New BPS Wilson loops in $\mathcal{N} = 4$ circular quiver Chern-Simons-matter theories

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ABSTRACT: We construct new families of 1/4 BPS Wilson loops in circular quiver $\mathcal{N} = 4$ superconformal Chern-Simons-matter (SCSM) theories in three dimensions. They are defined as the holonomy of superconnections that contain non-trivial couplings to scalar and fermions, and cannot be reduced to block-diagonal matrices. Consequently, the new operators cannot be written in terms of double-node Wilson loops, as the ones considered so far in the literature. For particular values of the couplings the superconnection becomes block-diagonal and we recover the known fermionic 1/4 and 1/2 BPS Wilson loops. The new operators are cohomologically equivalent to bosonic 1/4 BPS Wilson loops and are then amenable of exact evaluation via localization techniques. Moreover, in the case of orbifold ABJM theory we identify the corresponding gravity duals for some of the 1/4 and 1/2 BPS Wilson loops.

Contents

1 Introduction

BPS Wilson loops (WLs) in 3D supersymmetric Chern-Simons-matter (SCSM) theories exhibit a rich spectrum of peculiar properties that need to be deeper understood.

Due to dimensional reasons not only scalar but also fermion matter together with ordinary gauge connections can be used to construct WLs as the holonomy of generalized (super)connections [1, 2]. This allows to define a large web of BPS operators with different degrees of preserved supersymmetry (SUSY).

Even if protected from UV divergent corrections, BPS WLs can still feature non-trivial vacuum expectation values that are often computed exactly by using localization techniques [3, 4]. Localization predictions can then be directly checked at weak coupling in perturbation theory. Moreover, in theories that allow for string theory or M-theory dual descriptions, BPS WLs in fundamental representation were shown to be dual to fundamental string or M2-brane configurations [5, 6]. Therefore, their expectation values at strong coupling can be computed using the holographic description and the matching with localization results expanded at strong coupling provides a crucial test of the AdS/CFT correspondence.

Given a 3D SCSM theory it is therefore crucial to construct and classify the whole spectrum of BPS WLs, identify their gravity duals and compute their expectation values in diverse regimes. In this paper we aim at giving a complete classification of BPS WLs in circular quiver $\mathcal{N} = 4$ SCSM theories with alternating Chern-Simons levels. Our classification generalizes the one of [7] and we are also able to identify the precise gravity dual configurations for a given sublclass of operators.

The present understanding on BPS WLs in SCSM theories can be summarized as follows. BPS WLs operators in SCSM were first introduced in the seminal work [1]. There, 1/2 BPS operators for $\mathcal{N} = 2$ SCSM theories and 1/3 BPS WLs in $\mathcal{N} = 3$ SCSM theories were defined as the holonomy of generalized connections that include couplings to the scalar fields of the theories.

Later on, the construction was generalized to models with a higher degree of supersymmetry. In $\mathcal{N} = 6$ ABJ(M) theories [8–10] with $U(N)_k \times U(M)_{-k}$ gauge group, dual to M-theory in $AdS_4 \times S^7/\mathbb{Z}_k$ background, the whole classification of BPS WLs was carried out in [7, 11] for timelike infinite straight lines in Minkowski spacetime and for maximal circles in Euclidean space. The most general WL is 1/6 BPS and corresponds to the holonomy of a superconnection that includes parametric couplings to both scalars and fermions. For particular values of the parameters it reduces to the bosonic 1/6 BPS WL with couplings only to scalar fields [12–14], which is supposed to be dual to smeared M2-branes [12], and to the fermionic 1/2 BPS WLs, dual to M2-/anti-M2-branes [2, 15]. The generalization to fermionic BPS WLs with a reduced number of preserved supercharges was considered in [16–20]. These are featured by non-trivial *latitude* angles in the internal R-symmetry space plus possibly latitude deformations of the Euclidean closed contour.

$\mathcal{N} = 4$ circular quiver SCSM theories with gauge group and levels $\prod_{\ell=1}^{r}[U(N_{2\ell-1})_k \times U(N_{2\ell})_{-k}]$ were introduced in [21, 22]. In the general case of different group ranks they can be obtained via decomposition of the $U(N)_k \times U(M)_{-k}$ ABJ theory with $N = \sum_{\ell=1}^{r} N_{2\ell-1}$ and $M = \sum_{\ell=1}^{r} N_{2\ell}$ [15, 23]. In the special case of equal ranks one obtains the $\mathcal{N} = 4$ orbifold ABJM theory with $[U(N)_k \times U(N)_{-k}]^r$ gauge group [23]. This theory has a dual description in terms of M-theory in $AdS_4 \times S^7/(\mathbb{Z}_{rk} \times \mathbb{Z}_r)$ background [23–25].

For a class of $\mathcal{N} = 4$ SCSM theories corresponding to circular quivers with alternating levels[1] 1/4 BPS WLs were constructed in [7], which are the holonomy of superconnections that include parametric couplings to scalars and fermions. This was carried out under the assumption that the superconnections can be always written as 2×2 block diagonal matrices. As a consequence the corresponding WLs, when traced, can be expressed as linear combinations of r double-node operators $W^{(\ell)}$ that are nothing but the holonomy of the 2×2 block matrices. Therefore, their study reduces to the study of the generic $W^{(\ell)}$ confined at two adjacent nodes. For a particular choice of the parameters they give rise to the bosonic 1/4 BPS WL that includes couplings only to scalars and the fermionic 1/2 BPS WL with couplings also to fermions [27, 28].

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[1]The $\mathcal{N} = 4$ SCSM theories with vanishing levels [26, 27] or corresponding to linear quivers [21] are not included in the discussion.

In this paper we provide a more general construction of BPS WLs in $\mathcal{N} = 4$ SCSM theories using two different strategies. To begin with, we consider $[\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}]^r$ ABJM orbifold models and construct BPS WLs by performing a direct decomposition of BPS WLs classified in ABJM theory. Although this does not provide a full classification of WLs in orbifold ABJM theory, it allows to obtain a class of BPS operators that is however larger than the one known so far. Moreover, this method has the virtue to automatically provide the M-theory dual description of these new operators as given by M2-/anti-M2-branes wrapping some particular circles in the internal $\mathrm{S}^7/(\mathbb{Z}_{rk} \times \mathbb{Z}_r)$ space. The second method is more systematic and leads to a full classification of 1/4 and 1/2 BPS WLs for general circular quivers with alternating Chen-Simons levels. It is based on directly imposing the invariance of the most general superconnection compatible with the symmetries of the theory under a given fraction of SUSY charges.

Following these strategies, we find 1/4 and 1/2 BPS WLs already constructed in the literature as the holonomy of double-node superconnections in [27, 28] and [7], but also new ones that correspond to non-block-diagonal superconnections and are thus not ascribable to straightforward generalizations of the previous ones. These new WLs are in general 1/4 BPS and they are enhanced to the usual 1/2 BPS for special choices of the parameters. As we will discuss in details, they are all cohomological equivalent to the bosonic 1/4 BPS operators whose expectation values can be in principle computed with localization techniques.

The paper is organized as follows. In section 2 we review the classification of known BPS WLs in ABJ(M) theory by parametrizing the couplings to matter in the most general way. In section 3 we determine the BPS WLs in the orbifold ABJM model that can be obtained by decomposition of BPS WLs in ABJM theory. In particular, we find new BPS WLs that were not present in the previous literature. Section 3.3 is devoted to a discussion of the M2-/anti-M2-brane duals of BPS WLs in $\mathcal{N} = 4$ orbifold ABJM theory. In section 4, by studying the invariance of the most general superconnection under a given fraction of SUSY transformations we give a complete classification of all possible 1/4 and 1/2 BPS WLs in $\mathcal{N} = 4$ circular quiver SCSM theories. Finally, a summary of our main results is contained in section 5 where we also comment on the expected perturbative results for the newly found WLs and their matching with the localization prediction. Technical details about the general derivation of section 4 are collected in appendix A. As a completion to the classification of BPS WLs in ABJ(M) and $\mathcal{N} = 4$ SCSM theories, in appendix B we study the overlapping of preserved supercharges for general 1/2 BPS WLs in both theories. We find no new pairs of different operators preserving the same set of supercharges beyond the ones already discovered in [27] and [15]. In the main body of the paper we focus on timelike linear WLs in Minkowski spacetime, whereas circular 1/4 BPS WLs in Euclidean space are discussed in appendix C.

2 Review of BPS WLs in ABJ(M) theory

In this section we review the general classification of 1/6 and 1/2 BPS WLs in ABJ(M) theory given in [7, 11]. We present it in a way that turns out to be preparatory for the study of BPS operators in $\mathcal{N} = 4$ SCSM theories.

The ABJ(M) lagrangian in Minkowski spacetime can be written as the sum of four terms

$$\mathcal{L}_{CS} = \frac{k}{4\pi}\varepsilon^{\mu\nu\rho}\text{Tr}\left(A_\mu\partial_\nu A_\rho + \frac{2\text{i}}{3}A_\mu A_\nu A_\rho - B_\mu\partial_\nu B_\rho - \frac{2\text{i}}{3}B_\mu B_\nu B_\rho\right)$$

$$\mathcal{L}_k = \text{Tr}\left(-D_\mu\bar{\phi}^I D^\mu\phi_I + \text{i}\bar{\psi}_I\gamma^\mu D_\mu\psi^I\right)$$

$$\mathcal{L}_p = \frac{4\pi^2}{3k^2}\text{Tr}\left(\phi_I\bar{\phi}^I\phi_J\bar{\phi}^J\phi_K\bar{\phi}^K + \phi_I\bar{\phi}^J\phi_J\bar{\phi}^K\phi_K\bar{\phi}^I + 4\phi_I\bar{\phi}^J\phi_K\bar{\phi}^I\phi_J\bar{\phi}^K - 6\phi_I\bar{\phi}^J\phi_J\bar{\phi}^I\phi_K\bar{\phi}^K\right)$$

$$\mathcal{L}_Y = \frac{2\pi\text{i}}{k}\text{Tr}\big(\phi_I\bar{\phi}^I\psi^J\bar{\psi}_J - 2\phi_I\bar{\phi}^J\psi^I\bar{\psi}_J - \bar{\phi}^I\phi_I\bar{\psi}_J\psi^J + 2\bar{\phi}^I\phi_J\bar{\psi}_I\psi^J$$

$$+ \varepsilon^{IJKL}\phi_I\bar{\psi}_J\phi_K\bar{\psi}_L - \varepsilon_{IJKL}\bar{\phi}^I\psi^J\bar{\phi}^K\psi^L\big) \tag{2.1}$$

where the totally anti-symmetric Levi-Civita tensors ε^{IJKL}, ε_{IJKL} are defined as $\varepsilon^{1234} = \varepsilon_{1234} = 1$. Here A_μ and B_μ are the connections of U(N) and U(M) gauge groups respectively, ϕ_I, ψ^I ($I = 1, 2, 3, 4$) are complex scalars and Dirac fermions in the bifundamental representation of the gauge group and in the fundamental representation of the SU(4) R-symmetry group, and $\bar{\phi}^I, \bar{\psi}_I$ their complex conjugates. Covariant derivatives are defined as

$$D_\mu\phi_I = \partial_\mu\phi_I + \text{i}A_\mu\phi_I - \text{i}\phi_I B_\mu$$

$$D_\mu\bar{\phi}^I = \partial_\mu\bar{\phi}^I - \text{i}\bar{\phi}^I A_\mu + \text{i}B_\mu\bar{\phi}^I$$

$$D_\mu\psi^I = \partial_\mu\psi^I + \text{i}A_\mu\psi^I - \text{i}\psi^I B_\mu$$

$$D_\mu\bar{\psi}_I = \partial_\mu\bar{\psi}_I - \text{i}\bar{\psi}_I A_\mu + \text{i}B_\mu\bar{\psi}_I \tag{2.2}$$

The ABJ(M) action is invariant under the following Poincaré SUSY transformations [9, 29–31]

$$\delta A_\mu = -\frac{2\pi}{k}\left(\phi_I\bar{\psi}_J\gamma_\mu\theta^{IJ} + \bar{\theta}_{IJ}\gamma_\mu\psi^J\bar{\phi}^I\right)$$

$$\delta B_\mu = -\frac{2\pi}{k}\left(\bar{\psi}_J\phi_I\gamma_\mu\theta^{IJ} + \bar{\theta}_{IJ}\gamma_\mu\bar{\phi}^I\psi^J\right)$$

$$\delta\phi_I = \text{i}\bar{\theta}_{IJ}\psi^J, \qquad \delta\bar{\phi}^I = \text{i}\bar{\psi}_J\theta^{IJ}$$

$$\delta\psi^I = \gamma^\mu\theta^{IJ}D_\mu\phi_J + \frac{2\pi}{k}\theta^{IJ}\left(\phi_J\bar{\phi}^K\phi_K - \phi_K\bar{\phi}^K\phi_J\right) + \frac{4\pi}{k}\theta^{KL}\phi_K\bar{\phi}^I\phi_L$$

$$\delta\bar{\psi}_I = -\bar{\theta}_{IJ}\gamma^\mu D_\mu\bar{\phi}^J - \frac{2\pi}{k}\bar{\theta}_{IJ}\left(\bar{\phi}^J\phi_K\bar{\phi}^K - \bar{\phi}^K\phi_K\bar{\phi}^J\right) - \frac{4\pi}{k}\bar{\theta}_{KL}\bar{\phi}^K\phi_I\bar{\phi}^L \tag{2.3}$$

with the θ^{IJ} parameters satisfying

$$\theta^{IJ} = -\theta^{JI}, \ \ (\theta^{IJ})^* = \bar{\theta}_{IJ}, \ \ \bar{\theta}_{IJ} = \frac{1}{2}\varepsilon_{IJKL}\theta^{KL} \tag{2.4}$$

$\mathcal{N} = 6$ SUSY is realized explicitly.

The whole spectrum of BPS WLs defined along the timelike infinite straight line $x^\mu = (\tau, 0, 0)$ can be described by a parametric family of operators [2, 7, 11]

$$W = \mathcal{P}\exp\left(-\text{i}\int d\tau L(\tau)\right) \tag{2.5}$$

corresponding to a generalized U($N|M$) superconnection that includes couplings to scalar and fermion matter fields[2]

$$L = \begin{pmatrix} A_0 + \frac{2\pi}{k} U^I{}_J \phi_I \bar{\phi}^J & \sqrt{\frac{4\pi}{k}}(\bar{\alpha}_I \psi_+^I + \bar{\gamma}_I \psi_-^I) \\ \sqrt{\frac{4\pi}{k}}(\bar{\psi}_{I-}\beta^I - \bar{\psi}_{I+}\delta^I) & B_0 + \frac{2\pi}{k} U^I{}_J \bar{\phi}^J \phi_I \end{pmatrix} \tag{2.6}$$

Here $U^I{}_J$ is a 4×4 matrix with constant complex entries, whereas the fermionic couplings are given by constant vectors $\bar{\alpha}_I$, $\bar{\gamma}_I$, β^I, δ^I in $\mathbb{C}^4$ ($\alpha^I = (\bar{\alpha}_I)^*$, $|\alpha|^2 = \bar{\alpha}_I \alpha^I$, etc.).

For special choices of the parameters, superconnection (2.6) gives rise to BPS WLs preserving a certain amount of SUSY, as we will review below for 1/6 and 1/2 BPS cases.

2.1 1/6 BPS WLs

An exhaustive classification of 1/6 BPS WLs in ABJ(M) theory has been given in [7, 11]. These are operators that preserve two real Poincaré supercharges plus two real superconformal charges out of the original $12 + 12$ real supercharges.

In order to make the classification clearer, it is convenient to define two projectors $P^I{}_J$ and $Q^I{}_J$ in the SU(4) R-symmetry space that satisfy

$$\begin{aligned} P^I{}_J + Q^I{}_J = \delta^I_J, \qquad P^I{}_J Q^J{}_K = Q^I{}_J P^J{}_K = 0 \\ P^I{}_J P^J{}_K = P^I{}_K, \quad Q^I{}_J Q^J{}_K = Q^I{}_K, \qquad P^I{}_I = Q^I{}_I = 2 \end{aligned} \tag{2.7}$$

and break R-symmetry as SU(4)$\rightarrow$SU(2)$_L\times$SU(2)$_R$. We can rewrite the two projectors as

$$P^I{}_J = \mu^I \bar{\mu}_J + \nu^I \bar{\nu}_J, \quad Q^I{}_J = \rho^I \bar{\rho}_J + \sigma^I \bar{\sigma}_J \tag{2.8}$$

in terms of four orthonormal vectors in $\mathbb{C}^4$ satisfying

$$\begin{aligned} \bar{\mu}_I \mu^I = \bar{\nu}_I \nu^I = \bar{\rho}_I \rho^I = \bar{\sigma}_I \sigma^I = 1 \\ \bar{\mu}_I \nu^I = \bar{\mu}_I \rho^I = \bar{\mu}_I \sigma^I = \bar{\nu}_I \rho^I = \bar{\nu}_I \sigma^I = \bar{\rho}_I \sigma^I = 0 \end{aligned} \tag{2.9}$$

The two projectors are not independent, as $Q^I{}_J$ can be expressed in terms of $P^I{}_J$ using the first equation in (2.7). Moreover, for fixed $P^I{}_J$ and $Q^I{}_J$ there is some freedom in the choice of the $\bar{\mu}_I$, $\bar{\nu}_I$, $\bar{\rho}_I$, $\bar{\sigma}_I$ vectors, which are always determined up to a SU(2)$_L\times$SU(2)$_R$ rotation.

All 1/6 BPS WLs can be classified according to the choices of the parameters in table 1. The $U^I{}_J$ couplings are given in terms of the two projectors written as in (2.8) and we have expressed also the fermionic couplings as linear combination of the $\bar{\mu}_I$, $\bar{\nu}_I$, $\bar{\rho}_I$, $\bar{\sigma}_I$ vectors with four arbitrary constant complex parameters p_I. Their particular decompositions follow from the constraints $\bar{\alpha}_I P^I{}_J = P^I{}_J \beta^J = \bar{\gamma}_I Q^I{}_J = Q^I{}_J \delta^J = 0$ [7, 11].

In general, 1/6 BPS WLs in the first four classes include non-trivial couplings both to scalars and fermions. For this reason they are called *fermionic* 1/6 BPS WLs, in contrast

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[2]In three-dimensional Minkowski spacetime, we use the gamma matrix $\gamma^\mu{}_\alpha{}^\beta = (i\sigma^2, \sigma^1, \sigma^3)$ with $\sigma^{1,2,3}$ being the Pauli matrices. For an arbitrary spinor θ_α, we define $\theta_\pm = \pm i u_\pm^\alpha \theta_\alpha$ with the Grassmann even spinors $u_\pm^\alpha = \frac{1}{\sqrt{2}}(\mp i, -1)$ [15, 32].

