SCUOLA DI DOTTORATO
UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA

Dipartimento di / Department of

Dottorato di Ricerca in / PhD program Economics Ciclo / Cycle 32

Curriculum in (se presente / if it is)

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ANNO ACCADEMICO / ACADEMIC YEAR 2019

Abstract

This dissertation contains three essays in macroeconomics and finance. Chapter 1 is a survey of the techniques used to solve heterogeneous-agent economies with financial frictions in continuous time. Chapter 2 studies the impact of zombie firms on the economy. Chapter 3 explores how the impact of nonbanks on the economy.

Dedication

To my family

Declaration

I declare that...

Acknowledgements

I want to thank

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Chapter 1

Macro-Finance Modeling in Continuous Time: A Survey

1.1 Introduction

The global financial crisis of 2007-08 has highlighted the importance of understanding the role of the financial sector in propagating risk in the economy. Existing business cycle models failed to predict and explain the depth and persistence of the global financial crisis (Lindé et al., 2016; Benes et al., 2014; Milne, 2009) creating the need for better tools. A novel strand of literature Macro-Finance combines methods used in macroeconomics and finance to build models solved in continuous time.
Macro-finance models share many similarities to business cycle models with financial frictions. The main amplification mechanism characterized by adverse feedback loops determined by the changes in asset prices, weak balance sheets of financially constrained agents and pecuniary externalities is maintained (Brunnermeier and Sannikov, 2016; **?**). Financial frictions also arise due to incomplete markets which makes the wealth distribution of agents relevant to characterize equilibrium outcomes (Brunnermeier and Sannikov, 2016) (Brunnermeier and Sannikov, 2016). Furthermore, between-sector heterogeneity has an impact on aggregate outcomes, but within-sector heterogeneity does not. [1] Lastly, exogenous shocks from the real sector spreads to other sectors within the economy.
Comparing the two models, discrete-time business cycle models with financial frictions apply a local solution method to study shocks that originate at the deterministic steady state. Whilst, continuous-time macro-finance models apply a global solution method that characterizes the equilibrium along the entire state space, and are thus able to capture nonlinearities and large deviations from the steady state (Brunnermeier and Sannikov, 2014). Although, discrete-time business cycle models can admit several state variables continous time macro-finance models suffer from the curse of dimensionality and limited numerical techniques to solve the resulting partial

[1] Fernández-Villaverde, Hurtado, and Nuno (2019) build a model that shows that within-sector heterogeneity matters for understanding the aggregate consequences of financial frictions. Specifically, how agents in a heterogeneous model with financial frictions, idiosyncratic individual shocks interact with exogenous aggregate shocks to generate time-varying levels of leverage and endogenous aggregate risk. They provide new tools (machine learning) to solve for the infinite-dimensional wealth distribution since standard dynamic programming techniques cannot be used.

differential equations in order to obtain a reasonable result.
The purpose of this chapter is to review the tools required to implement and solve this methodology for a nonspecialist reader and provide a brief review of the literature. It is structured as follows. Section 1.1 Addresses feedback given from the reviewers. Section 1.2 provides a brief introduction to continous-time macro-finance models. Section 1.3 derives the Basak and Cuoco (1998) model with logarithmic utility solved in closed form and, the Brunnermeier and Sannikov (2016) model with Constant Relative Risk Aversion (CRRA) utility solved using numerical methods. Finally, section 1.4 concludes.

1.2 Solving Continuous-Time Macro-Finance Models

1.2.1 Basak and Cuoco (1998)

This section presents a detailed derivation of Basak and Cuoco (1998) using the Hamilton Jacobi Bellman approach. The model has no price effects or endogenous risk. Return processes, utility maximizations and market clearing conditions are used to find an equilibrium characterization. The model is then solved in closed-form and plotted. The results show that after a negative shock experts' wealth share η_t is reduced more than the output due to leverage and risk premia and Sharpe ratio to rise.

A. Environment

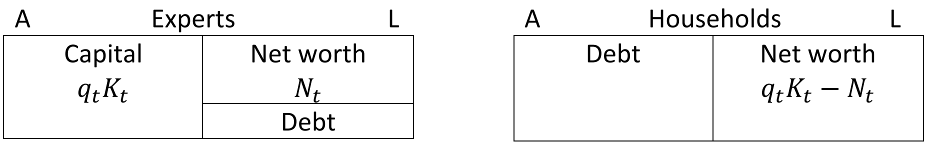


Figure 1.1: Balance Sheet Representation of the Basak and Cuoco (1998) Model
Notes: Adapted from Brunnermeier and Sannikov (2016)

The model is an infinite-horizon stochastic production economy with heterogeneous agents and financial frictions. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space that satisfies the usual conditions. Time is continuous with $t \in [0, \infty)$. The model is populated by a continuum of households and experts. There are two goods: the final consumption good and physical capital. Figure 1 provides a sketch of the balance sheet of these agents in equilibrium.

Technology
The production technology is linear and shows that experts are productive at holding capital. The capital held by each expert produces the following output:

$$y_t = ak_t$$

where a is the productivity parameter and k_t is the physical capital measured in efficiency units.
Households do not hold any capital.

Physical capital evolves according to:

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)dt + \sigma dZ_t, \tag{1.1}$$

where ι_t is the investment per unit of capital, δ is the depreciation rate, σ is a positive constant that captures the exogenous volatility of capital growth and Z_t is the Brownian motion linked to the capital quality shocks. Experts can convert $\iota_t k_t$ units of the final consumption good (output) into $k_t\Phi(\iota_t)$ units of physical capital using the following log investment function:

$$\Phi(\iota_t) = \frac{\log(\kappa\iota_t + 1)}{\kappa} \tag{1.2}$$

where κ is the adjustment cost parameter and function $\Phi(\iota_t)$ represents technological illiquidity or adjustment costs and satisfies $\Phi(0) = 0, \Phi'(0) = 1, \Phi'(.) > 0$ and $\Phi''(.) \leq 0$.

The Brownian shock is the only source of exogenous aggregate uncertainty in the model. It does not affect the productivity of experts, but it does impact the quantity of capital held for example, negative cash flow or demand shocks.

Preferences
There are two types of agents households and experts, Both types have logarithmic utility with the same discount rate ρ. The lifetime utility V_t of any agent is given by the recursion:

$$V_t = \mathbb{E}_t\left[\int_t^\infty e^{-\rho s}\log c_s\, ds\right],$$

where c_t is the consumption rate of the aggregate good at time t.

Markets
The following markets exist:

1. Physical capital q_t and the final goods in which experts and households trade continuously, and

2. Risk-free debt r_t

Financial Friction
Experts can only issue risk-free debt, but not equity to households. The financial friction can be motivated by a "skin in the game" constraint.

Capital Price and Returns
Physical capital. Return on capital managed by experts is

$$dr_t^k \equiv \underbrace{\frac{(a-\iota_t)k_t}{q_t k_t}}_{\text{dividend}} dt + \underbrace{\frac{d(q_t k_t)}{q_t k_t}}_{\text{capital gains}}, \tag{1.3}$$

where the first term is the dividend yield net of investment earned from capital, and the second term is the capital gains rate, which comprises of the price and capital quantity fluctuations.
In view of the uncertainty driving the economy, the evolution of the capital price process is conjectured as follows

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dZ_t, \tag{1.4}$$

where μ^q is the capital price growth and σ_t^q is the endogenous risk, i.e., the sensitivity of the capital price growth to the aggregate capital quality shock dZ_t. Both objects are determined endogenously in equilibrium.
Applying Ito's product rule [2] on equation (1.1) and equation (1.3) the capital return of experts in equation (1.5) can be written as

$$dr_t^k = \mu_t^R dt + \underbrace{(\underbrace{\sigma}_{\text{exogenous}} + \underbrace{\sigma_t^q}_{\text{amplification}})}_{\text{total diffusion risk} = \sigma^R} \underbrace{dZ_t}_{\text{real shock}}, \tag{1.5}$$

where

$$\mu_t^R \equiv \frac{a - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q,$$

is the expected return conditional on no aggregate shocks.

B. Agents' Problems

Households

Households cannot hold capital and can only invest in the experts short-term risk-free debt. They begin with net worth $\underline{n}_0$ obtained from the ownership of a fraction of aggregate capital which they straight away sell to experts. Households maximize their life-time utility function by choosing their consumption rate $\underline{c}_t$. The state variable for this problem is the net worth of household's given by $\underline{n}_t$. A household solves:

$$\underline{V}_0 = \max_{\{\underline{c}_t\}} \mathbb{E}\left[\int_0^\infty e^{-\rho t} \log(\underline{c}_t) dt\right]$$

subject to the law of motion of wealth:

$$\frac{d\underline{n}_t}{\underline{n}_t} = (r_t - \hat{\underline{c}}_t) dt \tag{1.6}$$

and the solvency constraint:

$$\underline{n}_t \geq 0 \tag{1.7}$$

Equation (1.6) is derived from the net wealth of the household as follows: $\underline{n}_t = \underline{b}_t$, i.e., net wealth is equal to assets (debt). Let $\underline{c}_t$ be the consumption of the household. Then $\underline{n}_t$ evolves as:

$$d\underline{n}_t = \underline{b}_t r_t dt - c_t dt$$

[2] Ito's product rule for two stochastic processes:

$$\frac{dX_tY_t}{X_tY_t} = (\mu_t^X + \mu_t^Y + \sigma_t^X \sigma_t^Y) dt + (\sigma_t^X + \sigma_t^Y) dZ_t$$

$$= (r_t - \underline{c}_t)dt$$

where the deterministic return on net wealth is equal to the return on risk-free debt,r_t less the consumption to net worth ratio $\underline{\hat{c}}_t = \underline{c}_t/\underline{n}_t$.

Experts

Experts maximize their life-time utility function by choosing their consumption rate c_t and portfolio weight of physical capital x_t. The state variable for this problem is the net worth of expert's given by n_t. An expert solves:

$$V_0 = \max_{\{c_t, x_t \geq 0\}} \mathbb{E}\left[\int_0^\infty e^{-\rho t} \log{(c_t)} dt\right]$$

subject to the law of motion of wealth:

$$\frac{dn_t}{n_t} = \left(r_t + x_t(\mu_t^R - r_t) - \hat{c}_t\right) dt + x_t \sigma_t^R dZ_t \tag{1.8}$$

and the solvency constraint:

$$n_t \geq 0 \tag{1.9}$$

Equation (1.8) is derived from the net wealth of the expert as follows: $n_t = q_t k_t - b_t$, i.e., net wealth is the difference between her assets (capital) and her liabilities (debt). Let c_t be the consumption of the expert. Then n_t evolves as:

$$dn_t = k_t dr_t^k - b_t r_t dt - c_t dt$$

$$= \left[(r_t + x_t(\mu_t^R - r_t))n_t - \hat{c}_t\right] dt + x_t \sigma_t^R n_t dZ_t$$

where x_t is the leverage ratio of the expert. The term $r_t + x_t(\mu_t^R - r_t)$ is the deterministic return on net wealth which is equal to the return on risk-free debt,r_t plus x_t times the excess return on leverage, $(\mu_t^R - r_t)$ less the consumption to net worth ratio $\hat{c}_t = c_t/n_t$. The term $x_t \sigma_t^R n_t$ reflects the risk of holding capital induced by the capital growth rate shock.

