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A gluing formula for Reidemeister-Turaev torsion

Stefano Borghini

Abstract

We extend Turaev's theory of Euler structures and torsion invariants on a 3-manifold M to the case of vector fields having generic behavior on ∂M. This allows to easily define gluings of Euler structures and to develop a completely general gluing formula for Reidemeister torsion of 3-manifolds. Lastly, we describe a combinatorial presentation of Euler structures via stream-spines, as a tool to effectively compute torsion.

Introduction

Reidemeister torsion is a classical topological invariant introduced by Reidemeister ([10]) in order to classify lens spaces. Significant improvements in the study of this invariant have been made by Milnor ([6]), who discovered connections between torsion and Alexander polynomial, and Turaev ([12]), who showed that the ambiguity in the definition of Reidemeister torsion could be fixed by means of Euler structures (i.e., equivalence classes of non-singular vector fields). Actually, to completely fix the ambiguity, Turaev introduced the additional notion of homology orientation, but we will not consider it (see Remark II.4).

Recently, Reidemeister torsion has proven its utility in a number of topics in 3-dimensional topology. For instance, Reidemeister torsion is the main tool in the definition of the Casson-Walker-Lescop invariants ([4]) and of Turaev's maximal abelian torsion ([14]), which in turn has been proved to be equivalent (up to sign) to the Seiberg-Witten invariants on 3-manifolds (if the first Betti number is $\neq 0$).

The aim of this paper is to describe the behavior of Reidemeister torsion on 3-manifolds with respect to gluings along a surface. This certainly is a problem of interest: to name a few examples of the importance of gluings, Heegaard splittings are one of the main ingredients in the construction of Heegaard Floer homology ([11]), and multiplicativity with respect to gluings is one of the fundamental axioms of Topological Quantum Field Theories ([1]).

Our reference model is the following. We consider a (closed) 3-manifold endowed with an Euler structure, and we split it into two submanifolds

M_1, M_2 along a surface S. As we have complete freedom in the choice of S, we need to define Euler structures of M_1, M_2 as equivalence classes of vector fields with a generic behavior on the boundary S.

The definition of Euler structure is the object of Section I. In particular, we describe the action of the first integer homology group on combinatorial and smooth Euler structures and we recover Turaev's reconstruction map Ψ, i.e., an equivariant bijection from combinatorial to smooth Euler structure.

In Section II we define Reidemeister torsion of a pair $(M, \mathfrak{e}^c)$, where M is a 3-manifold and $\mathfrak{e}^c$ is a combinatorial Euler structure. If $\mathfrak{e}^s$ is a smooth Euler structure, the torsion of $(M, \mathfrak{e}^s)$ is defined as the torsion of $(M, \Psi^{-1}(\mathfrak{e}^s))$. We emphasize that we need a way to explicitly invert the reconstruction map in order to effectively compute torsion (this will be the subject of Section IV).

Section III is devoted to the proof of Theorem III.2, informally stated below:

Theorem. *Reidemeister torsion acts multiplicatively with respect to gluings. Namely, given a smooth compact oriented closed 3-manifold M and an embedded surface S splitting M into two submanifolds M_1, M_2:*

- *a representative of an Euler structure $\mathfrak{e}$ on M induces Euler structures $\mathfrak{e}_1, \mathfrak{e}_2$ on M_1, M_2;*
- *the Reidemeister torsion of $(M, \mathfrak{e})$ is the product of Reidemeister torsions of $(M_1, \mathfrak{e}_1)$ and $(M_2, \mathfrak{e}_2)$, times a corrective term $\mathfrak{T}$ coming from the homologies.*

This theorem greatly extends a preceding result due to Turaev ([14, Lemma VI.3.2]), which holds in his very special setting only (S is a union of tori and Euler structures are equivalence classes of vector fields everywhere transversal to the boundary). The closure of M is not a necessary hypothesis, we have assumed it only to simplify notations and the proof (an extension to the case with boundary is stated, without proof, in Remark III.3). In the end of Section III we show some computations, aimed at simplify the term $\mathfrak{T}$.

Finally, in Section IV we describe a combinatorial encoding of Euler structures in order to explicitly invert the reconstruction map Ψ. The key tool will be a generalized version of standard spines (the stream-spines described in [9]), that allows to encode vector fields with generic behavior on the boundary.

In our work, we have focused on the abelian version of Reidemeister torsion (in order to simplify the algebraic machinery); all the results extend with minimal modifications to the non-abelian case. Section I, II, IV follow the exposition and the ideas of [2], where we have a first extension of Turaev's theory to the case of vector fields with simple boundary tangencies.

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I Euler structures

We consider generic vector fields on a 3-manifold M and we show that their behavior on the boundary ∂M is fixed by the choice of a boundary pattern $\mathcal{P}$. We define the sets $\mathfrak{Eul}^c(M,\mathcal{P})$ of combinatorial Euler structures (equivalence classes of singular integer 1-chains) and $\mathfrak{Eul}^s(M,\mathcal{P})$ of smooth Euler structures (equivalence classes of generic vector fields). We describe the action of the first integer homology group $H_1(M)$ and the construction of the equivariant bijection $\Psi : \mathfrak{Eul}^c(M,\mathcal{P}) \to \mathfrak{Eul}^s(M,\mathcal{P})$.

I.1 Generic vector fields

We first introduce the object of our investigation:

Notation. In what follows, with the word *3-manifold* we will always understand a smooth compact oriented manifold of dimension 3.

Let M be a 3-manifold and $\mathfrak{v}$ a non-singular vector field on M. In general, there is a wide range of possible behaviors of $\mathfrak{v}$ on the boundary ∂M. However, through an easy adjustment of Whitney's results ([15]), one can prove that, up to a small modification of the field $\mathfrak{v}$, the local models for the pair $(\partial M, \mathfrak{v})$ are the three in Fig. 1 only.

Therefore, given a non-singular vector field $\mathfrak{v}$ on M, $\mathfrak{v}$ can be slightly modified to obtain a new vector field with the following properties:

1. $\mathfrak{v}$ is still non-singular on M;
2. $\mathfrak{v}$ is transverse to ∂M in each point, except for a union $G \subset \partial M$ of circles, in which $\mathfrak{v}$ is tangent to ∂M;

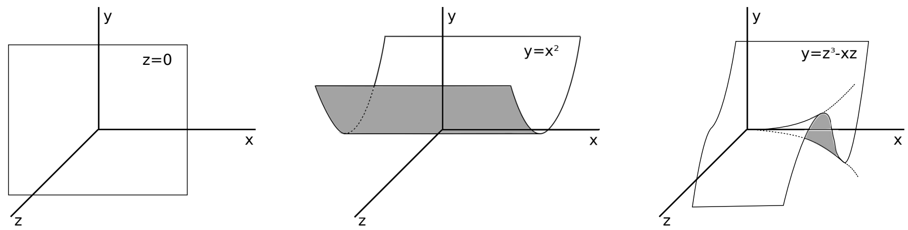

Fig. 1: Possible configurations of the boundary in a neighborhood of a point $p \in \partial M$. The coordinates are chosen in such a way that p coincides with the origin and the vector field is headed in the z direction.

3. $\mathfrak{v}$ is tangent to G in a finite set Q of points only.

A vector field on M satisfying conditions 1,2,3 is called *generic*. A generic vector field $\mathfrak{v}$ induces a partition $\mathcal{P} = (W, B, V, C, Q^+, Q^-)$ on ∂M where:

- $W \cup B = \partial M \setminus G$ is the set of *regular* points (Fig. 1-left), i.e. the points in which $\mathfrak{v}$ is transverse to ∂M. W is the *white part*, i.e., the set of the points in ∂M for which $\mathfrak{v}$ is directed inside M; B is the *black part*, i.e., the set of the points in ∂M for which $\mathfrak{v}$ is directed outside M. W and B are the interior of compact surfaces embedded in ∂M, and $\partial W = \partial B = G$.
- $V \cup C = G \setminus Q$ is the set of *fold* points (Fig. 1-center). V is the *convex* part, i.e., the set of points in G for which $\mathfrak{v}$ is directed towards B; C is the *concave* part, i.e., the set of points in G for which $\mathfrak{v}$ is directed towards W. The names (convex and concave) are justified by the cross-section in Fig. 2. V and C are disjoint unions of circles and open segments, and $\partial V = \partial C = Q$.
- $Q^+ \cup Q^- = Q$ is the set of *cuspidal* points (Fig. 1-right). Q^+ is the set of points where $\mathfrak{v}$ is directed towards C; Q^- is the set of points where $\mathfrak{v}$ is directed towards V.

Such a partition $\mathcal{P}$ is called a *boundary pattern* on ∂M. A generic vector field $\mathfrak{v}$ and a boundary pattern $\mathcal{P}$ are said to be *compatible* if $\mathcal{P}$ is induced by $\mathfrak{v}$, up to a diffeomorphism of M.

Remark I.1. A more general work, due to Morin ([8]), generalizes the results of Whitney in every dimension. This would probably allow to extend the results of Sections I, II, III to dimensions greater than 3.

I.2 Euler structures

A combing is a pair $[M, \mathfrak{v}]$, where M is a 3-manifold and $\mathfrak{v}$ is a generic vector field on M, viewed up to diffeomorphism of M and homotopy of $\mathfrak{v}$.

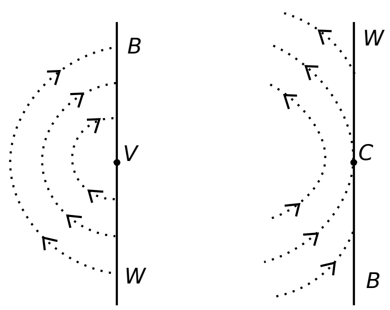

Fig. 2: Convex (on the left) and concave (on the right) points on the boundary.

We denote by $\mathfrak{Comb}$ the set of all combings. Notice that, under a homotopy of $\mathfrak{v}$, the boundary pattern on ∂M changes by an isotopy. Therefore, to a combing $[M, \mathfrak{v}]$ is associated a pair $(M, \mathcal{P})$ viewed up to diffeomorphism of M, and $\mathfrak{Comb}$ naturally splits as the disjoint union of subsets $\mathfrak{Comb}(M, \mathcal{P})$ of combings on M compatible with $\mathcal{P}$.

Two classes $[M, \mathfrak{v}_1], [M, \mathfrak{v}_2] \in \mathfrak{Comb}(M, \mathcal{P})$ are said to be *homologous* if $\mathfrak{v}_1, \mathfrak{v}_2$ are obtained from each other by homotopy through vector fields compatible with $\mathcal{P}$ and modifications supported in closed interior balls (that is, up to homotopy, $\mathfrak{v}_1, \mathfrak{v}_2$ coincide outside a ball contained in $\mathrm{Int}\, M$).

The quotient of $\mathfrak{Comb}(M, \mathcal{P})$ through the equivalence relation of homology is denoted by $\mathfrak{Eul}^s(M, \mathcal{P})$, and its elements are called *smooth Euler structures.*

Proposition I.2. *$\mathfrak{Eul}^s(M, \mathcal{P})$ is non-empty if and only if $\chi(M) - \chi(W) - \chi(V) - \chi(Q^+) = 0$.*

Proof. This result will be an immediate consequence of Proposition I.4 and Theorem I.5 below. A direct proof can be enstablished in a way similar to [2, Prop. 1.1], as an application of the Hopf theorem. $\square$

Let $H_1(M)$ be the first integer homology group of M. It is a standard fact of obstruction theory (see [12, § 5.2] for more details) that the map

$$\alpha^s : \mathfrak{Eul}^s(M, \mathcal{P}) \times \mathfrak{Eul}^s(M, \mathcal{P}) \to H_1(M),$$

which associates to a pair $(\mathfrak{e}_1, \mathfrak{e}_2)$ the first obstruction $\alpha_s(\mathfrak{e}_1, \mathfrak{e}_2) \in H_1(M)$ to their homotopy, is well defined. The map α^s defines an action of $H_1(M)$ on $\mathfrak{Eul}^s(M, \mathcal{P})$.

Recall that every 3-manifold admits a cellularization; this is a consequence of the Hauptvermutung or of Theorem II.2 below. A finite cellularization $\mathcal{C}$ of M is called *suited* to $\mathcal{P}$ if points in Q^+ and Q^- are 0-cells of $\mathcal{C}$ and $G = V \cup C \cup Q^+ \cup Q^-$ is a subcomplex. Let such a $\mathcal{C}$ be given. Denote by $E_{\mathcal{C}}$ the union of the cells of $M \setminus (W \cup V \cup Q^+)$. An *Euler chain* is an integer singular 1-chain ξ in M such that

$$\partial \xi = \sum_{e \subset E_{\mathcal{C}}} (-1)^{\dim(e)} \cdot x_e \qquad (1)$$

where $x_e \in e$ for all e.

Given two Euler chains ξ, ξ' with boundaries $\partial\xi = \sum(-1)^{\dim(e)} x_e$, $\partial\xi' = \sum(-1)^{\dim(e)} y_e$, we say that ξ, ξ' are *homologous* if, chosen for each $e \in E_{\mathcal{C}}$ a path α_e from x_e to y_e, the 1-cycle

$$\xi - \xi' + \sum_{e \in E_{\mathcal{C}}} (-1)^{\dim(e)} \alpha_e$$

represents the class 0 in $H_1(M)$.

