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MEASURES OF RISK: valuation and capital adequacy in illiquid markets and systemic risk

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ABSTRACT

MEASURES OF RISK:
VALUATION AND CAPITAL ADEQUACY IN ILLIQUID MARKETS
AND SYSTEMIC RISK

Maria ARDUCA

In this thesis, we study market consistent valuation and capital adequacy in markets with frictions, and duality for systemic risk measures. All along the thesis, we focus on uniperiodal market models. In the first chapter, we consider a market with convex transaction costs, convex portfolio constraints and convex acceptance set that reflects the preferences of an agent who acts as a buyer in the market. We define the set of *market consistent prices* for every conceivable payoff, where consistent is meant with respect to the market and the preferences of the buyer. We show that the supremum of this set coincides with the well-known superreplication price, this giving to this functional an interpretation that goes beyond the classical hedging explanation. We develop an extension of the Fundamental Theorem of Asset Pricing in a context where arbitrages are replaced by acceptable deals (i.e. replicable payoffs with nonpositive price belonging to the acceptance set) and prices are not necessarily linear. This allows to characterize, under suitable assumptions, the set of market consistent prices of any payoff. In the second chapter, we consider an abstract economy with transaction costs both at initial time and at maturity, and portfolio constraints. We do not assume convexity a priori, tough some results hold only under convexity assumptions. An external regulator fixes the acceptance set, that is the set of possible agent's capital positions that he deems acceptable from a risk perspective. We define capital adequacy rules that generalize the coherent risk measures of Artzner, Delbaen, Eber and Heath [9] in that they represent the minimum amount that the agent has to invest in the market in order to reach the acceptability requirements. The chapter aims to study the properties of these general risk measures. In particular, we establish conditions on the portfolios ensuring that they are lower semicontinuous, and we compare these conditions with no-acceptable deal type assumptions. In convex and quasi convex case, we also provide dual representations. In the third chapter we establish dual representations for systemic risk measures. We model interactions among a finite number of institutions through an aggregation function, and we assume that a regulator fixes a set of acceptable aggregated positions. Systemic risk is estimated through the minimum amount of capital that has to be injected in the system (before or after aggregation) in order to make the aggregated position acceptable. Hence, we deal with systemic risk measures of both "first allocate, then aggregate" and "first aggregate, then allocate" type. In both cases, we provide a detailed analysis of the corresponding systemic acceptance sets and their support functions. Our general results cover some specific cases already studied in the literature. The same approach delivers a simple and self-contained proof of the dual representation of utility-based risk measures for univariate positions.

"L'importante nella vita non è fare qualcosa, ma nascere e lasciarsi amare."

Chiara Corbella Petrillo

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CONTENTS

To Richi

INTRODUCTION

This thesis in (one of) the fruit(s) of a two years research activity played between Milan and Zurich, thanks to the support of the PhD program in Statistics and Mathematical Finance of the University of Milano Bicocca and the collaboration with the Center for Finance and Insurance of the University of Zürich. The content of this thesis is part of a project conducted jointly with my supervisor, Cosimo Munari, and partially with Pablo Koch-Medina.

In the next few pages, the reader will be provided with a general overview of the framework and the problems motivating this thesis. We are going to present some standard concepts from Mathematical Finance from a point of view that shall facilitate the reading of the thesis. We do not mean to be neither exhaustive nor technical. In fact, we try to keep the language and the model as simple as possible so as to facilitate the understanding of the key ideas. The attempt to capture the essential structure behind standard topics in Mathematical Finance such as Fundamental Theorems and Risk Measures has guided us during the entire project. The focus on uniperiodal models helps in this direction, as one does not need to deal with technicalities that come out in discrete multiperiodal or continuous time settings.

In the last part of this introduction we summarize the contents chapter by chapter. We postpone to the introductions of the single chapters the related references and the comparison with the literature.

FRICTIONLESS ARBITRAGE THEORY

Consider two fixed time instants: we refer to the first one as *initial time*, and to the second one as *maturity*. Assume that there exist N pre-fixed types of contracts that can be exchanged by a subject (*agent*) at initial time. For every $i = 1, \dots, N$, contract i consists of paying $p_i \in \mathbb{R}$ units of a fixed unit of account (e.g. a currency) at initial time, and receiving the payoff S_i at maturity. We call these N available contracts *basic securities* (or simply *securities*) and we refer to the place where the exchange of the securities happens as the *market*. The real value p_i is called the *initial price* (or simply *price*) of the ith security. Typically the payoffs S_i are modeled through random variables on a probability space that describes uncertainty at maturity. For the sake of generality, we fix a topological vector space $\mathcal{X}$ that plays the role of the space of every conceivable payoff and we assume that S_i's are elements of $\mathcal{X}$. Moreover, we assume that $\mathcal{X}$ is equipped with a compatible order $\geq$ such that the positive con $\mathcal{X}_+$ is closed, and that every rational agent prefers the elements of the positive cone $\mathcal{X}_+$ to 0.

We assume that an agent who acts in this market is allowed to buy any quantity of any basic security and that the price of $\lambda_i \in \mathbb{R}$ units of basic security i is λ_i times the unitary price, i.e. $\lambda_i p_i$. Moreover, we assume that he is allowed to buy every combination of the securities. Such combinations are called *portfolios* and are described through vectors $(\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$, where λ_i represents the number of units of the ith security. The price for buying a portfolio is assumed to be the combination of the prices of the acquired securities. To be precise, we assign to every portfolio $\lambda \in \mathbb{R}^N$ the initial price

$$V_0(\lambda) = \sum_{i=1}^{N} \lambda_i p_i.$$

We do not distinguish between buying and selling prices, as we assume that there is no bid-ask spread. The total payoff received at maturity by the agent who has bought the portfolio λ at initial time is the following:

$$V_1(\lambda) = \sum_{i=1}^{N} \lambda_i S_i.$$

Every rational agent is attracted by particular portfolios known as *arbitrage opportunities* (or simply *arbitrages*), which can be bought at nonpositive price and have final nonzero payoff which is in $\mathcal{X}_+$.[1] We read in Föllmer and Schied [49]:

> The existence of such an arbitrage opportunity may be regarded as a market inefficiency in the sense that certain assets are not priced in a reasonable way. In real-world markets, arbitrage opportunities are rather hard to find. If such an opportunity would show up, it would generate a large demand, prices would adjust, and the opportunity would disappear.

For this reason, most of the theory of financial markets has been developed under the assumption that there are no arbitrage opportunities. In this case, we say that the *No Arbitrage condition* holds, which means that:

$$\lambda \in \mathbb{R}^N, \;\; V_0(\lambda) \leq 0, \;\; V_1(\lambda) \in \mathcal{X}_+ \quad \Longrightarrow \quad V_1(\lambda) = 0. \tag{NoA}$$

From now on, we assume that (NoA) holds, and that $S_1 \in \mathcal{X}_+ \setminus \{0\}$. This automatically implies $p_1 > 0$. Under these assumptions, whenever two portfolios have the same payoff, their prices must coincide, for otherwise it would be possible to buy the cheaper, sell the more expensive and buy a positive number of units of S_1 so as to create an arbitrage. We say that the *law of one price holds*, i.e.

$$\lambda,\, \mu \in \mathbb{R}^N, \quad V_1(\lambda) = V_1(\mu) \quad \Longrightarrow \quad V_0(\lambda) = V_0(\mu). \tag{L1P}$$

The validity of (L1P) allows to get rid of portfolios and functionals V_0 and V_1, and to work at the level of the payoffs. This is what we do in what follows (and in Chapter 1).

We define the finite dimensional vector subspace of $\mathcal{X}$ of *attainable payoffs*:

$$\mathcal{S} := \{V_1(\lambda) \,:\, \lambda \in \mathbb{R}^N\} = \operatorname{span}(S_1, \ldots, S_N).$$

Thanks to (L1P), the following linear functional is well defined for every $Z \in \mathcal{S}$:

$$\pi(Z) := V_0(\lambda) \;\text{ for } \lambda \in \mathbb{R}^N \text{ such that } Z = V_1(\lambda).$$

For $Z \in \mathcal{S}$, $\pi(Z)$ represents the price (expressed in units of the fixed unit of account) that the agent has to pay at initial time so as to buy a portfolio whose payoff at maturity is Z. Note that (NoA) holds if and only if π is strictly positive on $\mathcal{S}$, in the sense that π takes strictly positive values on every nonzero $Z \in \mathcal{S} \cap \mathcal{X}_+$. But the characterization of (NoA) we are mostly interested in is the following:

$$\text{(NoA)} \quad \Longleftrightarrow \quad \mathcal{X}_+ \cap \big(\ker(\pi) - \mathcal{X}_+\big) = \{0\}.$$

As the reader will appreciate, the difference $\ker(\pi) - \mathcal{X}_+$ is the real subject of our discussion. It coincides with the set of all payoffs X such that there exists an attainable payoff $Z \in \mathcal{S}$ with initial price 0 and such that Z *superreplicates* X (we say that a payoff X *superreplicates* a payoff Y if $X \geq Y$). This means that by paying 0 at initial time, and possibly throwing away something, we obtain the payoff X. The (NoA) condition states that this remaining X cannot be positive, unless it is equal to 0. Since $\ker(\pi) - \mathcal{X}_+$ is a convex cone, this set is usually called the *cone of superreplicable payoffs at zero cost*. Note that by positivity of π, it coincides with $\{Z \in \mathcal{S} \,:\, \pi(Z) \leq 0\} - \mathcal{X}_+$. From a topological point of view, a straightforward application of the Dieudonné Theorem B.1.9 shows that the difference of $\ker(\pi)$ and $\mathcal{X}_+$ is closed (see Dieudonné [39] for the original result). Both of them are indeed convex closed cones, $\ker(\pi)$ is finite dimensional and their intersection is $\{0\}$ due to (NoA).[2] Closedness of $\ker(\pi) - \mathcal{X}_+$ has two important consequences that we are going to discuss: one regarding the regularity of the so called *superreplication price*, the other regarding strictly positive extensions of π to the payoff space $\mathcal{X}$.

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[1] In Chapter 1, the term *arbitrage opportunity* indicates the payoff $V_1(\lambda)$ of a portfolio λ such that $V_0(\lambda) \leq 0$ and $V_1(\lambda) \in \mathcal{X}_+ \setminus \{0\}$, instead of the portfolio itself. This is not misleading since, due to the model specification of that chapter, there is a bijection between portfolios and payoffs.

[2] In multiperiodal or continuous time models, the dimension of $\mathcal{S}$ is generally not finite, since there is a random component not only in the basic securities' payoffs but also in the portfolios. In these cases, (NoA) has been often replaced by stronger assumptions that fall under the name of "no free lunch", and that consist in assuming that some type of closure of $\ker(\pi) - \mathcal{X}_+$ has $\{0\}$-intersection with $\mathcal{X}_+$. This is the approach used e.g. in Clark [33] and Kreps [69]. The first assumes that $\operatorname{cl}\big(\ker(\pi) - \mathcal{X}_+\big) \cap \mathcal{X}_+ = \{0\}$, while the second requires that one cannot find $X \in \mathcal{X}$ and nets $(X_\alpha) \subset \mathcal{X}$ and $(Z_\alpha) \subset \mathcal{S}$ such that $X_\alpha \to X$, $X_\alpha \leq Z_\alpha$ and $\liminf_\alpha \pi(Z_\alpha) \leq 0$.

For a payoff $X \in \mathcal{X}$, the superreplication price is defined as the infimum of the prices of portfolios whose payoff superreplicate X:

$$\pi^+(X) := \inf\{V_0(\lambda) \,:\, \lambda \in \mathbb{R}^N,\ V_1(\lambda) \geq X\}. \tag{1}$$

It is typically used in hedging problems: suppose an agent must deliver at maturity a payoff X and wants to set up a trading strategy so as to be covered against potential losses. The value $\pi^+(X)$ represents the minimum amount of the fixed unit of account that he has to invest at initial time in the market in order to have a final payoff that superreplicates the claim that he must deliver. Actually, it turns out that the superreplication price plays a key role also in pricing problems. The reader will find a detailed discussion about this in Chapter 1.

Since we have assumed that (NoA) holds, the superreplication price of $X \in \mathcal{X}$ can equivalently be expressed as

$$\pi^+(X) = \inf\{\pi(Z) \,:\, Z \in \mathcal{S},\ Z \geq X\}.$$

Now, the linearity of our model allows to concentrate the pricy part of every portfolio on the first basic security. Indeed $\mathcal{S} = \ker(\pi) + \mathbb{R}S_1$, and $\pi(Z + mS_1) = mp_1$ for every $Z \in \ker(\pi)$ and $m \in \mathbb{R}$. Hence, for every $X \in \mathcal{X}$,

$$\pi^+(X) = \inf\left\{m \in \mathbb{R} \,:\, m\frac{S_1}{p_1} - X \in -\ker(\pi) + \mathcal{X}_+\right\}.$$

This characterization brings to light that closedness of $\ker(\pi) - \mathcal{X}_+$ implies that, whenever $\pi^+(X)$ is real valued, the infimum in the definition (and in the characterizations) of $\pi^+(X)$ is attained and one can then identify the best portfolio(s) superreplicating X. Moreover, it follows that sublevels of π^+ are closed sets, showing that π^+ is lower semicontinuous, a feature that may turn out to be useful in applications, where one generally deals with approximations rather than exact payoffs.

In order to investigate the other consequence of closedness of $\ker(\pi) - \mathcal{X}_+$, we need to introduce duality. From now on, assume that $\mathcal{X}$ is locally convex, and let $\mathcal{X}'$ be the topological dual of $\mathcal{X}$. It is well known (see Proposition A.1.10) that closed convex cones admit a representation as intersections of half spaces defined through functionals that are nonnegative on the cone itself. Let us apply then this representation to $-(\ker(\pi) - \mathcal{X}_+) = \ker(\pi) + \mathcal{X}_+$. Note that the set of nonnegative functionals on this set consists of positive functionals that extend π:

$$\ker(\pi) + \mathcal{X}_+ = \bigcap_{\psi \in \mathcal{X}'_+,\ \psi|_{\mathcal{S}} = \pi} \{X \in \mathcal{X} \,:\, \psi(X) \geq 0\}. \tag{2}$$

The essence of the Fundamental Theorem of Asset Pricing is to find among the functionals appearing in (2), a strictly positive functional (i.e. $\psi(X) > 0$ whenever $X \in \mathcal{X}_+ \setminus \{0\}$). Such a functional is a good candidate for pricing payoffs belonging or not to $\mathcal{S}$: if $X \in \mathcal{S}$, then the new price $\psi(X)$ coincides with the market price $\pi(X)$; if $X \notin \mathcal{S}$, the price $\psi(X)$ is typically considered reasonable in the sense that the (NoA) condition keeps holding in the extended market where X is added to attainable payoffs in the following way: $\mathcal{S}' = \mathcal{S} + \mathbb{R}X$, $\pi'(Z + mX) = \pi(Z) + m\psi(X)$ for every $Z \in \mathcal{S}$ and $m \in \mathbb{R}$. We refer the reader to Chapter 1 for a detailed discussion on the problem of pricing non attainable payoffs in a way that is *consistent* with the market, and for a different justification of why and how these functionals could work as pricing rules.

The existence of a strictly positive functional that extends π is obtained through a separation process. Due to (NoA), the cone of superreplicable claims intersects the positive cone only in 0. Using standard Hahn-Banach separation, one finds for every nonzero $X \in \mathcal{X}_+$ a functional ψ_X that extends π and is strictly positive on X. Then the point is how to build a unique extension of π that is strictly positive on $\mathcal{X}_+ \setminus \{0\}$. This was first done by Yan [94] in spaces of random variables with an *ad hoc* technique for probability spaces, and by Kreps [69] in general topological vector spaces satisfying suitable separability properties. In both cases, the author constructs the desired functional as an infinite convex combination of a proper countable subfamily of $\{\psi_X\}_{X \in \mathcal{X}_+ \setminus \{0\}}$. Again, we refer to Chapter 1 for more on this topic.

ARBITRAGE THEORY IN THE PRESENCE OF FRICTIONS

We introduce this section using the words of Pennanen [79]:

> The market model considered so far describes perfectly liquid markets where the unit price of a security does not depend on whether we are buying or selling nor on the quantity of the traded amount. In reality, different unit prices are associated with buying and selling and, moreover, as the traded quantities increase, the prices start to move against us. This is often referred to as illiquidity.

Like in the cited paper, after having described a perfectly liquid market, we make the model more realistic by introducing frictions like (not necessarily proportional) transaction costs, illiquidity and portfolio constraints. The general framework is like before: a one period market where N basic securities are traded. The substantial difference in the model is that we drop linearity. For instance the cost of buying n units of basic security i could exceed n times the cost of buying one unit, since the number of units of security i at the lowest price is finite, and «when buying more, one gets the second lowest price and so on» ([79]). Moreover, the price of a combination of securities or portfolios could be strictly less than the combination of the prices since purchases and sales could compensate and this would reduce transaction costs. These facts justify the choice of a convex function $V_0 : \mathbb{R}^N \to \mathbb{R}$ to model portfolios prices at initial time. Moreover, we assume that $V_0(0) = 0$. Since buying $-\lambda$ coincides with selling λ, it comes that the selling price of the portfolio λ is $-V_0(-\lambda)$, which needs not coincide with $V_0(\lambda)$.

We assume that trading occurs at maturity too, and the agent in possession of a portfolio is forced to liquidate it. The gain from this liquidation is described through an element of the abstract payoff space $\mathcal{X}$. The function $V_1 : \mathbb{R}^N \to \mathcal{X}$ assigns to each portfolio its liquidation value. We assume that $V_1(0) = 0$ and V_1 is concave for the same reasons for which V_0 is convex. The case where there is no trading at maturity and each security delivers its terminal contractual payoff is covered by taking V_1 linear.

Finally, we take into account the possibility that the agent could have no access to every combination of the basic securities due to constraints such as borrowing and short selling restrictions. To model this fact, we assume that he is allowed to buy only portfolios in a subset $\mathcal{P} \subset \mathbb{R}^N$. It makes sense to assume that $\mathcal{P}$ is convex as the agent can decide to buy combinations of admissible portfolios. Moreover we assume that the zero portfolio belongs to $\mathcal{P}$.

Arbitrages, and the (NoA) condition, are defined like in the case of a perfectly liquid market. Simple examples show that in this context, the absence of arbitrages is not a sufficient condition for (L1P) to hold, as the potential gain of buying and selling two portfolios with the same liquidation value and different initial price could vanish due to transaction costs. See for instance Example 2.1.4. One could assume in principle that the law of one price holds, and get rid of portfolios in favor of prices defined directly on liquidation values. This is the case in the model of Chapter 1, where (L1P) holds because V_1 is linear and injective.

For our analysis, it is convenient to define the set of attainable payoffs with maximum cost $m \in \mathbb{R}$:

$$\mathcal{V}_m := \{V_1(\lambda) \,:\, \lambda \in \mathcal{P},\ V_0(\lambda) \leq m\}.$$

Like we have done for a perfectly liquid market, we characterize (NoA) as follows:

$$\text{(NoA)} \quad \Longleftrightarrow \quad \mathcal{X}_+ \cap \big(\mathcal{V}_0 - \mathcal{X}_+\big) = \{0\}.$$

The difference $\mathcal{V}_0 - \mathcal{X}_+$ is the set of superreplicable claims at zero (or lower) cost. It is a convex set that contains 0. In case of conic constraints (i.e. $\mathcal{P}$ conic) and proportional transaction costs (i.e. V_0 and V_1 positively homogeneous), it is also conic. In the spirit of the linear case, one could address the problem of closedness of this set but, due to the lack of linearity things are more complicated than for linear markets. One way to prove the desired closedness is to use some Dieudonné Theorem's extension under suitable assumptions on the market. This is what we do in Theorem 2.3.13. Otherwise one has to try with a more ad-hoc approach and guess sufficient conditions so that accumulation points belong to $\mathcal{V}_0 - \mathcal{X}_+$. One of the goals of this thesis is to identify these assumptions.

Now, suppose we are lucky enough to have that $\mathcal{V}_m - \mathcal{X}_+$ is closed for every $m \in \mathbb{R}$. Do we have the same benefits from this closedness as in liquid markets (i.e. regularity of the superreplication price and existence of a strictly positive functional that "extends" the market)? Concerning lower semicontinuity of π^+ (defined in equation (1)) the answer is a priori negative. Indeed, since

$$\pi^+(X) = \inf\{m \in \mathbb{R} \,:\, X \in \mathcal{V}_m - \mathcal{X}_+\},$$

without further assumptions ensuring a continuous dependence of the sets $\mathcal{V}_m - \mathcal{X}_+$ on m, nothing can be said about π^+.

Concerning the second question, for ease of exposition, in this introduction we treat the dual representation of $\mathcal{V}_0 - \mathcal{X}_+$ only in case $\mathcal{P}$ is a cone and V_0 and V_1 are positively homogeneous and hence $\mathcal{V}_0 - \mathcal{X}_+$ is a convex cone. It follows from Proposition A.1.10 that

$$\mathcal{X}_+ - \mathcal{V}_0 = \bigcap_{\psi \in \mathcal{D}^*} \{X \in \mathcal{X} \,:\, \psi(X) \geq 0\},$$

where

$$\begin{aligned} \mathcal{D}^* &:= \left\{ \psi \in \mathcal{X}'_+ \,:\, \sup_{\lambda \in \mathcal{P},\, V_0(\lambda) \leq 0} \psi(V_1(\lambda)) = 0 \right\} \\ &= \{\psi \in \mathcal{X}'_+ \,:\, \psi(V_1(\lambda)) \leq 0 \quad \forall \lambda \in \mathcal{P} \text{ such that } V_0(\lambda) \leq 0\}. \end{aligned}$$

Functionals in the dual set of $\mathcal{X}_+ - \mathcal{V}_0$ are those and only those that assign nonpositive price to the liquidation value of portfolios with nonpositive initial price.[3] Although this property establishes a compatibility among the market and the functional, we think that it is not sufficient to interpret functionals in $\mathcal{D}^*$ as pricing rules that extend π from $\mathcal{S}$ to $\mathcal{X}$. We expect that such a functional satisfies $\psi(V_1(\lambda)) \leq V_0(\lambda)$ for every $\lambda \in \mathcal{P}$ so that new prices are compatible with the bid ask spread of the market. It turns out that

$$\mathcal{D} := \{\psi \in \mathcal{X}'_+ \,:\, \psi(V_1(\lambda)) \leq V_0(\lambda) \;\; \forall \lambda \in \mathcal{P}\} \subset \mathcal{D}^*, \tag{3}$$

but in general the inclusion can be strict (simple examples can be constructed).

It is not hard to see that $\mathcal{D}$ coincides with $\mathcal{D}^*$ in the presence of a traded security, let us say the first basic security, unconstrained and perfectly liquid. In our model, this would mean that $\mathcal{P} = \mathcal{P} + \mathbb{R}\mathbf{e}_1$ and $V_0(\lambda + m\mathbf{e}_1) = V_0(\lambda) + mp_1$ and $V_1(\lambda + m\mathbf{e}_1) = V_1(\lambda) + mS_1$ for every $\lambda \in \mathcal{P}$ and $m \in \mathbb{R}$, where $p_1 > 0$ and $S_1 \in \mathcal{X}_+ \setminus \{0\}$. Note that in this case we would have

$$\mathcal{V}_m - \mathcal{X}_+ = \frac{m}{p_1} S_1 + \mathcal{V}_0 - \mathcal{X}_+,$$

hence the closedness of the set of superreplicable claims at zero cost would imply that π^+ is lower semicontinuous and that the infimum in the definition of $\pi^+(X)$ is actually a minimum, when it is real valued.

Like pointed out in Kabanov [65], Kabanov et al. [64] and Schachermayer [90], there are models where such a linear direction does not necessarily exist. In particular, those papers consider multicurrency markets or markets with physical delivery that, due to their intrinsically multivariate nature, are modeled by way of *solvency cones*, which coincide with the negative of the set of portfolios freely available on the market, and vector-valued portfolios. They assume proportional transaction costs, a generalization of their model towards convex markets has been done in Lepinette and Molchanov [71]. We follow instead the approach of Pennanen (see e.g. [76, 78, 79, 81]) based on direct modeling of pricing functionals defined on "single valued" portfolios, which arises as the natural generalization of the frictionless case. Actually, we take from this author the

[3] Le us make a brief parallelism with pricing systems as they have been defined in that part of the literature developed around the concept of solvency cones. Consider e.g. Schachermayer [90], where the author describes a market with proportional transaction costs and physical delivery and does not assume the existence of a reference asset like cash into which the other assets can be liquidated. The set that he calls $-\widehat{K}(\Pi)$ consists of portfolios available at price zero, and its dual cone

$$\widehat{K}^*(\Pi) := \{w \in \mathbb{R}^N \,:\, \langle w, \lambda \rangle \leq 0 \quad \forall \lambda \in -\widehat{K}(\Pi)\}$$

is defined as the set of *consistent price systems*. Note that if we identify $-\widehat{K}^*(\Pi)$ with the set of portfolios available at zero cost in our model, that is $\{\lambda \in \mathcal{P} \,:\, V_0(\lambda) \leq 0\}$, then for every $\psi \in \mathcal{D}^*$, assumed that V_1 is linear and $\mathcal{P} = \mathbb{R}^N$, we have that $\psi \circ V_1 : \mathbb{R}^N \to \mathbb{R}$ is a consistent pricing system in the sense of [90].

key idea of focusing on the set of *superreplicable processes* instead of superreplicable payoffs. Instead of working with $\mathcal{V}_0 - \mathcal{X}_+$, we consider the following subset of the product space $\mathcal{X} \times \mathbb{R}$:

$$\mathcal{C} := \{(X, m) \in \mathcal{X} \times \mathbb{R} \,:\, \text{exists } \lambda \in \mathcal{P} \text{ such that } V_0(\lambda) \leq -m,\ V_1(\lambda) \geq X\}.$$

The fact that we have added one linear direction to the payoff space $\mathcal{X}$ will allow to compare directly the real values $\psi(V_1(\lambda))$ and $V_0(\lambda)$ and hence to find the desired pricing functionals in $\mathcal{D}$. The set $\mathcal{C}$ is convex and in case of proportional transaction costs and portfolio constraints it is also conic. Again, it is possible to characterize the absence of arbitrages in terms of $\mathcal{C}$:

$$\text{(NoA)} \quad \Longleftrightarrow \quad \mathcal{C} \cap (\mathcal{X}_+ \times \mathbb{R}_+) = \{0\}.$$

Since for every $X \in \mathcal{X}$ the superreplication price admits the following representation

$$\pi^+(X) = \inf\{m \in \mathbb{R} \,:\, (X, -m) \in \mathcal{C}\},$$

it is easy to see that whenever $\mathcal{C}$ is closed, then π^+ is lower semicontinuous and the infimum in (1) is attained. Moreover, the closedness of $\mathcal{C}$ allows to obtain useful dual representations. As above, for clarity of exposition, we again focus the discussion about duality on the conic case. For a treatment of the genuinely convex case, we refer the reader to Chapters 1 and 2. So, let $\mathcal{P}$ be conic, and V_0 and V_1 be positively homogeneous. Assuming that $\mathcal{C}$ is closed, its dual representation would be an intersection over functionals in the dual product space $\mathcal{X}' \times \mathbb{R}$. Moreover, we assume that it is possible to reduce these functionals to those whose real component is equal to 1 (see Chapter 1 for details). It turns out that

$$\mathcal{C} = \bigcap_{\psi \in \mathcal{D}} \{(X, m) \in \mathcal{X} \times \mathbb{R} \,:\, \psi(X) - m \geq 0\}$$

where $\mathcal{D}$ is like in (3). Hence the functionals appearing in the representation of $\mathcal{C}$ are exactly those that extend the market in a way that is compatible with the bid-ask spread as we have required. Like in the case of perfectly liquid markets, if $\mathcal{X}$ is a suitable space of random variables or has suitable separability properties, then one could find a strictly positive functional in $\mathcal{D}$.

Hence, it would be interesting to find sufficient conditions for $\mathcal{C}$ to be closed. This is what we do in Chapters 1 and 2. We conclude this section noting that if the market is perfectly liquid, $\mathcal{C}$ is closed if and only if $\ker(\pi) - \mathcal{X}_+$ is so, showing that the "$\mathcal{C}$-approach" for illiquid markets actually generalizes the standard approach for perfectly liquid markets.

ACCEPTABLE DEALS THEORY

So far, we have described perfectly liquid markets and then we have generalized the model in order to include the more realistic case of illiquid markets. Now, we move towards a different generalization, which is not about the market (that for this section may be assumed to be with or without frictions) but rather regards the concept of superreplication. We use the words of Pennanen [81] to introduce the problem:

> In reality, one rarely looks for superhedging strategies when trading in practice. Instead, one (more or less quantitatively) sets bounds on acceptable levels of "risk" when taking positions in the market and when quoting prices.

Similarly, Cherny [32] says:

> When a trader sells a contract, he/she would charge for it a price, with which he/she would be able to superreplicate the contract. In theory the superreplication is typically understood almost surely, but in practice an agent looks for an offsetting position such that the risk of his/her overall portfolio would stay within the limits prescribed by his/her management (the almost sure superreplication is virtually impossible in practice).

The idea is to relax the superreplication condition by imposing suitable constraints on acceptable replication errors. This is achieved by fixing a subset $\mathcal{A} \subset \mathcal{X}$ called *acceptance set*, which has the property that X is preferred to Y if $X - Y \in \mathcal{A}$. In this sense $\mathcal{A}$ may be interpreted as a set of admissible superreplication errors. Of course, it makes sense that the positive cone is a subset of $\mathcal{A}$, and $\mathcal{A}$ could be required to be convex in line with the diversification principle.

Now, one could come up with a natural question: who decides the preference criterion? That is, who decides $\mathcal{A}$? In contrast to classical arbitrage theory, where it is common agreement that if $X - Y \in \mathcal{X}_+$ then X is preferred over Y, here different agents may come up with different ways to define preferences and acceptable errors. Thus it is the agent himself who models the acceptance set $\mathcal{A}$ on his subjective preferences.

Now, assume that $\mathcal{P}$, V_0 and V_1 are given like above, and an acceptance set $\mathcal{A}$ has been fixed. The ideas that we have developed with the positive cone as acceptance set still hold. In particular, arbitrages are replaced by *acceptable deal opportunities* (or *acceptable deals*), i.e. portfolios $\lambda \in \mathcal{P}$ such that $V_0(\lambda) \leq 0$ and $V_1(\lambda) \in \mathcal{A} \setminus \{0\}$[4] , the convex set corresponding to $\mathcal{C}$ is defined as follows

$$\mathcal{C}_{\mathcal{A}} := \{(X, m) \in \mathcal{X} \times \mathbb{R} \,:\, \text{exists } \lambda \in \mathcal{P} \text{ such that } V_0(\lambda) \leq -m,\ V_1(\lambda) - X \in \mathcal{A}\},$$

and the new superreplication price of $X \in \mathcal{X}$ is:

$$\pi^+_{\mathcal{A}}(X) := \inf\{V_0(\lambda) \,:\, \lambda \in \mathcal{P},\ V_1(\lambda) - X \in \mathcal{A}\} = \inf\{m \in \mathbb{R} \,:\, (X, -m) \in \mathcal{C}_{\mathcal{A}}\}.$$

In Chapter 1, we show that in the absence of suitable acceptable deal opportunities, the set $\mathcal{C}_{\mathcal{A}}$ is closed, this fact implying that $\pi^+_{\mathcal{A}}$ is lower semicontinuous, that the infimum in its definition is actually a minimum, and that $\mathcal{C}_{\mathcal{A}}$ admits a representation over functionals that extend the market in a consistent way (i.e. in $\mathcal{D}$ for the positively homogeneous case). Moreover, like for the positive cone, following the approach of [69], it is possible to determine conditions on the space $\mathcal{X}$ and on $\mathcal{A}$ such that there exists a pricing functional that takes strictly positive values on every nonzero element of $\mathcal{A}$.

Risk measures

The model just described is also suitable for applications to financial regulation. Consider a uniperiodal market modeled as above through $\mathcal{P}$, V_0 and V_1. As said above, according to the properties of these primary elements, the market will have frictions or not. Assume that an acceptance set $\mathcal{A} \subset \mathcal{X}$ is fixed, but let us change the interpretation of $\mathcal{A}$ with respect to the previous section. Let $X \in \mathcal{X}$ describes the capital position of an agent (or an institution). For example, it may correspond to the net asset value of an institution, or the P&L profile of a payoff. Suppose that a financial regulator wants to evaluate the goodness of X. The regulator fixes the set $\mathcal{A}$ of capital positions that he deems acceptable, and compares the agent's profile with them. In particular, he states the problem in capital adequacy terms: if the position is not acceptable, is it possible to make it acceptable through an appropriate management action? If yes, which is the minimum cost for doing so? In Artzner et al. [9], the authors say:

> For an unacceptable risk (i.e., a position with an unacceptable future net worth), one remedy may be to alter the position. Another remedy is to look for some commonly accepted instruments that, when added to the current position, make its future value acceptable to the regulator/supervisor.

In order to implement this remedy using the basic securities of our model as *commonly accepted instruments*, one naturally comes up with the following functional:

$$\rho(X) := \inf\{V_0(\lambda) \,:\, \lambda \in \mathcal{P},\quad X + V_1(\lambda) \in \mathcal{A}\}, \tag{4}$$

which is nothing else than the generalization of the standard risk measures axiomatically defined right in [9]. Note that, up to a sign, also $\pi^+_{\mathcal{A}}$ falls in the large class of functionals of this type. For an agent with capital position X at maturity, $\rho(X)$ represents then the minimum amount of the fixed unit of account that he has to invest in the market at initial time so that his final position at maturity after having liquidated his portfolio is acceptable. As said in [9], this value could be used as a measure of the riskiness of X:

[4] Like for arbitrages, in Chapter 1 acceptable deals are defined as the payoffs $V_1(\lambda)$ of portfolios λ such that $V_0(\lambda) \leq 0$ and $V_1(\lambda) \in \mathcal{A} \setminus \{0\}$. Thanks to the bijection between portfolios and payoffs of that chapter's model, acceptable deals as portfolios correspond to acceptable deals as payoffs. In the model of Chapter 2, instead, such a bijection does not hold in general, but the focus there is on the absence of acceptable deals rahter than on their definition, and absence of acceptable deals as defined in Chapter 2 is equivalent to the absence of acceptable deals as portfolios.

> The current cost of getting enough of this or these instrument(s) is a good candidate for a measure of risk of the initially unacceptable position.

Since the cited seminal paper of 1999, the risk measures related literature has been developed a lot (for an overview, see the introductions of Chapters 2 and 3). Although generalizations in different directions of the cash-additive single asset risk measures have been studied, to the best of our knowledge the problem has never been treated in such a generality, i.e. allowing for general (non necessary proportional) transaction costs both at initial time and at maturity, portfolio constraints and general acceptance sets. For this reason, Chapter 2 tackles a systematic study of functionals of type (4).

Risk measures have also been generalized so as to obtain functionals suitable for measuring the risk not only of a single agent or institution, but of a connected system of individuals. To this end, an aggregation function appears in defining the so called *systemic risk measures*, which has the role of modeling interactions among the elements of the system. We study these type of risk measures in Chapter 3.

Unfortunately, in applications one does not really know the real capital position of the institution (or of the system), but can only estimate approximations. This fact clarifies the importance of proving continuity properties of functionals of type (4). In particular, lower semicontinuity plays a key role in linearization of convex problems, making them more tractable for applications. It is known indeed that, at least in the presence of (quasi) convexity, lower semicontinuous functions admits a dual representation as supremum of affine functions. A considerable part of this thesis is devoted to study dual representation of different types of risk measures and to provide conditions ensuring the lower semicontinuity of functionals of type (4) (such as the closedness of the set $\mathcal{C}_{\mathcal{A}}$, as explained above).

STRUCTURE OF THE THESIS

The rest of this thesis is divided in three chapters. Each of them can be read independently of each other and regardless of this introduction. This is why the reader will find some overlap between the content of the last few pages and the introductions or the model descriptions of the following chapters. There are only few references among different chapters which help compare the results. A common appendix at the end collects the necessary mathematical background, and the List of Symbols on page 117 contains the used notations.

In Chapter 1, we work in a reference space a random variables and we consider a uniperiodal market model with convex transaction costs at initial time, convex constraints and convex acceptance set that reflects the preferences of an agent who acts as a buyer in the market. As we assume that the law of one price holds, we abandon the modeling through portfolios and we conduct our study on payoffs. The focus of this chapter is the problem of assigning prices to financial contracts. To this end, we introduce a definition of *market consistent prices*, where consistent is meant with respect to the market and the preferences of the buyer. In order to provide a dual representation of these prices, we develop an extension of the Fundamental Theorem of Asset Pricing in a context where arbitrages are replaced by acceptable deals. The chapter is based on the submitted paper Arduca and Munari [6].

In Chapter 2 the reference model space is abstract. We consider a uniperiodal market with transaction costs both at initial time and at maturity, and portfolio constraints. We do not assume convexity a priori, tough some results hold only under convexity assumptions. The acceptance set too is not required to be convex in general, and it is interpreted as the set of positions that are deemed acceptable by an external regulator. We define capital adequacy rules like in (4), and the objective of the entire chapter is to study their properties. As the law of one price needs not hold, we work all along the chapter at the level of portfolios instead of payoffs like in Chapter 1. In particular, we establish conditions on the portfolios ensuring the the functional is lower semicontinuous, and we compare these conditions with no-acceptable deal type assumptions. In the convex and quasi convex case, we also provide a dual representation of the functionals of interest. This chapter collects the results achieved jointly with Munari in an ongoing project.

Chapter 3 is devoted to the dual representation of systemic risk measures defined on product spaces of random variables. We consider a uniperiodal model and a finite number of institutions that are connected to each other. We model interactions through an aggregation function, and we assume that an external regulator fixes a set of acceptable aggregated positions. Systemic risk is estimated as the minimum amount of money that has to be injected in the system (before or after aggregation) in order to make the aggregated position acceptable. The goal of the chapter is to develop a unifying approach to obtain dual representations of systemic risk measures, which is not related to a particular choice of the acceptance set or the aggregation function. Our general results cover some specific cases already studied in literature. This chapter is based on Arduca et al. [5].

CHAPTER 1

MARKET-CONSISTENT PRICING WITH ACCEPTABLE RISK

The goal of every pricing theory in finance is to address the basic question: *Which reasonable price(s) can be assigned to payoffs of financial contracts?*

The classical framework of arbitrage pricing theory tackles the pricing problem starting from the fundamental notion of arbitrage-free prices. Since the pioneering contributions of Black and Scholes [23], Merton [72], Cox and Ross [35] and Harrison and Kreps [59], this framework has successfully been extended in several directions. A prominent line of research has worked to the construction of what may be broadly called a general theory of "subjective pricing". This has been achieved by investigating the pricing problem under suitable relaxations of the classical notion of an arbitrage opportunity. A key contribution in this direction is the theory of "good deal pricing". To recall the main ideas behind it, we start by providing a brief overview of arbitrage pricing theory in a static setting.

ARBITRAGE PRICING

Consider a one-period financial market where a finite number of securities are traded at the initial date and deliver their payoff at the terminal date. For ease of exposition, we assume that the market is frictionless in the sense that there are neither transaction costs nor portfolio constraints. We denote by $\mathcal{L}$ a vector space of random variables over a fixed probability triple $(\Omega, \mathcal{F}, \mathbb{P})$. The elements of $\mathcal{L}$ represent the payoffs of relevant financial contracts at the terminal date. The set of positive payoffs is denoted by $\mathcal{L}_+$. For definiteness, we assume that $\mathcal{L}$ is the space of all integrable random variables. The set of payoffs that can be fully replicated by trading in the market is a vector space $\mathcal{M} \subset \mathcal{L}$. (This assumption requires the payoffs of the basic traded securities to be integrable. Should this fail, we can always ensure integrability after an appropriate change of probability measure. Moreover, the new probability measure can always be taken to have bounded Radon-Nikodym derivative with respect to the original one). We assume that to each replicable payoff we can assign a certain price at the initial date through a linear functional $\pi : \mathcal{M} \to \mathbb{R}$. For every replicable payoff $X \in \mathcal{M}$ the quantity $\pi(X)$ can be interpreted as its replication cost. The interesting and more realistic situation is when the market is incomplete in the sense that there exist payoffs that cannot be fully replicated by trading in the market.

The classical arbitrage pricing theory provides an answer to the initial motivating pricing problem under the assumption that arbitrage opportunities cannot be encountered in the market. An *arbitrage opportunity* is any replicable payoff that is positive (but not null) and can be acquired without cost. The absence of arbitrage opportunities is clearly equivalent to the strict positivity of the pricing functional. This condition is typically justified from an economical perspective by arguing that, should an arbitrage opportunity exist, there would be infinite demand for it so that its price would increase until the opportunity would eventually vanish. In an arbitrage-free market, the range of reasonable prices for a payoff $X \in \mathcal{L}$ is usually assumed to consist of the so-called arbitrage-free prices. A candidate price $p \in \mathbb{R}$ is said to be an *arbitrage-free price* for X if the linear extension of π to the enlarged marketed space $\mathcal{M} + \mathbb{R}X$ obtained by assigning to X the value p is strictly positive. The formalization of this notion goes back to Harrison and Kreps [59]. The usual interpretation is that p is arbitrage free for X if the arbitrage-free market for the basic traded securities can be extended in a frictionless way by adding the new security with unitary price p and unitary payoff X without creating arbitrage opportunities. This interpretation is, however, at odds with the fact that introducing a new security in the market will generally alter the prices of the existing securities, as Kreps remarks in [69]:

> Of course, this [...] interpretation cannot be made in general, as the introduction of a market for x [X] creates new economic opportunities for agents and may thereby change the prices of bundles in M [$\mathcal{M}$].

Another way to define a range of reasonable prices is the following. We say that p is a *market-consistent price* for X if

- for every $Z \in \mathcal{M}$ such that $Z - X \in \mathcal{L}_+ \setminus \{0\}$ we have $p < \pi(Z)$,
- for every $Z \in \mathcal{M}$ such that $X - Z \in \mathcal{L}_+ \setminus \{0\}$ we have $p > \pi(Z)$.

This notion is used in Koch-Medina and Munari [67]. The above conditions stipulate that p is a market-consistent price for X from a buyer's, respectively seller's, perspective: An agent interested in buying, respectively selling, the payoff X at the price p cannot find any replicable payoff in the market that is more attractive than X from a buyer's, respectively seller's, perspective and can be bought at a lower price, respectively sold at a higher price. In other words, it is not foolish to transact X for the price p given the alternatives offered by the market. In spite of the different interpretation, it turns out that arbitrage-free prices and market-consistent prices are equivalent concepts in an arbitrage-free market. Now, the key question becomes: *How to characterize arbitrage-free prices or equivalently market-consistent prices?*

The first important result in this direction is to show that the set of arbitrage-free prices is an interval whose bounds can be expressed in terms of the *superreplication price* defined for $X \in \mathcal{L}$ by

$$\pi^+(X) = \inf\{\pi(Z)\,;\, Z \in \mathcal{M},\, Z - X \in \mathcal{L}_+\}.$$

Result 1. *The set of arbitrage-free prices $\Pi(X)$ for a payoff $X \in \mathcal{L}$ is an interval with upper bound $\pi^+(X)$ and lower bound $-\pi^+(-X)$.*

Note that the superreplication price corresponds to the lowest monetary amount that has to be invested in the market in order to cover the underlying payoff in every contingency. Thus, it is the minimum price accepted by a seller who wants to finance a strategy to hedge in every future state the claim of the buyer. At the same time, if we take the buyer's point of view, by the stated result it is the maximum price such that is not foolish given the alternatives in the market. Throughout this chapter, we focus on the latter perspective, since the focus is on pricing rather than on hedging problems.

According to the above result, the problem of characterizing arbitrage-free prices boils down to determining whether superreplication prices are themselves arbitrage-free prices or not. It turns out that the superreplication price of a replicable payoff is always an arbitrage-free price. This implies that a replicable payoff has a unique arbitrage-free price, which coincides with its replication cost. In contrast, the superreplication price of a nonreplicable payoff is never arbitrage free. In this case, there exist infinitely many arbitrage-free prices.

Result 2. *If the market is arbitrage free, then for a payoff $X \in \mathcal{L}$ the following statements hold:*

- *If $X \in \mathcal{M}$, then $-\pi^+(-X) = \pi^+(X) = \pi(X)$ and $\Pi(X) = \{\pi(X)\}$.*
- *If $X \notin \mathcal{M}$, then $-\pi^+(-X) < \pi^+(X)$ and $\Pi(X) = (-\pi^+(-X), \pi^+(X))$.*

To obtain a more concrete description of arbitrage-free prices, which can be effectively used in both theory and practice, one needs a convenient representation of superreplication prices. The key tool to achieve this is the celebrated Fundamental Theorem of Asset Pricing. Here, we record one of its equivalent formulations.

Result 3. *If the market is arbitrage free, then there exists a family $\mathcal{D}$ of strictly-positive bounded random variables D such that $\mathbb{E}_{\mathbb{P}}[DX] = \pi(X)$ for every replicable payoff $X \in \mathcal{M}$. Moreover,*

- *$\pi^+(X) = \sup\{\mathbb{E}_{\mathbb{P}}[DX] \,:\, D \in \mathcal{D}\}$ for every payoff $X \in \mathcal{L}$.*
- *$\Pi(X) = \{\mathbb{E}_{\mathbb{P}}[DX] \,:\, D \in \mathcal{D}\}$ for every payoff $X \in \mathcal{L}$.*

The elements of $\mathcal{D}$ are known in the literature under different names including *pricing densities*, *stochastic discount factors*, *price deflators*. By definition, any pricing density can be used to represent the initial price of each replicable payoff by way of an expectation applied directly to the payoff itself. This provides a strictly-positive linear extension of the pricing functional beyond the space of replicable payoffs. These extensions deliver a representation of superreplication prices together with the desired concrete characterization of arbitrage-free prices. These representations can be exploited to tackle a variety of concrete pricing problems in theory and practice.

GOOD DEAL PRICING

The representation of superreplication prices in Result 3 is the starting point of the "good deal pricing" literature. The notion of a good deal is a generalization of that of an arbitrage opportunity. Broadly speaking, a *good deal* is any (nonzero) replicable payoff that belongs to a set $\mathcal{A} \subset \mathcal{L}$ of sufficiently attractive payoffs and can be acquired without cost. In the literature, this set is sometimes called the acceptance set and is typically, but not always, assumed to contain the positive cone $\mathcal{L}_+$. In this case, every arbitrage opportunity is also a good deal. It is the agent's task to specify the threshold to attractive payoffs based on his or her individual preferences. The common assumption in the "good deal pricing" literature is that the absence of arbitrage opportunities is replaced by the more general absence of good deals. This leads to tighter pricing bounds that are called *good deal bounds*. One can distinguish between two fundamental research directions in the field.

A first strand of literature starts by imposing suitable constraints on pricing densities. The goal is to restrict the set of pricing densities thereby obtaining a smaller, more tractable, interval of arbitrage-free prices. The following example from Cherny [32] motivates the intent to restrict the interval of prices:

> Consider a contract that with probability $\frac{1}{2}$ yields nothing and with probability $\frac{1}{2}$ yields 1000 USD. The no arbitrage price interval for this contract is (0, 1000). But if the price of the contract is, for instance, 15 USD, then everyone would be willing to buy it, and the demand would not match the supply. Thus, 15 USD is an unrealistic price because it yields a good deal, i.e., a trade that is attractive to most market participants. The technique of the no-good deal pricing is based on the assumption that good deals do not exist.

A variety of constraints have been considered in the literature including constraints based on Sharpe ratios in Cochrane and Saa Requejo [34] and Bion-Nadal and Di Nunno [22], gain-loss ratios in Bernardo and Ledoit [18], utility functions in Černý [28], and general conic constraints in Cherny [32]. The resulting good deal bounds can be expressed in terms of the restricted superreplication price defined for every payoff $X \in \mathcal{L}$ by

$$\pi^+_{\mathcal{E}}(X) = \sup\{\mathbb{E}_{\mathbb{P}}[DX] \,:\, D \in \mathcal{E}\}$$

for a given family of pricing densities $\mathcal{E} \subset \mathcal{D}$. The rationale for discarding the portion of arbitrage-free prices exceeding the given good deal bounds is that pricing at those levels would allow for good deals with respect to a suitable acceptance set $\mathcal{A} \subset \mathcal{L}$. A second strand of literature starts from the very definition of superreplication price and relaxes the superreplication condition by imposing suitable constraints on acceptable replication errors. This is achieved by tightening the superreplication price into the more restrictive pricing bound defined for a payoff $X \in \mathcal{L}$ by

$$\pi^+_{\mathcal{A}}(X) = \inf\{\pi(Z) \,:\, Z \in \mathcal{M},\ Z - X \in \mathcal{A}\}$$

for a given acceptance set $\mathcal{A} \subset \mathcal{L}$. Provided that the acceptance set contains the positive cone $\mathcal{L}_+$, the result is again a restriction of the interval of arbitrage-free prices. A pricing theory for general acceptance sets was developed in Jaschke and Küchler [60] and Staum [91]. Special acceptance sets have also been studied including sets based on test probabilities in Carr et al. [26], and utility functions in Černý and Hodges [32] and Arai [2].

The two approaches are dual to each other in the sense that one can build a formal one-to-one correspondence between sets of pricing densities and acceptance sets. The advantage of the first approach is that it immediately entails a dual characterization of the restricted set of arbitrage-free prices. However, it may not be easy to understand which acceptance set corresponds to a given constraint on the pricing densities. Conversely, the advantage of the second approach is that it starts with the explicit choice of a set of acceptable replication errors. However, a dual characterization of the corresponding set of arbitrage-free prices has to be explicitly obtained. This requires a suitable extension of the Fundamental Theorem of Asset Pricing.

OUR CONTRIBUTION

The goal of this chapter is to extend the existing literature by building a general theory of "good deal pricing" in a one-period financial market. The theory is general in the sense that we consider general convex acceptance sets containing all positive random variables, and we allow for general convex transaction costs and portfolio constraints. Throughout this chapter, good deals are named *acceptable deals*.

A first relevant feature of our approach is that the reference space $\mathcal{L}$ is taken to be the natural modeling framework, namely the vector space of random variables on $(\Omega, \mathcal{F}, \mathbb{P})$, and the acceptance set $\mathcal{A}$ is assumed to be a generic convex subset of $\mathcal{L}$ containing all positive random variables. This is different from the bulk of the "good deal pricing" literature where regularity conditions on acceptable payoffs are stipulated upfront in view of the application of special mathematical results, e.g. duality theory. The advantage of our approach is that we are able to highlight where and why a restriction to a subset of $\mathcal{L}$ is needed, e.g. to apply duality theory, and what are its consequences in terms of the original pricing problem. The assumption that $\mathcal{A}$ contains the positive cone ensures that our good deal bounds are truly tighter than the classical arbitrage-free bounds.

A second relevant feature is that the pricing rule π is no longer linear but convex and, similarly, the set of admissible replicable payoffs $\mathcal{M}$ is no longer a vector space but a convex set. This allows us to incorporate (proportional and nonproportional) transaction costs and portfolio constraints. The presence of frictions has a variety of important implications, some of which are highlighted later in the chapter. From the perspective of our motivating pricing problem, the most important implication is that, in a market with frictions, the classical notion of an arbitrage-free price becomes ambiguous because it is not clear how to extend the market preserving the prices of the basic traded securities together with the absence of arbitrage opportunities or, more generally, good deals. Cherny [32] considers a linear extension like in the case of frictionless markets, but we do not find adequate reasons for this choice. At the same time, the above formulation of market-consistent prices can be unambiguously adapted to our general market model by simply substituting the positive cone $\mathcal{L}_+$ with the acceptance set $\mathcal{A}$. More precisely, we say that $p \in \mathbb{R}$ is a *market-consistent price (with respect to $\mathcal{A}$)* for a payoff $X \in \mathcal{L}$ if

- for every $Z \in \mathcal{M}$ such that $Z - X \in \mathcal{A} \setminus \{0\}$ we have $p < \pi(Z)$,
- for every $Z \in \mathcal{M}$ such that $X - Z \in \mathcal{A} \setminus \{0\}$ we have $p > \pi(Z)$.

We argue that market-consistent prices constitute the natural reasonable prices for buyers and sellers who have full access to the market and are prepared to accept a replication error belonging to $\mathcal{A}$. To the best of our knowledge, a clear definition of reasonable prices beyond the classical arbitrage pricing theory has never been explicitly stated. The goal of this chapter is to provide an answer to the question: *How to characterize market-consistent prices with acceptable risk?*

As a first simple step, we show that Result 1 still holds provided we replace the superreplication price by the pricing bound defined for every payoff $X \in \mathcal{L}$ by

$$\pi^+_{\mathcal{A}}(X) = \inf\{\pi(Z) \, : \, Z \in \mathcal{M}, \, Z - X \in \mathcal{A}\}$$

where π is now a convex functional and $\mathcal{M}$ is a convex set. We establish a direct and a dual characterization of market-consistent prices with acceptable risk in the spirit of Result 2 and Result 3 above.

As for the direct characterization, it is worth highlighting that only part of Result 1 can be extended to our setting. Indeed, while the market consistency of $\pi^+_{\mathcal{A}}(X)$ forces the payoff X to belong to the set $\mathcal{M}$ of admissible replicable payoffs, the converse implication no longer holds. In particular, if X belongs to $\mathcal{M}$, then its replication cost $\pi(X)$ may fail to be market consistent. This reveals a fundamental difference between frictionless markets and markets with frictions and extends to a "good deal pricing" setting the classical findings of Bensaid et al. [17].

A dual characterization of market-consistent prices akin to Result 3 is more challenging to obtain and requires extending the Fundamental Theorem of Asset Pricing to our setting. To this end, a preliminary step is to identify the appropriate generalization of a pricing density. This is given by the so-called *strictly-consistent pricing densities*, i.e. the Riesz densities of those linear functionals that belong to the domain of the conjugate function of π and are strictly positive on $\mathcal{A}$. We think that strict positivity on $\mathcal{A}$ is the correct way of generalizing strict positivity of functionals in arbitrage theory, as the set $\mathcal{A}$ plays the role of the positive cone. From a financial perspective, these functionals can be interpreted as the pricing rules of suitable frictionless complete markets where the basic traded securities are "priced" in accordance with their bid-ask spreads and every (nonzero) acceptable payoff has a strictly positive "price". The Fundamental Theorem of Asset Pricing provides sufficient conditions for the existence of strictly-consistent pricing densities and can be used to derive a dual representation of $\pi^+_{\mathcal{A}}(X)$ and a corresponding dual characterization of market-consistent prices. In the case of a conic acceptance set, the key condition is that the market admits no *scalable acceptable deal*, i.e. no acceptable deal that remains an acceptable deal independently of how it is rescaled. This is a very weak condition that may be satisfied even though the market admits good deals. In the case of a nonconic acceptance set, the absence of scalable acceptable deals is no longer sufficient and we have to require the absence of scalable acceptable deals with respect to a suitably enlarged acceptance set. This extends to a "good deal pricing" setting the Fundamental Theorem established by Pennanen [76].

EMBEDDING IN THE LITERATURE

The natural term of comparison for our work are the papers belonging to the second branch of the "good deal pricing" literature as presented above. The focus of most of those papers is on frictionless markets and/or specific acceptance sets but there are three contributions, namely Jaschke and Küchler [60], Staum [91] and Cherny [32] where a general theory of "good deal pricing" in markets with frictions is presented. To highlight the link to our work we provide a brief description of each of our main references (with special emphasis on their formulations of the Fundamental Theorem of Asset Pricing) in order of publication.

- The focus of Carr et al. [26] is on one-period frictionless markets with finite probability space and convex polyhedral acceptance sets defined in terms of test probability measures. The authors establish a Fundamental Theorem of Asset Pricing (Theorem 1) characterizing the absence of a special type of good deals that is specific to the polyhedral structure of their acceptance sets.

- The focus of Jaschke and Küchler [60] is on multi-period markets with proportional frictions and conic convex acceptance sets. The reference model space is abstract. The authors establish a Fundamental Theorem of Asset Pricing (Corollary 8) characterizing the absence of a strong form of good deals under a suitable closedness assumption. No sufficient condition for the closedness assumption to hold is provided. For more details, see Subsection 1.5.2.

- The focus of Černý and Hodges [29] is on multi-period frictionless markets with convex acceptance sets. The reference model space is abstract. The authors establish a Fundamental Theorem of Asset Pricing (Theorem 2.5) characterizing the absence of good deals under the assumption that the model space is an L^p space with $1 < p < \infty$ and that the acceptance set is boundedly generated. The latter condition is seldom met in infinite dimensional model spaces.

- The focus of Staum [91] is on multi-period markets with convex frictions and convex acceptance sets. The reference model space is abstract. The author establishes a Fundamental Theorem of Asset Pricing (Theorem 6.2) characterizing the absence of a generalized type of good deals. Unfortunately, the proof of the Fundamental Theorem contains a major flaw invalidating the entire result. For more details, see Subsection 1.5.2.

- The focus of Cherny [32] is on multi-period markets with convex frictions and conic convex acceptance sets. They establish a Fundamental Theorem of Asset Pricing (Theorem 3.1) which heavily relies on the weak closedness of the barrier cone of the acceptance set, and that characterizes the absence of good deals with the existence of a pricing density positive (not strictly positive) on the acceptance set and nonpositive on the set of replicable payoffs with nonpositive cost. The setting is the space of all random variables on a given probability space, but they require that the market is such that it is possible to restrict the dual set of interest to a tractable space.

- The focus of Arai [2] is on multi-period frictionless markets with convex acceptance sets defined in terms of utility functions. The reference model space is an Orlicz space. The author establishes dual representations of the corresponding good deal bounds.

We believe we bring some contribution to existing literature. First, we explicitly start from the definition of a market-consistent price with acceptable risk. This is an important step from the point of view of the economical interpretation of the theory to be developed. Second, we work under general convex transaction costs and portfolio constraints. Finally, we identify a generalization of strictly positive pricing densities to the good deal setting and we provide a version of the Fundamental Theorem of Asset Pricing that does not suffer from the flaws in Staum [91] and extends the results in Černý and Hodges [29] beyond the frictionless setting and beyond boundedly-generated acceptance sets, the results in Jaschke and Küchler [60] beyond the conic setting and the results in Cherny [32] beyond conic acceptance sets. It is worth noting that our choice to work in a one-period model allows us to improve the main result of Jaschke and Küchler [60] also in the conic setting. In particular, we are able to provide sufficient conditions for the existence of *strictly-consistent* pricing densities instead of simple *consistent* pricing densities as considered in that paper. This is crucial to establish our desired characterization of market-consistent prices.

1.1 THE MARKET MODEL

For definitions and notations we refer to Appendix A. We consider a one-period financial market where uncertainty about the terminal state of the economy is captured by a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Every payment is denominated in a fixed unit of account, which for simplicity we call money. The *payoff* of a financial contract at the terminal date is modeled by a random variable (better, an equivalence class of random variables) $X \in L^0(\mathbb{P})$. The positive and the negative value of any random variable correspond to the absolute inflow and outflow of money specified by the contract at the terminal date. The elements of $\mathbb{R}$ are identified with constant random variables and are therefore interpreted as *risk-free payoffs*. In what follows, we will find convenient to apply expectations to generic random variables regardless of their integrability. We do this like explained in Appendix A.

1.1.1 THE BASIC TRADED SECURITIES

We assume that N *basic securities* are traded in the market. The terminal payoff of the ith basic security is represented by a (not necessarily positive) random variable $S_i \in L^0(\mathbb{P})$. This is equivalent to a cash delivery that agents receive in exchange for their claims. In particular, no physical delivery occurs so that agents do not have to liquidate their positions in the market at the terminal date to convert them into cash. To avoid dealing with redundant securities, we assume throughout that $S_1, \dots, S_N$ are linearly independent. Through their trading activity, agents can set up portfolios of basic securities at the initial date. A *portfolio* of basic securities is represented by a vector $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$. We adopt the standard convention according to which a positive entry refers to a long position and a negative entry to a short position. The corresponding ask prices are described by a function $V_0 : \mathbb{R}^N \to (-\infty, \infty]$. We assume that V_0 may take nonfinite values to allow for the presence of unfeasible portfolios. Throughout the chapter we assume that V_0 is convex and lower semicontinuous and satisfies $V_0(0) = 0$.

Example 1.1.1. *In a frictionless market the bid-ask spread associated with every basic security is zero so that every unit of the ith basic security can be bought or sold for the same price $p_i \in \mathbb{R}$. The associated pricing rule is linear and given by*

$$V_0(\lambda) = \sum_{i=1}^{N} \lambda_i p_i$$

for every $\lambda \in \mathbb{R}^N$. In a market with proportional transaction costs every unit of the ith basic security can be bought for the price $p_i^b \in \mathbb{R}$ and sold for the price $p_i^s \in \mathbb{R}$. It is natural to assume that $p_i^b \geq p_i^s$ so that the corresponding bid-ask spread is nonnegative. The associated pricing rule is sublinear and given by

$$V_0(\lambda) = \sum_{\lambda_i \geq 0} \lambda_i p_i^b + \sum_{\lambda_i < 0} \lambda_i p_i^s$$

for every $\lambda \in \mathbb{R}^N$. In a market with nonproportional transaction costs the unitary buying and selling prices for the ith basic security vary with the volume traded according to some functions $p_i^b, p_i^s : \mathbb{R}_+ \to \mathbb{R}$. Again, it makes sense to assume that $p_i^b(x) \geq p_i^s(x)$ for every $x \in \mathbb{R}_+$ so that the corresponding bid-ask spread is nonnegative. As carefully explained in Pennanen [79], in limit order markets it is natural to assume that p_i^b is convex and p_i^s is concave. Moreover, we assume that $\partial^+ p_i^s(0) \leq \partial^+ p_i^b(0)$. The associated pricing rule is convex and given by

$$V_0(\lambda) = \sum_{\lambda_i \geq 0} p_i^b(\lambda_i) - \sum_{\lambda_i < 0} p_i^s(-\lambda_i)$$

for every $\lambda \in \mathbb{R}^N$.

We have shown three examples of pricing rules that are defined as a sum of convex functions of the single components. This means that $V_0(\lambda) = \sum_{i=1}^N V_0(\lambda_i \mathbf{e}_i)$. It may happen, tough, that the pricing functional is properly subadditive in the components, in the sense that when buying different securities, one incurs in sales. Our model covers also this case.

Each portfolio of basic securities generates a payoff at the terminal date. These special payoffs are described by the map $V_1 : \mathbb{R}^N \to L^0(\mathbb{P})$ defined by

$$V_1(\lambda) := \sum_{i=1}^N \lambda_i S_i.$$

The linearity of V_1 is consistent with our one-period setting where no trading occurs at the terminal date and each security delivers its terminal state-contingent contractual payoff. The vector space spanned by the payoffs of the basic securities is denoted by $\mathcal{S}$, i.e.

$$\mathcal{S} := \{V_1(\lambda) \,:\, \lambda \in \mathbb{R}^N\}.$$

The elements of $\mathcal{S}$ represent payoffs of financial contracts that can be replicated by trading in the market of the basic securities and will therefore be referred to as *replicable payoffs*. It is important to point out that, by finite dimensionality, there exists a unique topology on $\mathcal{S}$ which makes $\mathcal{S}$ a Hausdorff topological vector space. Every topological property related to $\mathcal{S}$ has to be understood with respect to such topology.

1.1.2 The pricing rule

For our later analysis it is convenient to associate an ask price directly to replicable payoffs. This is possible because we have assumed that no basic security is redundant so that the payoffs $S_1, \ldots, S_N$ are linearly independent. Indeed, under this assumption, two portfolios having the same payoffs must coincide and, hence, command the same ask price

$$V_1(\lambda) = V_1(\mu) \quad \Longrightarrow \quad V_0(\lambda) = V_0(\mu).$$

In the introduction of this thesis, we have called this effect "law of one price". It allows us to introduce a pricing rule $\pi : \mathcal{S} \to (-\infty, \infty]$ by setting for every replicable payoff $X \in \mathcal{S}$

$$\pi(X) := V_0(\lambda),$$

where $\lambda \in \mathbb{R}^N$ is any portfolio satisfying $X = V_1(\lambda)$. The quantity $\pi(X)$ can thus be unambiguously interpreted as the replication cost of X. The properties of π that we need in the sequel are recorded in the next proposition. In particular, we highlight a characterization of the asymptotic cone of the set of replicable payoffs that can be acquired without cost in terms of the asymptotic function π^∞ (see Appendix B).

Proposition 1.1.2. *The map π is convex, lower semicontinuous, satisfies $\pi(0) = 0$ and one has*

$$\{X \in \mathcal{S} \,:\, \pi(X) \leq 0\}^\infty = \{X \in \mathcal{S} \,:\, \pi^\infty(X) \leq 0\}.$$

Proof. It follows from the convexity of V_0 and the linearity of V_1 that π is convex. Moreover, we clearly have $\pi(0) = V_0(0) = 0$. To show lower semicontinuity, take a sequence $(X_n) \subset \mathcal{S}$ and $X \in \mathcal{S}$ and assume that $X_n \to X$. By definition of $\mathcal{S}$, we find a sequence $(\lambda_n) \subset \mathbb{R}^N$ and $\lambda \in \mathbb{R}^N$

such that $X_n = V_1(\lambda_n)$ for every $n \in \mathbb{N}$ and $X = V_1(\lambda)$. As $S_1, \dots, S_N$ are linearly independent, we infer that $\lambda_n \to \lambda$. As a consequence, the lower semicontinuity of V_0 implies that

$$\pi(X) = V_0(\lambda) \leq \liminf_{n\to\infty} V_0(\lambda_n) = \liminf_{n\to\infty} \pi(X_n).$$

This delivers the desired lower semicontinuity. Since $\{X \in \mathcal{S} \,:\, \pi(X) \leq 0\}$ contains the zero payoff, the last equality is a direct consequence of Proposition B.2.8. $\square$

By virtue of the last proposition and Corollaries B.1.7 and B.2.7, we may indifferently use recession or asymptotic cones and functions:

$$\operatorname{rec}\left(\{X \in \mathcal{S} \,:\, \pi(X) \leq 0\}\right) = \{X \in \mathcal{S} \,:\, \pi(X) \leq 0\}^\infty, \qquad \operatorname{rec}(\pi) = \pi^\infty.$$

All along this chapter, whenever the identification recession-asymptotic holds, we will opt for the asymptotic notation as it looks lighter than the recession's one. We recall that the recession function $\operatorname{rec}(\pi) = \pi^\infty$ is the smallest sublinear map dominating π . This means that π^∞ can be interpreted as the best approximation of π among all the pricing rules in markets with proportional transaction costs where bid-ask spreads are larger than the ones in the original market.

1.1.3 The set of admissible replicable payoffs

We model portfolio constraints such as borrowing and short selling restrictions on specific basic securities by restricting the set of admissible portfolios to a subset $\mathcal{P} \subset \mathbb{R}^N$. Throughout we assume that $\mathcal{P}$ is closed, convex, and contains the zero portfolio. This implies that $\operatorname{rec}(\mathcal{P}) = \mathcal{P}^\infty$ due to Corollary B.1.7.

Example 1.1.3. *The case of no short selling corresponds to $\mathcal{P} = \mathbb{R}^N_+$. The case of no short selling with caps on the long positions corresponds to $\mathcal{P} = [0, \overline{\lambda}_1] \times \cdots \times [0, \overline{\lambda}_N]$ for a suitable portfolio $\overline{\lambda} \in \mathbb{R}^N$ with strictly-positive components. In a similar fashion one can include caps on the short positions as well.*

We denote by $\mathcal{M}$ the set of all replicable payoffs generated by admissible portfolios of basic securities, i.e.

$$\mathcal{M} := \{V_1(\lambda) \,:\, \lambda \in \mathcal{P}\} \subset \mathcal{S}.$$

Every payoff in $\mathcal{M}$ is called an *admissible replicable payoff*. The properties of $\mathcal{M}$ that are needed in the sequel are recorded in the next proposition. In particular, we show that the asymptotic cone of $\mathcal{M}$ (which coincides with its recession cone) consists of all the replicable payoffs associated to portfolios that are admissible regardless of their size.

Proposition 1.1.4. *The set $\mathcal{M}$ is closed, convex, contains the null payoff, and satisfies*

$$\mathcal{M}^\infty = \{V_1(\lambda) \,:\, \lambda \in \mathcal{P}^\infty\}.$$

Proof. Since $\mathcal{P}$ is convex and contains the zero portfolio, it readily follows from the linearity of V_1 that $\mathcal{M}$ is convex and contains the null payoff. Now, take a sequence $(X_n) \subset \mathcal{M}$ and $X \in \mathcal{S}$ and assume that $X_n \to X$. By definition of $\mathcal{S}$ and $\mathcal{M}$, we find a sequence $(\lambda_n) \subset \mathcal{P}$ and $\lambda \in \mathbb{R}^N$ such that $X_n = V_1(\lambda_n)$ for every $n \in \mathbb{N}$ and $X = V_1(\lambda)$. As $S_1, \dots, S_N$ are linearly independent, we infer that $\lambda_n \to \lambda$. As a result, the closedness of $\mathcal{P}$ yields that $\lambda \in \mathcal{P}$, showing that $X \in \mathcal{M}$. This establishes that $\mathcal{M}$ is closed. Hence we have that $\operatorname{rec}(\mathcal{M}) = \mathcal{M}^\infty$. The representation of $\mathcal{M}^\infty$ is a direct consequence of the linearity of V_1 by applying the definition of $\operatorname{rec}(\mathcal{M})$ and Corollary B.1.7. $\square$

Remark 1.1.5 (From buyer to seller)**.** In chapter we take the perspective of a buyer. In particular, π is interpreted as an ask pricing functional and $\mathcal{M}$ as a set of admissible replicable payoffs for a buyer. To switch to the seller's perspective one has simply to consider the pricing rule $X \mapsto -\pi(-X)$ and the set of admissible replicable payoffs $-\mathcal{M}$.

Remark 1.1.6 (From payoffs to portfolios)**.** Our results will be formulated in terms of payoffs instead of portfolios. To this effect, we only need to fix a finite dimensional vector space $\mathcal{S}$ of replicable payoffs, a pricing rule $\pi : \mathcal{S} \to (-\infty, \infty]$ satisfying the properties in Proposition 1.1.2, and a set $\mathcal{M}$ of admissible replicable payoffs satisfying the properties in Proposition 1.1.4. It is important to note that, in this case, we can always specify a set of linearly independent payoffs $S_1, \ldots, S_N \in L^0(\mathbb{P})$, a pricing functional $V_0 : \mathbb{R}^N \to (-\infty, \infty]$ and a set of admissible portfolios $\mathcal{P}$ such that

(i) $\mathcal{S} = \left\{ \sum_{i=1}^N \lambda_i S_i \, : \, \lambda \in \mathbb{R}^N \right\}$,

(ii) $\pi(X) = V_0(\lambda)$ for all $X \in \mathcal{S}$ and $\lambda \in \mathbb{R}^N$ such that $X = \sum_{i=1}^N \lambda_i S_i$,

(iii) $\mathcal{M} = \left\{ \sum_{i=1}^N \lambda_i S_i \, : \, \lambda \in \mathcal{P} \right\}$.