WL type	Choice of the parameters
$W^{\mathrm{I}}_{1/6}[\bar{\mu}_I, \bar{\nu}_I, p_I]$	$U^I_J = \mu^I \bar{\mu}_J + \nu^I \bar{\nu}_J - (1 - 2\bar{\alpha}_K \beta^K)(\delta^I_J - \mu^I \bar{\mu}_J - \nu^I \bar{\nu}_J) - 2\beta^I \bar{\alpha}_J$ $\bar{\alpha}_I = p_1 \bar{\rho}_I + p_2 \bar{\sigma}_I, \quad \beta^I = p_3 \rho^I + p_4 \sigma^I, \quad \bar{\gamma}_I = \delta^I = 0$
$W^{\mathrm{II}}_{1/6}[\bar{\mu}_I, \bar{\nu}_I, p_I]$	$U^I_J = (1 - 2\bar{\gamma}_K \delta^K)(\mu^I \bar{\mu}_J + \nu^I \bar{\nu}_J) + 2\delta^I \bar{\gamma}_J - (\delta^I_J - \mu^I \bar{\mu}_J - \nu^I \bar{\nu}_J)$ $\bar{\gamma}_I = p_1 \bar{\mu}_I + p_2 \bar{\nu}_I, \quad \delta^I = p_3 \mu^I + p_4 \nu^I, \quad \bar{\alpha}_I = \beta^I = 0$
$W^{\mathrm{III}}_{1/6}[\bar{\mu}_I, \bar{\nu}_I, p_I]$	$U^I_J = -\delta^I_J + 2\mu^I \bar{\mu}_J + 2\nu^I \bar{\nu}_J$ $\bar{\alpha}_I = p_1 \bar{\rho}_I + p_2 \bar{\sigma}_I, \quad \bar{\gamma}_I = p_3 \bar{\mu}_I + p_4 \bar{\nu}_I, \quad \beta^I = \delta^I = 0$
$W^{\mathrm{IV}}_{1/6}[\bar{\mu}_I, \bar{\nu}_I, p_I]$	$U^I_J = -\delta^I_J + 2\mu^I \bar{\mu}_J + 2\nu^I \bar{\nu}_J$ $\beta^I = p_1 \rho^I + p_2 \sigma^I, \quad \delta^I = p_3 \mu^I + p_4 \nu^I, \quad \bar{\alpha}_I = \bar{\gamma}_I = 0$
$W^{\mathrm{bos}}_{1/6}[\bar{\mu}_I, \bar{\nu}_I]$	$U^I_J = -\delta^I_J + 2\mu^I \bar{\mu}_J + 2\nu^I \bar{\nu}_J, \quad \bar{\alpha}_I = \bar{\gamma}_I = \beta^I = \delta^I = 0$

Table 1. The four types of *fermionic* WLs and the *bosonic* 1/6 BPS WL in ABJ(M) theory. In writing the U^I_J matrices we have used $\rho^I \bar{\rho}_J + \sigma^I \bar{\sigma}_J = \delta^I_J - \mu^I \bar{\mu}_J - \nu^I \bar{\nu}_J$, which follows from the first identity in (2.7).

with $W^{\mathrm{bos}}_{1/6}$ that, including only couplings to scalars, is called *bosonic* 1/6 BPS WL. All the 1/6 BPS WLs preserve the same set of Poincaré supercharges,[3] which correspond to

$$P^I_K P^J_L \theta^{KL}_-, \quad Q^I_K Q^J_L \theta^{KL}_+ \tag{2.10}$$

Different classes of WLs give rise to independent operators, as they cannot be mapped one into the other by R-symmetry rotations. In fact, $W^{\mathrm{I}}_{1/6}$ and $W^{\mathrm{II}}_{1/6}$ operators break the original SU(4) R-symmetry down to SU(2)$_L$ and SU(2)$_R$, respectively. For $W^{\mathrm{III}}_{1/6}$ and $W^{\mathrm{IV}}_{1/6}$, the R-symmetry is broken completely. Finally, $W^{\mathrm{bos}}_{1/6}$ is invariant under the SU(2)$_L \times$ SU(2)$_R$ subgroup.

After fixing P^I_J, Q^I_J in the R-symmetry space, i.e., fixing the preserved supercharges (2.10), each of the four types of fermionic WLs depends on four independent complex parameters p_I, while the bosonic WL is totally fixed. The bosonic WL can be obtained from any of the four fermionic WLs by setting $p_I = 0$, $I = 1, 2, 3, 4$, and so it is just one particular representative of the four families of 1/6 BPS WLs.

It is important to recall that all the fermionic 1/6 BPS WLs are cohomologically equivalent to the bosonic 1/6 BPS WL, being expressible as [2, 7]

$$W^{\mathrm{I,II,III,IV}}_{1/6}[\bar{\mu}_I, \bar{\nu}_I, p_I] = W^{\mathrm{bos}}_{1/6}[\bar{\mu}_I, \bar{\nu}_I] + \mathcal{Q} V^{\mathrm{I,II,III,IV}}[\bar{\mu}_I, \bar{\nu}_I, p_I] \tag{2.11}$$

where $\mathcal{Q}$ is a linear combination of preserved supercharges (2.10). This property turns out to be very important when computing vacuum expectation values (vev) of Euclidean circular BPS WLs in the ABJ(M) theory compactified on S^3. In fact, using the $\mathcal{Q}$ supercharge in (2.11) to localize the path integral, the cohomological relation implies that at quantum level all the vev are identical and equal to $\langle W^{\mathrm{bos}}_{1/6} \rangle$ computed by a matrix model [4, 33, 34].

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[3]For WLs on an infinite straight line, Poincaré and conformal supercharges are separately and similarly preserved. Therefore, it is sufficient to discuss WLs invariance under Poincaré supercharges.

2.2 1/2 BPS WLs

For special values of the parameters in table 1, SUSY gets enhanced and we obtain operators that preserve half of the supersymmetries [2]. Precisely, this happens in Class I by setting $\beta^I = \frac{\alpha^I}{|\alpha|^2}$, $|\alpha|^2 \neq 0$ and in Class II for $\delta^I = \frac{\gamma^I}{|\gamma|^2}$, $|\gamma|^2 \neq 0$. In our notations of table 1 this corresponds to choosing

$$p_3 = \frac{\bar{p}_1}{p_1 \bar{p}_1 + p_2 \bar{p}_2} \qquad p_4 = \frac{\bar{p}_2}{p_1 \bar{p}_1 + p_2 \bar{p}_2} \qquad p_1 \bar{p}_1 + p_2 \bar{p}_2 \neq 0 \qquad (2.12)$$

We denote the corresponding 1/2 BPS WLs as $W^{\mathrm{I}}_{1/2}[\bar{\alpha}_I]$ and $W^{\mathrm{II}}_{1/2}[\bar{\gamma}_I]$, respectively. They are manifestly invariant under a residual SU(3) R-symmetry subgroup.

Being particular cases of $W^{\mathrm{I,II}}_{1/6}[\bar{\mu}_I, \bar{\nu}_I, p_I]$ families, these operators have to preserve supercharges (2.10). However, due to their SU(3) invariance, the corresponding sets of preserved supercharges contain also supercharges that are SU(3) rotations of (2.10). It is easy to prove that this enlarges the set of preserved SUSY's to

$$\begin{aligned} W^{\mathrm{I}}_{1/2}[\bar{\alpha}_I]: & \quad \bar{\alpha}_J \theta^{IJ}_+, \quad \varepsilon_{IJKL}\alpha^J \theta^{KL}_- \\ W^{\mathrm{II}}_{1/2}[\bar{\gamma}_I]: & \quad \bar{\gamma}_J \theta^{IJ}_-, \quad \varepsilon_{IJKL}\gamma^J \theta^{KL}_+ \end{aligned} \qquad (2.13)$$

As long as we consider the two 1/2 BPS WLs as particular representatives of Class I and Class II fermionic WLs, the corresponding parameters are necessarily orthogonal, i.e. $\bar{\alpha}_I \gamma^I = 0$. However, at this point nothing prevents from relaxing these conditions and freely rotating the two vectors in the SU(4) R-symmetry space. From (2.13) it then follows that for a given representative of Class I selected by choosing a specific vector $\bar{\alpha}_I$ in $\mathbb{C}^4$, it is possible to select a representative in Class II corresponding to $\bar{\gamma}_I = \bar{\alpha}_I$ that preserves the complementary set of supercharges. For ABJM theory, they are in fact dual to a pair of M2-/anti-M2-branes placed at the same position in $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$ spacetime [15]. For a more general discussion on the overlapping of preserved supercharges for different values of the parameters we refer to appendix B.

From identity (2.11) and R-symmetry, it follows that $W^{\mathrm{I}}_{1/2}[\bar{\alpha}_I]$ is cohomologically equivalent to the bosonic 1/6 BPS WL $W^{\mathrm{bos}}_{1/6}[\bar{\mu}_I, \bar{\nu}_I]$ for arbitrary $\bar{\mu}_I$, $\bar{\nu}_I$ satisfying $\bar{\alpha}_I \mu^I = \bar{\alpha}_I \nu^I = \bar{\mu}_I \nu^I = 0$, $\bar{\mu}_I \mu^I = \bar{\nu}_I \nu^I = 1$. Analogously, $W^{\mathrm{II}}_{1/2}[\bar{\gamma}_I]$ is cohomologically equivalent to the bosonic 1/6 BPS WL $W^{\mathrm{bos}}_{1/6}[\bar{\gamma}_I/|\gamma|^2, \bar{\nu}_I]$ for arbitrary $\bar{\nu}_I$ that satisfies $\bar{\gamma}_I \nu^I = 0$, $\bar{\nu}_I \nu^I = 1$.

3 BPS WLs in $\mathcal{N} = 4$ orbifold ABJM theory

As is well-known, a particular realization of Chern-Simons-matter theory with $\mathcal{N} = 4$ supersymmetry can be obtained by orbifold projection of the ABJM theory [23]. Precisely, starting from the ABJM theory with gauge group and levels $\mathrm{U}(rN)_k \times \mathrm{U}(rN)_{-k}$ and taking a $\mathbb{Z}_r$ quotient one obtains the $\mathcal{N} = 4$ orbifold ABJM theory corresponding to a circular quiver $[\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}]^r$ with alternating levels.

The study of BPS WLs in orbifold ABJM theory was initiated in [27, 28] and refined in [7], both for straight line contours in Minkowski and circular contours in Euclidean space. The presently known classification includes one bosonic and four classes of fermionic WLs obtained by assuming that the corresponding superconnection can be always written as a 2×2 block diagonal matrix. As a consequence, when traced, the WL can be written as a linear combination of WLs connecting two adjacent quiver nodes.

Aimed at extending this classification, here we use an alternative approach to construct WLs in orbifold ABJM that does not require making any particular ansatz on the structure of the superconnection. Precisely, we construct a class of Wilson operators by performing the orbifold decomposition of WLs of the ABJM theory.

The orbifold decomposition breaks the original SU(4) R-symmetry to SU(2)$_L \times$SU(2)$_R$. Correspondingly, we choose the following decomposition of the R-symmetry indices

$$I = 1, 2, 4, 3 \; \rightarrow \; i = 1, 2, \; \hat{i} = \hat{1}, \hat{2} \tag{3.1}$$

A generic vector $\bar{\alpha}_I$ in $\mathbb{C}^4$ is then decomposed as

$$\bar{\alpha}_I \; \rightarrow \; \bar{\alpha}_i, \bar{\alpha}_{\hat{i}} \tag{3.2}$$

with complex conjugates $\alpha^i = (\bar{\alpha}_i)^*, \alpha^{\hat{i}} = (\bar{\alpha}_{\hat{i}})^*$. We also define $|\alpha|^2 = \bar{\alpha}_i \alpha^i + \bar{\alpha}_{\hat{i}} \alpha^{\hat{i}}$.

In the U$(rN)_k \times$ U$(rN)_{-k}$ ABJM model we consider the most general superconnection of the form (2.6). Applying the orbifold decomposition the supermatrix gets decomposed for $r \geq 3$ as

$$L = \begin{pmatrix} \mathcal{A}^{(1)} & f_1^{(1)} & h_1^{(1)} & 0 & 0 & \cdots & h_2^{(2r-1)} & f_2^{(2r)} \\ f_2^{(1)} & \mathcal{B}^{(2)} & f_1^{(2)} & h_1^{(2)} & 0 & \cdots & 0 & h_2^{(2r)} \\ h_2^{(1)} & f_2^{(2)} & \mathcal{A}^{(3)} & f_1^{(3)} & \ddots & & 0 & 0 \\ 0 & h_2^{(2)} & f_2^{(3)} & \mathcal{B}^{(4)} & \ddots & \ddots & & \vdots \\ 0 & & \ddots & \ddots & \ddots & \ddots & h_1^{(2r-3)} & 0 \\ \vdots & & & \ddots & \ddots & \ddots & f_1^{(2r-2)} & h_1^{(2r-2)} \\ h_1^{(2r-1)} & 0 & \cdots & 0 & h_2^{(2r-3)} & f_2^{(2r-2)} & \mathcal{A}^{(2r-1)} & f_1^{(2r-1)} \\ f_1^{(2r)} & h_1^{(2r)} & 0 & \cdots & 0 & h_2^{(2r-2)} & f_2^{(2r-1)} & \mathcal{B}^{(2r)} \end{pmatrix} \tag{3.3}$$

and for $r = 2$ as

$$L = \begin{pmatrix} \mathcal{A}^{(1)} & f_1^{(1)} & h_1^{(1)} + h_2^{(3)} & f_2^{(4)} \\ f_2^{(1)} & \mathcal{B}^{(2)} & f_1^{(2)} & h_1^{(2)} + h_2^{(4)} \\ h_1^{(3)} + h_2^{(1)} & f_2^{(2)} & \mathcal{A}^{(3)} & f_1^{(3)} \\ f_1^{(4)} & h_1^{(4)} + h_2^{(2)} & f_2^{(3)} & \mathcal{B}^{(4)} \end{pmatrix} \tag{3.4}$$

where we have defined

$$\mathcal{A}^{(2\ell-1)} = A_0^{(2\ell-1)} + \frac{2\pi}{k}\left(U^i{}_j\phi^{(2\ell-2)}_i\bar{\phi}^j_{(2\ell-2)} + U^{\hat{i}}{}_{\hat{j}}\phi^{(2\ell-1)}_{\hat{i}}\bar{\phi}^{\hat{j}}_{(2\ell-1)}\right)$$

$$\mathcal{B}^{(2\ell)} = B_0^{(2\ell)} + \frac{2\pi}{k}\left(U^i{}_j\bar{\phi}^j_{(2\ell)}\phi^{(2\ell)}_i + U^{\hat{i}}{}_{\hat{j}}\bar{\phi}^{\hat{j}}_{(2\ell-1)}\phi^{(2\ell-1)}_{\hat{i}}\right)$$

$$f_1^{(2\ell-1)} = \sqrt{\frac{4\pi}{k}}\left(\bar{\alpha}_i\psi^i_{(2\ell-1)+} + \bar{\gamma}_i\psi^i_{(2\ell-1)-}\right), \qquad f_1^{(2\ell)} = \sqrt{\frac{4\pi}{k}}\left(\bar{\psi}^{(2\ell)}_{\hat{i}-}\beta^{\hat{i}} - \bar{\psi}^{(2\ell)}_{\hat{i}+}\delta^{\hat{i}}\right)$$

$$f_2^{(2\ell-1)} = \sqrt{\frac{4\pi}{k}}\left(\bar{\psi}^{(2\ell-1)}_{i-}\beta^i - \bar{\psi}^{(2\ell-1)}_{i+}\delta^i\right), \qquad f_2^{(2\ell)} = \sqrt{\frac{4\pi}{k}}\left(\bar{\alpha}_{\hat{i}}\psi^{\hat{i}}_{(2\ell)+} + \bar{\gamma}_{\hat{i}}\psi^{\hat{i}}_{(2\ell)-}\right)$$

$$h_1^{(2\ell-1)} = \frac{2\pi}{k}U^{\hat{i}}{}_j\phi^{(2\ell-1)}_{\hat{i}}\bar{\phi}^j_{(2\ell)}, \qquad h_1^{(2\ell)} = \frac{2\pi}{k}U^{\hat{i}}{}_j\bar{\phi}^j_{(2\ell)}\phi^{(2\ell+1)}_{\hat{i}}$$

$$h_2^{(2\ell-1)} = \frac{2\pi}{k}U^i{}_{\hat{j}}\phi^{(2\ell)}_i\bar{\phi}^{\hat{j}}_{(2\ell-1)}, \qquad h_2^{(2\ell)} = \frac{2\pi}{k}U^i{}_{\hat{j}}\bar{\phi}^{\hat{j}}_{(2\ell+1)}\phi^{(2\ell)}_i \qquad (3.5)$$

In superconnections (3.3) and (3.4) we have the usual gauge and scalar field couplings in the diagonal blocks, fermion fields in the next-to-diagonal blocks, and also new scalar field couplings in the next-to-next-to-diagonal blocks. The novelty here is the presence of these next-to-next-to-diagonal blocks.[4] These new blocks, present both in the bosonic and fermionic WLs, together with generically non-vanishing $f_{1,2}^{(2\ell-1)}$ and $f_{1,2}^{(2\ell)}$ blocks define superconnections that are outside the general class of 2×2 block diagonal superconnections considered so far in the literature [7, 27, 28, 35]. They reduce to those ones for the particular choice of the couplings $U^{\hat{i}}{}_j = U^i{}_{\hat{j}} = 0$ and either $f_{1,2}^{(2\ell-1)} = 0$ or $f_{1,2}^{(2\ell)} = 0, \forall\,\ell$.

In the next two subsections we give explicit examples of the construction of BPS WLs in orbifold ABJM theory from orbifold reduction of 1/6 and 1/2 BPS operators in ABJM theory.

3.1 From 1/6 BPS WLs in ABJM theory

We begin by considering the decomposition of the BPS WLs reviewed in subsection 2.1. Decomposing the constant vector parameters of the general superconnection (2.6) according to (3.2) and imposing constraints (2.9) we obtain generic WLs $W[\bar{\mu}_i, \bar{\mu}_{\hat{i}}, \bar{\nu}_i, \bar{\nu}_{\hat{i}}, p_I]$ with

$$\bar{\mu}_i\mu^i + \bar{\mu}_{\hat{i}}\mu^{\hat{i}} = \bar{\nu}_i\nu^i + \bar{\nu}_{\hat{i}}\nu^{\hat{i}} = 1, \qquad \bar{\mu}_i\nu^i + \bar{\mu}_{\hat{i}}\nu^{\hat{i}} = 0 \qquad (3.6)$$

and p_I still labeling four constant complex numbers.