Define $\mathfrak{Eul}^c(M,\mathcal{P})_{\mathcal{C}}$ as the set of homology classes of Euler chains. The following result was proved by Turaev (see [12, § 1.2]) in his framework, but extends to our setting without significant modifications.

Proposition I.3. *If $\mathcal{C}'$ is a subdivision of $\mathcal{C}$, then there exists a canonical $H_1(M)$-isomorphism $\mathfrak{Eul}^c(M,\mathcal{P})_{\mathcal{C}} \to \mathfrak{Eul}^c(M,\mathcal{P})_{\mathcal{C}'}$.*

Thus, the set $\mathfrak{Eul}^c(M,\mathcal{P})$ is canonically defined up to $H_1(M)$-isomorphism, independently of the cellularization. The elements of $\mathfrak{Eul}^c(M,\mathcal{P})$ are called *combinatorial Euler structure* of M compatible with $\mathcal{P}$.

Proposition I.4. *$\mathfrak{Eul}^c(M,\mathcal{P})$ is non-empty if and only if $\chi(M)-\chi(W)-\chi(V)-\chi(Q^+)=0$.*

Proof. This follows immediately from the observation that the algebraic number of points appearing on the right side of (1) is $\chi(M)-\chi(W)-\chi(V)-\chi(Q^+)$. □

It is easy to obtain an action of $H_1(M)$ on $\mathfrak{Eul}^c(M,\mathcal{P})$: it is the one induced by the map

$$\alpha^c : \mathfrak{Eul}^c(M,\mathcal{P}) \times \mathfrak{Eul}^c(M,\mathcal{P}) \to H_1(M)$$

defined by $\alpha^c(\mathfrak{e}_1,\mathfrak{e}_2)=[\mathfrak{e}_1-\mathfrak{e}_2]$.

I.3 Reconstruction map

A fundamental result is that the combinatorial and differentiable approach are equivalent, as stated by the following theorem.

Theorem I.5. *There exists a canonical $H_1(M)$-equivariant isomorphism*

$$\Psi : \mathfrak{Eul}^c(M,\mathcal{P}) \to \mathfrak{Eul}^s(M,\mathcal{P})$$

The map Ψ in the theorem is called *reconstruction map*, and it is explicitly constructed in the proof of the theorem.

Proof. The proof follows the scheme of [2, Thm. 1.4], which in turn is an extension of [12, § 6].

Let M' be the manifold obtained by attaching the collar $\partial M \times [0,+\infty)$ along ∂M, in such a way that $\partial M \times \{0\}$ is identified with ∂M. Consider a cellularization $\mathcal{C}$ of M: $\mathcal{C}$ extends to a "cellularization" $\mathcal{C}'$ on M' by attaching a cone to every cell of ∂M and then removing the vertex. Notice that some of the cells of $\mathcal{C}'$ have ideal vertices, thus $\mathcal{C}'$ is not a proper cellularization.

We set the following hypotheses:

(Hp1) $\mathcal{C}$ is suited with $\mathcal{P}$;

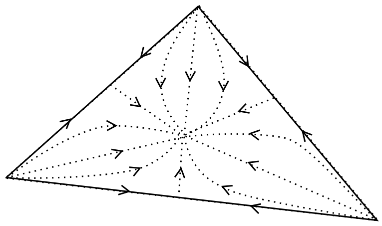

Fig. 3: $\mathfrak{w}_{\mathcal{T}'}$ on a triangle.

(Hp2) $\mathcal{C}$ is obtained by face-pairings on a finite number of polyhedra, and the projection of each polyhedron to M is smooth.

Such a cellularization certainly exists: for instance, a triangulation $\mathcal{T}$ of M satisfies (Hp2), and up to subdivision we can suppose that $\mathcal{T}$ is suited with $\mathcal{P}$.

Thanks to property (Hp2), we can recover the "first barycentric subdivision" of $\mathcal{C}'$, denoted by $\mathcal{C}''$. Its vertices are the points $\{p_\sigma\}_{\sigma\in\mathcal{C}'}$, where p_σ is inside the open cell σ for all $\sigma \in \mathcal{C}'$. Moreover, it is well defined a canonical vector field $\mathfrak{w}_{\mathcal{C}'}$ with the following properties:

- $\mathfrak{w}_{\mathcal{C}'}$ has singularities, of index $(-1)^{\dim\sigma}$, in the points p_σ only;
- the orbits of $\mathfrak{w}_{\mathcal{C}'}$ start (asintotically) from a point p_σ and end (asymptotically) in a point $p_{\sigma'}$ with $\sigma' \subset \sigma$.

Fig. 3 shows the behavior of $\mathfrak{w}_{\mathcal{C}'}$ on a triangle. The exact definition of $\mathfrak{w}_{\mathcal{C}'}$ is given in [3] for triangulations, but extends to our cellularization without complications. From now on, $\mathfrak{w}_{\mathcal{C}'}$ will be called *fundamental field* of the cellularization $\mathcal{C}'$.

Let N be the star of ∂M in $\mathcal{C}''$; identify N with $\partial M \times (-1, 1)$, in such a way that $\partial M \times (-1, 0] = M \cap N$.

Given a map $h : \partial M \to (-1, 1)$, we denote by M_h the manifold

$$M_h = (M \setminus N) \cup \{(x, t) \in \partial M \times (-1, 1) \cong N : t \leq h(x)\}.$$

Notice that M and M_h are isomorphic. We want to choose h in such a way that $\mathfrak{w}_{\mathcal{C}'}$ has no singularities on ∂M_h and induces on $\partial M_h \cong \partial M$ the boundary pattern $\mathcal{P}$.

Let $Q = Q^+ \cup Q^-$ and $G = V \cup C \cup Q$ (recall that G is a disjoint union of circles and $Q \subset G$ is a finite union of points). Denote by $U \subset \partial M$ the star of G in $\mathcal{C}''_\partial$ (where $\mathcal{C}''_\partial$ is the restriction of $\mathcal{C}''$ to ∂M). We have a diffeomorphism $U \cong G \times (-1, 1)$ such that:

$$G \cong G \times \{0\} \quad ; \quad U \cap W \cong G \times (-1, 0) \quad ; \quad U \cap B \cong G \times (0, 1)$$

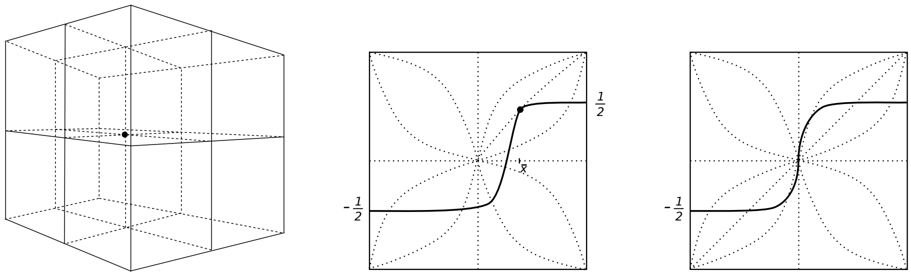

Fig. 4: The star of a vertex in G (left); the function $g_{\bar{x}}$ (center) and g_0 (right).

On $\partial M \setminus U$ we define h by:

$$h(p) = \begin{cases} -\frac{1}{2} & \text{, if } p \in W \setminus U \\ \frac{1}{2} & \text{, if } p \in B \setminus U \end{cases}$$

Obviously $\mathfrak{w}_{\mathcal{C}'}$ points outside M_h on W and inside M_h on B, as wished.

It remains to define h on U. To simplify the exposition, we are going to make the following hypothesis on the cellularization $\mathcal{C}$:

(Hp3) The star in $\mathcal{C}'$ of each 0-cell p is formed by eight 3-cells, arranged in such a way that the star in $\mathcal{C}''$ of p has the form shown in Fig. 4-left.

It is clear that a cellularization satisfying (Hp1), (Hp2), (Hp3) exists: again, one starts from a triangulation $\mathcal{T}$ of M suited with $\mathcal{P}$. By unifying or subdividing some of the simplices of $\mathcal{T}$, one obtains a cellularization (that still satisfies (Hp1), (Hp2)) such that the star in $\mathcal{C}'$ of each 0-cell p is formed by four 3-cells, disposed in the right way (namely, the boundary of each 3-cell does not contain both the convex and concave line incident in p). The extension of this cellularization to a "cellularization" $\mathcal{T}'$ of M' satisfies (Hp3).

We also need to define a preliminary continuous function $g : [-\frac{1}{3}, \frac{1}{3}] \times [-1, 1] \to [-\frac{1}{2}, \frac{1}{2}]$ as follows. Consider the square $[-1, 1] \times [-1, 1]$ and the fundamental field of its obvious cellularization (4 vertices, 4 edges and one 2-cell). For $\bar{x} \in [-\frac{1}{3}, \frac{1}{3}] \setminus \{0\}$, we impose the one-variable function $g_{\bar{x}}(t) = g(\bar{x}, t)$ to be an increasing function with all derivatives zero in $-1, 1$, with $g_{\bar{x}}(1) = \frac{1}{2}, g_{\bar{x}}(-1) = -\frac{1}{2}$ and with the property that the fundamental field is tangent to the curve $t \mapsto (t, g_{\bar{x}}(t))$ for $t = \bar{x}$ only. For $\bar{x} = 0$: $g_0(t) = g(0, t)$ is a strictly increasing function with all derivatives zero in $-1, 1$, with $g_0(1) = \frac{1}{2}, g_0(-1) = -\frac{1}{2}, g_0(0) = 0$ and never tangent to the fundamental field. It is clear that such a function g exists: we show $g_{\bar{x}}$ in Fig. 4-center,right. We will avoid its explicit construction, that is not very significant.

Let $T \subset G$ be the star of Q in $\mathcal{C}''_G$, where $\mathcal{C}''_G$ is the restriction of $\mathcal{C}''$ to G (T is just a disjoint union of segments). Let $U_V, U_C, U_Q \subset U$ be the stars of V, C, Q in $\mathcal{C}''_\partial$. Identify U_V, U_C, U_G with $V \times (-1, 1), C \times (-1, 1), T \times (-1, 1)$ consistently with the identification of U with $G \times (-1, 1)$.

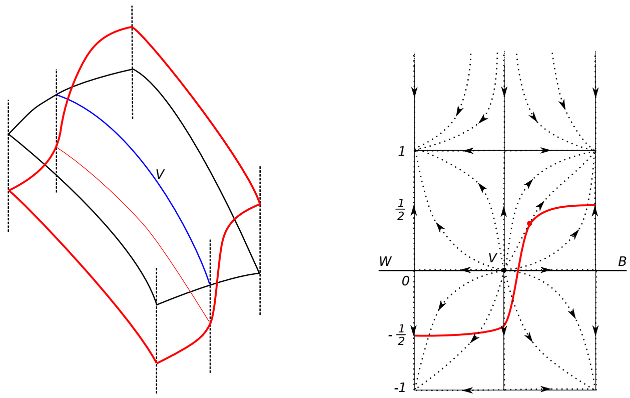

Fig. 5: The field $\mathfrak{w}_{C'}$ has convex tangency on V.

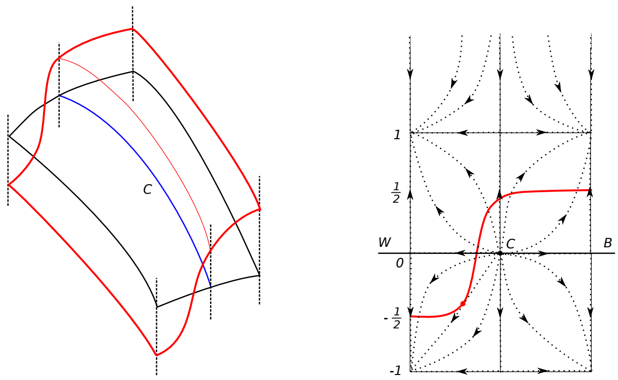

Fig. 6: The field $\mathfrak{w}_{C'}$ has concave tangency on C.

On $U \setminus U_Q$, define h as follows:

$$h(s,t) = \begin{cases} g(\frac{1}{3}, t) & \text{, if } (s,t) \in U_V \setminus U_Q \cong (V \setminus T) \times (-1,1) \\ g(-\frac{1}{3}, t) & \text{, if } (s,t) \in U_C \setminus U_Q \cong (C \setminus T) \times (-1,1). \end{cases}$$

It is clear that h induces the wished pattern: to the points of $V \setminus T$ corresponds a convex point in ∂M_h (Fig. 5), while to the points in $C \setminus T$ corresponds a concave point in ∂M_h (Fig. 6).