Moreover, under the stipulated properties of π and $\mathcal{M}$, the functional V_0 is convex, lower semicontinuous, and satisfies $V_0(0) = 0$, and the set $\mathcal{P}$ is convex, closed, and contains the null portfolio. This fact will be repeatedly use throughout the chapter.

Remark 1.1.7 (On the market model)**.** Let $\mathcal{S} \subset L^0(\mathbb{P})$ be a vector space of replicable payoffs. The range of market models that are compatible with $\mathcal{S}$ depends on the dimensionality of $\mathcal{S}$. In the finite dimensional case, the eligible models are a standard one-period market or a multi-period market where the only admissible trading strategies are of buy-and-hold type. In the infinite dimensional case, we may consider any (discrete or continuous) multi-period market. Many results in chapter do not require $\mathcal{S}$ to be finite dimensional. However, the finite dimensionality of $\mathcal{S}$ will play a decisive role in a key closedness result, namely Theorem 1.3.14, that is the basis for our general versions of the Fundamental Theorem of Asset Pricing. For this reason, we have opted to formulate the entire chapter in the setting of a standard one-period market.

1.2 THE PRICING PROBLEM

In this section we introduce the notion of an acceptance set and the key related notion of a market-consistent price. This extends the classical concept of an arbitrage-free price beyond the setting of a frictionless market and beyond standard superreplication. In the frictionless literature, arbitrage-free prices are commonly defined as those prices that allow to enlarge the original market in a frictionless way without creating arbitrage opportunities; see, e.g., Harrison and Kreps [59]. In a market with frictions this approach becomes problematic because it is not clear how the market could and should be enlarged in the first place. We therefore follow a different path and adapt to our market model the notion of a market-consistent price studied in a frictionless setting in Koch-Medina and Munari [67].

Consider an agent interested in buying a financial contract with payoff $X \in L^0(\mathbb{P})$. The agent's problem is to determine a range of "reasonable" prices at which he should be prepared to acquire said contract. The fundamental idea behind arbitrage pricing is that the agent will compare X with the replicable payoffs offered by the market and will use the corresponding market prices to benchmark whether a candidate buying price is too high or not. More precisely, the agent will look for all the replicable payoffs that are deemed *better* than X from a buyer's perspective and use the corresponding market prices to set up an upper bound on prices. To this effect, one has to specify a criterion that allows to compare and rank payoffs. In the classical arbitrage pricing theory, a replicable payoff $Z \in \mathcal{S}$ is better than X (from a buyer's perspective) if it delivers a higher payoff in every future contingency, i.e.

$$Z - X \in L^0(\mathbb{P})_+.$$

In this case, one often says that Z superreplicates X. Based on this criterion, agents will use superreplicating replicable payoffs to determine the range of prices at which they are willing to purchase the contract. The same criterion lies at the heart of another key notion of the classical arbitrage pricing theory, namely that of an arbitrage opportunity, i.e., a nonzero replicable payoff $Z \in \mathcal{S}$ such that $\pi(Z) \leq 0$ and $Z \in L^0(\mathbb{P})_+$. In line with the above ranking criterion, an arbitrage opportunity is a desirable payoff that can be acquired at zero price.

1.2.1 THE ACCEPTANCE SET

In chapter, we relax the criterion behind arbitrage pricing and assume that agents are prepared to accept a suitable superreplication error when comparing payoffs. This leads us to replace the positive cone by a larger set $\mathcal{A} \subset L^0(\mathbb{P})$, which we call the acceptance set, and postulate that a replicable payoff $Z \in \mathcal{S}$ is considered better than a given payoff $X \in L^0(\mathbb{P})$ (from a buyer's perspective) whenever

$$Z - X \in \mathcal{A}.$$

Since $\mathcal{A}$ may contain nonpositive payoffs, it may happen that Z fails to superreplicate X is some future contingency. A sound acceptance set should stipulate a reasonable tradeoff between positive and negative outcomes so that the failure of superreplication is truly acceptable. Note that, in contrast to the "homogeneous" setting of the classical arbitrage pricing theory, different agents may naturally come up with different ways to define acceptability.

The formal definition of an acceptance set is as follows. We assume that the notion of acceptability is well behaved with respect to aggregation in the sense that every convex combination of acceptable payoffs remains acceptable, that the null payoff is acceptable and that every payoff dominating a given acceptable payoff is automatically acceptable. The last property corresponds to the usual monotonicity requirement stipulated in risk measure theory; see, e.g., Artzner et al. [9]. A direct consequence of our definition is that every positive payoff is acceptable. We refer to Section 1.6 for a list of concrete acceptance sets.

Definition 1.2.1. A set $\mathcal{A} \subset L^0(\mathbb{P})$ is called an *acceptance set* if it is convex, contains 0 and

$$\mathcal{A} + L^0(\mathbb{P})_+ \subset \mathcal{A}.$$

From now on we fix an acceptance set $\mathcal{A}$. Every element of $\mathcal{A}$ is called an *acceptable payoff*. Note that, in line with our pricing problem, we do not restrict the acceptance set to belong to any "nice" subspace of $L^0(\mathbb{P})$ as commonly done in risk measures theory.

1.2.2 MARKET-CONSISTENT PRICES

As previously recalled, the fundamental idea behind arbitrage pricing is that agents interested in buying a certain payoff will compare it with the replicable payoffs offered by the market and use the corresponding market prices to assess whether a candidate buying price is too high or not. The resulting reasonable prices are known in the literature under different names including fair prices, arbitrage-free prices, and market-consistent prices. The natural extension of this notion to our setting is recorded in the next definition.

Definition 1.2.2. For a payoff $X \in L^0(\mathbb{P})$ we say that $p \in \mathbb{R}$ is a *market-consistent (buyer) price for X (with respect to $\mathcal{A}$)* whenever:

(1) $p < \pi(Z)$ for every replicable payoff $Z \in \mathcal{M}$ such that $Z - X \in \mathcal{A} \setminus \{0\}$;

(2) $p \leq \pi(X)$ whenever $X \in \mathcal{M}$.

We denote by $\mathrm{MCP}(X)$ the set of market consistent prices for X.

The range of market-consistent prices with respect to $\mathcal{A}$ is the natural range of reasonable prices for a buyer who has full access to the market of basic securities and is willing to take superreplication risk according to the acceptance set $\mathcal{A}$. Indeed, if a price is not market consistent for a given payoff, then the buyer can always invest that amount (or less) to purchase an admissible replicable payoff that is strictly preferable from the perspective of $\mathcal{A}$. In particular, note that every market-consistent price is also an arbitrage-free price in the classical sense. This is because the acceptance set $\mathcal{A}$ contains the positive cone $L^0(\mathbb{P})_+$.

Remark 1.2.3. Note that condition (2) is automatically implied by condition (1) if for every replicable payoff $X \in \mathcal{M}$ there exist $U \in \mathcal{A} \setminus \{0\}$ and $c \in \mathbb{R}$ such that $X + \frac{1}{n}U \in \mathcal{M}$ and $\pi(X + \frac{1}{n}U) \leq \pi(X) + \frac{1}{n}c$ for every $n \in \mathbb{N}$. In particular, this holds if $\mathcal{A}$ and $\mathcal{M}$ have nonzero intersection and $\mathcal{M}$ and π are conic.

Remark 1.2.4 (From buyer to seller)**.** Consistently with Remark 1.1.5, for a payoff $X \in L^0(\mathbb{P})$ we say that $p \in \mathbb{R}$ is a market-consistent seller price for X (with respect to $\mathcal{A}$) whenever:

(1) $p > -\pi(-Z)$ for every $Z \in -\mathcal{M}$ such that $X - Z \in \mathcal{A} \setminus \{0\}$;

(2) $p \geq -\pi(-X)$ whenever $X \in -\mathcal{M}$.

It is readily seen that p is a market-consistent buyer price for X if and only if $-p$ is a market consistent seller price for $-X$. As a result, a characterization of market-consistent buyer prices will immediately deliver a corresponding characterization of market-consistent seller prices.

1.2.3 Acceptable deals

The goal of chapter is to establish a characterization of market-consistent prices in the presence of our general market frictions and acceptance set. In the classical arbitrage pricing setting, this relies on the celebrated Fundamental Theorem of Asset Pricing and is possible provided the market is free of arbitrage opportunities. In this section we introduce the notion of an acceptable deal, which constitutes the natural generalization of an arbitrage opportunity to our setting. In the next sections we provide a direct and a dual characterization of market-consistent prices under the assumption that suitable acceptable deals do not exist. The dual characterization is based on an appropriate extension of the Fundamental Theorem of Asset Pricing.

In the classical frictionless theory an arbitrage opportunity is a nonzero replicable payoff that is positive and can be acquired at zero cost. In the presence of portfolio constraints, one has to additionally require that the payoff be admissible. We say that an arbitrage opportunity is scalable if it remains an arbitrage opportunity regardless of its size. Clearly, this specification only makes sense in the presence of nonconic frictions. The corresponding arbitrage opportunities with respect to $\mathcal{A}$ are called (scalable) acceptable deals.

Definition 1.2.5. We say that a nonzero replicable payoff $X \in \mathcal{S}$ is:

(1) an *arbitrage opportunity* if $X \in \mathcal{M} \cap L^0(\mathbb{P})_+$ and $\pi(X) \leq 0$.

(2) a *scalable arbitrage opportunity* if tX is an arbitrage opportunity for every $t > 0$.

(3) an *acceptable deal (with respect to $\mathcal{A}$)* if $X \in \mathcal{M} \cap \mathcal{A}$ and $\pi(X) \leq 0$.

(4) a *scalable acceptable deal (with respect to $\mathcal{A}$)* if tX is an acceptable deal for every $t > 0$.

Remark 1.2.6. (i) The notion of acceptable deal has appeared, together with a number of variations, under different names in the literature, among which the most common is *good deal*. This term was invented in Cochrane and Saa-Requeio [34] and used for assets with high Sharpe ratio of the excess return and hence deemed desirable. The same terminology is used in Jaschke and Küchler [60]. Here, they define good deals of *first* and *second kind*. By relating their setting to ours as explained in Subsection 1.5.2, the absence of good deals of the first kind corresponds to our absence of acceptable deals, while the absence of good deals of second kind corresponds to $\mathcal{A} \cap \{Z \in \mathcal{M} \,:\, \pi(Z) < 0\} = \emptyset$. In Cherny [32], a good deal is defined as an attainable payoff (at zero cost) such that the monetary risk measure induced by $\mathcal{A}$ takes strictly negative valued on it, while in Černý and Hodges [29], good deals are defined as our acceptable deals (but their acceptance set cannot be identified with our since, e.g., it does not contain 0). Acceptable deals are called *acceptable opportunities* in Carr et al. [26]. The notion of a scalable acceptable deal is a direct extension of that of a *scalable arbitrage opportunity* introduced by Pennanen [76] and, in a frictionless setting, corresponds to the notion of a *scalable good deal* in Baes et al. [11].

(ii) In the classical interpretation, an arbitrage opportunity constitutes an anomaly in a well-functioning and transparent market because every rational agent will seek to exploit it thereby raising its demand until prices will also rise and the arbitrage opportunity will eventually vanish. This provides an economic foundation for assuming the absence of such opportunities. The situation is different when we consider acceptable deals because there may be no consensus across agents in the identification of a common criterion of acceptability.

Since $\mathcal{M}$, π and $\mathcal{A}$ are convex, recession analysis is the natural language for characterizing the concept of scalability, which is the content of the next proposition. We have already said that $\mathrm{rec}(\mathcal{M})$ and $\mathrm{rec}(\pi)$ coincide with $\mathcal{M}^\infty$ and π^∞ respectively, using the said topology on $\mathcal{S}$. Note that $\mathcal{A}^\infty$ is not well defined unless we fix a topology on $L^0(\mathbb{P})$.

Proposition 1.2.7. *A nonzero replicable payoff $X \in \mathcal{S}$ is:*

(i) a scalable arbitrage opportunity if and only if $X \in \mathcal{M}^\infty \cap L^0(\mathbb{P})_+$ and $\pi^\infty(X) \leq 0$.

(ii) a scalable acceptable deal (with respect to $\mathcal{A}$) if and only if $X \in \mathcal{M}^\infty \cap \mathrm{rec}(\mathcal{A})$ and $\pi^\infty(X) \leq 0$.

Proof. The statements are straightforward using Definition B.1.6 and thanks to the identifications $\mathrm{rec}(\mathcal{M}) = \mathcal{M}^\infty$ and $\mathrm{rec}(\pi) = \pi^\infty$. □

For later use, it is convenient to name the set of scalable acceptable deals together with the null payoff:

$$\mathcal{G} := \{Z \in \mathcal{M}^\infty \cap \operatorname{rec}(\mathcal{A}) \;:\; \pi^\infty(Z) \leq 0\} \subset \mathcal{S}. \tag{1.1}$$

As mentioned above, the absence of (scalable) acceptable deals will play a key role in our study of market-consistent prices. We conclude this section by highlighting a number of financially interesting situations where the absence of scalable acceptable deals automatically holds. The easy proof is omitted.

Proposition 1.2.8. *Assume that one of the following conditions holds:*

(i) $\operatorname{rec}(\mathcal{A}) = L^0(\mathbb{P})_+$ *and there exists no scalable arbitrage opportunity.*

(ii) $\mathcal{M}^\infty \subset \mathcal{S}_+$ *and there exists no scalable arbitrage opportunity.*

(iii) $\mathcal{M}^\infty = \{0\}$.

Then, there exists no scalable acceptable deal.

Remark 1.2.9 (On scalable acceptable deals)**.** We collect some observations about the above result.

(i) Unless $\mathcal{A}$ coincides with $L^0(\mathbb{P})_+$ in the first place, the condition $\operatorname{rec}(\mathcal{A}) = L^0(\mathbb{P})_+$ can hold only if $\mathcal{A}$ is not conic. We refer to Section 1.6 for concrete examples where this holds.

(ii) The condition $\mathcal{M}^\infty \subset \mathcal{S}_+$ is typically implied by portfolio constraints of limited short-selling type. For instance, if the payoffs of the basic securities are positive and the set of admissible portfolios is bounded from below so that short-selling is possible but restricted for each security, then the desired condition follows from Proposition 1.1.4.

(iii) The condition $\mathcal{M}^\infty = \{0\}$ is equivalent to the boundedness of $\mathcal{M}$ due to Proposition B.1.3. By linearity of V_1, this is also equivalent to the boundedness of the set of admissible portfolios $\mathcal{P}$. Note that, in this case, there exists no scalable acceptable deal regardless of the choice of $\mathcal{A}$ and π.

1.3 DIRECT CHARACTERIZATION OF MARKET-CONSISTENT PRICES

In this section we establish a first characterization of market-consistent prices, which we call "direct" to distinguish it from the "dual" characterization obtained in the next section. The key observation is that the set of market-consistent prices is an interval unbounded to the left. To obtain our desired characterization we therefore have to discover under which conditions the right extreme of the interval is itself a market-consistent price. This extreme coincides with the appropriate generalization to our setting of the classical superreplication price. In other words, we have to verify under which conditions the generalized superreplication price is a market-consistent price. In a frictionless market where the acceptance set is given by the positive cone this occurs if and only if the underlying payoff is replicable. In our general market setting this is no longer true.

We start by fixing a reference payoff space $\mathcal{X}$ that is assumed to contain all the replicable payoffs.

Assumption 1.3.1. We denote by $\mathcal{X}$ a linear subspace of $L^0(\mathbb{P})$ containing $\mathcal{S}$.

The next definition records the announced generalization to our setting of the classical superreplication price, which is also known in the literature under the name of superhedging price. For better comparability, we maintain the same terminology here.

Definition 1.3.2. For a payoff $X \in \mathcal{X}$ we define the *superreplication price of* X by

$$\pi^+(X) := \inf\{\pi(Z) \;:\; Z \in \mathcal{M},\; Z - X \in \mathcal{A}\}.$$

Remark 1.3.3 (From buyer to seller)**.** We will see below that the superreplication price constitutes the natural threshold for the market-consistent buyer prices. The corresponding threshold from a seller's perspective is given for every payoff $X \in \mathcal{X}$ by the subreplication price

$$\pi^-(X) = \sup\{-\pi(-Z) \;:\; Z \in -\mathcal{M},\; X - Z \in \mathcal{A}\}.$$

It is immediate to verify that $\pi^-(X) = -\pi^+(-X)$.

We collect some simple properties of π^+ in the next proposition.

Proposition 1.3.4. *The map* $\pi^+ : \mathcal{X} \to [-\infty, +\infty]$ *is convex and for any* $Z \in \mathcal{M}$ *we have that* $\pi^+(Z) \leq \pi(Z)$. *Moreover* $\pi^+(0) \leq 0$.

Proof. Convexity of π^+ is a consequence of the properties of $\mathcal{A}$, $\mathcal{M}$ and π. Now, for $Z \in \mathcal{M}$, we have that $\pi^+(Z) \leq \pi(Z)$ since $Z - Z = 0 \in \mathcal{A}$. Since by assumption $\pi(0) = 0$, the last assertion follows from the previous one. $\square$

We start by highlighting that the set of market-consistent prices is an interval that is unbounded to the left and bounded to the right by the superreplication price.

Proposition 1.3.5. *For every payoff* $X \in \mathcal{X}$ *the set* $\mathrm{MCP}(X)$ *is an interval such that:*

(i) $\inf \mathrm{MCP}(X) = -\infty$.

(ii) $\sup \mathrm{MCP}(X) = \pi^+(X)$.

Proof. It is clear that $(-\infty, p) \subset \mathrm{MCP}(X)$ for every market-consistent price $p \in \mathrm{MCP}(X)$. Now, take any $p \in (-\infty, \pi^+(X))$ and note that, by definition of π^+, we have $p < \pi(Z)$ for every $Z \in \mathcal{M}$ such that $Z - X \in \mathcal{A}$. This shows that p is a market-consistent price for X and implies that $\pi^+(X) \leq \sup \mathrm{MCP}(X)$. Conversely, take an arbitrary market-consistent price $p \in \mathrm{MCP}(X)$. If $Z \in \mathcal{M}$ is such that $Z - X \in \mathcal{A}$, then $\pi(Z) \geq p$. Taking the infimum over such Z's and the supremum over such p's delivers the inequality $\pi^+(X) \geq \sup \mathrm{MCP}(X)$. This shows that $\pi^+(X)$ is the supremum of the set $\mathrm{MCP}(X)$. $\square$

It follows from the preceding proposition that establishing a characterization of market-consistent prices is tantamount to establishing a characterization of when the superreplication price is itself a market-consistent price. The next theorem provides a solution to this problem. It turns out that the superreplication price may or may not be market consistent, so that the corresponding set of market-consistent prices may or may not be a closed interval. Figure 1.1 furnishes an intuition of the possibilities for attainable replicable payoffs.

Theorem 1.3.6 (Direct characterization of market consistent prices)**.** *Assume that* $X \in \mathcal{X}$ *is such that* $\pi^+(X) \in \mathbb{R}$. *Then, we have* $\mathrm{MCP}(X) \neq \emptyset$ *and the following statements hold:*

(i) $\pi^+(X) \notin \mathrm{MCP}(X)$ *if and only if* $\pi^+(X) = \pi(Z)$ *for some* $Z \in \mathcal{M}$ *such that* $Z - X \in \mathcal{A} \setminus \{0\}$, *if and only if*
$$(X + \mathcal{A}) \cap \{Z \in \mathcal{M} \, : \, \pi(Z) \leq \pi^+(X)\} \not\subset \{X\}. \tag{1.2}$$
In particular, in this case the infimum in the definition of $\pi^+(X)$ *is attained.*

(ii) $\pi^+(X) \in \mathrm{MCP}(X)$ *if and only if*
$$(X + \mathcal{A}) \cap \{Z \in \mathcal{M} \, : \, \pi(Z) \leq \pi^+(X)\} \subset \{X\}.$$

(iii) If $X \notin \mathcal{M}$, *then both* $\pi^+(X) \in \mathrm{MCP}(X)$ *and* $\pi^+(X) \notin \mathrm{MCP}(X)$ *can hold, the latter being equivalent to the attainment of the infimum in the definition of* $\pi^+(X)$.

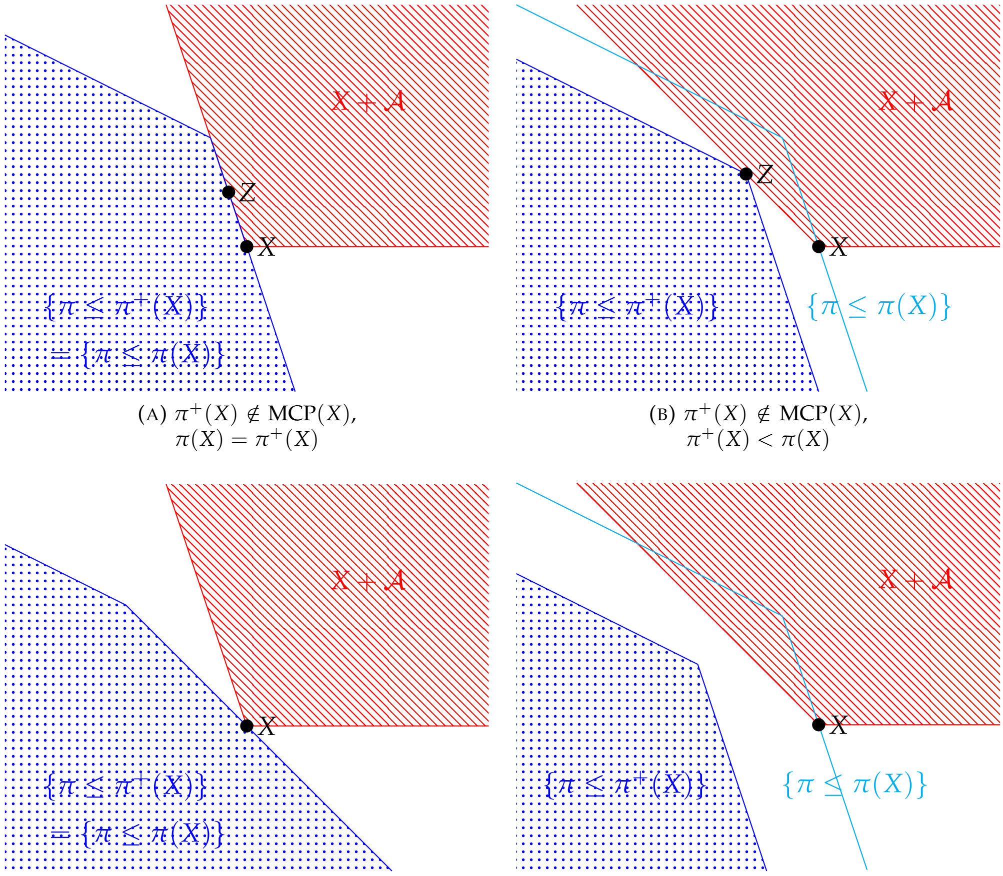
(iv) If $X \in \mathcal{M}$, *then both* $\pi^+(X) \in \mathrm{MCP}(X)$ *and* $\pi^+(X) \notin \mathrm{MCP}(X)$ *can hold, and in both situations either* $\pi^+(X) = \pi(X)$ *or* $\pi^+(X) < \pi(X)$ *can hold. Moreover the following statements hold:*

(a) $X \notin (X + \mathcal{A}) \cap \{Z \in \mathcal{M} \, : \, \pi(Z) \leq \pi^+(X)\}$ *if and only if* $\pi^+(X) < \pi(X)$.

(b) $X \in (X + \mathcal{A}) \cap \{Z \in \mathcal{M} \, : \, \pi(Z) \leq \pi^+(X)\}$ *if and only if* $\pi^+(X) = \pi(X)$.

Finally, if $\pi^+(X) \in MCP(X)$, *(a) holds if and only if the infimum in the definition of* $\pi^+(X)$ *is not attained, and (b) holds if and only if the infimum in the definition of* $\pi^+(X)$ *is attained.*

Proof. By assumption, $\pi^+(X) \in \mathbb{R}$, showing that $\mathrm{MCP}(X) \neq \emptyset$. Now, by definition of market consistent price and by the inequality $\pi^+(Z) \leq \pi(Z)$ for every $Z \in \mathcal{M}$, $\pi^+(X)$ is not market consistent if and only if there is $Z \in \mathcal{M}$ such that $Z - X \in \mathcal{A} \setminus \{0\}$ and $\pi(Z) \leq \pi^+(X)$. But, by definition of $\pi^+(X)$, the inequality $\pi^+(X) \leq \pi(Z)$ holds too, proving the first equivalence in *(i)* and showing that the infimum in the definition of $\pi^+(X)$ is attained at Z. Second equivalence in



(A) $\pi^+(X) \notin \mathrm{MCP}(X)$, $\pi(X) = \pi^+(X)$

(B) $\pi^+(X) \notin \mathrm{MCP}(X)$, $\pi^+(X) < \pi(X)$

(C) $\pi^+(X) \in \mathrm{MCP}(X)$, $\pi(X) = \pi^+(X)$

(D) $\pi^+(X) \in \mathrm{MCP}(X)$, $\pi^+(X) < \pi(X)$

FIGURE 1.1: Reciprocal positions of $X + \mathcal{A}$ and $\{Z \in \mathcal{M} \,:\, \pi(Z) \leq \pi^+(X)\}$ for $X \in \mathcal{M}$.

(i) is immediate after observing that the inequality in (1.2) can be replaced by equality. Statement *(ii)* is counternominal to statement *(i)*.

Now, assume that $X \notin \mathcal{M}$. Example 1.3.7 (and Examples 1.3.8 (ii) and (iii)) shows that both the situation may hold. To complete the proof of *(iii)*, it remains to show that if $\pi^+(X) = \pi(Z)$ for $Z \in \mathcal{M}$ such that $Z - X \in \mathcal{A}$, then $\pi^+(X)$ cannot be market consistent for X. But this is obvious by definition of $MCP(X)$ since $X \neq Z$.

Now, assume that $X \in \mathcal{M}$. Example 1.3.7 shows that all the enumerated possibilities may happen, except for $\pi^+(X) = \pi(X)$ if $\pi^+(X) \in \mathrm{MCP}(X)$. For this case, see Example 1.3.9 (i). (For other examples, see also Example 1.3.8 (i) for $\pi^+(X) < \pi(X)$ and $\pi^+(X) \in \mathrm{MCP}(X)$, Example 1.3.9 (i) for $\pi^+(X) < \pi(X)$ and $\pi^+(X) \notin \mathrm{MCP}(X)$, Example 1.3.9 (ii) for $\pi^+(X) = \pi(X)$ and $\pi^+(X) \notin \mathrm{MCP}(X)$). The equivalences in *(a)* and *(b)* are clear, and the last equivalences follow using point *(ii)* and the proof is concluded. $\square$

We show by way of example that each of the situations enumerated in the last theorem can hold. In the first example $\mathcal{A}$ is convex nonconic, while π and $\mathcal{M}$ are conic.

Example 1.3.7. *Let $\Omega = \{\omega_1, \omega_2\}$ and assume that $\mathcal{F}$ is the power set of Ω and that $\mathbb{P}$ is specified by $\mathbb{P}(\omega_1) = \mathbb{P}(\omega_2) = \frac{1}{2}$. In this setting, we take $\mathcal{X} = L^0(\mathbb{P})$ and identify every element of $\mathcal{X}$ with a vector of $\mathbb{R}^2$. Set $\mathcal{S} = \mathbb{R}^2$ and $\mathcal{M} = \mathbb{R}^2_+$ and consider the linear pricing rule $\pi(x,y) = x$ for every $(x,y) \in \mathcal{S}$, and acceptance set defined by*

$$\mathcal{A} = \{(x,y) \in \mathcal{X} \,:\, \mathbb{E}_{\mathbb{P}}[u(X)] \geq 0\},$$

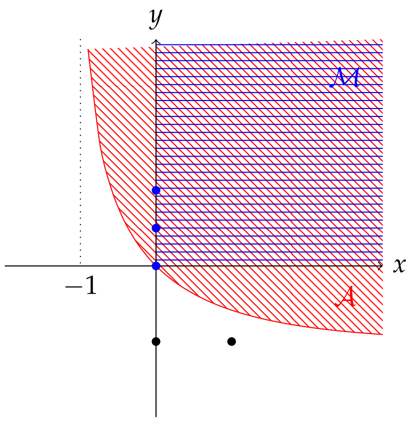


FIGURE 1.2: Example 1.3.7

where $u : \mathbb{R} \to \mathbb{R}$ *is the utility function defined by* $u(x) := 1 - e^{-x \log 2}$. *It is not difficult to show that*

$$\mathcal{A} = \left\{ (x,y) \in \mathcal{X} \,:\, x > -1,\ y \geq -\frac{\log(2 - e^{-x \log 2})}{\log 2} \right\}.$$

Note that the requirements in Assumption 1.3.1 are fulfilled.

For any $X = (x,y) \in \mathcal{X}$, *the superreplication price is*

$$\pi^+(X) = \begin{cases} 0 & \text{if } x < 1 \\ x - 1 & \text{if } x \geq 1 \end{cases}$$

and the infimum in the definition of $\pi^+(X)$ *is attained if and only if* $x < 1$.

- *For* $X = (0,-1) \notin \mathcal{M}$ *we have* $\pi^+(X) \notin \mathrm{MCP}(X)$ *since*
$$(X + \mathcal{A}) \cap \{Z \in \mathcal{M} \,:\, \pi(Z) \leq \pi^+(X)\} = \{(0,t) \in \mathbb{R}^2 \,:\, t \geq 0\} \not\subset \{X\}.$$
- *For* $X = (1,-1) \notin \mathcal{M}$ *we have* $\pi^+(X) \in \mathrm{MCP}(X)$ *since*
$$(X + \mathcal{A}) \cap \{Z \in \mathcal{M} \,:\, \pi(Z) \leq \pi^+(X)\} = \emptyset \subset \{X\}.$$
- *For* $X = (0,0) \in \mathcal{M}$ *we have* $\pi^+(X) = 0 = \pi(X)$ *and* $\pi^+(X) \notin \mathrm{MCP}(X)$ *since*
$$(X + \mathcal{A}) \cap \{Z \in \mathcal{M} \,:\, \pi(Z) \leq \pi^+(X)\} = \{(0,t) \in \mathbb{R}^2 \,:\, t \geq 0\} \not\subset \{X\}.$$
- *For* $X = \left(0, \frac{1}{2}\right) \in \mathcal{M}$ *we have* $\pi^+(X) = 0 < \frac{1}{2} = \pi(X)$ *and* $\pi^+(X) \notin \mathrm{MCP}(X)$ *since*
$$(X + \mathcal{A}) \cap \{Z \in \mathcal{M} \,:\, \pi(Z) \leq \pi^+(X)\} = \left\{ (0,t) \in \mathbb{R}^2 \,:\, t \geq -\frac{\log\left(2 - e^{-\frac{1}{2}\log 2}\right)}{\log 2} \right\} \not\subset \{X\}.$$
- *For* $X = (1,0) \in \mathcal{M}$ *we have* $\pi^+(X) = 0 < 1 = \pi(X)$ *and* $\pi^+(X) \in \mathrm{MCP}(X)$ *since*
$$(X + \mathcal{A}) \cap \{Z \in \mathcal{M} \,:\, \pi(Z) \leq \pi^+(X)\} = \emptyset \subset \{X\}.$$

In the next two examples we chose a conic acceptance set.

Example 1.3.8. *Consider the setting of the previous example. Set* $\mathcal{S} = \mathbb{R}^2$ *and consider the convex pricing rule* $\pi(x,y) = e^x - 1$ *for every* $(x,y) \in \mathcal{S}$, *and the conic acceptance set defined by*

$$\mathcal{A} = \{(x,y) \in \mathbb{R}^2 \,:\, y \geq \max\{-x, 0\}\}.$$

Note that the requirements in Assumption 1.3.1 are fulfilled.

(i) *Set* $\mathcal{M} = \mathcal{S} = \mathbb{R}^2$ *and* $X = (0,0) \in \mathcal{M}$. *Since* $\pi(X) = 0$ *and*

$$\pi^+(X) = \inf\{e^x - 1 \,:\, (x,y) \in \mathcal{A}\} = -1,$$

the infimum is not attained, $\pi^+(X) < \pi(X)$ *and* $(X + \mathcal{A}) \cap \{Z \in \mathcal{M} \,:\, \pi(Z) \leq \pi^+(X)\} = \emptyset$. *This shows that* $\pi^+(X) \in \mathrm{MCP}(X)$.

(ii) *Set* $\mathcal{M} = \{(x,y) \in \mathbb{R}^2 \,:\, y \geq 0\}$ *and* $X = (0,-1) \notin \mathcal{M}$. *It holds that the infimum*

$$\pi^+(X) = \inf\{e^x - 1 \,:\, (x,y) \in (X + \mathcal{A}) \cap \mathcal{M}\} = -1$$

is not attained, and $(X + \mathcal{A}) \cap \{Z \in \mathcal{M} \,:\, \pi(Z) \leq \pi^+(X)\} = \emptyset$. *This shows that* $\pi^+(X) \in \mathrm{MCP}(X)$.

(iii) *Set* $\mathcal{M} = \{(x,y) \in \mathbb{R}^2 \,:\, x \geq 0\}$ *and* $X = (-1,0) \notin \mathcal{M}$. *It holds that the infimum*

$$\pi^+(X) = \inf\{e^x - 1 \,:\, (x,y) \in (X + \mathcal{A}) \cap \mathcal{M}\} = \min\{e^x - 1 \,:\, x \geq 0\} = 0$$

is attained, and $(X + \mathcal{A}) \cap \{Z \in \mathcal{M} \,:\, \pi(Z) \leq \pi^+(X)\} = \{(0,t) \in \mathbb{R}^2 \,:\, t \geq 0\}$. *This shows that* $\pi^+(X) \notin \mathrm{MCP}(X)$.

Example 1.3.9. *Consider the setting of the previous examples. Set* $\mathcal{S} = \mathcal{M} = \mathbb{R}^2$ *and consider the acceptance set defined by*

$$\mathcal{A} = \{(x,y) \in \mathbb{R}^2 \,:\, y \geq \max\{-x, 0\}\}.$$

Note that both $\mathcal{A}$ *and* $\mathcal{M}$ *are cones and the requirements in Assumption 1.3.1 are fulfilled.*

(i) *Define the conic pricing rule* $\pi(x,y) = \max\{2x + y, x + 2y\}$ *for every* $(x,y) \in \mathbb{R}^2$. *A direct inspection shows that for every payoff in* $\mathcal{X}$ *the attainability condition* $\pi^+(X) = \pi(Z)$ *for* $Z \in (X + \mathcal{A}) \cap \mathcal{M}$ *is fulfilled. Set* $X = (-2,1) \in \mathcal{M}$ *and observe that* $\pi^+(X) = \pi(X) = 0$ *and*

$$(\mathcal{A} + X) \cap \{Z \in \mathcal{M} \,:\, \pi(Z) \leq \pi^+(X)\} = \{X\},$$

hence $\pi^+(X) \in \mathrm{MCP}(X)$. *Next, take* $Y = (1,-2) \in \mathcal{M}$. *In this case, an explicit calculation shows that*

$$\pi^+(Y) = \inf_{x \in \mathbb{R}} \max\{2x - 2 + \max\{1 - x, 0\}, x - 4 + 2\max\{1 - x, 0\}\} = -\frac{3}{2}.$$

At the same time, for $Z = (-\frac{1}{2}, -\frac{1}{2}) \in (\mathcal{A} + Y) \cap \mathcal{M}$ *we have* $\pi(Z) = -\frac{3}{2}$, *showing that* $(\mathcal{A} + Y) \cap \{Z \in \mathcal{M} \,:\, \pi(Z) \leq \pi^+(Y)\} \not\subset \{Y\}$, *hence* $\pi^+(Y) \notin \mathrm{MCP}(Y)$. *Note also that* $\pi(Y) = 0$ *so that* $\pi(Y) > \pi^+(Y)$.

(ii) *Define the conic pricing rule* $\pi(x,y) = \max\{x + y, x + 2y\}$ *for every* $(x,y) \in \mathbb{R}^2$. *A direct inspection shows that for every payoff in* $\mathcal{X}$ *the attainability condition* $\pi^+(X) = \pi(Z)$ *for* $Z \in (X + \mathcal{A}) \cap \mathcal{M}$ *is fulfilled. Set* $X = (1,-1) \in \mathcal{M}$ *and observe that* $\pi^+(X) = 0$ *and*

$$(\mathcal{A} + X) \cap \{Z \in \mathcal{M} \,:\, \pi(Z) \leq \pi^+(X)\} = \{tX \,:\, t \in [0,1]\}$$

so that $\pi^+(X) \notin \mathrm{MCP}(X)$. *Note also that* $\pi(X) = 0$ *so that* $\pi(X) = \pi^+(X)$.

Theorem 1.3.6 unveils a stark contrast between our general setting and the classical frictionless setting. In our case, for a replicable payoff X belonging to $\mathcal{M}$, the market price $\pi(X)$ may be strictly larger than the superreplication price $\pi^+(X)$. This fact is in line with the findings in Bensaid et al. [17], where the focus was on a multi-period Cox-Ross-Rubinstein model with proportional transaction costs and no portfolio constraints and the acceptance set was taken to be the positive cone. As explained in that paper, the counterintuitive inequality $\pi(X) > \pi^+(X)$ is, in fact, reasonable to expect because trading is costly and it may therefore «pay to weigh the benefits of replication against those of potential savings on transaction costs». What also follows from the previous result and was only implicitly highlighted in [17] is that, contrary to the frictionless case, for a replicable payoff X belonging to $\mathcal{M}$, both the market price $\pi(X)$ and the superreplication price $\pi^+(X)$ may fail to be market consistent! This is again triggered by transaction costs, which may allow the existence of replicable payoffs $Z \in \mathcal{M}$ satisfying $Z - X \in \mathcal{A} \setminus \{0\}$ and $\pi(Z) - \pi(X) \leq 0$ even if the market admits no acceptable deals.

Motivated by the above discussion, we provide sufficient conditions for the market price of a payoff in $\mathcal{M}$ to be market consistent and, hence, to coincide with the corresponding superreplication price. More precisely, we show that this holds for every payoff with "zero bid-ask spread" provided the market admits no acceptable deals.

Proposition 1.3.10. *If there exists no acceptable deal, then $\pi(X) = \pi^+(X) \in \mathrm{MCP}(X)$ for every replicable payoff $X \in \mathcal{M} \cap (-\mathcal{M})$ such that $\pi(-X) = -\pi(X)$. In particular, if $\mathcal{M}$ and π are linear and there exists no acceptable deal, then $\pi(X) = \pi^+(X) \in \mathrm{MCP}(X)$ for every replicable payoff $X \in \mathcal{M}$.*

Proof. Take an arbitrary $X \in \mathcal{M} \cap (-\mathcal{M})$ such that $\pi(-X) = -\pi(X)$. Since $\pi^+(X)$ is the supremum of the set $\mathrm{MCP}(X)$ and $\pi^+(X) \leq \pi(X)$, it suffices to show that $\pi(X) \in \mathrm{MCP}(X)$. To this effect, take any replicable payoff $Z \in \mathcal{M}$ satisfying $Z - X \in \mathcal{A} \setminus \{0\}$. Note that $\frac{1}{2}Z - \frac{1}{2}X = \frac{1}{2}(Z - X) + \frac{1}{2}0 \in \mathcal{A} \cap \mathcal{M}$. As a result, the absence of acceptable deals implies that

$$0 < \pi\left(\frac{1}{2}Z - \frac{1}{2}X\right) \leq \frac{1}{2}\pi(Z) + \frac{1}{2}\pi(-X) = \frac{1}{2}\pi(Z) - \frac{1}{2}\pi(X).$$

This yields $\pi(X) < \pi(Z)$ and proves that $\pi(X)$ is a market-consistent price for X. □

1.3.1 THE SET $\mathcal{C}$

The characterization of market-consistent prices established in Theorem 1.3.6 allows for different cases, but it may be simplified if the infimum in the definition of superreplication price is attained for every payoff with finite superreplication price. The remainder of this section is devoted to finding sufficient conditions for this attainability property to hold. In particular, we look for economically meaningful conditions involving the underlying financial primitives, namely the acceptance set $\mathcal{A}$, the pricing rule π, and the set of admissible replicable payoffs $\mathcal{M}$.

For technical reasons, we need to require that the restriction of the acceptance set to $\mathcal{X}$ is closed with respect to a fixed linear topology on $\mathcal{X}$. This implies that the natural choice $\mathcal{X} = L^0(\mathbb{P})$ is possible only if the chosen acceptance set is closed with respect to, e.g., the topology of convergence in probability. This strong closedness property is satisfied by the positive cone and few other acceptance sets but typically fails to hold. As a result, the choice of the space $\mathcal{X}$ will generally depend on the underlying acceptance set. Hence, together with Assumption 1.3.1, we require the following.

Assumption 1.3.11. We assume that $\mathcal{X}$ is equipped with a Hausdorff topology which makes this space a topological linear space, and that $\mathcal{A} \cap \mathcal{X}$ is closed with respect to the given topology.

Note that under Assumption 1.3.11, by virtue of Corollary B.1.7, the asymptotic and the recession cone of $\mathcal{A} \cap \mathcal{X}$ coincide,

$$(\mathcal{A} \cap \mathcal{X})^\infty = \mathrm{rec}(\mathcal{A} \cap \mathcal{X}) = \mathrm{rec}(\mathcal{A}) \cap \mathcal{X}.$$

Like for π and $\mathcal{M}$, we prefer the asymptotic notation better than the recession one.

A key role in our analysis will be played by the set

$$\mathcal{C} := \{(X, m) \in \mathcal{X} \times \mathbb{R} \, : \, \exists Z \in \mathcal{M} \text{ such that } Z - X \in \mathcal{A}, \; \pi(Z) \leq -m\}.$$

It is immediate to verify that the epigraph of π^+ is related to $\mathcal{C}$ as follows:

$$\mathrm{epi}(\pi^+) \supset \{(X, m) \, : \, (X, -m) \in \mathcal{C}\}.$$

Even though without further assumptions the opposite inclusion may fail, it turns out that in the classic representation of superreplication prices as $\pi^+(X) = \inf\{m \in \mathbb{R} \, : \, (X, m) \in \mathrm{epi}(\pi^+)\}$, up to a sign we can replace the epigraph of π^+ with the set $\mathcal{C}$. This is the content of the next proposition.

Proposition 1.3.12. *For every payoff $X \in \mathcal{X}$ the superreplication price of X can be expressed as*

$$\pi^+(X) = \inf\{m \in \mathbb{R} \, : \, (X, -m) \in \mathcal{C}\}.$$

Proof. For every $m \in \mathbb{R}$ we have $(X, -m) \in \mathcal{C}$ if and only if there exists a replicable payoff $Z \in \mathcal{M}$ such that $Z - X \in \mathcal{A}$ and $\pi(Z) \leq m$. As a result, we get

$$\begin{aligned}\pi^+(X) &= \inf\{\pi(Z) \,:\, Z \in \mathcal{M},\ Z - X \in \mathcal{A}\}\\ &= \inf\{m \in \mathbb{R} \,:\, \exists Z \in \mathcal{M} \text{ such that } Z - X \in \mathcal{A},\ \pi(Z) \leq m\}\\ &= \inf\{m \in \mathbb{R} \,:\, (X, -m) \in \mathcal{C}\}.\end{aligned}$$

This establishes the desired equality. □

The preceding representation of superreplication prices suggests that a strategy to tackle the attainability problem is to look for conditions ensuring that the set $\mathcal{C}$ is closed (with respect to the natural product topology on $\mathcal{X} \times \mathbb{R}$, where $\mathcal{X}$ is equipped with the topology fixed at the beginning of this section and $\mathbb{R}$ with its canonical Borel topology). This is the content of our next result, which will play a critical role in the next section as well. Before, we need a preparatory lemma. Recall that the set $\mathcal{G}$ is defined in (1.1) and coincides with the set of scalable acceptable deals together with the null payoff. In the case that $\mathcal{G}$ is a linear subspace of $\mathcal{S}$, we denote by $\mathcal{G}_{\mathcal{S}}^{\perp}$ its orthogonal complement in $\mathcal{S}$, with reference to a fixed norm that induces the unique Hausdorff topology making $\mathcal{S}$ a topological vector space.

Lemma 1.3.13. *If $\mathcal{G}$ is a linear space, then for any $(X, m) \in \mathcal{C}$ there is $Z \in \mathcal{G}_{\mathcal{S}}^{\perp}$ such that $\pi(Z) \leq -m$ and $Z - X \in \mathcal{A}$.*

Proof. For any $(X, m) \in \mathcal{C}$, we find $Y \in \mathcal{M}$ such that $\pi(Y) \leq -m$ and $Y - X \in \mathcal{A}$. Define Y_0 as the orthogonal projection of Y on $\mathcal{G}$ and $Z := Y - Y_0 \in \mathcal{G}_{\mathcal{S}}^{\perp}$ (the operation of projecting is meant in the finite dimensional space $\mathcal{S}$). Now, by B.2.5, we have that $\pi(Z) = \pi(Y - Y_0) \leq \pi(Y) \leq -m$ as $-Y_0 \in \mathcal{G}$ and so $\pi^\infty(-Y_0) \leq 0$. Moreover, $Z - X = (Y - X) - Y_0 \in \mathcal{A}$ by virtue of Proposition B.1.5 since $Y - X \in \mathcal{A} \cap \mathcal{X}$ and $-Y_0 \in \text{rec}(\mathcal{A}) \cap \mathcal{X} = (\mathcal{A} \cap \mathcal{X})^\infty$. This concludes the proof. □

Theorem 1.3.14. *If the set $\mathcal{G}$ is a linear space, then $\mathcal{C}$ is closed and $(0, n) \notin \mathcal{C}$ for some $n \in \mathbb{N}$. In particular, the same thesis hold if there exists no scalable acceptable deal.*

Proof. We first establish closedness. To this end, take a net $(X_\alpha, m_\alpha) \subset \mathcal{C}$ indexed on the directed set $(A, \succeq)$ and a point $(X, m) \in \mathcal{X} \times \mathbb{R}$ and assume that $(X_\alpha, m_\alpha) \to (X, m)$. By Lemma 1.3.13, we find a net $(Z_\alpha) \subset \mathcal{G}_{\mathcal{S}}^{\perp}$ such that $Z_\alpha - X_\alpha \in \mathcal{A}$ and $\pi(Z_\alpha) \leq -m_\alpha$ for every $\alpha \in A$. Now, suppose that (Z_α) has no convergent subnet, and denote by $\|\cdot\|$ a norm on $\mathcal{S}$ which induces the unique Hausdorff topology which makes $\mathcal{S}$ a topological vector space. In this case, we find a subnet of (Z_α) consisting of nonzero elements with strictly-positive diverging norms. (Indeed, it suffices to consider the index set $B = \{(\alpha, n) \,:\, \alpha \in A,\ n \in \mathbb{N},\ \|Z_\alpha\| > n\}$ equipped with the direction defined by $(\alpha, n) \succeq (\beta, m)$ if and only if $\alpha \succeq \beta$ and $m \geq n$ and take $Z_{(\alpha,n)} = Z_\alpha$ for every $(\alpha, n) \in B$). We still denote this subnet by (Z_α) for convenience. Since the unit sphere in $\mathcal{S}$ is compact, we can assume without loss of generality that

$$\frac{Z_\alpha}{\|Z_\alpha\|} \to Z$$

for a suitable nonzero $Z \in \mathcal{M}^\infty$ by definition of asymptotic cone. Clearly, Z belongs to $\mathcal{G}_{\mathcal{S}}^{\perp}$ too. As (X_α) is a convergent net by assumption,

$$\frac{Z_\alpha - X_\alpha}{\|Z_\alpha\|} \to Z.$$

This implies that $Z \in (\mathcal{A} \cap \mathcal{X})^\infty$ again by definition of asymptotic cone. We claim that $\pi^\infty(Z) \leq 0$. Otherwise, we must find $\lambda > 0$ such that $\pi(\lambda Z) > 0$. Without loss of generality we may assume that $\|Z_\alpha\| > \lambda$ for every $\alpha \in A$. Since (m_α) is a convergent net, we can use the lower semicontinuity and convexity of π to get

$$0 < \pi(\lambda Z) \leq \liminf_\alpha \pi\left(\frac{\lambda Z_\alpha}{\|Z_\alpha\|}\right) \leq \liminf_\alpha \frac{\lambda \pi(Z_\alpha)}{\|Z_\alpha\|} \leq \liminf_\alpha \frac{-\lambda m_\alpha}{\|Z_\alpha\|} = 0. \tag{1.3}$$

This shows that $\pi^\infty(Z) \leq 0$ must hold. As a result, it follows that Z is a scalable acceptable deal which belongs to $\mathcal{G}_{\mathcal{S}}^{\perp}$, which is impossible. To avoid this contradiction, the net (Z_α) must admit a convergent subnet, which we still denote by (Z_α) for convenience. By closedness of $\mathcal{M}$, the limit Z

also belongs to $\mathcal{M}$. As we clearly have $Z_\alpha - X_\alpha \to Z - X$, it follows that $Z - X \in \mathcal{A}$ by closedness of $\mathcal{A} \cap \mathcal{X}$. Moreover,

$$\pi(Z) \leq \liminf_\alpha \pi(Z_\alpha) \leq \liminf_\alpha -m_\alpha = -m$$

by lower semicontinuity and convexity of π. This shows that $(X, m) \in \mathcal{C}$ and establishes that $\mathcal{C}$ is closed.

As a second step, we show that $(0, n) \notin \mathcal{C}$ for some $n \in \mathbb{N}$. To this effect, assume to the contrary that for every $n \in \mathbb{N}$ there exists $Z_n \in \mathcal{A} \cap \mathcal{G}_\mathcal{S}^\perp$ such that $\pi(Z_n) \leq -n$. If the sequence (Z_n) is bounded, then we may assume without loss of generality that $Z_n \to Z$ for some $Z \in \mathcal{A} \cap \mathcal{M}$. The lower semicontinuity of π implies $\pi(Z) \leq \liminf_{n\to\infty} \pi(Z_n) = -\infty$, which cannot hold. Hence, the sequence (Z_n) must be unbounded. As argued above, we find a suitable subsequence, which we still denote by (Z_n), that has strictly-positive divergent norms satisfying $\frac{Z_n}{\|Z_n\|} \to Z$ for some nonzero $Z \in (\mathcal{A} \cap \mathcal{X})^\infty \cap \mathcal{M}^\infty$. We claim that $\pi^\infty(Z) \leq 0$. Otherwise, we must find $\lambda > 0$ such that $\pi(\lambda Z) > 0$. Without loss of generality we may assume that $\|Z_n\| > \lambda$ for every $n \in \mathbb{N}$. The lower semicontinuity and convexity of π imply

$$0 < \pi(\lambda Z) \leq \liminf_{n\to\infty} \pi\left(\frac{\lambda Z_n}{\|Z_n\|}\right) \leq \liminf_{n\to\infty} \frac{\lambda \pi(Z_n)}{\|Z_n\|} \leq \liminf_{n\to\infty} \frac{-\lambda n}{\|Z_n\|} \leq 0. \tag{1.4}$$

This shows that $\pi^\infty(Z) \leq 0$ must hold. As a result, it follows that Z is a scalable acceptable deal, contradicting the fact that $Z \in \mathcal{G}_\mathcal{S}^\perp$. To avoid this contradiction, we must have $(0, n) \notin \mathcal{C}$ for some $n \in \mathbb{N}$.

Finally, if there are no scalable acceptable deals, then $\mathcal{G} = \{0\}$ and hence it is a linear space. $\square$

Remark 1.3.15. (i) Closedness of $\mathcal{C}$ in the case that $\mathcal{G}$ is a linear space may also be obtained by applying a generalization of the famous Dieudonné theorem about the closure of the difference of convex closed sets. Indeed, $\mathcal{C}$ can be equivalently written as

$$\mathcal{C} = \text{hypo}(-\pi) - ((\mathcal{A} \cap \mathcal{X}) \times \mathbb{R}_+),$$

and the hypograph of $-\pi$ is finite dimensional, providing the local compactness needed for this type of results. Closedness of $\mathcal{C}$ is ensured by the original Dieudonné theorem in Dieudonné [39] if $\mathcal{G} = \{0\}$ (see also Theorem B.1.9 in Appendix B), and by its generalization provided e.g. by Theorem 1.1.8 in Zălinescu [95] if $\mathcal{G}$ is a vector space.

(ii) Consider the case where $\mathcal{M} = \mathcal{S}$ and V_0 is linear on $\mathbb{R}^N$ (and consequently π is linear on $\mathcal{S}$), which correspond to a perfectly liquid market. Moreover, assume that there $U \in \mathcal{X}_+ \cap \mathcal{M}$ is such that $\pi(U) = 1$. Since in this case by linearity $\mathcal{C}$ can be reduced to

$$\mathcal{C} = \{(X, m) \in \mathcal{X} \times \mathbb{R} \, : \, mU + X \in \ker(\pi) - (\mathcal{A} \cap \mathcal{X})\},$$

it is clear that $\mathcal{C}$ is closed in the product space $\mathcal{X} \times \mathbb{R}$ if and only if $\ker(\pi) - (\mathcal{A} \cap \mathcal{X})$ is closed in $\mathcal{X}$. If $\mathcal{A}$ is taken to be the positive cone $L^0(\mathbb{P})_+$, this difference coincides with the set of superreplicable claim at zero cost. Thus the problem of the closure of $\mathcal{C}$ is nothing else than a generalization of the problem of the closedness of the cone of superreplicable claims at zero cost in frictionless markets, which is the hard core of the proofs of the fundamental theorems of asset pricing.

(iii) We have already said that the absence of scalable acceptable deals is interpreted as the absence of payoff any size of which is attractive and is available on the market at zero (or negative) cost. Similarly, one could wonder how to interpret the assumption "$\mathcal{G}$ is a linear space". Apart from being a nice mathematical generalization, this condition can be related to the absence of some type of *strong* scalable acceptable deals. Indeed if $\mathcal{G}$ is linear, every scalable acceptable deal X is such that $\pi^\infty(X) = \pi^\infty(-X) = 0$, and hence also $\pi(X) = -\pi(X) = 0$. Thus its price cannot become strictly negative.

The next example shows a case where $\mathcal{C}$ is not closed, and a case where $\mathcal{C}$ is closed even if $\mathcal{G}$ is not linear, the latter proving that the converse of the closedness result in Theorem 1.3.14 does not hold.

Example 1.3.16. *Let $\Omega = \{\omega_1, \omega_2, \omega_3\}$ and assume that $\mathcal{F}$ is the power set of Ω and that $\mathbb{P}$ is specified by $\mathbb{P}(\omega_i) = \frac{1}{3}$. We take $\mathcal{X} = L^0(\mathbb{P})$ and identify every element of $L^0(\mathbb{P})$ with a vector of $\mathbb{R}^3$. Let $\mathcal{M}$*

coincide with $\mathcal{S} = \{(x,y,z) \in \mathbb{R}^3 \,:\, x = 0\}$ *and let* $\pi : \mathcal{M} \to \mathbb{R}$ *be defined as* $\pi\big((0,y,z)\big) = y$. *Note that* $\mathcal{M} = \mathcal{M}^\infty$, $\pi = \pi^\infty$ *and* $\ker(\pi) = \operatorname{span}\{(0,0,1)\}$.

(i) *Consider the closed convex conic acceptance set*

$$\mathcal{A} = \Big\{(x,y,z) \in \mathbb{R}^3\,;\, x^2 + y^2 + 6xy + 2\sqrt{6}xz + 2\sqrt{6}yz \geq 0,\ \sqrt{3}x + \sqrt{3}y + \sqrt{2}z \geq 0\Big\},$$

obtained by rotating the cone $\mathcal{A}' = \{(x,y,z) \in \mathbb{R}^3 \,:\, x^2 + y^2 \leq 3z^2,\ z \geq 0\}$ *by* $\pi/3$ *around the direction* $(-1,1,0)$. *Clearly,* $\mathcal{A} = \mathcal{A} \cap \mathcal{X}$ *and* $\mathcal{A} = \mathcal{A}^\infty$. *We show that* $\mathcal{C}$ *is not closed. For every* $n \in \mathbb{N}$ *define* $X_n = \left(1 - \frac{1}{n}, -1, 0\right)$ *and note that* $(X_n, 0) \in \mathcal{C}$ *because* $Z_n = (0,0,n^2) \in \mathcal{M}$ *satisfies* $\pi(Z_n) = 0$ *and* $Z_n - X_n \in \mathcal{A}$. *Clearly, we have* $(X_n, 0) \to (X, 0)$ *with* $X = (1,-1,0)$. *However, a simple verification shows that* $(X,0)$ *does not to belong to* $\mathcal{C}$*, proving that* $\mathcal{C}$ *is not closed.*

(ii) *Let* $\mathcal{A}$ *(and hence* $\mathcal{A} \cap \mathcal{X}$ *and* $\mathcal{A}^\infty$*) coincide with the positive cone* $\mathbb{R}^3_+$. *In this case* $\mathcal{G} = \{(0,0,t) \in \mathbb{R}^3 \,:\, t > 0\}$, *which is not a linear space, and* $\ker(\pi) - \mathcal{A} = \{(x,y,z) \in \mathbb{R}^3 \,:\, x \leq 0,\ y \leq 0\}$. *From Remark 1.3.15 (ii) we have that* $\mathcal{C}$ *is closed.*

Thanks to Theorem 1.3.14 and the representation of π^+ in terms of $\mathcal{C}$, we can provide sufficient conditions for the infimum in the definition of $\pi^+(X)$ to be attained. This is the content of the next proposition.

Proposition 1.3.17. *If* $\mathcal{G}$ *is linear, then* π^+ *is a proper lower semicontinuous function and for every* $X \in \mathcal{X}$ *with* $\pi^+(X) < \infty$ *there exists a replicable payoff* $Z \in \mathcal{M}$ *such that* $Z - X \in \mathcal{A}$ *and* $\pi^+(X) = \pi(Z)$. *In particular, the same holds if there are no scalable acceptable deals.*

Proof. In Proposition 1.3.4 we have shown that π^+ is convex. Moreover, since $\mathcal{C}$ is closed by Theorem 1.3.14, thanks to the representation in Proposition 1.3.12 and Proposition A.1.8, π^+ is lower semicontinuous. Since $(0,n) \notin \mathcal{C}$ for some $n \in \mathbb{N}$ by Theorem 1.3.14, it follows that $\pi^+(0) > -\infty$. As a result, π^+ is finite at 0, as $\pi^+(0) \leq 0$ by Proposition 1.3.4 Being convex and lower semicontinuous, the map π^+ can never attain the value $-\infty$ on the space $\mathcal{X}$. Indeed, if $\pi^+(X) = -\infty$ for some $X \in \mathcal{X}$, then we readily see that $\pi^+(\lambda X) = -\infty$ for every $\lambda \in (0,1)$ by convexity. However, this implies $\pi^+(0) \leq \liminf_{\lambda \downarrow 0} \pi^+(\lambda X) = -\infty$ by lower semicontinuity, which cannot hold. This shows that π^+ is proper. Now, for every $X \in \mathcal{X}$ with $\pi^+(X) < \infty$, we have $\pi^+(X) \in \mathbb{R}$. Thanks to the representation obtained in Proposition 1.3.12 and the closedness of $\mathcal{C}$, the couple $(X, -\pi^+(X))$ belongs to $\mathcal{C}$, hence we find $Z \in \mathcal{M}$ such that $\pi(Z) = \pi^+(X)$ and $Z - X \in \mathcal{A}$. $\square$

The next result is just an application of the direct characterization of market consistent prices (Theorem 1.3.6) under assumptions ensuring that the attainability in the definition of π^+ is satisfied.

Corollary 1.3.18 (Direct characterization of market consistent prices)**.** *Assume that* $\mathcal{G}$ *is linear or there are no scalable acceptable deals. Then for every* $X \in \mathcal{X}$ *such that* $\pi^+(X) < \infty$ *we have* $\mathrm{MCP}(X) \neq \emptyset$ *and the following statements hold:*

1. *If* $X \notin \mathcal{M}$, *then* $\pi^+(X) \notin \mathrm{MCP}(X)$.
2. *If* $X \in \mathcal{M}$, *then both* $\pi^+(X) \in \mathrm{MCP}(X)$ *and* $\pi^+(X) \notin \mathrm{MCP}(X)$ *can hold. Moreover:*
   (a) *If* $\pi^+(X) \in \mathrm{MCP}(X)$, *then* $\pi^+(X) = \pi(X)$.
   (b) *If* $\pi^+(X) \notin \mathrm{MCP}(X)$, *then both* $\pi^+(X) = \pi(X)$ *and* $\pi^+(X) < \pi(X)$ *can hold.*

1.4 Dual characterization of market-consistent prices

In this section we establish a dual characterization of market-consistent prices based on a general version of the Fundamental Theorem of Asset Pricing in the setting of pricing with acceptable risk. As done in the previous section, we fix a reference payoff space and we assume that throughout this section Assumptions 1.3.1 and 1.3.11 hold. For the time being, we do not need to equip this space with any further special structure. Later on, to be able to apply duality theory, we will have to consider a convenient topology.

1.4.1 Pricing densities

Dual characterization of market-consistent prices will be expressed in terms of suitable dual elements, called pricing densities, that generalize the classical stochastic discount factors in frictionless markets. Here we introduce these dual elements and we investigate their existence's impact on (scalable) acceptable deals.

Definition 1.4.1. A random variable $D \in L^0(\mathbb{P})$ is called a *pricing density* if the following conditions hold:

(1) $DX \in L^1(\mathbb{P})$ for every replicable payoff $X \in \mathcal{S}$.

(2) $\sup\{\mathbb{E}_{\mathbb{P}}[DX] - \pi(X) \,:\, X \in \mathcal{M}\} < \infty$.

In addition, we say that a pricing density $D \in L^0(\mathbb{P})$ is:

(3) *consistent (with $\mathcal{A}$ and $\mathcal{X}$)* if $\inf\{\mathbb{E}_{\mathbb{P}}[DX] \,:\, X \in \mathcal{A} \cap \mathcal{X}\} > -\infty$.

(4) *strictly consistent (with $\mathcal{A}$ and $\mathcal{X}$)* if $\mathbb{E}_{\mathbb{P}}[DX] > 0$ for every nonzero payoff $X \in \mathcal{A} \cap \mathcal{X}$.

Remark 1.4.2. Functionals that "extend" the market and are consistent with the acceptance set appear under different names and specifications in the literature about pricing with acceptable risk. In Jaschke and Küchler [60], they are called *consistent price systems* and correspond to consistent pricing densities. Being their model conic, the supremum in (2) and the infimum in (3) are zero (see Subsection 1.5.2 for a comparison with their setting). In Staum [91], *consistent pricing kernels* are consistent pricing densities with the same supremum and infimum zero, which additionally are strictly positive, in the sense that belong to $\mathcal{X}'_{++}$. Cherny [32] works with *risk neutral measures*, that are probabilities $\mathbb{Q}$ absolutely continuous with respect to $\mathbb{P}$ such that $\mathbb{E}_{\mathbb{Q}}[X] \leq 0$ for every *attainable P&L X* and such that are consistent with $\mathcal{A}$, with infimum in (3) equal to zero due to conicity of $\mathcal{A}$. In Černý and Hodges, the market is perfectly liquid, hence the so called *no-good deal pricing functionals* equal π on $\mathcal{S}$, and moreover are strictly consistent with $\mathcal{A}$. Note that their acceptance set, though, cannot be identified with our as it does not contain 0. Also the setting of Carr et al. [26] is frictionless and unconstrained. Here *state price functions* correspond, up to a normalization, with strictly positive pricing densities, and *representative state pricing function* correspond to pricing densities strictly consistent with $\mathrm{cone}(\mathcal{A})$.

To illustrate the financial interpretation of the above notion and its connection to stochastic discount factors, consider a pricing density $D \in L^0(\mathbb{P})$ and define the vector space $\mathcal{L} = \{X \in L^0(\mathbb{P}) \,:\, DX \in L^1(\mathbb{P})\}$. Note that $\mathcal{L}$ contains $\mathcal{S}$. Moreover, define the linear functional $\psi : \mathcal{L} \to \mathbb{R}$ by

$$\psi(X) = \mathbb{E}_{\mathbb{P}}[DX].$$

By definition of a pricing density, there exists a constant $\gamma_{\pi,\mathcal{M}} \geq 0$ such that for every replicable payoff $X \in \mathcal{M}$ we have $\psi(X) \leq \pi(X) + \gamma_{\pi,\mathcal{M}}$. In particular, for every replicable payoff $X \in \mathcal{M} \cap (-\mathcal{M})$

$$-\pi(-X) - \gamma_{\pi,\mathcal{M}} \leq \psi(X) \leq \pi(X) + \gamma_{\pi,\mathcal{M}}.$$

The functional ψ can therefore be viewed as the pricing rule of an "artificial" frictionless market where every payoff in $\mathcal{L}$ is "replicable" and prices for the admissible replicable payoffs respect, up to a suitable enlargement, the bid-ask spread associated to their "true" market prices. If D is such that $\gamma_{\pi,\mathcal{M}} = 0$, no enlargement is needed. This happen for instance when

$$\sup_{X \in \mathrm{cone}(\mathcal{M})} \{\mathbb{E}_{\mathbb{P}}[DX] - \mathrm{cone}(\pi)(X)\} < \infty.$$

In this case, D qualifies as a pricing density for the "conified" market with conic pricing rule $\mathrm{cone}(\pi)$ and conic set of admissible payoffs $\mathrm{cl}(\mathrm{cone}(\mathcal{M}))$. This is automatically satisfied if both π and $\mathcal{M}$ are conic in the first place. In particular, this holds if the pricing rule π is linear and $\mathcal{M} = \mathcal{S}$, in which case ψ is a genuine extension of π beyond the space of replicable payoffs. This is in line with the classical interpretation of stochastic discount factors in frictionless markets.

The consistency of D implies that prices of acceptable payoffs in the "artificial" frictionless market with pricing rule ψ are bounded from below by a constant $\gamma_{\mathcal{A}} \leq 0$ so that for every $X \in \mathcal{A} \cap \mathcal{X} \cap \mathcal{L}$

$$\psi(X) \geq \gamma_{\mathcal{A}}.$$

A simple situation where these "artificial" prices are nonnegative is when

$$\inf_{X\in\mathrm{cone}(\mathcal{A})\cap\mathcal{X}} \mathbb{E}_{\mathbb{P}}[DX] > -\infty.$$

In this case, D qualifies as a consistent pricing density with respect to the conic acceptance set $\mathrm{cone}(\mathcal{A})$. This is clearly satisfied if $\mathcal{A}$ is a cone in the first place. In particular, if $\mathcal{A}$ is taken to be the positive cone, then (strict) consistency boils down to the (strict) positivity of ψ.

We summarize the above discussion in the following proposition, which highlights the role of conicity in simplifying the formulation of a consistent pricing density.

Proposition 1.4.3. *Let $D \in L^0(\mathbb{P})$ be a pricing density. Then, the following statements hold:*

(i) $\mathbb{E}_{\mathbb{P}}[DX] \leq \pi(X)$ *for every $X \in \mathcal{M}^\infty$ such that π is conic on* $\mathrm{cone}(X)$.

(ii) $\mathbb{E}_{\mathbb{P}}[DX] = \pi(X)$ *for every $X \in \mathcal{M}^\infty \cap (-\mathcal{M})^\infty$ such that π is linear on* $\mathrm{span}(X)$.

If D is consistent, then the following statement holds:

(iii) $\mathbb{E}_{\mathbb{P}}[DX] \geq 0$ *for every* $X \in \mathrm{rec}(\mathcal{A}) \cap \mathcal{X}$.

Proof. Let $D \in L^0(\mathbb{P})$ be a pricing density and take an arbitrary $X \in \mathcal{X}$. Since $\mathrm{span}(X) = \mathrm{cone}(X) \cup \mathrm{cone}(-X)$, it is clear that *(i)* implies *(ii)*. To prove *(i)*, assume that $X \in \mathcal{M}^\infty$ and π is conic on $\mathrm{cone}(X)$. Then, by definition of a pricing density,

$$\sup_{n\in\mathbb{N}}\{n(\mathbb{E}_{\mathbb{P}}[DX] - \pi(X))\} = \sup_{n\in\mathbb{N}}\{\mathbb{E}_{\mathbb{P}}[D(nX)] - \pi(nX)\} < \infty.$$

This is only possible if $\mathbb{E}_{\mathbb{P}}[DX] - \pi(X) \leq 0$, showing the desired claim. Finally, to establish *(iii)*, assume that D is consistent and $X \in \mathrm{rec}(\mathcal{A})$. Then, by definition of consistency,

$$\inf_{n\in\mathbb{N}}\{n\mathbb{E}_{\mathbb{P}}[DX]\} = \inf_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}[D(nX)] > -\infty.$$

This is only possible if $\mathbb{E}_{\mathbb{P}}[DX] \geq 0$, proving the desired claim and concluding the proof. □

Remark 1.4.4 (From pricing densities to risk-neutral probabilities)**.** In a market where some admissible replicable payoff is frictionless, every positive pricing density can be equivalently represented in terms of a probability measure. To see this, let $D \in L^0(\mathbb{P})$ be a (strictly) positive pricing density and consider a strictly-positive payoff $U \in \mathcal{M}^\infty \cap (-\mathcal{M})^\infty$ such that π is linear along $\mathrm{span}(U)$ and satisfies $\pi(U) > 0$. It follows from the preceding proposition that $\mathbb{E}_{\mathbb{P}}[DU] = \pi(U)$. Then, we find a probability measure $\mathbb{Q}$ that is absolutely continuous with (equivalent to) $\mathbb{P}$ and satisfies $\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{DU}{\pi(U)}$. In this case,

$$\frac{\mathbb{E}_{\mathbb{P}}[DX]}{\pi(U)} = \mathbb{E}_{\mathbb{Q}}\left[\frac{X}{U}\right]$$

for every $X \in L^0(\mathbb{P})$ such that $DX \in L^1(\mathbb{P})$. This shows that the action of a pricing density can be equivalently formulated in terms of an expectation under $\mathbb{Q}$ applied to payoffs expressed in units of the reference payoff U. The probability $\mathbb{Q}$ thus plays the role of an (equivalent) risk-neutral probability from the classical frictionless theory.