C. Equilibrium

Notation. Denote the set of experts by $\mathbb{I} = [0,1]$ and index individual experts by $i \in \mathbb{I}_t$. Similarly, denote the set of households by $\mathbb{J} = [1,2]$ and index individual households by $j \in \mathbb{J}_t$.
Equilibrium definition. For any initial endowment of capital $\{k_0^i, \underline{k}_0^j : i \in \mathbb{I}_0, j \in \mathbb{J}_0\}$ such that

$$\int_{\mathbb{I}_0} k_0^i di + \int_{\mathbb{J}_0} \underline{k}_0^j dj = K_0$$

an equilibrium is a set of stochastic functions on the filtered probability space defined by aggregate shock $\{Z_t : t \geq 0\}$: capital price $\{q_t\}$, risk-free rate $\{r_t(.)\}$, households' decisions $\{\underline{c}_t\}$ and experts' decisions $\{c_t, \iota_t, x_t\}$ such that

1. Initial net worths satisfy $n_0^i = q_0 k_0^i$ and $n_0^j = q_0 k_0^j$ for all $i \in \mathbb{I}_0$ and $j \in \mathbb{J}_0$.

2. Agents optimize

(a) Households: Given stochastic functions $\{q_t, r_t\}$, decisions $\{\underline{c}_t^j\}$ solve households' problem for all $j \in \mathbb{J}$

(b) Experts: Given stochastic functions $\{q_t, r_t\}$, decisions $\{c_t^i, \iota_t^i, x_t^i\}$ solve experts' problem for all $i \in \mathbb{I}$

3. Markets clear

(a) Goods

$$\int_{\mathbb{I}_t} c_t^i di + \int_{\mathbb{J}_t} \underline{c}_t^j dj = \int_{\mathbb{I}_t} (a - \iota_t^i) k_t^i di$$

(b) Capital

$$\int_{\mathbb{I}_t} k_t^i di = K_t$$

(c) Risk-free bonds by Walras' Law

4. The law of motion of aggregate capital is

$$dK_t = \Big(\int_{\mathbb{I}_t} \Phi(\iota_t^i) k_t^i di - \delta) K_t dt + \sigma K_t dZ_t \Big)$$

D. Discussion of assumptions

Capital quality shocks. Shocks to the quantity of capital are used instead of the usual productivity shocks. This increases tractability and is considered the norm in macro-finance literature. The two properties of (i) shocks to the quantity of capital and (ii) a linear production function enables the economy to scale with the capital stock. Particularly, it decreases the state space by removing the aggregate capital stock as a state variable. Capital is then measured in efficiency of units and the shocks represent news about its quality. Hence, a positive capital quality shock mimics a persistent positive productivity shock that produces a higher utilization rate.

Financial friction. Experts cannot issue any equity. The micro-foundation for for no equity issuance exists in the literature. It can be justified as a "skin in the game" constraints that matches the incentives of firm insiders to those of outside equity holders (Jensen and Meckling, 1979; Bolton and Scharfstein, 1990; DeMarzo and Sannikov, 2006). This constraint results in incomplete markets limiting the agents' ability to write contracts conditional on the aggregate capital quality shock. Market incompleteness connects the allocation of productive resources to the allocation of risk since agents need to bear the risk associated with capital in order to use it for production.

E. Equilibrium Solution

Aggregate state

To recap the only individual state variable of an agent's problem is his net worth.This is due to the fact that the shocks to the quantity of capital and the linear production function enables the economy to scale with the capital stock, therefore eliminating capital k_t as a state variable for experts. The state space can be further simplified to a single aggregate state variable using the following logic. (i) Since all agents' decisions will be linear in their net worth, the net worth heterogeneity within sectors will be irrelevant. So, it reduces the set of aggregate states to experts N_t and households' $\underline{N}_t$ aggregate net worth. (ii) The total capital in the economy K_t and the net worth share of experts can be used as states of the economy.

$$\eta_t \equiv \frac{N_t}{\underline{N}_t + N_t}$$

This is possible as capital is in positive net supply, therefore aggregate net worth is equal to the total value of capital stock, i.e., $\underline{N}_t + N_t = q_t K_t$. (iii) The aggregate of capital K_t is not a state variable due to scale invariance, i.e., allocations do not depend on the level of the capital stock. This follows from the linear production technology and the linearity of decisions in net worth. Hence, the only aggregate state is the expert sector's wealth share η_t.
Moving forward, I implement recursive notation where the time subindexes are suppressed. All equilibrium objects are functions of the aggregate state of the economy η_t, but this dependence is left implicit.

Risks
The economy is subject to only one type of aggregate risk. It relates to the exogenous real shock and its amplification mechanism. The total risk or total volatility arises from the exogenous and endogenous component. The exogenous risk or fundamental risk refers to the direct impact of the shock while the behaviour of the agent remains static. They are small and occur frequently. The endogenous risk is the the additional sensitivity generated by agents' endogenous responses, which create amplification. The risks are defined with reference to a variable of interest for example, returns on capital or output growth. Since this economy can be solved in closed form as detailed below there is no amplification mechanism.

Real investment
The return on capital for experts is maximized by choosing the investment rate that solves

$$\max_{\iota} q\Phi(\iota) - \iota.$$

The first-order condition $q\Phi'(\iota) = 1$ also known as the marginal Tobin's Q equates the marginal benefit of investment, i.e., $q\Phi'(\iota)$ to its marginal cost, i.e., a unit of final goods. The investment decision is a completely static problem in that it only depends on the current capital price as the investment process has no delays.

Households' Consumption Decision
The following characterizes the optimal consumption decision.
Households' optimal consumption c satisfies:

$$\underline{c} = \rho \underline{n} \tag{1.10}$$

where ρ is the time preference rate and $\underline{n}$ is the household's net worth. Optimal consumption for households is proportional to their wealth.
See derivation 1 in Appendix A

Experts' Consumption and Portfolio Decision

Consumption decisions are independent of asset returns since log utility is used. The portfolio decisions, i.e., capital holdings and issuance of short-term debt can be summarized by the capital portfolio share for experts $x \equiv \frac{1}{\eta}$. The optimal portfolio share of capital for experts x satisfies:

$$\mu^R - r = \underbrace{x(\sigma + \sigma^q)^2}_{\text{diffusion risk compensation}} \tag{1.11}$$

where the left-hand side is the market excess return (conditional on no aggregate shocks) of capital after short-term debt, and the right-hand side is the excess return required by experts to hold capital. Diffusion risk compensation can be expressed as risk price (the sensitivity of the expert's net worth to this risk) $x(\sigma + \sigma^q)$ multiplied by the risk quantity (the sensitivity of the asset return to diffusion risk) $(\sigma + \sigma^q)$. It is associated with the expert's stochastic discount factor. The following lemma characterizes the optimal consumption and portfolio decision.

Experts' optimal consumption c and optimal portfolio weight on capital x satisfies:

$$c = \rho n \tag{1.12}$$

$$x = \frac{\mu^R - r}{(\sigma + \sigma^q)^2} \tag{1.13}$$

For the optimal consumption equation (1.12), ρ is the time preference rate and n is the expert's net worth. Optimal consumption for experts is proportional to their wealth.

For the optimal portfolio choice equation (1.13), the numerator is the market excess return and the denominator is the diffusion risk compensation.

See derivation 2 in Appendix A

Market clearing

Experts hold all the capital in the economy. The total wealth in the economy is $qK = N + \underline{N}$ since capital is the only asset in positive net supply.

Output goods market clearing. The market clearing for output goods (scaled by total capital) can be given as:

$$\rho q + \iota(q) = a \tag{1.14}$$

where the left-hand side is the aggregate demand, and the right-hand side is the aggregate supply. The first time on the left-hand side is the aggregate consumption of all agents given by ρqK, which is the preference rate ρ multiplied by the total wealth qK. The second term on the left-hand side represents the aggregate investment $\iota(q)K$. The right-hand side is the total production per unit of capital.

Capital market clearing. The market clearing for capital (scaled by total wealth) is:

$$x\eta = 1 \tag{1.15}$$

where $xN = x\eta qK$ is the total wealth invested in capital by experts.

Simplifications

The output goods market clearing equation (1.14) leads to a constant value for the price of capital q, therefore $\mu^q = \sigma^q = 0$. Applying the investment function in equation (1.2), the optimal investment rate becomes $\iota = q - 1/\kappa$. Then applying the output goods market clearing condition leads to the capital price of

$$q = \frac{1+\kappa a}{1+\kappa\rho} \tag{1.16}$$

where the price converges to 1 as $\kappa \to 0$, i.e., the investment technology is fully elastic. The price q converges to a/ρ as $\kappa \to \infty$

Since q is constant the capital return of experts equation(1.9) becomes:

$$dr^k = \mu^R dt + \sigma dZ_t, \tag{1.17}$$

where

$$\mu^R \equiv \frac{a-\iota}{q} + \Phi(\iota) - \delta,$$

is the expected return conditional on no aggregate shocks.

Evolution of the aggregate state

Evolution of aggregate state variable. The wealth share of experts $\eta = N/qK$ characterizes the dynamics of the economy. Given the dynamic budget constraint of experts, the aggregate law of motion for N is derived as:

$$\frac{dN}{N} = rdt + x(\mu^R - r)dt - \frac{C}{N}dt + x\sigma dZ_t \tag{1.18}$$

The dynamics of the state variable also depend on the law of motion of the aggregate physical capital:

$$\frac{dK}{K} = (\Phi(\iota) - \delta)dt + \sigma dZ_t \tag{1.19}$$

Given the laws of motion of N, q and K the law of motion for η is derived and summarized in the following lemma [3].

The expert sector's wealth share dynamics are:

$$\frac{d\eta}{\eta} = \mu^\eta dt + \sigma^\eta dZ_t$$

where

$$\eta\mu^\eta = \eta\frac{(\eta-1)^2}{\eta^2}\sigma^2 \tag{1.20}$$

$$\eta\sigma^\eta = (1-\eta)\sigma \tag{1.21}$$

See derivation 3 in Appendix A

[3] Ito's ratio rule for two stochastic processes is applied, where:

$$\frac{dX_t/Y_t}{X_t/Y_t} = (\mu_t^X - \mu_t^Y + (\sigma_t^Y)^2 - \sigma_t^X\sigma_t^Y)dt + (\sigma_t^X - \sigma_t^Y)dZ_t$$

A Closed Form Solution
The portfolio share of experts can be derived from the capital market clearing condition in equation(1.15) and is given by:

$$x = \frac{1}{\eta}.$$

The asset pricing equation can be obtained by rewriting the expert's optimal portfolio share in equation(1.13) derived from the fist-order condition as follows:

$$\mu^R - r = \frac{\sigma^2}{\eta}$$

where the right-hand side is obtained by substituting $x = 1/\eta$ and setting $\sigma^q = 0$ since q is constant leading to no endogenous risk in the model. The risk-free rate can then be found by rearranging the asset pricing equation to yield:

$$r = \underbrace{\frac{a - \iota}{q}}_{\rho} + \underbrace{\frac{1}{\kappa}\log\left(\frac{1 + \kappa a}{1 + \kappa\rho}\right)}_{\Phi(\iota)} - \delta - \frac{\sigma^2}{\eta}$$

where the dividend yield is equal to the time preference rate ρ from the goods market clearing condition. The investment function $\Phi(\iota)$ is found by substituting in $\iota = q - 1/\kappa$ which simplifies to $\Phi(\iota) = \log(q)/\kappa$ and the price of capital calculated in equation(1.16) is inserted. The asset pricing equation and the risk-free rate is then plugged into the law of motion of N. Ito's Lemma is then used to derive the the experts' wealth share N/qK.
Therefore, four equations characterize the dynamics of the economy:

1. The capital price $q = \dfrac{1 + \kappa a}{1 + \kappa\rho}$
2. The risk-free rate $r = \rho + \dfrac{1}{\kappa}\log\left(\dfrac{1 + \kappa a}{1 + \kappa\rho}\right) - \delta - \dfrac{\sigma^2}{\eta}$
3. Diffusion of the state variable $\eta\sigma^\eta = (1 - \eta)\sigma$
4. Drift of the state variable $\eta\mu^\eta = \eta\dfrac{(\eta - 1)^2}{\eta^2}\sigma^2$

Numerical approach
Since q is constant, there is no need to express $q(\eta)$ as a function of η. Define a finite grid over the state variable η and plot the dynamics (1 - 4) of the economy using MATLAB.