Imposing that the new operators in the $\mathcal{N} = 4$ theory preserve some amount of SUSY provides further constraints on the complex parameters. We find that the solution $\bar{\mu}_{\hat{i}} = 0$, $\bar{\nu}_i = 0$ leads to fermionic BPS WLs preserving two Poincaré and two conformal super-charges.[5] The resulting operators, obtained from the four classes of fermionic WLs in table 1, and the corresponding parameters are classified in table 2.

It can be shown that all the four types of fermionic BPS WLs preserve the same set of supercharges corresponding to

$$\bar{\mu}_i\bar{\nu}_{\hat{j}}\theta^{i\hat{j}}_-, \quad \varepsilon_{ij}\varepsilon_{\hat{k}\hat{l}}\mu^i\nu^{\hat{k}}\theta^{j\hat{l}}_+ \qquad (3.7)$$

---

[4]Non-vanishing next-to-next-to-diagonal blocks have been considered in [27] only for 1/2 BPS WLs in $\mathcal{N} = 4$ SCSM theories with some quiver nodes corresponding to vanishing CS levels.

[5]An alternative solution would be $\bar{\mu}_i = 0$, $\bar{\nu}_{\hat{i}} = 0$, which leads to equivalent WLs. Without loss of generality we discuss only one case.

WL type	Choice of the parameters
$W_{1/4}^{\mathrm{I}}[\bar{\mu}_i, 0, 0, \bar{\nu}_{\hat{i}}, p_I]$	$U^i{}_j = \mu^i \bar{\mu}_j - (1 - 2\bar{\alpha}_k \beta^k - 2\bar{\alpha}_{\hat{k}} \beta^{\hat{k}})(\delta^i_j - \mu^i \bar{\mu}_j) - 2\beta^i \bar{\alpha}_j$ $U^{\hat{i}}{}_{\hat{j}} = \nu^{\hat{i}} \bar{\nu}_{\hat{j}} - (1 - 2\bar{\alpha}_k \beta^k - 2\bar{\alpha}_{\hat{k}} \beta^{\hat{k}})(\delta^{\hat{i}}_{\hat{j}} - \nu^{\hat{i}} \bar{\nu}_{\hat{j}}) - 2\beta^{\hat{i}} \bar{\alpha}_{\hat{j}}$ $U^i{}_{\hat{j}} = -2\beta^i \bar{\alpha}_{\hat{j}}, \quad U^{\hat{i}}{}_j = -2\beta^{\hat{i}} \bar{\alpha}_j, \quad \bar{\gamma}_i = \bar{\gamma}_{\hat{i}} = \delta^i = \delta^{\hat{i}} = 0$ $\bar{\alpha}_i = p_1 \varepsilon_{ij} \mu^j, \quad \bar{\alpha}_{\hat{i}} = p_2 \varepsilon_{\hat{i}\hat{j}} \nu^{\hat{j}}, \quad \beta^i = p_3 \varepsilon^{ij} \bar{\mu}_j, \quad \beta^{\hat{i}} = p_4 \varepsilon^{\hat{i}\hat{j}} \bar{\nu}_{\hat{j}}$
$W_{1/4}^{\mathrm{II}}[\bar{\mu}_i, 0, 0, \bar{\nu}_{\hat{i}}, p_I]$	$U^i{}_j = (1 - 2\bar{\gamma}_k \delta^k - 2\bar{\gamma}_{\hat{k}} \delta^{\hat{k}})\mu^i \bar{\mu}_j + 2\delta^i \bar{\gamma}_j - (\delta^i_j - \mu^i \bar{\mu}_j)$ $U^{\hat{i}}{}_{\hat{j}} = (1 - 2\bar{\gamma}_k \delta^k - 2\bar{\gamma}_{\hat{k}} \delta^{\hat{k}})\nu^{\hat{i}} \bar{\nu}_{\hat{j}} + 2\delta^{\hat{i}} \bar{\gamma}_{\hat{j}} - (\delta^{\hat{i}}_{\hat{j}} - \nu^{\hat{i}} \bar{\nu}_{\hat{j}})$ $U^i{}_{\hat{j}} = 2\delta^i \bar{\gamma}_{\hat{j}}, \quad U^{\hat{i}}{}_j = 2\delta^{\hat{i}} \bar{\gamma}_j, \quad \bar{\alpha}_i = \bar{\alpha}_{\hat{i}} = \beta^i = \beta^{\hat{i}} = 0$ $\bar{\gamma}_i = p_1 \bar{\mu}_i, \quad \bar{\gamma}_{\hat{i}} = p_2 \bar{\nu}_{\hat{i}}, \quad \delta^i = p_3 \mu^i, \quad \delta^{\hat{i}} = p_4 \nu^{\hat{i}}$
$W_{1/4}^{\mathrm{III}}[\bar{\mu}_i, 0, 0, \bar{\nu}_{\hat{i}}, p_I]$	$U^i{}_j = -\delta^i_j + 2\mu^i \bar{\mu}_j, \quad U^{\hat{i}}{}_{\hat{j}} = -\delta^{\hat{i}}_{\hat{j}} + 2\nu^{\hat{i}} \bar{\nu}_{\hat{j}}$ $U^i{}_{\hat{j}} = U^{\hat{i}}{}_j = \beta^i = \beta^{\hat{i}} = \delta^i = \delta^{\hat{i}} = 0$ $\bar{\alpha}_i = p_1 \varepsilon_{ij} \mu^j, \quad \bar{\alpha}_{\hat{i}} = p_2 \varepsilon_{\hat{i}\hat{j}} \nu^{\hat{j}}, \quad \bar{\gamma}_i = p_3 \bar{\mu}_i, \quad \bar{\gamma}_{\hat{i}} = p_4 \bar{\nu}_{\hat{i}}$
$W_{1/4}^{\mathrm{IV}}[\bar{\mu}_i, 0, 0, \bar{\nu}_{\hat{i}}, p_I]$	$U^i{}_j = -\delta^i_j + 2\mu^i \bar{\mu}_j, \quad U^{\hat{i}}{}_{\hat{j}} = -\delta^{\hat{i}}_{\hat{j}} + 2\nu^{\hat{i}} \bar{\nu}_{\hat{j}}$ $U^i{}_{\hat{j}} = U^{\hat{i}}{}_j = \bar{\alpha}_i = \bar{\alpha}_{\hat{i}} = \bar{\gamma}_i = \bar{\gamma}_{\hat{i}} = 0$ $\beta^i = p_1 \varepsilon^{ij} \bar{\mu}_j, \quad \beta^{\hat{i}} = p_2 \varepsilon^{\hat{i}\hat{j}} \bar{\nu}_{\hat{j}}, \quad \delta^i = p_3 \mu^i, \quad \delta^{\hat{i}} = p_4 \nu^{\hat{i}}$
$W_{1/4}^{\mathrm{bos}}[\bar{\mu}_i, 0, 0, \bar{\nu}_{\hat{i}}]$	$U^i{}_j = -\delta^i_j + 2\mu^i \bar{\mu}_j, \quad U^{\hat{i}}{}_{\hat{j}} = -\delta^{\hat{i}}_{\hat{j}} + 2\nu^{\hat{i}} \bar{\nu}_{\hat{j}}$ $U^i{}_{\hat{j}} = U^{\hat{i}}{}_j = \bar{\alpha}_i = \bar{\alpha}_{\hat{i}} = \bar{\gamma}_i = \bar{\gamma}_{\hat{i}} = \beta^i = \beta^{\hat{i}} = \delta^i = \delta^{\hat{i}} = 0$

Table 2. The four types of *fermionic* WLs and the *bosonic* 1/4 BPS WL in circular quiver $\mathcal{N} = 4$ SCSM theories with alternating levels. We have set $\bar{\mu}_{\hat{i}} = \bar{\nu}_i = 0$, and thus $\bar{\mu}_i \mu^i = \bar{\nu}_{\hat{i}} \nu^{\hat{i}} = 1$.

The important observation is that the classes of fermionic 1/4 BPS WLs found here through the orbifolding projection are more general than the ones constructed in [7]. In fact, for generic parameters superconnections (3.3), (3.4) are not block-diagonal and cannot be mapped simply by R-symmetry rotations to the block-diagonal matrices previously considered in the literature [7, 15, 27, 28, 35].

In order to better clarify this point, we focus on particular WL representatives in each class selected by choosing for instance $\bar{\mu}_i = (0, 1)$ and $\bar{\nu}_{\hat{i}} = (0, 1)$. As long as we keep the four p_I parameters generically different from zero, from (3.5) we read that superconnections in Class I have non-trivial entries

$$
\begin{aligned}
f_1^{(2\ell-1)} &= p_1 \sqrt{\frac{4\pi}{k}} \psi_{(2\ell-1)+}^1 & f_1^{(2\ell)} &= p_4 \sqrt{\frac{4\pi}{k}} \bar{\psi}_{\hat{1}-}^{(2\ell)} \\
f_2^{(2\ell-1)} &= p_3 \sqrt{\frac{4\pi}{k}} \bar{\psi}_{1-}^{(2\ell-1)} & f_2^{(2\ell)} &= p_2 \sqrt{\frac{4\pi}{k}} \psi_{(2\ell)+}^{\hat{1}} \\
h_1^{(2\ell-1)} &= -\frac{4\pi}{k} p_1 p_4 \, \phi_{\hat{1}}^{(2\ell-1)} \bar{\phi}_{(2\ell)}^1 & h_1^{(2\ell)} &= -\frac{4\pi}{k} p_1 p_4 \, \bar{\phi}_{(2\ell)}^1 \phi_{\hat{1}}^{(2\ell+1)} \\
h_2^{(2\ell-1)} &= -\frac{4\pi}{k} p_2 p_3 \, \phi_1^{(2\ell)} \bar{\phi}_{(2\ell-1)}^{\hat{1}} & h_2^{(2\ell)} &= -\frac{4\pi}{k} p_2 p_3 \, \bar{\phi}_{(2\ell+1)}^{\hat{1}} \phi_1^{(2\ell)}
\end{aligned}
\tag{3.8}
$$

that prevent the superconnections to be written as block diagonal matrices. For Class II we find a similar pattern. Although operators in Class III and IV always have $h_{1,2}^{(2\ell-1)} = h_{1,2}^{(2\ell)} = 0$, still they have non-vanishing $f_1^{(2\ell-1)}$, $f_2^{(2\ell)}$ and $f_2^{(2\ell-1)}$, $f_1^{(2\ell)}$ respectively,

$$\text{Class III:} \quad f_1^{(2\ell-1)} = \sqrt{\frac{4\pi}{k}}\left(p_1\,\psi_{(2\ell-1)+}^1 + p_3\,\psi_{(2\ell-1)-}^2\right), \quad f_2^{(2\ell)} = \sqrt{\frac{4\pi}{k}}\left(p_2\,\psi_{(2\ell)+}^{\hat{1}} + p_4\,\psi_{(2\ell)-}^{\hat{2}}\right)$$

$$\text{Class IV:} \quad f_2^{(2\ell-1)} = \sqrt{\frac{4\pi}{k}}\left(p_1\,\bar{\psi}_{1-}^{(2\ell-1)} - p_3\,\bar{\psi}_{2+}^{(2\ell-1)}\right), \qquad f_1^{(2\ell)} = \sqrt{\frac{4\pi}{k}}\left(p_2\,\bar{\psi}_{\hat{1}-}^{(2\ell)} - p_4\,\bar{\psi}_{\hat{2}+}^{(2\ell)}\right)$$

$$(3.9)$$

which do not allow to make them block-diagonal.

However, it can be easily realized that by choosing $p_2 = p_4 = 0$ we have $h_{1,2}^{(2\ell-1)} = h_{1,2}^{(2\ell)} = f_{1,2}^{(2\ell)} = 0$, and in all the cases the superconnections become block-diagonal

$$L = \text{diag}(L_1, L_2, \ldots, L_r), \qquad L_\ell = \begin{pmatrix} \mathcal{A}^{(2\ell-1)} & f_1^{(2\ell-1)} \\ f_2^{(2\ell-1)} & \mathcal{B}^{(2\ell)} \end{pmatrix} \qquad (3.10)$$

The corresponding traced WL

$$\text{Tr}\,\mathcal{P}\exp\left(-\mathrm{i}\int d\tau L(\tau)\right) \qquad (3.11)$$

can then be written as a linear combination of double-node operators $W^{(\ell)}$, which are the holonomy of superconnections L_ℓ with $\ell = 1, 2, \cdots, r$.

Similarly, by choosing $p_1 = p_3 = 0$, in all the cases we have $h_{1,2}^{(2\ell-1)} = h_{1,2}^{(2\ell)} = f_{1,2}^{(2\ell-1)} = 0$ and the superconnections become block-diagonal

$$\tilde{L} = \text{diag}(\tilde{L}_1, \tilde{L}_2, \ldots, \tilde{L}_r), \quad \tilde{L}_\ell = \begin{pmatrix} \mathcal{B}^{(2\ell)} & f_1^{(2\ell)} \\ f_2^{(2\ell)} & \mathcal{A}^{(2\ell+1)} \end{pmatrix} \qquad (3.12)$$

Fermionic 1/4 BPS WLs with block-diagonal superconnections have been considered in [7]. Our classification generalizes the previous one, providing the previously found 1/4 BPS WLs but also an infinite set of new 1/4 BPS operators.

To complete this picture we still need to consider the decomposition of the bosonic 1/6 BPS WL $W_{1/6}^{\text{bos}}[\bar{\mu}_I, \bar{\nu}_I]$ in table 1. We obtain an operator $W_{1/4}^{\text{bos}}[\bar{\mu}_i, 0, 0, \bar{\nu}_{\hat{i}}]$ with superconnections (3.3) or (3.4) and parameters given in table 2. It is a bosonic 1/4 BPS WL that preserves supercharges (3.7). It can also be got from $W_{1/4}^{\text{I,II,III,IV}}[\bar{\mu}_i, 0, 0, \bar{\nu}_{\hat{i}}, p_I]$ by setting $p_I = 0$. The bosonic 1/4 BPS WL has been constructed in [27, 28] and coincides with the present ones up to a R-symmetry rotation.

Finally, it is easy to see that cohomological equivalence (2.11) survives the decomposition. Therefore, fermionic 1/4 BPS WLs in table 2 are equivalent to the bosonic 1/4 BPS WL up to a $\mathcal{Q}$-exact term

$$W_{1/4}^{\text{I,II,III,IV}}[\bar{\mu}_i, 0, 0, \bar{\nu}_{\hat{i}}, p_I] = W_{1/4}^{\text{bos}}[\bar{\mu}_i, 0, 0, \bar{\nu}_{\hat{i}}] + \mathcal{Q}V^{\text{I,II,III,IV}}[\bar{\mu}_i, 0, 0, \bar{\nu}_{\hat{i}}, p_I] \qquad (3.13)$$

Therefore, their vev computed on the three sphere are expected to coincide.

3.2 From 1/2 BPS WLs in ABJM theory

As a particular case of the previous analysis we now consider the decomposition of the 1/2 BPS WLs $W_{1/2}^{\rm I}[\bar{\alpha}_I]$, $W_{1/2}^{\rm II}[\bar{\gamma}_I]$ in ABJM theory. This turns out to be particularly helpful for studying possible enhancement of SUSY for operators in table 2, and for finding out the cases for which gravity duals are known, as we discuss in the next section.

Decomposing $W_{1/2}^{\rm I}[\bar{\alpha}_I]$ in table 1, we obtain an operator $W^{\rm I}[\bar{\alpha}_i, \bar{\alpha}_{\hat{i}}]$ with superconnection (3.3) or (3.4) and parameters

$$U^i{}_j = \delta^i_j - \frac{2\alpha^i\bar{\alpha}_j}{|\alpha|^2}, \quad U^{\hat{i}}{}_{\hat{j}} = \delta^{\hat{i}}_{\hat{j}} - \frac{2\alpha^{\hat{i}}\bar{\alpha}_{\hat{j}}}{|\alpha|^2}, \quad U^i{}_{\hat{j}} = -\frac{2\alpha^i\bar{\alpha}_{\hat{j}}}{|\alpha|^2}, \qquad U^{\hat{i}}{}_j = -\frac{2\alpha^{\hat{i}}\bar{\alpha}_j}{|\alpha|^2} \tag{3.14}$$

$$\beta^i = \frac{\alpha^i}{|\alpha|^2}, \qquad \beta^{\hat{i}} = \frac{\alpha^{\hat{i}}}{|\alpha|^2}, \qquad |\alpha|^2 = \bar{\alpha}_i\alpha^i + \bar{\alpha}_{\hat{i}}\alpha^{\hat{i}} \neq 0, \qquad \bar{\gamma}_i = \bar{\gamma}_{\hat{i}} = \delta^i = \delta^{\hat{i}} = 0$$

This operator can be seen as a special case of the fermionic 1/4 BPS WLs $W_{1/4}^{\rm I}[\bar{\mu}_i, 0, 0, \bar{\nu}_{\hat{i}}, p_I]$ in table 2, obtained by choosing the parameters as in (2.12). It can be 1/2 or 1/4 BPS depending on the parameters. In fact, we can distinguish three cases.

1) When $\bar{\alpha}_i\alpha^i \neq 0$, $\bar{\alpha}_{\hat{i}} = 0$, we obtain a 1/2 BPS WL $W_{1/2}^{\rm I}[\bar{\alpha}_i, 0]$, with preserved supercharges

$$\bar{\alpha}_i\theta_+^{i\hat{k}}, \quad \varepsilon_{ij}\alpha^i\theta_-^{j\hat{k}} \qquad \hat{k} = \hat{1}, \hat{2} \tag{3.15}$$

as follows from (2.13). For $\bar{\alpha}_i = \delta^1_i$ it coincides with the ψ_1-loop in [27, 28], alternatively called W_1 in [15], whereas for $\bar{\alpha}_i = \delta^2_i$ it is the W_2 operator of [15]. It is cohomologically equivalent to the bosonic 1/4 BPS WL

$$W_{1/4}^{\rm bos}\left[\varepsilon_{ij}\alpha^j/\sqrt{\bar{\alpha}_k\alpha^k}, 0, 0, \bar{\nu}_{\hat{i}}\right] \tag{3.16}$$

for arbitrary $\bar{\nu}_{\hat{i}}$, with $\bar{\nu}_{\hat{i}}\nu^{\hat{i}} = 1$.