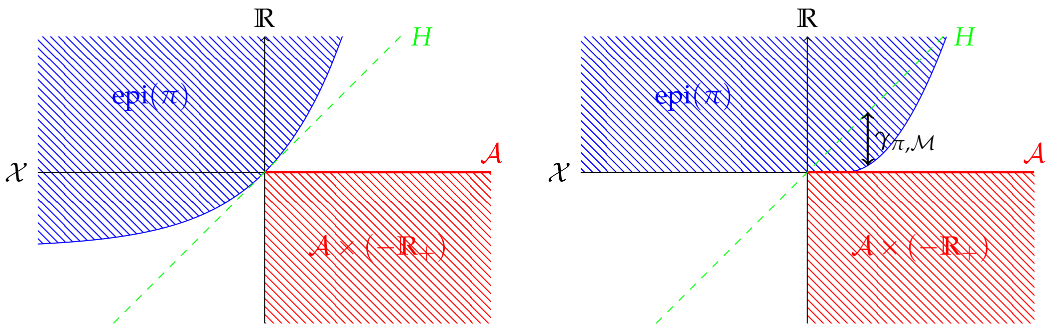
Now we try to highlight the link between the existence of strictly-consistent pricing densities and the absence of acceptable deals. Before stating our result, we furnish a geometric intuition, with reference to Figure 1.3. If D is a strictly consistent pricing density, by definition we have

$$\gamma_{\mathcal{A}} := \inf_{X\in\mathcal{A}\cap\mathcal{X}} \mathbb{E}_{\mathbb{P}}[DX] = 0 \quad \text{and} \quad \mathbb{E}_{\mathbb{P}}[DX] > 0 \text{ for } X \in \mathcal{A}\cap\mathcal{X}\setminus\{0\} \tag{1.5}$$

$$\gamma_{\pi,\mathcal{M}} := \sup_{X\in\mathcal{M}} \{\mathbb{E}_{\mathbb{P}}[DX] - \pi(X)\} < \infty. \tag{1.6}$$

In the product space $\mathcal{X} \times \mathbb{R}$, we consider the hyperplane

$$H := \{(X, m) \in \mathcal{X} \times \mathbb{R} \,:\, m = \mathbb{E}_{\mathbb{P}}[DX]\}.$$



(A) $\gamma_{\pi,\mathcal{M}} = 0$ (B) $\gamma_{\pi,\mathcal{M}} > 0$

FIGURE 1.3: Strictly consistent pricing densities and acceptable deals

From (1.5), it is clear that

$$(\mathcal{A} \cap \mathcal{X}) \times (-\mathbb{R}_+) \setminus \{(0,0)\} \subset H^{--} := \{(X,m) \in \mathcal{X} \times \mathbb{R} \,:\, m < \mathbb{E}_{\mathbb{P}}[DX]\}.$$

If $\gamma_{\pi,\mathcal{M}} = 0$, by virtue of (1.6) it also holds that

$$\text{epi}(\pi) \subset H^{+} := \{(X,m) \in \mathcal{X} \times \mathbb{R} \,:\, m \geq \mathbb{E}_{\mathbb{P}}[DX]\}.$$

Since $H^+ \cap H^{--} = \emptyset$, whenever $\gamma_{\pi,\mathcal{M}} = 0$ we have that $(\mathcal{A} \cap \mathcal{X}) \times (-\mathbb{R}_+) \cap \text{epi}(\pi) = \{(0,0)\}$, which is equivalent to the absence of acceptable deals. But if $\gamma_{\pi,\mathcal{M}} > 0$, then the inclusion $\text{epi}(\pi) \subset H^+$ fails, and it is not possible to conclude right away that the model has no acceptable deals.

The next proposition shows that the market is always free of scalable acceptable deals whenever a strictly-consistent pricing density exists. As we have anticipated, under the additional assumption $\gamma_{\pi,\mathcal{M}} = 0$, the market is also free of acceptable deals.

Proposition 1.4.5. *(i) If there exists a strictly-consistent pricing density, then there exists no scalable acceptable deal.*

(ii) If there exists a strictly-consistent pricing density $D \in L^0(\mathbb{P})$ such that $\mathbb{E}_{\mathbb{P}}[DX] \leq \pi(X)$ for every payoff $X \in \mathcal{M}$, then there exists no acceptable deal.

Proof. Let $D \in L^0(\mathbb{P})$ be a strictly-consistent pricing density and take a nonzero payoff $X \in \mathcal{A} \cap \mathcal{M}^\infty$. We have to show that $\pi^\infty(X) > 0$. To this effect, note that, by definition of a pricing density,

$$\sup_{n\in\mathbb{N}}\{n(\mathbb{E}_{\mathbb{P}}[DX] - \pi^\infty(X))\} = \sup_{n\in\mathbb{N}}\{\mathbb{E}_{\mathbb{P}}[D(nX)] - \pi^\infty(nX)\} \leq \sup_{n\in\mathbb{N}}\{\mathbb{E}_{\mathbb{P}}[D(nX)] - \pi(nX)\} < \infty,$$

where we used that π^∞ dominates π. This is only possible if $\mathbb{E}_{\mathbb{P}}[DX] - \pi^\infty(X) \leq 0$. As a result, we obtain $\pi^\infty(X) \geq \mathbb{E}_{\mathbb{P}}[DX] > 0$. This yields *(i)*. Next, assume that $\mathbb{E}_{\mathbb{P}}[DX] \leq \pi(X)$ for every payoff $X \in \mathcal{M}$ and take a nonzero payoff $X \in \mathcal{A} \cap \mathcal{M}$. Then, we immediately see that $\pi(X) \geq \mathbb{E}_{\mathbb{P}}[DX] > 0$, showing *(ii)*. $\square$

The next example shows that, contrary to the classical frictionless setting, the existence of a strictly-consistent pricing density does not generally imply that the market be free of acceptable deals. In view of the preceding proposition, this may occur only if either the pricing rule or the set of admissible replicable payoffs fails to be conic and $\gamma_{\pi,\mathcal{M}}$ is strictly positive. We provide an example in both cases.

Example 1.4.6. *Let $\Omega = \{\omega_1, \omega_2\}$ and assume that $\mathcal{F}$ is the power set of Ω and that $\mathbb{P}$ is specified by $\mathbb{P}(\omega_1) = \mathbb{P}(\omega_2) = \frac{1}{2}$. In this simple setting, we take $\mathcal{X} = L^0(\mathbb{P})$ and identify every element of $\mathcal{X}$ with a vector of $\mathbb{R}^2$. Set $\mathcal{S} = \mathbb{R}^2$ and consider the acceptance set defined by*

$$\mathcal{A} = \{(x,y) \in \mathbb{R}^2 \,:\, y \geq \max\{-x, 0\}\}.$$

Note that $\mathcal{A}$ is conic. We consider the following two situations.

(i) Set $\pi(x,y) = x + y^2$ for every $(x,y) \in \mathbb{R}^2$ and $\mathcal{M} = \mathbb{R}^2$. Note that $\mathcal{M}$ is conic while π is not. Note also that the requirements in Assumption 1.3.1 are met. It is clear that $D = (2,4)$ is a strictly-consistent pricing density. In particular, we have

$$\sup_{X \in \mathcal{M}} \{\mathbb{E}_{\mathbb{P}}[DX] - \pi(X)\} = \sup_{y \in \mathbb{R}} \{2y - y^2\} = 1.$$

However, $X = (-1,1) \in \mathcal{A} \cap \mathcal{M}$ satisfies $\pi(X) = 0$ and is thus an acceptable deal.

(ii) Set $\pi(x,y) = x + y$ for every $(x,y) \in \mathbb{R}^2$ and $\mathcal{M} = \{(x,y) \in \mathbb{R}^2 \, : \, x \geq -1, \, 0 \leq y \leq 1\}$. Note that π is conic while $\mathcal{M}$ is not. Note also that the requirements in Assumption 1.3.1 are met. It is clear that $D = (2,4)$ is a strictly-consistent pricing density. In particular, we have

$$\sup_{X \in \mathcal{M}} \{\mathbb{E}_{\mathbb{P}}[DX] - \pi(X)\} = \sup_{0 \leq y \leq 1} y = 1.$$

However, $X = (-1,1) \in \mathcal{A} \cap \mathcal{M}$ satisfies $\pi(X) = 0$ and is thus an acceptable deal.

1.4.2 DUAL REPRESENTATION OF $\mathcal{C}$

In this section we study from a dual perspective the subset $\mathcal{C}$ of the product space $\mathcal{X} \times \mathbb{R}$, which will play a key role in determining the dual representation of market consistent prices. For this analysis, the reference payoff space $\mathcal{X}$ has to be equipped with a topological structure and paired with a suitable dual space $\mathcal{X}'$ with pairing defined through expectation. In particular, the explicit choice of the dual space $\mathcal{X}'$ will allow us to obtain a flexible characterization of market consistent prices expressed in terms of expectations and densities belonging to the chosen dual space. To this effect, it is critical to impose that $\mathcal{A} \cap \mathcal{X}$ is closed with respect to a specific weak topology, namely $\sigma(\mathcal{X}, \mathcal{X}')$. At first sight, this assumption may seem rather restrictive unless $\mathcal{X}$ is a normed space and $\mathcal{X}'$ is its norm dual, in which case $\sigma(\mathcal{X}, \mathcal{X}')$-closedness is equivalent to norm closedness by convexity; see e.g. Theorem 5.98 in Aliprantis and Border [1]. It turns out that, for the common payoff spaces and the standard acceptance sets, the required closedness holds even with respect to the most restrictive choice $\mathcal{X}' = L^\infty(\mathbb{P})$; see Proposition 1.6.2 below.

Hence, together with Assumptions 1.3.1 and 1.3.11, we require the following.

Assumption 1.4.7. We denote by $\mathcal{X}'$ a linear subspaces of $L^0(\mathbb{P})$. We assume that $\mathcal{X}$ and $\mathcal{X}'$ contain $L^\infty(\mathbb{P})$ and satisfy $XY \in L^1(\mathbb{P})$ for all $X \in \mathcal{X}$ and $Y \in \mathcal{X}'$. These space are in separating duality through

$$(X,Y) \mapsto \mathbb{E}_{\mathbb{P}}[XY].$$

We equip $\mathcal{X}$ and $\mathcal{X}'$ with the weakest linear topologies $\sigma(\mathcal{X}, \mathcal{X}')$ and $\sigma(\mathcal{X}', \mathcal{X})$ associated with the above mapping. Unless otherwise stated, all the topological properties on $\mathcal{X}$ and $\mathcal{X}'$ refer to such topologies (in particular, the topology on $\mathcal{X}$ of Assumption 1.3.11 coincides with $\sigma(\mathcal{X}, \mathcal{X}')$, and hence $\mathcal{A} \cap \mathcal{X}$ is assumed to be $\sigma(\mathcal{X}, \mathcal{X}')$-closed).

First, we recall the definition of the set $\mathcal{C}$:

$$\mathcal{C} := \{(X,m) \in \mathcal{X} \times \mathbb{R} \, : \, \exists Z \in \mathcal{M} \text{ such that } Z - X \in \mathcal{A}, \, \pi(Z) \leq -m\}.$$

Moreover, we define the set of consistent pricing densities belonging to $\mathcal{X}'$:

$$\mathcal{D} := \{D \in \mathcal{X}' \, : \, D \text{ is a consistent pricing density}\}.$$

Similarly, the set of strictly-consistent pricing densities belonging to $\mathcal{X}'$ is denoted by

$$\mathcal{D}_{str} := \{D \in \mathcal{X}' \, : \, D \text{ is a strictly-consistent pricing density}\}.$$

In addition, it is convenient to introduce the maps $\gamma_{\mathcal{M},\pi} : \mathcal{X}' \to (-\infty, \infty]$, $\gamma_{\mathcal{A}} : \mathcal{X}' \to [-\infty, \infty)$ defined by

$$\gamma_{\pi,\mathcal{M}}(Y) := \sup_{X \in \mathcal{M}} \{\mathbb{E}_{\mathbb{P}}[XY] - \pi(X)\},$$

$$\gamma_{\mathcal{A}}(Y) := \inf_{X \in \mathcal{A} \cap \mathcal{X}} \mathbb{E}_{\mathbb{P}}[XY].$$

Note that $\gamma_{\pi,\mathcal{M}}$ coincides with the conjugate function of the restriction to $\mathcal{M}$ of the pricing rule π whereas $\gamma_{\mathcal{A}}$ is the support function $\sigma_{\mathcal{A}\cap\mathcal{X}}$ of the set $\mathcal{A}\cap\mathcal{X}$. These maps appear in the definition of a consistent pricing density.

The next proposition records the fundamental properties of the set $\mathcal{C}$. In particular, we show that consistent pricing densities appear naturally in the barrier cone of $\mathcal{C}$ (see Definition A.1.9).

Proposition 1.4.8. *The sets $\mathcal{C}$ and $\mathcal{D}$ are convex and the following statements hold:*

(i) $-((\mathcal{A}\cap\mathcal{X})\times\mathbb{R}_+)\subset\mathcal{C}$.

(ii) $-\operatorname{bar}(\mathcal{C})\subset\mathcal{X}'_+\times\mathbb{R}_+$ *and* $\mathcal{D}\subset\mathcal{X}'_+$.

(iii) $\sigma^{\mathcal{C}}(Y,1)=\gamma_{\pi,\mathcal{M}}(Y)-\gamma_{\mathcal{A}}(Y)$ *for every* $Y\in\mathcal{X}'$.

(iv) $\mathcal{D}=\{Y\in\mathcal{X}'_+\,:\,\sigma^{\mathcal{C}}(Y,1)<\infty\}=\{Y\in\mathcal{X}'_+\,:\,(Y,1)\in-\operatorname{bar}(\mathcal{C})\}$.

Proof. The convexity of $\mathcal{C}$ is clear. Now, take an arbitrary $(X,m)\in-((\mathcal{A}\cap\mathcal{X})\times\mathbb{R}_+)$ and set $Z=0\in\mathcal{M}$. Then, we clearly have $Z-X=-X\in\mathcal{A}$ as well as $\pi(Z)=0\leq -m$, showing that $(X,m)\in\mathcal{C}$. This establishes the thesis. Next, take any $(Y,r)\in-\operatorname{bar}(\mathcal{C})$ and note that

$$\sup_{m\in\mathbb{N}}\{-m\mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\{Y<0\}}Y]\}+\sup_{n\in\mathbb{N}}\{-nr\}=\sup_{m,n\in\mathbb{N}}\{\mathbb{E}_{\mathbb{P}}[-m\mathbb{1}_{\{Y<0\}}Y]-nr\}\leq\sigma^{\mathcal{C}}(Y,r)<\infty,$$

where we used that $-(m\mathbb{1}_{\{Y<0\}},n)\in-((\mathcal{A}\cap\mathcal{X})\times\mathbb{R}_+)\subset\mathcal{C}$ by monotonicity of $\mathcal{A}$ and by point *(i)*. This shows that (Y,r) must belong to $\mathcal{X}'_+\times\mathbb{R}_+$ and yields *(ii)*. An explicit calculation shows that

$$\begin{aligned}\sigma^{\mathcal{C}}(Y,1)&=\sup_{m\in\mathbb{R}}\sup_{Z\in\mathcal{M},\,\pi(Z)\leq -m}\sup_{X\in Z-\mathcal{A}\cap\mathcal{X}}\{\mathbb{E}_{\mathbb{P}}[XY]+m\}=\sup_{Z\in\mathcal{M}}\sup_{X\in Z-\mathcal{A}\cap\mathcal{X}}\{\mathbb{E}_{\mathbb{P}}[XY]-\pi(Z)\}\\&=\sup_{Z\in\mathcal{M}}\{\mathbb{E}_{\mathbb{P}}[ZY]-\pi(Z)\}+\sup_{X\in-(\mathcal{A}\cap\mathcal{X})}\mathbb{E}_{\mathbb{P}}[XY]=\gamma_{\pi,\mathcal{M}}(Y)-\gamma_{\mathcal{A}}(Y)\end{aligned}$$

for every $Y\in\mathcal{X}'$. This establishes *(iii)* and *(iv)* and implies that $\mathcal{D}$ is convex by convexity of $\sigma^{\mathcal{C}}$. □

The next theorem presents the dual representation of the closure of $\mathcal{C}$, understood with respect to the natural product topology on $\mathcal{X}\times\mathbb{R}$, where $\mathcal{X}$ is equipped with the topology fixed at the beginning of this section and $\mathbb{R}$ with its canonical Borel topology.

Theorem 1.4.9. *There exists $n>0$ such that $(0,n)\notin\operatorname{cl}(\mathcal{C})$ if and only if $\operatorname{cl}(\mathcal{C})\neq\mathcal{X}\times\mathbb{R}$ and the following representation holds:*

$$\operatorname{cl}(\mathcal{C})=\bigcap_{Y\in\mathcal{D}}\{(X,m)\in\mathcal{X}\times\mathbb{R}\,:\,\mathbb{E}_{\mathbb{P}}[XY]+m\leq\gamma_{\pi,\mathcal{M}}(Y)-\gamma_{\mathcal{A}}(Y)\}. \tag{1.7}$$

Proof. Assume that $(0,n)\notin\operatorname{cl}(\mathcal{C})$. Hence $\mathcal{X}\times\mathbb{R}$ strictly contains $\operatorname{cl}(\mathcal{C})$. We can apply to $\operatorname{cl}(\mathcal{C})$ the dual representation in Proposition A.1.10 to get

$$\operatorname{cl}(\mathcal{C})=\bigcap_{(Y,r)\in\mathcal{X}'\times\mathbb{R}}\{(X,m)\in\mathcal{X}\times\mathbb{R}\,:\,\mathbb{E}_{\mathbb{P}}[XY]+mr\leq\sigma^{\mathcal{C}}(Y,r)\}. \tag{1.8}$$

We claim that $-\operatorname{bar}(\mathcal{C})\cap(\mathcal{X}'\times(0,\infty))\neq\emptyset$. Since $(0,n)\notin\mathcal{C}$ by assumption, there must exist $(Y,r)\in-\operatorname{bar}(\mathcal{C})$ satisfying

$$nr=\mathbb{E}_{\mathbb{P}}[0\cdot Y]+nr>\sigma^{\mathcal{C}}(Y,r)\geq 0.$$

This establishes the desired claim. Now, recall that $-\operatorname{bar}(\mathcal{C})\subset\mathcal{X}'_+\times\mathbb{R}_+$. Since $\sigma^{\mathcal{C}}$ is sublinear and $-\operatorname{bar}(\mathcal{C})$ is a convex cone, it follows that

$$\operatorname{cl}(\mathcal{C})=\bigcap_{Y\in\mathcal{X}'_+}\{(X,m)\in\mathcal{X}\times\mathbb{R}\,:\,\mathbb{E}_{\mathbb{P}}[XY]+m\leq\sigma^{\mathcal{C}}(Y,1)\}.$$

The desired representation is now a direct consequence of point *(iii)* of Proposition 1.4.8. Viceversa, assume that (1.7) holds and take $Y\in\mathcal{D}$, which exists since $\operatorname{cl}(\mathcal{C})\neq\mathcal{X}\times\mathbb{R}$. Observe that $(0,n)\in\operatorname{cl}(\mathcal{C})$ if and only if $n\leq\sigma_{\mathcal{C}}(Y,1)<\infty$. This delivers the desired result. □

Remark 1.4.10. If $(0,n) \in \mathrm{cl}(\mathcal{C})$ for every $n > 0$, then $\mathcal{D}$ is empty, and the closure of $\mathcal{C}$ has the following representation:

$$\mathrm{cl}(\mathcal{C}) = \left(\bigcap_{Y \in \mathcal{X}'_+} \{X \in \mathcal{X} \,:\, \mathbb{E}_{\mathbb{P}}[XY] \leq \sigma^{\mathcal{M}\cap\mathcal{X}}(Y) - \gamma_{\mathcal{A}}(Y)\} \right) \times \mathbb{R}.$$

This is the case in the next example. However, from a financial point of view, this case in not of interest since it would mean that the agent would have the chance to receive an arbitrary high amount of money in change of acquiring desirable positions.

Example 1.4.11. *Consider a two state economy, so that $\mathcal{X} = \mathbb{R}^2$. Let $\mathcal{A} = \{(x,y) \in \mathbb{R}^2; \min\{x, x+y\} \geq 0\}$ and $\mathcal{M} = \{(-z,z) \in \mathbb{R}^2 \,:\, z \in \mathbb{R}\}$. Suppose the pricing functional is defined on $\mathcal{M}$ as $\pi((-z,z)) = z$. It is easy to see that $\mathcal{C} = \{(x,y,m) \in \mathbb{R}^2 \times \mathbb{R} \,:\, x+y \geq 0\}$, which contains $(0,0,n)$ for every $n > 0$, and $\sigma^{\mathcal{C}}(Y,r) = \infty$ whenever $r > 0$.*

Of course, representation (1.7) is actually a representation of the set $\mathcal{C}$ whenever it turns out to be closed in the first place. In Theorem 1.3.14 we have shown that this holds if $\mathcal{G}$ is a linear space.

Corollary 1.4.12 (Dual representation of $\mathcal{C}$)**.** *If $\mathcal{G}$ is a linear space (in particular, if there exists no scalable acceptable deal), then the following representation of $\mathcal{C}$ holds:*

$$\mathcal{C} = \bigcap_{Y \in \mathcal{D}} \{(X,m) \in \mathcal{X} \times \mathbb{R} \,:\, \mathbb{E}_{\mathbb{P}}[XY] + m \leq \gamma_{\pi,\mathcal{M}}(Y) - \gamma_{\mathcal{A}}(Y)\}.$$

Proof. It is enough to apply Theorems 1.3.14 and 1.4.9. ☐

We close this section presenting a set of assumptions under which the dual elements appearing in the representation of $\mathcal{C}$ may be reduced to strictly consistent pricing densities. This happens when $\mathcal{A}$ is a cone and at least one strictly consistent pricing density exists in $\mathcal{X}'$. Note that our next proposition is based on the progressive application of three results of this and the preceding section: first Proposition 1.4.5 to show that there are no scalable acceptable deals, then Theorem 1.3.14 to affirm that $\mathcal{C}$ is closed and $(0,n) \notin \mathcal{C}$ for some $n \in \mathbb{N}$, and then Theorem 1.4.9 for the dual representation of $\mathcal{C}$.

Proposition 1.4.13. *If $\mathcal{A}$ is a cone and $\mathcal{D}_{str}$ is not empty, the following representation holds:*

$$\mathcal{C} = \bigcap_{D \in \mathcal{D}_{str}} \{(X,m) \in \mathcal{X} \times \mathbb{R} \,:\, \mathbb{E}_{\mathbb{P}}[XD] + m \leq \gamma_{\pi,\mathcal{M}}(D)\}. \tag{1.9}$$

Proof. Let D^* belong to $\mathcal{D}_{str}$. By Proposition 1.4.5, scalable acceptable deals do not exists. Hence, we invoke Corollary 1.4.12 to affirm that representation (1.7) holds for $\mathcal{C}$. Since $\gamma_{\mathcal{A}}(D) = 0$ for every $D \in \mathcal{D}$ by conicity of $\mathcal{A}$, we only need to establish the inclusion "$\supset$" of (1.9). To this end, take an arbitrary $(X,m) \in \mathcal{X} \times \mathbb{R}$ such that $\mathbb{E}_{\mathbb{P}}[XD] + m \leq \gamma_{\pi,\mathcal{M}}(D)$ for every $D \in \mathcal{D}_{str}$. Now, take any $D \in \mathcal{D}$. For every $\lambda \in (0,1)$, it is immediate to verify using conicity of $\mathcal{A}$ that $\lambda D^* + (1-\lambda)D \in \mathcal{D}_{str}$ so that

$$\begin{aligned}
\lambda(\mathbb{E}_{\mathbb{P}}[XD^*] + m) + (1-\lambda)(\mathbb{E}_{\mathbb{P}}[XD] + m) &= \mathbb{E}_{\mathbb{P}}[X(\lambda D^* + (1-\lambda)D)] + m \\
&\leq \gamma_{\pi,\mathcal{M}}(\lambda D^* + (1-\lambda)D) \\
&\leq \lambda\gamma_{\pi,\mathcal{M}}(D^*) + (1-\lambda)\gamma_{\pi,\mathcal{M}}(D).
\end{aligned}$$

Letting $\lambda \downarrow 0$ delivers $\mathbb{E}_{\mathbb{P}}[XD] + m \leq \gamma_{\pi,\mathcal{M}}(D)$ and shows the desired inclusion. ☐

1.4.3 DUAL REPRESENTATION OF SUPERREPLICATION PRICES

In this section we show that the dual representation of set $\mathcal{C}$ can be used for convenient expressions of superreplication prices. This fact will be exploited in the next section for the dual characterization of market consistent prices.

Proposition 1.4.14. *If $\mathcal{G}$ is a linear space, the following representation holds for every payoff $X \in \mathcal{X}$:*

$$\pi^+(X) = \sup_{D \in \mathcal{D}} \{\mathbb{E}_{\mathbb{P}}[DX] - \gamma_{\pi,\mathcal{M}}(D) + \gamma_{\mathcal{A}}(D)\}. \tag{1.10}$$

Proof. Take any payoff $X \in \mathcal{X}$. It follows from Proposition 1.3.12 and Corollary 1.4.12 that

$$\begin{aligned}\pi^+(X) &= \inf\{m \in \mathbb{R} \,:\, (X, -m) \in \mathcal{C}\} \\ &= \inf\{m \in \mathbb{R} \,:\, \mathbb{E}_{\mathbb{P}}[DX] - m \leq \gamma_{\pi,\mathcal{M}}(D) - \gamma_{\mathcal{A}}(D),\ \forall D \in \mathcal{D}\} \\ &= \inf\{m \in \mathbb{R} \,:\, m \geq \mathbb{E}_{\mathbb{P}}[DX] - \gamma_{\pi,\mathcal{M}}(D) + \gamma_{\mathcal{A}}(D),\ \forall D \in \mathcal{D}\} \\ &= \sup\{\mathbb{E}_{\mathbb{P}}[DX] - \gamma_{\pi,\mathcal{M}}(D) + \gamma_{\mathcal{A}}(D) \,:\, D \in \mathcal{D}\}.\end{aligned}$$

This establishes (1.10). □

Similarly to the preceding section, we close by treating the case where $\mathcal{A}$ is a cone and there exists at least one strictly consistent pricing density. Thanks to the special representation of $\mathcal{C}$, it turns out that the supremum is taken over strictly consistent pricing densities also in representing the superreplication price.

Proposition 1.4.15. *Assume that $\mathcal{A}$ is a cone and $\mathcal{D}_{str}$ is not empty. In this case, the following representation holds:*

$$\pi^+(X) = \sup_{D \in \mathcal{D}_{str}} \{\mathbb{E}_{\mathbb{P}}[DX] - \gamma_{\pi,\mathcal{M}}(D)\}. \tag{1.11}$$

Proof. It is enough to replicate the proof of Proposition 1.4.14, using representation (1.9) for $\mathcal{C}$. □

Example 1.4.20 shows that if $\mathcal{A}$ fails to be a cone, the representation of π^+ is no longer valid.

1.4.4 Dual characterization of market consistent prices

In a preceding discussion, we have interpreted pricing densities as pricing rules of a frictionless "artificial" market which respect, up to an enlargement, the bid-ask spread of the original market. In this section, we aim to use (some of) these fictitious pricing rules to characterize market consistent prices.

We start by showing that strictly-consistent pricing densities can be used to define special market-consistent prices.

Proposition 1.4.16. *If $\mathcal{D}_{str}$ is not empty, then for every payoff $X \in \mathcal{X}$ and for every $D \in \mathcal{D}_{str}$*

$$\mathbb{E}_{\mathbb{P}}[DX] - \gamma_{\pi,\mathcal{M}}(D) \in \mathrm{MCP}(X).$$

Proof. Let $D \in \mathcal{D}_{str}$ and $X \in \mathcal{X}$. Note that for every replicable payoff $Z \in \mathcal{M}$ such that $Z - X \in \mathcal{A} \setminus \{0\}$ we have

$$\begin{aligned}\pi(Z) &\geq \mathbb{E}_{\mathbb{P}}[DZ] - \gamma_{\pi,\mathcal{M}}(D) \\ &= \mathbb{E}_{\mathbb{P}}[D(Z - X)] + \mathbb{E}_{\mathbb{P}}[DX] - \gamma_{\pi,\mathcal{M}}(D) \\ &> \mathbb{E}_{\mathbb{P}}[DX] - \gamma_{\pi,\mathcal{M}}(D)\end{aligned}$$

by strict consistency. Note also that $\mathbb{E}_{\mathbb{P}}[DX] - \gamma_{\pi,\mathcal{M}}(D) \leq \pi(X)$ in the case that $X \in \mathcal{M}$. This shows that $\mathbb{E}_{\mathbb{P}}[DX] - \gamma_{\pi,\mathcal{M}}(D)$ is a market-consistent price for X. □

Note that, since $\mathrm{MCP}(X)$ is an interval unbounded from below, under the assumptions of the last proposition, we have that

$$\mathrm{MCP}(X) \supset \{p \in \mathbb{R} \,:\, \exists D \in \mathcal{D}_{str} \text{ such that } p \leq \mathbb{E}_{\mathbb{P}}[DX] - \gamma_{\pi,\mathcal{M}}(D)\}. \tag{1.12}$$

To establish our desired dual characterization of market-consistent prices we have to determine whether or not the converse inclusion also holds. The next theorem provides an answer to this more challenging task under the assumption that the acceptance set is conic. In particular, it features a dual representation of superreplication prices in terms of strictly-consistent pricing densities that extends the well-known representation via stochastic discount factors in a frictionless setting where the acceptance set is taken to be the positive cone.

Theorem 1.4.17 (Dual characterization of market consistent prices)**.** *If $\mathcal{A}$ is a cone and $\mathcal{D}_{str}$ is not empty, the following statements hold for any $X \in \mathcal{X}$:*

(i) If $\pi^+(X) \notin \mathrm{MCP}(X)$, then the supremum in (1.11) *is not attained and*

$$\mathrm{MCP}(X) = (-\infty, \pi^+(X)) = \{p \in \mathbb{R} \,:\, \exists D \in \mathcal{D}_{str} \text{ such that } p \leq \mathbb{E}_{\mathbb{P}}[DX] - \gamma_{\pi,\mathcal{M}}(D)\}. \tag{1.13}$$

(ii) If $\pi^+(X) \in \mathrm{MCP}(X)$, then the following statements hold and both can happen:

(a) the supremum in (1.11) *is attained if and only if*

$$\mathrm{MCP}(X) = (-\infty, \pi^+(X)] = \{p \in \mathbb{R} \,:\, \exists D \in \mathcal{D}_{str} \text{ such that } p \leq \mathbb{E}_{\mathbb{P}}[DX] - \gamma_{\pi,\mathcal{M}}(D)\}.$$

(b) the supremum in (1.11) *is not attained if and only if*

$$\begin{aligned}\mathrm{MCP}(X) &= (-\infty, \pi^+(X)] \\ &\supsetneq \{p \in \mathbb{R} \,:\, \exists D \in \mathcal{D}_{str} \text{ such that } p \leq \mathbb{E}_{\mathbb{P}}[DX] - \gamma_{\pi,\mathcal{M}}(D)\} = (-\infty, \pi^+(X)).\end{aligned}$$

This situation can occur even if both π and $\mathcal{M}$ are conic and there exists no acceptable deal.

Proof. Recall from Proposition 1.3.5 that $\pi^+(X)$ is the supremum of the set $\mathrm{MCP}(X)$. Assume that $\pi^+(X)$ does not belong to $\mathrm{MCP}(X)$. In view of Proposition 1.4.16, to complete the proof we only have to show the inclusion "$\subset$" in (1.13). To this effect, take an arbitrary market-consistent price $p \in \mathrm{MCP}(X)$ and note that we must have $p < \pi^+(X)$. Hence, it follows from the representation (1.10) that $p < \mathbb{E}_{\mathbb{P}}[DX] - \gamma_{\pi,\mathcal{M}}(D)$ for a suitable $D \in \mathcal{D}_{str}$. This delivers the desired inclusion.

If $\pi^+(X)$ belongs to $\mathrm{MCP}(X)$, then the inclusion in (1.12) is an equality if and only if the supremum in (1.11) is attained. This delivers equivalences in *(a)* and *(b)*. We refer to Example 1.4.18 for a concrete situation where *(a)* and *(b)* happen. This concludes the proof. $\square$

The following examples complement the proof. We show that, contrary to the standard frictionless setting, for a replicable payoff with market-consistent superreplication price, the supremum in the dual representation of the corresponding superreplication price need not be attained. In view of the above result, this implies that a dual characterization of market-consistent prices in terms of strictly-consistent pricing densities may not be possible for replicable payoffs. Interestingly, this can occur even if both the pricing rule and the set of admissible payoffs are conic and there exists no acceptable deal.

Example 1.4.18. *Let $\Omega = \{\omega_1, \omega_2\}$ and assume that $\mathcal{F}$ is the power set of Ω and that $\mathbb{P}$ is specified by $\mathbb{P}(\omega_1) = \mathbb{P}(\omega_2) = \frac{1}{2}$. In this simple setting, we take $\mathcal{X} = \mathcal{X}' = L^0(\mathbb{P})$ and identify every element of $L^0(\mathbb{P})$ with a vector of $\mathbb{R}^2$. Take $\mathcal{A} = \mathbb{R}^2_+$ and $\mathcal{M} = \{(x,y) \in \mathbb{R}^2\,;\, 0 \leq y \leq -x\}$ and define*

$$\pi(x,y) = \begin{cases} \sqrt{x^2+xy} & \text{if } (x,y) \in \mathcal{M}, \\ \infty & \text{otherwise.} \end{cases}$$

(To verify that π is convex, one can observe that π is continuous on $\mathcal{M}$ and that the Hessian matrix of π in the interior of $\mathcal{M}$ is given by

$$H\pi(x,y) = \begin{bmatrix} \frac{y^2}{4(x^2+xy)^{3/2}} & -\frac{xy}{4(x^2+xy)^{3/2}} \\ -\frac{xy}{4(x^2+xy)^{3/2}} & \frac{x^2}{4(x^2+xy)^{3/2}} \end{bmatrix}$$

and has nonnegative eigenvalues, namely 0 and $\frac{1}{4}(x^2+y^2)(x^2+xy)^{-3/2}$). Both $\mathcal{A}$ and $\mathcal{M}$ are cones and π is conic. Moreover, there exists no acceptable deal and all the requirements in the stipulated assumptions are fulfilled. A direct inspection shows that strictly-consistent pricing densities $D \in \mathcal{X}'$ exist (for instance, take $D = (1,2)$) and satisfy $\gamma_{\pi,\mathcal{M}}(D) = 0$ by conicity. Now, set $X = (-1,1) \in \mathcal{M}$. It is immediate to verify that if $Z \in \mathcal{M}$ and $Z - X \in \mathcal{A}$, then $Z = X$. This implies that $\pi^+(X) = \pi(X) = 0 \in \mathrm{MCP}(X)$. We show that there is no $D = (d_1, d_2) \in \mathcal{D}_{str}$ such that $\mathbb{E}_{\mathbb{P}}[DX] = 0$. Indeed, we would otherwise have $d_1 = d_2$ and taking $Z_\lambda = (-1, \lambda) \in \mathcal{M}$ for $\lambda \in (0,1)$ would deliver

$$\sup_{0<\lambda<1} \{\mathbb{E}_{\mathbb{P}}[DZ_\lambda] - \pi(Z_\lambda)\} \leq 0 \implies d_1 \geq \sup_{0<\lambda<1} \frac{2}{\sqrt{1-\lambda}} = \infty.$$

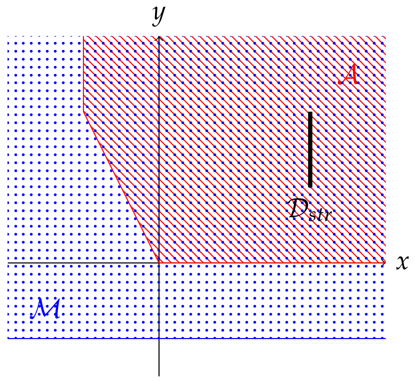


FIGURE 1.4: Example 1.4.20

As a result, the supremum in (1.11) *is not attained.*

Now, set $Y = 0 \in \mathcal{M}$ and note that, like before, $\pi^+(Y) = \pi(Y) = 0 \in \mathrm{MCP}(Y)$. It is clear that $\mathbb{E}_{\mathbb{P}}[DY] = 0$ for every $D \in \mathcal{D}_{str}$. In particular, the supremum in (1.11) *is attained.*

Motivated by the preceding example, we provide a sufficient condition on replicable payoffs under which it is possible to derive a dual characterization of the corresponding market-consistent prices in terms of strictly-consistent pricing densities. The condition is automatically met in frictionless markets.

Proposition 1.4.19. *If $\mathcal{A}$ is a cone and there exists a strictly-consistent pricing density $D \in \mathcal{X}'$ such that $\gamma_{\pi,\mathcal{M}}(D) = 0$, then for every payoff $X \in \mathcal{X}$ such that $\pi^+(X) \in \mathrm{MCP}(X)$ and such that $X \in \mathcal{M} \cap (-\mathcal{M})$ and π is linear on $\mathrm{span}(X)$ we have*

$$\mathrm{MCP}(X) = \{p \in \mathbb{R} \,:\, \exists D \in \mathcal{D}_{str} \text{ such that } p \leq \mathbb{E}_{\mathbb{P}}[DX]\}.$$

Proof. It follows from Proposition 1.4.5 that $\mathcal{A} \cap \{Z \in \mathcal{M} \,:\, \pi(Z) \leq 0\} = \{0\}$. Now, take a payoff $X \in \mathcal{X}$ such that $\pi^+(X) \in \mathrm{MCP}(X)$ and assume that $X \in \mathcal{M} \cap (-\mathcal{M})$ and π is linear on $\mathrm{span}(X)$. By Theorem 1.3.6 we have $\pi^+(X) = \pi(X)$. Moreover, by Proposition 1.4.3, we know that $\pi(X) = \mathbb{E}_{\mathbb{P}}[DX]$. As a result, the supremum in (1.11) is attained and the desired statement follows from Theorem 1.4.17. $\square$

One may wonder whether the conclusions of Theorem 1.4.17 still hold if the acceptance set is not assumed to be conic. The next example shows that conicity of $\mathcal{A}$ is necessary for both the dual representation of superreplication prices and the dual characterization of market-consistent prices to hold.

Example 1.4.20. *Let $\Omega = \{\omega_1, \omega_2\}$ and assume that $\mathcal{F}$ is the power set of Ω and that $\mathbb{P}$ is specified by $\mathbb{P}(\omega_1) = \mathbb{P}(\omega_2) = \frac{1}{2}$. In this simple setting, we take $\mathcal{X} = \mathcal{X}' = L^0(\mathbb{P})$ and identify every element of $L^0(\mathbb{P})$ with a vector of $\mathbb{R}^2$. Define $\pi(x,y) = \max\{x, x+y\}$ for every $(x,y) \in \mathbb{R}^2$ and set*

$$\mathcal{M} = \{(x,y) \in \mathbb{R}^2 \,:\, y \geq -1\}, \quad \mathcal{A} = \{(x,y) \in \mathbb{R}^2 \,:\, y \geq \max\{-2x, 0\},\ x \geq -1\}.$$

Note also that all the requirements in Assumption 1.4.7 are fulfilled and there exists no acceptable deal. It is not difficult to verify that strictly-consistent pricing densities exist, namely

$$\mathcal{D}_{str} = \left\{D = (d_1, d_2) \in \mathbb{R}^2_+ \,:\, d_1 = 2,\ 1 < d_2 \leq 2\right\}.$$

In this case, we have

$$\gamma_{\pi,\mathcal{M}}(D) = \sup_{X \in \mathcal{M}} \{\mathbb{E}_{\mathbb{P}}[DX] - \pi(X)\} = \max\left\{\sup_{-1 \leq y < 0}\left\{\frac{1}{2}d_2 y\right\}, \sup_{y \geq 0}\left\{\left(\frac{1}{2}d_2 - 1\right)y\right\}\right\} = 0.$$

If $X = (x, y) \notin \mathcal{M}$, computations show that its superreplication price is

$$\pi^+(X) = \begin{cases} x - 1 & \text{if } y \leq -2 \\ x + \frac{y}{2} & \text{if } -2 < y < -1 \end{cases}$$

and by Theorem 1.3.6 we see that $\mathrm{MCP}(X) = (-\infty, \pi^+(X))$*. Moreover the representation*

$$\sup_{D \in \mathcal{D}_{str}} \{\mathbb{E}_{\mathbb{P}}[DX] - \gamma_{\pi,\mathcal{M}}(D)\} = \sup_{D \in \mathcal{D}_{str}} \mathbb{E}_{\mathbb{P}}[DX] = x + \frac{y}{2},$$

coincides with $\pi^+(X)$ if and only if $-2 \leq y < -1$. Thus for every $(x, y) \notin \mathcal{M}$ with $y < -2$ we have that

$$\sup_{D \in \mathcal{D}_{str}} \{DX - \pi^*(D)\} = x + \frac{y}{2} < x - 1 = \pi^+(X).$$

This shows that both the representation of the superreplication price and the characterization of market-consistent prices obtained in Theorem 1.4.17 can fail when $\mathcal{A}$ is not conic (even for a payoff outside $\mathcal{M}$).

1.5 FUNDAMENTAL THEOREM OF ASSET PRICING

The dual characterization of market-consistent prices established in Theorem 1.4.17 requires to know that a strictly-consistent pricing density exists in the first place. This section is devoted to the investigation of this problem. In line with the previous section, we are especially interested in economically meaningful conditions for the existence of such a pricing density expressed in terms of the underlying financial primitives, namely the acceptance set $\mathcal{A}$, the pricing rule π, and the set of admissible replicable payoffs $\mathcal{M}$. This will lead us to establishing a general version of the Fundamental Theorem of Asset Pricing in our setting. To do so, the reference payoff space and its dual have to be equipped with a special topological structure. As illustrated below, our framework will prove to be general and flexible enough to accommodate a variety of concrete important examples.

Hence throughout this section, together with Assumptions 1.3.1,1.3.11 and 1.4.7, we assume the following.

Assumption 1.5.1. $\mathcal{X}'$ is the norm dual of a suitable normed space in $L^0(\mathbb{P})$ and $\sigma(\mathcal{X}', \mathcal{X})$ is weaker than the weak-star topology on $\mathcal{X}'$.

The key tool to determine the existence of strictly-consistent pricing densities is the following version of the well-known Kreps-Yan Theorem.

Theorem 1.5.2 (Kreps-Yan Theorem). *Let $\mathcal{L}$ and $\mathcal{L}'$ be subsets respectively of $\mathcal{X}$ and $\mathcal{X}'$, and assume that the following properties hold:*

(1) Σ-completeness: For every sequence $(Y_n) \subset \mathcal{L}'$ there exist a sequence $(\lambda_n) \subset (0, \infty)$ and $Y \in \mathcal{L}'$ such that $\sum_{k=1}^n \lambda_k Y_k \to Y$.

(2) Countable separation: There exists a sequence $(Y_n) \subset \mathcal{L}' \cap (\mathrm{bar}(\mathrm{cone}(\mathcal{L})))$ such that for every nonzero $X \in \mathcal{L}$ we have $\mathbb{E}_{\mathbb{P}}[XY_n] > 0$ for some $n \in \mathbb{N}$.

Then, there exists $Y \in \mathcal{L}'$ such that $\mathbb{E}_{\mathbb{P}}[XY] > 0$ for every nonzero $X \in \mathcal{L}$.

Proof. By the countable separation property, there exists a sequence $(Y_n) \subset \mathcal{L}' \cap \mathrm{bar}(\mathrm{cone}(\mathcal{L}))$ such that for every nonzero $X \in \mathcal{L}$ we have $\mathbb{E}_{\mathbb{P}}[XY_n] > 0$ for some $n \in \mathbb{N}$. In particular, note that $\mathbb{E}_{\mathbb{P}}[XY_n] \geq 0$ for all $X \in \mathcal{L}$ and $n \in \mathbb{N}$ because $(Y_n) \subset \mathrm{bar}(\mathrm{cone}(\mathcal{L}))$. Moreover, by the Σ-completeness property, there exist a sequence $(\lambda_n) \subset (0, \infty)$ and $Y \in \mathcal{L}'$ such that $\sum_{k=1}^n \lambda_k Y_k \to Y$. Since the series converges in the $\sigma(\mathcal{X}', \mathcal{X})$-topology, it is immediate to see that $\mathbb{E}_{\mathbb{P}}[XY] > 0$ for every nonzero $X \in \mathcal{L}$. □

Remark 1.5.3. (i) We actually need less than the Σ-completeness property as it is state in point *(1)*. Indeed it is enough to have Σ-completeness with respect to the sequence (Y_n) of point *(2)*. We have stated the theorem in this way so that it is possible to compare it with the Kreps-Yan Theorem in Jouini et al. [63], and since in the case where we apply this theorem it happens that *(1)* holds.

(ii) The above result can be generalized for every pair of vector spaces $\mathcal{X}$ and $\mathcal{X}'$ equipped with a bilinear mapping $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{X}' \to \mathbb{R}$. In this respect, our statement is a minor adaptation of the abstract version of the Kreps-Yan Theorem obtained by Jouini et al. [63]. In that paper, the set $\mathcal{L}$ was assumed to be a convex cone satisfying $\mathcal{L} \cap (-\mathcal{L}) = \{0\}$ and $\mathcal{L} - \mathcal{L} = \mathcal{X}$ and the dual set $\mathcal{L}'$ was taken to satisfy

$$\mathcal{L}' = \mathrm{bar}(\mathcal{L}) = \{Y \in \mathcal{X}' \,:\, \langle X, Y \rangle \geq 0, \, \forall X \in \mathcal{L}\}.$$

Incidentally, note that the pointedness condition $\mathcal{L} \cap (-\mathcal{L}) = \{0\}$ is automatically implied by the countable separation property (regardless of the special choice of $\mathcal{L}$). Our formulation is motivated by the choice $\mathcal{L} = \mathcal{A} \cap \mathcal{X}$ and $\mathcal{L}' = \mathcal{D}$. In particular, in that paper's setting, even in the case where $\mathcal{A}$ and, hence, $\mathcal{L}$ is a cone, the set $\mathcal{L}'$ must be taken to be a subset of the barrier cone of $\mathcal{L}$.

(iii) The merit of Jouini et al. [63] is to have extracted the key underlying mathematical structure behind the original works by Kreps [69] and Yan [94]. We refer to Cassese [27], Gao and Xanthos [54], and Rokhlin [88, 87] for a variety of concrete examples where the above Σ-completeness and countable separation properties hold.

The preceding result gives us a pair of sufficient conditions for the existence of a strictly-consistent pricing density. This follows at once by applying the result to $\mathcal{L} = \mathcal{A} \cap \mathcal{X}$ and $\mathcal{L}' = \mathcal{D}$. It is therefore left to verify that the corresponding Σ-completeness and countable separation property are fulfilled. This is exactly the content of the fundamental theorem of asset pricing, which basically consists in an application of the Kreps-Yan Theorem.

Before stating our fundamental theorems, we highlight a useful equivalent condition to the absence of scalable acceptable deals in the presence of a conic acceptance set satisfying a suitable pointedness condition. We state it in a lemma, as it will be used as a technical tool in proving the fundamental theorems. Note that condition *(i)* means that there exist no acceptable deals that remain acceptable payoffs once they are diminished by arbitrary multiples of any given acceptable payoff.

Lemma 1.5.4. *Let $\mathcal{A}$ be a cone with $\mathcal{A} \cap (-\mathcal{A}) = \{0\}$. The following statements are equivalent:*

(i) For every nonzero $X \in \mathcal{A} \cap \mathcal{X}$ there exists $\lambda > 0$ such that $(\lambda X, 0) \notin \mathcal{C}$.

(ii) There exists no scalable acceptable deal.

Proof. If *(i)* holds, then for every nonzero $X \in \mathcal{A} \cap \mathcal{X}$ we find $\lambda > 0$ such that $\lambda X \notin \{Z \in \mathcal{M} : \pi(Z) \leq 0\}$ or equivalently $X \notin \{Z \in \mathcal{M}^\infty : \pi^\infty(Z) \leq 0\}$ by Proposition 1.1.2. This yields *(ii)*. Conversely, assume that *(ii)* holds and let $\|\cdot\|$ be a norm on $\mathcal{S}$ inducing the unique Hausdorff topology that makes $\mathcal{S}$ a topological vector space. First, we claim that

$$\{Z \in \mathcal{M} \cap \mathcal{A} \,:\, \pi(Z) \leq 0\} \text{ is bounded.} \tag{1.14}$$

Otherwise, for every $n \in \mathbb{N}$ we find $Z_n \in \mathcal{M} \cap \mathcal{A}$ such that $\pi(Z_n) \leq 0$ and $\|Z_n\| \geq n$. Since the unit sphere in $\mathcal{S}$ is compact, there exists a nonzero $Z \in \mathcal{S}$ such that $\frac{Z_n}{\|Z_n\|} \to Z$. Note that $Z \in \mathcal{M}^\infty \cap (\mathcal{A} \cap \mathcal{X})^\infty$ by definition of recession cone. Note also that $\pi^\infty(Z) \leq 0$ must hold, for otherwise $\widetilde{\lambda} > 0$ exists such that $\pi(\widetilde{\lambda} Z) > 0$, and the lower semicontinuity and convexity of π yields

$$0 < \pi(\widetilde{\lambda} Z) \leq \liminf_{n \to \infty} \pi\left(\widetilde{\lambda} \frac{Z_n}{\|Z_n\|}\right) \leq \liminf_{n \to \infty} \frac{\widetilde{\lambda}}{\|Z_n\|} \pi(Z_n) \leq 0.$$

This shows that Z is a scalable acceptable deal, contradicting *(ii)*. To avoid this, we must have (1.14). Now, assume that *(i)* fails to hold so that we find a nonzero $X \in \mathcal{A} \cap \mathcal{X}$ such that for every $\lambda > 0$ there exists $Z_\lambda \in \mathcal{M}$ with $\pi(Z_\lambda) \leq 0$ and $Z_\lambda - \lambda X \in \mathcal{A}$. In particular, note that $Z_\lambda \in \mathcal{A}$ and $\frac{Z_\lambda}{\lambda} \in \mathcal{A} + X$ for every $\lambda > 0$. Since $(\mathcal{A} + X) \cap \mathcal{S}$ is closed and does not contain the zero payoff by assumption on $\mathcal{A}$, the norm $\|\cdot\|$ must be bounded from below by a suitable $\varepsilon > 0$ on the set $(\mathcal{A} + X) \cap \mathcal{S}$. In particular, $\frac{\|Z_\lambda\|}{\lambda} \geq \varepsilon$ for every $\lambda > 0$. This implies that $\{Z_\lambda \,:\, \lambda > 0\}$ is unbounded. However, this is against (1.14). It then follows that *(i)* must hold. □

Remark 1.5.5. (i) If $\mathcal{A}$ is taken to be the positive cone, the pointedness condition $\mathcal{A} \cap (-\mathcal{A}) = \{0\}$ automatically holds and condition *(i)* is equivalent to the "no scalable arbitrage" condition in Pennanen [76]. Note that the same result holds under the weaker pointedness condition $\mathcal{A}\cap(-\mathcal{A})\cap\mathcal{X}=\{0\}$, which is necessary for condition *(i)* to hold.

(ii) Note that, since we have assumed that $\mathcal{A}\cap(-\mathcal{A})=\{0\}$, then the absence of scalable acceptable deals is equivalent to $\mathcal{G}$ being a linear space. In this case, indeed, the dimension of $\mathcal{G}$ is forced to be zero.

We are now ready to state the first fundamental theorem. Showing that Σ-completeness holds turns out to be quite simple, as it is a direct consequence of the fact that, by assumption, the space $\mathcal{X}'$ is a norm dual and $\sigma(\mathcal{X}',\mathcal{X})$ is weaker than the corresponding weak-star topology. On the other side, establishing the countable separation property requires some effort and can only be accomplished under suitable conditions. The importance of these conditions lies in the fact that they provide concrete situations where the dual characterization of market-consistent prices obtained in Theorem 1.4.17 actually holds.

We start by dealing with the basic situation where $\mathcal{A}$ is the positive cone.

Theorem 1.5.6 (Fundamental Theorem of Asset Pricing)**.** *Let $\mathcal{A}=L^0(\mathbb{P})_+$ and assume that there exists no scalable arbitrage opportunity. Then, there exists a strictly consistent pricing density in $\mathcal{X}'$.*

Proof. We divide the proof in two steps, so as to prove separately the two needed properties for applying Theorem 1.5.2 to $\mathcal{L}=\mathcal{A}\cap\mathcal{X}$ and $\mathcal{L}'=\mathcal{D}$. The existence of a strictly consistent pricing density in $\mathcal{X}'$ is then ensured by the said theorem.

Step 1 (Σ-completeness). For every sequence $(Y_n)\subset\mathcal{D}$ there exists a sequence $(\lambda_n)\subset(0,\infty)$ and $Y\in\mathcal{D}$ such that $\sum_{k=1}^n\lambda_kY_k\to Y$.

Recall that $\mathcal{D}\subset\mathcal{X}'_+$ by Proposition 1.4.8 and note that $\sigma^{\mathcal{C}}(Y,1)\geq 0$ for every $Y\in\mathcal{D}$. Moreover, recall that $\mathcal{X}'$ is a norm dual and denote by $\|\cdot\|_{\mathcal{X}'}$ the corresponding dual norm. Let $S_n=\sum_{k=1}^n\alpha_kY_k$ and $\alpha_n=(1+\|Y_n\|_{\mathcal{X}'})^{-1}(1+\sigma^{\mathcal{C}}(Y_n,1))^{-1}2^{-n}>0$ for every $n\in\mathbb{N}$. Since $\mathcal{X}'$ is complete with respect to its norm topology, we have $S_n\to Z$ for a suitable $Z\in\mathcal{X}'$ with respect to said topology. Hence, by our standing assumptions, we also have $S_n\to Z$ with respect to the reference topology $\sigma(\mathcal{X}',\mathcal{X})$. To conclude the proof, note that $\sum_{k=1}^n\alpha_k\to r$ for some $r>0$ and

$$\sigma^{\mathcal{C}}(Z,r)\leq\liminf_{n\to\infty}\sum_{k=1}^n\alpha_k\sigma^{\mathcal{C}}(Y_k,1)<\infty$$

by lower semicontinuity and sublinearity of $\sigma^{\mathcal{C}}$. This yields $(Z,r)\in-\mathrm{bar}(\mathcal{C})$. The desired statement follows by setting $\lambda_n=\frac{\alpha_n}{r}>0$ for every $n\in\mathbb{N}$ and $Y=\frac{Z}{r}\in\mathcal{D}$.

Step 2 (Countable separation). There exists a sequence $(Y_n)\subset\mathcal{D}$ such that

$$\text{for every nonzero } X\in\mathcal{X}_+ \text{ there exists } n\in\mathbb{N} \text{ such that } \mathbb{E}_{\mathbb{P}}[XY_n]>0. \tag{1.15}$$

As a preliminary step, note that for every nonzero $X\in\mathcal{X}_+$ there exists $\lambda>0$ such that $(\lambda X,0)\notin\mathcal{C}$ by Lemma 1.5.4. Since $\mathcal{C}$ is closed and $(0,n)\notin\mathcal{C}$ for some $n\in\mathbb{N}$ by Theorem 1.3.14, we can use the representation of $\mathcal{C}$ recorded in Corollary 1.4.12 to find an element $Y_X\in\mathcal{D}$ such that $\mathbb{E}_{\mathbb{P}}[\lambda XY_X]>\sigma^{\mathcal{C}}(Y_X,1)\geq 0$. Equivalently, we have that

$$\text{for every nonzero } X\in\mathcal{X}_+ \text{ there exists } Y_X\in\mathcal{D} \text{ such that } \mathbb{E}_{\mathbb{P}}[XY_X]>0. \tag{1.16}$$

To establish (1.15), we start by showing that the family $\mathscr{G}=\{\{Y>0\}\,:\,Y\in\mathcal{D}\}$ is nonempty and closed under countable unions. That $\mathscr{G}$ is nonempty follows from (1.16). To show that $\mathscr{G}$ is closed under countable unions, take an arbitrary sequence $(Y_n)\subset\mathcal{D}\setminus\{0\}$. By Step 1, we find a sequence $(\lambda_n)\subset(0,\infty)$ and an element $Y\in\mathcal{D}$ such that $S_n=\sum_{k=1}^n\lambda_kY_k\to Y$. It is easy to see that

$$\{Y>0\}=\bigcup_{n\in\mathbb{N}}\{Y_n>0\}\quad\mathbb{P}\text{-almost surely.} \tag{1.17}$$

Indeed, consider first the event $E=\{Y>0\}\cap\bigcap_{n\in\mathbb{N}}\{Y_n=0\}$. We must have $\mathbb{P}(E)=0$ for otherwise

$$0<\mathbb{E}_{\mathbb{P}}[\mathbb{1}_EY]=\lim_{n\to\infty}\mathbb{E}_{\mathbb{P}}[\mathbb{1}_ES_n]=0.$$

As a result, the inclusion "$\subset$" in (1.17) must hold. Next, we claim that $\mathbb{P}(Y \geq S_n) = 1$ for every $n \in \mathbb{N}$. If not, we find $k \in \mathbb{N}$ and $\varepsilon > 0$ such that the event $E = \{Y \leq S_k - \varepsilon\}$ satisfies

$$0 < \varepsilon\mathbb{P}(E) \leq \mathbb{E}_{\mathbb{P}}[\mathbb{1}_E(S_k - Y)] \leq \lim_{n\to\infty} \mathbb{E}_{\mathbb{P}}[\mathbb{1}_E(S_n - Y)] = 0.$$

This delivers the inclusion "$\supset$" in (1.17) and shows that $\mathscr{G}$ is closed under countable unions as desired.

Now, set $s = \sup\{\mathbb{P}(E) \ : \ E \in \mathscr{G}\}$. Take any sequence $(Y_n) \subset \mathcal{D}$ such that $\mathbb{P}(Y_n > 0) \uparrow s$. By closedness under countable unions, there must exist $Y^* \in \mathcal{D}$ such that $\{Y^* > 0\} = \bigcup_{n\in\mathbb{N}}\{Y_n > 0\}$ $\mathbb{P}$-almost surely. Take an arbitrary nonzero $X \in \mathcal{A} \cap \mathcal{X}$ and assume that $\mathbb{E}_{\mathbb{P}}[XY_n] = 0$ for every $n \in \mathbb{N}$. This would imply that $\mathbb{E}_{\mathbb{P}}[XY^*] = 0$ and, thus, the element $\frac{1}{2}Y^* + \frac{1}{2}Y_X \in \mathcal{D}$ would satisfy

$$\mathbb{P}\left(\frac{1}{2}Y^* + \frac{1}{2}Y_X > 0\right) \geq \mathbb{P}(Y^* > 0) + \mathbb{P}(\{Y^* = 0\} \cap \{Y_X > 0\}) > \mathbb{P}(Y^* > 0) = s,$$

which cannot hold. In conclusion, we must have $\mathbb{E}_{\mathbb{P}}[XY_n] > 0$ for some $n \in \mathbb{N}$, showing (1.15). $\square$

Remark 1.5.7. (i) Note that instead of proving countable separation and applying the Kreps-Yan Theorem, we could have shown that $s = 1$ in the second step, and this would have immediately implied that D^* is a strictly consistent pricing density. This approach heavily relies on the fact that we have chosen the positive cone as acceptance set. We have preferred to derive the result by applying the Kreps-Yan Theorem since this argument can be generalized to other choices of $\mathcal{A}$ (see Theorem 1.5.9).

(ii) Countable separation can also be obtained applying the well known Halmos-Savage theorem in [56]. Indeed recall that $\mathcal{D} \subset \mathcal{X}'_+ \subset L^1(\mathbb{P})_+$. By (1.16), the collection $(Y_X)_{X\in\mathcal{X}_+\setminus\{0\}}$ induces a family of finite measures on $(\Omega, \mathcal{F}, \mathbb{P})$ that is equivalent to $\mathbb{P}$ (meaning that $\mathbb{P}(E) = 0$ if and only if $\mathbb{E}_{\mathbb{P}}[\mathbb{1}_E Y_X] = 0$ for every nonzero $X \in \mathcal{X}_+$). Note that this is no more valid if we consider a larger acceptance set instead of the positive cone. By the Halmos-Savage Theorem, there is a countable subfamily $(Y_n)_n \subset (Y_X)_{X\in\mathcal{X}_+\setminus\{0\}}$ that is equivalent to $\mathbb{P}$, ensuring that for every nonzero $X \in \mathcal{X}_+$ we have $\mathbb{E}_{\mathbb{P}}[XY_n] > 0$ for some n. Actually, the proof of the celebrated Halmos-Savage theorem in [56], relies on an argument similar to the exhaustion procedure we have used in proving countable separation.

(iii) The above theorem is stated under the conditions $\mathcal{S} \subset \mathcal{X}$. It is not difficult to derive a formulation of the Fundamental Theorem without imposing any condition on $\mathcal{S}$. In this case, assume that there exists no scalable arbitrage opportunity. We can always find a probability measure $\mathbb{Q}$ that is equivalent to $\mathbb{P}$ and satisfies $\frac{d\mathbb{Q}}{d\mathbb{P}} \in L^\infty(\mathbb{P})$ and $\mathcal{S} \subset L^1(\mathbb{Q})$. Namely, we have to impose

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{c}{1 + \sum_{i=1}^N |S_i|}$$

for a suitable normalizing constant $c > 0$. Let $\mathcal{X} = L^1(\mathbb{Q})$ and $\mathcal{X}' = L^\infty(\mathbb{Q})$. A direct application of the above theorem (with $\mathbb{Q}$ replacing $\mathbb{P}$) yields the existence of $D_{\mathbb{Q}} \in L^\infty(\mathbb{Q})$ such that $D_{\mathbb{Q}}$ is strictly positive with respect to $\mathbb{Q}$ and

$$\sup_{X\in\mathcal{M}} \{\mathbb{E}_{\mathbb{Q}}[D_{\mathbb{Q}}X] - \pi(X)\} < \infty.$$

As a result, the random variable $D_{\mathbb{P}} = \frac{d\mathbb{Q}}{d\mathbb{P}} D_{\mathbb{Q}}$ satisfies the following properties:

(1) $D_{\mathbb{P}}X \in L^1(\mathbb{P})$ for every $X \in \mathcal{S}$,

(2) $\mathbb{E}_{\mathbb{P}}[D_{\mathbb{P}}X] > 0$ for every nonzero $X \in L^1(\mathbb{Q})_+$,

(3) $\sup\{\mathbb{E}_{\mathbb{P}}[D_{\mathbb{P}}X] - \pi(X) \ : \ X \in \mathcal{M}\} < \infty$.

This shows that $D_{\mathbb{P}}$ is a pricing density with respect to $L^0(\mathbb{P})$ that is strictly consistent with $L^0(\mathbb{P})_+$. Note that the fact that in point (2) we can derive that $\mathbb{E}_{\mathbb{P}}[D_{\mathbb{P}}X] > 0$ for every nonzero $X \in L^1(\mathbb{P})_+$ depends on the choice of the positive cone as acceptance set.

We aim to extend the Fundamental Theorem of Asset Pricing to general acceptance sets beyond the positive cone. One may wonder whether the proof of the above theorem can be adapted to achieve this objective. However, as remarked, our proof builds on a suitable application of the exhaustion argument underpinning the classical Halmos-Savage Theorem in [56] that breaks down in the presence of nonpositive acceptable payoffs. We are therefore forced to pursue a different strategy. Inspired by the original work by Kreps [69], we work under a suitable separability assumption. To be able to state our desired result for a general acceptance set, Theorem 1.5.2 suggests that a convenient "conification" of the acceptance set is necessary. This leads to considering the modified acceptance set

$$\mathcal{K}(\mathcal{A}) := \mathrm{cl}(\mathrm{cone}(\mathcal{A}) \cap \mathcal{X}) + L^0(\mathbb{P})_+,$$

where we have denoted by cl the closure operator with respect to the reference topology $\sigma(\mathcal{X}, \mathcal{X}')$. As a preliminary step to our main result, the next lemma collects some relevant properties of $\mathcal{K}(\mathcal{A})$.

Lemma 1.5.8. *The set $\mathcal{K}(\mathcal{A})$ is a conic acceptance set satisfying $\mathcal{K}(\mathcal{A}) \cap \mathcal{X} = \mathrm{cl}(\mathrm{cone}(\mathcal{A}) \cap \mathcal{X})$. In particular, if $\mathcal{A}$ is conic, then $\mathcal{K}(\mathcal{A}) \cap \mathcal{X} = \mathcal{A} \cap \mathcal{X}$.*

Proof. It is readily seen that $\mathcal{K}(\mathcal{A})$ is a conic acceptance set. Note that $\mathcal{K}(\mathcal{A}) \cap \mathcal{X} = \mathrm{cl}(\mathrm{cone}(\mathcal{A}) \cap \mathcal{X}) + \mathcal{X}_+$, and the last set coincides with $\mathrm{cl}(\mathrm{cone}(\mathcal{A}) \cap \mathcal{X})$ since it is monotone in $\mathcal{X}$. To see this monotonicity, recall that $\mathcal{A}$, and hence $\mathrm{cone}(\mathcal{A})$, is monotone in $L^0(\mathbb{P})$. It follows that $\mathrm{cone}(\mathcal{A}) \cap \mathcal{X}$ is monotone in $\mathcal{X}$ and its closure is monotone as well. □

We are finally in a position to state the announced version of the Fundamental Theorem of Asset Pricing for a general acceptance set.

Theorem 1.5.9 (Fundamental Theorem of Asset Pricing)**.** *Let the norm predual of $\mathcal{X}'$ be separable with respect to its norm topology and assume that one of the following sets of conditions holds:*

(i) $\mathcal{A}$ is a cone with $\mathcal{A} \cap (-\mathcal{A}) = \{0\}$ and there exists no scalable acceptable deal.

(ii) $\mathcal{K}(\mathcal{A}) \cap (-\mathcal{K}(\mathcal{A})) = \{0\}$ and there exists no scalable acceptable deal with respect to $\mathcal{K}(\mathcal{A})$.

Then, there exists a strictly-consistent pricing density in $\mathcal{X}'$.

Proof. It follows from Lemma 1.5.8 that $\mathcal{K}(\mathcal{A})$ is a conic acceptance set such that $\mathcal{K}(\mathcal{A}) \cap \mathcal{X}$ is closed and coincides with $\mathrm{cl}(\mathrm{cone}(\mathcal{A}) \cap \mathcal{X})$. Note that every pricing density that is (strictly) consistent with $\mathcal{K}(\mathcal{A})$ is also (strictly) consistent with $\mathcal{A}$. As a result, it suffices to prove the statement under *(i)*. Hence, assume that *(i)* holds. Like we have done for the other fundamental theorem, we divide the proof in two steps, each one showing that one of the properties required in Theorem 1.5.2 holds for $\mathcal{L} = \mathcal{A} \cap \mathcal{X}$ and $\mathcal{L}' = \mathcal{D}$. The existence of a strictly consistent pricing density in $\mathcal{X}'$ is then ensured by the said theorem.

Step 1 (Σ-completeness). For every sequence $(Y_n) \subset \mathcal{D}$ there exists a sequence $(\lambda_n) \subset (0, \infty)$ and $Y \in \mathcal{D}$ such that $\sum_{k=1}^{n} \lambda_k Y_k \to Y$.

See the proof of this step in Theorem 1.5.6.

Step 2 (Countable separation). There exists a sequence $(Y_n) \subset \mathcal{D}$ such that

$$\textit{for every nonzero } X \in \mathcal{A} \cap \mathcal{X} \textit{ there exists } n \in \mathbb{N} \textit{ such that } \mathbb{E}_{\mathbb{P}}[XY_n] > 0. \tag{1.18}$$

As a preliminary step, note that for every nonzero $X \in \mathcal{A} \cap \mathcal{X}$ there exists $\lambda > 0$ such that $(\lambda X, 0) \notin \mathcal{C}$ by Lemma 1.5.4. Since $\mathcal{C}$ is closed and $(0, n) \notin \mathcal{C}$ for some $n \in \mathbb{N}$ by Theorem 1.3.14, we can use the representation of $\mathcal{C}$ recorded in Corollary 1.4.12 to find an element $Y_X \in \mathcal{D}$ such that $\mathbb{E}_{\mathbb{P}}[\lambda X Y_X] > \sigma^{\mathcal{C}}(Y_X, 1) \geq 0$. Equivalently, we have that

$$\text{for every nonzero } X \in \mathcal{A} \cap \mathcal{X} \text{ there exists } Y_X \in \mathcal{D} \text{ such that } \mathbb{E}_{\mathbb{P}}[XY_X] > 0. \tag{1.19}$$

Recall that $\mathcal{X}'$ is a norm dual and denote by $\|\cdot\|_{\mathcal{X}'}$ the corresponding dual norm. For every nonzero $X \in \mathcal{A} \cap \mathcal{X}$ consider the rescaled couple

$$(Z_X, r_X) = \left(\frac{Y_X}{\|Y_X\|_{\mathcal{X}'}}, \frac{1}{\|Y_X\|_{\mathcal{X}'}} \right) \in -\mathrm{bar}(\mathcal{C}).$$

As the norm predual of $\mathcal{X}'$ is separable by assumption, the unit ball in $\mathcal{X}'$ is weak-star metrizable by Theorem 6.30 in Aliprantis and Border [1]. Being weak-star compact by virtue of the Banach-Alaoglu Theorem, see e.g. Theorem 6.21 in Aliprantis and Border [1], the unit ball together with any of its subsets is therefore weak-star separable. In particular, this is true for $\{Z_X : X \in (\mathcal{A} \cap \mathcal{X}) \setminus \{0\}\}$. Since our reference topology on $\mathcal{X}'$, namely $\sigma(\mathcal{X}', \mathcal{X})$, was assumed to be weaker than the weak-star topology, it follows that $\{Z_X : X \in (\mathcal{A} \cap \mathcal{X}) \setminus \{0\}\}$ is also separable with respect to $\sigma(\mathcal{X}', \mathcal{X})$. Let $\{Z_{X_n} : n \in \mathbb{N}\}$ be a countable dense subset. Then, for every nonzero $X \in \mathcal{A} \cap \mathcal{X}$, it follows immediately from (1.19) that we must have $\mathbb{E}_{\mathbb{P}}[XY_{X_n}] > 0$ for some $n \in \mathbb{N}$ by density. This delivers (1.18). □

Remark 1.5.10. (i) The separability of the norm predual of $\mathcal{X}'$ is typically ensured by suitable assumptions on the underlying σ-field. For concreteness, consider the case where $\mathcal{X}' = L^\infty(\mathbb{P})$, which is interesting because it delivers bounded pricing densities. In this case, the norm predual is $L^1(\mathbb{P})$. A simple sufficient condition for separability is that $\mathcal{F}$ is countably generated. A characterization of separability in the nonatomic setting can be found, e.g., in Theorem 13.16 in Aliprantis and Border [1]. It is worthwhile highlighting that separability of the predual may hold even if the reference payoff space $\mathcal{X}$ is not separable with respect to a pre-specified natural topology. Consider for instance the case where $\mathcal{F}$ is countably generated and $\mathcal{X}$ is an Orlicz space with a non Δ_2 Orlicz function (for details and terminology see the next section). If the probability space is nonatomic, then $\mathcal{X}$ fails to coincide with its Orlicz heart, and by Theorem 1, section 3.5 in Rao and Ren [83], $\mathcal{X}$ is not separable. Nonetheless we can pair this choice of $\mathcal{X}$ with $\mathcal{X}' = L^\infty(\mathbb{P})$ that is the norm dual of the separable space $L^1(\mathbb{P})$.

(ii) The case where $\mathcal{A}$ is a cone is the relevant one in light of Theorem 1.4.17. In this case, the above sets of assumptions are equivalent due to Lemma 1.5.8. In the general convex case, to establish the existence of a strictly-consistent pricing density we had to "conify" the acceptance set $\mathcal{A}$ so as to obtain another acceptance set $\mathcal{K}(\mathcal{A})$ satisfying the same standing assumptions. As mentioned above, this was suggested by the statement of Theorem 1.5.2. A more direct way to see that a "conification" is necessary is to observe that every strictly-consistent pricing density is automatically strictly consistent for the acceptance set $\mathcal{K}(\mathcal{A})$. Incidentally, note that this is also true for the more natural "conified" acceptance set $\operatorname{cone}(\mathcal{A})$. However, the problem with $\operatorname{cone}(\mathcal{A})$ is that the intersection $\operatorname{cone}(\mathcal{A}) \cap \mathcal{X}$, or equivalently $\operatorname{cone}(\mathcal{A} \cap \mathcal{X})$, need not be closed and, hence, our standing assumptions need not hold.