Baseline parameters
The model is solved by using the following parameter values in table(2.1).

Table 1.1: Parameter values in Basak and Cuoco (1998)

Parametes	**Description**	**Model**
Production		
a	Experts productivity constant	11%
σ	Capital efficiency shock	10%
δ	Experts depreciation rate	3%
κ	Investment adjustment cost	10
Experts and households		
ρ	Discount rate	6%

Notes: Adapted from Brunnermeier and Sannikov (2016)

F. Economic Insights

General results

The interpretation of Figure(1.2) is as follows. After a negative shock experts' wealth share η is reduced by more than the output due to leverage. On the other hand a positive shock increases the wealth share of experts. The top left panel shows that there is no endogenous risk in the economy since the price of capital q is constant. Therefore, there are no price effects that would normally yield interesting results. The top right panel shows that when a negative shock occurs experts require a higher compensation, i.e., risk premium to hold the risky asset capital. The risk-free rate $r = \rho + \Phi(\iota) - \delta - \sigma^2/\eta$ tends to $-\infty$ and this increases the Sharpe ratio (the market price of risk - the performance of the risky asset compared to a risk-free asset, after adjusting for its risk which is obtained from the optimal portfolio choice of experts) of the asset which incentives the experts to continue holding risky capital.

Since agents have the same discount rate, in the long run the expert sector becomes so large that it overwhelms the entire economy. This is illustrated in bottom right panel. The drift of the wealth share of experts is always positive. This further highlights the fact that the expert sector has an advantage over the household sector as they can only invest in the risky asset.

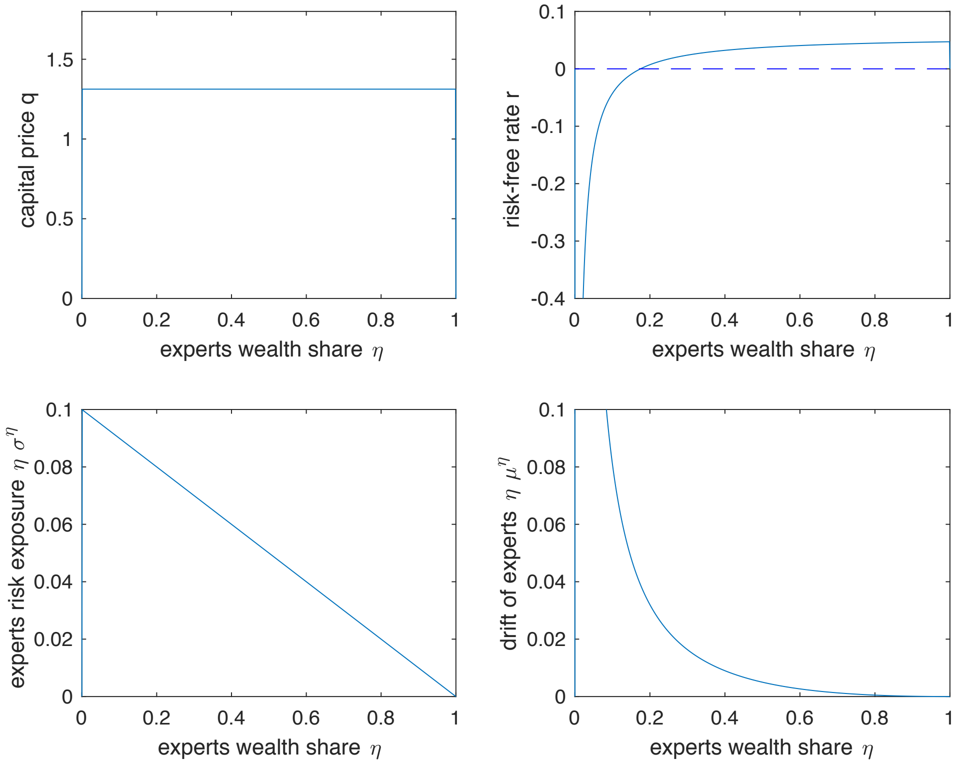


Figure 1.2: Capital price, risk-free rate, experts' risk exposure and drifk in Basak and Cuoco (1998)

1.2.2 Brunnermeier and Sannikov (2016)

This section presents a detailed derivation of Brunnermeier and Sannikov (2016) model using the Hamilton-Jacobi-Bellman (HJB) approach. This model differs from the one presented in the handbook in that experts cannot issue equity. Return processes, utility maximizations and market clearing conditions are used to find an equilibrium characterization. The model is then solved using numerical methods, and the results are plotted. There are two heterogeneous agents in the economy experts and households where experts are more productive than households. The economy is subject to instability and sometimes enters crisis periods. Risk is endogenous and asset price correlations are high during crisis. When a low shock hits the economy experts take more leverage making the economy more susceptible to systemic volatility spikes i.e. *volatility paradox*.

A. Environment

A Experts	L
Capital $\psi_t q_t K_t$	Inside Equity N_t fraction χ_t
	Outside Equity
	Debt

A Households	L
Debt	Net worth $q_t K_t - N_t$
Outside Equity	
Capital $(1 - \psi_t) q_t K_t$	

Figure 1.3: Balance Sheet Representation of the Brunnermeier and Sannikov (2016) Model
Notes: Adapted from Brunnermeier and Sannikov (2016)

The model is an infinite-horizon stochastic production economy with heterogeneous agents and financial frictions. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space that satisfies the usual conditions. Time is continuous with $t \in [0, \infty)$. The model is populated by a continuum of households and experts. There are two goods: the final consumption good and physical capital. Figure 1 provides a sketch of the balance sheet of these agents in equilibrium.

Technology
The production technology is linear and shows that experts are more productive than households at holding capital. The capital held by each expert and household produces the following output:

$$y_t = ak_t \qquad \text{and} \qquad \underline{y}_t = \underline{ak}_t$$

where a for experts and $\underline{a}$ for households is the productivity parameter with $a > \underline{a}$ and k_t and $\underline{k}_t$ is the physical capital measured in efficiency units. The underbar notation is used to denote parameters and variables associated with the household sector.
Physical capital evolves according to:

$$\begin{aligned} \text{Experts:} \qquad & \frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)dt + \sigma dZ_t \\ \text{Households:} \qquad & \frac{d\underline{k}_t}{\underline{k}_t} = (\Phi(\underline{\iota}_t) - \underline{\delta})dt + \sigma dZ_t \end{aligned} \tag{1.22}$$

where ι_t and $\underline{\iota}_t$ is the investment per unit of capital, δ and $\underline{\delta}$ is the depreciation rates with $\underline{\delta} > \delta$, σ is a positive constant that captures the exogenous volatility of capital growth and Z_t is the Brownian motion linked to the capital quality shocks. Experts and households can convert $\iota_t k_t$ and $\underline{\iota}_t \underline{k}_t$ units of the final consumption good (output) into $k_t \Phi(\iota_t)$ and $\underline{k}_t \Phi(\underline{\iota}_t)$ units of physical capital using the following log investment function:

$$\Phi(\iota_t) = \frac{\log(\kappa \iota_t + 1)}{\kappa} \tag{1.23}$$

where κ is the adjustment cost parameter and function $\Phi(\iota_t)$ represents technological illiquidity or adjustment costs and satisfies $\Phi(0) = 0, \Phi'(0) = 1, \Phi'(.) > 0$ and

$\Phi''(.) \leq 0$.
The Brownian shock is the only source of exogenous aggregate uncertainty in the model. It does not affect the productivity of experts, but it does impact the quantity of capital held for example, negative cash flow or demand shocks.

Preferences
Experts and households have Constant Relative Risk Aversion (CRRA) preferences with the same risk aversion γ, but different discount rates ρ and $\underline{\rho}$. The lifetime utility of each type of agent is given by the recursion:

$$\text{Experts:} \qquad V_t = \mathbb{E}_t \left[\int_t^\infty e^{-\rho s} \frac{c_s^{1-\gamma}}{1-\gamma} ds \right]$$

$$\text{Households:} \qquad \underline{V}_t = \mathbb{E}_t \left[\int_t^\infty e^{-\underline{\rho} s} \frac{\underline{c}_s^{1-\gamma}}{1-\gamma} ds \right].$$

Markets
The following markets exist:

1. Physical capital q_t and the final goods in which experts and households trade continuously
2. Risk-free debt r_t

Financial Friction
The friction in the economy arises from the fact that experts can only issue risk-free debt to households. This can be motivated by a "skin in the game" constraint.

Capital Price and Returns
Physical capital. Return on capital managed by:

$$\begin{aligned} &\text{Experts:} \qquad dr_t^k \equiv \underbrace{\frac{(a-\iota_t)k_t}{q_t k_t}}_{\text{dividend}} dt + \underbrace{\frac{d(q_t k_t)}{q_t k_t}}_{\text{capital gains}} \\ &\text{Households:} \qquad d\underline{r}_t^k \equiv \underbrace{\frac{(\underline{a}-\underline{\iota}_t)\underline{k}}{q\underline{k}_t}}_{\text{dividend}} dt + \underbrace{\frac{d(q_t \underline{k}_t)}{q_t \underline{k}_t}}_{\text{capital gains}} \end{aligned} \tag{1.24}$$

where the first term is the dividend yield net of investment earned from capital, and the second term is the capital gains rate, which comprises of the price and capital quantity fluctuations. Returns on capital managed by households $d\underline{r}_t^k$ are defined analogously using their parameters.
The evolution of the capital price process is conjectured as follows

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dZ_t, \tag{1.25}$$

where μ_t^q is the capital price growth and σ_t^q is the endogenous risk, i.e., the sensitivity of the capital price growth to the aggregate capital quality shock dZ_t. Both objects

are determined endogenously in equilibrium.
Applying Ito's product rule [4] to the equations in (1.22) and (1.25), the respective agent's capital return equations (1.24) can be written as:

$$\text{Experts:} \qquad dr_t^k = \mu_t^R dt + \underbrace{(\underbrace{\sigma}_{\text{exogenous}} + \underbrace{\sigma_t^q}_{\text{amplification}})}_{\text{total diffusion risk } = \sigma_t^R} \underbrace{dZ_t}_{\text{real shock}}$$

$$\text{Households:} \qquad d\underline{r}_t^k = \underline{\mu}_t^R dt + \underbrace{(\underbrace{\sigma}_{\text{exogenous}} + \underbrace{\sigma_t^q}_{\text{amplification}})}_{\text{total diffusion risk } = \underline{\sigma}_t^R} \underbrace{dZ_t}_{\text{real shock}}, \tag{1.26}$$

where

$$\text{Experts:} \qquad \mu_t^R \equiv \frac{a - \iota_t}{q_t} + \Phi(\iota_t) - \delta + \mu_t^q + \sigma\sigma_t^q$$

$$\text{Households:} \qquad \underline{\mu}_t^R \equiv \frac{\underline{a} - \underline{\iota}_t}{q_t} + \Phi(\underline{\iota}_t) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q,$$

is the expected return for experts and households conditional on no aggregate shocks.

