singularity for $\sigma = \sigma_+$; with a single O4-plane singularity for $t > 0$, $\sigma_- < \sigma < \sigma_+$, or for $t < 0$, $\sigma < \sigma_+$.

  With an O8-plane singularity for $-1 < t < 0$; for $\sigma > \sigma_-$ it also has an O4-plane singularity, for $\sigma = \sigma_-$ it has an orbifold singularity, for $\sigma < \sigma_-$ no other singularity.

  With an O8/O4-plane singularity for $t = -1$; for $\sigma < 0$ it also has an O4-plane singularity, for $\sigma = -1$ an orbifold singularity, for $\sigma < -1$ no other singularity.

4.2.4 $\kappa_2 = 0$

In this case we need to adjust the form of the solution as presented in 4.2.1, by replacing the factor κ_2 in front of the line element of Σ_2 by a constant m, to be specified below. The functions that determine the solution read:

$$q = 3x^4 - 4x^3 + n\,, \qquad u_1 = -24x\,, \qquad u_2 = 3\tilde{\ell}^2(1-x) + nx - x^4 \tag{4.69}$$

where the coordinate $x = y/L^2$, with L^2 related to the constant parameters appearing in (4.51) as:

$$L^2 = \frac{6\ell^2\kappa_1\nu_2}{F_0^2\ell^2\mu - 12\nu_1\nu_2}\,. \tag{4.70}$$

For n, $\tilde{\ell}$ and m we have the following relations:

$$\tilde{\ell} = \frac{\ell}{L^4 F_0}\,, \qquad n = \frac{6\nu_1}{\kappa_1 L^6 F_0^2} + 3\ell^2\,, \qquad m = -\frac{4F_0^2 L^6}{\nu_2}\,. \tag{4.71}$$

Finally, $g_s^2 = 2/(\sqrt{3}F_0^2 L^8)$.

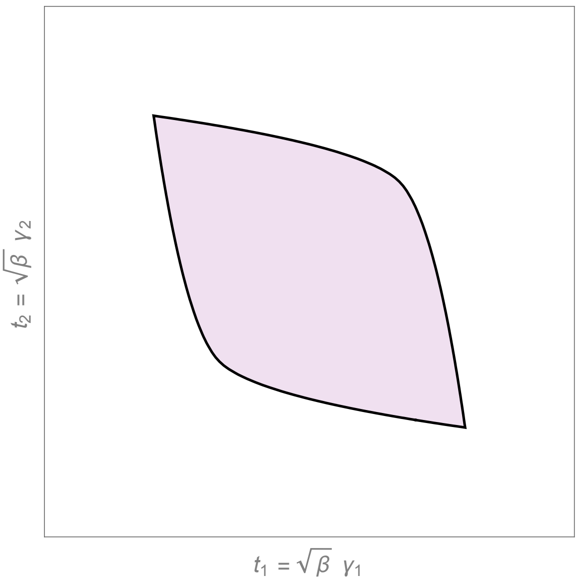

Figure 2. The parameter space of regular solutions with $F_0 = 0$ for the product base case.

In this case:

- $\kappa_1 = +1$: there is a solution with an O6-plane smeared along the T^2, for $\tilde{\ell} \neq 0$. This becomes an O8-plane for $\tilde{\ell} = 0$. For $n > 1$ it also has an O4-plane singularity; for $n = 1$ it has an orbifold singularity; for $n < 1$, $n \neq 0$ no other singularity.

- $\kappa_1 = -1$: there is a solution with a single O4-plane singularity for $n < 1$.

4.2.5 $F_0 = 0$

Similarly to section 4.1.5, we define the coordinate $x = \frac{\nu_2}{\ell^2} y$; this differs from the x defined in section 4.1.2 by a factor γ_2, recalling the definitions in (4.59). Let us define $t_i \equiv \gamma_i \sqrt{\beta}$. Taking the limit $\beta \to 0$ with t_i kept constant yields now the solution (4.52) with

$$q = \frac{t_1}{t_2} x^4 + \frac{2}{3}\left(1 + \frac{t_1}{t_2}\right) x^3 + \frac{1}{2} x^2 - \frac{t_2^2}{6}\,, \qquad u_1 = 2 + 4x\,, \qquad u_2 = 2 + 4\frac{t_1}{t_2} x\,. \tag{4.72}$$

Moreover $L^2 = \ell^2 \kappa_1 t_2 / (\nu_1 t_1)$, $g_s = 2L^3/\ell$.