2) When $\bar{\alpha}_i = 0$, $\bar{\alpha}_{\hat{i}}\alpha^{\hat{i}} \neq 0$, we have the 1/2 BPS WL $W_{1/2}^{\rm I}[0, \bar{\alpha}_{\hat{i}}]$ with preserved supercharges

$$\bar{\alpha}_{\hat{j}}\theta_+^{i\hat{j}}, \quad \varepsilon_{\hat{j}\hat{k}}\alpha^{\hat{j}}\theta_-^{i\hat{k}} \qquad i = 1, 2 \tag{3.17}$$

In [15] this operator was called $W_{\hat{1}}$ for $\bar{\alpha}_{\hat{i}} = \delta^{\hat{1}}_{\hat{i}}$ and $W_{\hat{2}}$ for $\bar{\alpha}_{\hat{i}} = \delta^{\hat{2}}_{\hat{i}}$. It is cohomologically equivalent to the bosonic 1/4 BPS WL

$$W_{1/4}^{\rm bos}\left[\bar{\mu}_i, 0, 0, \varepsilon_{\hat{i}\hat{j}}\alpha^{\hat{j}}/\sqrt{\bar{\alpha}_{\hat{k}}\alpha^{\hat{k}}}\right] \tag{3.18}$$

for arbitrary $\bar{\mu}_i$ with $\bar{\mu}_i\mu^i = 1$.

3) More interestingly, when $\bar{\alpha}_i\alpha^i \neq 0$ and $\bar{\alpha}_{\hat{i}}\alpha^{\hat{i}} \neq 0$, we have the 1/4 BPS WL $W_{1/4}^{\rm I}[\bar{\alpha}_i, \bar{\alpha}_{\hat{i}}]$ with preserved supercharges

$$\bar{\alpha}_i\bar{\alpha}_{\hat{j}}\theta_+^{i\hat{j}}, \quad \varepsilon_{ij}\varepsilon_{\hat{k}\hat{l}}\alpha^i\alpha^{\hat{k}}\theta_-^{j\hat{l}} \tag{3.19}$$

It is cohomologically equivalent to the bosonic 1/4 BPS WL

$$W_{1/4}^{\rm bos}\left[\varepsilon_{ij}\alpha^j/\sqrt{\bar{\alpha}_k\alpha^k}, 0, 0, \varepsilon_{\hat{i}\hat{j}}\alpha^{\hat{j}}/\sqrt{\bar{\alpha}_{\hat{k}}\alpha^{\hat{k}}}\right] \tag{3.20}$$

Repeating a similar analysis for the 1/2 BPS WL $W_{1/2}^{\mathrm{II}}[\bar{\gamma}_I]$ in ABJM theory we obtain an operator $W^{\mathrm{II}}[\bar{\gamma}_i, \bar{\gamma}_{\hat{i}}]$ in $\mathcal{N} = 4$ SCSM theories. This is just a particular case of the fermionic 1/4 BPS WL $W_{1/4}^{\mathrm{II}}[\bar{\mu}_i, 0, 0, \bar{\nu}_{\hat{i}}, p_I]$ in table 2 corresponding to parameters (2.12). When $\bar{\gamma}_{\hat{i}} = 0$, $\bar{\gamma}_i \gamma^i \neq 0$ we have a 1/2 BPS WL, $W_{1/2}^{\mathrm{II}}[\bar{\gamma}_i, 0]$, which for $\bar{\gamma}_i = \delta_i^2$ coincides with the ψ_2-loop of [27] (or $\tilde{W}_2$ in [15]). The corresponding conserved supercharges are

$$\bar{\gamma}_i \theta_-^{i\hat{k}}, \quad \varepsilon_{ij} \gamma^i \theta_+^{j\hat{k}} \qquad \hat{k} = \hat{1}, \hat{2} \tag{3.21}$$

For $\bar{\gamma}_i \sim \varepsilon_{ij} \alpha^j$ they coincide with the ones in (3.15). This is the degeneracy of WLs discovered in [27] and further elaborated in [15, 36, 37]. When $\bar{\gamma}_i = 0$, $\bar{\gamma}_{\hat{i}} \gamma^{\hat{i}} \neq 0$, we have a 1/2 BPS WL, $W_{1/2}^{\mathrm{II}}[0, \bar{\gamma}_{\hat{i}}]$ that preserves supercharges

$$\bar{\gamma}_{\hat{j}} \theta_-^{i\hat{j}}, \quad \varepsilon_{\hat{j}\hat{k}} \gamma^{\hat{j}} \theta_+^{i\hat{k}} \qquad i = 1, 2 \tag{3.22}$$

For $\bar{\gamma}_{\hat{i}} \sim \varepsilon_{\hat{i}\hat{j}} \alpha^{\hat{j}}$ they are degenerate with the preserved supercharges of $W_{1/2}^{\mathrm{I}}[0, \bar{\alpha}_{\hat{i}}]$ defined above. Finally, when $\bar{\gamma}_i \gamma^i \neq 0$ and $\bar{\gamma}_{\hat{i}} \gamma^{\hat{i}} \neq 0$, we obtain a 1/4 BPS WL, $W_{1/4}^{\mathrm{II}}[\bar{\gamma}_i, \bar{\gamma}_{\hat{i}}]$ with preserved supercharges

$$\bar{\gamma}_i \bar{\gamma}_{\hat{j}} \theta_-^{i\hat{j}}, \quad \varepsilon_{ij} \varepsilon_{\hat{k}\hat{l}} \gamma^i \gamma^{\hat{k}} \theta_+^{j\hat{l}} \tag{3.23}$$

All these WLs are cohomologically equivalent to bosonic 1/4 BPS WLs that can be easily determined.

3.3 M2-/anti-M2-brane duals

According to the AdS/CFT correspondence, 1/2 BPS WLs in $\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}$ ABJM theory are dual to M2-/anti-M2-branes in $\mathrm{AdS}_4 \times \mathrm{S}^7/\mathbb{Z}_k$ background [2, 15]

$$ds^2 = R^2 \left(\frac{1}{4} ds_{\mathrm{AdS}_4}^2 + ds_{\mathrm{S}^7/\mathbb{Z}_k}^2 \right) \tag{3.24}$$

More precisely, choosing the AdS_4 metric in the form

$$ds_{\mathrm{AdS}_4}^2 = u^2(-dt^2 + dx_1^2 + dx_2^2) + \frac{du^2}{u^2} \tag{3.25}$$

and the S^7 embedded in $\mathbb{C}^4$ as

$$
\begin{aligned}
ds_{\mathrm{S}^7}^2 = \frac{1}{4} \Bigg[& d\beta^2 + \cos^2 \frac{\beta}{2} \big(d\theta_1^2 + \sin^2 \theta_1 d\varphi_1^2 \big) + \sin^2 \frac{\beta}{2} \big(d\theta_2^2 + \sin^2 \theta_2 d\varphi_2^2 \big) \\
& + \sin^2 \frac{\beta}{2} \cos^2 \frac{\beta}{2} (d\chi + \cos\theta_1 d\varphi_1 - \cos\theta_2 d\varphi_2)^2 \\
& + \left(\frac{1}{2} d\zeta + \cos^2 \frac{\beta}{2} \cos\theta_1 d\varphi_1 + \sin^2 \frac{\beta}{2} \cos\theta_2 d\varphi_2 + \frac{1}{2} \cos\beta d\chi \right)^2 \Bigg]
\end{aligned}
\tag{3.26}
$$

with the $\mathbb{Z}_k$ identification $\zeta \sim \zeta - \frac{8\pi}{k}$, the 1/2 BPS operator $W_{1/2}^{\mathrm{I}}[\bar{\alpha}_I]$ reviewed in subsection 2.2 is dual to a M2-brane embedded as $t = \sigma^0, x_1 = x_2 = 0, u = \sigma^1, \zeta = \sigma^2$ and localized at a point specified by the complex vector [15]

$$
\begin{aligned}
\alpha^I = (\bar{\alpha}_I)^* = \Bigg(& \cos\frac{\beta}{2} \cos\frac{\theta_1}{2} e^{-\frac{\mathrm{i}}{4}(2\phi_1 + \chi + \zeta)}, \cos\frac{\beta}{2} \sin\frac{\theta_1}{2} e^{-\frac{\mathrm{i}}{4}(-2\phi_1 + \chi + \zeta)}, \\
& \sin\frac{\beta}{2} \cos\frac{\theta_2}{2} e^{-\frac{\mathrm{i}}{4}(2\phi_2 - \chi + \zeta)}, \sin\frac{\beta}{2} \sin\frac{\theta_2}{2} e^{-\frac{\mathrm{i}}{4}(-2\phi_2 - \chi + \zeta)} \Bigg)
\end{aligned}
\tag{3.27}
$$

satisfying $\alpha^I \bar{\alpha}_I = 1$. The α^I vector allows to identify the Killing spinors in $\mathrm{AdS}_4 \times \mathrm{S}^7 / \mathbb{Z}_k$ background corresponding to Poincaré and conformal supercharges preserved by the M2-brane solution. One finds that they coincide with supercharges (2.13) preserved by $W_{1/2}^{\mathrm{I}}[\bar{\alpha}_I]$.[6]

Similarly, The 1/2 BPS WL $W_{1/2}^{\mathrm{II}}[\bar{\gamma}_I]$ is dual to an anti-M2-brane that wraps a circle specified by a $\bar{\gamma}_I$ vector similar to the one in (3.27). This configuration preserves supercharges in the second line of (2.13). When $\bar{\alpha}^I = \bar{\gamma}^I$ the brane and the anti-brane preserve complementary sets of supercharges [15], in agreement with the field theory result.

The $\mathcal{N} = 4$ orbifold ABJM theory is dual to M-theory in $\mathrm{AdS}_4 \times \mathrm{S}^7 / (\mathbb{Z}_{rk} \times \mathbb{Z}_r)$ background [23–25]. Since its 1/2 and 1/4 BPS WLs $W_{1/2}^{\mathrm{I}}[\bar{\alpha}_i, 0]$, $W_{1/2}^{\mathrm{I}}[0, \bar{\alpha}_{\hat{\imath}}]$ and $W_{1/4}^{\mathrm{I}}[\bar{\alpha}_i, \bar{\alpha}_{\hat{\imath}}]$ can be obtained from the orbifold decomposition of the 1/2 BPS WL $W_{1/2}^{\mathrm{I}}[\bar{\alpha}_I]$ in ABJM theory, we can easily identify the M2-brane duals of these operators from the dual brane configurations in ABJM theory.

The background can be still described by metrics (3.25), (3.26) with the quotient $\mathbb{Z}_{rk} \times \mathbb{Z}_r$ realized by the identification $\zeta \sim \zeta - \frac{8\pi}{rk}$, $\chi \sim \chi - \frac{4\pi}{r}$ and $\zeta \sim \zeta - \frac{4\pi}{r}$. Decomposing the R-symmetry indices as in (3.2) we find that a M2-brane localized at

$$
\alpha^i = \left(\cos\frac{\beta}{2} \cos\frac{\theta_1}{2} e^{-\frac{\mathrm{i}}{4}(2\phi_1 + \chi + \zeta)}, \cos\frac{\beta}{2} \sin\frac{\theta_1}{2} e^{-\frac{\mathrm{i}}{4}(-2\phi_1 + \chi + \zeta)} \right)
$$
$$
\alpha^{\hat{\imath}} = \left(\sin\frac{\beta}{2} \sin\frac{\theta_2}{2} e^{-\frac{\mathrm{i}}{4}(-2\phi_2 - \chi + \zeta)}, \sin\frac{\beta}{2} \cos\frac{\theta_2}{2} e^{-\frac{\mathrm{i}}{4}(2\phi_2 - \chi + \zeta)} \right) \tag{3.28}
$$

with $\bar{\alpha}_i \alpha^i + \bar{\alpha}_{\hat{\imath}} \alpha^{\hat{\imath}} = 1$ and generically $\bar{\alpha}_i \alpha^i, \bar{\alpha}_{\hat{\imath}} \alpha^{\hat{\imath}} \neq 0$, is dual to the 1/4 BPS operator $W_{1/4}^{\mathrm{I}}[\bar{\alpha}_i, \bar{\alpha}_{\hat{\imath}}]$ introduced in subsection 3.2. The $(\bar{\alpha}_i, \bar{\alpha}_{\hat{\imath}})$ parameters select the set of supercharges preserved by the corresponding M2-brane configuration [15], which turn out to coincide with (3.19). Choosing $\beta = 0$, that is $\bar{\alpha}_{\hat{\imath}} = 0$, the set of preserved supercharges enhances to (3.15) and the M2-brane configuration is dual to $W_{1/2}^{\mathrm{I}}[\bar{\alpha}_i, 0]$. Similarly, choosing $\beta = \pi$ ($\bar{\alpha}_i = 0$) the set of preserved supercharges coincides with (3.17) and we obtain the dual configuration of $W_{1/2}^{\mathrm{I}}[0, \bar{\alpha}_{\hat{\imath}}]$.

In a similar fashion, $W_{1/4}^{\mathrm{II}}[\bar{\gamma}_i, \bar{\gamma}_{\hat{\imath}}]$, $W_{1/2}^{\mathrm{II}}[\bar{\gamma}_i, 0]$ and $W_{1/2}^{\mathrm{II}}[0, \bar{\gamma}_{\hat{\imath}}]$ operators, arising from the orbifold reduction of WL $W_{1/2}^{\mathrm{II}}[\bar{\gamma}_I]$ in ABJM theory, are dual to anti-M2-branes that wrap circles specified by vectors $(\bar{\gamma}_i, \bar{\gamma}_{\hat{\imath}})$ of the form (3.28).

It is remarkable that the orbifold decomposition provides explicit gravity duals not only for 1/2 BPS operators but also for the 1/4 BPS WLs $W_{1/4}^{\mathrm{I}}[\bar{\alpha}_i, \bar{\alpha}_{\hat{\imath}}]$ and $W_{1/4}^{\mathrm{II}}[\bar{\gamma}_i, \bar{\gamma}_{\hat{\imath}}]$, even in the absence of SUSY enhancement.

For generic fermionic 1/6 WLs in ABJM theory the precise gravity duals are not known. Consequently, we are not able to identify precise gravity duals for the generic fermionic 1/4 WLs in $\mathcal{N} = 4$ orbifold ABJM theory obtained by orbifold reduction.

To conclude this section we stress that the orbifold decomposition that leads from $\mathrm{U}(rN)_k \times \mathrm{U}(rN)_{-k}$ ABJM to $\mathcal{N} = 4$ orbifold ABJM theory can be generalized to a suitable decomposition of the $\mathrm{U}(N)_k \times \mathrm{U}(M)_{-k}$ ABJ theory to obtain a $\mathcal{N} = 4$ SCSM theory with circular quiver $\prod_{\ell=1}^{r} [\mathrm{U}(N_{2\ell-1})_k \times \mathrm{U}(N_{2\ell})_{-k}]$ [15]. Therefore, the general construction

---

[6]For details on this identification we refer the reader to section 4.3 of [15].
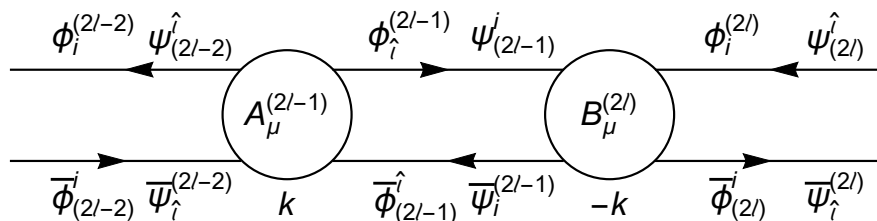
Figure 1. A section of quiver diagram of the $\mathcal{N} = 4$ circular quiver SCSM theories with alternating levels. We have $\ell = 1, 2, \cdots, r$.

of WLs presented above can be applied also in the more general case of $\mathcal{N} = 4$ SCSM quiver theories. The structure of the superconnections is still the one in (3.3), (3.4) with definitions (3.5). However, since for general circular quiver $\mathcal{N} = 4$ SCSM theories the M-theory dual description is not known, we cannot identify the gravity duals of the corresponding BPS WLs.

4 BPS WLs in $\mathcal{N} = 4$ SCSM theories: the general approach

Given the previous construction of BPS WLs obtained by direct orbifold decomposition of BPS WLs in ABJM, the natural question arises whether the set of operators in table 2 exhausts the whole spectrum of possible 1/4 and 1/2 BPS WLs. To answer this question we now approach the problem by applying a more systematic procedure, which consists in studying directly the SUSY variation of a generic (super)connection L and imposing [32]

$$\delta L = \partial_\tau G + \mathrm{i}[L, G] \tag{4.1}$$

for a suitable supermatrix G.

To this end, we consider a $\mathcal{N} = 4$ SCSM theory with gauge group and alternating levels $\prod_{\ell=1}^{r} [\mathrm{U}(N_{2\ell-1})_k \times \mathrm{U}(N_{2\ell})_{-k}]$. A section of its quiver is given in figure 1.

Superconformal transformations are related to Poincaré supercharges $P^{i\hat{i}}$, $\bar{P}_{i\hat{i}}$ and conformal supercharges $S^{i\hat{i}}$, $\bar{S}_{i\hat{i}}$ as

$$\delta = \mathrm{i}(\bar{\theta}^{i\hat{i}} P_{i\hat{i}} + \bar{\vartheta}^{i\hat{i}} S_{i\hat{i}}) = \mathrm{i}(\bar{P}_{i\hat{i}} \theta^{i\hat{i}} + \bar{S}_{i\hat{i}} \vartheta^{i\hat{i}}) \tag{4.2}$$

The corresponding SUSY transformations on the fields are given in appendix A (see eq. (A.1)).

In what follows we still restrict to WLs defined along the timelike infinite straight line $x^\mu = (\tau, 0, 0)$ in Minkowski spacetime.