B. Agents' Problems

Households

Households maximize their life-time utility function by choosing their consumption rate $\underline{c}_t$ and portfolio weight of physical capital $\underline{x}_t$. The state variable for this problem is the net worth of household's given by $\underline{n}_t$. For a given $\underline{n}_s$ a household solves:

$$\underline{V}_0 = \max_{\{\underline{c}_t, \underline{x}_t \geq 0\}} \mathbb{E}\left[\int_0^\infty e^{-\underline{\rho} t} \frac{\underline{c}_t^{1-\gamma}}{1-\gamma} dt\right]$$

subject to the law of motion of wealth:

$$\frac{d\underline{n}_t}{\underline{n}_t} = (r_t + \underline{x}_t(\underline{\mu}_t^R - r_t) - \hat{\underline{c}}_t)dt + \underline{x}_t \underline{\sigma}_t^R dZ_t \tag{1.27}$$

and the solvency constraint:

$$\underline{n}_t \geq 0 \tag{1.28}$$

Equation(1.27) is derived from the net wealth of the household as follows: $\underline{n}_t = q_t \underline{k}_t + \underline{b}_t$, i.e., net wealth is the sum of her assets capital and short-term lending. Let $\underline{c}_t$ be the consumption of the household. Then $\underline{n}_t$ evolves as:

$$d\underline{n}_t = \underline{k}_t d\underline{r}_t^k + \underline{b}_t r_t dt - \underline{c}_t$$

$$\left[(r_t + \underline{x}_t(\underline{\mu}_t^R - r_t))\underline{n}_t - \hat{\underline{c}}_t\right] dt + \underline{x}_t \underline{\sigma}_t^R \underline{n}_t dZ_t$$

where $\underline{x}_t$ is the leverage ratio of the household. The term $r_t + \underline{x}_t(\underline{\mu}_t^R - r_t)$ is the deterministic return on net wealth which is equal to the return on the risk-free debt, r_t plus $\underline{x}_t$ times the excess return on leverage $(\underline{\mu}_t^R - r_t)$ less the consumption to net worth ratio $\hat{\underline{c}}_t$. The term $\underline{x}_t \underline{\sigma}_t^R \underline{n}_t$ reflects the risk of holding capital induced by the capital growth rate shock.

[4] Ito's product rule for two stochastic processes:

$$\frac{dX_tY_t}{X_tY_t} = (\mu_t^X + \mu_t^Y + \sigma_t^X \sigma_t^Y)dt + (\sigma_t^X + \sigma_t^Y)dZ_t$$

Experts

Experts maximize their life-time utility function by choosing their consumption rate c_t and portfolio weight of physical capital x_t. The state variable for this problem is the net worth of household's given by n_t. For a given n_s an expert solves:

$$V_0 = \max_{\{c_t, x_t \geq 0\}} \mathbb{E}\left[\int_0^\infty e^{-\rho t}\,\frac{c_t^{1-\gamma}}{1-\gamma}dt\right]$$

subject to the law of motion of wealth:

$$\frac{dn_t}{n_t} = (r_t + x_t(\mu_t^R - r_t) - \hat{c}_t)dt + x_t\sigma_t^R dZ_t \tag{1.29}$$

and the solvency constraint:

$$n_t \geq 0 \tag{1.30}$$

Equation(1.29) is derived from the net wealth of the expert as follows: $n_t = q_t k_t - b_t$, i.e., net wealth is the difference between her assets (capital) and and her liabilities (debt). Let c_t be the consumption of the expert. Then n_t evolves as:

$$\begin{aligned} dn_t &= k_t dr_t^k - b_t r_t dt - c_t dt \\ &= \Big[(r_t + x_t(\mu_t^R - r_t))n_t - \hat{c}_t\Big]\,dt + x_t\sigma_t^R n_t dZ_t \end{aligned}$$

where x_t is the leverage ratio of the expert. The term $r_t + x_t(\mu_t^R - r_t)$ is the deterministic return on net wealth which is equal to the return on risk-free debt,r_t plus x_t times the excess return on leverage, $(\mu_t^R - r_t)$ less the consumption to net worth ratio $\hat{c}_t = c_t/n_t$. The term $x_t\sigma_t^R n_t$ reflects the risk of holding capital induced by the capital growth rate shock.

C. Equilibrium

Notation. Denote the set of experts by $\mathbb{I} = [0,1]$ and index individual experts by $i \in \mathbb{I}_t$. Similarly, denote the set of households by $\mathbb{J} = [1,2]$ and index individual households by $j \in \mathbb{J}_t$.

Equilibrium definition. For any initial endowment of capital $\{k^i 0, \underline{k}_0^j : i \in \mathbb{I}_0, j \in \mathbb{J}_0\}$ such that

$$\int_{\mathbb{I}_0} k_0^i di + \int_{\mathbb{J}_0} \underline{k}_0^j dj = K_0$$

an equilibrium is a set of stochastic functions on the filtered probability space defined by aggregate shock $\{Z_t : t \geq 0\}$: capital price $\{q_t\}$, risk-free rate $\{r_t(.)\}$, households' decisions $\{\underline{c}_t, \underline{\iota}_t, \underline{x}_t\}$, experts' decisions $\{c_t, \iota_t, x_t\}$ and value function $\{V^i(.)\}$ for $i \in \mathbb{I}$ and $\{\underline{V}^j(.)\}$ for $j \in \mathbb{J}$ such that

1. Initial net worths satisfy $n_0^i = q_0 k_0^i$ and $n_0^j = q_0 k_0^j$ for all $i \in \mathbb{I}_0$ and $j \in \mathbb{J}_0$.
2. Agents optimize
   (a) Households: Given stochastic functions $\{q_t, r_t, \underline{V}_t^j(.)\}$, decisions $\{\underline{c}_t, \underline{\iota}_t, \underline{x}_t\}$ solve households' problem for all $j \in \mathbb{J}$
   (b) Experts: Given stochastic functions $\{q_t, r_t(.), V_t^i(.)\}$, decisions $\{c_t^i, \iota_t^i, x_t^i\}$ solve experts' problem for all $i \in \mathbb{I}$

3. Markets clear

   (a) Goods
$$\int_{\mathbb{I}_t} c_t^i di + \int_{\mathbb{J}_t} \underline{c}_t^j dj = \int_{\mathbb{I}_t} (a - \iota_t^i) k_t^i di + \int_{\mathbb{J}_t} (\underline{a} - \underline{\iota}_t^j) \underline{k}_t^j dj$$

   (b) Capital
$$\int_{\mathbb{I}_t} k_t^i di + \int_{\mathbb{J}_t} \underline{k}_t^j dj = K_t$$

   (c) Risk-free bonds by Walras' Law

4. The law of motion of aggregate capital is

$$dK_t = \left(\int_{\mathbb{I}_t} \Phi(\iota_t^i) k_t^i di - \delta + \int_{\mathbb{J}_t} \Phi(\underline{\iota}_t^j) \underline{k}_t^j dj - \underline{\delta} \right) K_t dt + \sigma K_t dZ_t$$

D. Discussion of assumptions

Productivity difference. Experts' higher productivity captures the advantage that the expert sector has over households in holding capital. Capital is efficiently allocated when it is in the hands of the most productive sector. It is important that households hold capital in order to capture fire-sales in the model and to enable households to speculate.
Capital quality shocks. Shocks to the quantity of capital are used instead of the usual productivity shocks. This increases tractability and is considered the norm in macro-finance literature. The two properties of (i) shocks to the quantity of capital and (ii) a linear production function enables the economy to scale with the capital stock. Particularly, it decreases the state space by removing the aggregate capital stock as a state variable. Capital is then measured in efficiency of units and the shocks represent news about its quality. Hence, a positive capital quality shock mimics a persistent positive productivity shock that produces a higher utilization rate.
Financial friction. Experts cannot issue any equity. The micro-foundation for no equity issuance exists in the literature. It can be justified as a "skin in the game" constraints that matches the incentives of firm insiders to those of outside equity holders (Jensen and Meckling, 1979; Bolton and Scharfstein, 1990; DeMarzo and Sannikov, 2006). This constraint results in incomplete markets limiting the agents' ability to write contracts conditional on the aggregate capital quality shock. Market incompleteness connects the allocation of productive resources to the allocation of risk since agents need to bear the risk associated with capital in order to use it for production.

E. Recursive equilibrium solution

Aggregate state
To recap the only individual state variable of an agent's problem is his net worth.This is due to the fact that the shocks to the quantity of capital and the linear production function enables the economy to scale with the capital stock, therefore eliminating capital k_t as a state variable for either the household and expert. The state space can be further simplified to a single aggregate state variable using the following logic. (i) Since all agents' decisions will be linear in their net worth, the net worth

heterogeneity within sectors will be irrelevant. So, it reduces the set of aggregate states to experts N_t and households' $\underline{N}_t$ aggregate net worth. (ii) The total capital in the economy K_t and the net worth share of experts can be used as states of the economy.

$$\eta_t \equiv \frac{N_t}{\underline{N}_t + N_t}$$

This is possible as capital is in positive net supply, therefore aggregate net worth is equal to the total value of capital stock, i.e., $\underline{N}_t + N_t = q_t K_t$. (iii) The aggregate of capital K_t is not a state variable due to scale invariance, i.e., allocations do not depend on the level of the capital stock. This follows from the linear production technology and the linearity of decisions in net worth. Hence, the only aggregate state is the expert sector's wealth share η_t.
Moving forward, I implement recursive notation where the time subindexes are suppressed. All equilibrium objects are functions of the aggregate state of the economy η_t, but this dependence is left implicit.

Risks
The economy is subject to only one type of aggregate risk. It relates to the exogenous real shock and its amplification mechanism. The total risk or total volatility arises from the exogenous and endogenous component. The exogenous risk or fundamental risk refers to the direct impact of the shock while the behaviour of the agent remains static. They are small and occur frequently. The endogenous risk is the the additional sensitivity generated by agents' endogenous responses, which create amplification. The risks are defined with reference to a variable of interest for example, returns on capital or output growth.

Value functions
Since agents have CRRA utility, guess and verify that the value function of each agent has the following form:

$$V(n,\zeta) = \frac{1}{\rho}\frac{(n\zeta)^{1-\gamma}}{1-\gamma}, \qquad V(\underline{n},\underline{\zeta}) = \frac{1}{\rho}\frac{(\underline{n}\underline{\zeta})^{1-\gamma}}{1-\gamma}$$

for some stochastic processes $\{\zeta, \underline{\zeta}\}$ representing time variations in the set of investment opportunities for experts and households respectively. The law of motion of for the wealth multipliers is postulated as the following Ito process:

$$\frac{d\zeta}{\zeta} = \underbrace{\mu^{\zeta}}_{-r} dt + \underbrace{\sigma^{\zeta}}_{-\varsigma} dZ_t, \qquad \frac{d\underline{\zeta}}{\underline{\zeta}} = \underbrace{\mu^{\underline{\zeta}}}_{-\underline{r}} dt + \underbrace{\sigma^{\underline{\zeta}}}_{\underline{\varsigma}} dZ_t$$

Since the bond price is e^{-r} and is equal to the expected stochastic discount factor (SDF) multiplied by 1,the drift of the SDF is equal to the risk-free rate. The drift of the SDFs for both experts and household equal $-r = -\underline{r}$ since both agents can trade the risk-free asset.