(iii) The pointedness conditions can be slightly weakened. Indeed, it suffices that $\mathcal{A} \cap (-\mathcal{A}) \cap \mathcal{X} = \{0\}$ and $\mathcal{K}(\mathcal{A}) \cap (-\mathcal{K}(\mathcal{A})) \cap \mathcal{X} = \{0\}$ hold, respectively. In view of Lemma 1.5.8, the latter condition is equivalent to $\operatorname{cl}(\operatorname{cone}(\mathcal{A}) \cap \mathcal{X}) \cap (-\operatorname{cl}(\operatorname{cone}(\mathcal{A}) \cap \mathcal{X})) = \{0\}$.

The following example helps appreciate the preceding version of the Fundamental Theorem of Asset Pricing by showing that, in the presence of a nonconic acceptance set, the conditions on the "conified" acceptance set stipulated above are necessary for the existence of a strictly-consistent pricing density.

Example 1.5.11. *Let $\Omega = \{\omega_1, \omega_2\}$ and assume that $\mathcal{F}$ is the power set of Ω and that $\mathbb{P}$ is specified by $\mathbb{P}(\omega_1) = \mathbb{P}(\omega_2) = \frac{1}{2}$. In this simple setting, we take $\mathcal{X} = \mathcal{X}' = L^0(\mathbb{P})$ and identify every element of $L^0(\mathbb{P})$ with a vector of $\mathbb{R}^2$. Set $\mathcal{S} = \mathcal{M} = \mathbb{R}^2$ and $\pi(x, y) = \max\{x, y\}$ for every $(x, y) \in \mathbb{R}^2$ and define*

$$\mathcal{A} = \mathbb{R}^2_+ \cup \{(x, y) \in \mathbb{R}^2 : x < 0, y \geq x^2\}.$$

In this case, $\mathcal{M}$ and π are conic but $\mathcal{A}$ is not. All the requirements in the stipulated assumptions are satisfied and so are all the conditions in Theorem 1.5.9 with $\mathcal{A}$ in place of $\mathcal{K}(\mathcal{A})$. However, there exists no strictly-consistent pricing density $D = (d_1, d_2)$. Indeed, we could otherwise take $X_\lambda = (-\lambda, \lambda^2) \in \mathcal{A}$ for every $\lambda > 0$ and use the definition of a consistent pricing density to obtain

$$\mathbb{E}_{\mathbb{P}}[DX_\lambda] > 0 \implies d_2\lambda > d_1$$

for every $\lambda > 0$, which would contradict the strict positivity, hence the strict consistency, of D. What goes wrong is that there exists a scalable acceptable deal with respect to $\mathcal{K}(\mathcal{A})$. To see this, it suffices to note that $\mathcal{K}(\mathcal{A}) = \{(x, y) \in \mathbb{R}^2 : y \geq 0\}$, which shows that $X = (-1, 0)$ is indeed a scalable acceptable deal with respect to $\mathcal{K}(\mathcal{A})$.

By combining the results obtained in this section, namely Proposition 1.4.5 and Theorems 1.5.6 and 1.5.9, it is possible to reformulate the Fundamental Theorem of Asset Pricing in the form of an equivalence between the absence of scalable acceptable deals and the existence of strictly-consistent pricing densities.

Corollary 1.5.12 (Fundamental Theorem of Asset Pricing). *Assume that either $\mathcal{A} = L^0(\mathbb{P})_+$ or the norm predual of $\mathcal{X}'$ is separable with respect to its norm topology and $\mathcal{A}$ is a cone with $\mathcal{A} \cap (-\mathcal{A}) = \{0\}$. Then, the following statements are equivalent:*

(i) There exists no scalable acceptable deal.

(ii) There exists a strictly-consistent pricing density.

In this case, the pricing density can be taken in $\mathcal{X}'$.

1.5.1 EXTENSION OF THE FUNDAMENTAL THEOREM

In the proofs of our fundamental theorems, the role played by the absence of scalable acceptable deals is to determine the validity of the dual representation of $\mathcal{C}$, together with the condition $(\lambda X, 0) \notin \mathcal{C}$ for some $\lambda > 0$ for any nonzero acceptable payoff X in $\mathcal{X}$, in the case where $\mathcal{A}$ is a cone and satisfies a suitable pointedness condition. These two facts combined ensure the existence of pricing densities in $\mathcal{D}$ that are strictly positive on elements of $\mathcal{A} \cap \mathcal{X}$ (see equations (1.16) and (1.19)). Actually, the same holds if we cope with the closure of $\mathcal{C}$ instead of $\mathcal{C}$ itself. This means that we can weaken the no-scalable acceptable deals assumption and replace it with a pair of requirements: the condition $(\lambda X, 0) \notin \mathrm{cl}(\mathcal{C})$ for some $\lambda > 0$ for any nonzero acceptable payoff X in $\mathcal{X}$, together with the validity of the dual representation of $\mathrm{cl}(\mathcal{C})$. By Theorem 1.3.14 the last condition is almost equivalent to $(0, n) \notin \mathrm{cl}(\mathcal{C})$ for some $n \in \mathbb{N}$. We provide a financial interpretation for these assumptions as absence of some type of "good deals". Namely, $(\lambda X, 0) \notin \mathrm{cl}(\mathcal{C})$ means that the market does not offer something that is almost better than λX at almost zero cost, while $(0, n) \notin \mathrm{cl}(\mathcal{C})$ means that it is not possible to gain almost n units of the fixed currency when buying almost acceptable admissible replicable payoffs

We state the announced general version of the fundamental theorem at the end of this section since we have opted for stressing first of all the role of the no scalable acceptable deal condition. One, of course, could proceed the other way round, and derive the two other fundamental theorems as corollaries of this stronger result.

Theorem 1.5.13 (Fundamental Theorem of Asset Pricing, strong version). *Assume that either $\mathcal{A} = L^0(\mathbb{P})_+$ or the norm predual of $\mathcal{X}'$ is separable with respect to its norm topology, and $\mathcal{A}$ is a cone such that $\mathcal{A} \cap (-\mathcal{A}) = \{0\}$. Moreover assume that the following conditions hold:*

(a) There exists $n \in \mathbb{N}$ such that $(0, n) \notin \mathrm{cl}(\mathcal{C})$.

(b) For every nonzero $X \in \mathcal{A} \cap \mathcal{X}$ there exists $\lambda > 0$ such that $(\lambda X, 0) \notin \mathrm{cl}(\mathcal{C})$.

Then there exists a strictly consistent pricing density in $\mathcal{X}'$.

Proof. Like in the proofs of the other fundamental theorems, we aim to apply the Kreps-Yan Theorem 1.5.2 to $\mathcal{L} = \mathcal{A} \cap \mathcal{X}$ and $\mathcal{L}' = \mathcal{D}$. Hence we only need to verify the two properties required in that theorem. Σ-completeness holds like in the other fundamental theorems. For proving countable separation, note that we can used the representation of $\mathrm{cl}(\mathcal{C})$ recorded in Theorem 1.4.9 since $(0, n) \notin \mathrm{cl}(\mathcal{C})$ for some $n \in \mathbb{N}$. Moreover, by assumption we have that for every nonzero $X \in \mathcal{A} \cap \mathcal{X}$, there is $\lambda > 0$ such that $(\lambda X, 0) \notin \mathrm{cl}(\mathcal{C})$. The proof proceeds exactly like in Theorems 1.5.6 and 1.5.9, as $\sigma^{\mathcal{C}} = \sigma^{\mathrm{cl}(\mathcal{C})}$ and $\mathrm{bar}(\mathcal{C}) = \mathrm{bar}(\mathrm{cl}(\mathcal{C}))$. □

Remark 1.5.14. (i) (Beyond the finite dimensionality of $\mathcal{S}$) The force of Theorem 1.5.13 lies in the fact that its proof does not rely on the fact that $\mathcal{S}$ is a finite dimensional subspace of $\mathcal{X}$. Indeed, in the previous sections we have used this assumption to prove that the absence of scalable acceptable deals implies that $\mathcal{C}$ is closed. Now, we do not need this step as we work with the closure of $\mathcal{C}$, which may or may not coincide with $\mathcal{C}$. Hence the extended fundamental theorem remains valid in more general models, for instance in discrete multiperiodal or continuous setting, where the space of portfolios is generally not identified with $\mathbb{R}^N$ and the set of replicable payoffs $\mathcal{S}$ may have infinite dimension. Note that, if we work in the more general setting where $\mathcal{S}$ is not required to be finite dimensional, then the preceding Fundamental Theorems (Theorems 1.5.6 and 1.5.9) fail, as they are based on the fact the absence of acceptable deals furnishes the closedness of $\mathcal{C}$.

(ii) We show that in markets with proportional transaction costs and one frictionless asset, assumption *(b)* implies *(a)*. Assume that $\mathcal{A}$, $\mathcal{M}$ and π are conic, and $U \in \mathcal{A} \cap \mathcal{M}$ exists such that the market has no frictions and no constraints in the direction U. This means that $\mathcal{M} = \mathcal{M} + \mathbb{R}U$ and $\pi(X + \lambda U) = \pi(X) + \lambda\pi(U)$ for every $X \in \mathcal{M}$ and $\lambda \in \mathbb{R}$. To show that *(b)* implies *(a)*, fix $n \in \mathbb{N}$ such that $n \geq \pi(U)$ and assume by contradiction that $(0, n) \in \mathrm{cl}(\mathcal{C})$. Hence we find nets $X_\alpha \to 0$, $m_\alpha \to n$ and $(Z_\alpha) \subset \mathcal{M}$ such that $\pi(Z_\alpha) \leq -m_\alpha$ and $Z_\alpha - X_\alpha \in \mathcal{A}$. Now, define nets $\widetilde{X}_\alpha = X_\alpha + U \to U$, $\widetilde{m}_\alpha = m_\alpha - n \to 0$, $(\widetilde{Z}_\alpha = Z_\alpha + U) \subset \mathcal{M}$ and note that $\pi(\widetilde{Z}_\alpha) \leq -\widetilde{m}_\alpha$ and $\widetilde{Z}_\alpha - \widetilde{X}_\alpha = Z_\alpha - X_\alpha \in \mathcal{A}$. This implies that $(U, 0) \in \mathrm{cl}(\mathcal{C})$ and contradicts assumption *(b)* since $\mathrm{cl}(\mathcal{C})$ is conic.

1.5.2 Fundamental Theorems of Asset Pricing with acceptable risk in the literature

General Fundamental Theorems of Asset Pricing in the context of pricing with acceptable risk were established in Jaschke and Küchler [60], Staum [91] and Cherny [32]. In these three papers, the set $\mathcal{M}$ was not assumed to be embedded into a finite-dimensional space to allow for applications to multi-period models. In the first two, the focus was on abstract payoff spaces beyond the setting of random variables. In this section we analyze the Fundamental Theorems proved in these papers and we compare them with our results. We also consider the Fundamental Theorem established in Pennanen [76], where the context is that of arbitrages and not of acceptable deals, and the market is assumed to have convex frictions and constraints.

The specific focus of Jaschke and Küchler [60] is on markets with proportional frictions admitting at least one frictionless asset with payoff $\mathbf{1}$. Together with the convex cone $\mathcal{A}$, they consider an other convex cone M, the set of «cash streams that can be generated by trading with zero initial endowment», such that $\mathbf{1} \in \mathcal{A} - M$. In order to make a comparison with our model, we try to explicit the «market prices implicit in M» mentioned in footnote 12 on page 194 [60]. To this end, we need an additional assumption, that is $\mathbf{1} \notin \mathrm{span}(M)$. Let $\mathcal{M} := M + \mathbb{R}\mathbf{1}$ be the space of admissible replicable payoffs, and π the pricing functional defined on $\mathcal{M}$ as $\pi(Z + \alpha\mathbf{1}) = \alpha$ for $Z \in M$ and $\alpha \in \mathbb{R}$. Clearly $M = \ker(\pi)$, which agrees with their statement: «M plays here the same role as the final pay-outs of self-financing strategies with initial endowment of 0 do in the classical theory». By Remark 1.3.15, our set $\mathcal{C}$ is closed if and only if $\mathcal{A} - M$ is closed.

They assume the absence of so called *good deals of second kind*. It is easy to observe that this condition has many characterizations: $-\mathbf{1} \notin \mathcal{A} - M$, or $(\mathbf{1}, 0) \notin \mathcal{C}$, or $(X, 0) \notin \mathcal{C}$ for every nonzero $X \in \mathcal{A}$, or $(0, n) \notin \mathcal{C}$ for some (every) $n \in \mathbb{N}$. Written in our terms, the absence of good deals of second kind corresponds to $\mathcal{A} \cap \{Z \in \mathcal{M}, : \ \pi(Z) < 0\} = \emptyset$. Their fundamental theorem (Corollary 8 [60]) states that if $\mathcal{A} - M$ is closed in the weak topology and there are no good deals of second kind, then there exists a functional ψ in the right polar cone of $\mathcal{A} - M$ (that is a functional positive $\mathcal{A}$ and negative on M) such that $\psi(\mathbf{1}) = 1$. The obtained ψ corresponds to a pricing density, since it is dominated by π on $\mathcal{M}$:

$$\psi(Z + \alpha\mathbf{1}) = \psi(Z) + \alpha \leq \alpha = \pi(Z + \alpha\mathbf{1})$$

for every $Z \in M$ and $\alpha \in \mathbb{R}$. Note that their fundamental theorem is expressed in terms of consistent (not strictly-consistent) pricing densities. As a result, it is not possible to apply it to derive a characterization of market-consistent prices similar to Theorem 1.4.17. The proof is a straightforward application of the bipolar theorem: the separation of $-\mathbf{1}$ from $\mathcal{A} - M$ gives the desired functional, up to a normalization. In addition, due to the generality of $\mathcal{M}$, the Fundamental Theorem is stated under an assumption that corresponds to the closedness of our set $\mathcal{C}$ but no concrete conditions for it to hold are provided. In particular, the absence of good deal of second kind does not determine closedness of $\mathcal{C}$ even in finite dimensional cases as shown in the next example.

Example 1.5.15. *Consider Example 1.3.16. We have already proved that $\mathcal{C}$ is not closed in this case. In order to show one of the equivalent conditions to the absence of good deals of second kind, we define $\mathbf{1} = (0, 1, 0)$, and $M = \ker(\pi) = \mathrm{span}\{(0, 0, 1)\}$. Since $-\mathbf{1} \notin \mathcal{A} - M = \mathcal{A} + M$, there are no good deals of second kind.*

Differently from [60], in the model of Staum [91], the pricing functional is not translation invariant with respect to any reference asset. The 1st Fundamental Theorem (Theorem 6.2 [91]) is stated under the assumption of convexity of $\mathcal{A}$ and π, and is based on Lemma 6.1 [91], whose thesis is the existence of a strictly positive functional nonnegative on $\mathcal{A}$ and dominated by π (in his notation, a *consistent pricing kernels*). These functionals correspond to strictly-positive (not strictly-consistent) pricing densities. As a result, it is not possible to apply the fundamental theorem to derive a characterization of market-consistent prices similar to Theorem 1.4.17 unless $\mathcal{A}$ is the positive cone.

He supplies to the lack of homogeneity via a conification process by considering an artificial market with acceptance set cone$(\mathcal{A})$ and pricing functional cone(π). The assumptions of Lemma 6.1 are lower semicontinuity of a functional that corresponds to π^+ in the conified market, and $\pi^+(X) > 0$ for any nonzero $X \in \mathcal{X}_+$. Note that under the assumption of lower semicontinuity of π^+, $\text{hypo}(-\pi^+) = \text{cl}(\mathcal{C})$, and the second assumption coincides then with $(X, 0) \notin \text{cl}(\mathcal{C})$ for every nonzero $X \in \mathcal{X}_+$. Moreover, since $\pi^+(0) = 0$, he has that $(0, n) \notin \text{cl}(\mathcal{C})$ for every $n > 0$. Sufficient conditions for the lower semicontinuity of the map π^+ are provided only in the space $\mathcal{X} = L^\infty(\mathbb{P})$ equipped with the canonical norm topology.

Unfortunately, the proof of the key instrumental Lemma 6.1 is flawed. On the one side, Zorn's Lemma is evoked to infer that a family of sets that is closed under countable unions admits a maximal element. However, this is not true as illustrated, for instance, by the family of all the countable subsets of $\mathbb{R}$. On the other side, it is tacitly assumed that, for a generic dual pair $(\mathcal{X}, \mathcal{X}')$, the series $\sum_{n\in\mathbb{N}} 2^{-n} Y_n$ converges in the topology $\sigma(\mathcal{X}', \mathcal{X})$ for every choice of $(Y_n) \subset \mathcal{X}'$, which cannot hold unless special assumptions are required of the pair $(\mathcal{X}, \mathcal{X}')$ as those stipulated in Assumption 1.5.1. The strategy in [91] was to replicate the exhaustion argument used in the classical proof of the Fundamental Theorem. Regardless of the above issues, this argument seems unlikely to work in an abstract setting beyond random variables because it heavily relies on the existence of a (dominating) probability measure. Moreover, even in that setting, it seems to work only when the acceptance set consists exclusively of positive random variables. This explains why we had to pursue a different strategy when dealing with more general acceptance sets in Theorem 1.5.9.

Markets with convex transaction costs and convex constraints are also studied in Cherny [32]. Like us, the author chooses as space of payoffs the space of random variables on a fixed probability space. He fixes a set $\mathcal{D}$ of probability measures which identifies by duality a coherent risk measure ρ and the corresponding conic convex acceptance set, and introduces suitable spaces (*weak and strong* L^1) depending on $\mathcal{D}$ that play the role of our $\mathcal{X}$ as they allow to restrict $L^0(\mathbb{P})$ to a space where it is possible to apply duality theory. It turns out that when the acceptance set is induced by expected shortfall or by expectation under $\mathbb{P}$, these spaces coincide with $L^1(\mathbb{P})$. The author does not explicit the underlying market, but he considers a convex subset of $L^0(\mathbb{P})$ of payoffs that he calls *attainable P&Ls*. This set corresponds to our $\{Z \in \mathcal{M} \,:\, \pi(Z) \leq 0\}$, and is assumed to be $\mathcal{D}$-*consistent*, meaning that whenever we find a consistent pricing density in the market restricted to the strong L^1 space, then we also find a consistent pricing density in the market on $L^0(\mathbb{P})$. This assumption supplies our $\mathcal{S} \subset \mathcal{X}$ and the repeated use of $\mathcal{A} \cap \mathcal{X}$ in spite of $\mathcal{A}$. A good deal is defined as an attainable P&L such that ρ takes a strictly negative value on it. The author establishes a Fundamental Theorem (Theorem 3.1) showing that when there are no good deals, one find a consistent (not necessarily strictly consistent) pricing density. The proof heavily relies on the assumption that $\mathcal{D}$ is weakly compact.

Finally, we compare our Theorem 1.5.13 to Theorem 5.2 of Pennanen [76] in case $\mathcal{X} = L^1(\mathbb{P})$ and $\mathcal{A} = L^0(\mathbb{P})_+$ (the author works in $L^0(\mathbb{P})$, but for the proof he reduces to the space of integrable functions). He considers the conified market cl(cone$(\mathcal{M})$), cl(cone(π)), and the corresponding set $\mathcal{C}$, and proves that, under the condition $\text{cl}(\mathcal{C}) \cap (\mathcal{X}_+ \times \mathbb{R}_+) = \{0\}$, one finds a strictly positive *market price deflator*. In our notation, this is a strictly positive (and hence strictly consistent) pricing density $D \in L^\infty(\mathbb{P})$ both for the conified market and for the original one, with $\gamma_{\pi,\mathcal{M}}(D) = 0$. The assumption of Theorem 1.5.13, when applied to the conified market, is equivalent to $\text{cl}(\mathcal{C}) \cap (\mathcal{X}_+ \times \mathbb{R}_+) = \{0\}$ as $\mathcal{C}$ is a cone. The intent of assuming $\text{cl}(\mathcal{C}) \cap (\mathcal{X}_+ \times \mathbb{R}_+) = \{0\}$, is to avoid those cases where the set $\mathcal{C}$ related to the original market does not intersect the positive cone but anyway is "tangent" to it. To catch the idea, consider the simple case where $\mathcal{X} = \mathbb{R}$, $\mathcal{M} = \mathcal{A} = \mathbb{R}_+$ and $\pi(X) = X^2$ for $X \geq 0$. Here, the original $\mathcal{C}$ is $\mathcal{C} = \{(X, m) \in \mathbb{R} \times (-\mathbb{R}_+);\ \text{if } X \geq 0 \text{ then } m \leq -X^2\}$. In this case, since there are no acceptable deals, Theorem 1.5.6 ensures the existence of a strictly consistent pricing density D for the original market, but $\gamma_{\pi,\mathcal{M}}(D) > 0$.

1.6 EXAMPLES

In this section we fix a class of reference payoff spaces and their duals such that the requirements of Assumptions 1.4.7 and 1.5.1 are fulfilled, and we discuss some concrete examples of acceptance sets. In particular, we highlight the weak-closedness property required in Assumption 1.4.7 of the chosen acceptance sets and provide some explicit features that consistent pricing densities must have for any specification of $\mathcal{M}$ and π. Note that, in order to apply our version of the Fundamental Theorem for general acceptance sets (Theorem 1.5.9), one also need a pointedness condition on the smallest cone containing the acceptance set.

The natural payoff spaces considered below belong to the broad class of Orlicz spaces. For an overview, see Appendix A. Throughout this section we assume that the underlying probability space $(\Omega, \mathcal{F}, \mathbb{P})$ is atomless, and we pair every Orlicz space $\mathcal{X} = L^{\Phi}(\mathbb{P})$ with $L^{\infty}(\mathbb{P})$. It is clear that Assumption 1.4.7 is satisfied. Moreover, the space $L^1(\mathbb{P})$ is known to be the norm predual of $L^{\infty}(\mathbb{P})$, and whenever $L^1(\mathbb{P})$ is separable (e.g. if $\mathcal{F}$ is countably generated), Assumption 1.5.1 is satisfied too, as $\sigma(L^{\infty}(\mathbb{P}), \mathcal{X}) \subset \sigma(L^{\infty}(\mathbb{P}), L^1(\mathbb{P}))$. As pointed out in Remark 1.5.10, this is compatible with choices of a non separable $\mathcal{X}$.

Before we start with our description of concrete acceptance sets, we highlight a number of sufficient conditions for the weak closedness required in Assumption 1.4.7 to hold. These conditions are easy to check and fulfilled by virtually all acceptance sets of interest. As a preliminary step, we recall the notion of law invariance and surplus invariance. Law invariance is a standard property in risk measure theory and stipulates that acceptability is only driven by the probability distribution of a payoff. Surplus invariance was introduced in Koch-Medina et al. [66] and thoroughly studied in Koch-Medina et al. [68] and Gao and Munari [53] and stipulates that acceptability is only driven by the downside profile of a payoff. For every random variable $X \in L^0(\mathbb{P})$ we denote by $\mathbb{P}_X$ the probability law of X under $\mathbb{P}$.

Definition 1.6.1. We say that $\mathcal{A}$ is *law invariant under* $\mathbb{P}$ if for all $X, Y \in L^0(\mathbb{P})$

$$X \in \mathcal{A},\ \mathbb{P}_X = \mathbb{P}_Y \implies Y \in \mathcal{A}.$$

We say that $\mathcal{A}$ is *surplus invariant* if for all $X, Y \in L^0(\mathbb{P})$

$$X \in \mathcal{A},\ X^- = Y^- \implies Y \in \mathcal{A}.$$

Proposition 1.6.2. *Let $\mathcal{X}$ be an Orlicz space and assume that one of the following conditions holds:*

(i) $\mathcal{A} \cap L^1(\mathbb{P})$ is closed with respect to the norm topology of $L^1(\mathbb{P})$.

(ii) $\mathcal{A}$ is law invariant under $\mathbb{P}$ and for every sequence $(X_n) \subset \mathcal{A} \cap \mathcal{X}$ and every $X \in \mathcal{X}$

$$X_n \to X \ \mathbb{P}\text{-almost surely},\ \sup_{n \in \mathbb{N}} |X_n| \in \mathcal{X} \implies X \in \mathcal{A} \cap \mathcal{X}.$$

(iii) $\mathcal{A}$ is surplus invariant and for every sequence $(X_n) \subset \mathcal{A} \cap \mathcal{X}$ and every $X \in \mathcal{X}$

$$X_n \to X \ \mathbb{P}\text{-almost surely},\ \sup_{n \in \mathbb{N}} |X_n| \in \mathcal{X} \implies X \in \mathcal{A} \cap \mathcal{X}.$$

Then, $\mathcal{A} \cap \mathcal{X}$ is closed with respect to $\sigma(\mathcal{X}, L^{\infty}(\mathbb{P}))$.

Proof. If *(i)* holds, then $\mathcal{A} \cap L^1(\mathbb{P})$ is $\sigma(L^1(\mathbb{P}), L^{\infty}(\mathbb{P}))$-closed by Theorem 5.98 in Aliprantis and Border [1]. Since $\mathcal{X}$ is contained in $L^1(\mathbb{P})$, the desired closedness follows immediately. Next, assume that *(ii)* holds. In this case, the set $\mathcal{A} \cap \mathcal{X}$ is norm closed. This is because every sequence in $\mathcal{X}$ that converges in norm admits a dominated subsequence that converges $\mathbb{P}$-almost surely. This follows from a straightforward extension of the proof of Theorem 13.6 in Aliprantis and Border [1] to the Orlicz setting. As a result, the desired closedness follows from Theorem 5.98 in Aliprantis and Border [1] when $\mathcal{X} = L^1(\mathbb{P})$ and from Proposition 1.1 in Svindland [92] when $\mathcal{X} = L^{\infty}(\mathbb{P})$. In all other cases it follows from Theorem 1.1 in Gao et al. [55]. Finally, if *(iii)* holds, the desired closedness follows from Theorem 1 in Gao and Munari [53]. $\square$

Moreover, we provide sufficient conditions for the pointedness condition of $\mathcal{A}$ required in Remark 1.5.5 to hold. This condition is necessary for applying Theorem 1.5.9.

Proposition 1.6.3. *Let $\mathcal{X}$ be an Orlicz space and assume that $\mathcal{A} \cap L^\Phi(\mathbb{P})$ is conic, law invariant and closed with respect to $\sigma(L^\Phi(\mathbb{P}), L^\infty(\mathbb{P}))$. Then, one of the following two alternatives holds:*

(i) $\mathcal{A} = \{X \in L^\Phi(\mathbb{P}) : \mathbb{E}_\mathbb{P}[X] \geq 0\}$.

(ii) $\mathcal{A} \cap (-\mathcal{A}) \cap L^\Phi(\mathbb{P}) = \{0\}$.

Proof. The result is a straightforward application of Proposition 5.8 in Bellini et al. [16]. □

1.6.1 EXPECTED SHORTFALL

A prominent example of acceptance set defined in terms of a risk measure is the one based on Expected Shortfall at some level $\alpha \in (0, 1)$. For a given random variable $X \in L^0(\mathbb{P})$ we define the Value at Risk of X at level α as the negative of the upper α-quantile of X, i.e.

$$\mathrm{VaR}_\alpha(X) := \inf\{x \in \mathbb{R} : \mathbb{P}(X + x < 0) \leq \alpha\} = -\inf\{x \in \mathbb{R} : \mathbb{P}(X \leq x) > \alpha\}.$$

The Expected Shortfall of X at level α is defined by

$$\mathrm{ES}_\alpha(X) := \frac{1}{\alpha} \int_0^\alpha \mathrm{VaR}_p(X) dp.$$

Intuitively speaking, $\mathrm{ES}_\alpha(X)$ coincides with the expectation of $-X$ conditional to the left tail beyond the upper α-quantile. Note that we always have $\mathrm{ES}_\alpha(X) \geq \mathrm{VaR}_\alpha(X) > -\infty$. It follows that the quantity $\mathrm{ES}_\alpha(X)$ is finite if and only if the negative part of X is integrable under $\mathbb{P}$. Next, set

$$\mathcal{A}_{\mathrm{ES}}(\alpha) := \{X \in L^0(\mathbb{P}) : \mathrm{ES}_\alpha(X) \leq 0\}.$$

In line with the above interpretation, the set $\mathcal{A}_{\mathrm{ES}}(\alpha)$ consists of all the payoffs that are positive on average on the left tail beyond their upper α-quantile.

The next proposition records the fact that $\mathcal{A}_{\mathrm{ES}}(\alpha)$ is eligible for being chosen as acceptance set in the model of this chapter, and satisfies the closedness requirement in Assumption 1.4.7 with respect to the pair $\mathcal{X} = L^\Phi(\mathbb{P})$, $\mathcal{X}' = L^\infty(\mathbb{P})$. Moreover, it shows a feature that pricing densities consistent with this acceptance set must have.

Proposition 1.6.4. *The set $\mathcal{A}_{\mathrm{ES}}(\alpha)$ is a conic acceptance set such that $\mathcal{A}_{\mathrm{ES}}(\alpha) \cap L^\Phi(\mathbb{P})$ is closed with respect to $\sigma(L^\Phi(\mathbb{P}), L^\infty(\mathbb{P}))$ for every Orlicz function Φ, and $\mathcal{A}_{\mathrm{ES}}(\alpha) \cap (-\mathcal{A}_{\mathrm{ES}}(\alpha)) \cap L^\Phi(\mathbb{P}) = \{0\}$. Moreover, every pricing density $D \in L^\infty(\mathbb{P})$ that is consistent with $\mathcal{A}_{\mathrm{ES}}(\alpha)$ satisfies $\frac{D}{\mathbb{E}_\mathbb{P}[D]} \leq \frac{1}{\alpha}$.*

Proof. Using the properties of the map ES_α, it can be verified that $\mathcal{A}_{\mathrm{ES}}(\alpha)$ is a conic acceptance set. Moreover,

$$\mathcal{A}_{\mathrm{ES}}(\alpha) \cap L^1(\mathbb{P}) = \{X \in L^1(\mathbb{P}) : \mathrm{ES}_\alpha(X) \leq 0\},$$

and it is well known that the acceptance set induced by expected shortfall in $L^1(\mathbb{P})$ is norm closed, implying that $\mathcal{A}_{\mathrm{ES}}(\alpha) \cap L^\Phi(\mathbb{P})$ is closed with respect to $\sigma(L^\Phi(\mathbb{P}), L^\infty(\mathbb{P}))$ by Proposition 1.6.2. The condition $\mathcal{A}_{\mathrm{ES}}(\alpha) \cap (-\mathcal{A}_{\mathrm{ES}}(\alpha)) \cap L^\Phi(\mathbb{P}) = \{0\}$ follows from Proposition 1.6.3. Finally, since every consistent pricing density D belongs to the barrier cone of $\mathcal{A}_{\mathrm{ES}}(\alpha) \cap L^\infty(\mathbb{P})$, it has to satisfy $\frac{D}{\mathbb{E}_\mathbb{P}[D]} \leq \frac{1}{\alpha}$ by Theorem 4.52 in Föllmer and Schied [49]. □

1.6.2 GAIN-LOSS RATIOS

Another prominent example of acceptance set defined in terms of a risk measure is the one based on the expectile at some level $\alpha \in (0, \frac{1}{2}]$. For a given random variable $X \in L^0(\mathbb{P})$ such that the positive or the negative part is integrable, we define the expectile of X at level α as the unique solution $e_\alpha(X) \in [-\infty, \infty]$ of the equation

$$\alpha\, \mathbb{E}_\mathbb{P}[(X - e_\alpha(X))^+] = (1 - \alpha)\, \mathbb{E}_\mathbb{P}[(e_\alpha(X) - X)^+].$$

For a random variable $X \in L^0(\mathbb{P})$ such that $\mathbb{E}_{\mathbb{P}}[X^+] = \mathbb{E}_{\mathbb{P}}[X^-] = \infty$, we set $e_\alpha(X) = -\infty$. Note that $e_\alpha(X)$ is finite if and only if X is integrable under $\mathbb{P}$, $e_\alpha(X) = -\infty$ if $\mathbb{E}_{\mathbb{P}}[X^-] = \infty$ and $e_\alpha(X) = \infty$ if $\mathbb{E}_{\mathbb{P}}[X^+] = \infty$ and $\mathbb{E}_{\mathbb{P}}[X^-] < \infty$. Now, set

$$\mathcal{A}_e(\alpha) := \{X \in L^0(\mathbb{P}) \,:\, e_\alpha(X) \geq 0\}.$$

It is not difficult to prove that

$$\mathcal{A}_e(\alpha) = \left\{X \in L^0(\mathbb{P}) \,:\, \frac{\mathbb{E}_{\mathbb{P}}[X^+]}{\mathbb{E}_{\mathbb{P}}[X^-]} \geq \frac{1-\alpha}{\alpha}\right\},$$

with the convention $\frac{\infty}{\infty} = -\infty$ and $\frac{0}{0} = \infty$. Note that for $\alpha = \frac{1}{2}$, $\mathcal{A}_e(\alpha)$ coincides with the set of random variables with positive expectation. The set $\mathcal{A}_e(\alpha)$ consists of all the payoffs such that the ratio between the expected inflow and the expected outflow of the corresponding payments is sufficiently large. In particular, note that $\frac{1-\alpha}{\alpha} \geq 1$ by assumption on α, which implies that the expected inflow must be at least large as the the expected outflow. This type of acceptability criterion was investigated in Bernardo and Ledoit [18], even though the link with expectiles was not discussed there.

With the next proposition we show that that $\mathcal{A}_e(\alpha)$ is a candidate acceptance set in our the model, and satisfies the closedness requirement in Assumption 1.4.7 with respect to $\mathcal{X} = L^\Phi(\mathbb{P})$ and $\mathcal{X}' = L^\infty(\mathbb{P})$. Moreover, we show a feature that pricing densities consistent with this acceptance set must have.

Proposition 1.6.5. *The set $\mathcal{A}_e(\alpha)$ is a conic acceptance set such that $\mathcal{A}_e(\alpha) \cap L^\Phi(\mathbb{P})$ is closed with respect to $\sigma(L^\Phi(\mathbb{P}), L^\infty(\mathbb{P}))$ for every Orlicz function Φ and $\mathcal{A}_e(\alpha) \cap (-\mathcal{A}_e(\alpha)) \cap L^\Phi(\mathbb{P}) = \{0\}$ unless $\alpha = \frac{1}{2}$. Moreover, every pricing density $D \in L^\infty(\mathbb{P})$ that is consistent with $\mathcal{A}_e(\alpha)$ satisfies $\frac{\operatorname{ess\,sup}(D)}{\operatorname{ess\,inf}(D)} \leq \frac{1-\alpha}{\alpha}$.*

Proof. From the definition of $\mathcal{A}_e(\alpha)$ and its characterization, it is easy to verify that $\mathcal{A}_e(\alpha)$ is a monotone set in $L^0(\mathbb{P})$ containing 0. By Proposition 7 in Bellini [15], the intersection

$$\mathcal{A}_e(\alpha) \cap L^1(\mathbb{P}) = \{X \in L^1(\mathbb{P}) \,:\, e_\alpha(X) \geq 0\}$$

is a convex cone, and hence $\mathcal{A}_e(\alpha)$ is a conic acceptance set. Using the characterization of $\mathcal{A}_e(\alpha)$, we see that $\mathcal{A}_e(\alpha) \cap L^1(\mathbb{P})$ is norm closed in $L^1(\mathbb{P})$, proving that $\mathcal{A}_e(\alpha) \cap L^\Phi(\mathbb{P})$ is closed with respect to $\sigma(L^\Phi(\mathbb{P}), L^\infty(\mathbb{P}))$ by Proposition 1.6.2. By Proposition 1.6.3, we have that $\mathcal{A}_e(\alpha) \cap (-\mathcal{A}_e(\alpha)) \cap L^\Phi(\mathbb{P}) = \{0\}$ unless $\alpha = \frac{1}{2}$.Finally, since every consistent pricing density D belongs to the barrier cone of $\mathcal{A}_e(\alpha)$, it has to satisfy $\frac{\operatorname{ess\,sup}(D)}{\operatorname{ess\,inf}(D)} \leq \frac{1-\alpha}{\alpha}$ by Proposition 8 [15]. □

1.6.3 Expected utility

Let $u : \mathbb{R} \to [-\infty, \infty)$ be a nonconstant, increasing, concave, right-continuous function satisfying $u(0) = 0$ and

$$\lim_{x \to -\infty} \frac{u(x)}{x} = \infty. \tag{1.20}$$

We interpret u as a classical von Neumann-Morgenstern utility function. The last condition requires that a rational agent with utility u does not asymptotically behave like a risk-neutral agent for large losses. The case of a risk-neutral agent is covered by Section 1.6.4. For a fixed level $\alpha \in (-\infty, 0]$ define

$$\mathcal{A}_u(\alpha) := \{X \in L^0(\mathbb{P}) \,:\, \mathbb{E}_{\mathbb{P}}[u(X)] \geq \alpha\}.$$

This set consists of all the payoffs that exhibit a minimal expected utility. In particular, the level α could coincide with some utility level, in which case $\mathcal{A}_u(\alpha)$ would consist of all the payoffs that are preferable, from the perspective of the utility function u, to a certain monetary loss. This type of acceptability criteria has been considered in a pricing context in Arai [2] and Arai and Fukasawa [3].

Like in the previous examples, we show that that $\mathcal{A}_u(\alpha)$ is an eligible acceptance set in our model. Moreover, we show that $\operatorname{rec}(\mathcal{A}_u(\alpha))$ coincides with the positive cone, and as a consequence every scalable acceptable deal is a scalable arbitrage.

Proposition 1.6.6. *The set $\mathcal{A}_u(\alpha)$ is an acceptance set such that $\mathcal{A}_u(\alpha) \cap L^\Phi(\mathbb{P})$ is closed with respect to $\sigma(L^\Phi(\mathbb{P}), L^\infty(\mathbb{P}))$ for every Orlicz function Φ and $(\mathcal{A}_u(\alpha) \cap L^\Phi(\mathbb{P}))^\infty = L^\Phi(\mathbb{P})_+$. Moreover, every pricing density $D \in L^\infty(\mathbb{P})$ that is consistent with $\mathcal{A}_u(\alpha)$ satisfies $\mathbb{E}_\mathbb{P}[u^*(\lambda D)] > -\infty$ for some $\lambda > 0$, where $u^*(y) := \inf_{x \in \mathbb{R}}\{xy - u(x)\}$.*

Proof. By definition, it is clear that $\mathcal{A}_u(\alpha)$ is an acceptance set. We show that $(\mathcal{A}_u(\alpha) \cap L^\Phi(\mathbb{P}))^\infty \subset L^\Phi(\mathbb{P})_+$. To this effect, take $X \in (\mathcal{A}_u(\alpha) \cap L^\Phi(\mathbb{P}))^\infty$ and assume that $\mathbb{P}(X < 0) > 0$. In this case, we find $\varepsilon > 0$ such that $\mathbb{P}(X \le -\varepsilon) > 0$. Set $E = \{X \le -\varepsilon\}$ and take $a, b \in \mathbb{R}$ such that $u(x) \le ax + b$ for every $x \in \mathbb{R}$. Then, for every $\lambda > 0$

$$\alpha \le \mathbb{E}_\mathbb{P}[u(\lambda X)] \le \mathbb{E}_\mathbb{P}[u(\lambda X)\mathbb{1}_E] + \mathbb{E}_\mathbb{P}[u(\lambda X)\mathbb{1}_{\{X \ge 0\}}] \le \mathbb{P}(E)u(-\lambda\varepsilon) + a\lambda\mathbb{E}_\mathbb{P}[X^+] + b.$$

However, this is not possible because the right-hand side above diverges to $-\infty$ as λ goes to ∞ due to (1.20). As a consequence, we must have $\mathbb{P}(X < 0) = 0$, showing that $(\mathcal{A}_u(\alpha) \cap L^\Phi(\mathbb{P}))^\infty \subset L^0(\mathbb{P})_+$. The other inclusion is obvious.

Since the map $X \mapsto \mathbb{E}_\mathbb{P}[u(X)]$ is upper semicontinuous on $L^1(\mathbb{P})$, the set

$$\mathcal{A}_u(\alpha) \cap L^1(\mathbb{P}) = \{X \in L^1(\mathbb{P}) \,:\, \mathbb{E}[u(X)] \ge \alpha\}$$

is norm closed in $L^1(\mathbb{P})$ and hence $\mathcal{A}_u(\mathbb{P}) \cap L^\Phi(\mathbb{P})$ is $\sigma(L^\Phi(\mathbb{P}), L^\infty(\mathbb{P}))$-closed by Proposition 1.6.2. Finally, since every consistent pricing density D belongs to the barrier cone of $\mathcal{A}_u(\alpha) \cap L^\infty(\mathbb{P})$, by Theorem 4.115 in Föllmer and Schied [49] it can be derived that

$$\sigma_{\mathcal{A}_u(\alpha) \cap L^\infty(\mathbb{P})}(D) = \sup_{\lambda > 0}\{\alpha + \mathbb{E}_\mathbb{P}[u^*(\lambda D)]\} > -\infty,$$

implying that $\mathbb{E}_\mathbb{P}[u^*(\lambda D)] > -\infty$ for some $\lambda > 0$ □

It is not difficult to find examples of functions u that fall within the considered class of utility functions, such that $\mathcal{A}_u(\alpha) \cap (-\mathcal{A}_u(\alpha)) \cap L^\Phi(\mathbb{P})$ contains nonzero random variables for some choices of α, hence the same holds for $\mathcal{K}(\mathcal{A}_u(\alpha))$. Thus, without further assumptions on u and α, Theorem 1.5.9 cannot be applied to this type of acceptance sets.

1.6.4 TEST PROBABILITIES

In this subsection, we consider the acceptability criterion used in Carr et al. [26]. We fix finitely many test probability measures $\mathbb{P}_1, \dots, \mathbb{P}_m$ absolutely continuous with respect to $\mathbb{P}$, and associated floors $f_1, \dots, f_m \in (-\infty, 0]$. We consider acceptable any random variable whose expected payoffs with respect to the test probabilities exceed the associated floors. Namely

$$\mathcal{A}_{test} := \{X \in L^0(\mathbb{P}) \,:\, \mathbb{E}_{\mathbb{P}_i}[X] \ge f_i \ \text{ for } i = 1, \dots, m\}.$$

For later purpose, we also define $\mathbf{I} := \{i \in \{1, \dots, m\} \,:\, f_i = 0\}$ and we assume that $\mathbf{I} \neq \emptyset$.

In the next proposition, we show that $\mathcal{A}_{test}$ is an acceptance set that fulfills the closedness condition required in Assumption 1.4.7, we characterize $\operatorname{rec}(\mathcal{A}_{test})$ and we show that consistent pricing densities must belong to the cone generated by the test probabilities whose floor is equal to zero.

Proposition 1.6.7. *The set $\mathcal{A}_{test}$ is an acceptance set such that*

$$\operatorname{rec}(\mathcal{A}_{test}) = \{X \in L^0(\mathbb{P}) \,:\, \mathbb{E}_{\mathbb{P}_i}[X] \ge 0 \text{ for } i = 1, \dots, m\}.$$

If $\frac{d\mathbb{P}_i}{d\mathbb{P}} \in L^\infty(\mathbb{P})$ for every i, then $\mathcal{A}_{test} \cap L^\Phi(\mathbb{P})$ is closed with respect to $\sigma(L^\Phi(\mathbb{P}), L^\infty(\mathbb{P}))$ for every Orlicz function Φ, and for every pricing density $D \in L^\infty(\mathbb{P})$ consistent with $\mathcal{A}_{test}$, there are $w_i \in \mathbb{R}_+$, $i \in \mathbf{I}$ such that

$$D = \sum_{i \in \mathbf{I}} w_i \frac{d\mathbb{P}_i}{d\mathbb{P}}. \tag{1.21}$$

Proof. It is immediate to verify that $\mathcal{A}_{test}$ is an acceptance set. The characterization of $\operatorname{rec}(\mathcal{A}_{test})$ is straightforward once we recall that $\operatorname{rec}(\mathcal{A}_{test}) = \bigcap_{t>0} t\mathcal{A}_{test}$. Now, assume that $\frac{d\mathbb{P}_i}{d\mathbb{P}} \in L^\infty(\mathbb{P})$ for every i. Since

$$\mathcal{A}_{test} \cap L^\Phi(\mathbb{P}) = \{X \in L^\Phi(\mathbb{P}) \,:\, \mathbb{E}_{\mathbb{P}_i}[X] \ge f_i \ \text{ for } i = 1, \dots, m\},$$

it is clear that this set is $\sigma(L^\Phi(\mathbb{P}), L^\infty(\mathbb{P}))$-closed. Finally, every consistent pricing density in $L^\infty(\mathbb{P})$ is such that $\mathbb{E}_\mathbb{P}[DX] \geq 0$ for every $X \in \text{cone}(\mathcal{A}_{test} \cap L^\Phi(\mathbb{P}))$ and

$$\text{cone}(\mathcal{A}_{test}) = \{X \in L^0(\mathbb{P}) \,:\, \mathbb{E}_{\mathbb{P}_i}[X] \geq 0 \text{ for every } i \in \mathbf{I}\}.$$

It follows that (1.21) holds. $\square$

Since $\mathcal{A}_{test} \cap L^\Phi(\mathbb{P})$ is not law invariant, it is not possible to apply Proposition 1.6.3 to $\mathcal{K}(\mathcal{A}_{test}) \cap L^\Phi(\mathbb{P})$. Without further assumptions on the test probabilities, we cannot conclude that the pointedness property required in Theorem 1.5.9 holds.

CHAPTER 2

RISK MEASURES BEYOND FRICTIONLESS MARKETS

Since 1999, when Artzner, Delbaen, Eber and Heath published the seminal paper [9] about risk measures, a considerable branch of Mathematical Finance research has been devoted to this field.

Coherent risk measures were introduced in [9] as capital requirement rules: the riskiness of the capital position of a financial subject is measured through the amount of capital that has to be invested in a reference instrument in order to make the position acceptable. The criterion of acceptability is specified by an external regulator/supervisor who fixes the set $\mathcal{A}$ of the positions that he deems acceptable from a risk perspective. Denoting by S_0 and S_1 the initial and final values of the reference instrument, the risk measure evaluated in the position X is defined as

$$\rho(X) = \inf\{mS_0 \,:\, m \in \mathbb{R},\;\; X + mS_1 \in \mathcal{A}\}.$$

In [9], the risk measure ρ is assumed to be a convex and positively homogeneous function defined on the space of random variables on a finite set. Right after [9], a number of articles came up with generalizations of coherent risk measures in various directions. In Delbaen [37], the author extends the reference payoff space to general probability spaces, in Föllmer and Schied [48], the authors drop positive homogeneity and consider convex risk measures, in Frittelli and Rosazza Gianin [50] the focus is on dual representations of convex risk measures defined on abstract topological vector spaces. Though it was not required in the seminal paper [9], the subsequent research identified the reference instrument $S = (S_0, S_1)$ with cash. This reduction has been justified with a discounting argument. E.g. in Delbaen [37], the author says:

> ...we are working in a model without interest rate, the general case can "easily" be reduced to this case by "discounting".

Similarly, in Frittelli and Rosazza Gianin [50], it is stated:

> For simplicity, we will consider market models without interest rates; it is immediate, however, to extend all the definitions and results tho the "real" case, by appropriately discounting.

As widely discussed in Munari [75], the discounting procedure actually presents some problems: in case of defaultable bonds it is simply not possible, moreover by discounting one could fall out the chosen payoff space, or one could lose the structure and the properties of the acceptance set. For these reasons, the theory of risk measures has been extended beyond the cash additive paradigm, hence allowing for risky and possibly defaultable reference assets. This is done in Farkas et al. [45] for L^∞ and in Farkas et al. [44] for abstract spaces.

Risk measures with multiple reference assets appeared in 2002 in the first edition of Föllmer and Schied [49] (see paragraph 4.5 in the first edition and paragraph 4.8 in the second and the third edition) and were investigated in Artzner at al. [8] in the context of finite dimensional spaces of random variables, and generalized to general topological vector spaces in Farkas et al. [46] and in Baes et al. [11]. Their set-valued counterpart was studied in Jouini et al. [62], which in turn triggered Hamel and Heyde [57] and Hamel et al. [58]. Instead of one single reference instrument, the authors of [8], [46] and [11] fix a vector space $\mathcal{M}$ of payoffs of eligible assets, and a linear pricing functional $\pi : \mathcal{M} \to \mathbb{R}$, which describes initial costs for buying (and selling) the assets. The resulting risk measure is

$$\rho(X) = \inf\{\pi(Z) \,:\, Z \in \mathcal{M},\;\; X + Z \in \mathcal{A}\}.$$

As said in [8], under a proper no arbitrage condition the multiasset case may be substantially reduced to a single asset model:

> ...the original risk measure is identical to a coherent risk measure defined with anyone of the original assets acting as the single eligible asset and with the new acceptance set obtained from the original acceptance set [$\mathcal{A}$] by augmenting it by the future values of portfolios of eligible assets which have initial value zero [$\mathcal{A} + \ker(\pi)$].

Like the reference security in single-asset risk measures was assumed to be perfectly liquid, the choice of a liner subspace $\mathcal{M}$ and a linear pricing functional π implies that the modeled market has no frictions and no constraints. A model that allows for bid-ask spreads, illiquidity and constraints would be much more realistic though. Despite these effects are rarely taken into account in the risk measures literature, they are widely investigated in the in the context of pricing (e.g. Bouchard et al. [24]), utility maximization (e.g. Çetin [97]), hedging and Fundamental Theorems (e.g. Pennanen [81], Pennanen [78], Kabanov et al. [64], Schachermayer [90], Astic and Touzi [10] and Jouini and Kallal [61]). In these papers, frictions are modeled through non linear (typically convex or sublinear) pricing functionals and constrained spaces of admissible portfolios or payoffs.

In this chapter, we aim to develop a risk measure theory for abstract spaces in markets with potential frictions. The first step towards a generalization of risk measures in this nonlinear direction was made by Frittelli and Scandolo [51], where general capital requirements where defined for general sets $\mathcal{M}$ and general functions π. We start one step before with respect to [51]: prices are defined on portfolios of basic securities and not on the liquidation values or on the payoffs generated by portfolios. Our model hence allows for cases where two portfolios have the same liquidation value at maturity but different price at initial time, this being compatible with the absence of arbitrages in nonlinear markets. We consider a uniperiodal economy where a finite number of basic securities are traded both at initial time and at maturity, hence we define pricing functionals at both instants. Note that at maturity the portfolios of basic securities do not simply pay off their contractual value, but are liquidated to proceed with a subsequent management action. This assumptions makes this model suitable to be used as first block of a discrete multiperiodal model. The generalized risk measure we define looks like

$$\rho(X) = \inf\{V_0(\lambda) \,:\, \lambda \in \mathcal{P},\ \ X + V_1(\lambda) \in \mathcal{A}\},$$

where $\mathcal{P}$ is the constrained set of admissible portfolios, V_0 denotes the buying price at initial time, V_1 captures the liquidation value at maturity and $\mathcal{A}$ is the acceptance set specified by the regulator. Moreover, while lower semicontinuity of the generalized risk measure was assumed in [51] when needed, we provide sets of sufficient assumptions on the market for the risk measure to be lower semicontinuous. In this regard, the choice of a finite number of basic securities is crucial.

We try to keep the model as general as possible: for this reason, we make few assumptions on the primitive elements (e.g. the acceptance set is only required to be monotone) and for each result we require only the desired additional properties.

The structure of this chapter is as follows: in Section 2.1 we describe the model in two steps. The first description is abstract, without references to any possible financial application, showing that the model is suitable to be applied in other contexts. Then we specify the capital adequacy interpretation that we use throughout the chapter. In Section 2.2, we investigate how the algebraic properties of the primitive elements influence those of the generalized risk measure, while in Section 2.3 we find sufficient conditions for the risk measure to be lower semicontinuous. As these conditions are related to the absence of some particular appealing portfolios, in Section 2.4 we investigate the relation among them and the absence of some generalized arbitrages called acceptable deals. The necessary mathematical background is collected in the appendices and we refer to the List of Symbols on page 117 for the necessary notation.

2.1 MODEL AND INTERPRETATION

We describe the model in two steps. First, we define primitive elements without giving them any interpretation. Then, we use the model to describe a capital adequacy problem. The reason of this separation is to suggest to the reader the idea that applications of this chapter may go beyond the one we are going to consider (for an example, see Remark 3.2.2).

2.1.1 THE GENERAL MODEL

Consider a dynamic one period setting, where an agent acts at initial time (time 0) to achieve some predetermined goal at maturity (time 1) trying to minimize the cost of his action.

Assume the agent can select his activity in a predetermined actions' space of finite dimension, say $N \geq 1$. Allowing for the existence of constraints for his choice, we model the set of available actions for the agent with a nonempty subset $\mathcal{P}$ of $\mathbb{R}^N$, which may happen to coincide with $\mathbb{R}^N$. The cost of any admissible action expressed in a given unit of measure is described through the function $V_0 : \mathcal{P} \to \mathbb{R}$, which will be called the *valuation operator at time* 0.

We describe all possible states of the agent at maturity as elements of a locally convex topological vector space $\mathcal{X}$. We assume that $\mathcal{X}$ is partially ordered by a vector order generated by the pointed convex cone $\mathcal{X}_+$ and denoted by $\geq$. Let $V_1 : \mathcal{P} \to \mathcal{X}$ be a function describing the additive actions' impact at maturity, meaning that if $X \in \mathcal{X}$ is the agent's state at time 1, the final effect of undertaking action $\lambda \in \mathcal{P}$ at time 0 is to transform his final position in $X + V_1(\lambda)$. We refer to V_1 as the *liquidation operator at time* 1.

Finally, we assume that a nonempty proper subset of desirable positions $\mathcal{A} \subset \mathcal{X}$ is specified. It has to be thought to as determined by external factors, like regulators, agents' utility criteria, prescribed management limits... We call this set the *acceptance set*. According to the rational principle "more is better", where "more" has to be understood in the sense of the partial order of $\mathcal{X}$, we require $\mathcal{A}$ to be monotone (i.e. if $X \in \mathcal{A}$ and $Y \geq X$, then $Y \in \mathcal{A}$).

The aim of the agent with maturity state $X \in \mathcal{X}$ is to determine the minimal cost of an admissible action $\lambda \in \mathcal{P}$ ensuring that its final modified position $X + V_1(\lambda)$ is in the acceptance set. We define the set valued map $M : \mathcal{X} \rightrightarrows \mathcal{P}$ as

$$M(X) := \{\lambda \in \mathcal{P} \,:\, X + V_1(\lambda) \in \mathcal{A}\} \subset \mathbb{R}^N,$$

the set of admissible actions that the agent may consider to undertake, and the minimal cost functional $\rho : \mathcal{X} \to [\infty, \infty]$ as

$$\rho(X) := \inf\{V_0(\lambda) \,:\, \lambda \in \mathcal{P},\ X + V_1(\lambda) \in \mathcal{A}\}.$$

Note that $\rho(X) = \inf_{\lambda \in M(X)} V_0(\lambda)$ and for this reason may be referred as an *optimal value function*, while M is often called *constrained set mapping*. For later use, we also define the so called *optimal* and *quasi-optimal set mappings* M^*, $M^*_\varepsilon : \mathcal{X} \rightrightarrows \mathcal{P}$, $\varepsilon > 0$:

$$\begin{aligned} M^*(X) &:= \{\lambda \in M(X) \,:\, V_0(\lambda) = \rho(X)\} = M(X) \cap \{V_0 \leq \rho(X)\} \\ M^*_\varepsilon(X) &:= \{\lambda \in M(X) \,:\, V_0(\lambda) < \rho(X) + \varepsilon\} = M(X) \cap \{V_0 < \rho(X) + \varepsilon\}. \end{aligned}$$

Furthermore, for every $m \in \mathbb{R}$, we define the set of liquidation values achieved with initial cost bounded by m:

$$\mathcal{V}_m := \{V_1(\lambda) \,:\, \lambda \in \mathcal{P},\ V_0(\lambda) \leq m\}.$$

Note that we have not assumed any property on the primitive elements $\mathcal{P}$, V_0, V_1 and $\mathcal{A}$, apart from monotonicity of $\mathcal{A}$. Without further assumptions, we derive that ρ is monotone decreasing as an extended-real valued function on $\mathcal{X}$, showing that to a higher state corresponds a lower cost to reach acceptability.

Proposition 2.1.1. *For every $X, Y \in \mathcal{X}$ such that $Y \geq X$ we have $\rho(Y) \leq \rho(X)$.*

Proof. Let $Y \geq X$. If $\lambda \in \mathcal{P}$ is such that $X + V_1(\lambda) \in \mathcal{A}$, monotonicity of $\mathcal{A}$ ensures that $Y + V_1(\lambda) \in \mathcal{A}$. The thesis follows. □

2.1.2 THE CAPITAL ADEQUACY INTERPRETATION

So far, we have not made any reference to a financial context. The aim of this paragraph is to make clear the interpretation of our model from the point of capital requirements rules. As a result it will be clarified that ρ is nothing else than a generalization to markets with constraints and frictions of the coherent risk measures introduced in the seminal paper Artzner et al. [9]. In order to keep language and interpretation more accessible, throughout the chapter we will constantly make reference to this financial interpretation.

We consider a uniperiodal financial market where N basic securities are traded at initial time and their maturities concide or are later than the final time. At time 0, an agent buys a portfolio of basic securities. Buying the portfolio $\lambda = (\lambda_1, \dots, \lambda_N) \in \mathbb{R}^N$ means that the agent receives from a counterparty λ_i units of security i for each i, and pays to the counterparty the value $V_0(\lambda)$, which is expressed in a fixed reference unit of account. To be precise, we are assuming that if $\lambda_i \geq 0$ the agent receives λ_i units of security i, while if $\lambda_i < 0$ the agent gives way (sells) $-\lambda_i$ units, and if $V_0(\lambda) \geq 0$ the agent gives away the amount $V_0(\lambda)$, while if $V_0(\lambda) < 0$ the agent receives the amount $-V_0(\lambda)$.

At time 1 the agent liquidates the portfolio λ he has bought, meaning that he sells λ_i units of security i for every i, and he receives the "amount" $V_1(\lambda)$ which is added to his capital position. Since the outcome of the basic securities cannot generally be determined at initial time, liquidation values as well as capital positions are typically described through random variables, and $\mathcal{X}$ coincides in all practical examples with a suitable space of random variables. If the maturity of some security is later than the terminal time, during the liquidation the agent may incur in transaction costs or illiquidity effects. By admitting V_1 to be nonlinear, we cover this case. Note that the choice of a linear map V_1 corresponds to the case where there is no trading of the basic securities at maturity and they simply deliver their terminal contractual payoff (for further details see Chapter 1), or there is a perfectly liquid market.

An external financial regulator fixes the acceptance set $\mathcal{A}$: the agent is safe whenever, after the liquidation of his portfolio, his position falls in $\mathcal{A}$. From this perspective, the interpretation of ρ as a risk measure is straightforward: given the capital position X of the agent, $\rho(X)$ is the minimum amount of unit of account that the agent has to invest in the basic securities at initial time in order to meet the requirement imposed by the regulator at maturity.

Remark 2.1.2 (From buyer to seller)**.** So far, we have taken the point of view of a buyer since from this side the interpretation of the risk measure ρ arises naturally, but it is possible to change perspective and assume that the agent acts on the market by selling portfolios. Let us make a quick comment about this point of view. Since $\mathcal{P}$ is the set of portfolios that the agent can purchase, and since V_0 is the buying pricing functional, which portfolios can be sold and at which price? Concerning the exchange of basic securities, we assume that selling λ is equivalent to buying $-\lambda$, so that the set of portfolios that can be sold is $\mathcal{P}^s := -\mathcal{P}$. Moreover, we assume that receiving an amount is equivalent to paying minus that amount. Thus the agent who sells $\lambda \in \mathcal{P}^s$ is purchasing $-\lambda \in \mathcal{P}$ paying the price $V_0(-\lambda)$ and thus receiving the amount $V_0^s(\lambda) := -V_0(-\lambda)$. Similarly it can be done for V_1.

Remark 2.1.3 (Comparison with the model of Chapter 1)**.** Incidentally, we take the opportunity to notice that the setting of the present chapter is actually a generalization of the type of economy studied in Chapter 1. There, we start working on $L^0(\mathbb{P})$, and then we restrict to a topological vector space $\mathcal{X}$, while here we start directly with $\mathcal{X}$. The setting of Chapter 1 corresponds to the case where $\mathcal{X}$ is a space of random variables on some probability space, the market satisfies some convexity requirement (namely $\mathcal{P}$ is convex closed and contains 0, V_0 is convex lower semicontinuous and $V_0(0) = 0$), there is no liquidation at maturity and the payoffs of the basic securities are independent (i.e. V_1 is linear and injective), $\mathcal{A}$ is convex and contains 0. Note that the set that here we denote by $\mathcal{A}$, corresponds to $\mathcal{A} \cap \mathcal{X}$ of Chapter 1.

2.1.3 COSTS DEFINED ON PAYOFFS

Considering the model we have just described, one could ask whether it may be simplified by dropping the set $\mathcal{P}$ of portfolios and by assigning initial costs directly to each final impact. Of course, this can be done whenever the following condition holds:

$$\lambda,\, \mu \in \mathcal{P},\;\; V_1(\lambda) = V_1(\mu) \quad \Longrightarrow \quad V_0(\lambda) = V_0(\mu). \tag{2.1}$$

Indeed, considering the set of addictive impacts of admissible actions $\mathcal{M} := \{V_1(\lambda) \,:\, \lambda \in \mathcal{P}\}$, thanks to (2.1) the functional $\pi : \mathcal{M} \to \mathbb{R}$ given by

$$\pi(Z) := V_0(\lambda) \text{ for any } \lambda \in \mathcal{P} \text{ s.t. } Z = V_1(\lambda)$$

is well defined. Condition (2.1) is known as *law of one price* (see e.g. Föllmer and Schied [49]) and arises as a natural consequence of absence of arbitrages in standard frictionless and no-constraints market models. Indeed, if the law of one price fails, one could set up a strategy consisting in buying the cheaper of two portfolios with same value at maturity and selling the other, this leading to the existence of an arbitrage. Clearly, this is a consequence of linearity of V_0 and V_1 as buying a portfolio and selling another means buying the difference, whose cost coincides with the difference of the costs. But in nonlinear models, there are very simple examples showing that the correlation between the absence of arbitrages and the law of one price as expressed in equation (2.1) is no longer valid, even without portfolio constraints.

Example 2.1.4. *Assume that $N = 2$, $\mathcal{P} = \mathbb{R}^2$ and $\mathcal{X} = \mathbb{R}$. For every $\lambda = (\lambda_1, \lambda_2) \in \mathcal{P}$, define V_0 and V_1 as follows:*

$$V_0(\lambda) = \max\left\{\lambda_2 + 2\lambda_1, \lambda_2 + \frac{1}{2}\lambda_1\right\}, \qquad V_1(\lambda) = \lambda_1 + \lambda_2$$

It is easy to check that this market has no arbitrage opportunities, since there are no λ's such that $V_0(\lambda) \leq 0$ and $V_1(\lambda) > 0$. But the law of one price (2.1) *is violated as*

$$V_1(0,0) = V_1(-1,1) = 0, \qquad V_0(0,0) = 0 \neq V_0(-1,1) = \frac{1}{2}.$$

If we fall within the model of Chapter 1 as explained in Remark 2.1.3, the law of one price (2.1) holds since V_1 is linear and injective. In particular $\mathcal{M}$ and π as defined here coincide with $\mathcal{M}$ and π of Chapter 1.

One could try to formulate a law of one price ad-hoc for market models with frictions and portfolio constraints, but this topic goes beyond the scope of this thesis.

2.2 Algebraic properties

In this section we prove a number of simple statements showing how algebraic properties of the primitive elements $\mathcal{P}$, V_0, V_1 and $\mathcal{A}$ influence the functional ρ.

Proposition 2.2.1. *The following statements hold:*

(i) Assume that $\mathcal{A}$ and $\mathcal{P}$ are closed under addition (meaning that $X + Y \in \mathcal{A}$ for any $X, Y \in \mathcal{A}$, and similarly for $\mathcal{P}$), V_0 is subadditive, and V_1 is superadditive. Then ρ is subadditive.

(ii) Assume that $\mathcal{A}$ and $\mathcal{P}$ are cones, V_0 and V_1 are positively homogeneous and $\rho(0) \in \mathbb{R}$. Then ρ is positively homogeneous.

(iii) Assume that $\mathcal{A}$ and $\mathcal{P}$ are convex cones, V_0 is sublinear, V_1 is superlinear and $\rho(0) \in \mathbb{R}$. Then ρ is sublinear.

(iv) Assume that $\mathcal{A}$ and $\mathcal{P}$ are convex, V_0 is convex and V_1 is concave. Then ρ is convex.

(v) Assume that $\mathcal{A}$ and $\mathcal{P}$ are convex, V_0 is quasi convex and V_1 is concave. Then ρ is quasi convex.

(vi) Assume that $\mathcal{A}$ and $\mathcal{P}$ are cones, $V_0(tX) \geq t\,V_0(X)$ and $V_1(tX) \leq t\,V_1(X)$ for $t > 1$. Then $\rho(tX) \geq t\,\rho(X)$ for $t > 1$ (and $\rho(tX) \leq t\,\rho(X)$ for $0 < t < 1$).

Proof. *(i)* Take $X, Y \in \mathcal{X}$ and $\lambda, \mu \in \mathcal{P}$ such that $X + V_1(\lambda) \in \mathcal{A}$ and $Y + V_1(\mu) \in \mathcal{A}$. Then

$$X + Y + V_1(\lambda + \mu) \geq \big(X + V_1(\lambda)\big) + \big(Y + V_1(\mu)\big) \in \mathcal{A},$$

and thus $\rho(X + Y) \leq V_0(\lambda + \mu) \leq V_0(\lambda) + V_0(\mu)$. Since λ and μ are arbitrary, the thesis follows.

(ii) Take $X \in \mathcal{X}$ and $t > 0$. Then

$$\begin{aligned}
\rho(tX) &= \inf\{V_0(\lambda) \,:\, \lambda \in \mathcal{P},\ tX + V_1(\lambda) \in \mathcal{A}\} \\
&= \inf\left\{V_0(\lambda) \,:\, \lambda \in \mathcal{P},\ X + V_1\left(\frac{\lambda}{t}\right) \in \mathcal{A}\right\} \\
&= \inf\{V_0(t\lambda) \,:\, \lambda \in \mathcal{P},\ X + V_1(\lambda) \in \mathcal{A}\} \\
&= t\,\rho(X).
\end{aligned}$$

(iii) Follows from points *(i)* and *(ii)*.

(iv) Take $X, Y \in \mathcal{X}$ and $\lambda, \mu \in \mathcal{P}$ such that $X + V_1(\lambda) \in \mathcal{A}$ and $Y + V_1(\mu) \in \mathcal{A}$, and $t \in [0,1]$. Then

$$tX + (1-t)Y + V_1(t\lambda + (1-t)\mu) \geq t\big(X + V_1(\lambda)\big) + (1-t)\big(Y + V_1(\mu)\big) \in \mathcal{A}, \tag{2.2}$$

thus $\rho(tX + (1-t)Y) \leq V_0(t\lambda + (1-t)\mu) \leq tV_0(\lambda) + (1-t)V_0(\mu)$. The thesis follows since λ and μ are arbitrary.

(v) Take $X, Y \in \mathcal{X}$ and $\lambda, \mu \in \mathcal{P}$ such that $X + V_1(\lambda) \in \mathcal{A}$ and $Y + V_1(\mu) \in \mathcal{A}$, and $t \in [0,1]$. Then equation (2.2) holds. Hence

$$\rho(tX + (1-t)Y) \leq V_0(t\lambda + (1-t)\mu) \leq \max\{V_0(\lambda), V_0(\mu)\}. \tag{2.3}$$

Now, if $\rho(tX + (1-t)Y) > \rho(X)$, than there exists $\bar{\lambda} \in \mathcal{P}$ such that $X + V_1(\bar{\lambda}) \in \mathcal{A}$ and $\rho(tX + (1-t)Y) > V_0(\bar{\lambda})$. From (2.3), it follows that for every $\mu \in \mathcal{P}$ with $Y + V_1(\mu) \in \mathcal{A}$, we have $\rho(tX + (1-t)Y) \leq V_0(\mu)$. Hence $\rho(tX + (1-t)Y) \leq \rho(Y)$. Analogously, we can prove that $\rho(tX + (1-t)Y) \leq \rho(X)$.

(vi) For $t > 1$ and $X \in \mathcal{X}$ we have:

$$\begin{aligned}
\rho(tX) &= \inf\{V_0(\lambda) \,:\, \lambda \in \mathcal{P},\, tX + V_1(\lambda) \in \mathcal{A}\} \\
&= \inf\left\{V_0(\lambda) \,:\, \lambda \in \mathcal{P},\, X + \frac{1}{t}V_1(\lambda) \in \mathcal{A}\right\} \\
&\geq \inf\left\{V_0(\lambda) \,:\, \lambda \in \mathcal{P},\, X + V_1\left(\frac{\lambda}{t}\right) \in \mathcal{A}\right\} \\
&= \inf\{V_0(t\lambda) \,:\, \lambda \in \mathcal{P},\, X + V_1(\lambda) \in \mathcal{A}\} \\
&\geq t\,\rho(X).
\end{aligned}$$

□

Remark 2.2.2. We wonder whether linearity of the liquidation operator V_1, corresponding to the case where there is no market at maturity as explained in Subsection 2.1.2, would imply some more algebraic properties of ρ. Actually, it is easy to observe that if V_1 is linear, points *(i)*, *(iii)*, *(iv)*, *(v)*, *(iv)* of Propositions 2.2.1 still hold true when removing the assumption of monotonicity of the set $\mathcal{A}$, which we take for granted for its meaningfulness in applications. Thus in our model, the absence of linearity of V_1 is not an issue for the purpose of proving some algebraic property of ρ, as for instance the combination V_1 concave (or subadditive) and $\mathcal{A}$ monotone supplies the lack of linearity.

2.2.1 A NOTE ON CONVEXITY AND QUASI CONVEXITY

It is well known that in the case where there is a unique basic security and it is perfectly liquid and unconstrained, convexity and quasi convexity of the risk measure ρ are equivalent (see e.g. Föllmer and Schied [49]). In this case ρ is typically called a *monetary risk measure*. Being the equivalence of convexity and quasi convexity a direct consequence of the translation invariance property of monetary risk measures, it is not surprising that it may fail in the presence of illiquidity. Risk measures that are quasi convex and not convex have been studied e.g. in Drapeau and Kupper [40] and Cerreia-Vioglio et al. [30]. For completeness, we also provide an example of a quasi convex but non convex risk measure ρ.