It only remains to define h on $U_Q \cong T \times (-1,1)$. Identify each connected component of T with $(-1,1)$ in such a way that $(-1,0) \subset C$, $(0,1) \subset V$. Now each connected component of U_Q is identified with the square $(-1,1) \times (-1,1)$ in such a way that:

$$U_Q \cap W \cong (-1,1) \times (-1,0) \quad ; \quad U_Q \cap B \cong (-1,1) \times (0,1) \quad ;$$

$$U_Q \cap U_C \cong (-1,0) \times (-1,1) \quad ; \quad U_Q \cap U_V \cong (0,1) \times (-1,1).$$

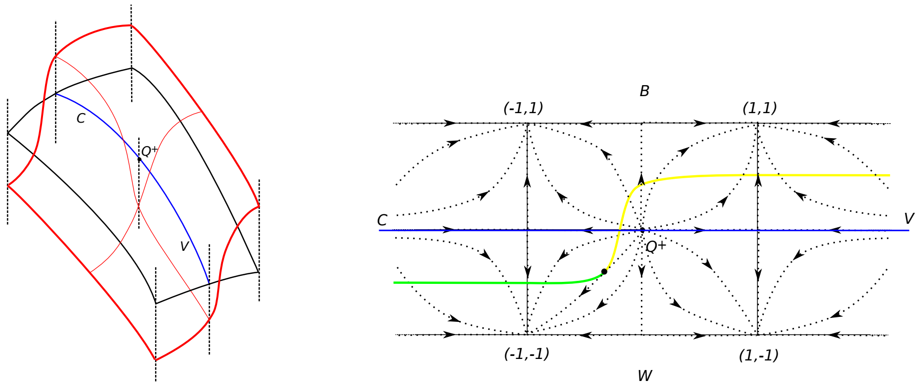

Fig. 7: Behavior of ∂M_h near Q^+ (left). View from above of $\mathfrak{w}_{\mathcal{C}'}$ near Q^+ (right): the field goes from a concave tangency line (green) to a convex tangency line (yellow) through a positive cuspidal point.

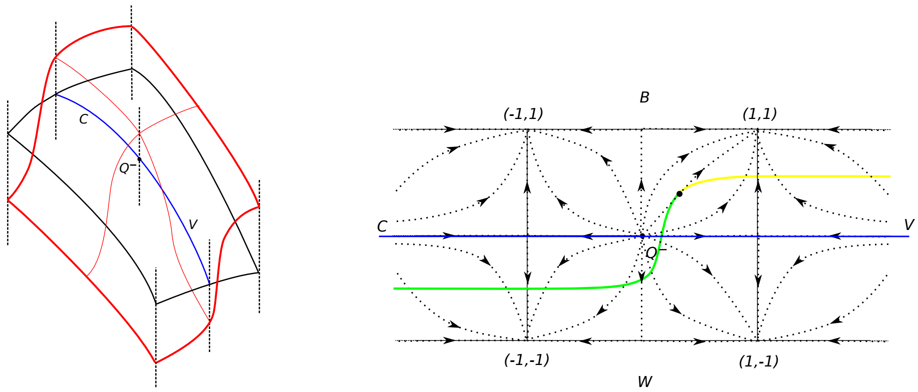

Fig. 8: Behavior of ∂M_h near Q^- (left). View from above of $\mathfrak{w}_{\mathcal{C}'}$ near Q^- (right): the field goes from a concave tangency line (green) to a convex tangency line (yellow) through a negative cuspidal point.

If U_{Q^+}, U_{Q^-} are the stars of Q^+, Q^- in $\mathcal{C}'_\partial$, we have $U_Q = U_{Q^+} \cup U_{Q^-}$. Set:

$$h(s,t) = \begin{cases} g\left(\frac{2}{3}\, g(-\frac{1}{3}, s), t\right) & \text{, if } (s,t) \in U_{Q^+} \cong (-1,1) \times (-1,1) \\ g\left(\frac{2}{3}\, g(\frac{1}{3}, s), t\right) & \text{, if } (s,t) \in U_{Q^-} \cong (-1,1) \times (-1,1). \end{cases}$$

The behavior of h near Q^+ and Q^- is described in Fig. 7 and Fig. 8. It is clear that we can choose g in such a way that $\mathfrak{w}_{\mathcal{T}'}$ is tangent to ∂M_h only on the green and yellow line (the best way to convince ourself about it is by choosing g in such a way that $g_{\bar{x}}$ is everywhere constant, except in a small neighborhood of $\bar{x}$, where it quickly increase from $-\frac{1}{2}$ to $\frac{1}{2}$). Notice that to each point in Q^+ corresponds a positive cuspidal point and to each point in Q^- corresponds a negative cuspidal point. Now h is a smooth function defined on all ∂M and $\mathfrak{w}_{\mathcal{T}'}$ induces the wished partition $\mathcal{P}$ on ∂M_h.

Remember that $\mathfrak{w}_{\mathcal{C}'}$ has singularities in the 0-cells p_σ of $\mathcal{C}''$. Consider a combinatorial Euler structure $\mathfrak{e}^c \in \mathfrak{Eul}^c(M,\mathcal{P})$, and a representative ξ of $\mathfrak{e}^c$. We can suppose that

$$\partial\xi = \sum_{\sigma\in E_{\mathcal{C}}} (-1)^{\dim\sigma} \cdot p_\sigma,$$

where $E_{\mathcal{C}}$ is the union of the cells of $\mathcal{C}$ in $M \setminus (W \cup V \cup Q^+)$. Notice that $\partial\xi$ consists exactly of the singularities of $\mathfrak{w}_{\mathcal{C}'}$ in M_h, each taken with its index. Moreover the sum of the indices of these singularities is zero, hence it is possible to modify the field on a neighborhood of the support of ξ in order to remove them. In this way, we obtain a non-singular vector field $\mathfrak{w}^\xi_{\mathcal{C}'}$ on $M_h \cong M$, representing a smooth Euler structure $\Psi(\mathfrak{e}^c) \in \mathfrak{Eul}^s(M,\mathcal{P})$. Turaev's proof that Ψ is well defined and $H_1(M)$-equivariant extends to our case without particular modifications. The $H_1(M)$-equivariance proves the bijectivity of Ψ. □

Remark I.6. The bijectivity of Ψ is obtained indirectly from the $H_1(M)$-equivariance, while the explicit construction of the inverse Ψ^{-1} is a harder task. In Section IV we will see how to invert Ψ using stream-spines.

Remark I.7. While hypotheses (Hp1), (Hp2) on the cellularization $\mathcal{C}$ are necessary, the hypothesis (Hp3) is not fundamental. The proof above can be repeated without using (Hp3): in the construction of h inside U_Q, one has to distinguish various cases, depending on the form of the stars of the cuspidal points.

Notation. Theorem I.5 allows us to ease the notation: if there is no ambiguity, we will write $\mathfrak{Eul}(M,\mathcal{P})$ to denote either $\mathfrak{Eul}^c(M,\mathcal{P})$ or $\mathfrak{Eul}^s(M,\mathcal{P})$; α to denote either α^c or α^s.

II Reidemeister torsion

Definitions in Sections II.1 and II.2 are known facts, preparatory to Section II.3, where Reidemeister torsion of a pair $(M,\mathcal{P})$ is defined. The main result is Proposition II.3, which shows that the ambiguity in the definition of Reidemeister torsion is fixed (up to sign) by the choice of an Euler structure $\mathfrak{e} \in \mathfrak{Eul}(M,\mathcal{P})$.

II.1 Torsion of a chain complex

Consider a finite chain complex over a field $\mathbb{F}$

$$C = \left(C_m \xrightarrow{\partial_m} C_{m-1} \xrightarrow{\partial_{m-1}} \cdots \xrightarrow{\partial_2} C_1 \xrightarrow{\partial_1} C_0 \right).$$

By *finite* we mean that every vector space C_i has finite dimension. We fix bases $\mathfrak{c} = (\mathfrak{c}_0,\ldots,\mathfrak{c}_m)$ and $\mathfrak{h} = (\mathfrak{h}_0,\ldots,\mathfrak{h}_m)$ of C and $H_*(C)$ respectively.

With this notation, we mean that $\mathfrak{c}_i$ (resp. $\mathfrak{h}_i$) is a basis of C_i (resp. $H_i(C)$) for all $i = 0, \dots, m$.

For all $i = 0, \dots, m$, we choose an arbitrary basis $\mathfrak{b}_i$ of the i-boundaries $B_i = \mathrm{Im}(\partial_{i+1})$, and we consider the short exact sequence:

$$0 \to B_i \to Z_i \to H_i(C) \to 0 \tag{2}$$

where $Z_i = \mathrm{Ker}(\partial_i)$ is the group of the i-cycles. By inspecting sequence (2), it is clear that a basis for Z_i is obtained by taking the union of $\mathfrak{b}_i$ and a lift $\widetilde{\mathfrak{h}}_i$ of $\mathfrak{h}_i$.

Now, consider the exact sequence:

$$0 \to Z_i \to C_i \xrightarrow{\partial_i} B_{i-1} \to 0 \tag{3}$$

By (3), $\mathfrak{b}_i\widetilde{\mathfrak{h}}_i\widetilde{\mathfrak{b}}_{i-1}$ (where $\mathfrak{b}_i\widetilde{\mathfrak{h}}_i$ is the basis of Z_i constructed above and $\widetilde{\mathfrak{b}}_{i-1}$ is a lift of $\mathfrak{b}_{i-1}$) is a basis for C_i.

We can define the torsion of the chain complex C as a sort of difference between the basis $\mathfrak{c}$ and the new basis of C obtained above.

Notation. Given two bases $\mathfrak{A}, \mathfrak{B}$ of the finite-dimensional vector space V, we denote by $[\mathfrak{A}/\mathfrak{B}]$ the determinant of the matrix that represents the change of basis from $\mathfrak{A}$ to $\mathfrak{B}$ (i.e., the matrix whose columns are the vectors of $\mathfrak{A}$ written in coordinates with respect to the basis $\mathfrak{B}$).

The *torsion* of the chain complex C is defined by:

$$\tau(C; \mathfrak{c}, \mathfrak{h}) = \prod_{i=0}^{m} [\mathfrak{b}_i\widetilde{\mathfrak{h}}_i\widetilde{\mathfrak{b}}_{i-1}/\mathfrak{c}_i]^{(-1)^{i+1}} \in \mathbb{F} \tag{4}$$

The torsion $\tau(C, \mathfrak{c}, \mathfrak{h})$ depends uniquely on the equivalence classes of the bases $\mathfrak{c}_i, \mathfrak{h}_i$, and does not depend on the choice of $\mathfrak{b}_i$ and of the lifts $\widetilde{\mathfrak{b}}_i, \widetilde{\mathfrak{h}}_i$.

The following is an interesting result, that will be fundamental in Section III.

Theorem II.1 (Milnor [7]). *Consider a short exact sequence of finite complexes*

$$0 \to C' \to C \to C'' \to 0$$

and the corresponding long exact sequence in homology

$$\mathcal{H} = \left(H_m(C') \to H_m(C) \to \cdots \to H_0(C) \to H_0(C'')\right).$$

$\mathcal{H}$ can be viewed as a finite acyclic chain complex. Fix bases $\mathfrak{h}', \mathfrak{h}, \mathfrak{h}''$ on $H_*(C'), H_*(C), H_*(C'')$ respectively. This gives a basis on $\mathcal{H}$; denote by $\tau(\mathcal{H})$ the torsion of $\mathcal{H}$, computed with respect to this basis. Choose compatible *bases $\mathfrak{c}', \mathfrak{c}, \mathfrak{c}''$ on C', C, C'' (by compatible, we mean that $\mathfrak{c}$ is the union of $\mathfrak{c}'$ and a lift of $\mathfrak{c}''$). With these hypothesis, the following formula holds:*

$$\tau(C; \mathfrak{c}, \mathfrak{h}) = \tau(\mathcal{H}) \cdot \tau(C'; \mathfrak{c}', \mathfrak{h}') \cdot \tau(C''; \mathfrak{c}'', \mathfrak{h}'')$$

II.2 Torsion of a pair

Let M be a smooth compact oriented manifold of arbitrary dimension and let $H_1(M)$ be its first integer homology group. To recover the definitions of Section II.1 we need a cellularization of M. The existence of a cellularization is granted by the following classical result:

Theorem II.2 (Whitehead)**.** *Every compact smooth manifold M admits a canonical PL-structure (in particular, M admits a cellularization, unique up to subdivisions).*

Let N be a compact submanifold of M. Consider a cellularization $\mathcal{C}$ of M such that N is a closed subcomplex of M. Assume that M is connected. If $\hat{M} \to M$ is the maximal abelian covering, $\mathcal{C}$ lifts to a cellularization of $\hat{M}$. Notice that $p^{-1}(N)$ is a closed subcomplex of $\hat{M}$, hence we can consider the cellular chain complex

$$C_*(M, N) = C_*^{\text{cell}}(\hat{M}, p^{-1}(N); \mathbb{Z})$$

The group $H_1(M)$ acts on $C_*(M, N)$ via the deck transformations, thus $C_*(M, N)$ can be viewed as a chain complex of $\mathbb{Z}[H_1(M)]$-modules and $\mathbb{Z}[H_1(M)]$-homomorphisms.