Markov Equilibrium
Market clearing. The market clearing for goods (scaled by total capital) can be written as:

$$\frac{C}{K} + \frac{\underline{C}}{\underline{K}} = \psi(a - \iota(q)) + (1-\psi)(\underline{a} - \iota(q)) \tag{1.31}$$

where $\psi = x\eta$ is the capital share managed by experts. The left-hand side of the equation(1.31) represents the aggregate demand, and the right-hand side is the aggregate supply. The first two terms on the left is the sum of the aggregate consumption of experts and households. The aggregate supply on the right-hand side is given by experts and households productivities less investment weighted by the share of capital that they manage. Equation(1.31) simplifies to:

$$(\hat{C}\eta + \underline{\hat{C}}(1-\eta))q = \psi(a-\iota) + (1-\psi)(\underline{a}-\iota)$$

where $\hat{C} = C/N$ and $\underline{\hat{C}} = \underline{C}/\underline{N}$ are the aggregate consumption to net worth ratio of experts and households respectively.
The market clearing of capital (scaled by total wealth) is:

$$x\eta + \underline{x}(1-\eta) = 1 \tag{1.32}$$

where $xN = x\eta qK$ is the total wealth invested in capital by experts, and $\underline{xN} = \underline{x}(1-\eta)qK$ is the total wealth invested in capital by households.
Consistency. Capital price dynamics need to be consistent with the dynamics of the aggregate state, i.e.

$$\begin{aligned} q\mu^q &= q_\eta \mu^\eta \eta + \frac{1}{2} q_{\eta\eta}(\sigma_\eta \eta)^2 \\ q\sigma^q &= q_\eta \sigma_\eta \eta \end{aligned} \tag{1.33}$$

Investment opportunities need to be consistent with the dynamics of the aggregate state, i.e.

$$\begin{aligned} \zeta\mu^\zeta &= \zeta_\eta \mu^\eta \eta + \frac{1}{2} \zeta_{\eta\eta}(\sigma^\eta \eta)^2 \\ \zeta\sigma^\zeta &= \zeta_\eta \sigma^\eta \eta \end{aligned} \tag{1.34}$$

where equations (1.33) and (1.34) are found by applying Ito's lemma to the function $q(\eta)$ and $\zeta(\eta)$. The same holds for household investment opportunities.
Markov equilibrium definition. A Markov equilibrium in η is a set of functions $f = f(\eta)$ for (i) prices $\{q, r\}$, (ii) individual controls for experts $\{c/n, x, \}$, households $\{\underline{c}/\underline{n}, \underline{x}, \}$, and the dynamics of experts' wealth share $\{\mu^\eta, \sigma^\eta\}$ such that:

1. Wealth multipliers $\{\zeta, \underline{\zeta}\}$ solve their respective Hamilton-Jacobi Bellman equations with optimal controls (ii) given prices (i).
2. Markets for output good, capital and risk-free bond clears:
   (a) Goods equation(1.31)
   (b) Capital equation(1.32)
   (c) Risk-free bond by Walras' Law.
3. The laws of motion for the state variable η are consistent with equilibrium functions.
4. Capital price dynamics are consistent with the dynamics of the aggregate state equation(1.33).
5. Investment opportunities are consistent with the dynamics of the aggregate state (1.34).

Real investment decision

The return on capital for experts and households is maximized by choosing the investment rate that solves

$$\max_{\iota} q\Phi(\iota) - \iota.$$

The first-order condition $q\Phi'(\iota) = 1$ also known as the marginal Tobin's Q equates the marginal benefit of investment, i.e., $q\Phi'(\iota)$ to its marginal cost, i.e., a unit of final goods. This implies that the optimal investment rate is a function of the price q_t, i.e.,

$$\iota = \underline{\iota} = \iota(q).$$

The investment decision is a completely static problem in that it only depends on the current capital price as the investment process has no delays.

Households' consumption and portfolio decision

The optimal portfolio share of capital for households $\underline{x}$ satisfies

$$\underline{\mu}^R - r \leq \underbrace{\gamma \underbrace{\underline{x}(\sigma + \sigma^q)^2}_{\text{wealth}} + (\gamma - 1)(\sigma + \sigma^q) \underbrace{\sigma^{\zeta}}_{\text{investment opportunity}}}_{\text{risk compensation}} \tag{1.35}$$

The left-hand side is the market excess return of capital over the risk-free rate, and the right-hand side is the compensation of risk required by the households to hold capital which stems from the wealth and investment opportunity risk. If $\underline{x} > 0$, the condition holds with equality and the excess return needs to compensation for the risk.

The optimal consumption for households is given by:

$$\underline{c} = \underline{n}\underline{\rho}^{\frac{1}{\gamma}}\underline{\zeta}^{1-\frac{1}{\gamma}} \tag{1.36}$$

The optimal capital portfolio share for households is given by:

$$\underline{x} = \frac{\underline{\mu}^R - r + (1-\gamma)(\sigma + \sigma^q)\sigma^{\underline{\zeta}}}{\gamma(\sigma + \sigma^q)^2} \tag{1.37}$$

See derivation 4 in Appendix A

Experts' consumption and portfolio decision

Similar to households, the optimal portfolio share of capital for experts x satisfies

$$\mu^R - r = \gamma x(\sigma + \sigma^q)^2 + (\gamma - 1)(\sigma + \sigma^q) \tag{1.38}$$

The left-hand side is the market excess return of capital over the risk-free rate, and the right-hand side is the compensation of risk required by the experts to hold capital which stems from the wealth and investment opportunity risk.

The optimal consumption for experts is given by:

$$c = n\rho^{\frac{1}{\gamma}}\zeta^{1-\frac{1}{\gamma}} \tag{1.39}$$

$$x = \frac{\mu^R - r + (1-\gamma)(\sigma + \sigma^q)\sigma^{\zeta}}{\gamma(\sigma + \sigma^q)^2} \tag{1.40}$$

See derivation 5 in Appendix A

Evolution of the aggregate state

Evolution of aggregate state variable. The wealth share of experts $\eta = N/qK$ characterizes the dynamics of the economy. Given the dynamic budget constraint of experts, the aggregate law of motion for N is derived as:

$$\frac{dN}{N} = (r + x(\mu^R - r) - \hat{C})dt + x(\sigma + \sigma^q)dZ_t \tag{1.41}$$

The dynamics of the state variable also depend on the law of motion of the aggregate physical capital:

$$\frac{dK}{K} = (\Phi(\iota) - \delta)dt + \sigma dZ_t \tag{1.42}$$

Given the laws of motion of N, q and K the law of motion for η is derived and summarized in the following lemma [5].

The expert sector's wealth share dynamics are:

$$\frac{d\eta}{\eta} = \mu^\eta dt + \sigma^\eta dZ_t$$

where

$$\mu^\eta = r + x(\mu^R - r) - \hat{C} - \Phi(\iota) + \delta - \mu^q - \sigma\sigma^q + (\sigma + \sigma^q)^2 - x(\sigma + \sigma^q)^2 \tag{1.43}$$

$$\sigma^\eta = (x - 1)(\sigma + \sigma^q)^2 \tag{1.44}$$

See derivation 6 in Appendix A

Numerical approach

The model will be solved recursively using the state variable defined in Lemma **??**. The equilibrium variables will be solved recursively in the state variable then iterated on the value function multiplier. From the Hamilton Jacobi Bellman equation of experts and households the finite difference method can be applied to:

$$\mu^\zeta = \frac{\zeta_\eta}{\zeta}\mu^\eta\eta + \frac{1}{2}\frac{\zeta_{\eta\eta}}{\zeta}(\sigma^\eta\eta)^2$$

for experts and an analogous equation for households.

Then Ito's lemma can be applied to find σ^q, σ^ζ and μ^q from:

$$\sigma^q = \frac{q_\eta}{q}\sigma^\eta\eta$$

$$\sigma^\zeta = \frac{\zeta_\eta}{\zeta}\sigma^\eta\eta$$

for experts and an analogous equation for households

$$\mu^q = \frac{q_\eta}{q}\mu^\eta\eta + \frac{1}{2}\frac{q_{\eta\eta}}{q}(\sigma^\eta\eta)^2$$

The system can be solved using MATLAB by:

[5] Ito's ratio rule for two stochastic processes is applied, where:

$$\frac{dX_t/Y_t}{X_t/Y_t} = (\mu_t^X - \mu_t^Y + (\sigma_t^Y)^2 - \sigma_t^X\sigma_t^Y)dt + (\sigma_t^X - \sigma_t^Y)dZ_t$$

1. Defining a finite grid over the state variable η and set a guess for ζ and $\underline{\zeta}$ at the initial iteration.

2. Given ζ and $\underline{\zeta}$ solve for all equilibrium variables and diffusions using first order conditions and market clearing conditions using the Newton-Raphson method.

3. Solve for the next iteration of ζ_{n+1} and $\underline{\zeta}_{n+1}$ using the method described in Brunnermeier and Sannikov (2016).

4. Iterate on step 2 and 3 until convergence.

Baseline parameters
The model is solved by using the following parameter values in table(**??**).

Table 1.2: Parameter values in Brunnermeier and Sannikov (2016)

Parametes	**Description**	**Model**
Production		
a	Experts productivity constant	11%
$\underline{a}$	Households productivity constant	2%
σ	Capital efficiency shock	10%
δ	Experts depreciation rate	3%
κ	Investment adjustment cost	10
γ	Relative Risk Aversion	2
Experts and households		
ρ	Experts discount rate	6%
$\underline{\rho}$	Households discount rate	6%

Notes: Adapted from Brunnermeier and Sannikov (2016)

F. Economic Insights

Price and Capital Allocation

The model is characterized by two regions determined by the capital allocation parameter ψ. Looking at the capital allocation ψ and capital price q functions in Figure 1.4 and the price volatility σ^q function in Figure 1.5, the first region occurs when $\psi = 1$. Here, experts own all the capital and the price volatility is equal to zero and the capital price is at its maximum. Households in this region do not have incentives to buy capital. However, when the wealth share of experts is low experts risk exposure increases without bound. The second region occurs when $\psi < 1$ also known as the fire-sale region. Here, both households and experts are willing to hold capital, i.e.,:

$$\left(\underbrace{\frac{\psi}{\eta}}_{\text{experts}} - \underbrace{\frac{1-\psi}{1-\eta}}_{\text{households}} \right) (\sigma + \sigma^q)^2 = \frac{a - \underline{a}}{q} \tag{1.45}$$

where, experts have relatively more exposed to risk than households. The goods market clearing also implies that more supply of capital needs more demand. In equilibrium as the net worth of experts is decreasing, capital is misallocated as it is held in the hands of the less productive agent in the economy, the price of capital is reduces and the endogenous risk, i.e. the price volatility is high. Therefore, when experts wealth share is low the economy is in crisis and households act as liquidity providers as their speculative motive is to sell the capital back to experts at higher prices in the future (Brunnermeier and Sannikov, 2016; **?**).