Example 2.2.3. *Consider the probability space $[0,1]$ with Lebesgue sigma algebra and Lebesgue measure and denote by L^∞ the space of bounded random variables modulo a.s. equivalence. Assume that $\mathcal{X} = L^\infty$ and $\mathcal{A}$ is the acceptance set induced by expected shortfall at level α with $\alpha = \frac{1}{2}$, i.e. $\mathcal{A} = \{X \in L^\infty \,:\, \mathrm{ES}_\alpha(X) \leq 0\}$. Let $N = 1$, $\mathcal{P} = \mathbb{R}$, $V_0(\lambda) = \lambda^{\frac{1}{3}}$ and $V_1(\lambda) = \lambda$. Note that V_0 is quasi convex, strictly increasing, and not convex. Proposition 2.2.1 ensures quasi convexity of ρ. For any $Z \in L^\infty$ we have that $\rho(Z) = V_0(\mathrm{ES}_\alpha(Z))$. In order to show that ρ is not convex, it is enough to find X and Y in L^∞ such that the expected shortfall is "linear" on the segment joining X and Y, while V_0 is not convex on $[\mathrm{ES}_\alpha(X), \mathrm{ES}_\alpha(Y)]$. Indeed in this case*

$\widehat{t} \in (0,1)$ *exists such that*

$$\begin{aligned}\rho(\widehat{t}X + (1-\widehat{t})Y) &= V_0(\mathrm{ES}_\alpha(\widehat{t}X + (1-\widehat{t})Y)) \\ &= V_0(\widehat{t}\,\mathrm{ES}_\alpha(X) + (1-\widehat{t})\,\mathrm{ES}_\alpha(Y)) \\ &> \widehat{t}\,V_0(\mathrm{ES}_\alpha(X)) + (1-\widehat{t})V_0(\mathrm{ES}_\alpha(Y)) \\ &= \widehat{t}\,\rho(X) + (1-\widehat{t})\,\rho(Y),\end{aligned}$$

showing that ρ is not convex. To this end, take $X = -\mathbb{1}_{[0,1/2]} + \mathbb{1}_{(1/2,1]}$ and $Y = -2\mathbb{1}_{[0,1/2]} + \mathbb{1}_{(1/2,1]}$. Expected shortfall for random variables of type $tX + (1-t)Y$, $t \in [0,1]$ is easily computed:

$$\mathrm{ES}_\alpha(tX + (1-t)Y) = 2 - t,$$

and it is linear in t. This concludes the example as $\mathrm{ES}_\alpha(X) = 1$, $\mathrm{ES}_\alpha(Y) = 2$ and V_0 is strictly concave on $\mathbb{R}_+$.

A challenging question is: how to characterize quasi convexity of our risk measures? We only provide a partial answer. As recalled in Definition A.1.3, ρ is quasi convex if and only if every set of type $\{\rho \le m\}$ is convex. Since

$$\begin{aligned}\{\rho \le m\} &= \{X \in \mathcal{X} \,:\, \forall \varepsilon > 0\ \exists \lambda_\varepsilon \in \mathcal{P} \text{ s.t. } X + V_1(\lambda_\varepsilon) \in \mathcal{A} \text{ and } V_0(\lambda_\varepsilon) \le m + \varepsilon\} \\ &= \bigcap_{\varepsilon > 0} (\mathcal{A} - \mathcal{V}_{m+\varepsilon}),\end{aligned}$$

for every $m \in \mathbb{R}$, the sets that actually matter for ρ to be quasi convex are those of type $\mathcal{A} - \mathcal{V}_m$. Clearly, if $\mathcal{A} - \mathcal{V}_m$ is convex for every $m \in \mathbb{R}$, then ρ is quasi convex. The validity of the opposite implication is not straightforward in general, but in Theorems 2.3.10, 2.3.12 and 2.3.14 we will see that under suitable conditions, $\{\rho \le m\}$ actually coincides with $\mathcal{A} - \mathcal{V}_m$, and hence the opposite implication holds as well. The sets $\mathcal{A} - \mathcal{V}_m$ turn out to be convex if $\mathcal{A}$, $\mathcal{P}$ and V_0 are convex, and V_1 is concave. But none of these four conditions is necessary, as we show in the next simple examples. The first example below (and Example 3.2.(ii) in Baes et al. [11]) shows another difference from the case of monetary risk measures, where convexity of the acceptance set is necessary for quasi convexity of ρ.

Example 2.2.4. *Let $\mathcal{X} = \mathbb{R}^3$ and define $\mathcal{A} \subset \mathcal{X}$ as follows*

$$\mathcal{A} := \{(x,y,z) \in \mathbb{R}^3_+ : \max\{x+y, x+z, y+z\} \ge 1\}.$$

The acceptance set $\mathcal{A}$ is not convex, as $(1,0,0)$, $(0,1,0)$, $(0,0,1)$ are in $\mathcal{A}$ but their convex combination $(1/3,1/3,1/3) \notin \mathcal{A}$. Clearly, $\mathcal{A}$ is monotone and closed.

Assume that $N = 2$ and $\mathcal{P} = \mathbb{R}^2$, and define

$$V_0 : \mathcal{P} \to \mathbb{R}, \quad V_0(\lambda_1, \lambda_2) = \lambda_2$$

$$V_1 : \mathcal{P} \to \mathcal{X}, \quad V_1(\lambda_1, \lambda_2) = (\lambda_1, -\lambda_1 + 2\lambda_2, \lambda_2).$$

Since the law of one price (2.1) *holds, we can define prices directly on the payoffs space $\mathcal{M} = \{V_1(\lambda) : \lambda \in \mathcal{P}\} = \{(x,y,z) \in \mathbb{R}^3 : x + y - 2z = 0\}$ as $\pi : \mathcal{M} \to \mathbb{R}$ defined by $\pi(x,y,z) = z$ (we maintain the coordinate system inherited from $\mathbb{R}^3$). Note that $\mathcal{V}_m = \mathcal{V}_0 + m(0,2,1)$ for every m, so that $\mathcal{A} - \mathcal{V}_m$ is convex iff $\mathcal{A} - \mathcal{V}_0$ is so. Moreover,*

$$\mathcal{V}_0 = \{X \in \mathcal{M} \,:\, \pi(X) \le 0\} = \left\{(x, 2z - x, z) \in \mathbb{R}^3 \,:\, z \le 0\right\}.$$

We claim that

$$\mathcal{A} - \mathcal{V}_0 = \{(x,y,z) \in \mathbb{R}^3 : z \ge 0,\ x + y \ge 0,\ x + y + z \ge 1\},$$

which is convex being the intersection of three half spaces. To show our claim, take $(x,y,z) \in \mathbb{R}^3$ such that $z \ge 0$, $x + y \ge 0$, $x + y + z \ge 1$. Then

$$(x,y,z) = \underbrace{(0, x+y, z)}_{\in \mathcal{A}} - \underbrace{(-x, x, 0)}_{\in \mathcal{V}_0} \in \mathcal{A} - \mathcal{V}_0$$

and we have proved the inclusion "$\supset$". For the other, consider the point

$$(x_{\mathcal{A}}, y_{\mathcal{A}}, z_{\mathcal{A}}) - (x, 2z - x, z)$$

where $(x_{\mathcal{A}}, y_{\mathcal{A}}, z_{\mathcal{A}}) \in \mathcal{A}$ and $z \leq 0$. The third component, $z_{\mathcal{A}} - z$, is clearly positive, the sum of the first two as well, since

$$\underbrace{x_{\mathcal{A}}}_{\geq 0} + \underbrace{y_{\mathcal{A}}}_{\geq 0} + \underbrace{-x - 2z + x}_{\geq 0} \geq 0,$$

and finally the sum of the three components is greater than 1:

$$\underbrace{x_{\mathcal{A}} + y_{\mathcal{A}} + z_{\mathcal{A}}}_{\geq 1} \underbrace{-x - 2z + x - z}_{\geq 0} \geq 1,$$

showing the other inclusion. We conclude that $\mathcal{A} - \mathcal{V}_m$ is convex for every $m \in \mathbb{R}$, and hence ρ is quasi convex. One could also note that in this case, as ρ can be written as

$$\rho(X) = \inf\{m \in \mathbb{R} \,:\, X + m(0, 2, 1) \in \mathcal{A} - \mathcal{V}_0\},$$

Lemma 2.5 in [44] ensures that ρ is convex.

Example 2.2.5. *Assume that $\mathcal{X} = \mathbb{R}$, $\mathcal{A} = \mathbb{R}_+$, $N = 2$, $\mathcal{P} = \mathbb{R}^2$ and V_1 and V_0 are defined as follows on $\lambda = (\lambda_1, \lambda_2) \in \mathcal{P}$:*

$$V_0(\lambda) = \begin{cases} \lambda_1 + \lambda_2 & \text{if } \lambda_2 > 0 \\ \lambda_1 + 2\lambda_2 & \text{if } \lambda_2 \leq 0 \end{cases}, \quad V_1(\lambda) = \min\{\lambda_1, \lambda_2\}.$$

We have that V_0 is not convex, but $\mathcal{A} - \mathcal{V}_m = \mathbb{R}_+$ for $m \leq 0$ and $\mathcal{A} - \mathcal{V}_m = [-m/2, +\infty)$ for $m > 0$, which is convex.

Example 2.2.6. *Assume that $\mathcal{X} = \mathbb{R}$, $\mathcal{A} = \mathbb{R}_+$, $N = 2$, $\mathcal{P} = \mathbb{R}^2$ and V_1 and V_0 are defined as follows on $\lambda = (\lambda_1, \lambda_2) \in \mathcal{P}$:*

$$V_0(\lambda) = V_1(\lambda) = \max\{\lambda_1, \lambda_2\}.$$

We have that V_1 is not concave, but $\mathcal{A} - \mathcal{V}_m = [-m, +\infty)$ for every $m \in \mathbb{R}$, which is convex.

Example 2.2.7. *Assume that $\mathcal{X} = \mathbb{R}$, $\mathcal{A} = \mathbb{R}_+$, $N = 1$, $\mathcal{P} = \mathbb{R} \setminus \{0\}$ and V_1 and V_0 are defined as follows on $\lambda \in \mathcal{P}$:*

$$V_0(\lambda) = V_1(\lambda) = \lambda.$$

We have that $\mathcal{P}$ is not concave, but $\mathcal{A} - \mathcal{V}_m = [-m, +\infty)$ for every $m \in \mathbb{R} \setminus \{0\}$ and $\mathcal{A} - \mathcal{V}_0 = (0, +\infty)$, which are convex.

2.3 CONTINUITY AND STABILITY PROPERTIES

In this section, we establish sufficient conditions for ρ to be lower semicontinuous or continuous, and for the map M_ε^* to be lower semicontinuous. In the first part, we focus on a particular set in the product space $\mathcal{X} \times \mathbb{R}$ which turns out to be strictly related to the epigraph of ρ, while in the last part we follow a completely different approach that holds if the acceptance set has nonempty interior.

2.3.1 THE ROLE OF THE CLOSEDNESS OF $\mathcal{C}$

First, note that ρ can be expressed as

$$\rho(X) = \inf\{m \in \mathbb{R} \,:\, (X, m) \in \mathcal{C}\} \tag{2.4}$$

where

$$\mathcal{C} := \{(X, m) \in \mathcal{X} \times \mathbb{R} \,:\, \exists \lambda \in \mathcal{P} \text{ such that } V_0(\lambda) \leq m \text{ and } V_1(\lambda) + X \in \mathcal{A}\}. \tag{2.5}$$

Equality (2.4) does not coincide with the standard representation of a function through its epigraph (A.1) since in general only the inclusion $\text{epi}(\rho) \supset \mathcal{C}$ holds. The set $\mathcal{C}$ may be interpreted (modulo temporal order) as the set of processes such that the initial amount m allows to set up a strategy that grants to fulfill the acceptability requirements at maturity.

The importance of the closedness of $\mathcal{C}$ in the product topology is highlighted by the following proposition.

Proposition 2.3.1. *If $\mathcal{C}$ is closed, the following statements hold:*

(i) *ρ is lower semicontinuous.*

(ii) *If $\rho(X) \in \mathbb{R}$ for some $X \in \mathcal{X}$, the infimum in* (2.4) *is attained:*

$$\rho(X) = \min\{m \in \mathbb{R} \,:\, (X, m) \in \mathcal{C}\}.$$

(iii) *The sets $\{\rho \leq m\}$ and $\mathcal{A} - \mathcal{V}_m$ coincide and are closed for every $m \in \mathbb{R}$.*

(iv) *If $\rho(X) \in \mathbb{R}$ for some $X \in \mathcal{X}$, the infimum defining ρ is attained:*

$$\rho(X) = \min\{V_0(\lambda) \,:\, \lambda \in \mathcal{P},\ X + V_1(\lambda) \in \mathcal{A}\} \qquad \text{i.e. } M^*(X) \neq \emptyset.$$

Proof. If $\mathcal{C}$ is closed, assertion *(i)* holds by Proposition A.1.8 (ii). In this case $\text{epi}(\rho) = \mathcal{C}$ and assertion *(ii)* is straightforward. Since the infimum in (2.4) is attained we have that

$$\{\rho \leq m\} = \{X \in \mathcal{X} \,:\, (X, m) \in \mathcal{C}\} = \mathcal{A} - \mathcal{V}_m,$$

which is closed by lower semicontinuity of ρ. This shows *(iii)*. For proving *(iv)*, note that if $\rho(X) \in \mathbb{R}$, then $(X, \rho(X)) \in \mathcal{C}$ by *(ii)*. This means that we find $\lambda \in \mathcal{P}$ such that $V_1(\lambda) + X \in \mathcal{A}$ and $V_0(\lambda) \leq \rho(X)$. By definition of $\rho(X)$ the opposite inequality holds too, hence $\rho(X) = V_0(\lambda)$. □

Remark 2.3.2 (Comparison with Theorem 1.3.14)**.** As we have illustrated in Remark 2.1.3, the model of Chapter 1 is a special case of the present setting, where law of one price (2.1) and suitable convexity assumptions hold, and V_1 is linear and injective. Moreover the set $\mathcal{C}$ defined in (2.5) corresponds, up to a sign, to the set $\mathcal{C}$ studied in Chapter 1. As a consequence of Theorem 1.3.14, $\mathcal{C}$ is closed in that context whenever there are no acceptable deals, i.e. when there are no nonzero X in $\mathcal{A}$ such that $X = V_1(\lambda)$ for some $\lambda \in \mathcal{P}$ with $V_0(\lambda) \leq 0$. Examples 2.3.3 and 2.3.4 below show that out of the setting of that chapter, the absence of acceptable deals is no more a sufficient condition for the set $\mathcal{C}$ to be closed. In particular, in Example 2.3.3 V_1 is not linear, and in Example 2.3.4 the law of one price fails.

Example 2.3.3. *Consider the simple case where $\mathcal{X} = \mathbb{R}$ and $\mathcal{A} = \mathbb{R}_+$. Moreover, assume that V_0 and V_1 are defined on $\mathcal{P} = \mathbb{R}$ as*

$$V_0(\lambda) = e^{\lambda} - 1, \quad V_1(\lambda) = \arctan(\lambda).$$

It it easy to see that $\mathcal{A} \cap \{V_1(\lambda) \,:\, \lambda \in \mathcal{P},\ V_0(\lambda) \leq 0\} = \{0\}$. For $n \in \mathbb{N}$, define $X_n := \arctan(n)$ and $m_n := e^{-n} - 1$. The couples (X_n, m_n) belong to $\mathcal{C}$, since $\lambda_n := -n$ is such that $V_0(\lambda_n) = m_n$ and $X_n + V_1(\lambda_n) = 0 \in \mathcal{A}$. Moreover, $(X_n, m_n) \to (\pi/2, -1)$, which does not belong to $\mathcal{C}$ since there are no $\lambda \in \mathcal{P}$ having $V_0(\lambda) \leq -1$.

Example 2.3.4. *Assume that $\mathcal{X} = \mathbb{R}^2$, $\mathcal{A} = \mathbb{R}^2_+$ and $\mathcal{P} = -\mathbb{R}_+ \times \mathbb{R}$. Moreover, assume that V_0 and V_1 are defined on $\mathcal{P}$ as*

$$V_0(\lambda_1, \lambda_2) = e^{\lambda_2} - 1, \quad V_1(\lambda_1, \lambda_2) = (\lambda_1, 0).$$

Note that $\mathcal{A} \cap \{V_1(\lambda) \,:\, \lambda \in \mathcal{P},\ V_0(\lambda) \leq 0\} = \{0\}$. For $n \in \mathbb{N}$, define $X_n := 0$ and $m_n := e^{-n} - 1$. The couples (X_n, m_n) belong to $\mathcal{C}$, since $\lambda_n := (0, -n)$ is such that $V_0(\lambda_n) = m_n$ and $X_n + V_1(\lambda_n) = 0 \in \mathcal{A}$. Moreover, $(X_n, m_n) \to (0, -1)$, which does not belong to $\mathcal{C}$ since there are no $\lambda \in \mathcal{P}$ having $V_0(\lambda) \leq -1$.

2.3.2 Closedness of the map $M : \mathcal{X} \rightrightarrows \mathcal{P}$

In this section, we follow the approach of Pennanen [78] to prove closedness of $\mathcal{C}$, and we provide in the final proposition a result that allows to extend the result in [78] to acceptance sets different from the positive cone in one period models.

In Pennanen [78] the setting is discrete multiperiodal. Considering only one period, it is easy to see that the set $\mathcal{C}$ considered there coincides with (2.5) up to a sign, where $\mathcal{X} = L^0(\mathbb{P})$ is the space of equivalence classes of random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$, equipped with the topology of convergence in probability and almost sure pointwise order. With reference to the notation of that paper, portfolio constraints (D_0, D_1) are modeled by $(\mathcal{P}, -\mathcal{P})$, while the cost process (S_0, S_1) coincides with $(V_0, -V_1(-\cdot))$ (note that in [78], S_1 is defined ω by ω, and hence we define $V_1(\lambda)$ as the equivalence class of $-S_1(-\lambda)$). The set $\mathcal{P}$ is assumed to be convex and closed and $0 \in \mathcal{P}$, the function V_0 is assumed to be convex and lower semicontinuous and $V_0(0) = 0$. Moreover we assume that $V_1(0) = 0$ and there are representatives of $V_1(\lambda)$ for every $\lambda \in \mathcal{P}$ such that for every ω the function $V_1(\cdot)(\omega)$ is concave and upper semicontinuous. Finally, we assume that the map $\omega \mapsto \text{hypo}(V_1(\cdot)(\omega))$ is $\mathcal{F}$-measurable (see Chapter 14 of [86]). The acceptance set $\mathcal{A}$ corresponds to the cone of positive random variables $L^0(\mathbb{P})_+$.

In Theorem 8 of [78] it is shown that $\mathcal{C}$ is closed provided that the convex cone

$$\mathcal{L}_0 := \mathcal{P}^\infty \cap \{\lambda \in \mathcal{P}^\infty \,:\, V_0^\infty(\lambda) \leq 0\} \cap \{\lambda \in \mathcal{P}^\infty \,:\, V_1^\infty(\lambda) \geq 0 \text{ a.s.}\} \tag{2.6}$$

is linear (the proof is an application of Theorem 5.2 in Pennanen [77]), where $\mathcal{P}^\infty$ is the recession or asymptotic cone of $\mathcal{P}$ (Definition B.1.6), V_0^∞ is the recession or asymptotic function of V_0 (Definition B.2.6), while, with an abuse of notation, $V_1^\infty : \mathcal{P} \to L^0(\mathbb{P})$ is pointwise defined as

$$V_1^\infty(\lambda)(\omega) := -(-V_1(\cdot)(\omega))^\infty(\lambda) \quad \forall \omega \in \Omega. \tag{2.7}$$

Since the proof of this result is based on a.s. procedures, the choice of the space $L^0(\mathbb{P})$ is crucial, as well as the fact that both the acceptance set and V_1 admit a pointwise representation (indeed $\mathcal{A} = \{X \in L^0(\mathbb{P}) : \; X \geq 0 \text{ a.s.}\}$), and V_1 is upper semicontinuous ω by ω, for suitable representatives. This fact allows to express $\mathcal{C}$ as

$$\mathcal{C} = \{(X, m) \,:\, \exists \lambda \in \mathcal{P} \text{ s.t. } (\lambda, m, X(\omega)) \in C(\omega) \text{ a.s.}\}, \tag{2.8}$$

where $C(\omega) := \{(\lambda, m, x) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \,:\, \lambda \in \mathcal{P},\; V_0(\lambda) \leq m,\; V_1(\lambda)(\omega) + x \geq 0\}$. Moreover, the proof heavily relies on convexity assumptions on V_0 and V_1.

The proof of Theorem 8 in [78] keeps working if we replace $\mathcal{A}$ with the set of random variables that are greater than some fixed $L \in L^0(\mathbb{P})$, as (2.8) holds true with $C(\omega) := \{(\lambda, m, x) \in \mathbb{R}^N \times \mathbb{R} \times \mathbb{R} \,:\, \lambda \in \mathcal{P},\; V_0(\lambda) \leq m,\; V_1(\lambda)(\omega) + x \geq L(\omega)\}$. Now, we provide a different and simplified proof of Theorem 8 in [78] that works in our uniperiodal setting. This will bring to light in which step the particular shape of $\mathcal{A}$ is used and opens a path to possible generalization. To this end, we consider the set $\mathcal{L}_0$ defined as in (2.6), and

$$\mathcal{N}_0 := \mathcal{L}_0 \cap (-\mathcal{L}_0).$$

Note that without further assumptions $\mathcal{L}_0$ is a cone and $\mathcal{N}_0$ is a linear space. We denote by $\mathcal{N}_0^\perp$ the orthogonal complement of $\mathcal{N}_0$ in the finite dimensional linear space $\text{span}(\mathcal{P})$. Before stating the result, we prove a projection lemma inspired by Theorem 5.2 in [77].

Lemma 2.3.5. *Let $\mathcal{X} = L^0(\mathbb{P})$ be the space of equivalence classes of random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. Assume that $\mathcal{P}$ is convex, closed and contains 0, V_0 is convex and lower semicontinuous and there are representatives of $V_1(\lambda)$ for every $\lambda \in \mathcal{P}$ such that $V_1(\cdot)(\omega)$ is concave and upper semicontinuous for every $\omega \in \Omega$. Then, if $(X, m) \in \mathcal{C}$, we find $\widetilde{\lambda} \in \mathcal{N}_0^\perp$ such that $(X, \widetilde{\lambda}) \in \mathcal{C}$.*

Proof. If $(X, m) \in \mathcal{C}$, there is $\lambda \in \mathcal{P}$ such that $V_0(\lambda) \leq m$ and $X + V_1(\lambda) \in \mathcal{A}$. Define λ_0 as the orthogonal projection of λ on $\mathcal{N}_0$ and $\widetilde{\lambda} := \lambda - \lambda_0 \in {\mathcal{N}_0}^\perp$. Then:

- $\widetilde{\lambda} \in \mathcal{P} + \mathcal{P}^\infty \subset \mathcal{P}$ since $\mathcal{P}$ is convex closed and $0 \in \mathcal{P}$, by virtue of Proposition B.1.5.
- $V_0(\widetilde{\lambda}) = V_0(\lambda - \lambda_0) \leq V_0(\lambda) \leq m$ since V_0 is convex and lower semicontinuous, and $V_0^\infty(-\lambda_0) \leq 0$, by virtue of Proposition B.2.5.
- $X + V_1(\widetilde{\lambda}) = X + V_1(\lambda - \lambda_0) \geq X + V_1(\lambda)$ since V_1 is concave and upper semicontinuous and $V_1^\infty(-\lambda_0) \geq 0$, by virtue of B.2.5.

Thanks to monotonicity of $\mathcal{A}$, we conclude that $(X, \widetilde{\lambda}) \in \mathcal{C}$. $\square$

Theorem 2.3.6 (Theorem 8 of [78], uniperiodal version). *Let $\mathcal{X} = L^0(\mathbb{P})$ be the space of equivalence classes of random variables on the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ equipped with the topology of the convergence in probability. Assume that $\mathcal{P}$ is convex, closed and contains 0, V_0 is convex and lower semicontinuous and there are representatives of $V_1(\lambda)$ for every $\lambda \in \mathcal{P}$ such that $V_1(\cdot)(\omega)$ is concave and upper semicontinuous for every $\omega \in \Omega$. Moreover, let $L \in L^0(\mathbb{P})$ be such that*

$$\mathcal{A} = \{X \in L^0(\mathbb{P}) : \; X \geq L \;\; \mathbb{P}\text{-}a.s.\}.$$

If the following condition holds

$$\mathcal{L}_0 \textit{ is a linear space}$$

then $\mathcal{C}$ is closed.

Proof. Throughout the proof, we consider the concave upper semicontinuous representatives of $V_1(\lambda)$ for $\lambda \in \mathcal{P}$. Let $\left((X_n, m_n)\right)_n \subset \mathcal{C}$ be a sequence that converges to $(X, m) \in L^0(\mathbb{P}) \times \mathbb{R}$. For all n, there is $\lambda_n \in \mathcal{P}$ such that $X_n + V_1(\lambda_n) \in \mathcal{A}$ and $V_0(\lambda_n) \leq m_n$. We way assume that $\lambda_n \in \mathcal{N}_0^{\perp} = \mathcal{L}_0^{\perp}$ by virtue of Lemma 2.3.5 and of the linearity condition. Moreover, eventually $m_n \leq m+1$. By fixing representatives for each equivalence class and possibly by passing to a subsequence, we find Ω' with probability 1 such that $X_n(\omega) \to X(\omega)$ for every $\omega \in \Omega'$. Let $M(\omega) := \sup_n X_n(\omega) < \infty$ for $\omega \in \Omega'$. We may also assume that for $\omega \in \Omega'$ we have $X_n(\omega) + V_1(\lambda_n)(\omega) \geq L(\omega)$.

For every $\omega \in \Omega'$ define $\mathcal{M}(\omega) := \{\lambda \in \mathcal{P} : M(\omega) - L(\omega) \geq -V_1(\lambda)(\omega)\}$. The sequence $(\lambda_n)_n$ is contained in $\mathcal{M}(\omega)$ because for every n it holds that

$$M(\omega) + V_1(\lambda_n)(\omega) \geq X_n(\omega) + V_1(\lambda_n)(\omega) \geq L(\omega),$$

and the inclusion remains valid for the recession cones:

$$(\lambda_n)_n^{\infty} \subset \bigcap_{\omega \in \Omega'} \mathcal{M}(\omega)^{\infty}.$$

As $\mathcal{M}(\omega)$ is the sub level of the convex lower semicontinuous function $-V_1(\cdot)(\omega)$, due to Corollary B.2.9

$$\mathcal{M}(\omega)^{\infty} = \{\lambda \in \mathbb{R}^N : (-V_1(\cdot)(\omega))^{\infty} \leq 0\}.$$

Since $(\lambda_n)_n \in \{V_0 \leq m+1\}$, from Corollary B.2.9 it follows that

$$(\lambda_n)_n^{\infty} \subset \{\lambda \in \mathcal{P} \, : \, V_0(\lambda) \leq m+1\}^{\infty} = \{\lambda \in \mathbb{R}^N \, : \, V_0^{\infty}(\lambda) \leq 0\}.$$

We have shown that $(\lambda_n)_n^{\infty} \subset \mathcal{L}_0 \cap \mathcal{L}_0^{\perp} = \{0\}$. This implies that $(\lambda_n)_n$ is a bounded sequence, and without losing generality we may assume that $\lambda_n \to \lambda \in \mathcal{P}$.

Being V_0 lower semicontinuous, $V_0(\lambda) \leq \lim_n m_n = m$. Since V_1 is upper semicontinuous, for every $\omega \in \Omega'$ we have that $X(\omega) + V_1(\lambda)(\omega) \geq \limsup(X_n(\omega) + V_1(\lambda_n)(\omega) \geq L(\omega)$, thus $X + V_1(\lambda) \in \mathcal{A}$, and we can conclude that $(X, m) \in \mathcal{C}$ and $\mathcal{C}$ is closed. $\square$

It is clear that the choice of $\mathcal{A}$ plays a role only in the last step of the proof. Indeed we have $X_n \to X$, $\lambda_n \to \lambda$, $X_n + V_1(\lambda_n) \in \mathcal{A}$ and we want to conclude that $X + V_1(\lambda) \in \mathcal{A}$. This property coincides with closedness of the set valued map M (see Definition A.3.1). The next proposition shows that M turns out to be closed in this case independently on the choice of $\mathcal{A}$, provided that it is closed in probability. Hence we can generalize the theorem to the class of closed acceptance sets.

We establish the result for abstract spaces, and we make use of the notion of upper semicontinuity for functions valued in ordered topological vector spaces (Definition A.2.2). It is easy to see that in $L^0(\mathbb{P})$, the upper semicontinuity of V_1 required in Theorem 2.3.6 means that for every $\lambda_n \to \lambda$ one finds random variables $Z_n \geq V_1(\lambda_n)$ such that $Z_n(\omega) \to V_1(\lambda)(\omega)$ for almost every ω. Indeed it is enough to take $Z_n := \max\{V_1(\lambda_n), V_1(\lambda)\}$. This shows that upper semicontinuity of V_1 required in Theorem 2.3.6 coincides with that of Definition A.2.2 by virtue of Proposition A.2.3, and hence the next proposition applies to the case studied in our last theorem.

Proposition 2.3.7. *Assume that $\mathcal{P}$ is closed, $\mathcal{A}$ is closed and V_1 is upper semicontinuous. Then M is closed at every $X \in \mathcal{X}$ (hence $M(X)$ is a closed set). If moreover V_0 is lower semicontinuous, then $M^*(X)$ is a closed set for every $X \in \mathcal{X}$. Finally, if moreover ρ is upper semicontinuous at X, then M^* is closed at X.*

Proof. Consider a net $(X_\alpha)_{\alpha\in A} \subset \mathcal{X}$ directed by $(A, \succeq)$ and $X \in \mathcal{X}$ such that $X_\alpha \underset{\alpha}{\to} X$. Let $(\lambda_\alpha)_{\alpha\in A}$ be a corresponding net in $\mathcal{P}$ such that $\lambda_\alpha \in M(X_\alpha)$, and let $\lambda \in \mathcal{P}$ be such that $\lambda_\alpha \underset{\alpha}{\to} \lambda$.

Since V_1 is upper semicontinuous at λ, we find a subnet $(\lambda_\beta) \subset (\lambda_\alpha)$ and $(Y_\beta) \subset \mathcal{X}$ such that $Y_\beta \to V_1(\lambda)$ and $V_1(\lambda_\beta) \le Y_\beta$. Clearly, the corresponding net (X_β) converges to X as a subnet of (X_α). By monotonicity of $\mathcal{A}$, we have that $X_\beta + Y_\beta \in \mathcal{A}$, and hence the limit $X + V_1(\lambda) \in \mathcal{A}$.

If V_0 is lower semicontinuous, then $M^*(X)$ is closed as it is the intersection of two closed sets. Now assume that ρ is upper semicontinuous at X and take nets $X_\alpha \to X$, $\lambda_\alpha \to \lambda$, such that $\lambda_\alpha \in M^*(X_\alpha)$ for every α. Being M closed, $\lambda \in M(X)$. Since

$$V_0(\lambda) \le \liminf_\alpha V_0(\lambda_\alpha) = \liminf_\alpha \rho(X_\alpha) \le \limsup_\alpha \rho(X_\alpha) \le \rho(X),$$

we have that $\lambda \in M^*(X)$. □

We have shown that upper semicontinuity of V_1 can provide closedness of the map M. This conveys a sort of stability, since it means that in order to secure X, we can approach it with capital positions X_α that are secured through actions λ_α, and undertake the action λ approximated by λ_α, if it exists.

Remark 2.3.8. It is worth mentioning that Pennanen has addressed the problem of lower semicontinuity of a function of ρ's type in convex market models also with acceptance sets of the form $\mathcal{A} := \{X \in L^0(\mathbb{P}) : \mathbb{E}[v_1(-X)] \le 0\}$ for a proper function $v_1 \colon \mathbb{R} \times \Omega \to \overline{\mathbb{R}}$ (see [80] and [79]). Reducing to one period and adopting the identification $(S_0, S_1) = (V_0, -V_1(.))$, $D_0 = \mathcal{P}$ and $D_1 = -\mathcal{P}$, in Theorems 5.1 and 6.1 of [80] it is shown that the *reservation value* π_0 coincides with $\rho(-\cdot)$ and is lower semicontinuous, provided that $\mathcal{L}_0$ is linear. The techniques used in this case though, being based on the theory of normal integrands, are related to this specific choice of $\mathcal{A}$, and provide lower semicontinuity of ρ without establishing closedness of $\mathcal{C}$.

2.3.3 CLOSEDNESS OF $\mathcal{C}$ WITH CONVEXITY

The aim of this section is to establish an analogue of Theorem 2.3.6 which holds beyond spaces of random variables. Since it is not possible to replicate neither [78]'s nor Theorem 2.3.6's strategy, we elaborate a new proof. The linearity condition we impose has be inspired by the cited paper. Consider indeed the following sets

$$\mathcal{L} := \mathcal{P}^\infty \cap \{\lambda \in \mathcal{P}^\infty : V_1(\lambda) \in \mathcal{A}^\infty\} \cap \{\lambda \in \mathcal{P}^\infty : V_0^\infty(\lambda) \le 0\}, \tag{2.9}$$

$$\mathcal{N} := \mathcal{L} \cap (-\mathcal{L}),$$

which will play a central role in our proof of closedness of $\mathcal{C}$, and observe that if V_1 is positively homogeneous, $\mathcal{L}$ is a cone and $\mathcal{N}$ is a linear space. Moreover, note that if $\mathcal{X} = L^0(\mathbb{P})$, $\mathcal{A}$ is bounded from below and V_1 is superlinear, then $\mathcal{A}^\infty = L^0(\mathbb{P})_+$, $V_1 = V_1^\infty$ and thus $\mathcal{L} = \mathcal{L}_0$.

The set $\mathcal{L}$ contains those portfolios any size of which is available on the market with nonpositive price, and with liquidation value any size of which is acceptable. We denote by $\mathcal{N}^\perp$ the orthogonal complement of $\mathcal{N}$ in span$(\mathcal{P})$. Inspired by Theorem 5.2 in [77], we use some convexity/homogeneity assumption to restrict to $\mathcal{N}^\perp$ the set where λ's in the definition of $\mathcal{C}$ are chosen.

Lemma 2.3.9. *If $\mathcal{A}$ is a convex and closed acceptance set containing 0, $\mathcal{P}$ is convex closed and contains 0, V_1 is superlinear and V_0 is convex and lower semicontinuous, then $\mathcal{N}$ is a linear space and if $(X, m) \in \mathcal{C}$ we find $\widetilde{\lambda} \in \mathcal{N}^\perp$ such that $(X, \widetilde{\lambda}) \in \mathcal{C}$.*

Proof. First, note that $\mathcal{L}$ is a cone, so $\mathcal{N}$ is a linear space. For $X \in \mathcal{X}$ and $\lambda \in \mathcal{P}$ such that $V_0(\lambda) \le m$ and $X + V_1(\lambda) \in \mathcal{A}$, define λ_0 as the orthogonal projection of λ on $\mathcal{N}$ and $\widetilde{\lambda} := \lambda - \lambda_0 \in \mathcal{N}^\perp$. Then:

- $\widetilde{\lambda} \in \mathcal{P} + \mathcal{P}^\infty \subset \mathcal{P}$ because $\mathcal{P}$ is convex closed and contains 0, by virtue of Proposition B.1.5.
- $V_0(\widetilde{\lambda}) = V_0(\lambda - \lambda_0) \le V_0(\lambda) \le m$ because V_0 is convex and lower semicontinuous and $V_0^\infty(-\lambda_0) \le 0$, by virtue of B.2.5;

- $X + V_1(\widetilde{\lambda}) \geq (X + V_1(\lambda)) + V_1(-\lambda_0) \in \mathcal{A} + \mathcal{A}^\infty \subset \mathcal{A}$ because V_1 is superlinear and $\mathcal{A}$ is convex closed and contains 0, by virtue of Proposition B.1.5.

This shows that $(X, \widetilde{\lambda}) \in \mathcal{C}$. □

The next theorem is the main result of the current section.

Theorem 2.3.10. *Assume that $\mathcal{A}$ is convex closed and contains* 0*, $\mathcal{P}$ is convex closed and contains* 0*, V_1 is superlinear and upper semicontinuous, and V_0 is convex and lower semicontinuous. If the following condition holds:*

$$\mathcal{L} \text{ is a linear space}$$

then $\mathcal{C}$ is closed.

Proof. Take a net $\big((X_\alpha, m_\alpha)\big)_\alpha \subset \mathcal{C}$ that converges to $(X, m) \in \mathcal{X} \times \mathbb{R}$. For all α, there is $\lambda_\alpha \in \mathcal{P}$ such that $X_\alpha + V_1(\lambda_\alpha) \in \mathcal{A}$ and $V_0(\lambda_\alpha) \leq m_\alpha$. We may assume that $\lambda_\alpha \in \mathcal{N}^\perp = \mathcal{L}^\perp$ by virtue of Lemma 2.3.9 and of the linearity condition.

Now, suppose that (λ_α) has no convergent subnets. In this case, we find a subnet of (λ_α) consisting of nonzero elements with diverging norms. (Indeed, it suffices to consider the index set $\{(\alpha, n) \,:\, \alpha \in A,\ n \in \mathbb{N},\ \|\lambda_\alpha\| > n\}$ equipped with the direction defined by $(\alpha, n) \succeq (\beta, m)$ if and only if $\alpha \succeq \beta$ and $m \geq n$ and take $\lambda_{(\alpha,n)} = \lambda_\alpha$ for every $(\alpha, n) \in A$). We still denote this subnet by (λ_α) for convenience and we may assume that $\|\lambda_\alpha\| \geq 1$, so that $\frac{\lambda_\alpha}{\|\lambda_\alpha\|} \in \mathcal{P}$. Hence $\lambda \neq 0$ exists in $\mathcal{P}^\infty \cap \mathcal{L}^\perp$ such that

$$\frac{\lambda_\alpha}{\|\lambda_\alpha\|} \to \lambda.$$

Being V_1 upper semicontinuous at λ, we find a subnet $\left(\frac{\lambda_\beta}{\|\lambda_\beta\|}\right)$ and $(Y_\beta) \subset \mathcal{X}$ such that

$$V_1\left(\frac{\lambda_\beta}{\|\lambda_\beta\|}\right) \leq Y_\beta$$

and $Y_\beta \to V_1(\lambda)$. Clearly, the corresponding net (X_β) converges to X. It follows that

$$Z_\beta := X_\beta + \|\lambda_\beta\|\, Y_\beta \geq X_\beta + \|\lambda_\beta\|\, V_1\left(\frac{\lambda_\beta}{\|\lambda_\beta\|}\right) \geq X_\beta + V_1(\lambda_\beta) \in \mathcal{A}.$$

Since $\mathcal{A}$ is monotone, (X_β) converges and the norms of (λ_α) diverge, we have that

$$\frac{Z_\beta}{\|\lambda_\beta\|} = \frac{X_\beta}{\|\lambda_\beta\|} + Y_\beta \to 0 + V_1(\lambda) \in \mathcal{A}^\infty.$$

Since $V_0(\lambda_\alpha) \leq m_\alpha \leq m + 1$ for each α, we have that

$$\lambda \in \{\mu \in \mathbb{R}^N \,:\, V_0(\mu) \leq m + 1\}^\infty \subset \{\mu \in \mathbb{R}^N \,:\, V_0{}^\infty(\mu) \leq 0\}$$

(last inclusion is due to Proposition B.2.8), and we have obtained that $\lambda \in \mathcal{L} \cap (\mathcal{L}^\perp) = \{0\}$, which contradicts $\lambda \neq 0$. Thus (λ_α) must have a convergent subnet. We may assume that $\lambda_\alpha \to \lambda \in \mathcal{P}$. Proposition 2.3.7 ensures that $X + V_1(\lambda) \in \mathcal{A}$, and lower semicontinuity of V_0 gives

$$V_0(\lambda) \leq \liminf_\alpha V_0(\lambda_\alpha) \leq \lim_\alpha m_\alpha = m,$$

showing that $\mathcal{C}$ is closed. □

Remark 2.3.11 (Comparison with the closedness result of Chapter 1)**.** We show that the closedness result for $\mathcal{C}$ obtained in Chapter 1 (in Theorem 1.3.14), is actually a corollary of our last theorem. Indeed, as explained in Remark 2.1.3, the model studied in Chapter 1 falls within the assumptions of Theorem 2.3.10, where the set $\mathcal{A}$ here is identified with the set $\mathcal{A} \cap \mathcal{X}$ of Chapter 1. It is not difficult to see that, thanks to linearity and injectivity of V_1,

$$\mathcal{L} \text{ is linear} \quad \Longleftrightarrow \quad \mathcal{G} \text{ is linear}$$

(recall that $\mathcal{G}$ is defined in Chapter 1, (1.1)). This implies that the closedness result is Theorem 1.3.14 is a corollary of Theorem 2.3.10.

2.3.4 CLOSEDNESS OF $\mathcal{C}$ WITHOUT CONVEXITY

We now drop some convexity assumptions of Theorem 2.3.10, and we prove an analogous result. The drawback is that, since convexity and superlinearity were crucial in the proof of Lemma 2.3.9, now we have to assume that $\mathcal{L}$ is reduced to $\{0\}$.

Theorem 2.3.12. *Assume that $\mathcal{A} \subset \mathcal{X}$ is closed, $\mathcal{P}$ is closed, contains 0 and is star shaped about 0, V_0 is lower semicontinuous, V_1 is upper semicontinuous and such that $V_1(t\lambda) \geq t\,V_1(\lambda)$ for $\lambda \in \mathcal{P}$ and $0 < t \leq 1$. Moreover assume the following condition holds:*

$$\mathcal{L} = \{0\}.$$

Then $\mathcal{C}$ is closed.

Proof. Consider a net $\big((X_\alpha, m_\alpha)\big)_{\alpha \in A} \subset \mathcal{C}$ and assume it converges to $(X, m) \in \mathcal{X} \times \mathbb{R}$. Possibly passing to a subnet, we can assume that $m_\alpha \leq m + 1$ for every α. For each $\alpha \in A$, let $\lambda_\alpha \in \mathcal{P}$ be such that $X_\alpha + V_1(\lambda_\alpha) \in \mathcal{A}$ and $V_0(\lambda_\alpha) \leq m_\alpha$.

If (λ_α) has no convergent subnets, by repeating the argument in the proof of Theorem 2.3.10 (except for assuming that λ_α and λ are in $\mathcal{L}^\perp$) we find a nonzero $\lambda \in \mathcal{L}$, which is impossible. Hence (λ_α) has a convergent subnet and we may assume that $\lambda_\alpha \to \lambda \in \mathcal{P}$. Like in Theorem 2.3.10, Proposition 2.3.7 ensures that $X + V_1(\lambda) \in \mathcal{A}$, and lower semicontinuity of V_0 gives $V_0(\lambda) \leq m$, showing that $\mathcal{C}$ is closed. ☐

Under a similar set of assumptions it is possible to apply a Dieudonné type theorem about the closedness of the algebraic difference of sets, so as to show that the sets $\mathcal{A} - \mathcal{V}_m$ are closed. Since

$$\{\rho \leq m\} = \bigcap_{\varepsilon > 0} (\mathcal{A} - \mathcal{V}_{m+\varepsilon}),$$

this would imply lower semicontinuity of ρ. By similarity we incorporate the result in this section, even though it does not provide sufficient conditions for $\mathcal{C}$ to be closed.

Theorem 2.3.13. *Assume that $\mathcal{A} \subset \mathcal{X}$ is closed, $\mathcal{P}$ is closed, V_0 is lower semicontinuous, V_1 is positively homogeneous and continuous on $\mathcal{P}$ and admits a positively homogeneous and continuous extension on $\mathrm{cl}(\mathrm{cone}(\mathcal{P})) \to \mathcal{X}$. Moreover assume that the following condition holds:*

$$\mathcal{L} = \{0\}.$$

Then $\mathcal{A} - \mathcal{V}_m$ is closed for every $m \in \mathbb{R}$ and ρ is lower semicontinuous.

Proof. If $\mathcal{V}_m$ is empty there is nothing to be proved. Otherwise, the result is a straightforward application of Theorem 1.59 of Barbu and Precupanu [13] applied to $E_1 = \mathbb{R}^N$, $E_2 = \mathcal{X}$, $A = \{\lambda \in \mathcal{P} \,:\, V_0(\lambda) \leq m\}$, $B = \mathcal{A}$ and $T = V_1$. Recall that by Corollary B.2.9, $\{\lambda \in \mathcal{P} \,:\, V_0(\lambda) \leq m\}^\infty \subset \{\lambda \in \mathcal{P}^\infty \,:\, V_0^\infty(\lambda) \leq 0\}$. ☐

We close this section by establishing sufficient conditions for $\mathcal{C}$ to be closed when $\mathcal{P}$ is a compact subset of $\mathbb{R}^N$. The next theorem is not a consequence of Theorem 2.3.12, since $V_1(t\lambda) \geq t\,V_1(\lambda)$ for $0 < t < 1$ is not required here. Note that under compactness hypothesis on $\mathcal{P}$, we do not need to require the condition $\mathcal{L} = \{0\}$ as it is automatically satisfied because $\mathcal{P}^\infty = \{0\}$.

Theorem 2.3.14. *Assume that $\mathcal{P}$ is compact, $\mathcal{A}$ is closed, V_0 is lower semicontinuous and V_1 is upper semicontinuous. Then $\rho > -\infty$ and $\mathcal{C}$ is closed. Moreover if $\rho(X) < \infty$ for some $X \in \mathcal{X}$, the infimum defining ρ is attained:*

$$\rho(X) = \min\{V_0(\lambda) \,:\, \lambda \in \mathcal{P},\ X + V_1(\lambda) \in \mathcal{A}\} \qquad \text{i.e. } M^*(X) \neq \emptyset$$

and the sets $\{\rho \leq m\}$ and $\mathcal{A} - \mathcal{V}_m$ coincide and are closed for every $m \in \mathbb{R}$.

Proof. Being V_0 a lower semicontinuous function defined on the compact set $\mathcal{P}$, it has a minimum. Thus $\rho > -\infty$. Let $\big((X_\alpha, m_\alpha)\big)_{\alpha \in A} \subset \mathcal{C}$ be a net converging to $(X, m) \in \mathcal{X} \times \mathbb{R}$. For each α, there is $\lambda_\alpha \in \mathcal{P}$ such that $X_\alpha + V_1(\lambda_\alpha) \in \mathcal{A}$ and $V_0(\lambda_\alpha) \leq m_\alpha$.

Since $\mathcal{P}$ is compact, we find a subnet, that for convenience we still denote by (λ_α) that converges to some $\lambda \in \mathcal{P}$. For Proposition 2.3.7, we have that $X + V_1(\lambda) \in \mathcal{A}$. Moreover $V_0(\lambda) \leq \liminf_\alpha V_0(\lambda_\alpha) \leq m$, showing that $(X, m) \in \mathcal{C}$. Like in Theorem 2.3.10 we can conclude that $\rho(X)$ is actually a minimum and that $\{\rho \leq m\} = \mathcal{A} - \mathcal{V}_m$ for every $m \in \mathbb{R}$. □

For an alternative proof of the lower semicontinuity of ρ on the interior of its domain under the assumption of Theorem 2.3.14 in case $\mathcal{X}$ is a metric vector space, we can apply Theorem 4.2.1(1) in [12].

2.3.5 A RESULT OF LOCAL CONTINUITY USING THE INTERIOR OF $\mathcal{A}$

In this section we follow a completely different path to determine conditions for ρ to be continuous and M^*_ε to be lower semicontinuous. This is done respectively in Theorems 2.3.16 and 2.3.19. This section has been inspired by Chapter 4 of Bank et al. [12] and Baes et al. [11].

The following technical lemma is the base for proving the continuity result.

Lemma 2.3.15. *Assume that $\mathcal{A}$ has nonempty interior, $\mathcal{P}$ is convex and V_1 is concave. Let $M_0 : \mathcal{X} \rightrightarrows \mathcal{P}$ be defined for $X \in \mathcal{X}$ as*

$$M_0(X) := \{\lambda \in \mathcal{P} \,:\, X + V_1(\lambda) \in \operatorname{int}(\mathcal{A})\}.$$

Then $M_0(X) \subset M(X) \subset \operatorname{cl}(M_0(X))$ whenever $M_0(X) \neq \emptyset$.

Proof. The first inclusion is trivial. For the second, take $\lambda \in M(X)$ and $\lambda_0 \in M_0(X)$. Since $\mathcal{A}$ is a convex set with nonempty interior, the segment joining $X + V_1(\lambda) \in \mathcal{A}$ and $X + V_1(\lambda_0) \in \operatorname{int}(\mathcal{A})$ is contained in the interior of $\mathcal{A}$, which is itself monotone as $\mathcal{A}$ is monotone. Being V_1 concave, we have that

$$X + V_1(t\lambda + (1-t)\lambda_0) \geq t(X + V_1(\lambda)) + (1-t)(X + V_1(\lambda_0)) \in \operatorname{int}(\mathcal{A}).$$

Hence the open segment joining λ and λ_0 in contained in $M_0(X)$ and we conclude that $\lambda \in \operatorname{cl}(M_0(X))$. □

Theorem 2.3.16. *Assume that $\mathcal{A}$ is convex with nonempty interior, $0 \in \mathcal{A}$, $\mathcal{P}$ is convex, V_0 is convex and continuous, and V_1 is concave. Let $X \in \mathcal{X}$ be such that $\rho(X) \in \mathbb{R}$ and $\lambda_0 \in \mathcal{P}$ exists such that $X + V_1(\lambda_0) \in \operatorname{int}(\mathcal{A})$. Then ρ is continuous at X.*

Proof. Since ρ is convex by Proposition 2.2.1, it is enough to prove the upper semicontinuity at X to conclude that ρ is actually continuous. Fix $\varepsilon > 0$ and $\lambda \in M(X)$ with $V_0(\lambda) < \rho(X) + \varepsilon$. For Lemma 2.3.15 and continuity of V_0, we find $\lambda_0^* \in M_0(X)$ such that $V_0(\lambda_0^*) < \rho(X) + \varepsilon$. Therefore, $\mathcal{U} + V_1(\lambda_0^*) \in \mathcal{A}$ for some $\mathcal{U} \in \mathscr{N}_X$, and $\rho(Y) \leq V_0(\lambda_0^*) < \rho(X) + \varepsilon$ for all $Y \in \mathcal{U}$, proving that ρ is upper semicontinuous at X. □

In what follows we aim to show that under the hypothesis of Theorem 2.3.16, the quasi-optimal set mapping M^*_ε is lower semicontinuous. First, we state two lemmas. The first is a technical one, and we omit the proof as it is given in Lemma 5.23 of [11] (there, it is stated in a global sense, but the proof is still valid when considering pointwise continuity).

Lemma 2.3.17. *For all maps $S_1, S_2 : \mathcal{X} \rightrightarrows \mathcal{P}$ the following statements hold:*

(i) Assume that S_1 is strongly lower semicontinuous and S_2 is lower semicontinuous at some $X_0 \in \mathcal{X}$. Then, the set-valued map $S : \mathcal{X} \rightrightarrows \mathcal{P}$ given by $S(X) = S_1(X) \cap S_2(X)$ is lower semicontinuous at X_0.

(ii) Assume that S_1 is strictly lower semicontinuous at some $X_0 \in \mathcal{X}$ and $S_1(X_0) \subset S_2(X_0) \subset \operatorname{cl}(S_1(X_0))$. Then, S_2 is lower semicontinuous at X_0.

Lemma 2.3.18. *If V_0 is upper semicontinuous, M is lower semicontinuous at $X_0 \in \mathcal{X}$ and ρ is lower semicontinuous at X_0, then M^*_ε is lower semicontinuous at X_0 for every $\varepsilon > 0$.*

Proof. Take $\varepsilon > 0$ and observe that for every $X \in \mathcal{X}$

$$M^*_\varepsilon(X) = M(X) \cap H_\varepsilon(X), \qquad H_\varepsilon(X) := \{\lambda \in \mathcal{P} : V_0(\lambda) < \rho(X) + \varepsilon\}.$$

We claim that H_ε is strictly lower semicontinuous at X_0. This concludes the proof due to Lemma 2.3.17. If $H_\varepsilon(X_0) = \emptyset$, there is nothing to be proved. Otherwise let $\lambda_0 \in \mathcal{P}$ and $\alpha \in \mathbb{R}$ be such that $V_0(\lambda_0) < \alpha < \rho(X_0) + \varepsilon$. Upper semicontinuity of V_0 ensures that there is a neighborhood $\mathcal{V}$ of λ_0 such that $V_0(\lambda) < \alpha$ if $\lambda \in \mathcal{V}$, and lower semicontinuity of ρ ensures that there is a neighborhood $\mathcal{U}$ of X_0 such that $\rho(X) > \alpha - \varepsilon$ if $X \in \mathcal{U}$. Hence $\mathcal{V} \subset H_\varepsilon(X)$ for all $X \in \mathcal{U}$ and we have proved strongly lower semicontinuity of H_ε at X_0. □

Theorem 2.3.19. *Assume that $\mathcal{A}$ is convex with nonempty interior, $0 \in \mathcal{A}$, $\mathcal{P}$ is convex, V_0 is convex and continuous, and V_1 is concave. Let $X \in \mathcal{X}$ be such that $\rho(X) \in \mathbb{R}$ and $\lambda_0 \in \mathcal{P}$ exists such that V_1 is continuous at λ_0 and $X + V_1(\lambda_0) \in \operatorname{int}(\mathcal{A})$. Then M^*_ε is lower semicontinuous at X_0 for each $\varepsilon > 0$.*

Proof. By Theorem 2.3.16, ρ is continuous at X. Once proved that M is lower semicontinuous at X, the thesis follow from Lemma 2.3.18.

Since $X + V_1(\lambda_0) \in \operatorname{int}(\mathcal{A})$ and $\mathcal{A}$ is open, there are $\mathcal{U} \in \mathscr{N}_X$ and $\mathcal{U}' \in \mathscr{N}_{V_1(\lambda_0)}$ such that $\mathcal{U} + \mathcal{U}' \subset \operatorname{int}(\mathcal{A})$. Then for continuity of V_1, there is $\mathcal{V}$ neighborhood of λ_0 such that $\mathcal{U} + V_1(\mathcal{V}) \subset \operatorname{int}(\mathcal{A})$. Thus for every $Y \in \mathcal{U}$ we have that $\mathcal{V} \subset M_0(Y)$, and we have obtained the strong lower semicontinuity of M_0 at X. Now note that $M_0(X) \subset M(X) \subset \operatorname{cl}(M_0(X))$ and apply Lemma 2.3.17 to get lower semicontinuity of M at X. □

2.4 NO ACCEPTABLE DEAL CONDITIONS VS $\mathcal{L}$-TYPE CONDITIONS

As we have pointed out in the introduction of this thesis, in frictionless unconstrained uniperiodal models, the absence of arbitrages is a sufficient condition for the cone of superreplicable claims at zero cost to be closed. We have also said that in a nonlinear world, due to the possible lack of translation invariance, we are interested in the set $\mathcal{C}$ in the product space $\mathcal{X} \times \mathbb{R}$ rather than in the set of superreplicable claims at zero cost. Examples 2.3.3 and 2.3.4 show that in markets with frictions the absence of arbitrages is not enough to conclude that $\mathcal{C}$ is closed, and in Sections 2.3.3 and 2.3.4 we have established sets of assumptions ensuring that $\mathcal{C}$ is closed. Namely, together with algebraic or topological properties of the elements of the model, we have required the set $\mathcal{L}$ to be either linear or equal to $\{0\}$. We will refer informally to these assumptions as "$\mathcal{L}$-type conditions". The scope of this section is to investigate the $\mathcal{L}$-type conditions by relating them to the more familiar "no acceptable deal" (NAD) conditions.

First, we give general definitions and we state some general results linking $\mathcal{L}$-type conditions and NAD conditions. Then, we use these results to investigate the case of a perfectly liquid market with conic acceptance set (Proposition 2.4.8) and to investigate the conditions $\mathcal{L}$ linear and $\mathcal{L} = \{0\}$ (Subsection 2.4.1).

Definition 2.4.1. For any couple of nonempty sets $\mathcal{Q} \subset \mathbb{R}^N$ and $\mathcal{B} \subset \mathcal{X}$ such that $\mathcal{B}$ is monotone, and for any couple of maps $T_0 : \mathcal{Q} \to \mathbb{R}$ and $T_1 : \mathcal{Q} \to \mathcal{X}$, we define the following set

$$\mathcal{L}(\mathcal{Q}; T_0; T_1, \mathcal{B}) := \{\lambda \in \mathcal{Q} \,:\, T_0(\lambda) \leq 0,\ T_1(\lambda) \in \mathcal{B}\},$$

and we say that the NAD$(\mathcal{Q}; T_0; T_1, \mathcal{B})$ (read "no-acceptable deal with respect to $(\mathcal{Q}, T_0, T_1, \mathcal{B})$") condition holds if the following implication holds true:

$$\lambda \in \mathcal{L}(\mathcal{Q}; T_0; T_1, \mathcal{B}) \quad \Longrightarrow \quad T_1(\lambda) = 0.$$

The set $\mathcal{L}(\mathcal{Q}; T_0; T_1, \mathcal{B})$ is nothing else than a generalization of the sets we have encountered in the previous section, indeed $\mathcal{L}_0$ defined in (2.6) coincides with $\mathcal{L}(\mathcal{P}^\infty; V_0^\infty; V_1^\infty, \mathcal{X}_+)$ and $\mathcal{L}$ defined in (2.9) coincides with $\mathcal{L}(\mathcal{P}^\infty; V_0^\infty; V_1, \mathcal{A}^\infty)$.

The condition NAD$(\mathcal{Q}; T_0; T_1, \mathcal{B})$ derives its name from the case where $\mathcal{Q} = \mathcal{P}$, $T_0 = V_0$, $T_1 = V_1$ and $\mathcal{B} = \mathcal{A}$, since NAD$(\mathcal{P}; V_0; V_1, \mathcal{A})$ means that we cannot find any nonzero element of the reference space $\mathcal{X}$ which is acceptable and in the mean time can be obtained through the liquidation of a portfolio with nonpositive initial cost. Such an element is typically called an *acceptable deal* (see Definition 1.2.5), hence NAD stands for "no acceptable deals".

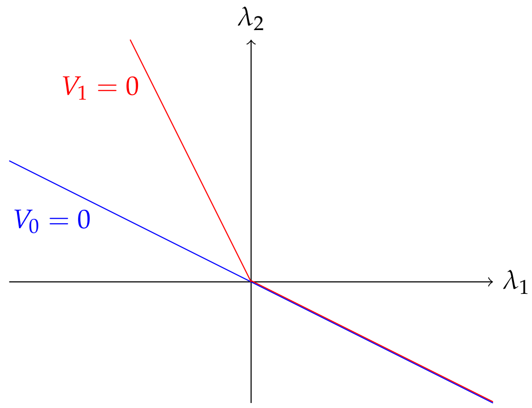


FIGURE 2.1: Example 2.4.4

Proposition 2.4.2. *Assume that $\mathcal{B}$ is monotone and contains 0 and $T_1(0) = 0$. Then $\mathrm{NAD}(\mathcal{Q}; T_0; T_1, \mathcal{B})$ holds in each of the following cases:*

(a) $\mathcal{L}(\mathcal{Q}; T_0; T_1, \mathcal{B}) = \{0\}$,

(b) *$\mathcal{Q}$ is convex, T_1 is concave, $\mathcal{B} \cap (-\mathcal{B}) = \{0\}$ and $\mathcal{L}(\mathcal{Q}; T_0; T_1, \mathcal{B}) = -\mathcal{L}(\mathcal{Q}; T_0; T_1, \mathcal{B})$ (in particular $\mathcal{L}(\mathcal{Q}; T_0; T_1, \mathcal{B})$ is linear).*

Proof. Case *(a)* is straightforward. Now, assume *(b)* holds and take $\lambda \in \mathcal{L}(\mathcal{Q}; T_0; T_1, \mathcal{B})$, so that $T_1(\lambda), T_1(-\lambda) \in \mathcal{B}$. For concavity $-T_1(\lambda) \geq T_1(-\lambda)$, hence $T_1(\lambda) \in -\mathcal{B}$ by monotonicity, implying that $T_1(\lambda) = 0$. □

We exhibit some examples, the first two showing that the converse in general does not hold true, and the third showing the necessity of $\mathcal{B} \cap (-\mathcal{B}) = \{0\}$ in *(b)* to draw the conclusion.

Example 2.4.3. *Consider Example 2.3.4. The condition $\mathrm{NAD}(\mathcal{P}; V_0; V_1, \mathcal{A})$ holds, and $\mathcal{L}(\mathcal{P}; V_0; V_1, \mathcal{A}) = \{0\} \times \mathbb{R}_-$.*

Example 2.4.4. *Consider the simple case where $\mathcal{X} = \mathbb{R}$, $\mathcal{A} = \mathbb{R}_+$, $N = 2$ and $\mathcal{P} = \mathbb{R}^2$. The operators $V_0 : \mathcal{P} \to \mathbb{R}$ and $V_1 : \mathcal{P} \to \mathcal{X}$ are defined as follows:*

$$V_0\big((\lambda_1, \lambda_2)\big) = \lambda_1 + 2\lambda_2, \quad V_1(\lambda_1, \lambda_2) = \min\{\lambda_1 + 2\lambda_2, 2\lambda_1 + \lambda_2\}$$

Note that V_0 is linear, hence $V_0 = V_0^\infty$, while V_1 is superlinear. Moreover, $\mathcal{A} = \mathcal{A}^\infty$ and $\mathcal{P} = \mathcal{P}^\infty$.

Since $V_0(\lambda_1, \lambda_2) \leq 0$ iff $\lambda_2 \leq -\frac{1}{2}\lambda_1$, and $V_1(\lambda_1, \lambda_2) \in \mathcal{A}$ iff

$$\begin{cases} \lambda_2 \geq -2\lambda_1 \\ \lambda_2 \geq -\frac{1}{2}\lambda_1 \end{cases},$$

it follows that $\mathcal{L}(\mathcal{P}; V_0; V_1, \mathcal{A}) = \{(2\lambda, -\lambda) : \lambda \geq 0\}$ which is not equal to $-\mathcal{L}(\mathcal{P}; V_0; V_1, \mathcal{A})$. Furthermore, it is clear that the condition $\mathrm{NAD}(\mathcal{P}; V_0; V_1, \mathcal{A})$ holds

Example 2.4.5. *Assume that $\mathcal{X} = \mathbb{R}^2$, $\mathcal{A}$ is the half space $\mathbb{R}_+ \times \mathbb{R}$, $N = 2$ and $\mathcal{P} = \mathbb{R}^2$. Let V_0 and V_1 be defined as follows:*

$$V_0 : \mathcal{P} \to \mathbb{R}, \quad V_0(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2.$$

$$V_1 : \mathcal{P} \to \mathcal{X}, \quad V_1(\lambda_1, \lambda_2) = \Big(g\big(R(\lambda_1, \lambda_2)\big), \lambda_2\Big)$$

where R is the counter clockwise rotation of $\pi/4$, and

$$g(x, y) = \begin{cases} \log(y+1) & y \geq 0 \\ y & y < 0 \end{cases}.$$

We have that $V_0(\lambda) \leq 0$ iff $\lambda_2 \leq -\lambda_1$, and $V_1(\lambda) \in \mathcal{A}$ iff $\lambda_2 \geq \lambda_1$. Hence $\mathcal{L}(\mathcal{P}; V_0; V_1, \mathcal{A}) = \{(\lambda_1, -\lambda_1) : \lambda_1 \in \mathbb{R}\}$, that is a nontrivial linear space. Now, if $\lambda = (\lambda_1, \lambda_2) \in \mathcal{L}(\mathcal{P}; V_0; V_1, \mathcal{A})$ and $\lambda_2 \neq 0$, then $V_1(\lambda) = \left(g\left(\frac{\lambda_1}{\sqrt{2}}, 0\right), \lambda_2\right) = (0, \lambda_2) \neq (0,0)$, violating the no acceptable deal condition.

The next propositions characterizes the condition $\mathrm{NAD}(\mathcal{Q}; T_0; T_1, \mathcal{B})$ in terms of $\mathcal{L}(\mathcal{Q}; T_0; T_1, \mathcal{B})$ in some situations.

Proposition 2.4.6. *Assume that $\mathcal{B}$ contains 0 and either the law of one price (2.1) holds together with $T_0(0) = 0$ and $T_1(0) = 0$, or $\mathcal{Q}$ and T_0 are convex, T_1 is concave and one of the following statements holds:*

1. *$\mathcal{B}$ is a cone and there exists $\lambda^* \in \mathcal{Q}$ such that $T_1(\lambda^*) \in \mathcal{B} \setminus (-\mathcal{X}_+)$;*
2. *$\mathcal{Q} = \mathbb{R}^N$, T_1 is monotone increasing and there exists $\lambda^* \in \mathbb{R}^N$ such that $T_1(\lambda^*) \notin -\mathcal{X}+$.*

Then the following equivalence holds:

$$\mathrm{NAD}(\mathcal{Q}; T_0; T_1, \mathcal{B}) \iff \mathcal{L}(\mathcal{Q}; T_0; T_1, \mathcal{B}) = \ker(T_0) \cap \ker(T_1).$$

Proof. We only prove the " $\Longrightarrow$ " implication. Since $0 \in \mathcal{B}$, it is clear that $\ker(T_0) \cap \ker(T_1)$ is contained in $\mathcal{L}(\mathcal{Q}; T_0; T_1, \mathcal{B})$. Then by definition of $\mathrm{NAD}(\mathcal{Q}; T_0; T_1, \mathcal{B})$, we only need to show that our sets of hypothesis ensure that every $\lambda \in \mathcal{L}(\mathcal{Q}; T_0; T_1, \mathcal{B})$ has $T_0(\lambda) = 0$. This fact is immediate if the law of one price (2.1) holds together with $T_0(0) = 0$ and $T_1(0) = 0$. Otherwise assume that $\mathcal{Q}$ and T_0 are convex, T_1 is concave and proceed by contradiction. Let $\lambda \in \mathcal{L}(\mathcal{Q}; T_0; T_1, \mathcal{B})$ be such that $T_0(\lambda) < 0$. Clearly $T_1(\lambda) = 0$. We divide the proofs of the two statements.

1. We find μ on the segment joining λ and λ^* such that $T_0(\mu) \leq 0$ and $T_1(\mu) \in \mathcal{B} \setminus \{0\}$, which contradicts the no acceptable deal hypothesis. Indeed, $t \in (0,1)$ can be chosen so that

$$T_0(\mu) \leq t\, T_0(\lambda) + (1-t)\, T_0(\lambda^*) \leq 0$$

where $\mu = t\,\lambda + (1-t)\lambda^*$. By concavity of T_1 we have that

$$T_1(\mu) \geq t\, T_1(\lambda) + (1-t)\, T_1(\lambda^*) = (1-t)\, T_1(\lambda^*) \in \mathcal{B}.$$

Since $\mathcal{B}$ is a monotone cone, $T_1(\mu) \in \mathcal{B}$ and $T_1(\mu)$ is nonzero because $T_1(\lambda^*) \notin -\mathcal{X}_+$.

2. Being T_0 continuous on $\mathbb{R}^N$, there is an open neighborhood U of λ on which T_0 is strictly negative. Consider the open nonempty set $\widetilde{U} := U \cap (\lambda + \mathbb{R}^N_{++})$. If $\mu \in \widetilde{U}$, then $T_1(\mu) \in \mathcal{B}$ because T_1 and $\mathcal{B}$ are monotone, implying that $\mu \in \mathcal{L}(\mathcal{Q}; T_0; T_1, \mathcal{B})$ and hence $T_1(\mu) = 0$. Now, T_1 is a concave function that equals 0 on the open set $\widetilde{U}$. Then it is dominated by 0 on $\mathbb{R}^N$. Indeed, if λ in $\mathbb{R}^N$ and μ in $\widetilde{U}$, for t strictly positive and sufficiently near to 0

$$0 = T_1\big(t\,\lambda + (1-t)\,\mu\big) \geq t\, T_1(\lambda) + (1-t)\, T_1(\mu) = t\, T_1(\lambda).$$

This fact contradicts the existence of $\lambda^* \in \mathbb{R}^N$ such that $T_1(\lambda^*)$ is not dominated by 0. ☐

The next proposition gives a set of assumptions that allows to invert Proposition 2.4.2.

Proposition 2.4.7. *Assume that $\mathcal{Q} = \mathbb{R}^N$, $\mathcal{B}$ is convex, closed and contains 0, T_1 is concave, monotone increasing and $T_1(0) = 0$, T_0 is convex and $T_0(0) = 0$, there is $\lambda^* \in \mathcal{Q}$ such that $T_1(\lambda^*) \notin -\mathcal{X}_+$, the kernel of T_0 or T_1 does not contain nontrivial segments. If $\mathrm{NAD}(\mathbb{R}^N; T_0; T_1, \mathcal{B})$ holds, then*

$$\mathcal{L}(\mathbb{R}^N; T_0; T_1, \mathcal{B}) = \{0\}.$$

Proof. By Proposition 2.4.6 it holds that $\mathcal{L}(\mathbb{R}^N; T_0; T_1, \mathcal{B}) = \ker(T_0) \cap \ker(T_1)$. Moreover by our assumptions $\mathcal{L}(\mathbb{R}^N; T_0; T_1, \mathcal{B})$ is convex and does not contain any nontrivial segment, hence it coincides with $\{0\}$. ☐

We close this section by focusing on the simple case where the market is perfectly liquid and $\mathcal{A}$ is a pointed cone.

Proposition 2.4.8. *If $\mathcal{P}$ is a linear subspace of $\mathbb{R}^N$, V_0 and V_1 are linear maps, $\mathcal{A}$ is a cone such that $\mathcal{A} \cap (-\mathcal{A}) = \{0\}$, there exists $\lambda^* \in \mathcal{P}$ such that $V_1(\lambda^*) \in \mathcal{A} \setminus \{0\}$, then the following statements are equivalent:*

a) $\mathrm{NAD}(\mathcal{P}; V_0; V_1, \mathcal{A})$

b) $\mathcal{L}(\mathcal{P}; V_0; V_1, \mathcal{A}) = \ker(V_0) \cap \ker(V_1)$

c) $\mathcal{L}(\mathcal{P}; V_0; V_1, \mathcal{A})$ *is a linear space*

d) $\mathcal{L}(\mathcal{P}; V_0; V_1, \mathcal{A}) = -\mathcal{L}(\mathcal{P}; V_0; V_1, \mathcal{A})$.

If in addition V_1 is injective, they are also equivalent to

e) $\mathcal{L}(\mathcal{P}; V_0; V_1, \mathcal{A}) = \{0\}$.

Proof. The implications $\implies c \implies d$ and $b \implies a$ are trivial. For $a \implies b$ see Proposition 2.4.6, while $d \implies b$ is a clear consequence of linearity of $\mathcal{P}$, V_0, V_1 and pointedness of $\mathcal{A}$. Finally $e \implies d$ is obvious, and if V_1 is injective, $b \implies e$. ☐

2.4.1 About the $\mathcal{L}$-type conditions of Theorems 2.3.10 and 2.3.12

The aim of this section is to compare the assumptions of the two main theorems of Section 2.3, namely $\mathcal{L} = \{0\}$ in Theorem 2.3.12 and $\mathcal{L}$ linear in Theorem 2.3.10, with different types of no acceptable deal conditions to investigate which implications in general hold true. We first state the results, and then we provide a number of examples.