Now, consider a field $\mathbb{F}$ and a *representation* φ, i.e., a ring homomorphism $\varphi : \mathbb{Z}[H_1(M)] \to \mathbb{F}$. The field $\mathbb{F}$ can be viewed as a $\mathbb{Z}[H_1(M)]$-module with the product $z \cdot f = f\varphi(z)$ (where $z \in \mathbb{Z}[H_1(M)], f \in \mathbb{F}$). Therefore we can consider the following chain complex over $\mathbb{F}$:

$$C_*^{\varphi}(M, N) = C_*(M, N) \otimes_{\mathbb{Z}[H_1(M)]} \mathbb{F}. \tag{5}$$

$C_*^{\varphi}(M, N)$ is called *φ-twisted chain complex* of (M, N). Its homology (the *φ-twisted homology*) is denoted by $H_*^{\varphi}(M, N)$.

A *fundamental family* $\mathfrak{f}$ of (M, N) is a choice of a lift for each cell in $M \setminus N$. $\mathfrak{f}$ is a basis of $C_*^{\varphi}(M, N)$, thus, chosen a basis $\mathfrak{h}$ on $H_*^{\varphi}(M, N)$, we can compute the torsion of the twisted complex $C_*^{\varphi}(M, N)$.

The *Reidemeister torsion* of (M, N) with respect to $\mathfrak{f}, \mathfrak{h}$ is defined by

$$\tau^{\varphi}(M, N; \mathfrak{f}, \mathfrak{h}) = \tau(C_*^{\varphi}(M, N); \mathfrak{f}, \mathfrak{h}) \in \mathbb{F}^*. \tag{6}$$

The fact that the definition of $\tau^{\varphi}(M, N; \mathfrak{f}, \mathfrak{h})$ does not depends on the choice of the cellularization $\mathcal{C}$ is classical (see [12, Lemma 3.2.3]).

The definitions above extend in a natural way to the case of a non-connected manifold M. Namely, the twisted chain complex extends by direct sum on the connected components and Reidemeister torsion extends by multiplicativity.

II.3 Torsion of a 3-manifold

Now we specialize on dimension 3. Consider a 3-manifold M, a boundary pattern $\mathcal{P} = (W, B, V, C, Q^+, Q^-)$ and a cellularization $\mathcal{C}$ of M suited with $\mathcal{P}$. If $\hat{M} \to M$ is the maximal abelian covering, $\mathcal{C}$ lifts to a cellularization of $\hat{M}$. Assume that M is connected (as in Section II.2, the definitions below will extend to the non-connected case in the obvious way).

Notation. Consider a submanifold N of M, which is also a subcomplex with respect to the cellularization $\mathcal{C}$ (for instance, this happens if $N = \overline{W}, \overline{B}, \overline{V}, \overline{C}, Q^+, Q^-$). Given a representation $\varphi : \mathbb{Z}[H_1(M)] \to \mathbb{F}$, we can compose it with the map $i_* : \mathbb{Z}[H_1(N)] \to \mathbb{Z}[H_1(M)]$ induced by the inclusion $i : N \hookrightarrow M$. This gives a representation on N, that we will still denote by φ, with a slight abuse of notation.

Consider a field $\mathbb{F}$ and a representation $\varphi : \mathbb{Z}[H_1(M)] \to \mathbb{F}$. The *$\varphi$-twisted chain complex* of M relative to $\mathcal{P}$ is the chain complex over $\mathbb{F}$ defined by

$$C_*^\varphi(M, \mathcal{P}) = C_*^\varphi\big(M, \overline{W}\big) \oplus C_*^\varphi\big(\overline{C}, Q^+\big). \tag{7}$$

Its homology is called *φ-twisted homology* and it is denoted by $H_*^\varphi(M, \mathcal{P})$. A basis of $H_*^\varphi(M, \mathcal{P}) = H_*^\varphi(M, \overline{W}) \oplus H_*^\varphi(\overline{C}, Q^+)$ is a pair $(\mathfrak{h}', \mathfrak{h}'')$, where $\mathfrak{h}'$ is a basis of $H_*^\varphi(M, \overline{W})$ and $\mathfrak{h}''$ is a basis of $H_*^\varphi(\overline{C}, Q^+)$

A *fundamental family* $\mathfrak{f}$ of $(M, \mathcal{P})$ is a pair $(\mathfrak{f}', \mathfrak{f}'')$, where $\mathfrak{f}'$ is a fundamental family of the pair $(M, \overline{W})$ and $\mathfrak{f}''$ is a fundamental family of the pair $(\overline{C}, Q^+)$.

$\mathfrak{f} = (\mathfrak{f}', \mathfrak{f}'')$ induces a combinatorial Euler structure on M relative to $\mathcal{P}$ as follows. Lifting the inclusion $\overline{C} \hookrightarrow M$, one obtains a map $\iota : \hat{C} \to \hat{M}$ (here we have denoted by $\hat{C}$ the maximal abelian covering of $\overline{C}$), equivariant with respect to the inclusion homomorphism $H_1(\overline{C}) \to H_1(M)$. Take a point $x_0 \in \hat{M}$ and a point x_σ inside each cell $\sigma \in \mathfrak{f}' \cup \iota(\mathfrak{f}'')$. Choose paths β_σ from x_0 to x_σ and consider the 1-chain

$$\epsilon = \sum_{\sigma \in \mathfrak{f}} (-1)^{\dim(\sigma)} \beta_\sigma.$$

The projection of ϵ on M is an Euler chain, thus it represents an Euler structure $\mathfrak{e} \in \mathfrak{Eul}^c(M, \mathcal{P})$.

Given an Euler structure $\mathfrak{e} \in \mathfrak{Eul}^c(M, \mathcal{P})$ and a basis $\mathfrak{h}$ of $H_*^\varphi(M, \mathcal{P})$, the *Reidemeister torsion* of M relative to $\mathcal{P}$ is

$$\tau^\varphi(M, \mathcal{P}; \mathfrak{e}, \mathfrak{h}) = \tau(C_*^\varphi(M, \mathcal{P}); \mathfrak{f}, \mathfrak{h}) \in \mathbb{F}^*/\{\pm 1\}$$

where $\mathfrak{f}$ is a fundamental family of $(M, \mathcal{P})$ that induces the Euler structure $\mathfrak{e}$.

Proposition II.3. *$\tau^\varphi(M,\mathcal{P};\mathfrak{e},\mathfrak{h})$ is well defined. Namely, it does not depend on the choice of the fundamental family and of the cellularization. Moreover:*

$$\tau^\varphi(M,\mathcal{P};\mathfrak{e}',\mathfrak{h}) = \varphi\left(\alpha(\mathfrak{e},\mathfrak{e}')\right)\cdot\tau^\varphi(M,\mathcal{P};\mathfrak{e},\mathfrak{h}) \tag{8}$$

Proof. The indipendence on the cellularization is a consequence of the independence on the cellularization of the Reidemeister torsion of the pair defined in Section II.2. Formula (8) is easily proved by choosing representatives $\sum(-1)^{\dim\sigma}\beta_\sigma$, $\sum(-1)^{\dim\sigma}\beta'_\sigma$ of $\mathfrak{e},\mathfrak{e}'$ such that $\beta'_\sigma=\beta_\sigma$ for all σ but one.

It remains to prove that τ^φ does not depend on the choice of the fundamental family. To this end, consider two fundamental families

$$\mathfrak{f}_1 = (\mathfrak{f}'_1,\mathfrak{f}''_1) = (\{\sigma_1,\ldots,\sigma_r\},\{\sigma_{r+1},\ldots,\sigma_s\})$$

$$\mathfrak{f}_2 = (\mathfrak{f}'_2,\mathfrak{f}''_2) = (\{\tilde\sigma_1,\ldots,\tilde\sigma_r\},\{\tilde\sigma_{r+1},\ldots,\tilde\sigma_s\})$$

inducing the same Euler structure $\mathfrak{e}$. Suppose that $\mathfrak{f}_1$ and $\mathfrak{f}_2$ are ordered in such a way that $p(\sigma_j)=p(\tilde\sigma_j)$ $\forall j=1,\ldots,s$ (here we have denoted by the same letter the covering maps $p:\hat M\to M$ and $p:\hat C\to C$).

Hence, we can write $\tilde\sigma_j = h_j\sigma_j$, where $h_j\in H_1(M)$ for $j=1,\ldots,r$, $h_j\in H_1(\overline{C})$ for $j=r+1,\ldots,s$. Recall the inclusion morphism $i_*:H_1(\overline{C})\to H_1(M)$. $\mathfrak{f}_1$ and $\mathfrak{f}_2$ induce the same Euler structure, thus

$$\prod_{j=1}^{r} h_j^{(-1)^{\dim\sigma_j}}\cdot\prod_{j=r+1}^{s} i_*(h_j)^{(-1)^{\dim\sigma_j}}$$

is trivial in $H_1(M)$. Now, the result follows from the following easy computation (recall that $\mathfrak{h}=(\mathfrak{h}',\mathfrak{h}'')\in H_*^\varphi(M,\overline{W})\oplus H_*^\varphi(\overline{C},Q^+)$):

$$\begin{aligned}
&\tau(C_*^\varphi(M,\mathcal{P});\mathfrak{f}_1,\mathfrak{h}) = \\
&= \tau\left(C_*^\varphi(M,\overline{W});\mathfrak{f}'_1,\mathfrak{h}'\right)\cdot\tau\left(C_*^\varphi(\overline{C},Q^+);\mathfrak{f}''_1,\mathfrak{h}''\right) = \\
&= \varphi\Big(\prod_{j=1}^{r} h_j^{(-1)^{\dim\sigma_j}}\Big)\cdot\varphi\Big(i_*\Big(\prod_{j=r+1}^{s} h_j^{(-1)^{\dim\sigma_j}}\Big)\Big)\cdot\tau\left(C_*^\varphi(M,\overline{W});\mathfrak{f}'_2,\mathfrak{h}'\right)\cdot \\
&\qquad\cdot\tau\left(C_*^\varphi(\overline{C},Q^+);\mathfrak{f}''_2,\mathfrak{h}''\right) = \\
&= \varphi\Big(\prod_{j=1}^{r} h_j^{(-1)^{\dim\sigma_j}}\cdot\prod_{j=r+1}^{s} i_*(h_j)^{(-1)^{\dim\sigma_j}}\Big)\cdot\tau(C_*^\varphi(M,\mathcal{P});\mathfrak{f}_2,\mathfrak{h}) = \\
&= \tau(C_*^\varphi(M,\mathcal{P});\mathfrak{f}_2,\mathfrak{h})
\end{aligned}$$

□

Remark II.4. The definition of Reidemeister torsion above shows an indeterminacy in the sign, due to the arbitrariness in the choice of an order and an orientation of the fundamental family. A refinement of torsion exists: by means of an *homology orientation*, one can rule out the sign indeterminacy (see [13, § 18]). We will not consider homology orientation in our work, for it will complicate much more than expected the discussion and results of Section III.

Remark II.5. For a boundary pattern $\mathcal{P} = (W, B, V, C, \emptyset, \emptyset)$, one can define an $H_1(M)$-equivariant bijection $\Theta : \mathfrak{Eul}(M, \mathcal{P}) \to \mathfrak{Eul}(M, \theta(\mathcal{P}))$, where $\theta(\mathcal{P}) = (W, B, V \cup C, \emptyset, \emptyset, \emptyset)$ (see [2, § 1.2]). In general, it does not exist a basis $\mathfrak{h}'$ of $H_*^\varphi(\overline{C})$ such that:

$$\tau^\varphi(M, \mathcal{P}; \mathfrak{e}, \mathfrak{h} \cup \mathfrak{h}') = \tau^\varphi(M, \theta(\mathcal{P}), \Theta(\mathfrak{e}), \mathfrak{h}).$$

Thus, our definition of Reidemeister torsion is not coherent with the one given in [2, § 2] (in the case of mixed concave and convex tangency circles).

Notation. If M is closed, then the only boundary pattern is the trivial $\mathcal{P}_0 = (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$, thus we will avoid to specify it (for instance, we will write $C_*^\varphi(M)$ instead of $C_*^\varphi(M, \mathcal{P}_0)$). Notice that $\chi(M) = 0$ (from Poincaré duality), hence propositions I.2 and I.4 are automatically satisfied, i.e., the set of Euler structures $\mathfrak{Eul}(M)$ is not empty.

III Gluings

We show how to naturally define gluings of Euler structures, and we develop a multiplicative gluing formula for Reidemeister torsion (Theorem III.2).

III.1 Gluing of Euler structures

Let M be a 3-manifold and $S \subset M$ an embedded surface, that divides M into two smooth submanifolds M_1, M_2. Assume that M is closed (but the extension to $\partial M \neq \emptyset$ is straightforward).