Regular solutions exist for $\kappa_a = +1$. The parameter space is obtained by considering the equations for the resultants in (4.62), after taking $\beta \to 0$ with t_i kept constant; this gives two cubics, and the allowed parameter space is enclosed by them. The result is shown in figure 2. (Points related by inversion $t_1 \leftrightarrow t_2$ correspond to the same solution.) These correspond to the solutions found in [20, section 4.5] and [21], more or less in the coordinates of the first reference. They were studied in more detail in [10, section 3] where they were referred to as $A^{p,q,r}$ solutions. For $t_1 = t_2$ we recover the case of section 4.1.5 for $\mathrm{KE}_4 = \mathbb{CP}^1 \times \mathbb{CP}^1$. At the boundary of the region in figure 2, M_6 no longer has the topology of an S^2-fibration over $\mathbb{CP}^1 \times \mathbb{CP}^1$. For example, the solution at $t_1 = -t_2 = \frac{\sqrt{3}}{2\sqrt{2}}$ is the Fubini-Study solution on $\mathbb{CP}^3$. We refer to [10, section 3] for a detailed discussion.

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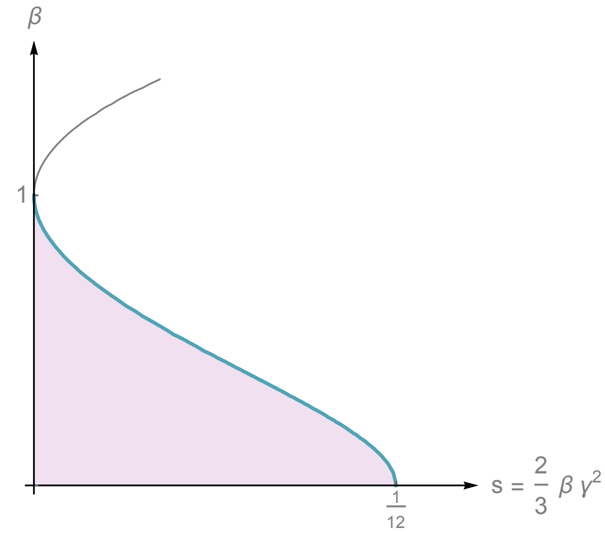

Figure 3. The allowed parameter space for regular solutions in the generic case with KE_4 base is the interior of the purple region.

4.3 Summary

We will now summarize the physically meaningful solutions we have found. This means we will exclude the solutions with smeared orientifolds, which are of dubious physical significance. The only fully-localized orientifolds we could find are O8-planes. We will first discuss solutions without any O8-planes, and then solutions with O8-planes.

Solutions with KE_4 base and without orientifolds. The parameter space is shown in figure 3, which is a different version of figure 1. In all this subsection the ψ periodicity is $\Delta\psi = 2\pi$, unless otherwise stated.

- The purple region corresponds to fully regular solutions. It is given by $\beta > 0$ and

$$\beta^6 - (2 + 9s)\beta^4 + (1 - 15s + 27s^2)\beta^2 + \frac{9}{4}s^2(1 - 12s) > 0\,. \qquad (4.73)$$

  Recall that s is related to β and γ by (4.42).

  Topologically, M_6 is an S^2-fibration over KE_4.

  These solutions realize the numerical solutions found in [7] and later generalized by [8, 10].

- The boundary $\beta = 0$ of the purple region gives $F_0 = 0$; these are IIA reductions of the $Y^{p,k}$ Sasaki-Einstein solutions in [19]. Indeed from (4.73) we see that $0 < s < \frac{1}{12}$, as derived there. The limit $s \to 0$ can be taken only after rescaling $s \to s/\nu^2$, in which case it leads to the IIA reduction [7, (2.10)] of the $M^{3,2}$ Sasaki-Einstein [22, 23], generalized by replacing $\mathbb{CP}^2$ with an arbitrary KE_4.

- The green boundary (obtained by replacing the inequality with equality in (4.73)) corresponds to solutions with a conical Calabi-Yau singularity. For $\mathrm{KE}_4 = \mathbb{CP}^2$, this is an orbifold singularity $\mathbb{C}^3/\mathbb{Z}_3$.