Proposition 2.4.9. *Assume that $\mathcal{P}^\infty \subset \mathcal{P}$, $V_1(0) = 0$, $V_0^\infty \geq V_0$. Then the implications in the following diagram hold true, and in general no other implication among the statements holds:*

$$\begin{tikzcd}
\mathrm{NAD}(\mathcal{P}^\infty; V_0^\infty; V_1, \mathcal{A}^\infty) \arrow[d, bend right, "\text{Ex 2.4.15}"', "/" marking] & \mathrm{NAD}(\mathcal{P}^\infty; V_0; V_1, \mathcal{A}^\infty) \arrow[l, Rightarrow] & \mathrm{NAD}(\mathcal{P}; V_0; V_1, \mathcal{A}^\infty) \arrow[l, Rightarrow] \\
\mathcal{L} = \{0\} \arrow[u, Rightarrow] \arrow[ur, "\text{Ex 2.4.16}", "/" marking] & \mathcal{L}(\mathcal{P}^\infty; V_0; V_1, \mathcal{A}^\infty) = \{0\} \arrow[l, Rightarrow] \arrow[u, Rightarrow] \arrow[ur, "\text{Ex 2.4.17}", "/" marking] & \mathcal{L}(\mathcal{P}; V_0; V_1, \mathcal{A}^\infty) = \{0\} \arrow[l, Rightarrow] \arrow[u, Rightarrow]
\end{tikzcd}$$

Proof. The arrows towards left hold true because $V_0 \leq V_0^\infty$ and $\mathcal{P}^\infty \subset \mathcal{P}$, while upward arrows hold because of Proposition 2.4.2 and $V_1(0) = 0 \in \mathcal{A}^\infty$. The examples provided in the references on the diagram show that no other implication can be proved without further assumptions. ☐

Remark 2.4.10. In Proposition 2.4.9, $\mathcal{A}^\infty$ can be replaced by any monotone nonempty set $\mathcal{B} \subset \mathcal{X}$ containing 0. We have used $\mathcal{A}^\infty$ so as to have the set $\mathcal{L}$ in the left corner of the diagram.

Proposition 2.4.11. *Assume that $\mathcal{X}_+ \subset \mathcal{A}^\infty \subset \mathcal{A}$ and $V_1(0) = 0$. Then the implications in the following diagram hold true, and in general no other implication among the statements holds:*

$$\begin{tikzcd}
\mathrm{NAD}(\mathcal{P}^\infty; V_0^\infty; V_1, \mathcal{X}_+) \arrow[d, bend right, "\text{Ex 2.4.15}"', "/" marking] & \mathrm{NAD}(\mathcal{P}^\infty; V_0^\infty; V_1, \mathcal{A}^\infty) \arrow[l, Rightarrow] & \mathrm{NAD}(\mathcal{P}^\infty; V_0^\infty; V_1, \mathcal{A}) \arrow[l, Rightarrow] \\
\mathcal{L}(\mathcal{P}^\infty; V_0^\infty; V_1, \mathcal{X}_+) = \{0\} \arrow[u, Rightarrow] \arrow[ur, "\text{Ex 2.4.18}", "/" marking] & \mathcal{L} = \{0\} \arrow[l, Rightarrow] \arrow[u, Rightarrow] \arrow[ur, "\text{Ex 2.4.19}", "/" marking] & \mathcal{L}(\mathcal{P}^\infty; V_0^\infty; V_1, \mathcal{A}) = \{0\} \arrow[l, Rightarrow] \arrow[u, Rightarrow]
\end{tikzcd}$$

Proof. The arrows towards left hold true because $\mathcal{X}_+ \subset \mathcal{A}^\infty \subset \mathcal{A}$, while upward arrows hold because of Proposition 2.4.2 and $V_1(0) = 0$. The examples provided in the references on the diagram show that no other implication can be proved without further assumptions. ☐

Remark 2.4.12. In Proposition 2.4.11, $\mathcal{P}^\infty$ and V_0^∞ can be replaced by any subset $\mathcal{P}$ of $\mathbb{R}^N$ containing 0 and any function $V_0 : \mathcal{P} \to \mathbb{R}$. We have used $\mathcal{P}^\infty$ and V_0^∞ so as the set $\mathcal{L}$ to appear in the diagram.

Proposition 2.4.13. *Assume that V_0 is convex and lower semicontinuous with $V_0(0) = 0$, V_1 is concave with $V_1(0) = 0$, $\mathcal{P}^\infty$ is convex and $\mathcal{A}^\infty \cap (-\mathcal{A}^\infty) = \{0\}$. Then the implications in the following diagram hold true, and in general no other implication among the statements holds:*

$$
\begin{tikzcd}
\mathrm{NAD}(\mathcal{P}^\infty; V_0^\infty; V_1, \mathcal{A}^\infty) \arrow[d, bend right, "\textit{Ex 2.4.4}"', "/" marking] & \mathrm{NAD}(\mathcal{P}^\infty; V_0; V_1, \mathcal{A}^\infty) \arrow[l, Rightarrow] \\
\mathcal{L} \textit{ linear} \arrow[u, Rightarrow] \arrow[ur, "\textit{Ex 2.4.16}", "/" marking] & \mathcal{L}(\mathcal{P}^\infty; V_0; V_1, \mathcal{A}^\infty) \textit{ linear} \arrow[l, Rightarrow] \arrow[u, Rightarrow]
\end{tikzcd}
$$

If moreover $\mathcal{A}$ is convex closed and $\mathcal{A} \cap (-\mathcal{A}) = \{0\}$, and V_1 is superlinear, the following diagram holds too:

$$
\begin{tikzcd}
\mathrm{NAD}(\mathcal{P}^\infty; V_0^\infty; V_1, \mathcal{A}^\infty) \arrow[d, bend right, "\textit{Ex 2.4.4}"', "/" marking] & \mathrm{NAD}(\mathcal{P}^\infty; V_0^\infty; V_1, \mathcal{A}) \arrow[l, Rightarrow] \\
\mathcal{L} \textit{ linear} \arrow[u, Rightarrow] \arrow[ur, "\textit{Ex 2.4.19}", "/" marking] & \mathcal{L}(\mathcal{P}^\infty; V_0^\infty; V_1, \mathcal{A}) \textit{ linear} \arrow[l, Rightarrow] \arrow[u, Rightarrow]
\end{tikzcd}
$$

Proof. Upward arrows hold because of Proposition 2.4.2, while leftward arrows in the top level of the first (second) diagram hold because $V_0^\infty \geq V_0$ ($\mathcal{A}^\infty \subset \mathcal{A}$).

Now, we claim that whenever V_0 is convex and lower semicontinuous with $V_0(0) = 0$, if $\mathcal{L}(\mathcal{P}^\infty; V_0; V_1, \mathcal{A}^\infty)$ is a cone, then it coincides with $\mathcal{L}$. Indeed, the inclusion "$\supset$" is trivial, while for the other it is enough to recall that $V_0^\infty(\lambda) = \sup_{t>0} \frac{V_0(t\lambda)}{t}$ by virtue of Proposition B.2.5. This shows the leftward arrow at the bottom level of the first diagram.

We also claim that whenever V_1 is superlinear and $\mathcal{A}$ is convex, closed and contains 0, if $\mathcal{L}(\mathcal{P}^\infty; V_0^\infty; V_1, \mathcal{A})$ is a cone, then it coincides with $\mathcal{L}$. Indeed, one inclusion is trivial, while for the other it is enough to recall that $\mathcal{A}^\infty = \bigcap_{t>0} t\mathcal{A}$ thanks to Corollary B.1.7. This shows the leftward arrow at the bottom level of the second diagram.

The examples provided in the references on the diagram show that no other implication can be proved without further assumptions. $\square$

Remark 2.4.14. In Proposition 2.4.13, V_0^∞ can be replaced by V_0. We have used V_0^∞ so as the set $\mathcal{L}$ to appear in the diagram.

Example 2.4.15. *Assume $\mathcal{X}$, N, $\mathcal{P}$, V_0, R and g are like in Example 2.4.5, and define*

$$V_1 : \mathcal{P} \to \mathbb{R}^2, \quad V_1(\lambda_1, \lambda_2) = \Big(g\big(R(\lambda_1, \lambda_2)\big), g\big(R(\lambda_1, \lambda_2)\big) \Big)$$

Moreover, let $\mathcal{A}$ coincide with the positive cone of $\mathbb{R}^2$. In this case we have $\mathcal{A} = \mathcal{A}^\infty$, $\mathcal{P} = \mathcal{P}^\infty$ and $V_0 = V_0^\infty$. Since $V_1(\lambda) \in \mathcal{A}$ holds iff $\lambda_2 \geq \lambda_1$, we have that $\mathcal{L} = \mathcal{L}(\mathcal{P}; V_0; V_1, \mathcal{A}) = \{(\lambda_1, -\lambda_1) \;:\; \lambda_1 \in \mathbb{R}\}$ like in Example 2.4.5. If $\lambda \in \mathcal{L}(\mathcal{P}; V_0; V_1, \mathcal{A})$, then $V_1(\lambda) = \left(g\left(\frac{\lambda_1}{\sqrt{2}}, 0\right), g\left(\frac{\lambda_1}{\sqrt{2}}, 0\right) \right) = (0,0)$, implying that $\mathrm{NAD}(\mathcal{P}; V_0; V_1, \mathcal{A})$ holds.

Example 2.4.16. *Consider the simple case where $\mathcal{X} = \mathbb{R}$, $\mathcal{A} = \mathbb{R}_+$, $N = 2$ and $\mathcal{P} = \mathbb{R}^2$. Define*

$$V_0 \colon \mathcal{P} \to \mathbb{R}, \quad V_0(\lambda_1, \lambda_2) = \max\left\{ \lambda_2 + 2\lambda_1, \lambda_2 + \frac{1}{2}\lambda_1 - 1 \right\}$$

and

$$V_1 \colon \mathcal{P} \to \mathcal{X}, \quad V_1(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2$$

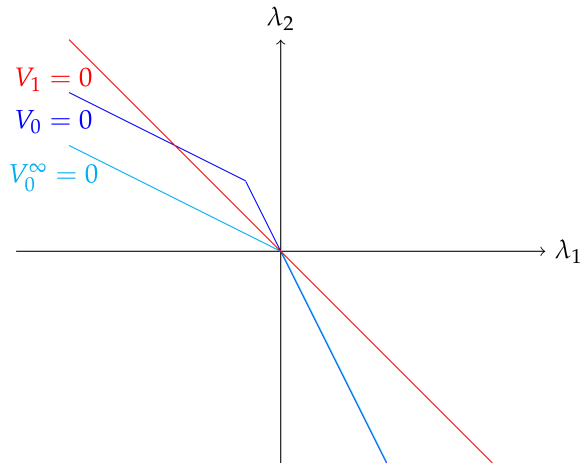


FIGURE 2.2: Example 2.4.16

We have that $\mathcal{A} = \mathcal{A}^\infty$ *and* $\mathcal{P} = \mathcal{P}^\infty$*. By Proposition B.2.5 we have*

$$\begin{aligned} V_0^\infty(\lambda_1, \lambda_2) &= \sup_{t>0} \frac{V_0(t\lambda_1, t\lambda_2)}{t} \\ &= \sup_{t>0} \max\left\{\lambda_2 + 2\lambda_1, \lambda_2 + \frac{1}{2}\lambda_1 - \frac{1}{t}\right\} \\ &= \max\left\{\lambda_2 + 2\lambda_1, \lambda_2 + \frac{1}{2}\lambda_1\right\} \geq V_0(\lambda_1, \lambda_2). \end{aligned}$$

As shown in Figure 2.2, we have $\mathcal{L} = \{0\}$*, hence* $\mathrm{NAD}(\mathcal{P}; V_0^\infty; V_1, \mathcal{A})$ *holds. Moreover* $\mathcal{L}(\mathcal{P}; V_0; V_1, \mathcal{A}) \supsetneq \{0\}$ *and on it* V_1 *does not take only value* 0*. Hence* $\mathrm{NAD}(\mathcal{P}; V_0; V_1, \mathcal{A})$ *does not hold.*

Example 2.4.17. *Consider the simple case where* $\mathcal{X} = \mathbb{R}$, $\mathcal{A} = \mathbb{R}_+$, $N = 2$*. Assume*

$$\mathcal{P} = \mathbb{R}_+^2 \cup \{(x, y) \in \mathbb{R}^2 \, : \, y \geq x^2\}$$

and for $(\lambda_1, \lambda_2) \in \mathcal{P}$ *define* $V_0(\lambda_1, \lambda_2) = \lambda_1 + \lambda_2$ *and* $V_1(\lambda_1, \lambda_2) = \lambda_2$*. Since* $\mathcal{P}^\infty = \mathbb{R}_+^2$, $\mathcal{A} = \mathcal{A}^\infty$ *and* $V_0 = V_0^\infty$ *on* $\mathcal{P}^\infty$*, we have* $\mathcal{L}(\mathcal{P}^\infty; V_0; V_1, \mathcal{A}) = \{0\}$*. Now, given* $(-1, 1) \in \mathcal{P}$*, we have that* $V_0(-1, 1) = 0$ *and* $V_1(-1, 1) = 1 \in \mathcal{A} \setminus \{0\}$*, violating the* $\mathrm{NAD}(\mathcal{P}; V_0; V_1, \mathcal{A})$ *condition.*

Example 2.4.18. *Assume that* $\mathcal{X} = \mathbb{R}^2$, $N = 2$, $\mathcal{P} = \mathbb{R}^2$ *and*

$$\mathcal{A} = \{(x_1, x_2) \in \mathbb{R}^2 \, : \, x_2 \geq \max\{-x_1, -x_1/2\}\},$$

$$V_0 : \mathcal{P} \to \mathbb{R}, \quad V_0(\lambda_1, \lambda_2) = \lambda_2 + 2\lambda_1$$

$$V_1 : \mathcal{P} \to \mathcal{X}, \quad V_1(\lambda_1, \lambda_2) = (\lambda_1, \lambda_2).$$

Being V_0 *linear and* $\mathcal{P}$ *and* $\mathcal{A}$ *conic,* $V_0 = V_0^\infty$, $\mathcal{A} = \mathcal{A}^\infty$ *and* $\mathcal{P} = \mathcal{P}^\infty$*.*

Since $V_1(\lambda) \geq 0$ *iff* $\lambda \in \mathbb{R}_+^2$*, we have that* $\mathcal{L}(\mathcal{P}; V_0; V_1, \mathcal{X}_+) = \{0\}$*. But*

$$\mathcal{L}(\mathcal{P}; V_0; V_1, \mathcal{A}) = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 \, : \, \lambda_1 + \lambda_2 \geq 0, \, 2\lambda_1 + \lambda_2 \leq 0\}$$

which is not contained in the kernel of V_1*, showing that* $\mathrm{NAD}(\mathcal{P}; V_0; V_1, \mathcal{A})$ *does not hold.*

Example 2.4.19. *Let* $\mathcal{X}$ *be the space of random variables on the finite probability space* $(\Omega, \mathcal{F}, \mathbb{P})$*, where* $\Omega = \{\omega_1, \omega_2\}$, $\mathcal{F} = 2^\Omega$, $p_i = \mathbb{P}(\{\omega_i\})$ *Suppose*

$$p_1 \geq p_2 > 0.$$

(note that $\mathcal{X}$ *can be identified with* $\mathbb{R}^2$ *as in the bulk of previous examples. Here we prefer to talk about random variables as we want to use the language of expectations and utility functions).*

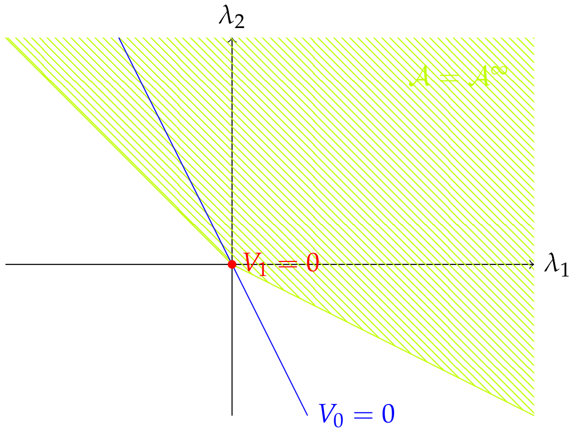


FIGURE 2.3: Example 2.4.18

For a fixed $\gamma > 0$, consider the real valued function u defined on $\mathbb{R}$ with its inverse u^{-1} defined on $(-\infty, 1)$

$$u : \mathbb{R} \to \mathbb{R}, \quad u(x) = 1 - e^{-\gamma x}$$
$$u^{-1} : (-\infty, 1) \to \mathbb{R} \quad u^{-1}(y) = -\frac{1}{\gamma}\log(1-y)$$

and the acceptance set $\mathcal{A} := \{X \in \mathcal{X} : \mathbb{E}[u(X)] \geq 0\}$. In the usual identification of $\mathcal{X}$ with $\mathbb{R}^2$,

$$\begin{aligned}
\mathcal{A} &= \{(x_1, x_2) \,:\, p_1\, u(x_1) + p_2\, u(x_2) \geq 0\} \\
&= \left\{(x_1, x_2) \,:\, u(x_2) \geq -\frac{p_1}{p_2} u(x_1), \;\; -\frac{p_1}{p_2} u(x_1) < 1\right\} \\
&= \left\{(x_1, x_2) \,:\, x_2 \geq u^{-1}\left(-\frac{p_1}{p_2} u(x_1)\right), \;\; x_1 > u^{-1}\left(-\frac{p_1}{p_2}\right)\right\} \\
&= \left\{(x_1, x_2) \,:\, x_2 \geq h(x_1), \;\; x_1 > -\frac{1}{\gamma}\log\left(1 + \frac{p_2}{p_1}\right)\right\} \\
&= \operatorname{epi}(h)
\end{aligned}$$

where

$$h : \left(-\frac{1}{\gamma}\log\left(1 + \frac{p_2}{p_1}\right), +\infty\right) \to \mathbb{R}, \quad h(x) = -\frac{1}{\gamma}\log\left(1 + \frac{p_1}{p_2} - \frac{p_1}{p_2} e^{-\gamma x}\right).$$

Note that h is convex, strictly decreasing, $h(0) = 0$, $h'(0) = -\frac{p_1}{p_2}$ and

$$\lim_{x \to +\infty} h(x) = -\frac{1}{\gamma}\log\left(1 + \frac{p_1}{p_2}\right).$$

In particular, note that if $p_1 = p_2 = \frac{1}{2}$, then $h'(0) = -1$ and $\mathcal{A} \cap (-\mathcal{A}) = \{0\}$. Since $\mathcal{A}$ is monotone and bounded from below, $\mathcal{A}^\infty$ equals to $\mathcal{X}_+$.

Valuation and liquidation operators V_0 and V_1 are defined on $\mathcal{P} = \mathbb{R}^2$ as follows:

$$V_0 : \mathcal{P} \to \mathbb{R}, \quad V_0(\lambda_1, \lambda_2) = \lambda_1 + \beta\lambda_2$$
$$V_1 : \mathcal{P} \to \mathcal{X}, \quad V_1(\lambda_1, \lambda_2) = (\lambda_1, \lambda_2)$$

where $\beta \in \mathbb{R}$ is such that $\frac{p_2}{p_1} > \beta > 0$. As Figure 2.4 clearly displays, $\mathcal{L} = \{0\}$, but V_1 does not vanish on

$$\mathcal{L}(\mathcal{P}; V_0; V_1, \mathcal{A}) = \{(\lambda_1, \lambda_2) \in \mathbb{R}^2 \,:\, \lambda_1 + \beta\lambda_2 \leq 0, \;\; \lambda_2 \geq h(\lambda_1)\} \supsetneq \{0\}$$

implying that the condition $\mathrm{NAD}(\mathcal{P}; V_0; V_1, \mathcal{A})$ does not hold.

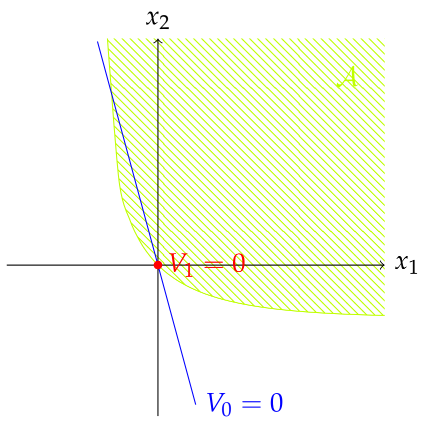


FIGURE 2.4: Example 2.4.19

2.5 DUAL REPRESENTATIONS

In the last section of this chapter, we focus on the case where ρ is convex or quasi convex and lower semicontinuous, and we establish a dual representation. The preceding sections provide sufficient conditions for ρ to have the mentioned properties.

2.5.1 CONVEX CASE

In this section we assume that ρ is convex and lower semicontinuous, and we investigate its dual representation.

The technique we are going to use to establish the dual representation consists in a straightforward application of the Fenchel-Moreau Theorem followed by an analysis of the penalty function ρ^*. Our proof mimics that of Proposition 3.9 in Frittelli and Scandolo [51]. There, portfolios λ are not specified and initial prices are defined on payoffs $V_1(\lambda)$. Our result is a generalization of Proposition 3.9 in [51] as it holds also if the law of one price (2.1) does not hold. In case it holds, our representation is a direct application of Proposition 3.9 in [51] by setting $\mathcal{M}$ and π like in Subsection 2.1.3.

Theorem 2.5.1. *Assume that ρ is proper, convex and lower semicontinuous. Then for every $X \in \mathcal{X}$ the following representation holds:*

$$\rho(X) = \sup_{\psi \in \mathcal{D}} \left\{ \psi(-X) + \sigma_{\mathcal{A}}(\psi) - V_0^{*,V_1}(\psi) \right\}$$

where $V_0^{*,V_1} : \mathcal{X}' \to (-\infty, \infty]$ is a convex and w^*-lower semicontinuous function defined as

$$V_0^{*,V_1}(\psi) := \sup_{\lambda \in \mathcal{P}} \{ \psi(V_1(\lambda)) - V_0(\lambda) \}$$

and

$$\mathcal{D} := \operatorname{bar}(\mathcal{A}) \cap \{ \psi \in \mathcal{X}'_+ \, : \, V_0^{*,V_1}(\psi) < \infty \}.$$

Proof. The Fenchel-Moreau representation of ρ (see Theorem A.1.6) is

$$\rho(X) = \sup_{\psi \in \mathcal{X}'_+} \{ \psi(-X) - \rho^*(-\psi) \}, \qquad \rho^*(-\psi) := \sup_{X \in \mathcal{X}} \{ -\psi(X) - \rho(X) \}$$

Now, we calculate ρ^*:

$$\begin{aligned}\rho^*(-\psi) &= \sup_{X\in\mathcal{X}} \{\psi(-X) - \inf\{V_0(\lambda) \,:\, \lambda \in \mathcal{P},\, X + V_1(\lambda) \in \mathcal{A}\}\} \\ &= \sup_{X\in\mathcal{X}} \sup\{\psi(-X) - V_0(\lambda) \,:\, \lambda \in \mathcal{P},\, X + V_1(\lambda) \in \mathcal{A}\} \\ &= \sup\{\psi(-Z + V_1(\lambda)) - V_0(\lambda) \,:\, \lambda \in \mathcal{P},\, Z \in \mathcal{A}\} \\ &= \sup\{\psi(-Z) + \psi(V_1(\lambda)) - V_0(\lambda) \,:\, \lambda \in \mathcal{P},\, Z \in \mathcal{A}\} \\ &= \sup_{z\in\mathcal{A}}\{\psi(-Z)\} + \sup_{\lambda\in\mathcal{P}}\{\psi(V_1(\lambda)) - V_0(\lambda)\},\end{aligned}$$

and we have achieved the desired representation. □

Remark 2.5.2. Representation of Theorem 2.5.1 can also be derived with a more geometric approach, based on the dual representation of sets (namely epigraphs) instead of functions. We do not mean to be too detailed in the description of this technique, since it substantially mimics the procedure applied in Chapter 1 for finding pricing densities and moreover Theorem 2.5.1 is already exhaustive. The key point is that whenever ρ is lower semicontinuous, from equation (2.4) we derive that $\mathrm{epi}(\rho) = \mathrm{cl}(\mathcal{C})$ by virtue of Proposition A.1.8 and

$$\rho(X) = \min\{m \in \mathbb{R} \,:\, (X, m) \in \mathrm{cl}(\mathcal{C})\}.$$

If ρ is also convex, we have the following representation by Proposition A.1.10:

$$\mathrm{cl}(\mathcal{C}) = \bigcap_{(\psi,\beta)\in\mathrm{bar}(\mathcal{C})} \{(X, m) \in \mathcal{X} \times \mathbb{R} \,:\, \psi(X) + \beta m \geq \sigma_{\mathcal{C}}(\psi, \beta)\}$$

where

$$\mathrm{bar}(\mathcal{C}) = \left\{(\psi, \beta) \in \mathcal{X}' \times \mathbb{R} \,:\, \psi \in \mathrm{bar}(\mathcal{A}),\ \sup_{\lambda\in\mathcal{P}}\{\psi(V_1(\lambda)) - \beta V_0(\lambda)\} < \infty\right\}$$

and

$$\sigma_{\mathcal{C}}(\psi, \beta) = \sigma_{\mathcal{A}}(\psi) - \sup_{\lambda\in\mathcal{P}}\{\psi(V_1(\lambda)) - \beta V_0(\lambda)\}.$$

If we assume further that $0 \geq \rho(0) > \infty$, then $(0, 0) \in \mathrm{cl}(\mathcal{C})$ and $(0, n) \notin \mathrm{cl}(\mathcal{C})$ for some $n < 0$. These facts allow to find $(\psi, \beta) \in \mathrm{bar}(\mathcal{C})$ with $\beta > 0$ and hence to reduce the representation as follows:

$$\mathrm{cl}(\mathcal{C}) = \bigcap_{\psi\in\mathcal{D}} \{(X, m) \in \mathcal{X} \times \mathbb{R} \,:\, \psi(X) + m \geq \sigma_{\mathcal{A}}(\psi) - V_0^{*,V_1}(\psi)\}.$$

Note that $\mathcal{D} = \{\psi \in \mathcal{X}' \,:\, (\psi, 1) \in \mathrm{bar}(\mathcal{C})\}$. The representation of ρ is now straightforward:

$$\begin{aligned}\rho(X) &= \min\{m \in \mathbb{R} \,:\, \psi(X) + m \geq \sigma_{\mathcal{A}}(\psi) - V_0^{*,V_1}(\psi) \text{ for every } \psi \in \mathcal{D}\} \\ &= \sup_{\psi\in\mathcal{D}} \left\{\psi(-X) + \sigma_{\mathcal{A}}(\psi) - V_0^{*,V_1}(\psi)\right\}.\end{aligned}$$

One could wonder whether the functionals ψ appearing in the representation act as pricing functionals defined on the whole space $\mathcal{X}$ that are, in some sense, consistent with the market and the acceptance set. This is indeed one of the ways dual representations have been interpreted in the literature about linear market models, for instance in Farkas et al. [46].

If ψ is intended as a pricing functional consistent with the market, we may expect that it assigns nonnegative price to acceptable positions, or at least there is a lower bound for these prices (i.e. $\psi \in \mathrm{bar}(\mathcal{A})$), and that it assigns to every payoff obtained through the liquidation of the basic securities a price that is sandwiched in the bid ask spread:

$$\psi(V_1(\cdot)) \leq V_0(\cdot) \quad \text{on } \mathcal{P}$$

and

$$-V_0(-\cdot) \leq \psi(-V_1(-\cdot)) \leq \psi(V_1(\cdot)) \leq V_0(\cdot) \quad \text{on } \mathcal{P} \cap (-\mathcal{P}).$$

Unfortunately, none of the inequalities holds in general for $\psi \in \mathcal{D}$. The central one holds if V_1 is linear, while the others are equivalent to $V_0^{*,V_1}(\psi) \leq 0$. In general, by definition of V_0^{*,V_1}, we only have that

$$\psi(V_1(\cdot)) - V_0^{*,V_1}(\psi) \leq V_0(\cdot) \quad \text{on } \mathcal{P}.$$

In the next proposition we claim that in presence of positive homogeneity, those ψ that appear in the dual representation induce prices on the payoffs of eligible portfolios that are dominated by V_0. But this fails if we relax the request of positive homogeneity, replacing it with convexity/concavity. If $\mathcal{P}$ and V_0 are convex and V_1 is positively homogeneous, we may use recession functions (Proposition 2.5.4) to regain homogeneity and we achieve the same type of dominance, only on $\mathcal{P}^\infty$, by means of pricing functional V_0^∞ and V_1.

Proposition 2.5.3. *Assume that $\mathcal{P}$ is a cone, and V_0 and V_1 are positively homogeneous. Then for $\psi \in \mathcal{X}'$*

$$V_0^{*,V_1}(\psi) = \begin{cases} 0 & \text{if } \psi \in \mathcal{D}_c \\ +\infty & \text{if } \psi \notin \mathcal{D}_c \end{cases}$$

where $\mathcal{D}_c := \{\psi \in \mathcal{X}'_+ \,:\, \psi(V_1(\lambda)) \leq V_0(\lambda) \text{ for every } \lambda \in \mathcal{P}\}$ is convex and w^*-closed. In this case, the dual representation of ρ is:

$$\rho(X) = \sup_{\psi \in \text{bar}(\mathcal{A}) \cap \mathcal{D}_c} \{\psi(-X) + \sigma_{\mathcal{A}}(\psi)\}. \tag{2.10}$$

Proof. Since $0 \in \mathcal{P}$ and V_0 and $V_1(0)$ take value 0 in 0, it is enough to show that if $\psi_0(V_1(\lambda_0)) > V_0(\lambda_0)$ for $\psi_0 \in \mathcal{X}'_+$ and $\lambda_0 \in \mathcal{P}$, then $V_0^{*,V_1}(\psi) = \infty$. This is the case since for $t > 0$ we have

$$V_0^{*,V_1}(\psi_0) \geq \psi_0(V_1(t\lambda_0)) - V_0(t\lambda_0) = t\big(\psi_0(V_1(\lambda_0)) - V_0(\lambda_0)\big) \xrightarrow[t \to \infty]{} \infty.$$

□

Proposition 2.5.4. *Assume that $\mathcal{P}$ is convex with $0 \in \mathcal{P}$, V_0 is convex with $V_0(0) = 0$ and V_1 is concave with $V_1(0) = 0$. Then for $\psi \in \mathcal{X}'$ the following equivalences hold:*

$$\begin{aligned} V_0^{*,V_1}(\psi) = 0 \quad &\Leftrightarrow \quad \psi(V_1(\lambda)) \leq V_0(\lambda) \quad \forall \lambda \in \mathcal{P} \\ &\Leftrightarrow \quad \exists \pi : \mathbb{R}^N \longrightarrow \mathbb{R} \text{ linear such that } \psi(V_1(\lambda)) \leq \pi(\lambda) \leq V_0(\lambda) \quad \forall \lambda \in \mathcal{P}. \end{aligned}$$

Proof. In this case V_0^{*,V_1} only takes nonnegative values, hence the first equivalence is trivial, while the second one holds due to "Sandwich Theorem" in Section 4 of [14]. □

Proposition 2.5.5. *Assume $\mathcal{P}$ is convex closed and contains 0, V_0 is convex and $V_0(0) = 0$, V_1 is positively homogeneous. If $\psi \in \mathcal{X}'_+$ is such that $V_0^{*,V_1}(\psi) < +\infty$, then*

$$\psi(V_1(\lambda)) \leq V_0^\infty(\lambda) \qquad \forall \lambda \in \mathcal{P}^\infty.$$

Proof. If $V_0^{*,V_1}(\psi) < +\infty$, for every $t > 0$ and $\lambda_0 \in \mathcal{P}^\infty$ the following holds:

$$t\big(\psi(V_1(\lambda_0)) - V_0^\infty(\lambda_0)\big) = \psi(V_1(t\lambda_0)) - V_0^\infty(t\lambda_0) \leq \sup_{\lambda \in \mathcal{P}^\infty} \{\psi(V_1(\lambda)) - V_0^\infty(\lambda)\} \leq V_0^{*,V_1}(\psi) < \infty.$$

□

Note that the last results are sort of generalizations of Lemma 3.10 in [51].

2.5.2 Quasi convex case

Here, we study the dual representation of ρ when it is quasi convex and lower semicontinuous. Throughout this section, we will make use of the following function defined for $X \in \mathcal{X}$ and $\psi \in \mathcal{X}'$:

$$\rho(X|\psi) := \inf\{\rho(Y) \,:\, Y \in \mathcal{X} \text{ such that } \psi(Y) \leq \psi(X)\}.$$

We start with a preparatory lemma.

Lemma 2.5.6. *For every $X \in \mathcal{X}$ and $\psi \in \mathcal{X}'$, the following identity holds:*

$$\rho(X|\psi) = \inf\big\{V_0(\lambda) \,:\, \lambda \in \mathcal{P},\ X + V_1(\lambda) \in \mathcal{A} + \{\psi \geq 0\}\big\}.$$

Proof. The result is achieved through simple computations:

$$\begin{aligned}
\rho(X|\psi) &= \inf\{\rho(Y) \,:\, Y \in \mathcal{X},\ \psi(Y - X) \leq 0\} \\
&= \inf\{\rho(X + Z) \,:\, Z \in \mathcal{X},\ \psi(Z) \leq 0\} \\
&= \inf\{\inf\{V_0(\lambda) \,:\, \lambda \in \mathcal{P},\ X + Z + V_1(\lambda) \in \mathcal{A}\} \,:\, Z \in \mathcal{X},\ \psi(Z) \leq 0\} \\
&= \inf\{V_0(\lambda) \,:\, \lambda \in \mathcal{P},\ Z \in \mathcal{X},\ \psi(Z) \leq 0,\ X + Z + V_1(\lambda) \in \mathcal{A}\} \\
&= \inf\{V_0(\lambda) \,:\, \lambda \in \mathcal{P},\ X + V_1(\lambda) \in \mathcal{A} + \{\psi \geq 0\}\}.
\end{aligned}$$

□

From Lemma 2.5.6, we derive that $\rho(\cdot|\psi)$ corresponds to the risk measure with the same set of admissible actions and valuation and liquidation operators, and with augmented acceptance set $\mathcal{A} + \{\psi \geq 0\}$. It thus indicates the minimum amount to be invested today in the basic securities in order to make our capital position X acceptable after adding something freely available in a complete market where buying prices are expressed through ψ. Furthermore, by writing $\rho(X|\psi)$ as

$$\rho(X|\psi) = \inf\big\{V_0(\lambda) \,:\, \lambda \in \mathcal{P},\ \mathcal{A} \cap \{\psi \leq \psi(X + V_1(\lambda))\} \neq \emptyset\big\},$$

we can also interpret it as the amount required today so that the value of the capital position X added to the liquidated portfolio of the basic securities will be grater at maturity than the value of an acceptable claim, in a world where values are obtained through the pricing functional ψ.

Now, we establish the dual representation of ρ.

Theorem 2.5.7. *Assume that ρ is proper, quasi convex and lower semicontinuous. Then if* $\mathrm{bar}(\mathcal{A})$ *is not empty, for every $X \in \mathcal{X}$ the following representation holds:*

$$\rho(X) = \sup_{\psi \in \mathrm{bar}(\mathcal{A})} \rho(X|\psi).$$

Otherwise $\rho(X) = \inf\{V_0(\lambda) : \lambda \in \mathcal{P}\}$ for every $X \in \mathcal{X}$.

Proof. The following representation

$$\rho(X) = \sup_{\psi \in \mathcal{X}'_+} \rho(X|\psi)$$

is just an application of dual representation of quasi convex functions (Theorem A.1.7). Now, assume that $\psi \notin \mathrm{bar}(\mathcal{A})$ and take $X \in \mathcal{X}$. We find $Y \in \mathcal{A}$ for which $\psi(X - Y) \geq 0$ holds, so that $X = Y + (X - Y) \in \mathcal{A} + \{\psi \geq 0\}$. Thus $\mathcal{A} + \{\psi \geq 0\}$ equals the space $\mathcal{X}$ and by virtue of Lemma 2.5.6, $\rho(\cdot|\psi)$ coincides with $\inf\{V_0(\lambda) \,:\, \lambda \in \mathcal{P}\}$ on $\mathcal{X}$. This shows the desired representation in case $\mathrm{bar}(\mathcal{A})$ is empty. Otherwise, take $\psi_1 \in \mathrm{bar}(\mathcal{A})$ and $\psi_0 \notin \mathrm{bar}(\mathcal{A})$, and observe that $\rho(\cdot|\psi_0) \leq \rho(\cdot|\psi_1)$ by Lemma 2.5.6, showing that for every $X \in \mathcal{X}$

$$\rho(X) = \sup_{\psi \in \mathcal{X}'_+} \rho(X|\psi) = \sup_{\psi \in \mathrm{bar}(\mathcal{A})} \rho(X|\psi).$$

□

Remark 2.5.8. The closure of $\mathcal{A} + \{\psi \geq 0\}$ is equal to $\{\psi \geq \sigma_{\mathcal{A}}(\psi)\}$. Indeed every element of $\mathcal{A} + \{\psi \geq 0\}$ is trivially contained in the closed set $\{\psi \geq \sigma_{\mathcal{A}}(\psi)\}$. On the other hand, take $X \in \mathcal{X}$ such that $\psi(X) \geq \sigma_{\mathcal{A}}(\psi)$ and a neighborhood $\mathcal{U}$ of X. If ψ is not identically zero, there is $Y \in \mathcal{U}$ such that $\psi(Y) > \psi(X)$ and hence $\psi(Y) \geq \psi(Z)$ for some $Z \in \mathcal{A}$. Thus $Y = Z + (Y - Z) \in \mathcal{A} + \{\psi \geq 0\}$ and $X \in \mathrm{cl}(\mathcal{A} + \{\psi \geq 0\})$. In general, the sum of $\mathcal{A}$ and $\{\psi \geq 0\}$ is not closed, but note that if $\mathcal{A}$ is a cone, $\mathcal{A} + \{\psi \geq 0\} = \{\psi \geq 0\}$ and the sum is automatically closed.

CHAPTER 3

DUAL REPRESENTATIONS FOR SYSTEMIC RISK MEASURES BASED ON ACCEPTANCE SETS

The study of risk measures for multivariate positions was first developed in the set-valued case by Jouini et al. [62], which triggered Hamel and Heyde [57], Hamel et al. [58] and Molchanov and Cascos [74]. Following a different approach, the real valued case was developed by Burgert and Rüschendorf [25], Rüschendorf [89] and Ekeland and Schachermayer [42]. In this literature, multivariate positions are typically interpreted as (random) portfolios of financial assets. In recent years, there has been significant interest in extending the theory of risk measures to a systemic risk setting, in which multivariate positions represent the (random) vector of future capital positions, i.e. assets net of liabilities, of financial institutions. In this setting, one considers a system of d financial institutions whose respective capital positions at a fixed future date is represented by a random vector

$$X = (X_1, \dots, X_d).$$

The bulk of the literature assumes that a macroprudential regulator specifies an "aggregation function"

$$\Lambda : \mathbb{R}^d \to \mathbb{R}$$

by means of which the system is summarized into a single (univariate) aggregated position $\Lambda(X)$. The simplest aggregation function is given by $\Lambda(x) = \sum_{i=1}^{d} x_i$ and corresponds to aggregating the entire system into a single consolidated balance sheet. The regulator also specifies a set $\mathcal{A}$ of "acceptable" aggregated positions: The level of systemic risk of the financial system is deemed acceptable whenever $\Lambda(X)$ belongs to the prescribed acceptance set $\mathcal{A}$. Two main classes of systemic risk measures based on aggregation functions and acceptance sets have been studied in the literature.

A first branch of the literature adopts a so-called "first allocate, then aggregate" approach, which is the macroprudential equivalent of the fundamental idea introduced in the context of microprudential regulation by Artzner et al. [9]: To ensure that the financial system has an acceptable level of systemic risk, the macroprudential regulator can require each of the member institutions to raise a suitable amount of capital. Such a requirement is represented by a vector $m = (m_1, \dots, m_d) \in \mathbb{R}^d$, where m_i corresponds to the amount of capital raised by institution i. This leads to a systemic risk measure of the form

$$\rho(X) = \inf \left\{ \sum_{i=1}^{d} m_i \ : \ m \in \mathbb{R}^d, \ \Lambda(X+m) \in \mathcal{A} \right\}.$$

The quantity $\rho(X)$ corresponds to the minimum amount of aggregate capital that needs to be injected into the financial system to ensure acceptability. This type of systemic risk measures has been studied in Feinstein et al. [47], Armenti et al. [7], Ararat and Rudloff [4], and Biagini et al. [19, 20] (where random allocations of the aggregate capital requirement are also considered).

A second branch of the literature advocates a "first aggregate, then allocate" approach and studies systemic risk measures of the form

$$\widetilde{\rho}(X) = \inf\{m \in \mathbb{R} \,:\, \Lambda(X) + m \in \mathcal{A}\}.$$

In this case, the quantity $\widetilde{\rho}(X)$ represents the minimal amount of capital that has to be added to the aggregated position to reach acceptability. In contrast to ρ, the operational interpretation of $\widetilde{\rho}$ is not straightforward since it is unclear how much each of the member institutions should contribute to the aggregate amount of required capital or, if the outcome of the above risk measure is interpreted as a bail-out cost, which institution should obtain which amount. Such systemic risk measures have been studied in Chen et al. [31], Kromer et al. [70], and Ararat and Rudloff [4].

The main objective of this chapter is to establish dual representations for the above systemic risk measures in a general setting with special emphasis on systemic risk measures of the "first allocate, then aggregate" type. By doing so, we provide a unifying perspective on the existing duality results in the literature. More precisely, we consider an arbitrary probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and assume that the multivariate positions belong to a space of d-dimensional random vectors $\mathcal{X}$ that is in duality with another space of d-dimensional random vectors $\mathcal{X}'$ through the pairing

$$(X, Z) \mapsto \sum_{i=1}^{d} \mathbb{E}_{\mathbb{P}}[X_i Z_i]$$

for $X \in \mathcal{X}$ and $Z \in \mathcal{X}'$. This setup is general enough to cover all the interesting examples encountered in the literature. Dual representations for ρ have been studied by Armenti et al. [7] and Biagini et al. [20] in the setting of Orlicz hearts and acceptance sets based on (multivariate) utility functions and by Ararat and Rudloff [4] in the setting of bounded random variables with only mild restrictions on the acceptance set. The strategy in [7] is to apply Lagrangian techniques while that of [4] is to rely on the dual representation of $\widetilde{\rho}$, which is tackled by using Fenchel-Moreau techniques for composed maps. Similarly to [20], the point of departure of this chapter is to observe that ρ can be written as

$$\rho(X) = \inf\{\pi(m) \,:\, m \in \mathbb{R}^d,\ X + m \in \Lambda^{-1}(\mathcal{A})\}$$

where the "acceptance set" $\Lambda^{-1}(\mathcal{A})$ and the "cost functional" $\pi : \mathbb{R}^d \to \mathbb{R}$ are given by

$$\Lambda^{-1}(\mathcal{A}) = \{X \in \mathcal{X} \,:\, \Lambda(X) \in \mathcal{A}\}, \quad \pi(m) = \sum_{i=1}^{d} m_i.$$

This shows that ρ belongs to the class of "multi-asset risk measures" introduced in Frittelli and Scandolo [51] and thoroughly studied in Farkas et al. [46]. Under suitable conditions on Λ and $\mathcal{A}$, the general duality results obtained in those papers yield the representation

$$\rho(X) = \sup\left\{\sigma(\mathbb{Q}) - \sum_{i=1}^{d} \mathbb{E}_{\mathbb{Q}_i}[X_i] \,:\, \mathbb{Q}_1, \ldots, \mathbb{Q}_d \ll \mathbb{P},\ \frac{d\mathbb{Q}}{d\mathbb{P}} \in \mathcal{X}'\right\}$$

where $\mathbb{Q} = (\mathbb{Q}_1, \ldots, \mathbb{Q}_d)$, $\frac{d\mathbb{Q}}{d\mathbb{P}} = \left(\frac{d\mathbb{Q}_1}{d\mathbb{P}}, \ldots, \frac{d\mathbb{Q}_d}{d\mathbb{P}}\right)$ and the objective function is given by

$$\sigma(\mathbb{Q}) = \inf_{X \in \Lambda^{-1}(\mathcal{A})} \sum_{i=1}^{d} \mathbb{E}_{\mathbb{Q}_i}[X_i].$$

The map σ corresponds to the (lower) support function of the systemic acceptance set $\Lambda^{-1}(\mathcal{A})$ and plays a fundamental role in the dual representation. The main technical contribution of this chapter is to provide a detailed analysis of these objects. In particular, we devote some effort to obtain a more explicit description of σ in terms of the primitives Λ and $\mathcal{A}$. It is worth noting that, in the systemic setting, the closedeness of the acceptance set $\Lambda^{-1}(\mathcal{A})$ does not necessarily imply the lower semicontinuity of ρ, which is a necessary condition for ρ to admit a dual representation. Hence, it is important to provide conditions on the primitives Λ and $\mathcal{A}$ to ensure that ρ is lower semicontinuous in the first place. To this effect, we rely on the abstract results in Farkas et al. [46], where an effort was made to derive properties of risk measures, such as lower semicontinuity, from the properties of the underlying acceptance sets. In particular, the duality results for acceptance sets obtained there build the starting point for our analysis of the systemic acceptance set $\Lambda^{-1}(\mathcal{A})$.

On the other side, dual representations for $\widetilde{\rho}$ have been studied by Chen et al. [31] in a finite-dimensional setting, by Ararat and Rudloff [4] in the setting of bounded random vectors, and by Kromer et al. [70] at our level of generality. In line with those papers, our starting point is to observe that $\widetilde{\rho}$ can be expressed as a standard cash-additive risk measure (namely the cash-additive risk measure associated with $\mathcal{A}$, which we denote by $\rho_{\mathcal{A}}$) applied to the aggregated univariate position as

$$\widetilde{\rho}(X) = \rho_{\mathcal{A}}(\Lambda(X)).$$

However, instead of working out the Fenchel-Moreau representation of a composition of maps, we exploit the standard dual representation for $\rho_{\mathcal{A}}$ to derive the desired representation in a direct way. In this case, conditions to ensure the lower semicontinuity of $\widetilde{\rho}$ are also easier to formulate.

Finally, to illustrate the convenience of our approach to duality based on acceptance sets, we provide a simple and self-contained proof of the dual representation for utility-based risk measures for univariate positions, which can be viewed as special systemic risk measures where $d = 1$ and Λ is a von Neumann-Morgenstern utility function.

The necessary mathematical background is collected in the appendices and we refer to the List of Symbols on page 117 for notations.

3.1 THE SETTING

In this section, we describe our setting.

3.1.1 BASIC NOTATION

Throughout, we fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and we refer to Appendix A for the necessary terminology and notation and to Aliprantis and Border [1] for a general presentation of Banach lattices.

A (nonzero) linear subspace $\mathcal{L} \subset L^0(\mathbb{P})$ is said to be *admissible* if it is a Banach lattice (with respect to the almost-sure partial order) such that $L^\infty(\mathbb{P}), \subset \mathcal{L} \subset L^1(\mathbb{P})$. In this case, we set

$$\mathcal{L}' := \{Z \in L^0(\mathbb{P}) \, : \, \mathbb{E}_{\mathbb{P}}\left[|XZ|\right] < \infty, \, \forall X \in \mathcal{L}\}.$$

Note that we always have $L^\infty(\mathbb{P}), \subset \mathcal{L}' \subset L^1(\mathbb{P})$.

Example 3.1.1. *The class of admissible spaces contains Orlicz spaces (and hence $L^p(\mathbb{P})$ spaces), which include the standard examples encountered in the literature. For an overview, see Appendix A. The following statements hold; see e.g. Edgar and Sucheston [41] or Meyer-Nieberg [73]:*

(1) $\mathcal{L} = L^\Phi(\mathbb{P})$ *is admissible and* $\mathcal{L}' = L^{\Phi^*}(\mathbb{P})$*;*

(2) $\mathcal{L} = H^\Phi(\mathbb{P})$ *is admissible if* Φ *is finite valued, in which case* $\mathcal{L}' = L^{\Phi^*}(\mathbb{P})$*;*

(3) $\mathcal{L} = L^p(\mathbb{P})$ *is admissible if* $p \in [1, \infty]$*, in which case* $\mathcal{L}' = L^{\frac{p}{p-1}}(\mathbb{P})$ *(with the convention* $\frac{1}{0} = \infty$ *and* $\frac{\infty}{\infty} = 1$*).*

Fix $m \in \mathbb{N}$. We always consider the standard inner product $\langle \cdot, \cdot \rangle : \mathbb{R}^m \times \mathbb{R}^m \to \mathbb{R}$ defined by

$$\langle a, b \rangle := \sum_{i=1}^{m} a_i b_i.$$

We say that a linear subspace $\mathcal{L} \subset L^0_m(\mathbb{P})$ is *admissible* whenever

$$\mathcal{L} = \mathcal{L}_1 \times \cdots \times \mathcal{L}_m$$

with admissible $\mathcal{L}_1, \ldots, \mathcal{L}_m \subset L^0(\mathbb{P})$. Note that, being the product of Banach lattices, the space $\mathcal{L}$ is also a Banach lattice. In particular, the lattice operations on $\mathcal{L}$ are understood component by

component. As above, we define
$$\mathcal{L}' := \mathcal{L}_1' \times \cdots \times \mathcal{L}_m'.$$
The pair $(\mathcal{L}, \mathcal{L}')$ is equipped with the bilinear form $(\cdot|\cdot) : \mathcal{L} \times \mathcal{L}' \to \mathbb{R}$ given by

$$(X|Z) := \mathbb{E}_{\mathbb{P}}[\langle X, Z \rangle] = \sum_{i=1}^{m} \mathbb{E}_{\mathbb{P}}[X_i Z_i].$$

The coarsest topology on $\mathcal{L}$ making the linear functional $X \mapsto (X|Z)$ continuous for every $Z \in \mathcal{L}'$ is denoted by $\sigma(\mathcal{L}, \mathcal{L}')$. Similarly, the coarsest topology on $\mathcal{L}'$ making the linear functional $Z \mapsto (X|Z)$ continuous for every $X \in \mathcal{L}$ is denoted by $\sigma(\mathcal{L}', \mathcal{L})$. Equipped with these topologies, $\mathcal{L}$ and $\mathcal{L}'$ are locally-convex topological vector spaces (which are also Hausdorff because the above form is separating).

3.1.2 Financial systems and systemic risk

We consider a one-period economy in which uncertainty at the terminal date is modeled by the probability space $(\Omega, \mathcal{F}, \mathbb{P})$. In this economy, we assume the existence of a *financial system* consisting of d member institutions (for completeness we also allow for the case $d = 1$). The possible terminal capital positions, i.e. assets net of liabilities, of these d institutions belong to an admissible space
$$\mathcal{X} = \mathcal{X}_1 \times \cdots \times \mathcal{X}_d \subset L_d^0(\mathbb{P}).$$
For every $X = (X_1, \ldots, X_d) \in \mathcal{X}$ the random variables $X_1, \ldots, X_d$ correspond to the capital positions of the various member institutions. Since $\mathcal{X}$ contains all bounded random vectors, the space $\mathbb{R}^d$ can be naturally viewed as a linear subspace of $\mathcal{X}$. We denote by $\mathbf{e}$ the constant random vector with all components equal to 1, i.e.

$$\mathbf{e} := (1, \ldots, 1) \in \mathbb{R}^d.$$

The impact of the financial system on systemic risk is measured through an *impact map*

$$S : \mathcal{X} \to \mathcal{E}$$

where $\mathcal{E}$ is a suitable admissible subspace of $L^0(\mathbb{P})$. Hence, for every $X \in \mathcal{X}$, the random variable $S(X)$ is interpreted as an indicator of the systemic risk posed by X; see Remark 3.1.5.

Definition 3.1.2. We say that S is *admissible* if it satisfies the following five properties:

(S1) *Discrimination*: S is not constant;

(S2) *Normalization*: $S(0) = 0$;

(S3) *Monotonicity*: $S(X) \geq S(Y)$ for all $X, Y \in \mathcal{X}$ such that $X \geq Y$;

(S4) *Concavity*: $S(\lambda X + (1-\lambda)Y) \geq \lambda S(X) + (1-\lambda)S(Y)$ for all $X, Y \in \mathcal{X}$ and $\lambda \in [0,1]$;

(S5) *Semicontinuity*: The map $X \mapsto \mathbb{E}_{\mathbb{P}}[S(X)W]$ is $\sigma(\mathcal{X}, \mathcal{X}')$-upper semicontinuous for every $W \in \mathcal{E}_+'$.

The next proposition provides a number of sufficient conditions for the technical assumption (S5) to hold. Recall that the lattice operations on $\mathcal{X}$ are performed component by component. Here, we use the standard notation for the limit superior of a sequence of random variables.

Definition 3.1.3. We say that S has the *Fatou property* if for every sequence $(X^n) \subset \mathcal{X}$ and every $X \in \mathcal{X}$
$$X^n \to X \text{ a.s.}, \ \sup_{n \in \mathbb{N}} |X^n| \in \mathcal{X} \implies S(X) \geq \limsup_{n \to \infty} S(X^n).$$
We say that S is *surplus invariant* if $S(X) = S(-X^-)$ for every $X \in \mathcal{X}$.

Proposition 3.1.4. *Assume that (S3) and (S4) hold. Then, (S5) holds in any of the following cases:*

(i) $\mathcal{X}_i'$ *is the norm dual of* $\mathcal{X}_i$ *for every* $i \in \{1, \dots, d\}$.

(ii) $\mathcal{X}_i = L^{\Phi_i}(\mathbb{P})$ *with* Φ_i^* *being* Δ_2 *(e.g.* $\mathcal{X}_i = L^\infty(\mathbb{P})$*),) for every* $i \in \{1, \dots, d\}$ *and* S *has the Fatou property.*

(iii) S *is surplus invariant and has the Fatou property.*

Proof. Throughout the proof fix $W \in \mathcal{E}_+'$ and define a functional $\varphi_W : \mathcal{X} \to \mathbb{R}$ by setting

$$\varphi_W(X) := \mathbb{E}_\mathbb{P}[S(X)W].$$

Note that φ_W is concave and nondecreasing by (S3) and (S4). Assume that *(i)* holds. In this case, we can apply the Extended Namioka-Klee Theorem from Biagini and Frittelli [21] to infer that φ_W is upper semicontinuous (in fact, continuous) with respect to the norm topology on $\mathcal{X}$. As the space $\mathcal{X}'$ coincides with the norm dual of $\mathcal{X}$ by assumption, it follows from Corollary 5.99 in Aliprantis and Border [1] that φ_W is also $\sigma(\mathcal{X}, \mathcal{X}')$-upper semicontinuous.

We make some preliminary observations before proceeding with the proof of *(ii)* and *(iii)*. First, we note that the Fatou property of S implies that φ_W is sequentially upper semicontinuous with respect to order convergence, i.e. dominated almost-sure convergence. Indeed, consider a sequence $(X^n) \subset \mathcal{X}$ that converges almost surely to some $X \in \mathcal{X}$ and such that $\sup_{n\in\mathbb{N}} |X^n| \leq M$ for some $M \in \mathcal{X}$. Since $|S(X^n)| \leq \max(|S(M)|, |S(-M)|)$ for every $n \in \mathbb{N}$ by (S3), it follows from the Fatou property of S and from the Fatou Lemma that

$$\varphi_W(X) \geq \mathbb{E}\Big[\limsup_{n\to\infty} S(X^n)W\Big] \geq \limsup_{n\to\infty} \mathbb{E}_\mathbb{P}[S(X^n)W] = \limsup_{n\to\infty} \varphi_W(X^n),$$

as claimed. Second, Theorem 2.6.4 in Meyer-Nieberg [73] tells us that, for every $i \in \{1, \dots, d\}$, the order-continuous dual of $\mathcal{X}_i$, i.e. the space of linear functionals that are continuous with respect to order convergence, coincides with $\mathcal{X}_i'$. This implies that the order-continuous dual of $\mathcal{X}$ also coincides with $\mathcal{X}'$. Denote by $\mathcal{X}_n^\sim$ the order-continuous dual of $\mathcal{X}$. We establish (S5) by showing that the upper semicontinuity of φ_W with respect to order convergence implies its $\sigma(\mathcal{X}, \mathcal{X}_n^\sim)$-upper semicontinuity.

Assume that *(ii)* holds. If $d = 1$, the desired assertion follows from Theorem 4.4 in Delbaen and Owari [38] (see also Theorem 3.2 in Delbaen [37] for the bounded case and Theorem 3.7 in Gao et al. [52] for the Orlicz case in a nonatomic setting). This result can be extended to a multivariate setting by using the results in Leung and Tantrawan [93]. We use their notation and terminology. Observe first that the constant vector $\mathbf{e}$ is a strictly-positive element in $\mathcal{X}_n^\sim$. Second, note that all the spaces $\mathcal{X}_i$'s are monotonically complete by Theorem 2.4.22 in [73], admit a special modular by Example 3.1 in [93], and their norm duals are order continuous by Remark 3.5 in [38]. This implies that $\mathcal{X}$ is also monotonically complete, admits a special modular, and its norm dual is order continuous. As a result, we can apply Theorem 3.4 in [93] to conclude that $\mathcal{X}$ satisfies property (P1) of that paper. This property implies that every concave functional on $\mathcal{X}$ that is upper semicontinuous with respect to order convergence, as our φ_W, is automatically $\sigma(\mathcal{X}, \mathcal{X}_n^\sim)$-upper semicontinuous. This delivers the desired result.

Finally, assume that *(iii)* holds. In this case, the functional φ_W is surplus invariant in the sense of Gao and Munari [53]. Since φ_W is concave and upper semicontinuous with respect to order convergence, we can apply Theorem 21 in [53] to infer that φ_W is $\sigma(\mathcal{X}, \mathcal{X}_n^\sim)$-upper semicontinuous. As above, this delivers the desired result. □

Remark 3.1.5. (i) In the literature, the impact map is typically derived from an *aggregation function*

$$\Lambda : \mathbb{R}^d \to \mathbb{R}$$

by setting $S(X) = \Lambda(X)$ for every $X \in \mathcal{X}$. We refer to the literature cited in the introduction for a discussion of concrete examples. Clearly, the choice of Λ limits the choice of the space $\mathcal{E}$ since, for instance, one needs to ensure that the random variables $\Lambda(X)$'s are integrable. This is typically done either by working in a space of bounded positions or by working in an Orlicz space where the Orlicz functions are defined in terms of Λ. To avoid having to worry about this aspect, we have defined the impact map as a map between abstract spaces.

(ii) If Λ is assumed to be nonconstant, nondecreasing, concave, and to satisfy $\Lambda(0) = 0$, then the corresponding S clearly fulfills properties (S1)-(S4). Moreover, as Λ is automatically continuous by concavity, S has automatically the Fatou property. Hence, we can use Proposition 3.1.4 to ensure property (S5).

We now assume that the regulator has defined acceptable levels of systemic risk by specifying a set

$$\mathcal{A} \subset \mathcal{E}$$

called the *acceptance set*: The financial system with capital positions $X \in \mathcal{X}$ is deemed to have an acceptable level of systemic risk if the systemic risk indicator $S(X)$ belongs to $\mathcal{A}$.

Definition 3.1.6. We say that $\mathcal{A}$ is *admissible* if it satisfies the following properties:

(A1) *Discrimination*: $S^{-1}(\mathcal{A})$ is a nonempty proper subset of $\mathcal{X}$;

(A2) *Normalization*: $0 \in \mathcal{A}$;

(A3) *Monotonicity*: $\mathcal{A} + \mathcal{E}_+ \subset \mathcal{A}$;

(A4) *Convexity*: $\lambda\mathcal{A} + (1-\lambda)\mathcal{A} \subset \mathcal{A}$ for every $\lambda \in [0, 1]$;

(A5) *Closedness*: $\mathcal{A}$ is $\sigma(\mathcal{E}, \mathcal{E}')$-closed.

The next proposition highlights a variety of situations where assumption (A5) is always satisfied.

Definition 3.1.7. We say that $\mathcal{A}$ is *Fatou closed* if for every sequence $(U_n) \subset \mathcal{A}$ and every $U \in \mathcal{E}$

$$U_n \to U \text{ a.s.}, \sup_{n \in \mathbb{N}} |U_n| \in \mathcal{E} \implies U \in \mathcal{A}.$$

We say that $\mathcal{A}$ is *law invariant* if for every $U \in \mathcal{A}$ and every $V \in \mathcal{E}$ with the same probability distribution as U we have $V \in \mathcal{A}$. Moreover, we say that $\mathcal{A}$ is *surplus invariant* if for every $U \in \mathcal{A}$ and every $V \in \mathcal{E}$ such that $V^- = U^-$ we have $V \in \mathcal{A}$.

Proposition 3.1.8. *Assume that (A3) and (A4) hold. Then, (A5) holds in any of the following cases:*

(i) *$\mathcal{E}'$ is the norm dual of $\mathcal{E}$ and $\mathcal{A}$ is norm closed.*

(ii) *$\mathcal{E} = L^{\Phi}(\mathbb{P})$ with Φ^* being Δ_2 (e.g. $\mathcal{E} = L^{\infty}(\mathbb{P})$,) and $\mathcal{A}$ is Fatou closed.*

(iii) *$\mathcal{E} = L^{\Phi}(\mathbb{P})$ with $(\Omega, \mathcal{F}, \mathbb{P})$ nonatomic and $\mathcal{A}$ is law invariant and Fatou closed.*

(iv) *$\mathcal{A}$ is surplus invariant and Fatou closed.*

Proof. The desired assertion holds under *(i)* by Theorem 5.98 in Aliprantis and Border [1]; under *(ii)* by Theorem 4.1 in Delbaen and Owari [38] (see also Theorem 3.2 in Delbaen [37] in the bounded case and Theorem 3.7 in Gao et al. [52] in the Orlicz case in a nonatomic setting); under *(iii)* by Corollary 4.6 in Gao et al. [55]; under *(iv)* by Theorem 8 in Gao and Munari [53]. □

3.2 "FIRST ALLOCATE, THEN AGGREGATE"-TYPE SYSTEMIC RISK MEASURES

In this section we focus on systemic risk measures of "first allocate, then aggregate" type. After discussing some conditions for their representability, we establish a general dual representation and provide a detailed analysis of the properties of the corresponding systemic acceptance sets and "penalty functions". Throughout the section we fix an admissible impact map S and an admissible acceptance set $\mathcal{A}$.

3.2.1 The systemic risk measure ρ

"First allocate, then aggregate"-type systemic risk measures are defined as follows (we adopt the usual convention $\inf \emptyset = \infty$):

Definition 3.2.1. We define the map $\rho : \mathcal{X} \to [-\infty, \infty]$ by setting

$$\rho(X) := \inf \left\{ \sum_{i=1}^{d} m_i \, : \, m \in \mathbb{R}^d, \, S(X+m) \in \mathcal{A} \right\}. \tag{3.1}$$

The above map determines the minimum amount of aggregate capital that can be allocated to the member institutions to ensure that the level of systemic risk of the financial system is acceptable. We start by observing that (3.1) can be rewritten as

$$\rho(X) = \inf\{\pi(m) \, : \, m \in \mathbb{R}^d, \, X + m \in S^{-1}(\mathcal{A})\}, \quad \pi(m) = \sum_{i=1}^{d} m_i. \tag{3.2}$$

As a result, ρ belongs to the broad class of risk measures introduced in Frittelli and Scandolo [51] and thoroughly studied in Farkas et al. [46]. We exploit this link in a systematic way. It is also easy to see that ρ belongs to the class of generalized risk measures studied in Chapter 2, with $\mathcal{P} = \mathbb{R}^d$, $V_0(m) = \pi(m)$, $V_1(m) = m$ and acceptance set $S^{-1}(\mathcal{A})$.

Remark 3.2.2. The systemic risk measure ρ furnishes an example where the model of Chapter 2 may have a different interpretation from that described in Subsection 2.1.2.

The first proposition collects some basic properties of the "systemic acceptance set" $S^{-1}(\mathcal{A})$ and of the risk measure ρ.

Proposition 3.2.3. *(i) The set $S^{-1}(\mathcal{A})$ is monotone, convex, $\sigma(\mathcal{X}, \mathcal{X}')$-closed, and contains* 0.

(ii) The systemic risk measure ρ is nonincreasing, convex, and satisfies $\rho(0) \le 0$. Moreover, ρ satisfies the multivariate version of cash-additivity, i.e.

$$\rho(X+m) = \rho(X) - \sum_{i=1}^{d} m_i$$

for every $X \in \mathcal{X}$ and every $m \in \mathbb{R}^d$.

Proof. (i) It is straightforward to prove that $S^{-1}(\mathcal{A})$ contains 0 and that it is monotone and convex. To show $\sigma(\mathcal{X}, \mathcal{X}')$-closedeness, it is enough to recall that $\mathrm{bar}(\mathcal{A}) \subset \mathcal{E}'_+$ and use Proposition A.1.10 to get

$$S^{-1}(\mathcal{A}) = \{X \in \mathcal{X} \, : \, \mathbb{E}_{\mathbb{P}}[S(X)W] \ge \sigma_{\mathcal{A}}(W), \, \forall W \in \mathrm{bar}(\mathcal{A})\}.$$

The claim follows immediately from (S5).

(ii) The stated properties of ρ are straightforward; see also Lemma 2 in Farkas et al. [46]. □

3.2.2 Properness and lower semicontinuity of ρ

In order to admit a dual representation, the risk measure ρ needs to be proper and lower semicontinuous. We highlight a number of sufficient conditions for this to be the case. We start with a simple characterization of properness provided we already know that ρ is lower semicontinuous. Recall that, by definition, ρ is proper if it never attains the value $-\infty$ and is finite at some point.

Proposition 3.2.4. *If ρ is $\sigma(\mathcal{X}, \mathcal{X}')$-lower semicontinuous, then ρ is proper if and only if $\rho(0) > -\infty$.*

Proof. We know that $\rho(0) < \infty$ by Proposition 3.2.3. As a result, the above equivalence follows from the fact that a $\sigma(\mathcal{X}, \mathcal{X}')$-lower semicontinuous convex map that assumes the value $-\infty$ cannot assume any finite value; see e.g. Proposition 2.2.5 in Zălinescu [95]. □

In contrast to the standard univariate (cash-additive) case, the closedeness of $S^{-1}(\mathcal{A})$ does not imply the lower semicontinuity of ρ; see Example 1 in Farkas et al. [46]. The purpose of the next result is to provide a number of sufficient conditions for ρ to be lower semicontinuous. The last two conditions are particularly easy to verify and often satisfied in the literature.

Proposition 3.2.5. *The following statements hold:*

(i) Assume that Ω is finite. If $\rho(0) > -\infty$, then ρ is finite valued and continuous.

(ii) Assume that $\mathcal{X}_i'$ is the norm dual of $\mathcal{X}_i$ for every $i \in \{1,\dots,d\}$. If ρ is finite valued, then ρ is $\sigma(\mathcal{X},\mathcal{X}')$-lower semicontinuous.

(iii) Assume that $\mathcal{X}_i'$ is the norm dual of $\mathcal{X}_i$ for every $i \in \{1,\dots,d\}$. If $S^{-1}(\mathcal{A})$ has nonempty interior in the norm topology and $\rho(0) > -\infty$, then ρ is finite valued and $\sigma(\mathcal{X},\mathcal{X}')$-lower semicontinuous.

(iv) Set $\mathcal{M}_0 := \left\{m \in \mathbb{R}^d \,:\, \sum_{i=1}^d m_i = 0\right\}$. If $S^{-1}(\mathcal{A}) \cap \mathcal{M}_0 = \{0\}$, then ρ is proper and $\sigma(\mathcal{X},\mathcal{X}')$-lower semicontinuous.

(v) If $\mathcal{A} \cap (-\mathbb{R}_+) = \{0\}$ and $S(m) \in (-\infty, 0)$ for every nonzero $m \in \mathcal{M}_0$, then ρ is proper and $\sigma(\mathcal{X},\mathcal{X}')$-lower semicontinuous.

Proof. (i) Since Ω is finite, $\mathbf{e}$ is an interior point of $\mathcal{X}_+$. Then, the desired result follows from Proposition 1 in Farkas et al. [46].

(ii) It follows from the Extended Namioka-Klee Theorem in Biagini and Frittelli [21] that ρ is lower semicontinuous (in fact, continuous) with respect to the norm topology. Then, ρ is also lower semicontinuous with respect to $\sigma(\mathcal{X},\mathcal{X}')$ by virtue of Corollary 5.99 in Aliprantis and Border [1].

(iii) Note that $\mathbf{e}$ is a strictly-positive element of $\mathcal{X}$, i.e. for every $Z \in \mathcal{X}_+' \setminus \{0\}$ we have

$$\mathbb{E}_{\mathbb{P}}[\langle \mathbf{e}, Z\rangle] = \sum_{i=1}^{d} \mathbb{E}_{\mathbb{P}}[Z_i] > 0.$$

Proposition 2 in [46] implies that ρ is finite valued so that *(ii)* can be applied.

(iv) For every $X \in \mathcal{X}$ it is not difficult to show that

$$\rho(X) = \inf\left\{r \in \mathbb{R} \,:\, X + \frac{r}{d}\mathbf{e} \in S^{-1}(\mathcal{A}) + \mathcal{M}_0\right\};$$

see Lemma 3 in [46]. Then, it follows from Proposition 3.2.3 that

$$S^{-1}(\mathcal{A}) + \mathcal{M}_0 - \frac{r}{d}\mathbf{e} \subset \{X \in \mathcal{X} \,:\, \rho(X) \le r\} \subset \mathrm{cl}\left(S^{-1}(\mathcal{A}) + \mathcal{M}_0 - \frac{r}{d}\mathbf{e}\right)$$

for every $r \in \mathbb{R}$, where cl denotes the closure operator with respect to $\sigma(\mathcal{X},\mathcal{X}')$. To establish the desired lower semicontinuity, we show that $S^{-1}(\mathcal{A}) + \mathcal{M}_0$ is $\sigma(\mathcal{X},\mathcal{X}')$-closed. To this effect, recall from Proposition 3.2.3 that $S^{-1}(\mathcal{A})$ is convex and $\sigma(\mathcal{X},\mathcal{X}')$-closed. Moreover, $\mathcal{M}_0$ is a finite-dimensional vector space and $S^{-1}(\mathcal{A}) \cap \mathcal{M}_0 = \{0\}$. The closedness criterion in Dieudonné [39] now implies that $S^{-1}(\mathcal{A}) + \mathcal{M}_0$ is $\sigma(\mathcal{X},\mathcal{X}')$-closed. Properness follows from Proposition 3.2.4.

(v) Let $m \in \mathcal{M}_0$. By assumption, we have $S(m) \in \mathcal{A}$ if and only if $m = 0$. This yields $S^{-1}(\mathcal{A}) \cap \mathcal{M}_0 = \{0\}$ and the desired statement immediately follows from point *(iv)*. □

3.2.3 THE DUAL REPRESENTATION OF ρ

We have already mentioned that, in view of (3.2), the risk measure ρ belongs to the class of risk measures studied in Farkas et al. [46]. The general results established in that paper can be exploited to derive a dual representation for ρ. This also follows from the general dual representation in Frittelli and Scandolo [51].