Consider an Euler chain ξ_1 on M_1, relative to a partition $\mathcal{P} = (W, B, V, C, Q^+, Q^-)$ on $\partial M_1 = S$. ξ_1 represents an Euler structure $\mathfrak{e}_1 \in \mathfrak{Eul}^c(M, \mathcal{P})$. Denote by $\mathcal{P}'$ the partition (B, W, C, V, Q^-, Q^+) (namely, we swap black and white part, convex and concave lines, positive and negative cuspidal points). $\mathcal{P}$ and $\mathcal{P}'$ are said to be *dual*. Consider an Euler structure $\mathfrak{e}_2 \in \mathfrak{Eul}^c(M, \mathcal{P}')$, represented by an Euler chain ξ_2. It is clear that $\xi = \xi_1 + \xi_2$ is an Euler chain on M. Denote by $\mathfrak{e}_1 \cup \mathfrak{e}_2 \in \mathfrak{Eul}^c(M)$ the Euler structure represented by ξ. We have defined a *gluing map*:

$$(\mathfrak{e}_1, \mathfrak{e}_2) \mapsto \mathfrak{e}_1 \cup \mathfrak{e}_2 : \mathfrak{Eul}^c(M_1, \mathcal{P}) \times \mathfrak{Eul}^c(M_2, \mathcal{P}') \to \mathfrak{Eul}^c(M). \tag{9}$$

It is easy to obtain a differentiable version of the gluing map. Consider Euler structures $\mathfrak{e}_1 \in \mathfrak{Eul}^s(M_1, \mathcal{P})$, $\mathfrak{e}_2 \in \mathfrak{Eul}^s(M_2, \mathcal{P}')$ represented by generic fields $\mathfrak{v}_1, \mathfrak{v}_2$ respectively. Up to homotopy, we can suppose that $\mathfrak{v}_1$ and $\mathfrak{v}_2$ coincide on S. Then $\mathfrak{v}_1$ and $\mathfrak{v}_2$ can be glued together (in a smooth way), giving a non-singular vector field $\mathfrak{v}$ on M. Again, denote by $\mathfrak{e}_1 \cup \mathfrak{e}_2 \in \mathfrak{Eul}^s(M)$ the Euler structure represented by $\mathfrak{v}$. Now we can define the differentiable analogous of map (9):

$$(\mathfrak{e}_1, \mathfrak{e}_2) \mapsto \mathfrak{e}_1 \cup \mathfrak{e}_2 : \mathfrak{Eul}^s(M_1, \mathcal{P}) \times \mathfrak{Eul}^s(M_2, \mathcal{P}') \to \mathfrak{Eul}^s(M). \tag{10}$$

The following lemma can be deduced directly from definitions:

Lemma III.1. The following diagram is commutative

$$\begin{array}{ccc} \mathfrak{Eul}^c(M_1, \mathcal{P}) \times \mathfrak{Eul}^c(M_2, \mathcal{P}') & \xrightarrow{\cup} & \mathfrak{Eul}^c(M) \\ \downarrow{\scriptstyle \Psi_1 \times \Psi_2} & & \downarrow{\scriptstyle \Psi} \\ \mathfrak{Eul}^s(M_1, \mathcal{P}) \times \mathfrak{Eul}^s(M_2, \mathcal{P}') & \xrightarrow{\cup} & \mathfrak{Eul}^s(M) \end{array}$$

III.2 Setting

Again, let M be a closed 3-manifold and $S \subset M$ an embedded surface, that splits M into two smooth submanifolds M_1, M_2.

Let $\mathfrak{v}$ be a non-singular vector field on M representing the Euler structure $\mathfrak{e} \in \mathfrak{Eul}^s(M)$. Consider the restrictions $\mathfrak{v}_1 = \mathfrak{v}|_{M_1}$, $\mathfrak{v}_2 = \mathfrak{v}|_{M_2}$. Up to a small modification of S or $\mathfrak{v}$, we can suppose that $\mathfrak{v}_1$ (then also $\mathfrak{v}_2$) is generic. Let $\mathcal{P} = (W, B, V, C, Q^+, Q^-)$ be the partition induced by $\mathfrak{v}_1$ on $\partial M_1 = S$. Then it is easy to check that $\mathfrak{v}_2$ induces on S the dual partition $\mathcal{P}' = (B, W, C, V, Q^-, Q^+)$. Thus, $\mathfrak{v}_1$ (resp. $\mathfrak{v}_2$) represents an Euler structure $\mathfrak{e}_1 \in \mathfrak{Eul}^s(M_1, \mathcal{P})$ (resp. $\mathfrak{e}_2 \in \mathfrak{Eul}^s(M_2, \mathcal{P}')$), and $\mathfrak{e} = \mathfrak{e}_1 \cup \mathfrak{e}_2$.

Choose bases $\mathfrak{h}, \mathfrak{h}_1, \mathfrak{h}_2$ on the twisted homologies $H_*^\varphi(M)$, $H_*^\varphi(M_1, \mathcal{P})$, $H_*^\varphi(M_2, \mathcal{P}')$ respectively.

Fix a cellularization $\mathcal{C}$ on M suited with $\mathcal{P}$, and choose a representation $\varphi : H_1(M) \to \mathbb{F}$. We have the following short exact sequences:

$$0 \to C_*^\varphi(Q) \to C_*^\varphi(\overline{V}) \oplus C_*^\varphi(\overline{C}) \to C_*^\varphi(G) \to 0 \tag{a}$$

$$0 \to C_*^\varphi(G) \to C_*^\varphi(\overline{W}) \oplus C_*^\varphi(\overline{B}) \to C_*^\varphi(S) \to 0 \tag{b}$$

$$0 \to C_*^\varphi(S) \to C_*^\varphi(M_1) \oplus C_*^\varphi(M_2) \to C_*^\varphi(M) \to 0 \tag{c}$$

$$0 \to C_*^\varphi(Q^+) \to C_*^\varphi(\overline{C}) \to C_*^\varphi(\overline{C}, Q^+) \to 0 \tag{d}$$

$$0 \to C_*^\varphi(Q^-) \to C_*^\varphi(\overline{V}) \to C_*^\varphi(\overline{V}, Q^-) \to 0 \tag{e}$$

$$0 \to C_*^\varphi(\overline{W}) \to C_*^\varphi(M_1) \to C_*^\varphi(M_1, \overline{W}) \to 0 \tag{f}$$

$$0 \to C_*^\varphi(\overline{B}) \to C_*^\varphi(M_2) \to C_*^\varphi(M_2, \overline{B}) \to 0 \tag{g}$$

It is clear that all the submanifolds appearing in the exact sequences above are also subcomplexes of $\mathcal{C}$ (because $\mathcal{C}$ is suited with $\mathcal{P}$), thus the twisted complexes are well defined.

Fix bases on the twisted homologies of the complexes above. We have complete freedom in the choice, except for the following requirements:

- The union of the bases of $H_*^\varphi(Q^+), H_*^\varphi(Q^-)$ gives the basis on $H_*^\varphi(Q) = H_*^\varphi(Q^+) \oplus H_*^\varphi(Q^-)$;
- the union of the bases of $H_*^\varphi(M_1, \overline{W})$ and $H_*^\varphi(\overline{C}, Q^+)$ gives the basis $\mathfrak{h}_1$ on $H_*^\varphi(M_1, \mathcal{P})$;
- the union of the bases of $H_*^\varphi(M_2, \overline{B})$ and $H_*^\varphi(\overline{V}, Q^-)$ gives the basis $\mathfrak{h}_2$ on $H_*^\varphi(M_2, \mathcal{P}')$;
- the basis of $H_*^\varphi(M)$ is $\mathfrak{h}$;

Denote by $\tau_a, \tau_b, \tau_c, \tau_d, \tau_e, \tau_f, \tau_g$ the torsions of the long exact sequences of homologies induced by the short exact sequences (a),(b),(c),(d),(e), (f),(g) respectively, computed with respect to the chosen bases.

III.3 A formula for gluings

Theorem III.2. *In the notations of Section III.2, the following gluing formula holds:*

$$\tau^\varphi(M; \mathfrak{e}, \mathfrak{h}) = \mathfrak{T}(\mathfrak{h}, \mathfrak{h}_1, \mathfrak{h}_2) \cdot \tau^\varphi(M_1, \mathcal{P}; \mathfrak{e}_1, \mathfrak{h}_1) \cdot \tau^\varphi(M_2, \mathcal{P}'; \mathfrak{e}_2, \mathfrak{h}_2)$$

where

$$\mathfrak{T}(\mathfrak{h}, \mathfrak{h}_1, \mathfrak{h}_2) = (\tau_a)^{-1} \cdot \tau_b \cdot (\tau_c)^{-1} \cdot \tau_d \cdot \tau_e \cdot \tau_f \cdot \tau_g$$

Proof. The idea of the proof is simple: we want to apply theorem II.1 on the exact sequences of Section III.2. To this end, we need to specify compatible bases (at least up to sign) on the twisted complexes. The parenthesis “at least up to sign” is meaningful: we remember that we are not considering homology orientations (see Remark II.4) and we have a sign indeterminacy in the torsion. In order to consider signs, one has to track the behavior of the homology orientations and to choose bases compatible also in the sign (notice that some of the morphisms in the Mayer-Vietoris exact sequences have a minus sign); this will complicate too much the proof and the results.

We start from the exact sequence (d); notice that there is only a fundamental family on Q^+ (because $\hat{Q}^+ \cong Q^+$). Now choose a fundamental family $\mathfrak{f}_1''$ on $(\overline{C}, Q^+)$. One easily sees that the union of these two fundamental families gives a fundamental family on $\overline{C}$ and these choices lead to compatible bases. The same approach works on (e), and we obtain compatible fundamental families on Q^-, $(\overline{V}, Q^-)$, $\overline{V}$. Denote by $\mathfrak{f}_2''$ the fundamental family on $(\overline{V}, Q^-)$.

Denote by $\hat{V}$, $\hat{C}$, $\hat{G}$ the maximal abelian coverings of $\overline{V}$, $\overline{C}$, G respectively. We have the natural inclusions $\hat{V} \hookrightarrow \hat{G}$, $\hat{C} \hookrightarrow \hat{G}$, and one easily sees that the union of the fundamental families on $(\overline{V}, Q^-)$, $(\overline{C}, Q^+)$ gives a fundamental family $\mathfrak{f}^G$ on G. If we chose on Q the fundamental family given by the union of the fundamental families of Q^+, Q^-, we have that the chosen bases are compatible with respect to the exact sequence (a).

Now consider the exact sequence (b) and the commutative diagram

$$\begin{array}{ccc} \hat{G} & \xrightarrow{i_1} & \hat{W} \\ {\scriptstyle i_2}\Big\downarrow & & {\scriptstyle j_1}\Big\downarrow \\ \hat{B} & \xrightarrow{j_2} & \hat{S}. \end{array}$$

Here $\hat{W}$, $\hat{B}$, $\hat{S}$ are the maximal abelian coverings of $\overline{W}$, $\overline{B}$, S. The morphisms are lifts of the corresponding inclusion, and they are equivariant with respect to the inclusion homomorphism in first integer homology. We already have a fundamental family $\mathfrak{f}^G$ on G. $i_1(\mathfrak{f}^G)$ is a family of cells in $\hat{W}$ such that each cell in $G \subset W$ lifts to exactly one cell in the family. Complete $i_1(\mathfrak{f}^G)$ to a fundamental family $\mathfrak{f}^W$ on $\hat{W}$ by adding a lift for each cell in $W \setminus G$. In the same way, starting from the family $i_2(\mathfrak{f}^G)$, we obtain a fundamental basis $\mathfrak{f}^B$ of B. Notice that $\mathfrak{f}^S = j_1(\mathfrak{f}^W) \cup j_2(\mathfrak{f}^B \setminus i_2(\mathfrak{f}^G))$ is a fundamental basis of S and that the bases are compatible.

It remains to analyze sequences (c),(f),(g). Consider the commutative diagram

$$\begin{array}{ccc} \hat{S} & \xrightarrow{r_1} & \hat{M}_1 \\ {\scriptstyle r_2}\Big\downarrow & & {\scriptstyle s_1}\Big\downarrow \\ \hat{M}_2 & \xrightarrow{s_2} & \hat{M}. \end{array}$$

As above, we want to complete $r_1(\mathfrak{f}^S)$ to a fundamental family of M_1. Consider the family in $r_1(\mathfrak{f}^S)$ of the cells that are lifts of cells in $S \setminus \overline{W} = B$, and complete it to a fundamental family $\mathfrak{f}_1'$ of $(M_1, \overline{W})$ such that $\mathfrak{f}_1 = (\mathfrak{f}_1', \mathfrak{f}_1'')$ is a fundamental family of $(M_1, \mathcal{P})$ representing the Euler structure $\mathfrak{e}_1$. In the same way we obtain a fundamental family $\mathfrak{f}_2 = (\mathfrak{f}_2', \mathfrak{f}_2'')$ of $(M_2, \mathcal{P}')$ representing the Euler structure $\mathfrak{e}_2$.

Now, $\mathfrak{f}_1' \cup r_1(j_1(\mathfrak{f}^W))$ is a fundamental basis on M_1 and $\mathfrak{f}_2' \cup r_2(j_2(\mathfrak{f}^B))$ is a fundamental basis on M_2. Choose on M the fundamental family $\mathfrak{f} = s_1(\mathfrak{f}_1') \cup s_2(\mathfrak{f}_2') \cup t(\mathfrak{f}^G)$ (where $t = s_1 \circ r_1 \circ j_1 \circ i_1 : \hat{G} \to \hat{M}$). One easily sees that $\mathfrak{f}$ induces the Euler structure $\mathfrak{e} = \mathfrak{e}_1 \cup \mathfrak{e}_2$ and that the chosen fundamental families are compatible with respect to the exact sequences (c),(f),(g).