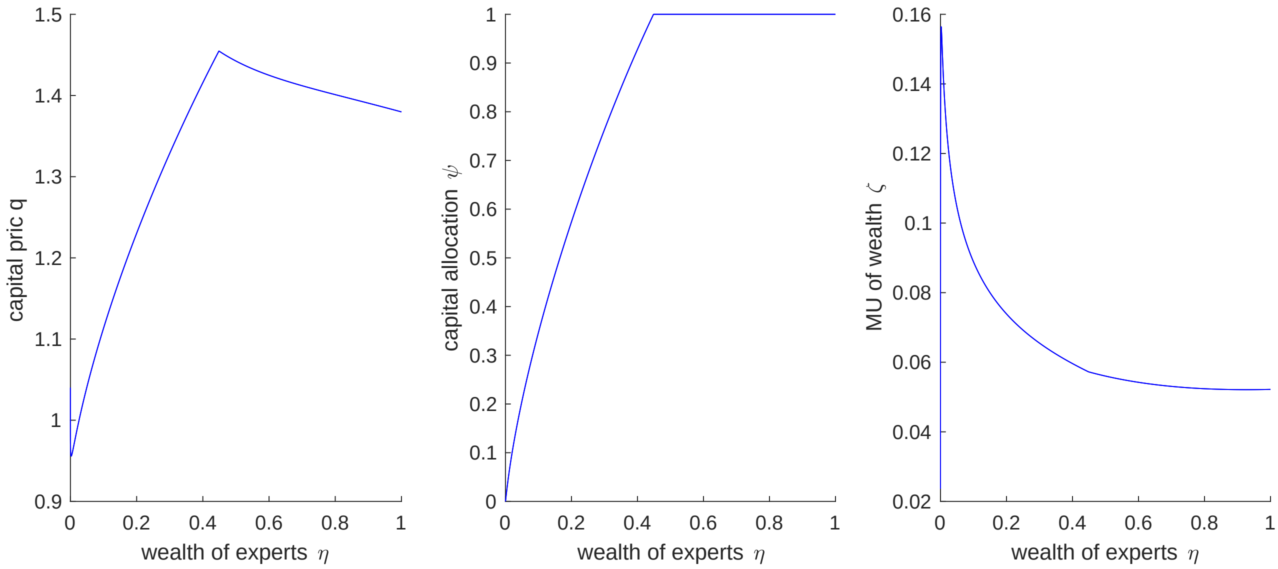


Figure 1.4: Capital price, capital allocation and experts marginal utility functions in Brunnermeier and Sannikov (2016)
Notes: Functions are calculated using code provided in Brunnermeier and Sannikov (2016).

Investment Opportunities

The marginal utility of wealth ζ of experts in Figure**??** shows the experts stochastic investment opportunities. The marginal utility of experts consumption decreases as the wealth of experts decrease. This is due to the positive relationship between the price of capital q and the wealth of experts η. Since the marginal utility of net worth is equal to the marginal utility of consumption, they are both constant at 0.05. At this point experts are indifferent between consuming and saving. In the range of $[0, 0.05]$ experts do not consume, and after that range they start to consume and channel any additional income to consumption (**?**).

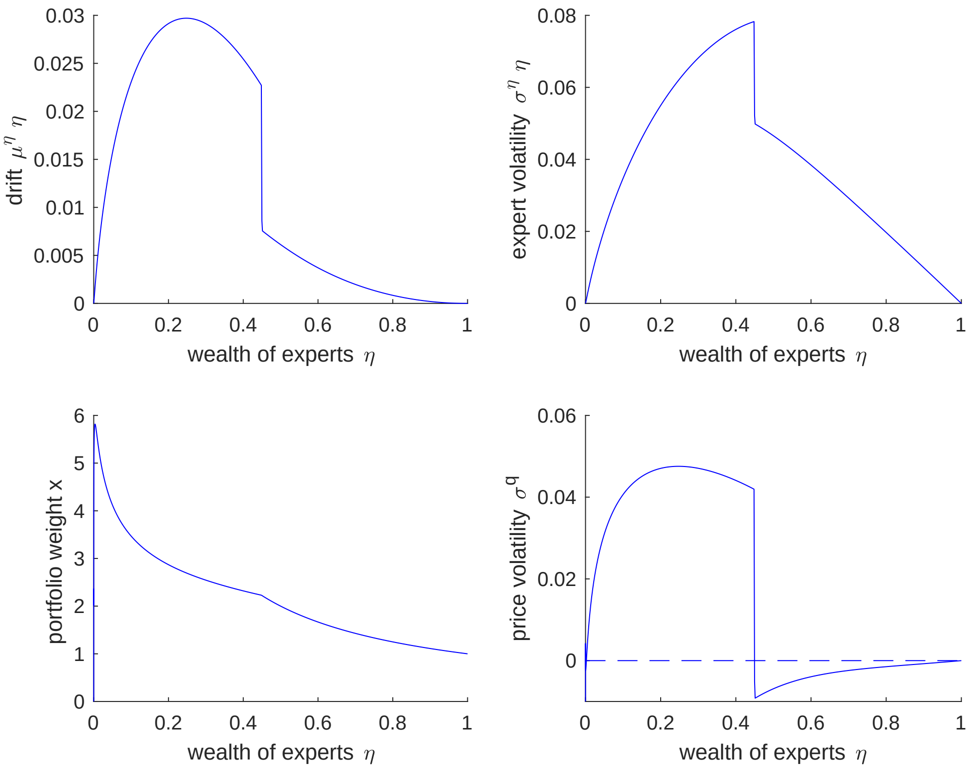


Figure 1.5: Drift, expert volatility, portfolio weight and price volatility functions in Brunnermeier and Sannikov (2016)
Notes: Functions are calculated using code provided in Brunnermeier and Sannikov (2016).

Endogenous Risk and Amplification
The top panel of Figure 1.5 depicts the drift μ^η and volatility σ^η of the wealth of experts which is the state variable in the economy. The bottom panel depicts the portfolio weight x on capital which is the physical-capital-to-net-worth ratio also known as expert leverage and the price volatility which is the endogenous risk in the economy. The drift is the deterministic growth of the state variable. It is increasing in the crisis (fire-sale) region because experts are highly leveraged and do not consume while households do consume. The drift declines as experts become less leveraged and the capital price increases. The drift remains positive from zero to the steady state. At the steady state experts begin consuming and the drift tends to zero. When a negative shock hits the economy the drift returns back to the steady state (**?**).
The exogenous volatility shock σ hits the wealth of experts η in the economy. The impact of the exogenous shock is amplified through the following spiral. The decline in experts' net worth decreases their holdings of physical capital which exerts downward pressure on the price of capital and reduces experts' net worth. The following equation illustrates the amplification mechanism:

$$\sigma^\eta = \frac{(x-1)}{\underbrace{1 - \frac{q_\eta(\eta)}{q(\eta)/\eta}(x-1)}_{\text{amplification}}}\sigma \tag{1.46}$$

where the numerator captures the leverage effect of the expert sector when there are no price movements. The leverage ratio $(x-1)$ measures the percentage drop in the wealth share of experts η as a result of a 1 percent drop in capital held by experts k before price fluctuations which increase with leverage. The denominator captures the amplification, i.e., *loss spiral* which is made up of the product of the elasticity of the capital price, i.e., the *market illiquidity* of capital and the leverage ratio. It represents the percentage loss in the price of capital as a result of the exogenous shock (Brunnermeier and Sannikov, 2016). The drop in the price of capital causes the balance sheet of experts to decline further which lead to further drops in the demand of capital putting more downward pressure on the price of capital q causing the vicious spiral mentioned above (**?**).

Volatility Paradox
The *volatility paradox* is the phenomena that as exogenous risk σ decreases, endogenous risk σ^q may rise. This is because low exogenous risk reduces the severity of financial frictions and increases the price of capital. Experts take on more leverage, thereby reducing their net worth reserves. This could result in the rise of the self-generated systemic risk, which makes the economy less stable (**?**).

G. Inefficiencies, Externalities and Macroprudential Policies

The incomplete market shown here gives rise to pecuniary externalities. They arise because individual market participants take prices as given, but in the aggregate they affect them. In this model the fire-sale is the pecuniary externality. In crisis experts sell assets to the household sector who cause an downward-sloping demand function (**?**). While experts are levering up before the crisis, they do not consider that in crisis their own fire-sales will depress prices of assets held by other experts. In aggregate this causes excess leverage since experts take fire-sale prices as given (**?**).
The authors demonstrate that good regulation, for example, a leverage constraint $x(\eta)$ in boom times encourages experts to retain earnings as a buffer to absorb losses in a crisis, and in crisis it enables experts be highly leveraged in order to stabilize the market. Some drawbacks of a leverage constraint is that it is less likely to bind in booms and leads to fire-sales in crisis. Specifically, experts cumulate more wealth η causing the steady state to increase reducing the probability of the economy going into crisis. This could possibly stabilize the system and increase welfare. However, in this context the effect is small, and creates inefficiencies such as capital misallocation and depressed prices that cause underinvestment.

1.3 Conclusion

This chapter has presented a brief review of the Continuous-time Macro-finance models. This new framework provides needed tools to study the non-linear effects of an economy. Compared to discrete-time modelling continuous-time modelling captures the economy between observations making it more realistic. They are more tractable as they enable more analytical steps by using methods developed in the finance literature such as continuous-time stochastic processes. The framework offers a new way to develop a better understanding of financial frictions in the macroeconomy.

The main idea highlighted throughout is that the wealth distribution across sectors in the economy matters for the level of economic activity (asset allocation) as well as the rates of earnings and risk exposures of various sectors. These earnings and risk exposures in turn drive the stochastic evolution of the wealth distribution. During a crisis endogenous risk is amplified and the economy is prone to instability regardless of the level of aggregate risk because leverage and risk-taking are endogenous. A volatility paradox possible opposite movement between exogenous risk and endogenous risk may arise.

Chapter 2

Zombie Firms: A Macroeconomic Model Approach

2.1 Introduction

This paper explores the role that zombie firms play to propagate slow economic recovery. Zombie firms are indebted firms that have persistent problems meeting their loan obligations, but are kept solvent by banks with speculative motives (Caballero et al., 2008). Banks lend to zombie firms to avoid recording huge losses against their net worth and to gamble on the financial resurrection of zombie firms (White, 2012). Studies by Gopinath, Kalemli-Özcan, Karabarbounis, and Villegas-Sanchez (2017), Andrews and Saia (2017) and Decker, Haltiwanger, Jarmin, and Miranda (2016) highlight that differing productivity, rising resource misallocation and decreasing business dynamism contribute to slow economic recovery. The empirical regularities of zombie firm dynamics in the economy can be summarized as follows:

- Zombie firms congest markets and limit the growth of more productive firms (Caballero et al., 2008).
- Financing zombie firms reduces investment, especially in sectors with good global growth opportunities (Decker et al., 2016; Kalemli-Ozcan et al., 2015).
- Financing zombie firms leads to lower productivity-enhancing capital reallocation (Adalet and Andrews, 2016).
- Zombie firms take on more debt and have higher cumulative default probability rates (Acharya et al., 2019).
- Lenders delay the deleveraging of zombie firms and roll over credit ‘evergreen’ in order to hide losses and gamble for resurrection (Storz et al., 2017; Peek and Rosengren, 2005).

I build on the model of financial crisis by Brunnermeier and Sannikov (2014), to study the impact of zombie firms on the economy. I do this by explicitly modelling the banking sector separately from firms. Therefore, the economy consists of three agents’ households, firms and banks. Firms are more productive in managing capital than households and banks. In addition to managing capital, banks provide financing to firms channelled from households. The financial frictions arise from the incompleteness of direct contracts similar to Holmstrom and Tirole (1997) where households cannot monitor borrowers, i.e. firms effectively, and banks can partially mitigate these frictions. The level of risk the banking sector can take depends on its overall health, which is described by the aggregate net worth of the firms and banks. Exogenous macroeconomic shocks affect the return on capital, where the price of capital depends on the net worth of firms and banks. Banks intermediate funds from households to firms and promise them the risk-free rate in return. There are two firm's non-zombie firms that are characterized as safe firms in the economy and zombie firms that are characterized as risky firms that are subject to default. Overall, the model is able to derive endogenous systemic risk dynamics demonstrated in liquidity spirals. Non-zombie firms obtain risk-free financing from the issuing of risk-free bonds, whilst zombie firms obtain bank loans. The level of default in the economy is exogenous and banks can choose whether or not to liquidate zombie firms. Banks also face bank regulation which impacts their financing decisions.

The findings in this model illustrate that in the event of a macroeconomic shock firms fire sell more assets which reduces the price of capital, lowers their holdings of physical capital and liquidity spirals further erode their net worth. In addition, the decline in bank net worth raises the cost of obtaining bank loans, lowering the aggregate productivity and pushing down the price of physical capital. This in turn impairs the net worth of the firms and banks further.