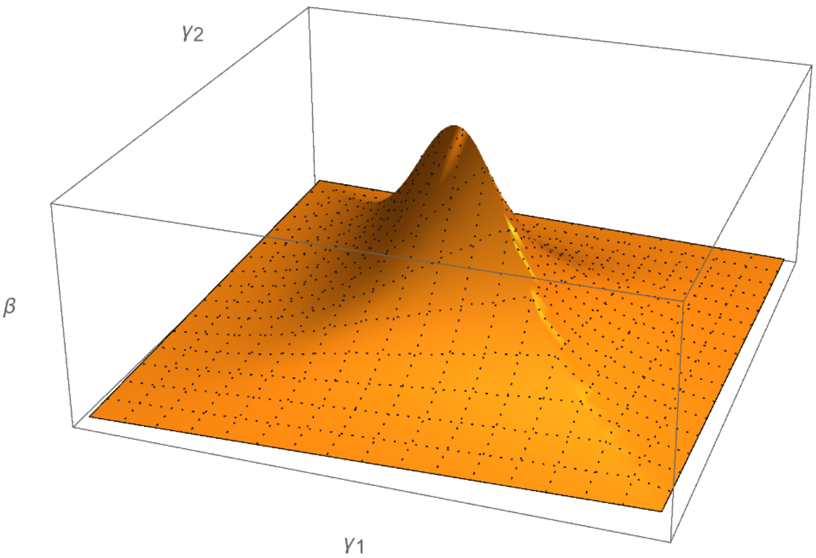

Figure 4. The allowed moduli space for regular solutions in the generic case with product base is between the plotted surface and the plane $\beta = 0$.

- The point at $\beta = 1$, $s = 0$ has two conical Calabi-Yau singularities. For $\mathrm{KE}_4 = \mathbb{CP}^2$, these are both orbifold singularities $\mathbb{C}^3/\mathbb{Z}_3$. In this case, however, it is also possible to take $\Delta\psi = 6\pi$, in which case the space becomes fully regular; this is the Guarino-Jafferis-Varela solution [11], whose generalization for arbitrary KE_4 was considered in [27].

Solutions with product base and without orientifolds. There are regular solutions with $\kappa_a = +1$. The parameter space is shown in figure 4, which is defined by certain branches of $\mathrm{res}(q', u_a) = 0$ (see (4.62)). This time we use the original parameters γ_1, γ_2, β. These solutions were discussed in [10, section 5] in detail, although they were only known numerically in that paper; thus we will be brief.

- The region in figure 4 between the plotted surface and $\beta = 0$ corresponds to regular solutions with the topology of an S^2-bundle over $\mathbb{CP}^1 \times \mathbb{CP}^1$.
- The limit $\beta \to 0$, with $t_a = \gamma_a\sqrt{\beta}$ kept constant, reproduces the $A^{p,q,r}$ massless solutions whose parameter space was shown in figure 2.
- The boundary of figure 4 corresponds to solutions with a $\mathbb{Z}_2$ orbifold singularity.
- The locus $\{\gamma_1 = \gamma_2\}$ is a particular case of the solutions with KE_4 base, where $\mathrm{KE}_4 = \mathbb{CP}^1 \times \mathbb{CP}^1$. At the boundary, there is a conifold/$\mathbb{Z}_2$ singularity.
- The intersection of the boundary with the locus $\{\gamma_1 = -\gamma_2\}$ (visible as the ridge in figure 2) corresponds to solutions with topology $\mathbb{CP}^3/\mathbb{Z}_2$. In this case one also has the option of taking $\Delta\psi = 4\pi$, thus making the topology directly $\mathbb{CP}^3$. These are the solutions studied in [9].
- The point $\beta = 1$, $\gamma_a = 0$ is a variant of the solution in [11], with $\mathbb{CP}^2$ replaced by $\mathbb{CP}^1 \times \mathbb{CP}^1$.

Solutions with O8-planes. There are many solutions which are regular except for a single O8-plane singularity. They occur for $\ell = 0$, both for KE_4 and for product base. Here is a list of possibilities:

- KE_4 base with $\kappa = +1$, and $\sigma > -9$ in (4.39).
- Product base with $\kappa_1 = +1$, $\kappa_2 = \pm 1$, and $\sigma > \sigma_-$; see (4.67), (4.68).
- Product base with $\kappa_1 = +1$, $\kappa_2 = 0$, and $n < 1$; see (4.69).

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