Therefore we can apply theorem II.1, obtaining seven equalities between torsions. The combination of them leads to the result; in the following calculation, all the torsions are computed with respect to the fundamental bases chosen above and the bases of the twisted homologies fixed in Section III.2:

$$\tau^\varphi(M; \mathfrak{e}, \mathfrak{h}) \underset{\text{(c)}}{=} (\tau_c)^{-1} \cdot (\tau^\varphi(S))^{-1} \cdot \tau^\varphi(M_1) \cdot \tau^\varphi(M_2) \underset{\text{(b),(f),(g)}}{=}$$

$$= \tau_b \cdot (\tau_c)^{-1} \cdot \tau_f \cdot \tau_g \cdot \tau^\varphi(G) \cdot \tau^\varphi(M_1, \overline{W}) \cdot \tau^\varphi(M_2, \overline{B}) \underset{\text{(a)}}{=}$$

$$= (\tau_a)^{-1} \cdot \tau_b \cdot (\tau_c)^{-1} \cdot \tau_f \cdot \tau_g \cdot (\tau^\varphi(Q))^{-1} \cdot \tau^\varphi(\overline{V}) \cdot \tau^\varphi(\overline{C}) \cdot \tau^\varphi(M_1, \overline{W}) \cdot \tau^\varphi(M_2, \overline{B}) \underset{\text{(d),(e)}}{=}$$

$$= (\tau_a)^{-1} \cdot \tau_b \cdot (\tau_c)^{-1} \cdot \tau_d \cdot \tau_e \cdot \tau_f \cdot \tau_g \cdot \tau^{\varphi}(M_1, \mathcal{P}; \mathfrak{e}_1, \mathfrak{h}_1) \cdot \tau^{\varphi}(M_2, \mathcal{P}'; \mathfrak{e}_2, \mathfrak{h}_2) \ \square$$

Remark III.3. Theorem III.2 extends easily to the case $\partial M \neq \emptyset$. One has to consider partitions $\mathcal{P}, \mathcal{P}_1, \mathcal{P}_2$ of $\partial M_1 \cap \partial M_2$, $\partial M \cap \partial M_1$, $\partial M \cap \partial M_2$ respectively; the resulting formula is:

$$\tau^{\varphi}(M, \mathcal{P}_1 \cup \mathcal{P}_2; \mathfrak{e}_1 \cup \mathfrak{e}_2, \mathfrak{h}) =$$
$$= \mathfrak{T}(\mathfrak{h}, \mathfrak{h}_1, \mathfrak{h}_2) \cdot \tau^{\varphi}(M_1, \mathcal{P} \cup \mathcal{P}_1; \mathfrak{e}_1, \mathfrak{h}_1) \cdot \tau^{\varphi}(M_2, \mathcal{P}' \cup \mathcal{P}_2; \mathfrak{e}_2, \mathfrak{h}_2).$$

Now the term $\mathfrak{T}(\mathfrak{h}, \mathfrak{h}', \mathfrak{h}'')$ contains other factors, coming from exact sequences involving elements of the partitions $\mathcal{P}_1$ and $\mathcal{P}_2$. We omit the details and the proof, that follows the same scheme as above.

III.4 Some computations

In what follows we will try to choose the bases of the twisted homologies wisely, in order to simplify the computation of $\mathfrak{T}(\mathfrak{h}, \mathfrak{h}_1, \mathfrak{h}_2)$. To this end, we notice that the exact sequences (a),(d),(e) are easy to compute in general, because we know exactly the involved chain complexes:

1. $Q = Q^+ \cup Q^-$ is a finite union of points. Let $Q^- = \{p_1, \dots, p_j\}$, $Q^+ = \{p_{j+1}, \dots p_k\}$. We have $H_*^{\varphi}(Q) = H_0^{\varphi}(Q) = \oplus_{i=1}^k \mathbb{F}\, p_i$, where we have identified Q with its maximal abelian covering.

2. G is a union of circles. Take one circle S; up to subdivision, S is a CW-complex with exactly one vertex p and one edge e. If $\varphi(H_1(S)) \neq 1$, then $C_*^{\varphi}(S)$ is acyclic (see [13, Lemma 6.2]), if $\varphi(H_1(S)) = 1$ then $H_0^{\varphi}(S) \cong H_1^{\varphi}(S) \cong \mathbb{F}$. We define a *canonical basis* on $H_*^{\varphi}(S)$ as the natural bases $\{\hat{p}, \hat{e}\}$, where $\hat{p}$ and $\hat{e}$ are lifts of p and e such that $\partial\hat{e} = t\hat{p} - \hat{p}$, and t is the generator of the action of $H_1(S)$ on $\hat{S}$ (see Fig. 9). The canonical basis on $H_*^{\varphi}(G)$ is the union of the canonical bases of all the circles in G.

3. $\overline{V}$ and $\overline{C}$ are unions of circles and segments. We have already studied the twisted homology of circles in point 2. Notice that segments retracts to points, hence their twisted homology is the same as point 1.

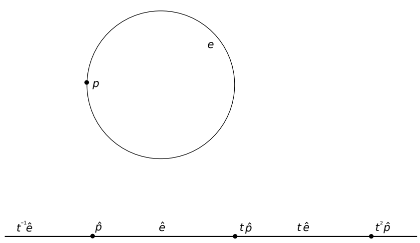

Fig. 9: A circle and its maximal abelian covering $\mathbb{R}$.

4. Now we study the pair $(\overline{C}, Q^+)$ (the same applies to $(\overline{V}, Q^-)$) and we fix a canonical basis, as already done for $H_*^\varphi(G)$. Each connected component S of $\overline{C}$ has one of the following four forms:

   - S is a circle: we have already studied this case in point 2, and we have already shown how to choose a canonical basis.
   - S is a segment and both points of ∂S belongs to Q^-: we have already studied it in point 3. Up to subdivisions, S is a CW-complex with exactly one edge e and two vertices $p_1, p_2 \in Q^-$ such that $\partial e = p_2 - p_1$. We have $H_0^\varphi(S, Q^+) = \mathbb{F}\, p_1$, thus a basis is formed by an element only. As a canonical basis for $H_*^\varphi(S)$ we chose $\{[p_1]\}$
   - S is a segment and both points of ∂S belong to Q^+: up to subdivisions, S is a CW-complex with exactly one edge e and two vertices p_1, p_2 such that $\partial e = p_2 - p_1$. We obtain $H_0^\varphi(\overline{C}, Q^+) = \{1\}$ and $H_1^\varphi(S, Q^+) = \mathbb{F}\, e$. In this case the canonical basis will be $\{[e]\}$.
   - S is a segment and ∂S is formed by a point in Q^+ and a point in Q^-: one easily checks that $C_*^\varphi(S, Q^+)$ is acyclic.

   The union of the canonical bases on the connected components gives the canonical basis on $H_*^\varphi(\overline{C}, Q^+)$.

These observations allow to easily compute torsions τ_a, τ_d, τ_e. We obtain the following:

Lemma III.4. Let $\{[p_1], \ldots, [p_r], [e_1], \ldots, [e_s]\}$ be the canonical basis of $(\overline{C}, Q^+)$ and $\{[p_{r+1}], \ldots, [p_u], [e_{s+1}], \ldots, [e_v]\}$ be the canonical basis of $(\overline{V}, Q^-)$. Equip $H_*^\varphi(G)$ with the canonical basis $\mathfrak{h}^G$. Let $\mathfrak{h}_1'', \mathfrak{h}_2''$ be generic bases of $H_*^\varphi(\overline{C}, Q^+)$, $H_*^\varphi(\overline{V}, Q^-)$. Then:

$$\{\mathfrak{h}_1'', \mathfrak{h}_2''\} = \{a_1[p_1], \ldots, a_u[p_u], b_1[e_1], \ldots b_v[e_v]\}$$

for opportune $a_i, b_j \in \mathbb{F}^*$. With respect to the bases $\mathfrak{h}^G, \mathfrak{h}_1', \mathfrak{h}_2'$ (regardless of the choice of the bases for the other twisted homologies), we have:

$$(\tau_a)^{-1} \cdot \tau_d \cdot \tau_e = \frac{a_1 \cdots a_u}{b_1 \cdots b_v}.$$

IV Combinatorial encoding of Euler structures

In Section IV.1 and IV.2 we recall the main results of [9]: in particular, we define stream-spines and we show that they encode vector fields on a 3-manifold. Using stream-spines, we show how to geometrically invert the reconstruction map Ψ (Theorem IV.6): this will give us a way to explicitly compute torsions.

IV.1 Stream-spines

A *stream-spine* P is a connected compact 2-dimensional polyedron such that a neighborhood of each point of P is homeomorphic to one of the five models in Fig. 10.

Specifically, a stream-spine P is formed by:

- some open surfaces, called *regions*, whose closure is compact and contained in P;
- some *triple lines*, to which three regions are locally incident;
- some *singular lines*, to which only one region is locally incident;
- some points, called *vertices*, to which six regions are incident;
- some points, called *spikes*, to which a triple line and a singular line are incident;

A *screw-orientation* on a triple line is an orientation of the line together with a cyclic ordering of the three regions incident on it, viewed up to a simultaneous reversal of both (see Fig. 11-left).

A stream-spine is said to be *oriented* if

- each triple line is endowed with a screw-orientation, so that at each vertex the screw-orientations are as in Fig. 11-center;
- each region is oriented, in such a way that no triple line is induced three times the same orientation by the regions incident to it.

Two oriented stream-spines are said to be *isomorphic* if there exists a PL-homomorphism between them preserving the orientations of the regions and the screw-orientations of the triple lines.

We denote by $\mathcal{S}_0$ the set of oriented stream-spines viewed up to isomorphism. An embedding of $P \in \mathcal{S}_0$ into a 3-manifold M is said to be *branched* if every region of P have a well defined tangent plane in every point, and the tangent planes at a singularity $p \in P$ to each region locally incident to p coincide (see Fig. 11-right for the geometric interpretation near a triple line; see [9, § 1.4] for an accurate definition of branching).

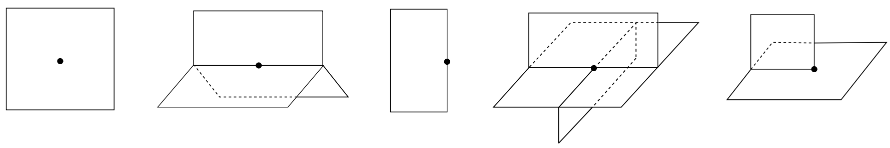

Fig. 10: Local models of a stream-spine

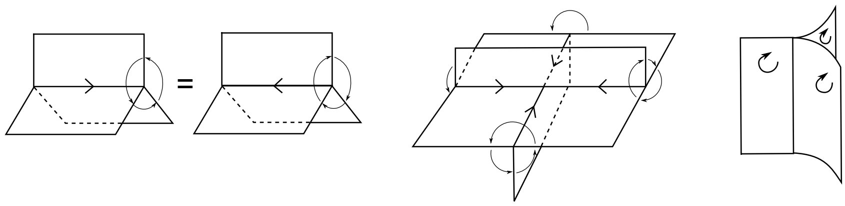

Fig. 11: Convention on screw orientation, compatibility at vertices and geometric interpretation of branching.

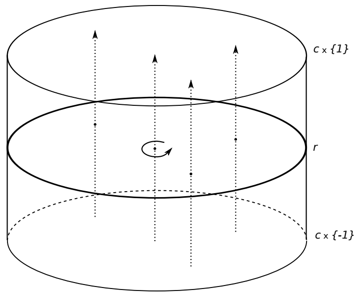

Fig. 12: Thickening c of the 2-cell r and the vector field $\mathfrak{v}^c$.

Proposition IV.1. *To each stream-spine $P \in \mathcal{S}_0$ is associated a pair $(\tilde{M}, \tilde{\mathfrak{v}})$, defined up to oriented diffeomorphism, where $\tilde{M}$ is a connected 3-manifold and $\tilde{\mathfrak{v}}$ is a vector field on $\tilde{M}$ whose orbits intersect $\partial\tilde{M}$ in both directions. Moreover, P embeds in a branched fashion in $\tilde{M}$ and the choice of a cellularization on P induces a cellularization $\tilde{\mathcal{C}}$ on $\tilde{M}$.*

Proof. The construction of $\tilde{M}$ and $\tilde{\mathfrak{v}}$ is carefully analyzed in [9, Prop. 1.2]. One start from the spine, thicken it to a PL-manifold $\hat{M}$ and then smoothen the angles to obtain a differentiable manifold $\tilde{M}$. $\tilde{\mathfrak{v}}$ is a vector field everywhere positively transversal to the spine.

It remains to show how to obtain the cellularization $\tilde{\mathcal{C}}$ from the cellularization of P. We will do it by thickening the 2-cells of P and then showing how to glue them together along the edges.

Pick a 2-cell r and thicken it to a cylinder $c \cong r \times [-1, 1]$. This identification is done in such a way that the original r is identified with $r \times \{0\}$, and the orientation of r (inherited from the branching of P) together with the positive orientation on the segment $[-1, 1]$ gives the positive orientation of $\mathbb{R}^3$ (see Fig. 12). The upper and lower faces $r \times \{1\}$ and $r \times \{-1\}$ will be part of the boundary (so they are not glued with any other quadrilateral); the side surface will be glued with the side surfaces of the other cylinders.