Definition 3.2.6. We denote by $\mathcal{D}$ the convex subset of $\mathcal{X}_+'$ defined by

$$\mathcal{D} := \{Z \in \mathcal{X}_+' \,:\, \mathbb{E}_{\mathbb{P}}[Z_1] = \cdots = \mathbb{E}_{\mathbb{P}}[Z_d] = 1\}.$$

Remark 3.2.7. Note that, with reference to Chapter 1, elements of $\mathcal{D}$ may be interpreted as *pricing densities*, as they extend the functional π defined on the subspace $\mathbb{R}^d$ to the entire space $\mathcal{X}$. Indeed, for $m \in \mathbb{R}^d$ and $Z \in \mathcal{D}$, we have that $\pi(m) = \sum_i m_i = \mathbb{E}_{\mathbb{P}}[\langle m, Z\rangle]$.

Theorem 3.2.8. *If ρ is proper and $\sigma(\mathcal{X},\mathcal{X}')$-lower semicontinuous, then* $\mathrm{bar}(S^{-1}(\mathcal{A}))\cap\mathcal{D}\neq\emptyset$ *and*

$$\rho(X)=\sup_{Z\in\mathcal{D}}\{\sigma_{S^{-1}(\mathcal{A})}(Z)-\mathbb{E}_{\mathbb{P}}[\langle X,Z\rangle]\}$$

for every $X\in\mathcal{X}$. The supremum can be restricted to $\mathcal{D}\cap\mathcal{X}'_{++}$ provided that $\mathrm{bar}(S^{-1}(\mathcal{A}))\cap\mathcal{D}\cap\mathcal{X}'_{++}\neq\emptyset$.

Proof. Note that the cost functional π in equation (3.2) is defined on $\mathbb{R}^d\subset\mathcal{X}$. It is easy to see that, for every $Z\in\mathcal{X}'$, the functional $X\mapsto\mathbb{E}_{\mathbb{P}}[\langle X,Z\rangle]$ is a positive extension of π to $\mathcal{X}$ if and only if Z belongs to $\mathcal{D}$. Since ρ is proper and $\sigma(\mathcal{X},\mathcal{X}')$-lower semicontinuous, it follows from Proposition 6 in [46] that the barrier cone of $S^{-1}(\mathcal{A})$ contains positive linear extensions of the cost functional π to $\mathcal{X}$, i.e. we have $\mathrm{bar}(S^{-1}(\mathcal{A}))\cap\mathcal{D}\neq\emptyset$. The desired representation is now a consequence of Theorem 3 in [46].

Now, assume we find $Z^*\in\mathrm{bar}(S^{-1}(\mathcal{A}))\cap\mathcal{D}\cap\mathcal{X}'_{++}$ and take any element $Z\in\mathrm{bar}(S^{-1}(\mathcal{A}))\cap\mathcal{D}$. For every $X\in\mathcal{X}$ and every $\lambda\in(0,1)$ we have $\lambda Z^*+(1-\lambda)Z\in\mathcal{D}$ and

$$\begin{aligned}\lambda(\sigma_{S^{-1}(\mathcal{A})}(Z^*)-\mathbb{E}_{\mathbb{P}}[\langle X,Z^*\rangle])+(1-\lambda)(\sigma_{S^{-1}(\mathcal{A})}(Z)-\mathbb{E}_{\mathbb{P}}[\langle X,Z\rangle])&\\ \leq\sup_{Z'\in\mathcal{D}\cap\mathcal{X}'_{++}}\{\sigma_{S^{-1}(\mathcal{A})}(Z')-\mathbb{E}_{\mathbb{P}}[\langle X,Z'\rangle]\}&\end{aligned}$$

by concavity of $\sigma_{S^{-1}(\mathcal{A})}$. Letting λ tend to 0 and taking a supremum over Z yields

$$\rho(X)\leq\sup_{Z'\in\mathcal{D}\cap\mathcal{X}'_{++}}\{\sigma_{S^{-1}(\mathcal{A})}(Z')-\mathbb{E}_{\mathbb{P}}[\langle X,Z'\rangle]\}.$$

The converse inequality is clear. This establishes the last assertion and concludes the proof. □

Remark 3.2.9. (i) We highlight the link between the dual representation in Theorem 3.2.8 and the standard Fenchel-Moreau representation; see also Remark 17 in Farkas et al. [46]. To see it, note that the map $-\sigma_{S^{-1}(\mathcal{A})}(-\cdot)+\delta_{\mathcal{D}}(-\cdot)$ is convex and lower semicontinuous and, if ρ is proper and $\sigma(\mathcal{X},\mathcal{X}')$-lower semicontinuous, it satisfies

$$\rho(X)=\sup_{Z\in\mathcal{X}'}\{\mathbb{E}_{\mathbb{P}}[\langle X,Z\rangle]+\sigma_{S^{-1}(\mathcal{A})}(-Z)-\delta_{\mathcal{D}}(-Z)\}$$

for every $X\in\mathcal{X}$ by Theorem 3.2.8. From the Fenchel-Moreau Theorem it follows that for every $Z\in\mathcal{X}'$

$$\rho^*(Z)=-\sigma_{S^{-1}(\mathcal{A})}(-Z)+\delta_{\mathcal{D}}(-Z)=\begin{cases}\sup_{X\in S^{-1}(\mathcal{A})}\mathbb{E}_{\mathbb{P}}[\langle X,Z\rangle] & \text{if } Z\in-\mathcal{D},\\ \infty & \text{otherwise.}\end{cases}$$

(ii) The dual elements in $\mathcal{D}$ can be naturally identified with d-dimensional vectors of probability measures on $(\Omega,\mathcal{F})$ that are absolutely continuous with respect to $\mathbb{P}$ or, in case they have strictly-positive components, equivalent to $\mathbb{P}$. This allows to reformulate the above dual representation in terms of probability measures. More concretely, denote by $\mathcal{Q}(\mathbb{P})$, respectively $\mathcal{Q}_e(\mathbb{P})$, the set of all d-dimensional vectors of probability measures over $(\Omega,\mathcal{F})$ that are absolutely continuous with respect to $\mathbb{P}$, respectively equivalent to $\mathbb{P}$. For every $\mathbb{Q}=(\mathbb{Q}_1,\dots,\mathbb{Q}_d)\in\mathcal{Q}(\mathbb{P})$ and for every $X\in\mathcal{X}$ we set

$$\frac{d\mathbb{Q}}{d\mathbb{P}}:=\left(\frac{d\mathbb{Q}_i}{d\mathbb{P}},\dots,\frac{d\mathbb{Q}_d}{d\mathbb{P}}\right),\quad \mathbb{E}_{\mathbb{Q}}[X]:=\mathbb{E}\left[\left\langle X,\frac{d\mathbb{Q}}{d\mathbb{P}}\right\rangle\right]=\sum_{i=1}^d\mathbb{E}_{\mathbb{Q}_i}[X_i].$$

Moreover, for every $\mathbb{Q}\in\mathcal{Q}(\mathbb{P})$ we define

$$\sigma(\mathbb{Q}):=\sigma_{S^{-1}(\mathcal{A})}\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right)=\inf_{X\in S^{-1}(\mathcal{A})}\mathbb{E}_{\mathbb{Q}}[X].$$

If ρ is proper and $\sigma(\mathcal{X},\mathcal{X}')$-lower semicontinuous, then for every $X \in \mathcal{X}$ we can write

$$\rho(X) = \sup_{\mathbb{Q}\in\mathcal{Q}(\mathbb{P}),\, \frac{d\mathbb{Q}}{d\mathbb{P}}\in\mathcal{X}'} \{\sigma(\mathbb{Q}) - \mathbb{E}_{\mathbb{Q}}[X]\}.$$

We can replace $\mathcal{Q}(\mathbb{P})$ by $\mathcal{Q}_e(\mathbb{P})$ in the above supremum provided that $\mathrm{bar}(S^{-1}(\mathcal{A})) \cap \mathcal{D} \cap \mathcal{X}'_{++} \neq \emptyset$.

The condition $\mathrm{bar}(S^{-1}(\mathcal{A})) \cap \mathcal{D} \cap \mathcal{X}'_{++} \neq \emptyset$ is necessary to be able to restrict the domain in the above dual representation to strictly-positive dual elements. In the terminology of convex analysis, this condition requires that the convex set $S^{-1}(\mathcal{A})$ admits a strictly-positive supporting functional that belongs to the special set $\mathcal{D}$. In the next proposition we show that this always holds if the acceptance set $\mathcal{A}$ is supported by a strictly-positive functional and the impact map S is bounded above by a strictly-increasing affine function of the consolidated capital position.

Proposition 3.2.10. *Assume that $\mathcal{X}_i = \mathcal{E}$ for every $i \in \{1,\dots,d\}$. Moreover, suppose that $\mathrm{bar}(\mathcal{A}) \cap \mathcal{E}'_{++} \neq \emptyset$ and there exist $a \in (0,\infty)$ and $b \in \mathbb{R}$ such that*

$$S(X) \leq a\sum_{i=1}^{d} X_i + b$$

for every $X \in \mathcal{X}$. Then, $\mathrm{bar}(S^{-1}(\mathcal{A})) \cap \mathcal{D} \cap \mathcal{X}'_{++} \neq \emptyset$.

Proof. Take $W \in \mathrm{bar}(\mathcal{A}) \cap \mathcal{E}'_{++}$ and set $Z = (aW,\dots,aW) \in \mathcal{D} \cap \mathcal{X}'_{++}$. Then, we easily see that

$$\sigma_{S^{-1}(\mathcal{A})}(Z) = \inf_{X\in S^{-1}(\mathcal{A})} \mathbb{E}_{\mathbb{P}}[\langle X,Z\rangle] \geq \inf_{X\in S^{-1}(\mathcal{A})} \mathbb{E}_{\mathbb{P}}[(S(X)-b)W] \geq \sigma_{\mathcal{A}}(W) - b\mathbb{E}_{\mathbb{P}}[W] > -\infty.$$

This delivers the desired assertion. □

3.2.4 CHARACTERIZING THE SYSTEMIC ACCEPTANCE SET $S^{-1}(\mathcal{A})$

Through the support function of the "systemic acceptance set" $S^{-1}(\mathcal{A})$, the dual representation of the systemic risk measure ρ in Theorem 3.2.8 highlights the dependence on the two fundamental underlying ingredients: The impact map S and the acceptance set $\mathcal{A}$. The aim of this subsection is to provide a dual description of the systemic acceptance set by using "penalty functions" that are related to (but different from) the support function $\sigma_{S^{-1}(\mathcal{A})}$ and to investigate the main properties of these maps. Our analysis is based on the following definition.

Definition 3.2.11. We define two maps $\alpha, \alpha^+ : \mathcal{X}' \to [-\infty,+\infty]$ by setting

$$\alpha(Z) := \sup_{W\in\mathrm{bar}(\mathcal{A})} \Big\{\sigma_{\mathcal{A}}(W) + \inf_{X\in\mathcal{X}}\{\mathbb{E}_{\mathbb{P}}[\langle X,Z\rangle] - \mathbb{E}_{\mathbb{P}}[S(X)W]\}\Big\},$$

$$\alpha^+(Z) := \sup_{W\in\mathrm{bar}(\mathcal{A})\cap(\mathcal{E}'_{++}\cup\{0\})} \Big\{\sigma_{\mathcal{A}}(W) + \inf_{X\in\mathcal{X}}\{\mathbb{E}_{\mathbb{P}}[\langle X,Z\rangle] - \mathbb{E}_{\mathbb{P}}[S(X)W]\}\Big\}.$$

Remark 3.2.12. (i) It is easy to see that α and α^+ are different in general. For example, if $d > 1$ and $\mathcal{X}_i = \mathcal{E}$ for every $i \in \{1,\dots,d\}$ and we set $S(X) = \sum_{i=1}^{d} X_i$ for every $X \in \mathcal{X}$ and $\mathcal{A} = \mathcal{E}_+$, then we have

$$\alpha = -\delta_{\mathcal{G}} \neq -\delta_{\mathcal{G}\cap(\mathcal{X}'_{++}\cup\{0\})} = \alpha^+$$

where $\mathcal{G} = \{Z \in \mathcal{X}'_+ \,:\, Z_1 = \cdots = Z_d\}$.

(ii) The above maps belong to the class of maps $\alpha_{\mathcal{K}} : \mathcal{X}' \to [-\infty,+\infty]$ defined by

$$\alpha_{\mathcal{K}}(Z) := \sup_{W\in\mathcal{K}} \Big\{\sigma_{\mathcal{A}}(W) + \inf_{X\in\mathcal{X}}\{\mathbb{E}_{\mathbb{P}}[\langle X,Z\rangle] - \mathbb{E}_{\mathbb{P}}[S(X)W]\}\Big\},$$

where $\mathcal{K}$ is a convex cone in $\mathrm{bar}(\mathcal{A})$ such that $\lambda\mathcal{K} + (1-\lambda)\,\mathrm{bar}(\mathcal{A}) \subset \mathcal{K}$ for every $\lambda \in [0,1]$. This will allow us to prove properties for α and α^+ simultaneously. In fact, all properties of α and α^+ we will consider are shared by the entire class.

The next theorem records the announced dual representation of the systemic acceptance set and shows why the above maps are natural "penalty functions".

Theorem 3.2.13. *The systemic acceptance set $S^{-1}(\mathcal{A})$ can be represented as*

$$S^{-1}(\mathcal{A}) = \bigcap_{Z\in\mathcal{X}'} \{X \in \mathcal{X} \,:\, \mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle] \geq \alpha(Z)\}.$$

If $\mathrm{bar}(\mathcal{A}) \cap \mathcal{E}'_{++} \neq \emptyset$*, then $S^{-1}(\mathcal{A})$ can also be represented as*

$$S^{-1}(\mathcal{A}) = \bigcap_{Z\in\mathcal{X}'} \{X \in \mathcal{X} \,:\, \mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle] \geq \alpha^{+}(Z)\}.$$

Proof. Let $\mathcal{K} \subset \mathrm{bar}(\mathcal{A})$ be a convex cone as in Remark 3.2.12. Note that, by concavity of $\sigma_{\mathcal{A}}$, we can equivalently rewrite the representation in Proposition A.1.10 applied to $\mathcal{A}$ as

$$\mathcal{A} = \bigcap_{W\in\mathcal{K}} \{U \in \mathcal{E} \,:\, \mathbb{E}_{\mathbb{P}}[UW] \geq \sigma_{\mathcal{A}}(W)\}.$$

Now, for each $W \in \mathcal{K} \subset \mathcal{E}'_{+}$ we consider the functional $\varphi_W : \mathcal{X} \to \mathbb{R}$ defined by

$$\varphi_W(X) := \mathbb{E}_{\mathbb{P}}[S(X)W].$$

As remarked in the proof of Proposition 3.1.4, the functional φ_W is concave by (S3) and (S4) and $\sigma(\mathcal{X}, \mathcal{X}')$-upper semicontinuous by (S5). Hence, it follows from the Fenchel-Moreau Theorem A.1.6 that

$$\varphi_W(X) = \inf_{Z\in\mathcal{X}'} \{\mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle] - (\varphi_W)^{\bullet}(Z)\}$$

for every $X \in \mathcal{X}$ (see Appendix A for the definition of $(\varphi_W)^{\bullet}$). As a result, we obtain

$$\begin{aligned}
S^{-1}(\mathcal{A}) &= \{X \in \mathcal{X} \,:\, S(X) \in \mathcal{A}\} \\
&= \{X \in \mathcal{X} \,:\, \mathbb{E}_{\mathbb{P}}[S(X)W] \geq \sigma_{\mathcal{A}}(W),\ \forall W \in \mathcal{K}\} \\
&= \{X \in \mathcal{X} \,:\, \mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle] - (\varphi_W)^{\bullet}(Z) \geq \sigma_{\mathcal{A}}(W),\ \forall W \in \mathcal{K},\ \forall Z \in \mathcal{X}'\} \\
&= \bigcap_{Z\in\mathcal{X}'} \left\{X \in \mathcal{X} \,:\, \mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle] \geq \sup_{W\in\mathcal{K}} \{\sigma_{\mathcal{A}}(W) + (\varphi_W)^{\bullet}(Z)\}\right\} \\
&= \bigcap_{Z\in\mathcal{X}'} \{X \in \mathcal{X} \,:\, \mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle] \geq \alpha_{\mathcal{K}}(Z)\}.
\end{aligned}$$

This delivers the desired representation when applied to $\mathcal{K} = \mathrm{bar}(\mathcal{A})$ and $\mathcal{K} = \mathrm{bar}(\mathcal{A}) \cap (\mathcal{E}'_{++} \cup \{0\})$. ☐

The next proposition collects some properties of the maps α and α^{+} and shows the relation between them. We recall that we denote by $\mathrm{dom}(\alpha)$ the domain of finiteness of α (similarly for α^{+}). In addition, we denote by cl the closure operator with respect to the topology $\sigma(\mathcal{X}', \mathcal{X})$.

Proposition 3.2.14. *The maps $\alpha, \alpha^{+} : \mathcal{X}' \to [-\infty, \infty]$ satisfy the following properties (the statements about α^{+} require that* $\mathrm{bar}(\mathcal{A}) \cap \mathcal{E}'_{++} \neq \emptyset$*):*

(i) *α and α^{+} take values in the interval* $[-\infty, 0]$.

(ii) *α and α^{+} are concave and positively homogeneous.*

(iii) *$\alpha^{+} \leq \alpha$ with equality on* $\mathrm{dom}(\alpha^{+})$.

(iv) $\mathrm{dom}(\alpha^{+}) \subset \mathrm{dom}(\alpha) \subset \mathrm{cl}(\mathrm{dom}(\alpha^{+})) \subset \mathcal{X}'_{+}$.

Proof. Throughout the proof we fix a convex cone $\mathcal{K} \subset \mathrm{bar}(\mathcal{A})$ as in Remark 3.2.12. The desired assertions will follow by taking $\mathcal{K} = \mathrm{bar}(\mathcal{A})$ and $\mathcal{K} = \mathrm{bar}(\mathcal{A}) \cap (\mathcal{E}'_{++} \cup \{0\})$.

(i) The representation of the systemic acceptance set $S^{-1}(\mathcal{A})$ established in the proof of Theorem 3.2.13 yields $\alpha_{\mathcal{K}}(Z) \leq \sigma_{S^{-1}(\mathcal{A})}(Z) \leq 0$ for every $Z \in \mathcal{X}'$.

(ii) To show that $\alpha_{\mathcal{K}}$ is concave, set for all $Z \in \mathcal{X}'$ and $W \in \mathcal{E}'$

$$\Phi(Z, W) := \inf_{X \in \mathcal{X}} \{\sigma_{\mathcal{A}}(W) + \mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle] - \mathbb{E}_{\mathbb{P}}[S(X)W]\}.$$

Being the infimum over the parameter X of a function that is clearly jointly concave in Z and W, we see that Φ is itself jointly concave. Since

$$\alpha_{\mathcal{K}}(Z) = \sup_{W \in \mathcal{K}} \Phi(Z, W)$$

for every $Z \in \mathcal{X}'$, we infer that $\alpha_{\mathcal{K}}$ is concave. To show that $\alpha_{\mathcal{K}}$ is positively homogeneous, note first that 0 always belongs to $\mathcal{K}$, so that $\alpha_{\mathcal{K}}(0) \geq 0$. Together with point *(i)*, this implies that $\alpha_{\mathcal{K}}(0) = 0$. Finally, for $Z \in \mathcal{X}'$ and $\lambda \in (0, \infty)$ we have

$$\begin{aligned}
\alpha_{\mathcal{K}}(\lambda Z) &= \sup_{W \in \mathcal{K}} \left\{\sigma_{\mathcal{A}}(W) + \inf_{X \in \mathcal{X}} \{\lambda \mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle] - \mathbb{E}_{\mathbb{P}}[S(X)W]\}\right\} \\
&= \lambda \sup_{W \in \mathcal{K}} \left\{\sigma_{\mathcal{A}}\left(\frac{1}{\lambda}W\right) + \inf_{X \in \mathcal{X}} \left\{\mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle] - \mathbb{E}_{\mathbb{P}}\left[S(X)\frac{1}{\lambda}W\right]\right\}\right\} \\
&= \lambda \sup_{W \in \mathcal{K}} \left\{\sigma_{\mathcal{A}}(W) + \inf_{X \in \mathcal{X}} \{\mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle] - \mathbb{E}_{\mathbb{P}}[S(X)W]\}\right\} \\
&= \lambda \alpha_{\mathcal{K}}(Z),
\end{aligned}$$

where we used that $\mathcal{K}$ is a cone. This shows that $\alpha_{\mathcal{K}}$ is positively homogeneous.

(iii) It is clear that $\alpha^+ \leq \alpha$. To show that $\alpha^+ = \alpha$ on $\mathrm{dom}(\alpha^+)$, take $Z \in \mathrm{dom}(\alpha^+)$ and note that

$$\alpha(Z) = \sup_{W \in \mathrm{bar}(\mathcal{A})} \Phi(Z, W), \quad \alpha^+(Z) = \sup_{W \in \mathrm{bar}(\mathcal{A}) \cap \mathcal{E}'_{++}} \Phi(Z, W).$$

Take $W^* \in \mathrm{bar}(\mathcal{A}) \cap \mathcal{E}'_{++}$ such that $\Phi(Z, W^*)$ is finite. For each $W \in \mathrm{bar}(\mathcal{A})$ set $W_\lambda = \lambda W + (1 - \lambda)W^*$ for $\lambda \in [0, 1)$. Note that $(W_\lambda) \subset \mathrm{bar}(\mathcal{A}) \cap \mathcal{E}'_{++}$, so that

$$\alpha^+(Z) \geq \Phi(Z, W_\lambda) \geq \lambda \Phi(Z, W) + (1 - \lambda)\Phi(Z, W^*) \xrightarrow{\lambda \uparrow 1} \Phi(Z, W).$$

Taking a supremum over W delivers $\alpha^+(Z) \geq \alpha(Z)$.

(iv) Note that $\mathrm{dom}(\alpha^+) \subset \mathrm{dom}(\alpha)$ by point *(iii)*. Since $\alpha \leq \sigma_{S^{-1}(\mathcal{A})}$ as proved in point *(i)*, we also have $\mathrm{dom}(\alpha) \subset \mathrm{bar}(S^{-1}(\mathcal{A})) \subset \mathcal{X}'_+$. As $\mathcal{X}'_+$ is $\sigma(\mathcal{X}', \mathcal{X})$-closed, it remains to show that $\mathrm{dom}(\alpha) \subset \mathrm{cl}(\mathrm{dom}(\alpha^+))$. To this effect, let $Z \in \mathrm{dom}(\alpha)$ and note that $\Phi(Z, W)$ must be finite for some $W \in \mathrm{bar}(\mathcal{A})$. Take $Z^* \in \mathrm{dom}(\alpha^+)$ and $W^* \in \mathrm{bar}(\mathcal{A}) \cap \mathcal{E}'_{++}$ such that $\Phi(Z^*, W^*)$ is finite. Then, for every $\lambda \in [0, 1]$ we have

$$\begin{aligned}
\alpha^+(\lambda Z + (1 - \lambda)Z^*) &\geq \Phi(\lambda Z + (1 - \lambda)Z^*, \lambda W + (1 - \lambda)W^*) \\
&\geq \lambda \Phi(Z, W) + (1 - \lambda)\Phi(Z^*, W^*) > -\infty
\end{aligned}$$

by the joint convexity of Φ. The claim follows by letting $\lambda \uparrow 1$. □

THE CASE WHERE S IS INDUCED BY Λ

As mentioned in Remark 3.1.5, the bulk of the literature has focused on the case where the impact function is based on an aggregation function $\Lambda : \mathbb{R}^d \to \mathbb{R}$. The last part of this subsection is devoted to provide an equivalent formulation of α and α^+ in this situation. We focus on the positive cone $\mathcal{X}'_+$ because both maps take nonfinite values elsewhere. For ease of notation, for every $Z \in \mathcal{X}'_+$ we set

$$E_+(Z) := \bigcup_{i=1}^{d} \{Z_i > 0\} \in \mathcal{F}.$$

Proposition 3.2.15. *Assume that $\mathcal{X}$ is closed with respect to multiplications by characteristic functions, i.e. for every $X \in \mathcal{X}$ and $E \in \mathcal{F}$ we have $(\mathbb{1}_E X_1, \ldots, \mathbb{1}_E X_d) \in \mathcal{X}$. Moreover, consider a nonconstant,* *nondecreasing, concave function* $\Lambda : \mathbb{R}^d \to \mathbb{R}$ *satisfying* $\Lambda(0) = 0$ *and assume that* $S(X) = \Lambda(X)$ *for every* $X \in \mathcal{X}$*. Then, the following statements hold for every nonzero* $Z \in \mathcal{X}'_+$*:*

(i) We have $\mathrm{bar}(\mathcal{A}) \cap \{W \in \mathcal{E}'_+ \,:\, W > 0 \text{ on } E_+(Z)\} \neq \emptyset$ *and*

$$\alpha(Z) = \sup_{W \in \mathrm{bar}(\mathcal{A}),\, W>0\, on\, E_+(Z)} \left\{\sigma_{\mathcal{A}}(W) + \mathbb{E}\left[\mathbb{1}_{\{W>0\}}\Lambda^\bullet\left(\frac{Z}{W}\right)W\right]\right\}.$$

(ii) If $\mathrm{bar}(\mathcal{A}) \cap \mathcal{E}'_{++} \neq \emptyset$*, then*

$$\alpha^+(Z) = \sup_{W \in \mathrm{bar}(\mathcal{A}) \cap \mathcal{E}'_{++}} \left\{\sigma_{\mathcal{A}}(W) + \mathbb{E}\left[\Lambda^\bullet\left(\frac{Z}{W}\right)W\right]\right\}$$

In both cases, the ratio $\frac{Z}{W}$ *is understood component by component.*

Proof. Let $\mathcal{K} \subset \mathrm{bar}(\mathcal{A})$ be a convex cone as in Remark 3.2.12 and fix a nonzero element $Z \in \mathcal{X}'_+$. Inspired by Ararat and Rudloff [4], we invoke Theorem 14.60 in Rockafellar and Wets [86] to get

$$\inf_{X \in \mathcal{X}} \{\mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle] - \mathbb{E}_{\mathbb{P}}[\Lambda(X)W]\} = \mathbb{E}\Big[\inf_{x \in \mathbb{R}^d}\{xZ - \Lambda(x)W\}\Big] \tag{3.3}$$

for every $W \in \mathcal{K}$ (this result requires that $\mathcal{X}$ be closed with respect to multiplications by characteristic functions). Recall that $\mathcal{K} \subset \mathcal{E}'_+$ and note that for every $W \in \mathcal{K}$ we have

$$\inf_{x \in \mathbb{R}^d}\{xZ - \Lambda(x)W\} = \begin{cases} \Lambda^\bullet(\frac{Z}{W})W & \text{on } \{W > 0\}, \\ 0 & \text{on } \{W = 0\} \cap (E_+(Z))^c, \\ -\infty & \text{on } \{W = 0\} \cap E_+(Z). \end{cases} \tag{3.4}$$

It follows from the definition of $\alpha_{\mathcal{K}}$ and (3.3) that

$$\alpha_{\mathcal{K}}(Z) = \sup_{W \in \mathcal{K}} \Big\{\sigma_{\mathcal{A}}(W) + \mathbb{E}\Big[\inf_{x \in \mathbb{R}^d}\{xZ - \Lambda(x)W\}\Big]\Big\}.$$

Clearly, no $W \in \mathrm{bar}(\mathcal{A})$ with $\mathbb{P}(\{W = 0\} \cap E_+(Z)) > 0$ contributes to the above supremum, so that

$$\begin{aligned} \alpha_{\mathcal{K}}(Z) &= \sup_{W \in \mathcal{K},\, W>0\, on\, E_+(Z)} \Big\{\sigma_{\mathcal{A}}(W) + \mathbb{E}\Big[\inf_{x \in \mathbb{R}^d}\{xZ - \Lambda(x)W\}\Big]\Big\} \\ &= \sup_{W \in \mathcal{K},\, W>0\, on\, E_+(Z)} \Big\{\sigma_{\mathcal{A}}(W) + \mathbb{E}\Big[\mathbb{1}_{\{W>0\}}\inf_{x \in \mathbb{R}^d}\{xZ - \Lambda(x)W\}\Big]\Big\} \\ &= \sup_{W \in \mathcal{K},\, W>0 \text{ on } E_+(Z)} \Big\{\sigma_{\mathcal{A}}(W) + \mathbb{E}\Big[\mathbb{1}_{\{W>0\}}\Lambda^\bullet(\tfrac{Z}{W})W\Big]\Big\}, \end{aligned}$$

where we used (3.4) in the last equality. The desired assertions follow by taking $\mathcal{K} = \mathrm{bar}(\mathcal{A})$ and $\mathcal{K} = \mathrm{bar}(\mathcal{A}) \cap (\mathcal{E}'_{++} \cup \{0\})$. □

3.2.5 CHARACTERIZING THE SUPPORT FUNCTION $\sigma_{S^{-1}(\mathcal{A})}$

As we have already noticed, the dual representation in Theorem 3.2.8 depends on the impact map S and the acceptance set $\mathcal{A}$ through the support function of the systemic acceptance set $S^{-1}(\mathcal{A})$. The goal of this subsection is to provide an equivalent description of the support function that relies on the "penalty functions" α and α^+. This is a direct consequence of the results in the preceding subsection. Here, we denote by $\mathrm{usc}(\alpha)$ the $\sigma(\mathcal{X}', \mathcal{X})$-upper semicontinuous hull of α, i.e. the smallest $\sigma(\mathcal{X}', \mathcal{X})$-upper semicontinuous map dominating α (similarly for α^+).

Theorem 3.2.16. *The support function* $\sigma_{S^{-1}(\mathcal{A})}$ *can be represented as*

$$\sigma_{S^{-1}(\mathcal{A})} = \mathrm{usc}(\alpha).$$

If $\mathrm{bar}(\mathcal{A})\cap\mathcal{E}'_{++}\neq\emptyset$*, then* $\sigma_{S^{-1}(\mathcal{A})}$ *can also be represented as*

$$\sigma_{S^{-1}(\mathcal{A})}=\mathrm{usc}(\alpha^+).$$

Proof. Let $\mathcal{K}\subset\mathrm{bar}(\mathcal{A})$ be a convex cone as in Remark 3.2.12. In the proof of Theorem 3.2.13 we established that

$$S^{-1}(\mathcal{A})=\bigcap_{Z\in\mathcal{X}'}\{X\in\mathcal{X}\,:\,\mathbb{E}_{\mathbb{P}}[\langle X,Z\rangle]\geq\alpha_{\mathcal{K}}(Z)\}. \tag{3.5}$$

This implies that $\alpha_{\mathcal{K}}\leq\sigma_{S^{-1}(\mathcal{A})}$. It follows from the $\sigma(\mathcal{X}',\mathcal{X})$-upper semicontinuity of $\sigma_{S^{-1}(\mathcal{A})}$ that we also have $\mathrm{usc}(\alpha_{\mathcal{K}})\leq\sigma_{S^{-1}(\mathcal{A})}$. In particular, $\mathrm{usc}(\alpha_{\mathcal{K}})$ never takes the value ∞. Moreover, note that $\mathrm{usc}(\alpha_{\mathcal{K}})(0)\geq\alpha_{\mathcal{K}}(0)=0$. As a result, Proposition 2.2.7 in Zălinescu [95] tells us that $\mathrm{usc}(\alpha_{\mathcal{K}})$ inherits concavity and positive homogeneity from $\alpha_{\mathcal{K}}$. Note that $\alpha_{\mathcal{K}}$ can be replaced by $\mathrm{usc}(\alpha_{\mathcal{K}})$ in (3.5). Since the only $\sigma(\mathcal{X}',\mathcal{X})$-upper semicontinuous map $\sigma:\mathcal{X}'\to[-\infty,\infty)$ that is concave and positively homogeneous and satisfies

$$S^{-1}(\mathcal{A})=\bigcap_{Z\in\mathcal{X}'}\{X\in\mathcal{X}\,:\,\mathbb{E}_{\mathbb{P}}[\langle X,Z\rangle]\geq\sigma(Z)\}$$

is precisely the support function of $S^{-1}(\mathcal{A})$, see e.g. Theorem 7.51 in Aliprantis and Border [1], we conclude that $\mathrm{usc}(\alpha_{\mathcal{K}})=\sigma_{S^{-1}(\mathcal{A})}$ must hold. The desired assertions follow by taking $\mathcal{K}=\mathrm{bar}(\mathcal{A})$ and $\mathcal{K}=\mathrm{bar}(\mathcal{A})\cap(\mathcal{E}'_{++}\cup\{0\})$. ☐

It is natural to ask whether taking the upper semicontinuous hull in Theorem 3.2.16 is redundant in the sense that α and/or α^+ are upper semicontinuous in the first place and, hence, coincide with the support function $\sigma_{S^{-1}(\mathcal{A})}$. As illustrated by the following example, the answer is negative in general.

Example 3.2.17. *Let* $(\Omega,\mathcal{F},\mathbb{P})$ *be nonatomic and consider the pairs given by* $(\mathcal{X},\mathcal{X}')=(L^\infty_d(\mathbb{P}),L^1_d(\mathbb{P}))$ *and* $(\mathcal{E},\mathcal{E}')=(L^\infty(\mathbb{P}),L^1(\mathbb{P}))$*. Fix* $\lambda\in(0,1)$ *and for every* $U\in L^0(\mathbb{P})$ *define the* Value at Risk *and* Expected Shortfall *of* U *at level* λ *by*

$$\mathrm{VaR}_\lambda(U):=\inf\{m\in\mathbb{R}\,:\,\mathbb{P}(U+m<0)\leq\lambda\},\quad \mathrm{ES}_\lambda(U):=\frac{1}{\lambda}\int_0^\lambda\mathrm{VaR}_\mu(U)\,d\mu.$$

Define $S:\mathcal{X}\to\mathcal{E}$ *and* $\mathcal{A}\subset\mathcal{E}$ *by setting*

$$S(X)=\sum_{i=1}^d -X_i^-,\quad \mathcal{A}=\{U\in\mathcal{E}\,:\,\mathrm{ES}_\lambda(U)\leq 0\}.$$

It is immediate to see that $S^{-1}(\mathcal{A})=\mathcal{X}_+$*, so that*

$$\sigma_{S^{-1}(\mathcal{A})}=-\delta_{\mathcal{X}'_+}=-\delta_{L^1_d(\mathbb{P})_+}.$$

To determine α*, take any* $Z\in\mathcal{X}'_+$ *and recall from Theorem 4.52 in Föllmer and Schied [49] that*

$$\sigma_{\mathcal{A}}=-\delta_{\mathrm{bar}(\mathcal{A})},\quad \mathrm{bar}(\mathcal{A})=\left\{W\in\mathcal{E}'_+\,:\,W\leq\frac{1}{\lambda}\mathbb{E}_{\mathbb{P}}[W]\right\}.$$

As a result, we infer that

$$\begin{aligned}\alpha(Z)&=\sup_{W\in\mathcal{E}'_+,\,W\leq\frac{\mathbb{E}_{\mathbb{P}}[W]}{\lambda}}\inf_{X\in\mathcal{X}}\mathbb{E}_{\mathbb{P}}\left[\sum_{i=1}^d(X_iZ_i+X_i^-W)\right]\\&=\sup_{W\in\mathcal{E}'_+,\,W\leq\frac{\mathbb{E}_{\mathbb{P}}[W]}{\lambda}}\inf_{X\in\mathcal{X}_+}\mathbb{E}_{\mathbb{P}}\left[\sum_{i=1}^d X_i(W-Z_i)\right].\end{aligned}$$

Now, if Z_j is not bounded for some $j \in \{1,\dots,d\}$, then $\mathbb{P}(W - Z_j < 0) > 0$ for every $W \in \mathrm{bar}(\mathcal{A})$ and

$$\inf_{X\in\mathcal{X}_+} \mathbb{E}_{\mathbb{P}}\left[\sum_{i=1}^{d} X_i(W - Z_i)\right] \leq \inf_{n\in\mathbb{N}} \mathbb{E}_{\mathbb{P}}[n\mathbb{1}_{\{W-Z_j<0\}}(W - Z_j)] = -\infty.$$

In this case, we have $\alpha(Z) = -\infty$. Otherwise, if Z is bounded, set $W = \max_{i\in\{1,\dots,d\}} \|Z_i\|_\infty \in \mathrm{bar}(\mathcal{A})$ and observe that

$$0 \geq \alpha(Z) \geq \inf_{X\in\mathcal{X}_+} \mathbb{E}_{\mathbb{P}}\left[\sum_{i=1}^{d} X_i(W - Z_i)\right] = 0.$$

In conclusion, we have

$$\alpha = -\delta_{\mathcal{X}_+} = -\delta_{L^\infty_d(\mathbb{P})_+}.$$

Since $L^\infty(\mathbb{P}) \neq L^1(\mathbb{P})$ when the underlying probability space is nonatomic, we conclude that $\sigma_{S^{-1}(\mathcal{A})} \neq \alpha$. The same conclusion holds for α^+ as well (note that $\mathrm{bar}(\mathcal{A}) \cap \mathcal{E}'_{++} \neq \emptyset$). This follows because, by Proposition 3.2.14, we always have $\alpha^+ \leq \alpha$. Alternatively, we can repeat the above argument and find that $\alpha^+ = \alpha$ in our situation.

Remark 3.2.18. By combining the dual representation in Theorem 3.2.8 and the representation of $\sigma_{S^{-1}(\mathcal{A})}$ obtained in Theorem 3.2.16, we see that

$$\rho(X) = \sup_{Z\in\mathcal{D}}\{\mathrm{usc}(\alpha)(Z) - \mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle]\} = \sup_{Z\in\mathcal{X}'}\{\mathrm{usc}(\alpha)(Z) - \delta_{\mathcal{D}}(Z) - \mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle]\} \tag{3.6}$$

for every $X \in \mathcal{X}$. If the equality $\sigma_{S^{-1}(\mathcal{A})} = \alpha$ holds, then we can drop the upper-semicontinuous hull in the representation (3.6) and obtain

$$\rho(X) = \sup_{Z\in\mathcal{D}}\{\alpha(Z) - \mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle]\} = \sup_{Z\in\mathcal{X}'}\{\alpha(Z) - \delta_{\mathcal{D}}(Z) - \mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle]\} \tag{3.7}$$

for every $X \in \mathcal{X}$. One may wonder whether the "simplified" representation (3.7) holds even if the equality $\sigma_{S^{-1}(\mathcal{A})} = \alpha$ does not hold. Note that $\mathrm{usc}(\alpha) - \delta_{\mathcal{D}}$ is concave and $\sigma(\mathcal{X}', \mathcal{X})$-upper semicontinuous and that $\alpha - \delta_{\mathcal{D}}$ is concave. As a result, the "simplified" representation holds if and only if

$$\mathrm{usc}(\alpha - \delta_{\mathcal{D}}) = \mathrm{usc}(\alpha) - \delta_{\mathcal{D}}.$$

The same holds with α^+ instead of α (provided that $\mathrm{bar}(\mathcal{A}) \cap \mathcal{E}'_{++} \neq \emptyset$). It is unclear whether this equality holds without additional assumptions on S and $\mathcal{A}$ because, in general, it is not possible to take an indicator function out of an upper-semicontinuous hull. For example, consider the simple situation where $\Omega = \{\omega\}$ and $d = 2$. In this case, we have the identification $(\mathcal{X}, \mathcal{X}') = (\mathbb{R}^2, \mathbb{R}^2)$. Consider the concave and positively homogeneous function f and the convex closed set $\mathcal{D}$ defined by

$$f = -\delta_{\mathcal{G}}, \quad \mathcal{G} = \{z \in \mathbb{R}^2 \,:\, 0 \leq z_1 < z_2\} \cup \{(0,0)\}, \quad \mathcal{D} = \{z \in \mathbb{R}^2 \,:\, z_1 = z_2 = 1\} = \{(1,1)\}.$$

Then, it is easy to see that

$$\mathrm{usc}(f - \delta_{\mathcal{D}}) = -\delta_{\emptyset} \neq -\delta_{\{(1,1)\}} = \mathrm{usc}(f) - \delta_{\mathcal{D}}.$$

3.2.6 Conditions for the identity $\sigma_{S^{-1}(\mathcal{A})} = \alpha$ to hold

We know from Theorem 3.2.16 that the support function of the systemic acceptance set $S^{-1}(\mathcal{A})$ always coincides with the upper semicontinuous hull of the penalty function α. However, as illustrated by Example 3.2.17, there are simple situations where the map α fails to be upper semicontinuous and, hence, the equality $\sigma_{S^{-1}(\mathcal{A})} = \alpha$ does not hold. In this subsection we establish a variety of sufficient conditions for this equality to hold. Clearly, one could also ask when $\sigma_{S^{-1}(\mathcal{A})} = \alpha^+$, which would automatically imply the statement for α. While it is easy to find examples where this holds, none of the conditions in this section apply to α^+.

As a first step, we highlight that the desired equality can be equivalently expressed in terms of a suitable minimax problem.

Lemma 3.2.19. *Let $Z \in \mathcal{X}'$ and define a map $K : \mathcal{X} \times \mathcal{E}' \to [-\infty, \infty]$ by setting*

$$K_Z(X, W) := \sigma_{\mathcal{A}}(W) + \mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle] - \mathbb{E}_{\mathbb{P}}[S(X)W].$$

The following statements are equivalent:

(a) $\sigma_{S^{-1}(\mathcal{A})} = \alpha$.

(b) α *is* $\sigma(\mathcal{X}', \mathcal{X})$*-upper semicontinuous.*

(c) For every $Z \in \mathcal{X}'$ *we have*

$$\inf_{X\in\mathcal{X}} \sup_{W\in\mathcal{E}'} K_Z(X, W) = \sup_{W\in\mathcal{E}'} \inf_{X\in\mathcal{X}} K_Z(X, W).$$

Proof. The equivalence between *(a)* and *(b)* is clear by Theorem 3.2.16. To establish equivalence with *(c)*, fix $Z \in \mathcal{X}'$ and note that

$$\alpha(Z) = \sup_{W\in\mathcal{E}'} \inf_{X\in\mathcal{X}} K_Z(X, W)$$

by definition of α. It remains to show that

$$\sigma_{S^{-1}(\mathcal{A})}(Z) = \inf_{X\in\mathcal{X}} \sup_{W\in\mathcal{E}'} K_Z(X, W).$$

Consider the auxiliary functions $f_Z : \mathcal{X} \to (-\infty, \infty]$ defined by

$$f_Z(X) := \mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle] + \delta_{S^{-1}(\mathcal{A})}(X)$$

and $F_Z : \mathcal{X} \times \mathcal{E} \to (-\infty, \infty]$ defined by

$$F_Z(X, U) := \mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle] + \delta_{\mathcal{A}-S(X)}(U).$$

Note that for every $X \in \mathcal{X}$ the map $F_Z(X, \cdot)$ is convex and lower semicontinuous and satisfies

$$\begin{aligned}
(F_Z(X,\cdot))^*(W) &= \sup_{U\in\mathcal{E}}\{\mathbb{E}_{\mathbb{P}}[UW] - F_Z(X,U)\} \\
&= \sup_{U\in\mathcal{E},\, U+S(X)\in\mathcal{A}} \{\mathbb{E}_{\mathbb{P}}[UW] - \mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle]\} \\
&= \sup_{V\in\mathcal{E}}\{\mathbb{E}_{\mathbb{P}}[(V - S(X))W] - \mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle]\} \\
&= -\sigma_{\mathcal{A}}(-W) - \mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle] + \mathbb{E}_{\mathbb{P}}[S(X)(-W)] \\
&= -K_Z(X, -W)
\end{aligned}$$

for every $W \in \mathcal{E}'$. As $F_Z(X, 0) = f_Z(X)$ for every $X \in \mathcal{X}$, we can apply Fenchel-Moreau to get

$$\sigma_{S^{-1}(\mathcal{A})}(Z) = \inf_{X\in\mathcal{X}} f_Z(X) = \inf_{X\in\mathcal{X}} \sup_{W\in\mathcal{E}'} \{\mathbb{E}_{\mathbb{P}}[0W] - (F_Z(X,\cdot))^*(W)\} = \inf_{X\in\mathcal{X}} \sup_{W\in\mathcal{E}'} K_Z(X, W).$$

This concludes the proof. □

The preceding lemma shows that, for every $Z \in \mathcal{X}'$, the identity $\sigma_{S^{-1}(\mathcal{A})}(Z) = \alpha(Z)$ is equivalent to the existence of a saddle value for the function K_Z. Unfortunately, the standard minimax theorems, see e.g. Fan [43], rely on compactness assumptions that do not hold in our setting. The remainder of this subsection is devoted to showing a number of situations where the identity holds or, equivalently, the above minimax problem has a solution.

THE LINEAR CASE

We start by proving the desired equality in the simple case where the impact map is given by the aggregated, or consolidated, capital position of all the d financial institutions. In this case, there is no restriction on the acceptance set.

Proposition 3.2.20. *Assume that $\mathcal{X}_i = \mathcal{E}$ for every $i \in \{1, \dots, d\}$. If $S(X) = \sum_{i=1}^d X_i$ for every $X \in \mathcal{X}$, then $\alpha = \sigma_{S^{-1}(\mathcal{A})}$.*

Proof. First of all, we show that for every $Z \in \mathcal{X}'_+$ we have

$$\sigma_{S^{-1}(\mathcal{A})}(Z) = \begin{cases} \sigma_{\mathcal{A}}(Z_1) & \text{if } Z_1 = \dots = Z_d, \\ -\infty & \text{otherwise.} \end{cases}$$

To see this, assume first that $\mathbb{P}(Z_i > Z_j) > 0$ for some distinct $i, j \in \{1, \dots, d\}$ and for every $n \in \mathbb{N}$ define a random vector $X^n \in \mathcal{X}$ by

$$X_k^n = \begin{cases} -n\mathbb{1}_{\{Z_i > Z_j\}} & \text{if } k = i, \\ n\mathbb{1}_{\{Z_i > Z_j\}} & \text{if } k = j, \\ 0 & \text{otherwise.} \end{cases}$$

Since $S(X^n) = 0 \in \mathcal{A}$ for every $n \in \mathbb{N}$, we clearly have

$$\sigma_{S^{-1}(\mathcal{A})}(Z) \leq \inf_{n \in \mathbb{N}} \mathbb{E}_{\mathbb{P}}[\langle X^n, Z \rangle] = \inf_{n \in \mathbb{N}} n\mathbb{E}_{\mathbb{P}}[\mathbb{1}_{\{Z_i > Z_j\}}(Z_j - Z_i)] = -\infty.$$

Next, assume that $Z_1 = \dots = Z_d$ and note that, in this case, we have

$$\sigma_{S^{-1}(\mathcal{A})}(Z) = \inf_{X \in S^{-1}(\mathcal{A})} \mathbb{E}_{\mathbb{P}}[S(X)Z_1] = \sigma_{\mathcal{A}}(Z_1).$$

This proves the above claim. Now, for every $Z \in \mathcal{X}'_+$ note that

$$\alpha(Z) = \sup_{W \in \text{bar}(\mathcal{A})} \left\{ \sigma_{\mathcal{A}}(W) + \inf_{X \in \mathcal{X}} \mathbb{E}_{\mathbb{P}} \left[\sum_{i=1}^d X_i(Z_i - W) \right] \right\} = \begin{cases} \sigma_{\mathcal{A}}(Z_1) & \text{if } Z_1 = \dots = Z_d \in \text{bar}(\mathcal{A}), \\ -\infty & \text{otherwise.} \end{cases}$$

This yields the desired assertion. □

THE CONIC CASE

Next, we deal with the case where S is positively homogeneous and $\mathcal{A}$ is a cone. In this case, we first show that α is given by a suitable indicator function and provide a general sufficient condition for the equality between $\sigma_{S^{-1}(\mathcal{A})}$ and α. At a later stage, we apply this general condition to a variety of concrete situations.

Lemma 3.2.21. *Assume that S is positively homogeneous and $\mathcal{A}$ is a cone. Then, we have $\alpha = -\delta_{\mathcal{G}}$ for*

$$\mathcal{G} := \{Z \in \mathcal{X}'_+ \ : \ \exists W \in \text{bar}(\mathcal{A}) \ : \ \mathbb{E}_{\mathbb{P}}[\langle X, Z \rangle] \geq \mathbb{E}_{\mathbb{P}}[S(X)W], \ \forall X \in \mathcal{X}\}.$$

Proof. Clearly, for every $Z \in \mathcal{G}$ there exists $W_Z \in \text{bar}(\mathcal{A})$ such that

$$\inf_{X \in \mathcal{X}} \{\mathbb{E}_{\mathbb{P}}[\langle X, Z \rangle] - \mathbb{E}_{\mathbb{P}}[S(X)W_Z]\} = \mathbb{E}_{\mathbb{P}}[\langle 0, Z \rangle] - \mathbb{E}_{\mathbb{P}}[S(0)W_Z] = 0.$$

As a result, for every $Z \in \mathcal{G}$ we have $0 \geq \alpha(Z) \geq \sigma_{\mathcal{A}}(W_Z) + 0 = 0$, showing that $\alpha(Z) = 0$. Now, fix $Z \in \mathcal{X}' \setminus \mathcal{G}$ and observe that, for every $W \in \text{bar}(\mathcal{A})$, we find $X_W \in \mathcal{X}$ such that $\mathbb{E}_{\mathbb{P}}[\langle X_W, Z \rangle] < \mathbb{E}_{\mathbb{P}}[S(X_W)W]$. Then,

$$\begin{aligned} \inf_{X \in \mathcal{X}} \{\mathbb{E}_{\mathbb{P}}[\langle X, Z \rangle] - \mathbb{E}_{\mathbb{P}}[S(X)W]\} &\leq \inf_{n \in \mathbb{N}} \{\mathbb{E}_{\mathbb{P}}[\langle nX_W, Z \rangle] - \mathbb{E}_{\mathbb{P}}[S(nX_W)W]\} \\ &= \inf_{n \in \mathbb{N}} \{n(\mathbb{E}_{\mathbb{P}}[\langle X_W, Z \rangle] - \mathbb{E}_{\mathbb{P}}[S(X_W)W])\} \\ &= -\infty. \end{aligned}$$

This implies that $\alpha(Z) = -\infty$ and concludes the proof. ☐

Lemma 3.2.22. *Assume that S is positively homogeneous and $\mathcal{A}$ is a cone. Moreover, assume that $S(\mathbf{e}) \in \mathbb{R}_+ \setminus \{0\}$ and that $\mathrm{bar}(\mathcal{A}) \cap \{W \in L^1(\mathbb{P}) \,:\, \|W\|_1 \leq 1\}$ is $\sigma(\mathcal{E}', \mathcal{E})$-compact. Then, $\sigma_{S^{-1}(\mathcal{A})} = \alpha$.*

Proof. Recall that $\sigma_{S^{-1}(\mathcal{A})} = \alpha$ holds if and only if α is $\sigma(\mathcal{X}', \mathcal{X})$-upper semicontinuous. Hence, by Lemma 3.2.21, it suffices to show that $\mathcal{G}$ is $\sigma(\mathcal{X}', \mathcal{X})$-closed. To this effect, take a net $(Z_\gamma) \subset \mathcal{G}$ converging to some $Z \in \mathcal{X}'$ in the topology $\sigma(\mathcal{X}', \mathcal{X})$. Note that $Z \in \mathcal{X}'_+$. By definition of $\mathcal{G}$, for each γ we find $W_\gamma \in \mathrm{bar}(\mathcal{A})$ such that

$$\mathbb{E}_{\mathbb{P}}[\langle X, Z_\gamma \rangle] \geq \mathbb{E}_{\mathbb{P}}[S(X) W_\gamma]$$

for every $X \in \mathcal{X}$. To establish the desired closedness, it is enough to show that (W_γ) admits a subnet that converges to some element of $\mathrm{bar}(\mathcal{A})$ in the topology $\sigma(\mathcal{E}', \mathcal{E})$. Note that $\mathrm{bar}(\mathcal{A}) = \{\sigma_{\mathcal{A}} \geq 0\}$ by conicity of $\mathcal{A}$, showing that $\mathrm{bar}(\mathcal{A})$ is $\sigma(\mathcal{E}', \mathcal{E})$-closed. Since $\mathrm{bar}(\mathcal{A}) \subset \mathcal{E}'_+$, we see that

$$\mathbb{E}_{\mathbb{P}}[\langle \mathbf{e}, Z_\gamma \rangle] \geq \mathbb{E}_{\mathbb{P}}[S(\mathbf{e}) W_\gamma] \geq 0,$$

or equivalently

$$\frac{\mathbb{E}_{\mathbb{P}}[\langle \mathbf{e}, Z_\gamma \rangle]}{S(\mathbf{e})} \geq \mathbb{E}_{\mathbb{P}}[W_\gamma] \geq 0,$$

for every γ. Since $\mathbb{E}_{\mathbb{P}}[\langle \mathbf{e}, Z_\gamma \rangle] \to \mathbb{E}_{\mathbb{P}}[\langle \mathbf{e}, Z \rangle]$, the net (W_γ) is bounded in $L^1(\mathbb{P})$ and, hence, by using the compactness assumption, it admits a convergent subnet in the topology $\sigma(\mathcal{E}', \mathcal{E})$. In view of the $\sigma(\mathcal{E}', \mathcal{E})$-closedness of $\mathrm{bar}(\mathcal{A})$, we infer that the limit belongs to $\mathrm{bar}(\mathcal{A})$. This concludes the proof. ☐

The next proposition describes a number of situations where we can ensure the above compactness condition and, thus, we can establish that $\sigma_{S^{-1}(\mathcal{A})} = \alpha$.

Proposition 3.2.23. *Assume that S is positively homogeneous and $\mathcal{A}$ is a cone. Moreover, assume that $S(\mathbf{e}) \in \mathbb{R}_+ \setminus \{0\}$. Then, $\sigma_{S^{-1}(\mathcal{A})} = \alpha$ in each of the following cases:*

(i) *Ω is finite.*

(ii) *$\mathcal{A}$ is polyhedral, i.e. there exist $W_1, \ldots, W_n \in \mathcal{E}'_+$ and $a_1, \ldots, a_n \in \mathbb{R}$ such that*

$$\mathcal{A} = \bigcap_{i=1}^{n} \{U \in \mathcal{E} \,:\, \mathbb{E}_{\mathbb{P}}[U W_i] \geq a_i\}.$$

(iii) *$\mathcal{A}$ is induced by Expected Shortfall, i.e. there exists $\lambda \in (0,1)$ such that*

$$\mathcal{A} = \{U \in \mathcal{E} \,:\, \mathrm{ES}_\lambda(U) \leq 0\}.$$

(iv) *$(\mathcal{E}, \mathcal{E}') = (L^\infty(\mathbb{P}), L^1(\mathbb{P}))$ and $\mathcal{A}$ is induced by Weighted Value at Risk, i.e. there exists a probability measure μ on $[0,1]$ such that*

$$\mathcal{A} = \left\{ U \in \mathcal{E} \,:\, \int_{[0,1]} \mathrm{ES}_\lambda(U)\, d\mu(\lambda) \leq 0 \right\}.$$

Proof. *(i)* In the case that Ω is finite, the space $\mathcal{E}'$ is finite dimensional and the compactness condition in Lemma 3.2.22 is clearly satisfied because $\mathrm{bar}(\mathcal{A}) = \{\sigma_{\mathcal{A}} \geq 0\}$ is always $\sigma(\mathcal{E}', \mathcal{E})$-closed.

(ii) If $\mathcal{A}$ is polyhedral, then it is easy to see that $\mathrm{bar}(\mathcal{A})$ is a finitely-generated convex cone, i.e. there exist $W_1, \ldots, W_n \in \mathcal{E}'_+$ such that

$$\mathrm{bar}(\mathcal{A}) = \left\{ \sum_{i=1}^{n} \lambda_i W_i \,:\, \lambda_1, \ldots, \lambda_n \in [0, \infty) \right\}.$$

Note that for all $\lambda_1, \dots, \lambda_n \in [0,\infty)$ we have

$$\left\| \sum_{i=1}^{n} \lambda_i W_i \right\|_1 = \sum_{i=1}^{n} \lambda_i \|W_i\|_1.$$

As a result, $\mathrm{bar}(\mathcal{A}) \cap \{W \in L^1(\mathbb{P}) \,:\, \|W\|_1 \leq 1\}$ is easily seen to be $\sigma(\mathcal{E}',\mathcal{E})$-compact and we can apply Lemma 3.2.22 to get the desired result.

(iii) If $\mathcal{A}$ is induced by Expected Shortfall as in Example 3.2.17, then

$$\mathrm{bar}(\mathcal{A}) = \left\{ W \in \mathcal{E}'_+ \,:\, W \leq \frac{1}{\lambda}\mathbb{E}_{\mathbb{P}}[W] \right\}.$$

As a result, we easily see that

$$\mathrm{bar}(\mathcal{A}) \cap \{W \in L^1(\mathbb{P}) \,:\, \|W\|_1 \leq 1\} \subset \{W \in L^\infty(\mathbb{P})_+ \,:\, W \leq \lambda^{-1}\}.$$

Since the set $\mathrm{bar}(\mathcal{A}) \cap \{W \in L^1(\mathbb{P}) \,:\, \|W\|_1 \leq 1\}$ is $\sigma(L^\infty(\mathbb{P}), L^1(\mathbb{P}))$-closed, it follows from the Banach-Alaoglu Theorem that it is even $\sigma(L^\infty(\mathbb{P}), L^1(\mathbb{P}))$-compact. As $\mathcal{E} \subset L^1(\mathbb{P})$, we automatically have $\sigma(\mathcal{E}',\mathcal{E})$-compactness and we conclude by applying Lemma 3.2.22.

(iv) If $\mathcal{A}$ is induced by Weighted Value at Risk, then $\mathrm{bar}(\mathcal{A}) \cap \{W \in L^1(\mathbb{P}) \,:\, \|W\|_1 \leq 1\}$ is norm closed in $L^1(\mathbb{P})$ by Exercise 4.6.6 [49], and hence $\sigma(L^1(\mathbb{P}), L^\infty(\mathbb{P}))$-closed. It follows from the Banach-Alaoglu Theorem that it is even $\sigma(L^1(\mathbb{P}), L^\infty(\mathbb{P}))$-compact and we may conclude by applying Lemma 3.2.22. □

THE CASE WHERE THE IMAGE OF S INTERSECTS THE INTERIOR OF $\mathcal{A}$

As a final step, we follow Rockafellar [84] to establish the identity $\sigma_{S^{-1}(\mathcal{A})} = \alpha$ under a suitable interiority condition, which also appears in Armenti et al. [7] and Biagini et al. [20].

Proposition 3.2.24. *(i) If there exists $X^* \in \mathcal{X}$ such that $S(X^*)$ belongs to the $\sigma(\mathcal{E},\mathcal{E}')$-interior of $\mathcal{A}$, then $\alpha = \sigma_{S^{-1}(\mathcal{A})}$.*

(ii) Assume that $\mathcal{E}'$ is the norm dual of $\mathcal{E}$. If there exists $X^* \in \mathcal{X}$ such that $S(X^*)$ belongs to the norm interior of $\mathcal{A}$, then $\alpha = \sigma_{S^{-1}(\mathcal{A})}$.

Proof. *(i)* By assumption, we find a $\sigma(\mathcal{E},\mathcal{E}')$-neighborhood of zero $\mathcal{U} \subset \mathcal{E}$ such that $S(X^*) + \mathcal{U} \subset \mathcal{A}$. Now, fix an element $Z \in \mathcal{X}'$ and define a map $\psi_Z : \mathcal{E} \to [-\infty,\infty]$ by setting

$$\psi_Z(U) := \inf_{X \in \mathcal{X}} F_Z(X,U).$$

Here, we have adopted the notation introduced in the proof of Lemma 3.2.19. It is easy to verify that F_Z is jointly convex and, hence, ψ_Z is convex. Note that

$$\psi_Z(U) \leq F_Z(X^*,U) = \mathbb{E}_{\mathbb{P}}[\langle X^*, Z\rangle]$$

for every $U \in \mathcal{U}$, so that ψ_Z is bounded from above on $\mathcal{U}$. In view of Lemma 3.2.19, the desired assertion follows from Theorem 17 in Rockafellar [84] (by taking $\varphi = \psi_Z$ and $F = F_Z$ in the notation of that result).

(ii) Since the norm topology on $\mathcal{E}$ is compatible with our bilinear form on $\mathcal{E} \times \mathcal{E}'$ under the assumption that $\mathcal{E}'$ is the norm dual of $\mathcal{E}$, we can repreat the same argument as in *(i)* by exploiting the fact that Theorem 17 in [84] holds under any compatible topology. □

3.3 "First aggregate, then allocate"-type systemic risk measures

In this short section we turn to systemic risk measures of "first aggregate, then allocate" type and their dual representations. Throughout the section we fix an admissible impact map S and an admissible acceptance set $\mathcal{A}$.

3.3.1 THE SYSTEMIC RISK MEASURE $\widetilde{\rho}$

"First aggregate, then allocate"-type systemic risk measures are defined as follows.

Definition 3.3.1. We define a map $\widetilde{\rho} : \mathcal{X} \to [-\infty, \infty]$ by setting

$$\widetilde{\rho}(X) = \inf\{m \in \mathbb{R} \,:\, S(X) + m \in \mathcal{A}\}.$$

The difference with respect to ρ is that, instead of injecting capital into the system in order to reach an acceptable level of systemic risk, one looks at the minimum level of the chosen systemic risk indicator that ensures acceptability. In particular, if the impact function is expressed in monetary terms, then $\widetilde{\rho}(X)$ can be interpreted as a bail-out cost for the "aggregated position" $S(X)$. For a thorough presentation of this type of systemic risk measures we refer to the literature cited in the introduction.

In what follows, we exploit the fact that $\widetilde{\rho}$ can be expressed as the composition between the impact map and the standard cash-additive risk measure $\rho_{\mathcal{A}} : \mathcal{E} \to [-\infty, \infty]$ given by

$$\rho_{\mathcal{A}}(X) := \inf\{m \in \mathbb{R} \,:\, X + m \in \mathcal{A}\}.$$

The next result records the key properties of $\widetilde{\rho}$. In particular, differently from the systemic risk measure ρ, we show that $\widetilde{\rho}$ is always lower semicontinuous under our standing assumptions on the impact map and the acceptance set.

Proposition 3.3.2. *The systemic risk measure $\widetilde{\rho}$ is convex, $\sigma(\mathcal{X}, \mathcal{X}')$-lower semicontinuous, and satisfies $\widetilde{\rho}(0) \leq 0$. Moreover, $\widetilde{\rho}$ is proper if and only if $\widetilde{\rho}(0) > -\infty$ if and only if $\mathcal{A} \cap (-\mathbb{R}_+) \neq -\mathbb{R}_+$.*

Proof. Convexity is clear by composition. To show lower semicontinuity, note that $\rho_{\mathcal{A}}$ is $\sigma(\mathcal{E}, \mathcal{E}')$-lower semicontinuous by the $\sigma(\mathcal{E}, \mathcal{E}')$-closedness of $\mathcal{A}$. Now, take $r \in \mathbb{R}$ and note that

$$\{X \in \mathcal{X} \,:\, \widetilde{\rho}(X) \leq r\} = S^{-1}(\{U \in \mathcal{E} \,:\, \rho_{\mathcal{A}}(U) \leq r\}).$$

Following the argument in the proof of Proposition 3.2.3 we can show that the above set is $\sigma(\mathcal{X}, \mathcal{X}')$-closed, which delivers the desired lower semicontinuity. To show properness, observe first that $\widetilde{\rho}(0) \leq 0$ because $S(0) = 0 \in \mathcal{A}$. The above equivalence can now be established as in the proof of Proposition 3.2.4. □

3.3.2 THE DUAL REPRESENTATION OF $\widetilde{\rho}$

The purpose of this subsection is to derive a dual representation of $\widetilde{\rho}$ and to compare it with the dual representation of ρ. In this case, the acceptability test is performed on $S(X)$ and the acceptance set is $\mathcal{A}$. This suggests to rely on the dual representation of $\rho_{\mathcal{A}}$ in order to achieve in a straightforward way the desired dual representation of $\widetilde{\rho}$. The following maps are the fundamental ingredients of the desired representation.

Definition 3.3.3. We define two maps $\widetilde{\alpha}, \widetilde{\alpha}^+ : \mathcal{X}' \to [-\infty, +\infty]$ by setting

$$\widetilde{\alpha}(Z) := \sup_{W \in \mathrm{bar}(\mathcal{A}),\, \mathbb{E}_{\mathbb{P}}[W]=1} \left\{\sigma_{\mathcal{A}}(W) + \inf_{X \in \mathcal{X}} \{\mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle] - \mathbb{E}_{\mathbb{P}}[S(X)W]\}\right\},$$

$$\widetilde{\alpha}^+(Z) := \sup_{W \in \mathrm{bar}(\mathcal{A}) \cap (\mathcal{E}'_{++} \cup \{0\}),\, \mathbb{E}_{\mathbb{P}}[W]=1} \left\{\sigma_{\mathcal{A}}(W) + \inf_{X \in \mathcal{X}} \{\mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle] - \mathbb{E}_{\mathbb{P}}[S(X)W]\}\right\}.$$

Remark 3.3.4. The above maps belong to the class of maps $\widetilde{\alpha}_{\mathcal{K}} : \mathcal{X}' \to [-\infty, +\infty]$ defined by

$$\widetilde{\alpha}_{\mathcal{K}}(Z) := \sup_{W \in \mathcal{K},\, \mathbb{E}_{\mathbb{P}}[W]=1} \left\{\sigma_{\mathcal{A}}(W) + \inf_{X \in \mathcal{X}} \{\mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle] - \mathbb{E}_{\mathbb{P}}[S(X)W]\}\right\},$$

where $\mathcal{K}$ is a convex cone in $\mathrm{bar}(\mathcal{A})$ such that $\lambda\mathcal{K} + (1-\lambda)\,\mathrm{bar}(\mathcal{A}) \subset \mathcal{K}$ for every $\lambda \in [0, 1]$; see also Remark 3.2.12. This will allow us to prove properties for $\widetilde{\alpha}$ and $\widetilde{\alpha}^+$ simultaneously. In fact, all properties of $\widetilde{\alpha}$ and $\widetilde{\alpha}^+$ we will consider are shared by the entire class.

Before we establish the desired dual representation we highlight some relevant properties of the above maps and point out their relationship with the penalty functions α and α^+. Recall that we denote by $\mathrm{dom}(\widetilde{\alpha})$ the domain of finiteness of $\widetilde{\alpha}$ (similarly for $\widetilde{\alpha}^+$), and by cl the closure operator with respect to the topology $\sigma(\mathcal{X}', \mathcal{X})$.

Proposition 3.3.5. *The maps $\widetilde{\alpha}, \widetilde{\alpha}^+ : \mathcal{X}' \to [-\infty, \infty]$ satisfy the following properties (the statements about $\widetilde{\alpha}^+$ require that $\mathrm{bar}(\mathcal{A}) \cap \mathcal{E}'_{++} \neq \emptyset$):*

(i) $\widetilde{\alpha}$ and $\widetilde{\alpha}^+$ take values in the interval $[-\infty, 0]$.

(ii) $\widetilde{\alpha}$ and $\widetilde{\alpha}^+$ are concave.

(iii) $\mathrm{dom}(\widetilde{\alpha}^+) \subset \mathrm{dom}(\widetilde{\alpha}) \subset \mathrm{cl}(\mathrm{dom}(\widetilde{\alpha}^+)) \subset \mathcal{X}'_+$.

(iv) α is the smallest positively homogeneous map dominating $\widetilde{\alpha}$, i.e. for every $Z \in \mathcal{X}'$

$$\alpha(Z) = \sup_{\lambda > 0} \frac{\widetilde{\alpha}(\lambda Z)}{\lambda}.$$

(v) α^+ is the smallest positively homogeneous map dominating $\widetilde{\alpha}^+$, i.e. for every $Z \in \mathcal{X}'$

$$\alpha^+(Z) = \sup_{\lambda > 0} \frac{\widetilde{\alpha}^+(\lambda Z)}{\lambda}.$$

Proof. *(i)-(ii), (iv)-(v)* Let $\mathcal{K} \subset \mathrm{bar}(\mathcal{A})$ be a convex cone as in Remark 3.3.4. It is clear that

$$\alpha_{\mathcal{K}}(Z) = \sup_{\lambda > 0} \frac{\widetilde{\alpha}_{\mathcal{K}}(\lambda Z)}{\lambda}$$

for every $Z \in \mathcal{X}'$. In particular, $\widetilde{\alpha}_{\mathcal{K}} \leq \alpha_{\mathcal{K}}$. It follows from the proof of Proposition 3.2.14 that $\widetilde{\alpha}_{\mathcal{K}}$ takes value into $[-\infty, 0]$. Moreover, the proof of the concavity of $\alpha_{\mathcal{K}}$ in that result can be repeated to establish the concavity of $\widetilde{\alpha}_{\mathcal{K}}$. The desired assertions follow by taking $\mathcal{K} = \mathrm{bar}(\mathcal{A})$ and $\mathcal{K} = \mathrm{bar}(\mathcal{A}) \cap (\mathcal{E}'_{++} \cup \{0\})$.

(iii) The assertion can be proved by repeating the proof of the corresponding statement in Proposition 3.2.14. □

We record the announced dual representation of $\widetilde{\rho}$ in the next result.

Theorem 3.3.6. *(i) If $\widetilde{\rho}$ is proper, then we have*

$$\widetilde{\rho}(X) = \sup_{Z \in \mathcal{X}'_+} \{\widetilde{\alpha}(Z) - \mathbb{E}_{\mathbb{P}}[\langle X, Z \rangle]\}$$

for every $X \in \mathcal{X}$. The supremum can be restricted to $\mathcal{X}'_{++}$ provided that $\mathrm{dom}(\widetilde{\alpha}) \cap \mathcal{X}'_{++} \neq \emptyset$.

(ii) Assume that $\mathrm{bar}(\mathcal{A}) \cap \mathcal{E}'_{++} \neq \emptyset$. If $\widetilde{\rho}$ is proper, then we have

$$\widetilde{\rho}(X) = \sup_{Z \in \mathcal{X}'_+} \{\widetilde{\alpha}^+(Z) - \mathbb{E}_{\mathbb{P}}[\langle X, Z \rangle]\}$$

for every $X \in \mathcal{X}$. The supremum can be restricted to $\mathcal{X}'_{++}$ provided that $\mathrm{dom}(\widetilde{\alpha}^+) \cap \mathcal{X}'_{++} \neq \emptyset$.