2.1.1 Literature Review

In the economy the financial sector performs vital functions. One primary function is the financial sector's ability to efficiently allocate capital by directing funds to firms that can use them most productively. Other important functions include maturity transformation, monitoring of borrowers and credit risk analysis. Only strong well capitalized banks can adequately perform vital functions. In contrast, when the banks are weakly capitalized a delay in restructuring of non-performing loans can be observed since liquidating non-performing loans could violate regulatory constraints and could lead to bank insolvency.

Eisfeldt and Rampini (2006), study the cyclical properties of reallocation. They find that illiquidity modelled as adjustment costs can explain the procyclicality of reallocation which negates its countercyclical benefits. Therefore, this friction delays reallocation at the point when the largest benefits are expected. Caballero, Hoshi, and Kashyap (2008), develop an entry and exit model which illustrates that limiting firm destruction by 'evergreening' loans depresses productivity by allowing inefficient firms to continue operating. This implies a stronger adjustment to shocks at the firm creation margin.

Efficient capital allocation is also linked to the analysis of loan liquidation and forbearance ('evergreening' of loans). Models that illustrate that a capital ratio is necessary to prevent weakly capitalized banks from lending to insolvent borrowers include Homar and van Wijnbergen (2017) and Bruche and Llobet (2014). Homar and van Wijnbergen (2017), study how recapitalizing banks with a large quantity of non-performing loans can avoid forbearance. Bruche and Llobet (2014), propose a voluntary system to prevent to prevent 'zombie' lending when banks are the only ones who can privately observe the loan quality. Inderst and Mueller (2008), use a noisy signal to analyse a bank's decision on whether or not to finance a risky project.

The above-mentioned models do not fully model the interaction of banks and firms, but focus rather on firms and entrepreneurs. The main contribution of this chapter is to provide a full macroeconomic model of credit and capital allocation focusing on the interaction between banks and firms. The literature highlights that proper allocation of capital is important for the growth of productive sectors, which is consistent with the evidence provided below. This chapter shows how the credit and capital allocation depends on the bank's and firm's capital structure, which is fundamental for current regulatory reforms.

This paper is related to studies that focus on financial frictions. Bernanke, Gertler, and Gilchrist (1998) model the financial accelerator where the standard debt contracts are based on the costly-state verification model of Townsend (1979). Kiyotaki and Moore (1997) construct a model where changes in collateral values propagate shocks to the economy. Gertler and Karadi (2011) and Gertler and Kiyotaki (2010) develop a models where financial intermediaries net worth is subject to deposit frictions, and they analyse unconventional monetary policy. Jermann and Quadrini (2012) create a model where production of firms is constrained by the amount of working capital borrowed from financial institutions. He and Krishnamurthy (2013), build a model where intermediaries are managed by bankers that face a wealth constraint that affects the intermediary's ability to increase leverage. Adrian and Boyarchenko (2013) construct a general equilibrium model with two types of intermediaries facing either a leverage constraint or equity constraint, and find that the overall financial sector has procyclical leverage.

Layout. The rest of the paper is organized as follows. Section 2.2 presents the baseline model which describes the model environment and equilibrium definition. Section 2.3 details the first order and market clearing conditions needed to solve the equilibrium. Section 2.4 presents the numerical results for a model without banks and for the model with firms, banks and households. Section 2.5 provide policy experiments on the regulation banks face. Finally, section 2.4 provide a summary of the chapter.

2.2 Model

2.2.1 Environment

The model is an infinite-horizon stochastic production economy with heterogeneous agents and financial frictions. Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a probability space that satisfies the usual conditions. Time is continuous with $t \in [0, \infty)$, and assume all stochastic processes are adapted. The model is populated by a continuum of three agents firms, banks and households. There are two goods: the final consumption good and physical capital. Figure 2.1 provides a sketch of the balance sheet of these agents in equilibrium.

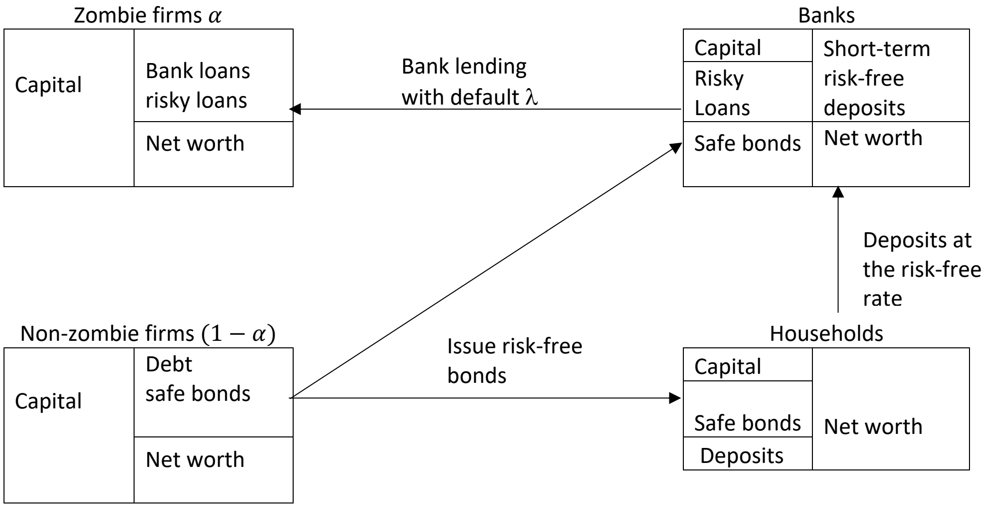


Figure 2.1: Balance sheet representative of the economy

A. Technology

The production technology is linear and reflects that firms are more efficient than households and banks at managing capital. The capital held by each firms produces the following output:

$$y_t = ak_t$$

where a is the productivity parameter and k_t is the physical capital measured in efficiency units. Capital held by households produces $\underline{y}_t = \underline{ak}_t$ output, and capital held by banks produce $\bar{y} = \bar{a}\bar{k}_t$ output, with $a > \underline{a} > \bar{a}$. The underbar and overbar hat notation is used to denote parameters and variables associated with the household and bank sector respectively.
Physical capital held by firms evolves according to:

$$\frac{dk_t}{k_t} = (\Phi(\iota_t) - \delta)dt + \sigma dZ_t \tag{2.1}$$

where ι_t is the investment per unit of capital, δ is the depreciation rate, σ is a positive constant that captures the exogenous volatility of capital growth and Z_t is the Brownian motion linked to the capital quality shocks.
Physical capital held by households $d\underline{k}_t/\underline{k}_t$ and banks $d\bar{k}_t/\bar{k}_t$, are defined analogously using their corresponding investment and depreciation rate, where $\bar{\delta} > \underline{\delta} > \delta$.
Firms can convert $\iota_t k_t$ units of the final consumption good (output) into $k_t\Phi(\iota_t)$ units of physical capital using the following log investment function:

$$\Phi(\iota_t) = \frac{\log(\kappa\iota_t + 1)}{\kappa} \tag{2.2}$$

where κ is the adjustment cost parameter and function $\Phi(\iota_t)$ represents technological illiquidity or adjustment costs and satisfies $\Phi(0) = 0, \Phi'(0) = 1, \Phi'(.) > 0$ and $\Phi''(.) \leq 0$.
Similarly, households and banks also convert their final consumption good into units of physical capital using equation (2.2).

The Brownian shock is the only source of exogenous aggregate uncertainty in the model. It does not affect the productivity of experts, but it does impact the quantity of capital held for example, negative cash flow or demand shocks.

B. Demographics

Firms and banks experience a shock τ and $\bar{\tau}$ that turns them into households and vice versa This assumption is introduced to keep the masses of firms and banks at unity. This limits the possibility that firms and banks take over all of the wealth in the economy and undo the effects of financial frictions as highlighted in Brunnermeier and Sannikov (2016).

C. Preferences

Firms, households and banks have log preferences with the same discount rate ρ. Logarithmic utility limits consumption to be non-negative.
The lifetime utility V_t for firms is given by the recursion:

$$V_t = \mathbb{E}_t \left[\int_0^\infty e^{-\rho t} \log c_s \, ds \right],$$

where c_t is the consumption rate of the aggregate good at time t. Similarly, the lifetime utility for households $\underline{V}_t$ and for banks $\bar{V}_t$ is given by the above recursion.

D. Financial structure

The financial structure in the economy is characterised by several financial frictions. The first financial friction is firms' inability to issue equity. This creates a financial friction that can be motivated by a "skin in the game" constraint (Jensen and Meckling, 1979; Bolton and Scharfstein, 1990; DeMarzo and Sannikov, 2006). The second friction is due to households' inability to directly lend to zombie firms, i.e., banks have a comparative advantage in lending activities over households as they can diversify across assets (Diamond, 1984). The third friction is between banks and zombie firms. Zombie firms have private knowledge of their production activities and transactions that banks do not possess. The idiosyncratic default shock that zombie firms incur is used by banks write-off loans, collect and resell collateral (He and Xiong, 2012). Banks also charge a premium to reach zombie firms and extend loans.

E. Markets

There are markets for physical capital, final goods, credit (bank loans) and risk-free deposits. Bank loans entitle the holder to a claim on borrowers' collateral capital in the case of default.
In the market for physical capital, capital is priced by postulating the following dynamic process:

$$\frac{dq_t}{q_t} = \mu_t^q dt + \sigma_t^q dZ_t \tag{2.3}$$

where μ_t^q is the capital growth and σ_t^q is the sensitivity of the capital price growth to the aggregate capital quality shock dZ_t.

2.2.2 Agents' problems

A. Households

The role of households in the economy is the following. Households can hold capital but less productively than firms in the economy. They act as savers in that they deposit funds in banks who act as financial intermediaries to zombie firms who receive loans from banks.
The total return that households earn from capital is given by:

$$d\underline{r}_t^k = \underline{\mu}_t^R dt + (\sigma + \sigma_t^q) dZ_t \tag{2.4}$$

where

$$\underline{\mu}_t^R = \underbrace{\frac{\underline{a} - \iota_t}{q_t}}_{\text{dividend yield}} + \underbrace{\Phi(\iota) - \underline{\delta} + \mu_t^q + \sigma\sigma_t^q}_{\text{capital gains rate}}$$

Households earn the risk-free rate r_t from deposits held in banks.
Households maximize their life-time utility function by choosing their consumption rate $\underline{c}_t$ and portfolio weight of physical capital $\underline{x}_t$. The state variable for this problem is the net worth of households given by $\underline{n}_t$. For a given $\underline{n}_s$ a household solves:

$$\underline{V}_0 = \max_{\{\underline{c}_t, \underline{x}_t \geq 0\}} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \log(\underline{c}_t) dt \right]$$

subject to the law of motion of wealth:

$$\frac{d\underline{n}_t}{\underline{n}_t} = (r_t + \underline{x}_t(\underline{\mu}_t^R - r_t) - \hat{\underline{c}}_t)dt + \underline{x}_t(\sigma + \sigma_t^q)dZ_t \tag{2.5}$$

Solvency constraint:

$$\underline{n}_t \geq 0 \tag{2.6}$$

where $\underline{V}_t(.)$ is the household's value function, r_t is the return on deposits, $\underline{x}_t$ is the portfolio weight on capital/ leverage, $\underline{\mu}_t^R$ is the return on capital, $\hat{\underline{c}}_t$ is the household's consumption to wealth ratio and $(\sigma + \sigma_t^q)$ is the total risk in the economy.