4.1 1/4 BPS WLs

In order to avoid clutter in the presentation we focus on the construction of the particular class of 1/4 WLs preserving Poincaré supercharges

$$\theta_+^{1\hat{1}}, \quad \theta_-^{2\hat{2}} \tag{4.3}$$

These correspond to supercharges (3.7) where we choose

$$\bar{\mu}_i = (0,1), \quad \bar{\nu}_{\hat{i}} = (0,1) \tag{4.4}$$

First, we observe that for this particular set of parameters, the bosonic 1/4 BPS WL $W_{1/4}^{\text{bos}}$ in table 2 has connection [27, 28]

$$L_{\text{bos}} = \text{diag}\left(\mathcal{A}_{\text{bos}}^{(1)}, \mathcal{B}_{\text{bos}}^{(2)}, \cdots, \mathcal{A}_{\text{bos}}^{(2r-1)}, \mathcal{B}_{\text{bos}}^{(2r)} \right) \tag{4.5}$$

$$\mathcal{A}_{\text{bos}}^{(2\ell-1)} = A_0^{(2\ell-1)} + \frac{2\pi}{k} \left(-\phi_1^{(2\ell-2)} \bar{\phi}_{(2\ell-2)}^1 + \phi_2^{(2\ell-2)} \bar{\phi}_{(2\ell-2)}^2 - \phi_{\hat{1}}^{(2\ell-1)} \bar{\phi}_{(2\ell-1)}^{\hat{1}} + \phi_{\hat{2}}^{(2\ell-1)} \bar{\phi}_{(2\ell-1)}^{\hat{2}} \right)$$

$$\mathcal{B}_{\text{bos}}^{(2\ell)} = B_0^{(2\ell)} + \frac{2\pi}{k} \left(-\bar{\phi}_{(2\ell)}^1 \phi_1^{(2\ell)} + \bar{\phi}_{(2\ell)}^2 \phi_2^{(2\ell)} - \bar{\phi}_{(2\ell-1)}^{\hat{1}} \phi_{\hat{1}}^{(2\ell-1)} + \bar{\phi}_{(2\ell-1)}^{\hat{2}} \phi_{\hat{2}}^{(2\ell-1)} \right)$$

To construct fermionic WLs we then consider the most general superconnection L_{fer} of the form (3.3), (3.4) where we now allow for the couplings to scalars and fermions in eqs. (3.5) to depend on the nodes. This choice enlarges the class of BPS WLs obtained by orbifold decomposition and in principle allows for finding more general BPS operators. Moreover, for computational convenience in (3.5) we redefine the scalar couplings as $U_{(2\ell-1)}{}^i{}_j = M_{(2\ell-1)}{}^i{}_j - (\sigma_3)^i{}_j$, $U_{(2\ell-1)}{}^{\hat{i}}{}_{\hat{j}} = M_{(2\ell-1)}{}^{\hat{i}}{}_{\hat{j}} - (\sigma_3)^{\hat{i}}{}_{\hat{j}}$, $U_{(2\ell-1)}{}^i{}_{\hat{j}} = M_{(2\ell-1)}{}^i{}_{\hat{j}}$, $U_{(2\ell-1)}{}^{\hat{i}}{}_j = M_{(2\ell-1)}{}^{\hat{i}}{}_j$, $U_{(2\ell)}{}^i{}_j = M_{(2\ell)}{}^i{}_j - (\sigma_3)^i{}_j$, $U_{(2\ell)}{}^{\hat{i}}{}_{\hat{j}} = M_{(2\ell)}{}^{\hat{i}}{}_{\hat{j}} - (\sigma_3)^{\hat{i}}{}_{\hat{j}}$, $U_{(2\ell)}{}^i{}_{\hat{j}} = M_{(2\ell)}{}^i{}_{\hat{j}}$, $U_{(2\ell)}{}^{\hat{i}}{}_j = M_{(2\ell)}{}^{\hat{i}}{}_j$ for all $\ell = 1, 2, \cdots, r$, with σ_3 being the third Pauli matrix. Given the structure of the quiver in figure 1 the superconnection entries then take the form

$$\mathcal{A}^{(2\ell-1)} = \mathcal{A}_{\text{bos}}^{(2\ell-1)} + \frac{2\pi}{k} \left(M_{(2\ell-1)}{}^i{}_j \phi_i^{(2\ell-2)} \bar{\phi}_{(2\ell-2)}^j + M_{(2\ell-1)}{}^{\hat{i}}{}_{\hat{j}} \phi_{\hat{i}}^{(2\ell-1)} \bar{\phi}_{(2\ell-1)}^{\hat{j}} \right)$$

$$\mathcal{B}^{(2\ell)} = \mathcal{B}_{\text{bos}}^{(2\ell)} + \frac{2\pi}{k} \left(M_{(2\ell)}{}^i{}_j \bar{\phi}_{(2\ell)}^j \phi_i^{(2\ell)} + M_{(2\ell)}{}^{\hat{i}}{}_{\hat{j}} \bar{\phi}_{(2\ell-1)}^{\hat{j}} \phi_{\hat{i}}^{(2\ell-1)} \right)$$

$$f_1^{(2\ell-1)} = i\sqrt{\frac{4\pi}{k}} \left(\bar{\alpha}_i^{(2\ell-1)} u_+ - \bar{\gamma}_i^{(2\ell-1)} u_- \right) \psi_{(2\ell-1)}^i, \quad f_1^{(2\ell)} = i\sqrt{\frac{4\pi}{k}} \bar{\psi}_{\hat{i}}^{(2\ell)} \left(u_- \beta_{(2\ell)}^{\hat{i}} + u_+ \delta_{(2\ell)}^{\hat{i}} \right)$$

$$f_2^{(2\ell-1)} = i\sqrt{\frac{4\pi}{k}} \bar{\psi}_i^{(2\ell-1)} \left(u_- \beta_{(2\ell-1)}^i + u_+ \delta_{(2\ell-1)}^i \right), \quad f_2^{(2\ell)} = i\sqrt{\frac{4\pi}{k}} \left(\bar{\alpha}_{\hat{i}}^{(2\ell)} u_+ - \bar{\gamma}_{\hat{i}}^{(2\ell)} u_- \right) \psi_{(2\ell)}^{\hat{i}}$$

$$h_1^{(2\ell-1)} = \frac{2\pi}{k} M_{(2\ell-1)}{}^{\hat{i}}{}_j \phi_{\hat{i}}^{(2\ell-1)} \bar{\phi}_{(2\ell)}^j, \qquad\qquad h_1^{(2\ell)} = \frac{2\pi}{k} M_{(2\ell)}{}^{\hat{i}}{}_j \bar{\phi}_{(2\ell)}^j \phi_{\hat{i}}^{(2\ell+1)}$$

$$h_2^{(2\ell-1)} = \frac{2\pi}{k} M_{(2\ell-1)}{}^i{}_{\hat{j}} \phi_i^{(2\ell)} \bar{\phi}_{(2\ell-1)}^{\hat{j}}, \qquad\qquad h_2^{(2\ell)} = \frac{2\pi}{k} M_{(2\ell)}{}^i{}_{\hat{j}} \bar{\phi}_{(2\ell+1)}^{\hat{j}} \phi_i^{(2\ell)} \tag{4.6}$$

To make the fermionic WL BPS, we apply SUSY transformations (A.1) to the entries in (4.6) and impose condition (4.1) with the choice

$$G = \begin{pmatrix} 0 & g_1^{(1)} & & & & & g_2^{(2r)} \\ g_2^{(1)} & 0 & g_1^{(2)} & & & & \\ & g_2^{(2)} & \ddots & \ddots & & & \\ & & \ddots & \ddots & g_1^{(2r-2)} & & \\ & & & g_2^{(2r-2)} & 0 & g_1^{(2r-1)} & \\ g_1^{(2r)} & & & & g_2^{(2r-1)} & 0 \end{pmatrix} \tag{4.7}$$

The detailed structure of the BPS constraints are given in (A.3). Solving them for the particular choice (4.3) of preserved supercharges, we obtain

$$\bar{\alpha}_i^{(2\ell-1)} = (\bar{\alpha}_1^{(2\ell-1)}, 0), \qquad\qquad \bar{\alpha}_{\hat{i}}^{(2\ell)} = (\bar{\alpha}_{\hat{1}}^{(2\ell)}, 0)$$

$$\beta_{(2\ell-1)}^i = (\beta_{(2\ell-1)}^1, 0), \qquad\qquad \beta_{(2\ell)}^{\hat{i}} = (\beta_{(2\ell)}^{\hat{1}}, 0)$$

$$\bar{\gamma}_i^{(2\ell-1)} = (0, \bar{\gamma}_2^{(2\ell-1)}), \qquad\qquad \bar{\gamma}_{\hat{i}}^{(2\ell)} = (0, \bar{\gamma}_{\hat{2}}^{(2\ell)})$$

$$\delta_{(2\ell-1)}^i = (0, \delta_{(2\ell-1)}^2), \qquad\qquad \delta_{(2\ell)}^{\hat{i}} = (0, \delta_{(2\ell)}^{\hat{2}})$$

$$M_{(2\ell-1)}{}^i{}_j = \text{diag}\left(2\bar{\alpha}_{\hat{1}}^{(2\ell-2)}\beta_{(2\ell-2)}^{\hat{1}}, -2\bar{\gamma}_{\hat{2}}^{(2\ell-2)}\delta_{(2\ell-2)}^{\hat{2}}\right)$$

$$M_{(2\ell-1)}{}^{\hat{i}}{}_{\hat{j}} = \text{diag}\left(2\bar{\alpha}_1^{(2\ell-1)}\beta_{(2\ell-1)}^1, -2\bar{\gamma}_2^{(2\ell-1)}\delta_{(2\ell-1)}^2\right)$$

$$M_{(2\ell)}{}^i{}_j = \text{diag}\left(2\bar{\alpha}_{\hat{1}}^{(2\ell)}\beta_{(2\ell)}^{\hat{1}}, -2\bar{\gamma}_{\hat{2}}^{(2\ell)}\delta_{(2\ell)}^{\hat{2}}\right)$$

$$M_{(2\ell)}{}^{\hat{i}}{}_{\hat{j}} = \text{diag}\left(2\bar{\alpha}_1^{(2\ell-1)}\beta_{(2\ell-1)}^1, -2\bar{\gamma}_2^{(2\ell-1)}\delta_{(2\ell-1)}^2\right)$$

$$M_{(2\ell-1)}{}^{\hat{i}}{}_j = \text{diag}\left(-2\bar{\alpha}_1^{(2\ell-1)}\beta_{(2\ell)}^{\hat{1}}, 2\bar{\gamma}_2^{(2\ell-1)}\delta_{(2\ell)}^{\hat{2}}\right)$$

$$M_{(2\ell-1)}{}^i{}_{\hat{j}} = \text{diag}\left(-2\bar{\alpha}_{\hat{1}}^{(2\ell)}\beta_{(2\ell-1)}^1, 2\bar{\gamma}_{\hat{2}}^{(2\ell)}\delta_{(2\ell-1)}^2\right)$$

$$M_{(2\ell)}{}^{\hat{i}}{}_j = \text{diag}\left(-2\bar{\alpha}_1^{(2\ell+1)}\beta_{(2\ell)}^{\hat{1}}, 2\bar{\gamma}_2^{(2\ell+1)}\delta_{(2\ell)}^{\hat{2}}\right)$$

$$M_{(2\ell)}{}^i{}_{\hat{j}} = \text{diag}\left(-2\bar{\alpha}_{\hat{1}}^{(2\ell)}\beta_{(2\ell+1)}^1, 2\bar{\gamma}_{\hat{2}}^{(2\ell)}\delta_{(2\ell+1)}^2\right) \tag{4.8}$$

with the remaining parameters subject to

$$\bar{\alpha}_1^{(2\ell-1)}\delta_{(2\ell-1)}^2 = \bar{\alpha}_1^{(2\ell-1)}\delta_{(2\ell-2)}^{\hat{2}} = \bar{\alpha}_1^{(2\ell-1)}\delta_{(2\ell)}^{\hat{2}} = 0$$

$$\bar{\alpha}_{\hat{1}}^{(2\ell)}\delta_{(2\ell)}^2 = \bar{\alpha}_{\hat{1}}^{(2\ell)}\delta_{(2\ell-1)}^{\hat{2}} = \bar{\alpha}_{\hat{1}}^{(2\ell)}\delta_{(2\ell+1)}^{\hat{2}} = 0$$

$$\bar{\gamma}_2^{(2\ell-1)}\beta_{(2\ell-1)}^1 = \bar{\gamma}_2^{(2\ell-1)}\beta_{(2\ell-2)}^{\hat{1}} = \bar{\gamma}_2^{(2\ell-1)}\beta_{(2\ell)}^{\hat{1}} = 0$$

$$\bar{\gamma}_{\hat{2}}^{(2\ell)}\beta_{(2\ell)}^1 = \bar{\gamma}_{\hat{2}}^{(2\ell)}\beta_{(2\ell-1)}^{\hat{1}} = \bar{\gamma}_{\hat{2}}^{(2\ell)}\beta_{(2\ell+1)}^{\hat{1}} = 0 \tag{4.9}$$

The most general solution depends on $8r$ parameters

$$\bar{\alpha}_1^{(2\ell-1)}, \ \bar{\alpha}_{\hat{1}}^{(2\ell)}, \ \beta_{(2\ell-1)}^1, \ \beta_{(2\ell)}^{\hat{1}}, \ \bar{\gamma}_2^{(2\ell-1)}, \ \bar{\gamma}_{\hat{2}}^{(2\ell)}, \ \delta_{(2\ell-1)}^2, \ \delta_{(2\ell)}^{\hat{2}}, \quad \ell = 1, 2, \cdots, r \tag{4.10}$$

constrained by (4.9). The solutions can be classified according to the number of free parameters they depend on, as we now describe.

Classification of 1/4 BPS WLs.

1) First, we find four classes of solutions that depend on $4r$ free complex parameters, two parameters for each even node and two for each odd one. They are explicitly given by

$$\begin{aligned}
\text{Class I:} \quad & \bar{\gamma}_2^{(2\ell-1)} = \bar{\gamma}_{\hat{2}}^{(2\ell)} = \delta_{(2\ell-1)}^2 = \delta_{(2\ell)}^{\hat{2}} = 0 \\
\text{Class II:} \quad & \bar{\alpha}_1^{(2\ell-1)} = \bar{\alpha}_{\hat{1}}^{(2\ell)} = \beta_{(2\ell-1)}^1 = \beta_{(2\ell)}^{\hat{1}} = 0 \\
\text{Class III:} \quad & \beta_{(2\ell-1)}^1 = \beta_{(2\ell)}^{\hat{1}} = \delta_{(2\ell-1)}^2 = \delta_{(2\ell)}^{\hat{2}} = 0 \\
\text{Class IV:} \quad & \bar{\alpha}_1^{(2\ell-1)} = \bar{\alpha}_{\hat{1}}^{(2\ell)} = \bar{\gamma}_2^{(2\ell-1)} = \bar{\gamma}_{\hat{2}}^{(2\ell)} = 0
\end{aligned} \tag{4.11}$$

These solutions are in general non-block-diagonal and generalize the ones in table 2. In fact, they reduce to the previous ones by imposing that the parameters are the same for every even (odd) node. For example, solutions (4.11) in Class I are parametrized by $(\bar{\alpha}_{\hat{1}}^{(2\ell)}, \beta_{(2\ell)}^{\hat{1}})$ at even nodes and $(\bar{\alpha}_{1}^{(2\ell-1)}, \beta_{(2\ell-1)}^{1})$ at odd nodes. When

$$\bar{\alpha}_1^{(2\ell-1)} = \bar{\alpha}_1, \quad \beta_{(2\ell-1)}^1 = \beta^1, \quad \bar{\alpha}_{\hat{1}}^{(2\ell)} = \bar{\alpha}_{\hat{1}}, \quad \beta_{(2\ell)}^{\hat{1}} = \beta^{\hat{1}} \qquad \ell = 1, 2, \cdots, r \qquad (4.12)$$

we obtain solution $W_{1/4}^{\mathrm{I}}[\bar{\mu}_i, 0, 0, \bar{\nu}_{\hat{i}}, p_I]$ in Class I of table 2 corresponding to $\bar{\mu}_i = (0,1)$, $\bar{\nu}_{\hat{i}} = (0,1)$.

2) Beyond these four classes, there are additional solutions that depend on a smaller set of parameters. For example, the solution obtained by setting

$$\begin{aligned}\bar{\gamma}_2^{(2\ell-1)} = \bar{\gamma}_{\hat{2}}^{(2\ell)} = \delta_{(2\ell)}^{\hat{2}} = 0 & \qquad \ell = 1, 2, \cdots, r \\ \bar{\alpha}_1^{(1)} = \bar{\alpha}_{\hat{1}}^{(2)} = \bar{\alpha}_{\hat{1}}^{(2r)} = \delta_{(2\ell-1)}^2 = 0 & \qquad \ell = 2, 3, \cdots, r\end{aligned} \qquad (4.13)$$

depends on $4r - 2$ complex parameters and corresponds to a new genuine WL that does not have a counterpart in table 2.

To conclude this section we prove that the most general 1/4 BPS fermionic operator W^{fer} corresponding to superconnection L_{fer} in (3.3) or (3.4) with assignments (4.6), (4.8) and (4.9) is cohomologically equivalent to the bosonic WL (4.5), that is

$$W^{\mathrm{fer}} - W_{1/4}^{\mathrm{bos}} = \mathcal{Q}V \qquad (4.14)$$

where $\mathcal{Q}$ is a linear combination of preserved supercharges (4.3).