Proof. Let $\mathcal{K} \subset \mathrm{bar}(\mathcal{A})$ be a convex cone as in Remark 3.3.4. Note that the dual representation in Proposition A.1.10 applied to $\mathcal{A}$ yields

$$\mathcal{A} = \bigcap_{W \in \mathcal{K}} \{U \in \mathcal{E} \,:\, \mathbb{E}_{\mathbb{P}}[UW] \geq \sigma_{\mathcal{A}}(W)\} = \bigcap_{W \in \mathcal{K}, \mathbb{E}_{\mathbb{P}}[W]=1} \{U \in \mathcal{E} \,:\, \mathbb{E}_{\mathbb{P}}[UW] \geq \sigma_{\mathcal{A}}(W)\}, \tag{3.8}$$

where we used the positive homogeneity of $\sigma_{\mathcal{A}}$ (together with the fact that $\mathcal{K} \subset \mathcal{E}'_+$). As a result, for every $U \in \mathcal{E}$ we get

$$\rho_{\mathcal{A}}(U) = \sup_{W \in \mathcal{K}, \mathbb{E}_{\mathbb{P}}[W]=1} \{\sigma_{\mathcal{A}}(W) - \mathbb{E}_{\mathbb{P}}[UW]\}.$$

Using the notation introduced in the proof of Theorem 3.2.13, we immediately get

$$\begin{aligned}\widetilde{\rho}(X) &= \sup_{W\in\mathcal{K},\,\mathbb{E}_{\mathbb{P}}[W]=1}\{\sigma_{\mathcal{A}}(W)-\mathbb{E}_{\mathbb{P}}[S(X)W]\}\\ &= \sup_{W\in\mathcal{K},\,\mathbb{E}_{\mathbb{P}}[W]=1}\ \sup_{Z\in\mathcal{X}'_+}\{\sigma_{\mathcal{A}}(W)-\mathbb{E}_{\mathbb{P}}[\langle X,Z\rangle]+(\varphi_W)^\bullet(Z)\}\\ &= \sup_{Z\in\mathcal{X}'_+}\ \sup_{W\in\mathcal{K},\,\mathbb{E}_{\mathbb{P}}[W]=1}\{\sigma_{\mathcal{A}}(W)-\mathbb{E}_{\mathbb{P}}[\langle X,Z\rangle]+(\varphi_W)^\bullet(Z)\}\\ &= \sup_{Z\in\mathcal{X}'_+}\{\widetilde{\alpha}_{\mathcal{K}}(Z)-\mathbb{E}_{\mathbb{P}}[\langle X,Z\rangle]\}\end{aligned}$$

for every $X\in\mathcal{X}$. If, in addition, $\mathrm{dom}(\widetilde{\alpha}_{\mathcal{K}})\cap\mathcal{X}'_{++}\neq\emptyset$, then we get

$$\widetilde{\rho}(X)=\sup_{Z\in\mathcal{X}'_{++}}\{\widetilde{\alpha}_{\mathcal{K}}(Z)-\mathbb{E}_{\mathbb{P}}[\langle X,Z\rangle]\}$$

for every $X\in\mathcal{X}$ by the same argument used to reduce the domain of the supremum in the proof of Theorem 3.2.8. The desired assertions now follow by taking $\mathcal{K}=\mathrm{bar}(\mathcal{A})$ and $\mathcal{K}=\mathrm{bar}(\mathcal{A})\cap(\mathcal{E}'_{++}\cup\{0\})$. ☐

Remark 3.3.7. (i) As in Remark 3.2.9, we highlight the link between the dual representation in Theorem 3.3.6 and the standard Fenchel-Moreau representation. We claim that, if $\widetilde{\rho}$ is proper, then

$$\widetilde{\rho}^*(Z)=-\,\mathrm{usc}(\widetilde{\alpha})(-Z)=-\,\mathrm{usc}(\widetilde{\alpha}^+)(-Z)$$

for every $Z\in\mathcal{X}'$ (where the last equality holds provided that $\mathrm{bar}(\mathcal{A})\cap\mathcal{E}'_{++}\neq\emptyset$). Here, we have denoted by $\mathrm{usc}(\widetilde{\alpha})$ the $\sigma(\mathcal{X}',\mathcal{X})$-upper semicontinuous hull of $\widetilde{\alpha}$ (similarly for $\widetilde{\alpha}^+$). To see this, note first that

$$\widetilde{\rho}(X)=\sup_{Z\in\mathcal{X}'}\{\widetilde{\alpha}(Z)-\mathbb{E}_{\mathbb{P}}[\langle X,Z\rangle]\}=\sup_{Z\in\mathcal{X}'}\{\mathrm{usc}(\widetilde{\alpha})(Z)-\mathbb{E}_{\mathbb{P}}[\langle X,Z\rangle]\}$$

for every $X\in\mathcal{X}$. The left-hand side equality holds because $\widetilde{\alpha}=-\infty$ outside $\mathcal{X}'_+$ by Proposition 3.3.5. The right-hand side equality follows from Theorem 2.3.1 in Zălinescu [95]. Since $\mathrm{usc}(\widetilde{\alpha})$ is concave and $\sigma(\mathcal{X}',\mathcal{X})$-upper semicontinuous, the desired claim is a consequence of the Fenchel-Moreau Theorem. The argument for $\widetilde{\alpha}^+$ is identical.

(ii) The dual elements in the above representation can be identified with d-dimensional vectors of probability measures on $(\Omega,\mathcal{F})$ that are absolutely continuous (or equivalent) with respect to $\mathbb{P}$ up to a normalizing vector that collects their expectations. This allows to express the above representation in terms of probability measures. Indeed, for every $w\in\mathbb{R}^d_+$ define

$$\mathcal{Q}^w(\mathbb{P}):=\{\mathbb{Q}\in\mathcal{Q}(\mathbb{P})\,:\,\mathbb{Q}_i=\mathbb{P}\text{ if }w_i=0,\ \forall i\in\{1,\dots,d\}\},\quad \mathcal{Q}^w_e(\mathbb{P})=\mathcal{Q}_e(\mathbb{P})\cap\mathcal{Q}^w(\mathbb{P}),$$

where we have used the notation from Remark 3.2.9. Then, if $\widetilde{\rho}$ is proper, we easily see that

$$\widetilde{\rho}(X)=\sup_{w\in\mathbb{R}^d_+,\,\mathbb{Q}\in\mathcal{Q}^w(\mathbb{P}),\,\frac{d\mathbb{Q}}{d\mathbb{P}}\in\mathcal{X}'}\left\{\widetilde{\alpha}\left(w_1\frac{d\mathbb{Q}_1}{d\mathbb{P}},\dots,w_d\frac{d\mathbb{Q}_d}{d\mathbb{P}}\right)-\sum_{i=1}^d w_i\mathbb{E}_{\mathbb{Q}_i}[X_i]\right\}$$

for every $X\in\mathcal{X}$. We can replace $\mathcal{Q}^w(\mathbb{P})$ by $\mathcal{Q}^w_e(\mathbb{P})$ in the above supremum provided that $\mathrm{dom}(\widetilde{\alpha})\cap\mathcal{X}'_{++}\neq\emptyset$. The same holds with $\widetilde{\alpha}^+$ instead of $\widetilde{\alpha}$ (provided that $\mathrm{bar}(\mathcal{A})\cap\mathcal{E}'_{++}\neq\emptyset$).

The condition $\mathrm{dom}(\widetilde{\alpha})\cap\mathcal{X}'_{++}\neq\emptyset$ is needed to restrict the domain in the above dual representation to strictly-positive dual elements (similarly for $\widetilde{\alpha}^+$). We conclude this section by providing a sufficient condition for this to hold; see also Proposition 3.2.10.

Proposition 3.3.8. *Assume that $\mathcal{X}_i=\mathcal{E}$ for every $i\in\{1,\dots,d\}$. Moreover, suppose that $\mathrm{bar}(\mathcal{A})\cap\mathcal{E}'_{++}\neq\emptyset$ and there exist $a\in(0,\infty)$ and $b\in\mathbb{R}$ such that*

$$S(X)\le a\sum_{i=1}^d X_i+b$$

for every $X \in \mathcal{X}$. Then, $\mathrm{dom}(\widetilde{\alpha}^+) \cap \mathcal{X}'_{++} \neq \emptyset$ *(and, a fortiori,* $\mathrm{dom}(\widetilde{\alpha}) \cap \mathcal{X}'_{++} \neq \emptyset$*).*

Proof. Take $W \in \mathrm{bar}(\mathcal{A}) \cap \mathcal{E}'_{++}$ and note that we can always assume that $\mathbb{E}_{\mathbb{P}}[W] = 1$ by conicity of $\mathrm{bar}(\mathcal{A})$. Setting $Z = (aW, \dots, aW) \in \mathcal{X}'_{++}$, we easily see that

$$\widetilde{\alpha}(Z) \geq \widetilde{\alpha}^+(Z) \geq \sigma_{\mathcal{A}}(W) + \inf_{X \in \mathcal{X}} \{\mathbb{E}_{\mathbb{P}}[\langle X, Z\rangle] - \mathbb{E}_{\mathbb{P}}[S(X)W]\} \geq \sigma_{\mathcal{A}}(W) - b\mathbb{E}_{\mathbb{P}}[W] > -\infty.$$

This delivers the desired assertion. □

3.4 Risk measures based on univariate utility functions

In this final section we provide a simple proof of the dual representation of shortfall risk measures, see Theorem 4.115 in Föllmer and Schied [49], that uses our general strategy to obtain dual representations. For ease of comparison, we focus on bounded positions.

Throughout the entire section we fix a nonconstant, concave, increasing function $u : \mathbb{R} \to \mathbb{R}$, which is interpreted as a standard von Neumann-Morgenstern utility function. We fix $u_0 \in \mathbb{R}$ such that $u(x) > u_0$ for some $x \in \mathbb{R}$ and define a map $\rho_u : L^\infty(\mathbb{P}), \to [-\infty, \infty]$ by

$$\rho_u(X) := \inf\{m \in \mathbb{R} \,:\, \mathbb{E}_{\mathbb{P}}[u(X+m)] \geq u_0\}.$$

Theorem 3.4.1. *The risk measure ρ_u is convex and $\sigma(L^\infty(\mathbb{P}), L^1(\mathbb{P}))$-lower semicontinuous. Moreover,*

$$\rho_u(X) = \sup_{\mathbb{Q} \ll \mathbb{P}} \left\{ \mathbb{E}_{\mathbb{Q}}[-X] + \sup_{\lambda > 0} \left\{ \frac{1}{\lambda} \left(u_0 + \mathbb{E}_{\mathbb{P}} \left[u^\bullet \left(\lambda \frac{d\mathbb{Q}}{d\mathbb{P}} \right) \right] \right) \right\} \right\}$$

for every $X \in L^\infty(\mathbb{P})$,.

Proof. It is well-known that ρ_u is convex and $\sigma(L^\infty(\mathbb{P}), L^1(\mathbb{P}))$-lower semicontinuous. To establish the above representation, note that ρ_u can be viewed as a "first allocate, then aggregate"-type systemic risk measure corresponding to the case $d = 1$ and the specifications

$$(\mathcal{X}, \mathcal{X}') = (L^\infty(\mathbb{P}), L^1(\mathbb{P})), \quad (\mathcal{E}, \mathcal{E}') = (L^\infty(\mathbb{P}), L^1(\mathbb{P})),$$

$$S(X) = u(X), \quad \mathcal{A} = \{U \in L^\infty(\mathbb{P}) \,:\, \mathbb{E}_{\mathbb{P}}[U] \geq u_0\}.$$

First of all, note that $\mathrm{bar}(\mathcal{A}) = \mathbb{R}_+$ and $\sigma_{\mathcal{A}}(\lambda) = \lambda u_0$ for every $\lambda \in \mathbb{R}_+$. Since $\mathrm{bar}(\mathcal{A}) \cap \mathcal{E}'_{++}$ is nonempty, we can work with α^+; see Definition 3.2.11. It follows from Proposition 3.2.15 that

$$\begin{aligned} \alpha^+(Z) &= \sup_{W \in \mathrm{bar}(\mathcal{A}) \cap \mathcal{E}'_{++}} \left\{ \sigma_{\mathcal{A}}(W) + \mathbb{E}\left[u^\bullet\left(\frac{Z}{W}\right) W \right] \right\} \\ &= \sup_{\lambda > 0} \left\{ \lambda u_0 + \mathbb{E}\left[u^\bullet\left(\frac{Z}{\lambda}\right) \lambda \right] \right\} \\ &= \sup_{\lambda > 0} \left\{ \frac{1}{\lambda}(u_0 + \mathbb{E}_{\mathbb{P}}[u^\bullet(\lambda Z)]) \right\} \end{aligned}$$

for every nonzero $Z \in \mathcal{X}'_+$. The representation of $S^{-1}(\mathcal{A})$ in Theorem 3.2.13 yields

$$S^{-1}(\mathcal{A}) = \bigcap_{Z \in \mathcal{X}'_+ \setminus \{0\}} \{X \in \mathcal{X} \,:\, \mathbb{E}_{\mathbb{P}}[XZ] \geq \alpha^+(Z)\} = \bigcap_{\mathbb{Q} \ll \mathbb{P}} \left\{ X \in \mathcal{X} \,:\, \mathbb{E}_{\mathbb{Q}}[X] \geq \alpha^+\left(\frac{d\mathbb{Q}}{d\mathbb{P}}\right) \right\},$$

where we used that $\mathrm{dom}(\alpha^+) \subset \mathcal{X}'_+$ and that α^+ is positively homogeneous; see Proposition 3.2.14. It remains to observe that

$$\rho_u(X) = \inf\{m \in \mathbb{R} \,:\, X + m \in S^{-1}(\mathcal{A})\}$$

for every $X \in \mathcal{X}$. □

APPENDIX A

BASICS IN ANALYSIS AND PROBABILITY

This appendix is a brief collection of definitions and results from Mathematical Analysis and Probability that are used throughout the thesis. It does not mean to be an exhaustive review of the involved topics, but just a quick way to refresh some standard knowledge. For this reason, we do not provide proofs. The interested reader will find proofs and details e.g. in Aliprantis and Border [1] and Zalinescu [95]. For more details about Orlicz spaces, we refer to Rao and Ren [83] and Meyer-Nieberg[73]. For the necessary notation, we refer to the List of Symbols on page 117.

A.1 CONVEX ANALYSIS

Let $\mathcal{X}$ be a locally convex Hausdorff topological vector space such that $\mathcal{X}'$ is its topological dual (or a space identified with it).

Remark A.1.1. In Chapters 1 and 3, we encounter the following situation. Let $\mathcal{X}, \mathcal{Y}$ be nonzero real vector spaces and let $\langle \cdot, \cdot \rangle : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ be a bilinear mapping. The weakest linear topology on $\mathcal{X}$ with respect to which the map $\langle \cdot, Y \rangle$ is continuous for every $Y \in \mathcal{Y}$ is denoted by $\sigma(\mathcal{X}, \mathcal{Y})$. The bilinear mapping is said to be separating for $\mathcal{X}$ if for all nonzero $X \in \mathcal{X}$ there exists $Y \in \mathcal{Y}$ such that $\langle X, Y \rangle \neq 0$. In this case, the topology $\sigma(\mathcal{X}, \mathcal{Y})$ is both Hausdorff and locally convex, and $\mathcal{Y}$ is identified with the topological dual of $\mathcal{X}$ equipped with $\sigma(\mathcal{X}, \mathcal{Y})$. Similarly, $\mathcal{Y}$ can be equipped with the $\sigma(\mathcal{Y}, \mathcal{X})$ topology. The results contained in this appendix are repeatedly applied to this type of topological paired spaces.

A number of definitions follow. Note that for the first three we actually need only the vector structure on $\mathcal{X}$. First, we define the types of sets and functions that we cope with all along the thesis.

Definition A.1.2. A subset $\mathcal{C} \subset \mathcal{X}$ is said:

(i) *Convex* if $t\mathcal{C} + (1-t)\mathcal{C} \subset \mathcal{C}$ for every $t \in [0, 1]$.

(ii) *Conic* (or a *cone*) if $t\mathcal{C} \subset \mathcal{C}$ for every $t \geq 0$.

(iii) *Closed under addition* if $\mathcal{C} + \mathcal{C} \subset \mathcal{C}$.

(iv) *Star shaped about* 0 if $t\mathcal{C} \subset \mathcal{C}$ for every $t \in [0, 1]$.

Definition A.1.3. Let $\mathcal{C} \subset \mathcal{X}$ be a nonempty set and consider a function $f : \mathcal{C} \to \overline{\mathbb{R}}$.

(i) f is said *proper* if there exists $X \in \mathcal{C}$ such that $f(X) \in \mathbb{R}$ and f never attains the value $-\infty$.

(ii) If $\mathcal{C}$ is convex, f is said *convex* if $f(tX + (1-t)Y) \leq tf(X) + (1-t)f(Y)$ for every $X, Y \in \mathcal{C}$, $t \in [0, 1]$ (equivalently, if $\mathrm{epi}(f)$ is a convex set).

(iii) If $\mathcal{C}$ is convex, f is said *concave* if $-f$ is convex (equivalently, if $\mathrm{hypo}(f)$ is a convex set).

(iv) If $\mathcal{C}$ is convex, f is said *quasi convex* if $f(tX + (1-t)Y) \leq \max\{f(X), f(Y)\}$ for every $X, Y \in \mathcal{C}$, $t \in [0, 1]$ (equivalently, if $\{X \in \mathcal{C} \,:\, f(X) \leq m\}$ is a convex set for every $m \in \mathbb{R}$).

(v) If $\mathcal{C}$ is conic, f is said *positively homogeneous* or *conic* if $f(tX) = tf(X)$ for every $X \in \mathcal{C}$, $t \geq 0$ (equivalently, if $\mathrm{epi}(f)$ is a conic set and $f(0) = 0$).

(vi) If $\mathcal{C}$ is closed under addition, f is said *subadditive* if $f(X+Y) \leq f(X) + f(Y)$ for every $X, Y \in \mathcal{C}$ (equivalently, if $\mathrm{epi}(f)$ is closed under addition).

(vii) If $\mathcal{C}$ is closed under addition, f is said *superadditive* if $-f$ is subadditive (equivalently, if $\mathrm{hypo}(f)$ is closed under addition).

(viii) If $\mathcal{C}$ is a convex cone, f is said *sublinear* if it is subadditive and positively homogeneous.

(ix) If $\mathcal{C}$ is a convex cone, f is said *superlinear* if it is superadditive and positively homogeneous.

We define conic hulls of sets and functions. For other sets and functions obtained through a "conification" procedure, see Appendix B.

Definition A.1.4. The *conic hull* of a set $\mathcal{C} \subset \mathcal{X}$ is the smallest cone containing $\mathcal{C}$ and is denoted by $\mathrm{cone}(\mathcal{C})$, that is

$$\mathrm{cone}(\mathcal{C}) := \bigcup_{t \geq 0} t\mathcal{C}.$$

The *conic hull* of a function $f : \mathcal{X} \to \overline{\mathbb{R}}$ such that $f(0) = 0$ is the largest conic function dominated by f and is denoted by $\mathrm{cone}(f)$, that is, for $X \in \mathcal{X}$,

$$\mathrm{cone}(f)(X) := \inf_{t>0} \frac{f(tX)}{t}.$$

The definition of Fenchel-Moreau conjugate function follows, together with the Fenchel-Moreau Theorem. We assume that the reader is familiar with the notion of lower semicontinuity for extended real valued functions on $\mathcal{X}$.

Definition A.1.5. Consider a function $f : \mathcal{X} \to (-\infty, \infty]$. The *convex conjugate* of f is the function $f^* : \mathcal{X}' \to (-\infty, \infty]$ defined by

$$f^*(\psi) := \sup_{X \in \mathcal{X}} \{\psi(X) - f(X)\}.$$

Consider a function $g : \mathcal{X} \to [-\infty, \infty)$. The *concave conjugate* of g is the function $g^\bullet : \mathcal{X}' \to [-\infty, \infty)$ defined by

$$g^\bullet(\psi) := \inf_{X \in \mathcal{X}} \{\psi(X) - g(X)\} = -(-g)^*(-\psi).$$

Theorem A.1.6 (Fenchel-Moreau)**.** *If a function $f : \mathcal{X} \to (-\infty, +\infty]$ is convex and lower semicontinuous, then*

$$f(X) = \sup_{\psi \in \mathcal{X}'} \{\psi(X) - f^*(\psi)\}$$

for every $X \in \mathcal{X}$, and if a function $g : \mathcal{X} \to [-\infty, \infty)$ is concave and upper semicontinuous, then

$$g(X) = \inf_{\psi \in \mathcal{X}'} \{\psi(X) - g^\bullet(\psi)\}$$

for every $X \in \mathcal{X}$.

The next theorem is the standard dual representation of quasi convex lower semicontinuous functions, which has been derived in Penot and Volle [82].

Theorem A.1.7 (Penot-Volle)**.** *Let $f : \mathcal{X} \longrightarrow (-\infty, +\infty]$ be a quasi convex lower semicontinuous function. Then*

$$f(X) = \sup_{\psi \in \mathcal{X}'} \inf\{f(Y) \,:\, \psi(Y) \leq \psi(X)\}.$$

for every $X \in \mathcal{X}$.

The next proposition regards epigraphs of lower semicontinuous functions. Recall that for any function $f : \mathcal{X} \to \overline{\mathbb{R}}$ the following holds for every $X \in \mathcal{X}$:

$$f(X) = \inf\{m \in \mathbb{R} \,:\, (X, m) \in \mathrm{epi}(f)\}. \tag{A.1}$$

Proposition A.1.8. *Let $\mathcal{G} \subset \mathcal{X} \times \mathbb{R}$ be such that if $(X, m) \in \mathcal{G}$, then $(X, n) \in \mathcal{G}$ for every $n \geq m$, and consider the function $f : \mathcal{X} \to \overline{\mathbb{R}}$ defined for every $X \in \mathcal{X}$ as*

$$f(X) := \inf\{m \in \mathbb{R} \,:\, (X, m) \in \mathcal{G}\}.$$

The following statements hold:

(i) If f is lower semicontinuous, then $\operatorname{epi}(f) = \operatorname{cl}(\mathcal{G})$ (closure in the product topology on $\mathcal{X} \times \mathbb{R}$).

(ii) If $\mathcal{G}$ is closed in the product topology on $\mathcal{X} \times \mathbb{R}$, then f is lower semicontinuous and $\operatorname{epi}(f) = \mathcal{G}$.

We now define support functions and barrier cones.

Definition A.1.9. Let $\mathcal{C} \subset \mathcal{X}$ be a nonempty set. The *support function of $\mathcal{C}$* and the *upper support function of $\mathcal{C}$* are the maps $\sigma_{\mathcal{C}} : \mathcal{X}' \to [-\infty, +\infty)$ and $\sigma^{\mathcal{C}} : \mathcal{X}' \to (-\infty, +\infty]$ defined for $\psi \in \mathcal{X}'$ by

$$\sigma_{\mathcal{C}}(\psi) := \inf_{X \in \mathcal{C}} \psi(X), \qquad \sigma^{\mathcal{C}}(\psi) := \sup_{X \in \mathcal{C}} \psi(X).$$

The *barrier cone of $\mathcal{C}$* is defined as follows:

$$\operatorname{bar}(\mathcal{C}) := \{\psi \in \mathcal{X}' \,:\, \sigma_{\mathcal{C}}(X) > -\infty\}.$$

Note that, since $\sigma^{\mathcal{C}}(\psi) = -\sigma_{\mathcal{C}}(-\psi)$, then

$$-\operatorname{bar}(\mathcal{C}) = \{\psi \in \mathcal{X}' \,:\, \sigma^{\mathcal{C}}(X) < \infty\}.$$

For every nonempty set $\mathcal{C} \subset \mathcal{X}$, $\sigma_{\mathcal{C}}$ is superlinear and $\sigma(\mathcal{X}', \mathcal{X})$-upper semicontinuous, $\sigma^{\mathcal{C}}$ is sublinear and $\sigma(\mathcal{X}', \mathcal{X})$-lower semicontinuous, and $\operatorname{bar}(\mathcal{C})$ is a convex cone. If moreover $\mathcal{C}$ is a cone, then $\operatorname{bar}(\mathcal{C})$ is $\sigma(\mathcal{X}', \mathcal{X})$-closed and

$$\sigma_{\mathcal{C}}(\psi) = \begin{cases} 0 & \text{if } \psi \in \operatorname{bar}(\mathcal{C}) \\ -\infty & \text{otherwise} \end{cases}, \qquad \sigma^{\mathcal{C}}(\psi) = \begin{cases} 0 & \text{if } \psi \in -\operatorname{bar}(\mathcal{C}) \\ \infty & \text{otherwise} \end{cases}.$$

Note that, unless $\mathcal{C}$ is conic, $\operatorname{bar}(\mathcal{C})$ may fail to be $\sigma(\mathcal{X}', \mathcal{X})$-closed. The *indicator function* of a set $\mathcal{A} \subset \mathcal{X}$ is the map $\delta_{\mathcal{A}} : \mathcal{X} \to [0, \infty]$ given by

$$\delta_{\mathcal{A}}(X) := \begin{cases} 0 & \text{if } X \in \mathcal{A}, \\ \infty & \text{otherwise.} \end{cases}$$

For every $\psi \in \mathcal{X}'$, we have $\sigma_{\mathcal{A}}(\psi) = (-\delta_{\mathcal{A}})^{\bullet}(\psi) = -\delta^{*}_{\mathcal{A}}(-\psi)$.

The following dual representation for closed convex sets is a direct consequence of the Hahn-Banach Separation Theorem.

Proposition A.1.10. *For every convex and closed set $\mathcal{C} \subset \mathcal{X}$ we have*

$$\mathcal{C} = \bigcap_{\psi \in \mathcal{X}'} \{X \in \mathcal{X} \,:\, \psi(X) \geq \sigma_{\mathcal{C}}(\psi)\} = \bigcap_{\psi \in \operatorname{bar}(\mathcal{C})} \{X \in \mathcal{X} \,:\, \psi(X) \geq \sigma_{\mathcal{C}}(\psi)\}.$$

A.2 Functions valued in ordered topological vector spaces

In this section, let $\mathcal{X}$ be an Hausdorff ordered locally convex topological vector space (for details, see [1]), and let us denote the compatible quasiorder with $\geq$ and the positive cone with $\mathcal{X}_+$, that is

$$\mathcal{X}_+ := \{X \in \mathcal{X} \,:\, X \geq 0\}.$$

If $\mathcal{X}'$ is the topological dual of $\mathcal{X}$, we assume that it is equipped with the natural order induced by the cone of *positive functionals*

$$\mathcal{X}'_+ := \{\psi \in \mathcal{X}' \,:\, \psi(X) \geq 0 \ \ \forall X \in \mathcal{X}_+\}.$$

As usual, a positive functional ψ is *strictly positive* if $\psi(X) > 0$ for every nonzero $X \in \mathcal{X}_+$. The set of strictly positive functionals is denoted by $\mathcal{X}'_{++}$. Similarly $X \in \mathcal{X}$ is said *strictly positive* if $\psi(X) > 0$ for every nonzero $\psi \in \mathcal{X}'_+$, and the set of strictly positive elements of $\mathcal{X}$ is denoted by $\mathcal{X}_{++}$. For a subset $\mathcal{L}$ of $\mathcal{X}$, $\mathcal{L}_+$ and $\mathcal{L}_{++}$ are defined as follows

$$\mathcal{L}_+ := \mathcal{L} \cap \mathcal{X}_+, \qquad \mathcal{L}_{++} := \mathcal{L} \cap \mathcal{X}_{++}.$$

Similarly for a subset $\mathcal{L}'$ of $\mathcal{X}'$.

We now define monotone sets and maps. Note that actually topology is not needed for these definitions.

Definition A.2.1. A set $\mathcal{C} \subset \mathcal{X}$ is *monotone* if $\mathcal{C} + \mathcal{X}_+ \subset \mathcal{C}$. A function $f : \mathcal{Y} \to \mathcal{X}$, where $\mathcal{Y}$ is ordered by $\succeq$, is *monotone increasing (decreasing)* if $f(X) \geq f(Y)$ whenever $Y \succeq X$ ($X \succeq Y$).

When it is clear from the context we do not distinguish between increasing and decreasing monotonicity. It is easy to show that if $\mathcal{C}$ is a monotone set, its barrier cone $\mathrm{bar}(\mathcal{C})$ is contained in the positive cone $\mathcal{X}'_+$, and if $f : \mathcal{X} \to \overline{\mathbb{R}}$ is a monotone increasing function, then the supremum in the Fenchel-Moreau convex representation and in the quasi convex representation can be taken over $\psi \in \mathcal{X}'_+$.

For functions valued in $\mathcal{X}$, the definition of *convex* (*concave, positively homogeneous, subadditive, superadditive, sublinear, superlinear*) function, extends naturally that of Definition A.1.3 by using the quasiorder of $\mathcal{X}$. For the scopes of this thesis, we also define upper semicontinuity for functions valued in $\mathcal{X}$.

Definition A.2.2. Let $\mathcal{C}$ be a nonempty subset of a topological space $\mathcal{Y}$, and take $Y_0 \in \mathcal{C}$. A function $f : \mathcal{C} \to \mathcal{X}$ is *upper semicontinuous at* Y_0 if for every neighborhood $\mathcal{U}$ of $f(Y_0)$, one finds a neighborhood $\mathcal{V}$ of Y_0 such that $f(\mathcal{V}) \subset \mathcal{U} - \mathcal{X}_+$. The function f is *upper semicontinuous* if it is upper semicontinuous at Y for every $Y \in \mathcal{C}$.

The next proposition gives some characterizations of upper semicontinuity if $\mathcal{Y}$ coincides with $\mathbb{R}^N$ (note that some of them are valid for every topological space). Since this type of upper semicontinuity is rarely used, in this case we furnish a proof of the proposition.

Proposition A.2.3. *Let $\mathcal{C}$ be a nonempty subset of $\mathbb{R}^N$. Consider a function $f : \mathcal{C} \to \mathcal{X}$ and $\lambda \in \mathcal{C}$. The following statements are equivalent:*

(i) f is upper semicontinuous at λ.

(ii) For every net (or for every sequence) $(\lambda_\alpha) \subset \mathcal{C}$, $\lambda_\alpha \to \lambda$ and for every $\mathcal{U} \in \mathscr{N}_{f(\lambda)}$, there is $\alpha_{\mathcal{U}}$ such that if $\alpha \succeq \alpha_{\mathcal{U}}$ one finds $Y^\alpha_{\mathcal{U}} \in \mathcal{U}$ such that $f(\lambda_\alpha) \leq Y^\alpha_{\mathcal{U}}$.

(iii) For every net (or for every sequence) $(\lambda_\alpha) \subset \mathcal{C}$, $\lambda_\alpha \to \lambda$, there is a subnet $(\lambda_\beta) \subset (\lambda_\alpha)$ and $(Y_\beta) \subset \mathcal{X}$ such that $Y_\beta \to f(\lambda)$ and $f(\lambda_\beta) \leq Y_\beta$.

If $\mathcal{X}$ is first countable, they are also equivalent to the following statement:

(iv) For every sequence $(\lambda_n) \subset \mathcal{C}$, $\lambda_n \to \lambda$, there is $(Y_n) \subset \mathcal{X}$ such that $Y_n \to f(\lambda)$ and $f(\lambda_n) \leq Y_n$.

Proof. It is clear that *(i)* implies *(ii)*.

We now show that *(ii)* implies *(iii)* for nets, and the same proof holds for sequences. Take a net $(\lambda_\alpha)_{\alpha \in A} \subset \mathcal{C}$, $\lambda_\alpha \to \lambda$. The set $B := \{(\alpha, \mathcal{U}) : \mathcal{U} \in \mathscr{N}_{f(\lambda)},\ \alpha \in A,\ \alpha \succeq \alpha_{\mathcal{U}}\}$ is directed by the binary relation $(\alpha, \mathcal{U}) \geq (\alpha', \mathcal{U}')$ iff $\alpha \succeq \alpha'$ and $\mathcal{U} \subseteq \mathcal{U}'$. It is easily verified that the net $(\lambda_{(\alpha,\mathcal{U})})_{(\alpha,\mathcal{U}) \in B}$ defined as $\lambda_{(\alpha,\mathcal{U})} := \lambda_\alpha$, is a subnet of $(\lambda_\alpha)_{\alpha \in A}$. Due to statement *(ii)*, for every $(\alpha, \mathcal{U}) \in B$, we find $Y_{(\alpha,\mathcal{U})} \in \mathcal{U}$ such that $f(\lambda_{(\alpha,\mathcal{U})}) \leq Y_{(\alpha,\mathcal{U})}$ and clearly $Y_{(\alpha,\mathcal{U})} \to f(\lambda)$.

It remains to prove that *(iii)* implies *(i)*. By contradiction, assume f is not upper semicontinuous at λ. Let $\mathcal{U}$ be a neighborhood of $f(\lambda)$ such that every neighborhood $\mathcal{V}$ of λ contains $\lambda_{\mathcal{V}}$ such that $f(\lambda_{\mathcal{V}})$ is not dominated by any point of $\mathcal{U}$. Take $(\mathcal{U}_n)_{n \in \mathbb{N}}$ a fundamental system of neighborhoods of λ and the corresponding $\lambda_{\mathcal{U}_n}$. By *(iii)* there is $Y \in \mathcal{U}$ such that $f(\lambda_{\mathcal{U}_k}) \leq Y$ for some k, which is a contradiction.

Now assume $\mathcal{X}$ is first countable. Clearly if *(iv)* holds, *(iii)* holds. We prove that *(ii)* implies *(iv)*. Take $(\lambda_n) \subset \mathcal{C}$, $\lambda_n \to \lambda$, and let $(\mathcal{U}_k)_k$ be a fundamental system of neighborhood of $f(\lambda)$ such that

$\mathcal{U}_k \supset \mathcal{U}_{k+1}$. From *(ii)*, we know that for every k, we find a corresponding n_k ($\alpha_{\mathcal{U}}$ in the statement) such that for $n \geq n_k$ there is $Y_k^n \in \mathcal{U}_k$ such that $f(\lambda_n) \leq Y_k^n$. We can assume that $(n_k)_k$ is strictly increasing. The sequence we are seeking is defined as $Y_n := Y_k^n$ if $n_k \leq n < n_{k+1}$. □

Incidentally, note that from *(iii)* it follows that for an upper semicontinuous map $f : \mathcal{C} \to \mathcal{X}$, the counterimage of closed monotone sets is closed, generalizing the standard notion of upper semicontinuity of real valued functions.

A.3 SET VALUED MAPPINGS

We use the notation $\rightrightarrows$ for set valued mappings, that is, functions that assign a subset of the image to every element of the domain. Here, we collect some definitions about continuity properties of set valued mappings. For a reference, see Aliprantis and Border [1] and Bank et al. [12].

Definition A.3.1. Let $F : \mathcal{Y} \rightrightarrows \mathcal{Z}$ be a set valued mapping between topological spaces, and let $Y_0 \in \mathcal{Y}$. We say that:

(i) F is *closed at* Y_0 if for $(Y_\alpha) \subseteq \mathcal{Y}$, $(Z_\alpha) \subseteq \mathcal{Z}$ with the properties $Y_\alpha \to Y_0$, $Z_\alpha \in F(Y_\alpha)$, $Z_\alpha \to Z_0 \in \mathcal{Z}$, it follows that $Z_0 \in F(Y_0)$.

(ii) F is *upper semicontinuous at* Y_0 if for every open set $\mathcal{U} \subseteq \mathcal{Z}$ containing $F(Y_0)$, there is a neighborhood $\mathcal{V}$ if Y_0 such that $F(Y) \subseteq \mathcal{U}$ whenever $Y \in \mathcal{V}$.

(iii) F is *lower semicontinuous at* Y_0 if for every open set $\mathcal{U} \subseteq \mathcal{Z}$ that has nonempty intersection with $F(Y_0)$, there is a neighborhood $\mathcal{V}$ if Y_0 such that $F(Y) \cap \mathcal{U} \neq \emptyset$ whenever $Y \in \mathcal{V}$.

(iv) F is *strongly lower semicontinuous at* Y_0 if for every $Z \in F(Y_0)$, there are neighborhoods $\mathcal{U}$ and $\mathcal{V}$ respectively of Z and Y_0 such that $\mathcal{U} \subseteq F(Y)$ whenever $Y \in \mathcal{V}$.

A.4 RANDOM VARIABLES

We fix a probability space $(\Omega, \mathcal{F}, \mathbb{P})$. A random variable is a Borel measurable function $X : \Omega \to \mathbb{R}$. The positive and negative part of X are

$$X^+ := \max\{X, 0\}, \quad X^- := \max\{-X, 0\},$$

and the modulus of X is

$$|X| := X^+ - X^-.$$

The elements of $\mathbb{R}$ are identified with constant random variables. We denote by $\mathbb{E}_{\mathbb{P}}$ the expectation with respect to $\mathbb{P}$ and use a similar notation when other probability measures are considered. If $\mathbb{Q}$ is a probability measure that is absolutely continuous with respect to $\mathbb{P}$, we denote by $\frac{d\mathbb{Q}}{d\mathbb{P}}$ its Radon-Nikodym derivative. In some chapters, we find convenient to apply expectations to generic random variables regardless of their integrability. We do this by setting

$$\mathbb{E}_{\mathbb{P}}[X] := \mathbb{E}_{\mathbb{P}}[X^+] - \mathbb{E}_{\mathbb{P}}[X^-],$$

where we adopt the convention $\infty - \infty = -\infty$. This means that a non-integrable negative part prevails over a non-integrable positive part.

We denote by $\mathscr{L}^0(\mathbb{P})$ the set of all random variables. It is an ordered vector space, equipped with the natural pointwise operations and partial order. Equipped with the topology of the pointwise convergence, it is a topological vector space that fails to be locally convex. We denote by $L^0(\mathbb{P})$ the set of equivalence classes of random variables with respect to almost-sure equality under $\mathbb{P}$. As usual, when the reference space is a subset of $L^0(\mathbb{P})$, we do not explicitly distinguish between an element of $L^0(\mathbb{P})$ and any of its representatives. We equip $L^0(\mathbb{P})$ with its canonical algebraic operations and almost sure partial order induced by $\mathbb{P}$. It is an ordered topological vector space when equipped with the topology of the convergence in probability, and it fails to be locally convex as well.

The positive cones of $\mathscr{L}^0(\mathbb{P})$ and $L^0(\mathbb{P})$ are the convex cones consisting of all positive random variables (or equivalence classes):

$$\mathscr{L}^0(\mathbb{P})_+ := \{X \in \mathscr{L}^0(\mathbb{P}) \,:\, X(\omega) \geq 0 \ \forall \omega \in \Omega\}, \quad L^0(\mathbb{P})_+ := \{X \in L^0(\mathbb{P}) \,:\, \mathbb{P}(X \geq 0) = 1\}.$$

Moreover we define the sets of strictly positive elements as follows:

$$\mathscr{L}^0(\mathbb{P})_{++} := \{X \in \mathscr{L}^0(\mathbb{P}) \,:\, X(\omega) > 0 \ \forall \omega \in \Omega\}, \quad L^0(\mathbb{P})_{++} := \{X \in L^0(\mathbb{P}) \,:\, \mathbb{P}(X > 0) = 1\}.$$

Similarly, the sets of positive and strictly positive elements from a given set $L \subset L^0(\mathbb{P})$ are defined by $L_+ := L \cap L^0(\mathbb{P})_+$ and $L_{++} = L \cap L^0(\mathbb{P})_{++}$, and the same for subsets of $\mathscr{L}^0(\mathbb{P})$. The space of m-dimensional random vectors (equivalence classes of m-dimensional random vectors) is denoted by $\mathscr{L}^0_m(\mathbb{P})$ ($L^0_m(\mathbb{P})$).

For a measurable set $A \in \mathcal{F}$, the indicator function on A is defined as:

$$\mathbb{1}_A(\omega) := \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A. \end{cases}$$

A.5 ORLICZ SPACES

A nonconstant function $\Phi : [0, \infty) \to [0, \infty]$ is called an Orlicz function if it is convex, left-continuous, increasing, finite on a right neighborhood of zero, and satisfies $\Phi(0) = 0$. The conjugate of Φ is the Orlicz function defined by

$$\Phi^*(u) := \sup_{t \in [0,\infty)} \{tu - \Phi(t)\}.$$

For every $X \in L^0(\mathbb{P})$ define the Luxemburg norm by

$$\|X\|_\Phi := \inf \left\{ \lambda \in (0, \infty) \,:\, \mathbb{E}_\mathbb{P} \left[\Phi \left(\frac{|X|}{\lambda} \right) \right] \leq 1 \right\}.$$

The corresponding Orlicz space is given by

$$L^\Phi(\mathbb{P}) := \{X \in L^0(\mathbb{P}) \,:\, \|X\|_\Phi < \infty\}.$$

The heart of $L^\Phi(\mathbb{P})$ is the space

$$H^\Phi(\mathbb{P}) := \left\{ X \in L^\Phi(\mathbb{P}) \,:\, \forall \lambda \in (0, \infty) \,:\, \mathbb{E}_\mathbb{P} \left[\Phi \left(\frac{|X|}{\lambda} \right) \right] < \infty \right\}.$$

These spaces are Banach lattices with respect to the Luxemburg norm. The classical Lebesgue spaces are special examples of Orlicz spaces. Indeed, if $\Phi(t) = t^p$ for $p \in [1, \infty)$ and $t \in [0, \infty)$, then $L^\Phi(\mathbb{P}) = H^\Phi(\mathbb{P}) = L^p(\mathbb{P})$ and the Luxemburg norm coincides with the usual p norm. Moreover, if we set $\Phi(t) = 0$ for $t \in [0, 1]$ and $\Phi(t) = \infty$ otherwise, then we have $L^\Phi(\mathbb{P}) = L^\infty(\mathbb{P})$ and the Luxemburg norm coincides with the usual esssup norm. Note that, in this case, $H^\Phi(\mathbb{P}) = \{0\}$.

We say that Φ satisfies the Δ_2 condition if there exist $s \in (0, \infty)$ and $k \in (0, \infty)$ such that $\Phi(2t) < k\Phi(t)$ for every $t \in [s, \infty)$. If Φ is Δ_2, then $L^\Phi = H^\Phi$. In a nonatomic setting. also the opposite implication holds. A well-known example of a nontrivial $H^\Phi(\mathbb{P})$ that is strictly contained in $L^\phi(\mathbb{P})$ is obtained by setting $\Phi(t) = \exp(t) - 1$ for $t \in [0, \infty)$.

The norm dual of $L^\Phi(\mathbb{P})$ cannot be identified with a subspace of $L^0(\mathbb{P})$ in general. However, if Φ is finite valued (otherwise $H^\Phi(\mathbb{P}) = \{0\}$), the norm dual of $H^\Phi(\mathbb{P})$ can always be identified with $L^{\Phi^*}(\mathbb{P})$. For the case $L^p(\mathbb{P})$, for $p \in [1, \infty)$, this is simply the well-known identification of the norm dual of $L^p(\mathbb{P})$ with $L^q(\mathbb{P})$ and $q = \frac{p}{p-1}$ (with the usual convention $\frac{1}{0} := \infty$).

APPENDIX B

ASYMPTOTIC CONES AND FUNCTIONS

In this appendix, we aim to collect the definitions of asymptotic cone and asymptotic function together with a number of elementary properties. The reference space $\mathcal{X}$ is a general Hausdorff topological vector space, and asymptotic cones and functions are defined for any subset of $\mathcal{X}$ and any extended real valued function on $\mathcal{X}$. In finite dimensional vector spaces the terms *horizon cone* and *horizon function* are sometimes used with the same meaning. For convex sets and functions, we also define the *recession cone* and the *recession function*, which in presence of closedness or lower semicontinuity coincide with the asymptotic cone and the asymptotic function.

For a wide treatment of asymptotic analysis in finite dimensional spaces see Rockafellar and Wets [86], and Rockafellar [85] for the convex case. The generalization to nonconvex and infinite dimensional sets has been firstly introduced in Dedieu [36]. Asymptotic cones in inifinite dimensional spaces have been widely studied in relation with asymptotical properties like asymptotical compactness and boundedness (see e.g. Zalinescu [96] and Barbu et al. [13]). The results stated in these pages are either trivial or may be found in the cited literature.

B.1 ASYMPTOTIC AND RECESSION CONES

Throughout this appendix, let $\mathcal{X}$ be a Hausdorff topological vector space. First, we define the asymptotic cone of a subset of $\mathcal{X}$. Note that this definition depends on the chosen topology.

Definition B.1.1. Let $\mathcal{C} \subset \mathcal{X}$ be a nonempty set. The *asymptotic cone of* $\mathcal{C}$ is defined as follows:

$$\mathcal{C}^\infty := \{X \in \mathcal{X} \,:\, \exists\, (X_\alpha)_\alpha \subset \mathcal{C},\ (t_\alpha)_\alpha \subset \mathbb{R}_+ \text{ such that } t_\alpha \to 0 \text{ and } t_\alpha X_\alpha \to X\}.$$

If $\mathcal{C} = \emptyset$, then its *asymptotic cone* is $\mathcal{C}^\infty := \{0\}$.

The next proposition states that the asymptotic cone of any set is closed and conic, and offers a characterization.

Proposition B.1.2. *The asymptotic cone of $\mathcal{C} \subset \mathcal{X}$ is a closed cone and the following equivalence holds:*

$$\mathcal{C}^\infty = \bigcap_{t>0} \mathrm{cl}([0,t]\mathcal{C}).$$

Proof. By definition it is immediate to see that $\mathcal{C}^\infty$ is a cone, and its closedness follows once proved the representation. Let $X = \lim_\alpha t_\alpha X_\alpha$ for $(X_\alpha)_\alpha \subset \mathcal{C}$ and $(t_\alpha)_\alpha \subset \mathbb{R}_+$, and let $t > 0$. Eventually, $t_\alpha < t$ and hence $t_\alpha X_\alpha \in [0,t]\mathcal{C}$, proving the inclusion "$\subset$". Not take $X \in \mathcal{C}^\infty$. For any neighborhood $\mathcal{U}$ of X and any $t > 0$, there is $X_{(\mathcal{U},t)} \in \mathcal{C}$ and $t_{(\mathcal{U},t)} \in [0,t]$ such that $t_{(\mathcal{U},t)} X_{(\mathcal{U},t)} \in \mathcal{U}$. By considering on the set $\{(\mathcal{U},t) \,:\, \mathcal{U} \text{ neighborhood of } X,\ t \in \mathbb{R}_+\}$ the partial order $(\mathcal{U},t) \succeq (\mathcal{U}',t')$ iff $\mathcal{U} \subset \mathcal{U}'$ and $t \le t'$, we have that the net $(t_{(\mathcal{U},t)})$ converges to 0 and $(t_{(\mathcal{U},t)} X_{(\mathcal{U},t)})$ converges to X, showing that the other inclusion holds too. □

The asymptotic cone of $\mathcal{C}$ may be regarded as what remains of $t\mathcal{C}$ as $t \to 0$. The following observation may help to visualize this fact. Immerse $\mathcal{X}$ in the Cartesian product $\mathbb{R} \times \mathcal{X}$ via the map $X \mapsto (1, X)$. Let $\mathcal{K}$ be the cone generated by the immersion of $\mathcal{C}$ in $\mathbb{R} \times \mathcal{X}$. As $\mathcal{K} = \{(t, X) \,:$

$t \geq 0,\ X \in t\mathcal{C}\}$, it intersects the plane with $\mathbb{R}$-coordinate 0 only in the vertex. But, when closing $\mathcal{K}$ in the product topology, some nonzero points with $\mathbb{R}$-coordinate 0 may be enclosed. Those points are precisely the nonzero elements of $\mathcal{C}^\infty$ (i.e. $X \in \mathcal{C}^\infty$ iff $(0, X) \in \mathrm{cl}(\mathcal{K})$), and represent the directions in which $\mathcal{C}$ is unbounded.

We have said that intuitively the asymptotic cone consists of those directions along which $\mathcal{C}$ is unbounded. The next proposition records the precise link between asymptotic cone and boundedness.

Proposition B.1.3. *The following statements hold for $\mathcal{C} \subset \mathcal{X}$:*

(i) If $\mathcal{C}$ is bounded, then $\mathcal{C}^\infty = \{0\}$.

(ii) If $\mathcal{C}^\infty = \{0\}$ and $\mathcal{C}$ has finite dimension, then $\mathcal{C}$ is bounded.

Proof. (i): If $\mathcal{C} = \emptyset$, there is nothing to prove. Otherwise take $X \in \mathcal{C}^\infty$ and $\mathcal{U}$ a neighborhood of 0. By boundedness, for $t > 0$ sufficiently small, $t\mathcal{C} \subset \mathcal{U}$. This delivers $X = 0$.

(ii): If $\mathcal{C}$ has finite dimension, any compatible norm on $\mathrm{span}(\mathcal{C})$ induces the original topology on $\mathcal{C}$. Assume by contradiction that $\mathcal{C}$ is unbounded, and let $(X_n)_n \subset \mathcal{C}$ be such that $\|X_n\| \geq n$. Possibly passing to a subsequence, by compactness of the finite dimensional sphere there is $X \in \mathrm{span}(\mathcal{C})$ such that $\|X\| = 1$ and

$$\frac{X_n}{\|X_n\|} \to X.$$

Hence $X \in \mathcal{C}^\infty$, which contradicts $\|X\| = 1$. □

Now we enumerate some properties of the asymptotic cone.

Proposition B.1.4. *Let $\mathcal{C} \subset \mathcal{X}$ be a nonempty subset. The following statements hold:*

(a) $\mathcal{C}^\infty = (\mathrm{cl}(\mathcal{C}))^\infty$.

(b) $\mathcal{C}^\infty = (\mathcal{C} - X)^\infty$ for every $X \in \mathcal{X}$.

(c) $\mathcal{C}^\infty = (t\mathcal{C})^\infty$ for evrery $t > 0$.

(d) If $\mathcal{C}$ is a cone, then $\mathcal{C}^\infty = \mathrm{cl}(\mathcal{C})$.

(e) If $\mathcal{C}$ is monotone, then $\mathcal{C}^\infty$ is monotone.

(f) If $\mathcal{C}$ is star shaped about 0, then $\mathcal{C}^\infty \subset \mathrm{cl}(\mathcal{C})$.

Proof. Assertion *(a)* follows from the characterization in Proposition B.1.2 since $\mathrm{cl}([0,t]\mathcal{C}) = [0,t]\,\mathrm{cl}(\mathcal{C})$, while *(b)* and *(c)* are easily proved using the definition of asymptotic cone. Now, assume $\mathcal{C}$ is a cone. Then $[0,t]\mathcal{C} = \mathcal{C}$ for every $t > 0$, hence $\mathcal{C}^\infty = \mathrm{cl}(\mathcal{C})$ and *(d)* is proved. Let $\mathcal{C}$ be a monotone set, and take $Y \geq X \in \mathcal{C}^\infty$. Hence we find nets $(X_\alpha) \subset \mathcal{C}$, $(t_\alpha) \subset \mathbb{R}_+$ such that $t_\alpha \to 0$, $t_\alpha X_\alpha \to X$, and

$$Y = X + (Y - X) = \lim_\alpha t_\alpha \left(X_\alpha + \frac{Y - X}{t_\alpha} \right),$$

which is in $\mathcal{C}^\infty$ since $Y - X \in \mathcal{X}_+$ and *(e)* is proved. Finally, if $\mathcal{C}$ is star shaped about 0, then $[0,1]\mathcal{C} \subset \mathcal{C}$, and *(f)* follows. □

We treat separately the case of a convex set $\mathcal{C}$.

Proposition B.1.5. *Let $\mathcal{C} \subset \mathcal{X}$ be a nonempty convex set. Then $\mathcal{C}^\infty$ is convex and for every $X_0 \in \mathcal{C}$ the following equalities hold:*

$$\begin{aligned} \mathcal{C}^\infty &= \bigcap_{t>0} t\big(\mathrm{cl}(\mathcal{C}) - X_0\big) && \text{(B.1)} \\ &= \{Z \in \mathcal{X} \,:\, X_0 + tZ \in \mathrm{cl}(\mathcal{C}) \ \ \forall t \geq 0\} && \text{(B.2)} \\ &= \{Z \in \mathcal{X} \,:\, X + tZ \in \mathrm{cl}(\mathcal{C}) \ \ \forall t \geq 0,\ \ \forall X \in \mathcal{C}\} && \text{(B.3)} \end{aligned}$$

and $\mathcal{C} + \mathcal{C}^\infty \subset \mathrm{cl}(\mathcal{C})$. If moreover $\mathcal{C}$ is closed, then

$$\mathcal{C}^\infty = \{Z \in \mathcal{X} \,:\, Z + \mathcal{C} \subset \mathcal{C}\}. \tag{B.4}$$

Proof. Since $\mathcal{C}$ is convex and $X_0 \in \mathcal{C}$, then $[0,t](\mathcal{C} - X_0) = t(\mathcal{C} - X_0)$ for $t > 0$. By point *(b)* of Proposition B.1.4, we have

$$\mathcal{C}^\infty = (\mathcal{C} - X_0)^\infty = \bigcap_{t>0} \mathrm{cl}\left([0,t](\mathcal{C} - X_0)\right) = \bigcap_{t>0} \mathrm{cl}\left(t(\mathcal{C} - X_0)\right) = \bigcap_{t>0} t\big(\mathrm{cl}(\mathcal{C}) - X_0\big),$$

proving (B.1). Equality in (B.2) is straightforward, and equality in (B.3) holds since the set in (B.2) is independent on the choice of $X_0 \in \mathcal{C}$. It follows from (B.3) that $\mathcal{C} + \mathcal{C}^\infty \subset \mathrm{cl}(\mathcal{C})$ holds, implying that the inclusion "$\subset$" in (B.4) holds too. For the other inclusion, take $Z \in \mathcal{X}$ such that $Z + \mathcal{C} \subset \mathcal{C}$. For $X \in \mathcal{C}$, we have that $nZ + X \in \mathcal{C}$ for $n \in \mathbb{N}$, and also for $n = 0$. By convexity, $X + tZ \in \mathcal{C}$ for every $t \geq 0$. □

For convex nonempty sets, we also define the recession cone as follows.

Definition B.1.6. Let $\mathcal{C} \subset \mathcal{X}$ be a nonempty convex set and let $X_0 \in \mathcal{C}$. The *recession cone of* $\mathcal{C}$ is defined as follows, and the definition does not depend on X_0:

$$\mathrm{rec}(\mathcal{C}) := \bigcap_{t>0} t(\mathcal{C} - X_0).$$

From the first characterization in Proposition B.1.5 it is clear that the recession cone and the asymptotic cone of a nonempty closed convex set coincide.

Corollary B.1.7. *If $\mathcal{C} \subset \mathcal{X}$ is a nonempty convex closed set, then*

$$\mathcal{C}^\infty = \mathrm{rec}(\mathcal{C}).$$

The next proposition shows the behavior of asymptotic cones with respect to sets lattice operation. We omit the proof as it is immediate using the definition or the characterizations of asymptotic cones.

Proposition B.1.8. *Let $\{C_i\}_i$ be a family of subsets of $\mathcal{X}$. The following statements hold:*

(i)

$$\Big(\bigcap_i C_i\Big)^\infty \subset \bigcap_i C_i^\infty,$$

with equality if C_i are convex and closed, and $\bigcap_i C_i \neq \emptyset$.

(ii)

$$\Big(\bigcup_i C_i\Big)^\infty \supset \bigcup_i C_i^\infty.$$

We now prove a simplified version of the Dieudonné Theorem about the closedness of the difference of closed sets, which is enough for the scopes of this thesis. For the original version, see Dieudonné [39], and for extensions see Barbu et al. [13].

Theorem B.1.9 (Dieudonné)**.** *Assume that $\mathcal{B}$ and $\mathcal{C}$ are closed subsets of $\mathcal{X}$ such that $\mathcal{B}$ is finite dimensional and*

$$\mathcal{B}^\infty \cap \mathcal{C}^\infty = \{0\}.$$

Then $\mathcal{B} - \mathcal{C}$ is closed.

Proof. There is nothing to prove if either $\mathcal{B}$ or $\mathcal{C}$ is empty. So let $\mathcal{B}$ and $\mathcal{C}$ be nonempty, and take $Z \in \mathcal{X}$ and nets $(X_\alpha) \subset \mathcal{B}$ and $(Y_\alpha) \subset \mathcal{C}$ such that $X_\alpha - Y_\alpha \to Z$. Denote by $\mathcal{M}$ the linear space generated by $\mathcal{B}$. If (X_α) has a convergent subnet, then possibly passing to a subnet $X_\alpha \to X \in \mathcal{B}$. Hence $Y_\alpha = (Y_\alpha - X_\alpha) + X_\alpha \to -Z + X \in \mathcal{C}$ and $Z \in \mathcal{B} - \mathcal{C}$. Now, assume that (X_α) has no convergent subnets and let $\|\cdot\|$ be a norm on $\mathcal{M}$ that induces the unique Hausdorff topology on $\mathcal{M}$ compatible with the linear structure. We find a subnet (that for convenience we still denote by X_α) such that $\|X_\alpha\| \to \infty$. Possibly passing to a subnet, we find a nonzero X such that

$$\frac{X_\alpha}{\|X_\alpha\|} \to X \in \mathcal{B}^\infty.$$

Moreover

$$\frac{Y_\alpha}{\|X_\alpha\|} = \frac{Y_\alpha - X_\alpha}{\|X_\alpha\|} + \frac{X_\alpha}{\|X_\alpha\|} \to 0 + X \in \mathcal{C}^\infty.$$

This fact contradicts that X is not 0 and the proof is concluded. □

B.2 ASYMPTOTIC AND RECESSION FUNCTIONS

We define the asymptotic function of an extended real valued function on $\mathcal{X}$. This definition derives from the observation that the asymptotic cone of an epigraph is itself an epigraph.

Definition B.2.1. Consider a function $f : \mathcal{X} \to \overline{\mathbb{R}}$ that is not identically ∞. The *asymptotic function of f* is defined as the extended real valued function $f^\infty : \mathcal{X} \to \overline{\mathbb{R}}$ such that

$$\operatorname{epi}(f^\infty) = \big(\operatorname{epi}(f)\big)^\infty.$$

It is possible to extend the definition of asymptotic function to functions defined on strict subsets of $\mathcal{X}$.

Definition B.2.2. Let $\mathcal{C} \subset \mathcal{X}$ be a nonempty set and consider a function $f : \mathcal{C} \to \overline{\mathbb{R}}$ that is not identically ∞. The *asymptotic function of f*, $f^\infty : \mathcal{X} \to \overline{\mathbb{R}}$, is defined as the asymptotic function of $\widetilde{f}$, where

$$\begin{array}{rccl} \widetilde{f}: & \mathcal{X} & \to & \overline{\mathbb{R}} \\ & X & \mapsto & \begin{cases} f(X) & X \in \mathcal{C} \\ +\infty & X \notin \mathcal{C} \end{cases} \end{array}$$

Proposition B.2.3. *Let $\mathcal{C} \subset \mathcal{X}$ be a nonempty set and let $f : \mathcal{C} \to \overline{\mathbb{R}}$ be not identically ∞. Then*

$$\operatorname{epi}(f^\infty) = \big(\operatorname{epi}(f)\big)^\infty$$

and $f^\infty(X) = \infty$ for $X \notin \mathcal{C}^\infty$.

Proof. The following holds:

$$\operatorname{epi}(f^\infty) = \operatorname{epi}(\widetilde{f}^\infty) = \big(\operatorname{epi}(\widetilde{f})\big)^\infty = \big(\operatorname{epi}(f)\big)^\infty.$$

Now, assume $X \notin \mathcal{C}^\infty$ and $(X, m) \in \operatorname{epi}(f^\infty)$ for some $m \in \mathbb{R}$. Hence $(X, m) \in \big(\operatorname{epi}(f)\big)^\infty$, so we find $(t_\alpha) \subset \mathbb{R}_+$ and $(X_\alpha) \subset \mathcal{C}$ such that $t_\alpha \to 0$ and $t_\alpha X_\alpha \to X$. It follows that $X \in \mathcal{C}^\infty$ which is a contradiction. □

The fact that Definition B.2.2 relies on B.2.1, allow us to state the next results for functions defined on $\mathcal{X}$. The reader could easily generalize them to strict subsets of $\mathcal{X}$.

Before stating the next proposition, we recall that for an extended real valued function, if the epigraph is a cone and the function is real valued in 0, then it is positively homogeneous.

Proposition B.2.4. *Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be not identically ∞. Then f^∞ is lower semicontinuous and $f^\infty(0)$ is either 0 or $-\infty$. If moreover $f^\infty(0) = 0$, then f^∞ is positively homogeneous.*

Proof. The epigraph of f^∞ is an asymptotic cone, hence is a closed cone, showing that f^∞ is lower semicontinuous and $f^\infty(tX) = tf^\infty(X)$ for $X \in \mathcal{X}$ and $t > 0$. If moreover $f^\infty(0) = 0$, then f^∞ is also positively homogeneous. □

We now consider the convex case.

Proposition B.2.5. *If $f : \mathcal{X} \to \overline{\mathbb{R}}$ is a convex function not identically ∞, then f^∞ is convex. If moreover f is proper and lower semicontinuous, then for every X_0 in the domain of f and $X \in \mathcal{X}$*

$$f^\infty(X) = \sup_{t>0} \frac{f(X_0 + tX) - f(X_0)}{t}.$$

Proof. If f is convex, the epigraph of f^∞ is convex by Proposition B.1.5, proving that f^∞ is convex as well. Now, assume that f is lower semicontinuous and that $f(X_0) \in \mathbb{R}$. By Corollary B.1.7,

$$\mathrm{epi}(f^\infty) = \bigcap_{t>0} t\Big(\mathrm{epi}(f) - \big(X_0, f(X_0)\big)\Big),$$

hence for $X \in \mathcal{X}$

$$\begin{aligned} f^\infty(X) &= \inf\{m \in \mathbb{R} \,:\, (X, m) \in \mathrm{epi}(f^\infty)\} \\ &= \inf\{m \in \mathbb{R} \,:\, (tX, tm) - (X_0, f(X_0)) \in \mathrm{epi}(f) \quad \forall t > 0\} \\ &= \inf\{m \in \mathbb{R} \,:\, tm + f(X_0) \geq f(X_0 + tX) \quad \forall t > 0\} \\ &= \sup_{t>0} \frac{f(X_0 + tX) - f(X_0)}{t}. \end{aligned}$$

□

Like for sets, one defines the recession function of a convex functions and shows that in presence of lower semicontinuity it coincides with the asymptotic function.

Definition B.2.6. Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be a convex function and fix X_0 in the domain of f. The recession function $\mathrm{rec}(f) : \mathcal{X} \to \overline{\mathbb{R}}$ is defined for every $X \in \mathcal{X}$ as follows, and the definition does not depend on X_0:

$$\mathrm{rec}(f)(X) := \sup_{t>0} \frac{f(X_0 + tX) - f(X_0)}{t}.$$

The recession function of a convex function f is the smallest sublinear map dominating f.

Corollary B.2.7. *If $f : \mathcal{X} \to \overline{\mathbb{R}}$ is a proper, convex, lower semicontinuous function, then*

$$\mathrm{rec}(f) = f^\infty.$$

We close this appendix by showing the relationship among sublevels of a function and its asymptotic function.

Proposition B.2.8. *Let $f : \mathcal{X} \to \overline{\mathbb{R}}$ be not identically ∞. Then for $r \in \mathbb{R}$:*

$$\{X \in \mathcal{X} : f(X) \leq r\}^\infty \subset \{X \in \mathcal{X} \,:\, f^\infty(X) \leq 0\}.$$

The above inclusion is an equality if f is convex, lower semicontinuous and proper, and $\{X \in \mathcal{X} \,:\, f(X) \leq r\}$ is not empty.

Proof. If $\{X \in \mathcal{X} \,:\, f(X) \leq r\}^\infty$ is empty, the result is trivial. So take X in this set. By definition of asymptotic cone, we find nets $(t_\alpha) \subset \mathbb{R}_+$ and $(X_\alpha) \subset \mathcal{X}$ such that $t_\alpha \to 0$, $t_\alpha X_\alpha \to X$ and $f(X_\alpha) \leq r$. This shows that the couple $(X, 0) \in (\mathrm{epi}(f))^\infty$ since $(X_\alpha, r) \in \mathrm{epi}(f)$ and $t_\alpha(X_\alpha, r) \to (X, 0)$. Hence $f^\infty(X) \leq 0$.

Now assume that f is convex and lower semicontinuous, and let $X_0 \in \mathcal{X}$ be such that $f(X_0) \leq r$. If $f^\infty(X) \leq 0$ for some $X \in \mathcal{X}$, by Proposition B.2.5

$$f(X_0 + tX) \leq f(X_0) \leq r \quad \text{for all } t > 0.$$

This shows that $X \in \bigcap_{t>0} t(\{Y \in \mathcal{X} \,:\, f(Y) \leq r\} - X_0)$ that coincides with the recession cone of $\{X \in \mathcal{X} \,:\, f(X) \leq r\}$. □

Corollary B.2.9. *Let $f : \mathcal{C} \to \overline{\mathbb{R}}$ be a not identically ∞ function defined on a nonempty set $\mathcal{C}$. Then for $r \in \mathbb{R}$:*

$$\{X \in \mathcal{C} : f(X) \leq r\}^\infty \subset \{X \in \mathcal{C}^\infty \,:\, f^\infty(X) \leq 0\}.$$

The above inclusion is an equality if $\mathcal{C}$ is convex and closed, f is convex, lower semicontinuous and proper, and $\{X \in \mathcal{C} \,:\, f(X) \leq r\}$ is not empty.

LIST OF SYMBOLS

iff	if and only if
$\emptyset$	empty set
$\inf\emptyset$	∞
$\sup\emptyset$	$-\infty$
$\mathbb{R}$	the real numbers
$\overline{\mathbb{R}}$	the extended real numbers, i.e. $\mathbb{R}\cup\{\infty,-\infty\}$
$\mathbb{R}_+$	the nonnegative real numbers, i.e. $x\in\mathbb{R}$ such that $x\geq 0$
$\mathbb{R}_{++}$	the strictly positive real numbers, i.e. $x\in\mathbb{R}$ such that $x>0$
$\mathbb{R}^n$	n-dimensional vectors with real components
$\mathbb{R}^n_+$	n-dimensional vectors with components in $\mathbb{R}_+$
$\mathbb{R}^n_{++}$	n-dimensional vectors with components in $\mathbb{R}_{++}$
$\mathbb{N}$	the natural numbers (excluding zero)
$\mathbf{e}$	the vector $(1,1,\dots,1)\in\mathbb{R}^n$
$\mathbf{e}_i$	the vector $(0,\dots,0,1,0,\dots,0)\in\mathbb{R}^n$ (ith component 1, other components 0)
$\min\{x,y\}$	the minimum between x and y
$\max\{x,y\}$	the maximum between x and y
x^+	the positive part of x, i.e. $\max\{x,0\}$
x^-	the negative part of x, i.e. $\max\{-x,0\}$
$\lvert x\rvert$	the modulus of x, i.e. $x^+ + x^-$
$A+B$	the algebraic sum of sets, i.e. $\{a+b\,:\,a\in A,\ b\in B\}$
$A-B$	the algebraic difference of sets, i.e. $A+(-B)=\{a-b\,:\,a\in A,\ b\in B\}$
$b+A$	the algebraic sum $\{b\}+A$
tA	the set $\{tX\,:\,X\in A\}$
$[x,y]A$	the set $\{tX\,:\,x\leq t\leq y,\ X\in A\}$
$\mathbb{R}X$	the linear space generated by X, i.e. $\{tX\,:\,t\in\mathbb{R}\}$
$A^\perp$	the orthogonal complement of A
$\mathrm{span}(A)$	the smallest linear space containing A
$\mathrm{conv}(A)$	the smallest convex set containing A
$\mathrm{cone}(A)$	the smallest cone containing A, i.e. $\bigcup_{t\geq 0} tA$
$\mathrm{rec}(A)$	the recession cone of A, i.e. $\bigcap_{t>0}(A-X)$ for $X\in A$
A^∞	the asymptotic cone of A
$\mathrm{dom}(f)$	the domain of finiteness of the function $f:\mathcal{D}\to\overline{\mathbb{R}}$, i.e. $\{X\in\mathcal{D}\,:\,f(X)\in\mathbb{R}\}$
$\mathrm{epi}(f)$	the epigraph of the function $f:\mathcal{D}\to\overline{\mathbb{R}}$, i.e. $\{(X,m)\in\mathcal{D}\times\mathbb{R}\,:\,f(X)\leq m\}$
$\mathrm{hypo}(f)$	the hypograph of the function $f:\mathcal{D}\to\overline{\mathbb{R}}$, i.e. $\{(X,m)\in\mathcal{D}\times\mathbb{R}\,:\,f(X)\geq m\}$
$\{f\leq m\}$	the sublevel of the function $f:\mathcal{D}\to\overline{\mathbb{R}}$, i.e. $\{X\in\mathcal{D}\,:\,f(X)\leq m\}$
$f(A)$	the image of A through the function f, i.e. $\{f(X)\,:\,X\in A\}$
$\partial^+ f(0)$	the right derivative of f at 0
$\mathrm{cone}(f)$	the conic hull of the function f, i.e. $\mathrm{cone}(f)(X)=\inf_{t>0}\frac{f(tX)}{t}$
$\mathrm{rec}(f)$	the recession function of the function f, i.e. $\mathrm{rec}(f)(X)=\sup_{t>0}\frac{f(tX)}{t}$
f^∞	the asymptotic function of f
$\mathrm{usc}(f)$	the upper semicontinuous hull of f
δ_A	the characteristic function of the set A, i.e. $\delta(X)=0$ if $X\in A$, $\delta(X)=\infty$ if $X\notin A$
$F:\mathcal{X}\rightrightarrows\mathcal{Y}$	set valued mapping
$(X_n)_n$	sequence (also denoted by (X_n))
$(X_\alpha)_\alpha$	net (also denoted by (X_α))
$\mathrm{cl}(A)$	the closure of A
$\mathrm{int}(A)$	the interior of A
$\mathcal{X}_+$	the positive cone of $\mathcal{X}$

$\mathcal{X}_{++}$	the set of strictly positive elements of $\mathcal{X}$
$\mathcal{X}'$	the topological dual of $\mathcal{X}$
$\mathcal{X}'_{+}$	the positive dual of $\mathcal{X}$, i.e. $\{\psi \in \mathcal{X}' \,:\, \psi(X) \geq 0 \;\forall X \in \mathcal{X}_{+}\}$
$\mathcal{X}'_{++}$	the strictly positive dual of $\mathcal{X}$, i.e. $\{\psi \in \mathcal{X}' \,:\, \psi(X) > 0 \;\forall X \in \mathcal{X}_{+},\ X \neq 0\}$
$\sigma(\mathcal{X}, \mathcal{Y})$	the weak topology on $\mathcal{X}$ with dual $\mathcal{Y}$
$\mathscr{N}_X$	the collection of neighborhoods of X
σ_A	the support function of the set A, i.e. $\sigma_A(\psi) = \inf_{X \in \mathcal{A}} \psi(X)$
σ^A	the upper support function of the set A, i.e. $\sigma^A(\psi) = \sup_{X \in \mathcal{A}} \psi(X)$
$\mathrm{bar}(A)$	the barrier cone of A, i.e. $\{\psi \in \mathcal{X}' \,:\, \sigma_A(\psi) > -\infty\}$
f^*	the convex conjugate function of f
$f^\bullet$	the concave conjugate function of f
$\mathscr{L}^0(\mathbb{P})$	the space of of random variables
$L^0(\mathbb{P})$	the space of equivalent classes of random variables
$L^0_m(\mathbb{P})$	the space of m-dimensional random vectors
$\mathbb{E}_\mathbb{P}$	the expectation under $\mathbb{P}$
$\mathbb{1}_A$	the indicator function on the set A
a.s.	almost surely

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