A natural vector field $\mathfrak{v}^c$ is defined on c: $\mathfrak{v}^c$ is the constant field whose orbits are rectilinear, directed from $r \times \{-1\}$ to $r \times \{1\}$, and orthogonal to $r \times \{0\}$ (see again Fig. 12).

c has a natural cellularization. Let $p_1, \dots, p_k$, $e_1, \dots e_k$ be the vertices and edges composing the boundary of r. Then the cells of c are the following:

1. the vertices are the points $p_i \times \{-1\}$ and $p_i \times \{1\}$, for $i = 1, \ldots, k$;
2. the edges are the lines $e_i \times \{-1\}$, $e_i \times \{1\}$, $p_i \times [-1, 1]$, for $i = 1, \ldots, k$;
3. the 2-cells are the faces $r \times \{-1\}$, $r \times \{1\}$ and $e_i \times [-1, 1]$ for $i = 1, \ldots, k$;
4. the only 3-cell is $r \times [-1, 1]$.

Now we shift our attention from the 2-cells to the edges of the cellularization of P. The edges will describe how to modify the side surfaces of the cylinders and how to glue them together.

Pick an edge e. Depending on the nature of e, we distinguish three cases:

- if e is a regular line (i.e., e is neither a singular nor a triple line), then it is contained in the boundary of two 2-cells r_1, r_2. The respective cylinders c_1, c_2 are simply glued together along the common face $e \times [-1, 1]$.
- if e is a singular line, then it is only contained in the boundary of one 2-cell r, thus no gluing is needed. We simply collapse the corresponding face $e \times [-1, 1]$ to the line $e \times \{0\}$ via the natural projection. Note that this collapse gives rise to a concave tangency line on the boundary (see Fig. 13-center);
- if e is a triple line, then there are three 2-cells r_1, r_2, r_3 containing the face $e \times [-1, 1]$. Recall that r_1, r_2, r_3 are oriented (with the orientation inherited from the spine) and that one, say r_1, induces on e the opposite orientation with respect to the other two (r_2, r_3). Subdivide the cell $e \times [-1, 1]$ in r_1 into two subcells $e \times [-1, 0]$ and $e \times [0, 1]$. Glue this two subcells with the corresponding cells on r_2 and r_3, as shown in Fig. 13-right. Note that this gluing gives rise to a convex tangency line.

Fig. 14 and Fig. 15 show what happens near vertices and spikes.

The gluing of the cylinders c, opportunely modified as explained above, and their vector fields $\mathfrak{v}^c$ gives rise to the pair $(\tilde{M}, \tilde{\mathfrak{v}})$ and to the cellularization $\tilde{\mathcal{C}}$. $\square$

IV.2 Combings

The main achievement of [9] is to show that stream-spines encode combings, so that they can be used as a combinatorial tool to study vector fields on 3-manifolds.

Proposition IV.1 gives us a map $\varphi : \mathcal{S}_0 \to \mathfrak{Comb}$. Unfortunately, this map is not surjective, as the image is formed only by combings $[M, \mathfrak{v}]$ where $\mathfrak{v}$ is

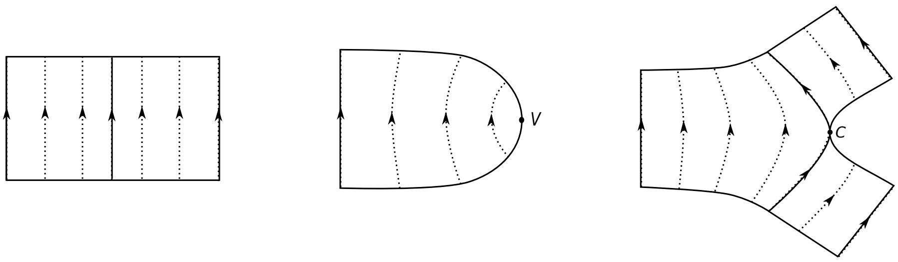

Fig. 13: Cross-section of gluings and modifications along regular (left), singular (center) and triple (right) line.

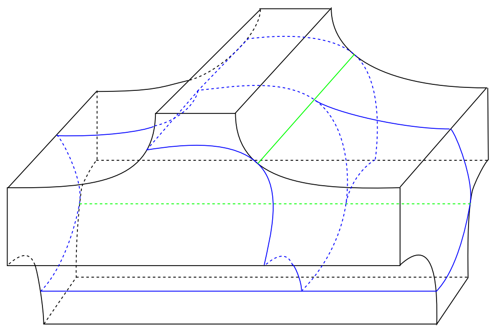

Fig. 14: Behavior of $\tilde{\mathfrak{v}}$ near a vertex. There are two concave lines corresponding to the two triple lines intersecting in the vertex. Notice that there is exactly one orbit of $\tilde{\mathfrak{v}}$ that is tangent to both the triple lines.

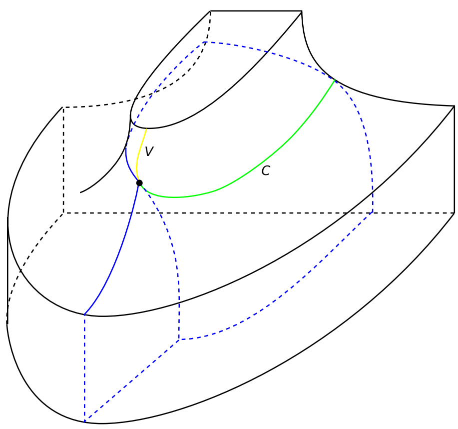

Fig. 15: Behavior of $\tilde{\mathfrak{v}}$ near a spike. Notice that $\tilde{\mathfrak{v}}$ goes from a concave tangency line (green) to a convex tangency line (yellow), or viceversa, through a cuspidal point.

a traversing field, i.e., a field whose orbits start and end on ∂M. Consider the subset $\mathcal{S} \subset \mathcal{S}_0$ of stream-spines P whose image $\varphi(P) = [\tilde{M}, \tilde{\mathfrak{v}}]$ contains at least one trivial sphere S_{triv} (i.e., a sphere in $\partial\tilde{M}$ that is split into one white disc and one black disc by a concave tangency circle). Denote by $\Phi(P)$ the combing $[M, \mathfrak{v}]$ obtained from $\varphi(P)$ by gluing to S_{triv} a trivial ball B_{triv} (i.e., a ball endowed with a vector field $\mathfrak{u}$ such that $(\partial B_{triv}, \mathfrak{u}|_{\partial B})$ is a trivial sphere) matching the vector fields. This gives a well defined map $\Phi : \mathcal{S} \to \mathfrak{Comb}$.

Theorem IV.2. $\Phi : \mathcal{S} \to \mathfrak{Comb}$ *is surjective.*

Remark IV.3. In [9] is also described a set of moves on stream-spines generating the equivalence relation induced by Φ. We will come back to this point in Section IV.4.

Remark IV.4. A restatement of the theorem is the following: given a non-singular vector field $\mathfrak{v}$ on a 3-manifold M, we can always find a sphere $S \subset M$ that splits $(M, \mathfrak{v})$ into a trivial ball B and a manifold $M \backslash B$ with a traversing field.

IV.3 Inverting the reconstruction map

Denote by $\mathcal{S}(M, \mathcal{P}) \subset \mathcal{S}$ the subset $\Phi^{-1}(\mathfrak{Comb}(M, \mathcal{P}))$. Φ restricts to a bijection $\mathcal{S}(M, \mathcal{P}) \to \mathfrak{Comb}(M, \mathcal{P})$. Composing Φ with the natural projection $\mathfrak{Comb}(M, \mathcal{P}) \to \mathfrak{Eul}^s(M, \mathcal{P})$, we obtain a map $\Xi^s : \mathcal{S}(M, \mathcal{P}) \to \mathfrak{Eul}^s(M, \mathcal{P})$.

We show in this section how to explicitly invert the reconstruction map via stream-spines. To do so, we will exhibit a map $\Xi^c : \mathcal{S}(M, \mathcal{P}) \to \mathfrak{Eul}^c(M, \mathcal{P})$ such that $\Xi^s = \Psi \circ \Xi^c$.

$$\begin{tikzcd}
 & & \mathfrak{Eul}^c(M,\mathcal{P}) \arrow[dd, "\Psi"] \\
\mathcal{S}(M,\mathcal{P}) \arrow[r, two heads, "\Phi"] \arrow[urr, bend left, "\Xi^c"] \arrow[drr, bend right, "\Xi^s"'] & \mathfrak{Comb}(M,\mathcal{P}) \arrow[dr, two heads] & \\
 & & \mathfrak{Eul}^s(M,\mathcal{P})
\end{tikzcd} \tag{11}$$

Take $P \in \mathcal{S}(M, \mathcal{P})$ and equip it with a cellularization. Recall from Proposition IV.1 that P induces a combing $\varphi(P) = [\tilde{M}, \tilde{\mathfrak{v}}]$ and a cellularization $\tilde{\mathcal{C}}$ on $\tilde{M}$.

Take a point p_u inside each cell $u \in \tilde{\mathcal{C}} \setminus \tilde{\mathcal{C}}_\partial$ (where $\tilde{\mathcal{C}}_\partial$ is the induced cellularization on $\partial\tilde{M}$), and denote by β_u the arc obtained by integrating $\tilde{\mathfrak{v}}$ in the positive direction, starting from p_u, until the boundary is reached. Consider the 1-chain:

$$\tilde{\xi}(P) = \sum_{u \in \mathcal{C}} (-1)^{\dim u} \cdot \beta_u.$$

Recall that $\Phi(P) = [M, \mathfrak{v}]$ is obtained from $[\tilde{M}, \tilde{\mathfrak{v}}]$ by gluing a trivial ball on a trivial sphere S_{triv} in $\partial\tilde{M}$. Thus we have a projection $\pi : \tilde{M} \to M$, obtained by collapsing S_{triv} to a point x_0, and a cellularization $\mathcal{C} = \pi(\tilde{\mathcal{C}})$ of M. It is easily seen that $\mathcal{C}$ is suited to the partition $\mathcal{P}$. Now consider the 1-chain $\xi(P) = \pi(\tilde{\xi}(P))$.

Lemma IV.5. $\xi(P)$ is a combinatorial Euler chain, and the class $[\xi(P)] \in \mathfrak{Eul}^c(M, \mathcal{P})$ does not depend on the cellularization chosen on P.

Proof. We first prove that $\xi(P)$ is an Euler chain. It is easily seen that $\partial\xi(P)$ contains, with the right sign, a point in each (open) cell of $\tilde{M}$, except for the cells of $W \cup V \cup Q^+$, as wished. It remains to prove that the resulting chain $\partial\xi(P)$ contains the singularity x_0 with coefficient 1. This coefficient is the sum of the coefficients of the cells in $B \cap S_{triv}$, and the conclusion follows from $\chi(B \cap S_{triv}) = \chi(\text{open disk}) = 1$.

The fact that $[\xi(P)]$ does not depend on the cellularization of P follows from the next theorem. $\square$

Theorem IV.6. $\Psi([\xi(P)]) = \Xi^s(P)$. *Thus the map that completes diagram* (11) *is defined by* $\Xi^c(P) = [\xi(P)]$.

Proof. Let $\mathfrak{w}_{\mathcal{C}}$ be the fundamental field of the cellularization $\mathcal{C}$. Recall from Theorem I.5 that the representative of $\Psi(\xi(P))$ is obtained by identifying M with a collared copy M_h of itself (the boundary of M_h is shown in red in Fig. 16), then applying a desingularization procedure to $\mathfrak{w}_{\mathcal{C}}$ in a neighborhood of $\xi(P)$. It should be noted that our cellularization $\mathcal{C}$ does not satisfy (Hp3) (in fact, the star at each spike differs from the one pictured in Fig. 4-left), thus the construction of h in Theorem I.5 does not apply directly. However, it is clear that a suitable function h can be defined (recall Remark I.7): the behavior of ∂M_h near regular, singular and triple line is shown in red in Fig. 16-right; the construction of h near spikes is a bit more complicated, but still analogous to the construction of h near cuspidal points in the proof of Theorem I.5.

It is easily seen that every connected component of the support S of $\xi(P)$ is contractible; therefore two different desingularizations of $\mathfrak{w}_{\mathcal{C}}$ represent the same Euler structure. Thus, it is enough to prove that $\mathfrak{v}$ is homologous to any desingularization of $\mathfrak{w}_{\mathcal{C}}$. In particular, it is enough to exhibit a desingularization that is everywhere antipodal to $\mathfrak{v}$.

We will do it in two steps:

- We prove that the set of points where $\mathfrak{w}_{\mathcal{C}}$ is antipodal to $\mathfrak{v}$ is contained in S;

- We provide a desingularization of $\mathfrak{w}_{\mathcal{C}}$ in a neighborhood of S to a field that is nowhere antipodal to $\mathfrak{v}$ in the neighborhood.