In order to prove it we first split the difference of the connections into a bosonic and a fermionic part

$$L_{\mathrm{fer}} - L_{\mathrm{bos}} \equiv L_B + L_F \qquad (4.15)$$

Then, following the prescription in [2, 28], we look for a supercharge $\mathcal{Q}$, one parameter κ and one matrix Λ satisfying

$$\begin{aligned}\kappa \Lambda^2 = L_B, \quad \mathcal{Q}\Lambda = L_F, \quad \mathcal{Q}L_{\mathrm{bos}} = 0 \\ \mathcal{Q}L_F = \partial_\tau(\mathrm{i}\kappa\Lambda) + \mathrm{i}[L_{\mathrm{bos}}, \mathrm{i}\kappa\Lambda]\end{aligned} \qquad (4.16)$$

A solution to these equations is given by $\mathcal{Q} = \mathrm{i}(P_+^{1\hat{1}} - P_-^{2\hat{2}})$, $\kappa = 1$ and

$$\Lambda = \begin{pmatrix} 0 & \lambda_1^{(1)} & & & & & \lambda_2^{(2r)} \\ \lambda_2^{(1)} & 0 & \lambda_1^{(2)} & & & & \\ & \lambda_2^{(2)} & \ddots & \ddots & & & \\ & & \ddots & \ddots & \lambda_1^{(2r-2)} & & \\ & & & \lambda_2^{(2r-2)} & 0 & \lambda_1^{(2r-1)} \\ \lambda_1^{(2r)} & & & & \lambda_2^{(2r-1)} & 0 \end{pmatrix} \qquad (4.17)$$

where we have defined

$$\lambda_1^{(2\ell-1)} = i\sqrt{\frac{4\pi}{k}}\big(\bar{\alpha}_1^{(2\ell-1)}\phi_{\hat{1}}^{(2\ell-1)} - \bar{\gamma}_2^{(2\ell-1)}\phi_{\hat{2}}^{(2\ell-1)}\big)$$

$$\lambda_1^{(2\ell)} = i\sqrt{\frac{4\pi}{k}}\big(\beta_{(2\ell)}^{\hat{1}}\bar{\phi}_{(2\ell)}^1 + \delta_{(2\ell)}^{\hat{2}}\bar{\phi}_{(2\ell)}^2\big)$$

$$\lambda_2^{(2\ell-1)} = i\sqrt{\frac{4\pi}{k}}\big(-\beta_{(2\ell-1)}^1\bar{\phi}_{(2\ell-1)}^{\hat{1}} - \delta_{(2\ell-1)}^2\bar{\phi}_{(2\ell-1)}^{\hat{2}}\big)$$

$$\lambda_2^{(2\ell)} = i\sqrt{\frac{4\pi}{k}}\big(-\bar{\alpha}_{\hat{1}}^{(2\ell)}\phi_1^{(2\ell)} + \bar{\gamma}_{\hat{2}}^{(2\ell)}\phi_2^{(2\ell)}\big) \tag{4.18}$$

4.2 1/2 BPS WLs

We begin by constructing 1/2 BPS WLs that preserve Poincaré supercharges

$$\theta_+^{1\hat{j}}, \quad \theta_-^{2\hat{j}}, \quad \hat{j} = \hat{1}, \hat{2} \tag{4.19}$$

Again, we consider ansatz (3.3) or (3.4) for the superconnection with definitions (4.6) and require the validity of the BPS condition (4.1) for a suitable choice of the matrix G of the form (4.7). Solving the corresponding constraints we obtain

$$\bar{\alpha}_i^{(2\ell-1)} = (\bar{\alpha}_1^{(2\ell-1)}, 0), \qquad \beta_{(2\ell-1)}^i = (\beta_{(2\ell-1)}^1, 0)$$

$$\bar{\gamma}_i^{(2\ell-1)} = (0, \bar{\gamma}_2^{(2\ell-1)}), \qquad \delta_{(2\ell-1)}^i = (0, \delta_{(2\ell-1)}^2) \tag{4.20}$$

$$\bar{\alpha}_{\hat{i}}^{(2\ell)} = \beta_{(2\ell)}^{\hat{i}} = \bar{\gamma}_{\hat{i}}^{(2\ell)} = \delta_{(2\ell)}^{\hat{i}} = 0$$

$$\bar{\alpha}_1^{(2\ell-1)}\delta_{(2\ell-1)}^2 = \bar{\gamma}_2^{(2\ell-1)}\beta_{(2\ell-1)}^1 = 0, \quad \bar{\alpha}_1^{(2\ell-1)}\beta_{(2\ell-1)}^1 + \bar{\gamma}_2^{(2\ell-1)}\delta_{(2\ell-1)}^2 = 1$$

$$M_{(2\ell-1)}{}^i{}_j = M_{(2\ell-1)}{}^{\hat{i}}{}_j = M_{(2\ell-1)}{}^i{}_{\hat{j}} = M_{(2\ell)}{}^i{}_j = M_{(2\ell)}{}^{\hat{i}}{}_j = M_{(2\ell)}{}^i{}_{\hat{j}} = 0$$

$$M_{(2\ell-1)}{}^{\hat{i}}{}_{\hat{j}} = \text{diag}\left(2\bar{\alpha}_1^{(2\ell-1)}\beta_{(2\ell-1)}^1, -2\bar{\gamma}_2^{(2\ell-1)}\delta_{(2\ell-1)}^2\right)$$

$$M_{(2\ell)}{}^{\hat{i}}{}_{\hat{j}} = \text{diag}\left(2\bar{\alpha}_1^{(2\ell-1)}\beta_{(2\ell-1)}^1, -2\bar{\gamma}_2^{(2\ell-1)}\delta_{(2\ell-1)}^2\right)$$

Notably, all the solutions correspond to block-diagonal superconnections $L_{1/2}$ with r independent blocks of the form

$$L_{1/2} = \begin{pmatrix} \mathcal{A}^{(1)} & f_1^{(1)} & & & & \\ f_2^{(1)} & \mathcal{B}^{(2)} & & & & \\ & & \ddots & \ddots & & \\ & & \ddots & \ddots & & \\ & & & & \mathcal{A}^{(2r-1)} & f_1^{(2r-1)} \\ & & & & f_2^{(2r-1)} & \mathcal{B}^{(2r)} \end{pmatrix} \tag{4.21}$$

In each block we have two possible choices of the parameters

$$\text{1st choice}: \quad \bar{\alpha}_1^{(2\ell-1)}\beta_{(2\ell-1)}^1 = 1, \quad \bar{\gamma}_2^{(2\ell-1)} = \delta_{(2\ell-1)}^2 = 0$$

$$\text{2nd choice}: \ \bar{\alpha}_1^{(2\ell-1)} = \beta_{(2\ell-1)}^1 = 0, \qquad \bar{\gamma}_2^{(2\ell-1)}\delta_{(2\ell-1)}^2 = 1 \tag{4.22}$$

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Therefore, there are in total 2^r different 1/2 BPS WLs, each depending on r free complex parameters. The 1/2 BPS WLs $W^{\mathrm{I}}_{1/2}[\bar{\alpha}_i, 0]$ and $W^{\mathrm{II}}_{1/2}[\bar{\gamma}_i, 0]$ in section 3.2, with respectively $\bar{\alpha}_i = (\bar{\alpha}_1, 0)$ and $\bar{\gamma}_i = (0, \bar{\gamma}_2)$, are just special cases of these general 1/2 BPS operators.

Similarly, we can construct operators that preserve Poincaré supercharges

$$\theta^{i\hat{1}}_+, \quad \theta^{i\hat{2}}_-, \quad i = 1, 2 \tag{4.23}$$

This time the couplings entering superconnection (3.3) or (3.4) with definitions (4.6) are fixed by the following set of constraints

$$
\begin{aligned}
&\bar{\alpha}^{(2\ell)}_{\hat{i}} = (\bar{\alpha}^{(2\ell)}_{\hat{1}}, 0), && \beta^{\hat{i}}_{(2\ell)} = (\beta^{\hat{1}}_{(2\ell)}, 0) && (4.24)\\
&\bar{\gamma}^{(2\ell-1)}_{\hat{i}} = (0, \bar{\gamma}^{(2\ell)}_{\hat{2}}), && \delta^{\hat{i}}_{(2\ell-1)} = (0, \delta^{\hat{2}}_{(2\ell)})\\
&\bar{\alpha}^{(2\ell-1)}_{i} = \beta^{i}_{(2\ell-1)} = \bar{\gamma}^{(2\ell-1)}_{i} = \delta^{i}_{(2\ell-1)} = 0\\
&\bar{\alpha}^{(2\ell)}_{\hat{1}} \delta^{\hat{2}}_{(2\ell)} = \bar{\gamma}^{(2\ell)}_{\hat{2}} \beta^{1}_{(\hat{2}\ell)} = 0, && \bar{\alpha}^{(2\ell)}_{\hat{1}} \beta^{\hat{1}}_{(2\ell)} + \bar{\gamma}^{(2\ell)}_{\hat{2}} \delta^{2}_{(\hat{2}\ell)} = 1\\
&M_{(2\ell-1)}{}^{\hat{i}}{}_{\hat{j}} = M_{(2\ell-1)}{}^{\hat{i}}{}_{j} = M_{(2\ell-1)}{}^{i}{}_{\hat{j}} = M_{(2\ell)}{}^{\hat{i}}{}_{\hat{j}} = M_{(2\ell)}{}^{\hat{i}}{}_{j} = M_{(2\ell)}{}^{i}{}_{\hat{j}} = 0\\
&M_{(2\ell-1)}{}^{i}{}_{j} = \operatorname{diag}\left(2\bar{\alpha}^{(2\ell-2)}_{\hat{1}} \beta^{\hat{1}}_{(2\ell-2)}, -2\bar{\gamma}^{(2\ell-2)}_{\hat{2}} \delta^{\hat{2}}_{(2\ell-2)}\right)\\
&M_{(2\ell)}{}^{i}{}_{j} = \operatorname{diag}\left(2\bar{\alpha}^{(2\ell)}_{\hat{1}} \beta^{\hat{1}}_{(2\ell)}, -2\bar{\gamma}^{(2\ell)}_{\hat{2}} \delta^{\hat{2}}_{(2\ell)}\right)
\end{aligned}
$$

The corresponding superconnection is also block-diagonal with r independent blocks

$$
\tilde{L}_{1/2} = \begin{pmatrix}
\mathcal{A}^{(1)} & & & & & f^{(2r)}_2 \\
& \mathcal{B}^{(2)} & f^{(2)}_1 & & & \\
& f^{(2)}_2 & \mathcal{A}^{(3)} & & & \\
& & & \ddots & \ddots & \\
& & & \ddots & \ddots & \\
f^{(2r)}_1 & & & & & \mathcal{B}^{(2r)}
\end{pmatrix} \tag{4.25}
$$

and in each block the parameters can be chosen in two different ways

$$
\begin{aligned}
\text{1st choice}: \quad &\bar{\alpha}^{(2\ell)}_{\hat{1}} \beta^{\hat{1}}_{(2\ell)} = 1, \; \bar{\gamma}^{(2\ell)}_{\hat{2}} = \delta^{\hat{2}}_{(2\ell)} = 0\\
\text{2nd choice}: \quad &\bar{\alpha}^{(2\ell)}_{\hat{1}} = \beta^{\hat{1}}_{(2\ell)} = 0, \qquad \bar{\gamma}^{(2\ell)}_{\hat{2}} \delta^{\hat{2}}_{(2\ell)} = 1
\end{aligned} \tag{4.26}
$$

Therefore, there are still 2^r different operators, each one depending on r free complex parameters. The 1/2 BPS WLs $W^{\mathrm{I}}_{1/2}[0, \bar{\alpha}_{\hat{i}}]$ with $\bar{\alpha}_{\hat{i}} = (\bar{\alpha}_{\hat{1}}, 0)$ and $W^{\mathrm{II}}_{1/2}[0, \bar{\gamma}_{\hat{i}}]$ with $\bar{\gamma}_{\hat{i}} = (0, \bar{\gamma}_{\hat{2}})$ in section 3.2 are special cases of this general class of 1/2 BPS WLs.

The 1/2 BPS WLs in this section correspond to superconnections that are simply direct sums of the double-node 1/2 BPS superconnections already introduced in [27, 28, 35]. There are in fact no new 1/2 BPS solutions.

Note that supercharges (4.3) are included in both (4.19) and (4.23), so the 1/2 BPS WLs just found are special cases of the fermionic 1/4 BPS WLs constructed in the previous subsection, as one can easily check.

5 Summary and discussion

The main result of this paper is the construction of new 1/4 BPS WLs in $\mathcal{N} = 4$ circular quiver SCSM theories with gauge group and levels $\prod_{\ell=1}^{r}[\mathrm{U}(N_{2\ell-1})_k \times \mathrm{U}(N_{2\ell})_{-k}]$.

First of all, we have considered $[\mathrm{U}(N)_k \times \mathrm{U}(N)_{-k}]^r$ $\mathcal{N} = 4$ orbifold ABJM models. By performing the orbifold decomposition of 1/2 and 1/6 BPS WLs in ABJ(M) we have not only recovered the already known WLs, but also found new WLs described by more general superconnections that are not block-diagonal like the ones considered so far in the literature [7, 15, 27, 28, 35]. As a consequence, the structure of the corresponding WLs is more general and not reducible to ABJ(M)-like double-node operators. These new WLs are 1/4 BPS and get enhanced to the already known 1/2 BPS for special values of the matter couplings appearing in the superconnection. They can be classified into four classes, and each class is parametrized by four free complex numbers. Setting all the parameters to zero we obtain the bosonic 1/4 BPS WL. Additionally, we have identified the corresponding M2-/anti-M2-brane duals. The novelty here is that we find the explicit M2 configuration dual to 1/4 BPS operators $W_{1/4}^{\mathrm{I}}[\bar{\alpha}_i, \bar{\alpha}_{\hat{i}}]$ and $W_{1/4}^{\mathrm{II}}[\bar{\gamma}_i, \bar{\gamma}_{\hat{i}}]$, beyond the already known duals of 1/2 BPS WLs [15].

These findings have been confirmed and generalized by using a more systematic approach that consists in studying the SUSY transformations of the most general superconnection in a $\mathcal{N} = 4$ SCSM theory and imposing its invariance under a given subset of supercharges. We have obtained the complete spectrum of 1/4 and 1/2 BPS WLs that contains a generalization of the four classes already mentioned plus some extra operators that fall outside these classes. The complete spectrum of fermionic 1/4 and 1/2 BPS WLs in $\mathcal{N} = 4$ circular quiver SCSM theories can be summarized as follows.

- *1/4 BPS WLs*: with the set of preserved supercharges being fixed, the most general 1/4 BPS WL is the holonomy of a superconnection (3.3) or (3.4) that depends on $8r$ complex parameters subject to $12r$ non-linear constraints. There are four classes of solutions to these constraints, and in each class the WL has $4r$ free complex parameters. There are also other solutions that do not fall into the four classes, and such WLs necessarily have fewer free parameters.

- *1/2 BPS WLs*: 1/2 BPS WLs always correspond to block diagonal superconnections with r blocks. Fixing the set of preserved supercharges, there are 2^r different choices of the parameters, and for each choice the operator depends on r free complex parameters.

Our investigation of the BPS nature of WLs has been carried out on operators defined on timelike straight lines in Minkowski spacetime. Performing a Wick rotation followed by a conformal transformation we can map them to WLs on circular contours in Euclidean space. For completeness, we report in appendix C the explicit forms of the circular 1/4 BPS WLs, which are particularly relevant for other purposes. Indeed, it is the expectation value of circular bosonic 1/4 BPS WL $W_{1/4}^{\mathrm{bos}}[\bar{\mu}_i, 0, 0, \bar{\nu}_{\hat{i}}]$ in Euclidean space that can be computed exactly using localization techniques [4]. In fact, given the circular bosonic 1/4 BPS WL

$$W_{1/4}^{\mathrm{bos}} = \mathrm{Tr}\mathcal{P}\mathrm{e}^{-\mathrm{i}\oint d\tau L_{\mathrm{bos}}} \tag{5.1}$$

where L_{bos} is of the form (C.4), being the theory invariant under $\text{SU}(2)_L \times \text{SU}(2)_R$ R-symmetry, the expectation value is independent of the particular choice of the $\bar{\mu}_i, \bar{\nu}_{\hat{i}}$ parameters. Since it is block diagonal with blocks $L_{\text{bos}}^{(a)}$, $a = 1, 2, \cdots, 2r$, the expectation value reduces to a sum over the blocks

$$\langle W_{1/4}^{\text{bos}} \rangle_1 = \sum_{a=1}^{2r} \langle W_{1/4}^{\text{bos},(a)} \rangle_1 \tag{5.2}$$

The subscript "1" indicates that the result is at framing one, the regularization scheme automatically selected by the localization procedure [4]. The weak coupling expansion of the (a)-block reads [36, 37]

$$\langle W_{1/4}^{\text{bos},(a)} \rangle_1 = N_a \left[1 + \frac{(-)^{a+1}\mathrm{i}\pi}{k} N_a + \frac{\pi^2}{6k^2}(-4N_a^2 + 3N_a N_{a-1} + 3N_a N_{a+1} + 1) + O\left(\frac{1}{k^3}\right) \right] \tag{5.3}$$

This immediately leads to an interesting feature of the full family of WLs that we have introduced. All the fermionic 1/4 and 1/2 BPS WLs are in the same $\mathcal{Q}$-cohomological class of $W_{1/4}^{\text{bos}}[\bar{\mu}_i, 0, 0, \bar{\nu}_{\hat{i}}]$, where $\mathcal{Q}$ is the charge used to localize the path integral. Therefore, if the cohomological equivalence in not broken by quantum anomalies, all the fermionic WLs should have the same framing one expectation value (5.2), (5.3).

The weak coupling expansion (5.3) has to match the result obtained from perturbation theory. Since the perturbative result is at trivial framing, this requires to correctly identify and remove the framing factor from the matrix model result. In the ABJ(M) case, for the bosonic 1/6 BPS WL this has been discussed in [38, 39] where it has been proved that both gauge and matter sectors contribute to build up the correct framing phase.

In the case of $\mathcal{N} = 4$ circular quiver SCSM theories, generalizing the results of [40, 41] we can compute the two-loop expectation values of all 1/2 and 1/4 BPS WLs. We expect the framing-zero results to depend non-trivially on the parameters [42]. Since the matrix model prediction (5.2), (5.3) is parameter independent, we conclude that a perturbative calculation done at framing one should enlighten a non-trivial conspiracy between gauge and matter sectors, whose parameter dependent contributions should cancel each other. We note that a similar pattern should arise also for parametric fermionic 1/6 BPS WLs in ABJ(M) theory listed in table 1. We will report a detailed study of this issue in [42].

In $\mathcal{N} = 4$ circular quiver SCSM theories a further related subtlety arises concerning degenerate WLs, that is fermionic 1/2 BPS operators in Classes I and II that preserve the same set of supercharges (see appendix B.2). Although at quantum level they are expected to have the same framing-one expectation value, being both $\mathcal{Q}$-equivalent to the bosonic 1/4 BPS WL, it has been proved that at framing-zero they start being different at three loops [36, 37], at least for quiver theories with different group ranks. More generally, an all-loop argument proves that they are necessarily different at any odd order, unless vanishing. This potential tension between the matrix model prediction and the perturbative calculation is presently an open question that definitively requires further investigation.

Finally, BPS WLs have been shown to be related to physical quantities like the energy radiated by a heavy quark slowly moving in a gauge background (Bremsstrahlung function) and the cusp anomalous dimension that governs the UV divergent behavior of WLs close to a cusp. In ABJM theory an exact prescription has been proposed [19, 43] that gives the

Bremsstrahlung function for 1/2 BPS quark configurations in terms of the framing phase of the 1/6 BPS bosonic WL. The cusp anomalous dimension and the Bremsstrahlung function are in principle amenable of exact computation via integrability, along the lines of the $\mathcal{N} = 4$ SYM case [44, 45]. This would require using the conjectural exact form of the interpolating $h(\lambda)$ function [46]. Therefore, BPS WLs turn out to be a potentially crucial tool to test the exact $h(\lambda)$ and, more generally, the integrability underlying the AdS/CFT correspondence (see [47, 48] for some preliminary results). It would be interesting to compute the cusp anomalous dimension for a cusp with generic fermionic 1/4 BPS operators on the two rays and study its dependence on the parameters in $\mathcal{N} = 4$ SCSM models. This would also open the possibility to define and compute the corresponding Bremsstrahlung function along the lines of [43] and [19, 49].

Acknowledgments

A   Technical details

Here we collect some technical details necessary to follow the general discussion of BPS WLs in section 4.

Given a generic $\mathcal{N} = 4$ SCSM theory with gauge group $\prod_{\ell=1}^{r}[\mathrm{U}(N_{2\ell-1})_k \times \mathrm{U}(N_{2\ell})_{-k}]$, using the field labeling of figure 1 the SUSY transformations read explicitly

$$\delta A_\mu^{(2\ell-1)} = -\frac{2\pi}{k}\left[\left(\phi_i^{(2\ell-2)}\bar{\psi}_{\hat{i}}^{(2\ell-2)} - \phi_{\hat{i}}^{(2\ell-1)}\bar{\psi}_i^{(2\ell-1)}\right)\gamma_\mu\epsilon^{i\hat{i}} + \bar{\epsilon}_{i\hat{i}}\gamma_\mu\left(\psi_{(2\ell-2)}^{\hat{i}}\bar{\phi}_{(2\ell-2)}^i - \psi_{(2\ell-1)}^i\bar{\phi}_{(2\ell-1)}^{\hat{i}}\right)\right]$$

$$\delta B_\mu^{(2\ell)} = -\frac{2\pi}{k}\left[\left(\bar{\psi}_{\hat{i}}^{(2\ell)}\phi_i^{(2\ell)} - \bar{\psi}_i^{(2\ell-1)}\phi_{\hat{i}}^{(2\ell-1)}\right)\gamma_\mu\epsilon^{i\hat{i}} + \bar{\epsilon}_{i\hat{i}}\gamma_\mu\left(\bar{\phi}_{(2\ell)}^i\psi_{(2\ell)}^{\hat{i}} - \bar{\phi}_{(2\ell-1)}^{\hat{i}}\psi_{(2\ell-1)}^i\right)\right]$$

$$\delta\phi_{\hat{i}}^{(2\ell-1)} = -\mathrm{i}\bar{\epsilon}_{i\hat{i}}\psi_{(2\ell-1)}^i, \quad \delta\bar{\phi}_{(2\ell-1)}^{\hat{i}} = -\mathrm{i}\bar{\psi}_i^{(2\ell-1)}\epsilon^{i\hat{i}}, \quad \delta\phi_i^{(2\ell)} = \mathrm{i}\bar{\epsilon}_{i\hat{i}}\psi_{(2\ell)}^{\hat{i}}, \quad \delta\bar{\phi}_{(2\ell)}^i = \mathrm{i}\bar{\psi}_{\hat{i}}^{(2\ell)}\epsilon^{i\hat{i}}$$

$$\delta\psi_{(2\ell-1)}^i = \gamma^\mu\epsilon^{i\hat{i}}D_\mu\phi_{\hat{i}}^{(2\ell-1)} + \vartheta^{i\hat{i}}\phi_{\hat{i}}^{(2\ell-1)} - \frac{4\pi}{k}\epsilon^{j\hat{j}}\left(\phi_{\hat{j}}^{(2\ell-1)}\bar{\phi}_{(2\ell)}^i\phi_j^{(2\ell)} - \phi_j^{(2\ell-2)}\bar{\phi}_{(2\ell-2)}^i\phi_{\hat{j}}^{(2\ell-1)}\right)$$
$$+ \frac{2\pi}{k}\epsilon^{i\hat{i}}\left(\phi_{\hat{i}}^{(2\ell-1)}\bar{\phi}_{(2\ell-1)}^{\hat{j}}\phi_{\hat{j}}^{(2\ell-1)} + \phi_{\hat{i}}^{(2\ell-1)}\bar{\phi}_{(2\ell)}^j\phi_j^{(2\ell)}\right.$$
$$\left. - \phi_{\hat{j}}^{(2\ell-1)}\bar{\phi}_{(2\ell-1)}^{\hat{j}}\phi_i^{(2\ell-1)} - \phi_j^{(2\ell-2)}\bar{\phi}_{(2\ell-2)}^j\phi_i^{(2\ell-1)}\right)$$

$$\delta\bar{\psi}_i^{(2\ell-1)} = -\bar{\epsilon}_{i\hat{i}}\gamma^\mu D_\mu\bar{\phi}_{(2\ell-1)}^{\hat{i}} + \bar{\vartheta}_{i\hat{i}}\bar{\phi}_{(2\ell-1)}^{\hat{i}} + \frac{4\pi}{k}\bar{\epsilon}_{j\hat{j}}\left(\bar{\phi}_{(2\ell-1)}^{\hat{j}}\phi_i^{(2\ell-2)}\bar{\phi}_{(2\ell-2)}^j - \bar{\phi}_{(2\ell)}^j\phi_i^{(2\ell)}\bar{\phi}_{(2\ell-1)}^{\hat{j}}\right)$$
$$- \frac{2\pi}{k}\bar{\epsilon}_{i\hat{i}}\left(\bar{\phi}_{(2\ell-1)}^{\hat{i}}\phi_{\hat{j}}^{(2\ell-1)}\bar{\phi}_{(2\ell-1)}^{\hat{j}} + \bar{\phi}_{(2\ell-1)}^{\hat{i}}\phi_j^{(2\ell-2)}\bar{\phi}_{(2\ell-2)}^j\right.$$
$$\left. - \bar{\phi}_{(2\ell-1)}^{\hat{j}}\phi_{\hat{j}}^{(2\ell-1)}\bar{\phi}_{(2\ell-1)}^{\hat{i}} - \bar{\phi}_{(2\ell)}^j\phi_j^{(2\ell)}\bar{\phi}_{(2\ell-1)}^{\hat{i}}\right)$$

$$\delta\psi_{(2\ell)}^{\hat{i}} = -\gamma^\mu\epsilon^{i\hat{i}}D_\mu\phi_i^{(2\ell)} - \vartheta^{i\hat{i}}\phi_i^{(2\ell)} - \frac{4\pi}{k}\epsilon^{j\hat{j}}\left(\phi_{\hat{j}}^{(2\ell+1)}\bar{\phi}_{(2\ell+1)}^{\hat{i}}\phi_j^{(2\ell)} - \phi_j^{(2\ell)}\bar{\phi}_{(2\ell-1)}^{\hat{i}}\phi_{\hat{j}}^{(2\ell-1)}\right)$$
$$- \frac{2\pi}{k}\epsilon^{i\hat{i}}\left(\phi_i^{(2\ell)}\bar{\phi}_{(2\ell-1)}^{\hat{j}}\phi_{\hat{j}}^{(2\ell-1)} + \phi_i^{(2\ell)}\bar{\phi}_{(2\ell)}^j\phi_j^{(2\ell)} - \phi_{\hat{j}}^{(2\ell+1)}\bar{\phi}_{(2\ell+1)}^{\hat{j}}\phi_i^{(2\ell)} - \phi_j^{(2\ell)}\bar{\phi}_{(2\ell)}^j\phi_i^{(2\ell)}\right)$$

$$\delta\bar{\psi}_{\hat{i}}^{(2\ell)} = \bar{\epsilon}_{i\hat{i}}\gamma^\mu D_\mu\bar{\phi}_{(2\ell)}^i - \bar{\vartheta}_{i\hat{i}}\bar{\phi}_{(2\ell)}^i + \frac{4\pi}{k}\bar{\epsilon}_{j\hat{j}}\left(\bar{\phi}_{(2\ell-1)}^{\hat{j}}\phi_{\hat{i}}^{(2\ell-1)}\bar{\phi}_{(2\ell)}^j - \bar{\phi}_{(2\ell)}^j\phi_{\hat{i}}^{(2\ell+1)}\bar{\phi}_{(2\ell+1)}^{\hat{j}}\right) \tag{A.1}$$
$$+ \frac{2\pi}{k}\bar{\epsilon}_{i\hat{i}}\left(\bar{\phi}_{(2\ell)}^i\phi_{\hat{j}}^{(2\ell+1)}\bar{\phi}_{(2\ell+1)}^{\hat{j}} + \bar{\phi}_{(2\ell)}^i\phi_j^{(2\ell)}\bar{\phi}_{(2\ell)}^j - \bar{\phi}_{(2\ell-1)}^{\hat{j}}\phi_{\hat{j}}^{(2\ell-1)}\bar{\phi}_{(2\ell)}^i - \bar{\phi}_{(2\ell)}^j\phi_j^{(2\ell)}\bar{\phi}_{(2\ell)}^i\right)$$

where the SUSY parameters are

$$\epsilon^{i\hat{i}} = \theta^{i\hat{i}} + x^\mu \gamma_\mu \vartheta^{i\hat{i}}, \qquad \bar{\epsilon}_{i\hat{i}} = \bar{\theta}_{i\hat{i}} - \bar{\vartheta}_{i\hat{i}} x^\mu \gamma_\mu \tag{A.2}$$

We apply these transformations to the most general superconnection of the form (3.3), (3.4) and impose that the result can be written as in (4.1) with a matrix G given in (4.7). This condition translates into a set of constraints on the superconnections entries (3.5) that read

$$\delta\mathcal{A}^{(2\ell-1)} = \mathrm{i}(f_1^{(2\ell-1)}g_2^{(2\ell-1)} + f_2^{(2\ell-2)}g_1^{(2\ell-2)} - g_1^{(2\ell-1)}f_2^{(2\ell-1)} - g_2^{(2\ell-2)}f_1^{(2\ell-2)})$$

$$\delta\mathcal{B}^{(2\ell-1)} = \mathrm{i}(f_1^{(2\ell)}g_2^{(2\ell)} + f_2^{(2\ell-1)}g_1^{(2\ell-1)} - g_1^{(2\ell)}f_2^{(2\ell)} - g_2^{(2\ell-1)}f_1^{(2\ell-1)})$$

$$\delta h_1^{(2\ell-1)} = \mathrm{i}(f_1^{(2\ell-1)}g_1^{(2\ell)} - g_1^{(2\ell-1)}f_1^{(2\ell)}), \quad \delta h_1^{(2\ell)} = \mathrm{i}(f_1^{(2\ell)}g_1^{(2\ell+1)} - g_1^{(2\ell)}f_1^{(2\ell+1)})$$

$$\delta h_2^{(2\ell-1)} = \mathrm{i}(f_2^{(2\ell)}g_2^{(2\ell-1)} - g_2^{(2\ell)}f_2^{(2\ell-1)}), \quad \delta h_2^{(2\ell)} = \mathrm{i}(f_2^{(2\ell+1)}g_2^{(2\ell)} - g_2^{(2\ell+1)}f_2^{(2\ell)})$$

$$\delta f_1^{(2\ell-1)} = \partial_\tau g_1^{(2\ell-1)} + \mathrm{i}(\mathcal{A}^{(2\ell-1)}g_1^{(2\ell-1)} - g_1^{(2\ell-1)}\mathcal{B}^{(2\ell)} + h_1^{(2\ell-1)}g_2^{(2\ell)} - g_2^{(2\ell-2)}h_1^{(2\ell-2)})$$

$$\delta f_1^{(2\ell)} = \partial_\tau g_1^{(2\ell)} + \mathrm{i}(\mathcal{B}^{(2\ell)}g_1^{(2\ell)} - g_1^{(2\ell)}\mathcal{A}^{(2\ell+1)} + h_1^{(2\ell)}g_2^{(2\ell+1)} - g_2^{(2\ell-1)}h_1^{(2\ell-1)})$$

$$\delta f_2^{(2\ell-1)} = \partial_\tau g_2^{(2\ell-1)} + \mathrm{i}(\mathcal{B}^{(2\ell)}g_2^{(2\ell-1)} - g_2^{(2\ell-1)}\mathcal{A}^{(2\ell-1)} + h_2^{(2\ell-2)}g_1^{(2\ell-2)} - g_1^{(2\ell)}h_2^{(2\ell-1)})$$

$$\delta f_2^{(2\ell)} = \partial_\tau g_2^{(2\ell)} + \mathrm{i}(\mathcal{A}^{(2\ell+1)}g_2^{(2\ell)} - g_2^{(2\ell)}\mathcal{B}^{(2\ell)} + h_2^{(2\ell-1)}g_1^{(2\ell-1)} - g_1^{(2\ell+1)}h_2^{(2\ell)})$$

$$h_1^{(2\ell-1)}g_1^{(2\ell+1)} - g_1^{(2\ell-1)}h_1^{(2\ell)} = h_2^{(2\ell-1)}g_2^{(2\ell-2)} - g_2^{(2\ell)}h_2^{(2\ell-2)} = 0$$

$$h_1^{(2\ell-2)}g_1^{(2\ell)} - g_1^{(2\ell-2)}h_1^{(2\ell-1)} = h_2^{(2\ell)}g_2^{(2\ell-1)} - g_2^{(2\ell+1)}h_2^{(2\ell-1)} = 0 \tag{A.3}$$

These constraints allow for non-trivial solutions depending on the number of preserved SUSY charges, as discussed in section 4 (see eqs. (4.8), (4.9) and (4.20)).

B Supercharge overlapping of 1/2 BPS WLs

This appendix is a supplement to [15], and we discuss the amount of overlapping supercharges for general 1/2 BPS WLs both in ABJ(M) theory and general $\mathcal{N} = 4$ circular quiver SCSM theories with alternating levels.

B.1 ABJ(M) theory

Given the two classes of 1/2 BPS WLs $W_{1/2}^{\mathrm{I}}[\bar{\alpha}_I]$, $W_{1/2}^{\mathrm{II}}[\bar{\gamma}_I]$ reviewed in subsection 2.2, we can distinguish three different cases.

- $W_{1/2}^{\mathrm{I}}[\bar{\alpha}_I]$ and $W_{1/2}^{\mathrm{I}}[\bar{\alpha}_I']$. When $\bar{\alpha}_I' \propto \bar{\alpha}_I$ they are in fact the same WL and trivially preserve the same supercharges. Otherwise, when $\bar{\alpha}_I' \not\propto \bar{\alpha}_I$, they share 1/3 of the preserved supercharges

$$\bar{\alpha}_I \bar{\alpha}_J' \theta_+^{IJ}, \quad \varepsilon_{IJKL}\alpha^I \alpha'^J \theta_-^{KL} \tag{B.1}$$

- $W_{1/2}^{\mathrm{I}}[\bar{\alpha}_I]$ and $W_{1/2}^{\mathrm{II}}[\bar{\gamma}_I]$. When $\bar{\alpha}_I \gamma^I = 0$, they share 2/3 of the preserved supercharges

$$\bar{\alpha}_I R^J_{\ K} \theta_+^{IK}, \quad \bar{\gamma}_I R^J_{\ K} \theta_-^{IK} \tag{B.2}$$

with $R^J_{\ K} = \delta^J_K - \frac{\alpha^J \bar{\alpha}_K}{|\alpha|^2} - \frac{\gamma^J \bar{\gamma}_K}{|\gamma|^2}$. Otherwise, when $\bar{\alpha}_I \gamma^I \neq 0$, there is no overlapping of preserved supercharges. In particular, for $\bar{\alpha}_I = \bar{\gamma}_I$ they preserve complementary sets of supercharges.

- $W^{II}_{1/2}[\bar{\gamma}_I]$ and $W^{II}_{1/2}[\bar{\gamma}'_I]$. When $\bar{\gamma}'_I \propto \bar{\gamma}_I$, they trivially preserve the same supercharges. Otherwise, when $\bar{\gamma}'_I \not\propto \bar{\gamma}_I$, they share $1/3$ of the preserved supercharges

$$\bar{\gamma}_I \bar{\gamma}'_J \theta^{IJ}_-, \quad \varepsilon_{IJKL} \gamma^I \gamma'^J \theta^{KL}_+ \tag{B.3}$$

All these results are consistent with the examples discussed in [15]. In particular, in ABJ(M) theory, there are no non-trivial cases of different 1/2 BPS WLs preserving the same set of supercharges.

B.2 $\mathcal{N} = 4$ circular quiver SCSM theories with alternating levels

As discussed in the main text, in this case we have four classes of 1/2 BPS WLs, $W^{I}_{1/2}[\bar{\alpha}_i, 0]$, $W^{I}_{1/2}[0, \bar{\alpha}_{\hat{i}}]$, $W^{II}_{1/2}[\bar{\gamma}_i, 0]$ and $W^{II}_{1/2}[0, \bar{\gamma}_{\hat{i}}]$. We need then to distinguish ten different cases.

- $W^{I}_{1/2}[\bar{\alpha}_i, 0]$ and $W^{I}_{1/2}[\bar{\alpha}'_i, 0]$. When $\bar{\alpha}'_i \propto \bar{\alpha}_i$ they are basically the same operator and trivially preserve the same supercharges. Otherwise, when $\bar{\alpha}'_i \not\propto \bar{\alpha}_i$ they do not share any preserved supercharges.

- $W^{I}_{1/2}[\bar{\alpha}_i, 0]$ and $W^{I}_{1/2}[0, \bar{\alpha}'_{\hat{i}}]$. They share $1/2$ of the preserved supercharges

$$\bar{\alpha}_i \bar{\alpha}'_{\hat{j}} \theta^{i\hat{j}}_+, \quad \varepsilon_{ij} \varepsilon_{\hat{k}\hat{l}} \alpha^i \alpha'^{\hat{k}} \theta^{j\hat{l}}_- \tag{B.4}$$

- $W^{I}_{1/2}[\bar{\alpha}_i, 0]$ and $W^{II}_{1/2}[\bar{\gamma}_i, 0]$. When $\bar{\alpha}_i \gamma^i = 0$, they preserve the same supercharges

$$\bar{\alpha}_i \theta^{i\hat{j}}_+ \sim \varepsilon_{ik} \gamma^k \theta^{i\hat{j}}_+, \quad \bar{\gamma}_i \theta^{i\hat{j}}_- \sim \varepsilon_{ik} \alpha^k \theta^{i\hat{j}}_- \tag{B.5}$$

  Otherwise, when $\bar{\alpha}_i \gamma^i \neq 0$, they do not share any preserved supercharge.

- $W^{I}_{1/2}[\bar{\alpha}_i, 0]$ and $W^{II}_{1/2}[0, \bar{\gamma}_{\hat{i}}]$. They share $1/2$ of the preserved supercharges

$$\varepsilon_{\hat{i}\hat{j}} \bar{\alpha}_k \gamma^{\hat{i}} \theta^{k\hat{j}}_+, \quad \varepsilon_{ij} \alpha^i \bar{\gamma}_{\hat{k}} \theta^{j\hat{k}}_- \tag{B.6}$$

- $W^{I}_{1/2}[0, \bar{\alpha}_{\hat{i}}]$ and $W^{I}_{1/2}[0, \bar{\alpha}'_{\hat{i}}]$. When $\bar{\alpha}'_{\hat{i}} \propto \bar{\alpha}_{\hat{i}}$, they trivially preserve the same supercharges. Otherwise, they do not share any preserved supercharge.