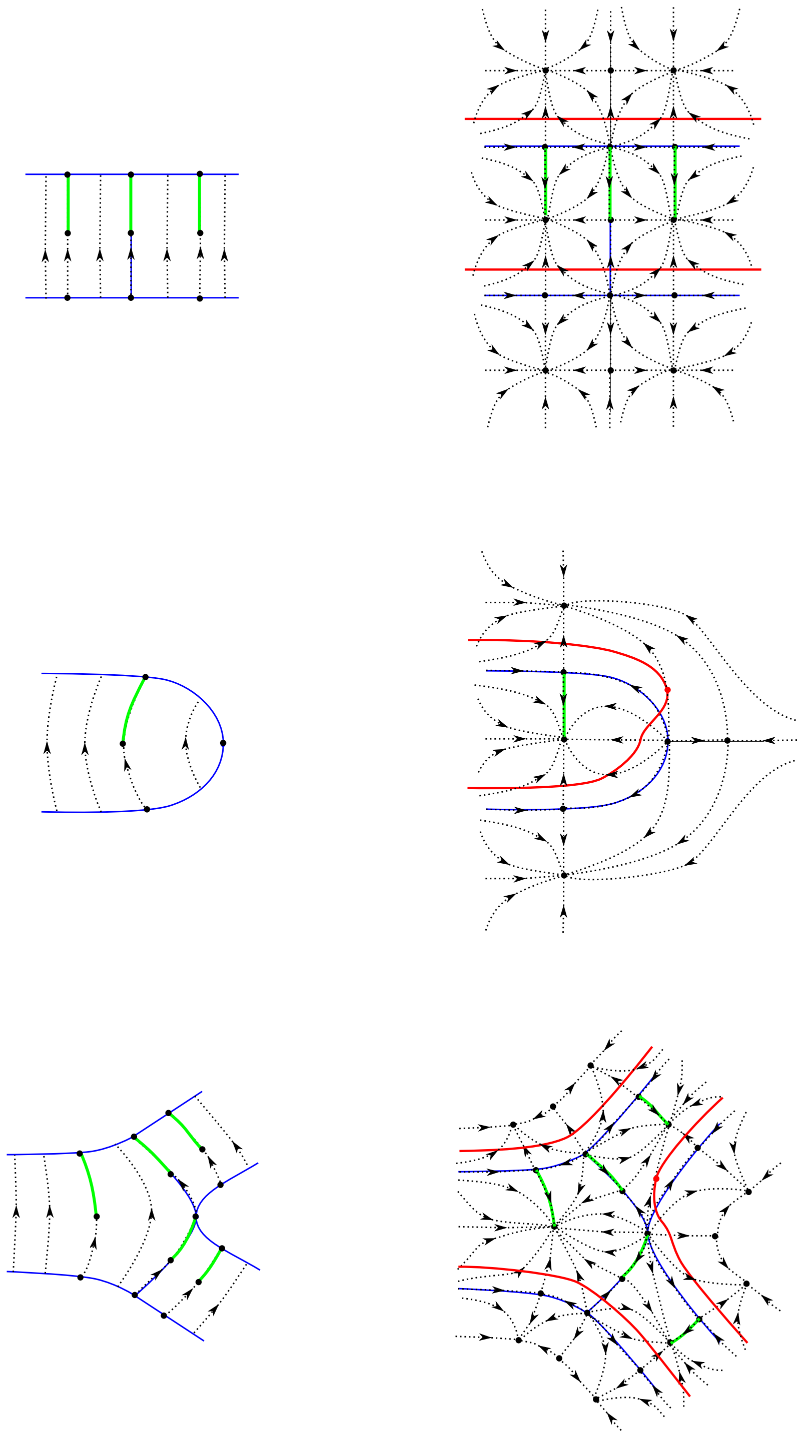

Fig. 16: Comparison between the field $\mathfrak{v}$ (left) and $\mathfrak{w}_{\tilde{\mathcal{C}}}$ (right) along regular, singular and triple lines. Notice that $\mathfrak{v}$ and $\mathfrak{w}_{\tilde{\mathcal{C}}}$ are antipodal only on S (in green).

We will prove the two claims working with $\tilde{M}$ (proving the formula on $\tilde{M}$ easily implies the formula on M). Notice that the cells of $\tilde{\mathcal{C}}$ are union of orbits of both $\mathfrak{w}_{\tilde{\mathcal{C}}}$ and $\mathfrak{v}$, hence we can analyze cells separately. Consider one of the cylinders c of the cellularization $\tilde{\mathcal{C}}$. Fig. 16 shows a cross-section of c and of the vector fields $\mathfrak{w}_{\tilde{\mathcal{C}}}$ and $\mathfrak{v}$: we see that they are antipodal only in S and it is easy to construct the wished desingularization. $\square$

Remark IV.7. In [9, § 3] is described how to explicitly invert the map Φ. Therefore Theorem IV.6 is an effective way to invert the reconstruction map: in details, one starts from a representative $\mathfrak{v}$ of a smooth Euler structure $\mathfrak{e}$, constructs the spine $P = \Phi^{-1}(\mathfrak{v})$ and applies Ξ^c to P.

IV.4 Standard stream-spines

We consider for a moment a standard spine P, i.e., a spine whose local models are the first, second and fourth of Fig. 10 only. This is the spine used in [2, § 3] to invert the reconstruction map for Euler structures relative to partitions without cuspidal points. It is easy to prove that one can transform each region of P in a 2-cell using sliding moves; hence the stratification of singularities gives a cellularization of P.

The same approach does not work with a stream-spine P, and we are left without a way to obtain a natural cellularization of P. In this section we show how to solve this problem by enriching the structure of a stream-spine with two new local models and a new sliding move.

A *standard stream-spine* P is a connected 2-polyedron whose local models are the five in Fig. 10, plus the two in Fig. 17; specifically, in addition to regular points, triple lines, singular lines, vertices, spikes, we allow:

1. some *bending lines* (Fig. 17-left), i.e., lines which are induced the same orientation by the two regions incident on it;

2. some *bending spikes* (Fig. 17-right), i.e., points where a singular, a triple and a bending line meet.

Moreover, we require the components of the stratification of singularities to be open cells. Denote by $\mathfrak{S}_0$ the set of standard stream-spines.

In addition to those described in [9, § 2.2], we define a new sliding move on $\mathfrak{S}_0$ as the one depicted in Fig. 18. Obviously, each standard stream-spine can be transformed into a stream-spine by applying the reversal of our sliding move to each bending line. This gives a natural map $\psi : \mathfrak{S}_0 \to \mathcal{S}_0$.

Consider now the set $\mathfrak{S}$ of standard stream-spines whose image is a stream-spine in $\mathcal{S}$.

Lemma IV.8. The restriction $\psi : \mathfrak{S} \to \mathcal{S}$ is surjective.

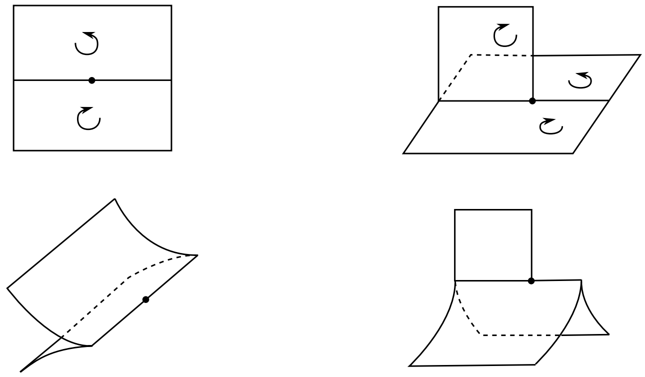

Fig. 17: New local models and their geometric interpretation.

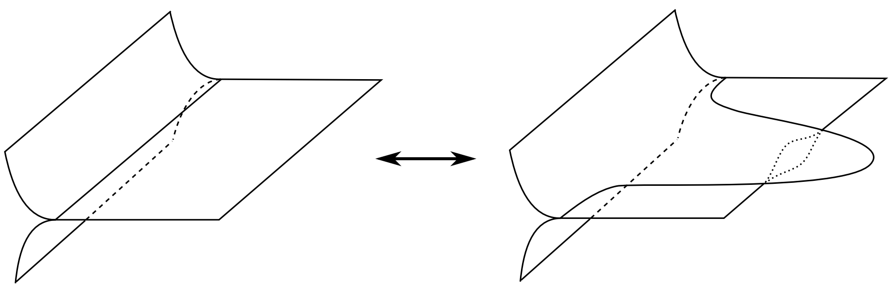

Fig. 18: The new sliding move consists in digging the triple line until the singular line is crossed.

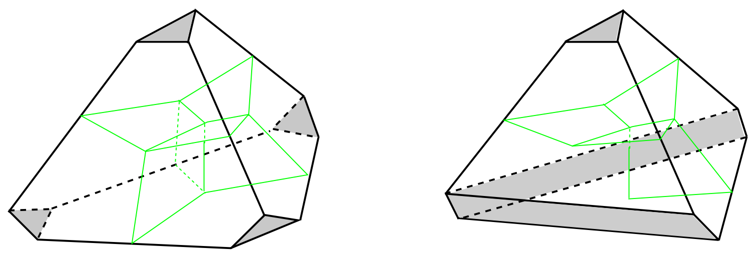

Fig. 19: Thickening of the singularities. The green part represents the immersion of the spine inside the manifold. The grey faces will form the boundary of the manifold; the white faces are glued with the white faces of other simplices.

Proof. It is enough to prove that each region of a stream-spine $P \in \mathcal{S}$ can be divided into a certain number of 2-cells by means of sliding moves. By definition, P contains a trivial sphere S, i.e., a sphere formed by two disks glued together along a triple line t, such that (1) the two disks induce the same orientation on t, and (2) P does not intersect the inner part of S. Consider a region r of P. If r contains no closed singular lines, the old sliding moves are enough to split r into 2-cells. If r contains a closed singular line s, we can slide t over other triple lines until we reach s (this can be done by means of the old sliding moves), then use our new sliding move to split s into a singular and a bending line. $\square$

Now we can repeat the arguments of Section IV.3 working with a spine $P \in \mathfrak{S}$ and the surjection $\Phi \circ \psi : \mathfrak{S} \to \mathfrak{Comb}$. The advantage is that now P is already endowed with a natural cellularization and we do not need to choose one.

It is easy to see how the thickening in the proof of Proposition IV.1 works near the new local models. On standard stream-spines we can even describe a different cellularization of $\tilde{M}$, more in the spirit of [2], by associating a simplex to each singularity:

- to each vertex we associate a truncated tetrahedron (Fig. 19-left), i.e., a simplex whose faces are four hexagons and four triangles;
- to each spike and to each bending spike we associate a tetrahedron with a different truncation (Fig. 19-right): its faces are one hexagon, four quadrilaterals and two triangles.

The simplices are then glued together as dictated by the spine (the ideas are the same as [5, Thm. 1.1.26]). The results of Section IV.3 can be recovered, without significant modifications, working with either the old or the new cellularization.

References

[1] M. Atiyah, *Topological Quantum Field Theories* . Publ. Math. IHES 68 (1989) 175-186.

[2] R. Benedetti, C. Petronio, *Reidemeister torsion of 3-dimensional Euler structures with simple boundary tangency and pseudo-Legendrian knots* . Manuscripta math. 106, 13-61, (2001).

[3] S. Halperin, D. Toledo, *Stiefel-Whitney homology classes* . Ann. of Math. (2) 96 (1972), 511-525.

[4] C. Lescop, *Global surgery formula for the Casson-Walker invariant* . Ann. of Math., Studies 140, Princeton University Press (1996).

[5] S. Matveev, *Algorithmic topology and classification of 3-manifolds* . Algorithms and Computation in Mathematics, Vol.9, Springer-Verlag, Berlin (2003).

[6] J. W. Milnor, *A duality theorem for Reidemeister torsion* . Ann. of Math., Vol.76, No.1 (1962).

[7] J. W. Milnor, *Whitehead torsion* . Bull. Amer. Math. Soc., Vol.72, No.3 (1966).

[8] B. Morin, *Formes canoniques des singularité d'une applicatione différentiable* . C. R. Acad. Sci. Paris 260 (1965), 6503-6506.

[9] C. Petronio, *Generic flows on 3-manifolds* . arXiv:1211.6445 (2013).

[10] K. Reidemeister, *Homotopieringe und Linsenraume* . Abh. Math. Sem. Univ. Hamburg, Vol.11 (1935).

[11] P. Ozsváth, Z. Szabó, *An introduction to Heegaard Floer homology* . Clay Math. Proc. Vol.5 (2006), 3-28.

[12] V. G. Turaev, *Euler structures, nonsingular vector fields, and torsions of Reidemeister type* . Math. USSR Izvestiya, Vol.34, No.3 (1990).

[13] V. G. Turaev, *Introduction to combinatorial torsion* . Notes taken by Felix Schlenk. Lectures in Mathematics ETH Zurich. Birkhäuser Verlag, Basel (2001).

[14] V. G. Turaev, *Torsions of 3-dimensional manifolds* . Progress in Mathematics, Vol.208, Birkhäuser Verlag, Basel (2002).

[15] H. Whitney, *On singularities of mappings of Euclidean spaces. I. Mapping of the plane into the plane* . Ann. of Math. Vol.62, No.3 (1955).