- $W^{I}_{1/2}[0, \bar{\alpha}_{\hat{i}}]$ and $W^{II}_{1/2}[\bar{\gamma}_i, 0]$. They share $1/2$ of the preserved supercharges

$$\varepsilon_{ij} \bar{\alpha}_{\hat{k}} \gamma^i \theta^{j\hat{k}}_+, \quad \varepsilon_{\hat{i}\hat{j}} \alpha^i \bar{\gamma}_k \theta^{k\hat{j}}_- \tag{B.7}$$

- $W^{I}_{1/2}[0, \bar{\alpha}_{\hat{i}}]$ and $W^{II}_{1/2}[0, \bar{\gamma}_{\hat{i}}]$. When $\bar{\alpha}_{\hat{i}} \gamma^{\hat{i}} = 0$, they preserve the same supercharges

$$\bar{\alpha}_{\hat{j}} \theta^{i\hat{j}}_+ \sim \varepsilon_{\hat{j}\hat{k}} \gamma^{\hat{k}} \theta^{i\hat{j}}_+, \quad \bar{\gamma}_{\hat{j}} \theta^{i\hat{j}}_- \sim \varepsilon_{\hat{j}\hat{k}} \alpha^{\hat{k}} \theta^{i\hat{j}}_- \tag{B.8}$$

  Otherwise, when $\bar{\alpha}_{\hat{i}} \gamma^{\hat{i}} \neq 0$, they do not share any preserved supercharge.

- $W^{II}_{1/2}[\bar{\gamma}_i, 0]$ and $W^{II}_{1/2}[\bar{\gamma}'_i, 0]$. When $\bar{\gamma}'_i \propto \bar{\gamma}_i$, they trivially preserve the same supercharges. Otherwise, when $\bar{\gamma}'_i \not\propto \bar{\gamma}_i$, they do not share any preserved supercharge.

- $W_{1/2}^{\mathrm{II}}[\bar\gamma_i, 0]$ and $W_{1/2}^{\mathrm{II}}[0, \bar\gamma'_{\hat\imath}]$. They share $1/2$ of the preserved supercharges

$$\bar\gamma_i\bar\gamma'_{\hat\jmath}\theta^{i\hat\jmath}_-, \quad \varepsilon_{ij}\varepsilon_{\hat k\hat l}\gamma^i\gamma'^{\hat k}\theta^{j\hat l}_+ \tag{B.9}$$

- $W_{1/2}^{\mathrm{II}}[0, \bar\gamma_{\hat\imath}]$ and $W_{1/2}^{\mathrm{II}}[0, \bar\gamma'_{\hat\imath}]$. When $\bar\gamma'_{\hat\imath} \propto \bar\gamma_{\hat\imath}$, they trivially preserve the same supercharges. Otherwise, they do not share any preserved supercharge.

These results are consistent with the examples in [15]. We conclude that the only non-trivial pairs of different $1/2$ BPS WLs preserving the same supercharges are

$$(W_{1/2}^{\mathrm{I}}[\bar\alpha_i, 0], W_{1/2}^{\mathrm{II}}[\bar\gamma_i, 0]) \quad \text{with} \quad \bar\alpha_i\gamma^i = 0$$

$$(W_{1/2}^{\mathrm{I}}[0, \bar\alpha_{\hat\imath}], W_{1/2}^{\mathrm{II}}[0, \bar\gamma_{\hat\imath}]) \quad \text{with} \quad \bar\alpha_{\hat\imath}\gamma^{\hat\imath} = 0 \tag{B.10}$$

C Circular 1/4 BPS WLs in Euclidean $\mathcal{N} = 4$ SCSM theories

Linear WLs have constant expectation values, so they cannot be used to check non-trivially the matching between matrix model, field theory and holographic predictions. Non-trivial expectation values can be obtained for WLs on closed contours. However, since there are no spacelike BPS WLs in Minkowski spacetime [50], we have to build-up circular BPS WLs in Euclidean space.

In this appendix, we generalize the procedure used for linear WLs in Minkowski spacetime to obtain the explicit form of euclidean circular $1/4$ BPS WLs in $\mathcal{N} = 4$ circular quiver SCSM theories.

We use coordinates $x^\mu = (x^1, x^2, x^3)$, metric $\delta_{\mu\nu} = \mathrm{diag}(+++)$ and choose gamma matrices as

$$\gamma^\mu{}_\alpha{}^\beta = (-\sigma^2, \sigma^1, \sigma^3) \tag{C.1}$$

For the circle $x^\mu = (\cos\tau, \sin\tau, 0)$, we define the following Grassmann even spinors

$$u_{+\alpha} = \frac{1}{\sqrt{2}}\begin{pmatrix} \mathrm{e}^{-\frac{\mathrm{i}\tau}{2}} \\ \mathrm{e}^{\frac{\mathrm{i}\tau}{2}} \end{pmatrix}, \qquad u_{-\alpha} = \frac{\mathrm{i}}{\sqrt{2}}\begin{pmatrix} -\mathrm{e}^{-\frac{\mathrm{i}\tau}{2}} \\ \mathrm{e}^{\frac{\mathrm{i}\tau}{2}} \end{pmatrix}$$

$$u_+^\alpha = \frac{1}{\sqrt{2}}\left(\mathrm{e}^{\frac{\mathrm{i}\tau}{2}}, -\mathrm{e}^{-\frac{\mathrm{i}\tau}{2}}\right), \quad u_-^\alpha = \frac{\mathrm{i}}{\sqrt{2}}\left(\mathrm{e}^{\frac{\mathrm{i}\tau}{2}}, \mathrm{e}^{-\frac{\mathrm{i}\tau}{2}}\right) \tag{C.2}$$

SUSY transformations and supercharges for $\mathcal{N} = 4$ SCSM theories in Euclidean space are formally the same as those in Minkowski spacetime (see eqs. (A.1), (4.2)).

To be definite, we construct BPS WLs preserving supercharges

$$\vartheta^{1\hat 1} = -\mathrm{i}\gamma_3\theta^{1\hat 1}, \quad \vartheta^{2\hat 2} = \mathrm{i}\gamma_3\theta^{2\hat 2} \tag{C.3}$$

We find the bosonic $1/4$ BPS WL W_{bos} with connection [27, 28]

$$L_{\mathrm{bos}} = \mathrm{diag}(\mathcal{A}_{\mathrm{bos}}^{(1)}, \mathcal{B}_{\mathrm{bos}}^{(2)}, \cdots, \mathcal{A}_{\mathrm{bos}}^{(2r-1)}, \mathcal{B}_{\mathrm{bos}}^{(2r)})$$

$$\mathcal{A}_{\mathrm{bos}}^{(2\ell-1)} = A_\mu^{(2\ell-1)}\dot x^\mu + \frac{2\pi\mathrm{i}}{k}\Big(\phi_1^{(2\ell-2)}\bar\phi_{(2\ell-2)}^1 - \phi_2^{(2\ell-2)}\bar\phi_{(2\ell-2)}^2$$

$$+ \phi_{\hat 1}^{(2\ell-1)}\bar\phi_{(2\ell-1)}^{\hat 1} - \phi_{\hat 2}^{(2\ell-1)}\bar\phi_{(2\ell-1)}^{\hat 2}\Big)|\dot x| \tag{C.4}$$

$$\mathcal{B}_{\mathrm{bos}}^{(2\ell)} = B_\mu^{(2\ell)}\dot x^\mu + \frac{2\pi\mathrm{i}}{k}\Big(\bar\phi_{(2\ell)}^1\phi_1^{(2\ell)} - \bar\phi_{(2\ell)}^2\phi_2^{(2\ell)} + \bar\phi_{(2\ell-1)}^{\hat 1}\phi_{\hat 1}^{(2\ell-1)} - \bar\phi_{(2\ell-1)}^{\hat 2}\phi_{\hat 2}^{(2\ell-1)}\Big)|\dot x|$$

For fermionic WLs we find superconnections L_{fer} of the form (3.3) or (3.4) with definitions

$$\mathcal{A}^{(2\ell-1)} = \mathcal{A}^{(2\ell-1)}_{\text{bos}} + \frac{4\pi\mathrm{i}}{k}\Big[-\bar{\alpha}^{(2\ell-2)}_{\hat{1}}\beta^{\hat{1}}_{(2\ell-2)}\phi^{(2\ell-2)}_1\bar{\phi}^1_{(2\ell-2)} + \bar{\gamma}^{(2\ell-2)}_{\hat{2}}\delta^{\hat{2}}_{(2\ell-2)}\phi^{(2\ell-2)}_2\bar{\phi}^2_{(2\ell-2)}$$
$$- \bar{\alpha}^{(2\ell-1)}_1\beta^1_{(2\ell-1)}\phi^{(2\ell-1)}_{\hat{1}}\bar{\phi}^{\hat{1}}_{(2\ell-1)} + \bar{\gamma}^{(2\ell-1)}_2\delta^2_{(2\ell-1)}\phi^{(2\ell-1)}_{\hat{2}}\bar{\phi}^{\hat{2}}_{(2\ell-1)}\Big]|\dot{x}|$$

$$\mathcal{B}^{(2\ell)} = \mathcal{B}^{(2\ell)}_{\text{bos}} + \frac{4\pi\mathrm{i}}{k}\Big[-\bar{\alpha}^{(2\ell)}_{\hat{1}}\beta^{\hat{1}}_{(2\ell)}\bar{\phi}^1_{(2\ell)}\phi^{(2\ell)}_1 + \bar{\gamma}^{(2\ell)}_{\hat{2}}\delta^{\hat{2}}_{(2\ell)}\bar{\phi}^2_{(2\ell)}\phi^{(2\ell)}_2$$
$$- \bar{\alpha}^{(2\ell-1)}_1\beta^1_{(2\ell-1)}\bar{\phi}^{\hat{1}}_{(2\ell-1)}\phi^{(2\ell-1)}_{\hat{1}} + \bar{\gamma}^{(2\ell-1)}_2\delta^2_{(2\ell-1)}\bar{\phi}^{\hat{2}}_{(2\ell-1)}\phi^{(2\ell-1)}_{\hat{2}}\Big]|\dot{x}|$$

$$f^{(2\ell-1)}_1 = \mathrm{i}\sqrt{\frac{4\pi}{k}}\Big(\bar{\alpha}^{(2\ell-1)}_i u_+ - \bar{\gamma}^{(2\ell-1)}_i u_-\Big)\psi^i_{(2\ell-1)}|\dot{x}|$$

$$f^{(2\ell)}_1 = -\mathrm{i}\sqrt{\frac{4\pi}{k}}\bar{\psi}_{\hat{i}}\Big(u_-\beta^{\hat{i}}_{(2\ell)} + u_+\delta^{\hat{i}}_{(2\ell)}\Big)|\dot{x}|$$

$$f^{(2\ell-1)}_2 = -\mathrm{i}\sqrt{\frac{4\pi}{k}}\bar{\psi}_i\Big(u_-\beta^i_{(2\ell-1)} + u_+\delta^i_{(2\ell-1)}\Big)|\dot{x}|$$

$$f^{(2\ell)}_2 = \mathrm{i}\sqrt{\frac{4\pi}{k}}\Big(\bar{\alpha}^{(2\ell)}_{\hat{i}} u_+ - \bar{\gamma}^{(2\ell)}_{\hat{i}} u_-\Big)\psi^{\hat{i}}_{(2\ell)}|\dot{x}|$$

$$h^{(2\ell-1)}_1 = \frac{4\pi\mathrm{i}}{k}\Big(\bar{\alpha}^{(2\ell-1)}_1\beta^{\hat{1}}_{(2\ell)}\phi^{(2\ell-1)}_{\hat{1}}\bar{\phi}^1_{(2\ell)} - \bar{\gamma}^{(2\ell-1)}_2\delta^{\hat{2}}_{(2\ell)}\phi^{(2\ell-1)}_{\hat{2}}\bar{\phi}^2_{(2\ell)}\Big)|\dot{x}|$$

$$h^{(2\ell)}_1 = \frac{4\pi\mathrm{i}}{k}\Big(\bar{\alpha}^{(2\ell+1)}_1\beta^{\hat{1}}_{(2\ell)}\bar{\phi}^1_{(2\ell)}\phi^{(2\ell+1)}_{\hat{1}} - \bar{\gamma}^{(2\ell+1)}_2\delta^{\hat{2}}_{(2\ell)}\bar{\phi}^2_{(2\ell)}\phi^{(2\ell+1)}_{\hat{2}}\Big)|\dot{x}|$$

$$h^{(2\ell-1)}_2 = \frac{4\pi\mathrm{i}}{k}\Big(\bar{\alpha}^{(2\ell)}_{\hat{1}}\beta^1_{(2\ell-1)}\phi^{(2\ell)}_1\bar{\phi}^{\hat{1}}_{(2\ell-1)} - \bar{\gamma}^{(2\ell)}_{\hat{2}}\delta^2_{(2\ell-1)}\phi^{(2\ell)}_2\bar{\phi}^{\hat{2}}_{(2\ell-1)}\Big)|\dot{x}|$$

$$h^{(2\ell)}_2 = \frac{4\pi\mathrm{i}}{k}\Big(\bar{\alpha}^{(2\ell)}_{\hat{1}}\beta^1_{(2\ell+1)}\bar{\phi}^{\hat{1}}_{(2\ell+1)}\phi^{(2\ell)}_1 - \bar{\gamma}^{(2\ell)}_{\hat{2}}\delta^2_{(2\ell+1)}\bar{\phi}^{\hat{2}}_{(2\ell+1)}\phi^{(2\ell)}_2\Big)|\dot{x}| \tag{C.5}$$

They depend on $8r$ parameters

$$\bar{\alpha}^{(2\ell-1)}_1,\ \bar{\alpha}^{(2\ell)}_{\hat{1}},\ \beta^1_{(2\ell-1)},\ \beta^{\hat{1}}_{(2\ell)},\ \bar{\gamma}^{(2\ell-1)}_2,\ \bar{\gamma}^{(2\ell)}_{\hat{2}},\ \delta^2_{(2\ell-1)},\ \delta^{\hat{2}}_{(2\ell)},\quad \ell = 1, 2, \cdots, r \tag{C.6}$$

subject to the following constraints

$$\bar{\alpha}^{(2\ell-1)}_1\delta^2_{(2\ell-1)} = \bar{\alpha}^{(2\ell-1)}_1\delta^{\hat{2}}_{(2\ell-2)} = \bar{\alpha}^{(2\ell-1)}_1\delta^{\hat{2}}_{(2\ell)} = 0$$
$$\bar{\alpha}^{(2\ell)}_{\hat{1}}\delta^2_{(2\ell)} = \bar{\alpha}^{(2\ell)}_{\hat{1}}\delta^{\hat{2}}_{(2\ell-1)} = \bar{\alpha}^{(2\ell)}_{\hat{1}}\delta^{\hat{2}}_{(2\ell+1)} = 0$$
$$\bar{\gamma}^{(2\ell-1)}_2\beta^1_{(2\ell-1)} = \bar{\gamma}^{(2\ell-1)}_2\beta^{\hat{1}}_{(2\ell-2)} = \bar{\gamma}^{(2\ell-1)}_2\beta^{\hat{1}}_{(2\ell)} = 0$$
$$\bar{\gamma}^{(2\ell)}_{\hat{2}}\beta^1_{(2\ell)} = \bar{\gamma}^{(2\ell)}_{\hat{2}}\beta^{\hat{1}}_{(2\ell-1)} = \bar{\gamma}^{(2\ell)}_{\hat{2}}\beta^{\hat{1}}_{(2\ell+1)} = 0 \tag{C.7}$$

As in Minkowski case, solutions can be classified in terms of the number of free parameters they depend on. First, we find four classes of solutions depending on $4r$ free complex parameters

$$\begin{aligned}
\text{Class I:} &\quad \bar{\gamma}^{(2\ell-1)}_2 = \bar{\gamma}^{(2\ell)}_{\hat{2}} = \delta^2_{(2\ell-1)} = \delta^{\hat{2}}_{(2\ell)} = 0\\
\text{Class II:} &\quad \bar{\alpha}^{(2\ell-1)}_1 = \bar{\alpha}^{(2\ell)}_{\hat{1}} = \beta^1_{(2\ell-1)} = \beta^{\hat{1}}_{(2\ell)} = 0\\
\text{Class III:} &\quad \beta^1_{(2\ell-1)} = \beta^{\hat{1}}_{(2\ell)} = \delta^2_{(2\ell-1)} = \delta^{\hat{2}}_{(2\ell)} = 0\\
\text{Class IV:} &\quad \bar{\alpha}^{(2\ell-1)}_1 = \bar{\alpha}^{(2\ell)}_{\hat{1}} = \bar{\gamma}^{(2\ell-1)}_2 = \bar{\gamma}^{(2\ell)}_{\hat{2}} = 0
\end{aligned} \tag{C.8}$$

In addition, we find extra solutions that do not fall into the previous four classes, being functions of a smaller number of parameters. All the considerations concerning the structure of superconnections being block or non-block-diagonal apply in this case exactly in the same manner as in Minkowski spacetime.

Also in the euclidean case fermionic 1/4 BPS WLs are cohomologically equivalent to the bosonic 1/4 BPS WLs. In fact, it is easy to prove that they satisfy the set of conditions (2.11) with

$$\mathcal{Q} = -\zeta(P^{1\hat{1}} - \mathrm{i}\gamma_3 S^{1\hat{1}}) - \zeta(P^{2\hat{2}} + \mathrm{i}\gamma_3 S^{2\hat{2}}), \quad \zeta^\alpha = (1,0), \quad \kappa = 2\mathrm{e}^{-\mathrm{i}\tau} \tag{C.9}$$

and Λ of the form (4.17) with

$$\lambda_1^{(2\ell-1)} = \sqrt{\frac{4\pi}{k}}\mathrm{e}^{\frac{\mathrm{i}\tau}{2}}\big(\mathrm{i}\bar{\alpha}_1^{(2\ell-1)}\phi_{\hat{1}}^{(2\ell-1)} + \bar{\gamma}_2^{(2\ell-1)}\phi_{\hat{2}}^{(2\ell-1)}\big)$$

$$\lambda_1^{(2\ell)} = \sqrt{\frac{4\pi}{k}}\mathrm{e}^{\frac{\mathrm{i}\tau}{2}}\big(\beta_{(2\ell)}^{\hat{1}}\bar{\phi}_{(2\ell)}^1 - \mathrm{i}\delta_{(2\ell)}^{\hat{2}}\bar{\phi}_{(2\ell)}^2\big)$$

$$\lambda_2^{(2\ell-1)} = \sqrt{\frac{4\pi}{k}}\mathrm{e}^{\frac{\mathrm{i}\tau}{2}}\big(-\beta_{(2\ell-1)}^1\bar{\phi}_{(2\ell-1)}^{\hat{1}} + \mathrm{i}\delta_{(2\ell-1)}^2\bar{\phi}_{(2\ell-1)}^{\hat{2}}\big)$$

$$\lambda_2^{(2\ell)} = \sqrt{\frac{4\pi}{k}}\mathrm{e}^{\frac{\mathrm{i}\tau}{2}}\big(-\mathrm{i}\bar{\alpha}_{\hat{1}}^{(2\ell)}\phi_1^{(2\ell)} - \bar{\gamma}_{\hat{2}}^{(2\ell)}\phi_2^{(2\ell)}\big) \tag{C.10}$$

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