B. Firms

The role of firms in the economy is the following. Firms have a competitive advantage over holding capital than households and banks. Their main role is to act as the most productive agents in the economy. Heterogeneity within the firm sector is introduced following He and Krishnamurthy (2013), where I assume that a fraction $0 < \varphi < 1$ of the firms are non-zombie firms and the remaining fraction $(1 - \varphi)$ are zombie firms. The wealth of the non-zombie and the zombie firm evolve differently between t and $t + \delta$. In the aggregate their wealth is pooled together [1].
Both normal and zombie firms have the same productivity parameter a, but the physical capital evolution in equation(2.1) differs in the following way:

$$\text{Non-zombie firms:} \quad \frac{dk_{n,t}}{k_{n,t}} = (\Phi(\iota_t) - \delta - \underbrace{p_n}_{=0})dt + \sigma dZ_t$$

$$\text{Zombie firms:} \quad \frac{dk_{z,t}}{k_{z,t}} = (\Phi(\iota_t) - \delta - p_z)dt + \sigma dZ_t \tag{2.7}$$

[1] The assumption to introduce heterogeneity and the pooling of wealth are used to keep the model as tractable as possible.

where subscript n and z reflect non-zombie and zombie firms respectively. Moreover, $p_z > p_n = 0$ represents the fact that zombie firms face a higher default probability in the economy and non-zombie firms do not default, i.e., $p = 0$. Following Klimenko, Pfeil, Rochet, and De Nicolo (2016), I assume that the defaulting probability has a postulated process equal to $p_j dt + \sigma dZ_t$, where p_j for $j = n, z$ and σ are exogenous variables.

Non-zombie Firms

Non-zombie firms hold capital in the economy and are financed by issuing risk-free short term debt that both households and banks can hold. They do not default in the economy and I assume that holding their debt represents a secure investment. The total return that non-zombie firms earns from holding capital is given by:

$$dr^k_{n,t} = \mu^R_{n,t} dt + (\sigma + \sigma^q_t) dZ_t \tag{2.8}$$

where

$$\mu^R_{n,t} = \underbrace{\frac{a - \iota_t}{q_t}}_{\text{dividend yield}} + \underbrace{\Phi(\iota) - (\delta + \underbrace{p_n}_{=0}) + \mu^q_t + \sigma\sigma^q_t}_{\text{capital gains rate}}$$

The law of motion of wealth for non-zombie firms is:

$$\frac{dn_{n,t}}{n_{n,t}} = (d_t(\mu^R_{n,t} - r_t) - \hat{c}_t)dt + d_t(\sigma + \sigma^q_t)dZ_t \tag{2.9}$$

where d_t is the portfolio share/leverage of short term debt, $\hat{c}_t$ is the consumption to wealth ratio, and $(\sigma + \sigma^q_t)$ is the total risk in the economy.

Zombie Firms

Zombie firms hold capital in the economy and are financed through collateralized bank loans. At time $t + dt$, they are exposed to an idiosyncratic shock ϕ which reduces their collateral, and is determined endogenously as:

$$\phi = b * (\eta)^{-c} \tag{2.10}$$

where b and c are constants that are set exogenously. The function is dependent on the state variable - the wealth share of firms which is a measure of the financial fragility of firms. The collateral shock is independent across zombie firms and a continuum of zombie firms can diversify the idiosyncratic risk.
The total return that zombie firms earn from holding capital is given by:

$$dr^k_{z,t} = \mu^R_{z,t} dt + (\sigma + \sigma^q_t) dZ_t \tag{2.11}$$

where

$$\mu^R_{z,t} = \underbrace{\frac{a - \iota_t}{q_t}}_{\text{dividend yield}} + \underbrace{\Phi(\iota) - (\delta + p_z) + \mu^q_t + \sigma\sigma^q_t}_{\text{capital gains rate}}.$$

The loan rate that zombie firms pay is r^l_t which is greater than the risk-free rate r_t as it includes a premium for default risk. Zombie firms also incur an external cost of $\phi\pi$ due to banks effort to reach zombie firms and extend loans where π is the

distance cost set exogenously and ϕ is the default probability.
The law of motion of wealth for non-zombie firms is:

$$\frac{dn_{z,t}}{n_{z,t}} = (l_t(\mu^R_{z,t} - \phi\pi - r^l_t) - \hat{c}_t)dt + (1-\phi)l_t(\sigma + \sigma^q_t)dZ_t \tag{2.12}$$

where l_t is the portfolio share/leverage of short term collateralized loans, $\hat{c}_t$ is the consumption to wealth ratio, ϕ is the default probability, and $(\sigma + \sigma^q_t)$ is the total risk in the economy.
Firms maximize their life-time utility function by choosing their consumption rate $\hat{c}_t$, their portfolio weigh on capital financed by bonds d_t and loans l_t. The state variable for this problem is the net worth of firms given by n_t. For a given n_s a firm solves:

$$V_0 = \max_{\{c_t, d_t \geq 0, l_t \geq 0\}} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \log{(c_t)} dt\right]$$

subject to the law of motion of wealth:

$$\frac{dn_t}{n_t} = \underbrace{\varphi d_t(\mu^R_{n,t} - r_t)dt}_{\text{non-zombie firms}} + \underbrace{(1-\varphi)(l_t(\mu^R_{z,t} - \phi\pi - r^l_t))dt}_{\text{zombie firms}} - \hat{c}_t dt + \\ (\underbrace{\varphi d_t}_{\text{non-zombie firms}} + \underbrace{(1-\varphi)(1-\phi)l_t}_{\text{zombie firms}})(\sigma + \sigma^q_t)dZ_t. \tag{2.13}$$

Solvency constraint:

$$n_t \geq 0 \tag{2.14}$$

where $V_t(.)$ is the firms value function, d_t and l_t are the portfolio weights on capital held by non-zombie firms financed by bonds and zombie firms financed by loans. $\mu^R_{n,t}$ and $\mu^R_{z,t}$ represents out-of-pocket financing by zombie and non-zombie firms. $\hat{c}_t$ is the firm's consumption to wealth ratio and $(\sigma+\sigma^q_t)$ is the total risk in the economy.

C. Banks

The role of banks in the economy is to channel funds from households to banks. Banks raise funds from households in the form of deposits and promise the risk-free rate r_t in return. Banks lend to zombie firms at a rate r^l_t which incorporates the risk premium and intermediation cost. At time $t+dt$ when the idiosyncratic default shock ϕ hits the economy, banks seize and resell collateral in the secondary market. Banks are also subject to a capital ratio requirement of x_{reg} that limits excess leverage from the banking sector.
The total return that banks earn from holding capital is given by:

$$d\bar{r}^k_t = \bar{\mu}^R_t dt + (\sigma + \sigma^q_t)dZ_t \tag{2.15}$$

where

$$\bar{\mu}^R_t = \underbrace{\frac{\bar{a} - \iota_t}{q_t}}_{\text{dividend yield}} + \underbrace{\Phi(\iota) - (\bar{\delta} + \mu^q_t + \sigma\sigma^q_t}_{\text{capital gains rate}}.$$

Banks maximize their life-time utility function by choosing their consumption rate $\bar{c}_t$, portfolio weight of physical capital $\bar{x}_t$, and loan leverage ratio x_t for loans to zombie firms. The state variable for this problem is the net worth of households given by $\bar{n}_t$. For a given $\bar{n}_s$ a household solves:

$$\bar{V}_0 = \max_{\{\bar{c}_t, \bar{x}_t \geq 0, x_t \geq 0\}} \mathbb{E}_0 \left[\int_0^\infty e^{-\rho t} \log(\bar{c}_t) dt \right]$$

subject to the law of motion of wealth:

$$\frac{d\bar{n}_t}{\bar{n}_t} = (r_t + \bar{x}_t(\bar{\mu}_t^R - r_t) + x(r_t^l - r_t) - \hat{\bar{c}}_t)dt + (\bar{x}_t + \phi x_t)(\sigma + \sigma_t^q)dZ_t \tag{2.16}$$

Solvency constraint:

$$\bar{n}_t \geq 0 \tag{2.17}$$

Bank regulation:

$$x_t \leq x_{\text{reg}} \tag{2.18}$$

where $\bar{V}_t(.)$ is the banks' value function, r_t is the risk-free rate, $\bar{x}_t$ is the portfolio weight on capital/ leverage, $\bar{\mu}_t^R$ is the return on capital, x_t is the loan leverage ratio, r_t^l is the loan rate, $\hat{\bar{c}}_t$ is the household's consumption to wealth ratio and $(\sigma + \sigma_t^q)$ is the total risk in the economy where banks are exposed to $(\bar{x}_t + \phi x_t)$ from holding capital and from the seizure of capital from defaulted zombie firms. The capital ratio requirement x_{reg} is the maximum bank leverage determined from the market clearing condition of the supply and demand of bank loans. When $x_{\text{reg}} > 1$, the bank absorbs deposits and transfers funds from households to firms.

2.2.3 Equilibrium

Notation. Denote the set of firms $\mathbb{F} = [0, 1)$, the set of bankers $\mathbb{B} = [1, 2)$, and the set of households $\mathbb{H} = [2, 3]$. The set of non-zombie and zombie firms is given by $\mathbb{F}^n$ and $\mathbb{F}^z$ respectively. Take the initial capital stock K_0 and its distribution among agents $\{k_0^f\}_{f \in F}$, $\{k_0^b\}_{b \in B}$ and $\{k_0^h\}_{h \in H}$ as given, with $\int_{\mathbb{F}} k_0^f df + \int_{\mathbb{B}} k_0^b db + \int_{\mathbb{H}} k_0^h dh = K_0$. Let $k_0^f > 0, k_0^b > 0$ and $k_0^h > 0$ so that all agents start with strictly positive net worth.
Equilibrium definition. An equilibrium is a set of aggregate stochastic processes adapted to the filtration generated by Z: the price of capital $\{q_t\}$, the risk-free rate $\{r_t\}$, the interest rate of bank loans $\{r_t^f f \in \mathbb{F}^z\}$ of zombie firms, aggregate wealth $\{N_t^f, N_t^b, N_t^h\}$, investment decisions $\{\iota_t^f, \iota_t^b, \iota_t^h\}$, asset holding decisions $\{x_t^b, x_t^h\}$ of banks and households, debt financing decisions of non-zombie firms $\{d_t^f f \in \mathbb{F}^n\}$, loan financing decisions $\{l_t^f f \in \mathbb{F}^z\}$ of zombie firms, bank lending $\{x_t^b\}$ and consumption $\{c_t^f, c_t^b, c_t^h\}$, such that:

1. Initial net worth satisfies $N_0^f = q_0 k_0^f, N_0^h = q_0 k_0^h$ and $N_0^b = q_0 k_0^b$.

2. Each firm, bank and household solves his or her problem taking prices and aggregate conditions as given.

3. Market Clearing:

Final goods:
$$\int_{\mathbb{I}} c_t^f df = \frac{1}{q_t} \int_{f \in \mathbb{F}^n} (a^f - \iota_t^f) n_t^f d_t^f df + \frac{1}{q_t} \int_{f \in \mathbb{F}^z} (a^f - \iota_t^f) n_t^f l_t^f df + \frac{1}{q_t} \int_{\mathbb{B}} (a^b - \iota_t^b) n_t^b x_t^b dj + \frac{1}{q_t} \int_{\mathbb{H}} (a^h - \iota_t^h) n_t^h x_t^h dh$$

Capital goods:
$$\frac{1}{q_t} \int_{f \in \mathbb{F}^n} n_t^f b_t^f df + \frac{1}{q_t} \int_{f \in \mathbb{F}^z} n_t^f l_t^f df + \frac{1}{q_t} \int_{\mathbb{B}} n_t^b x_t^b db + \frac{1}{q_t} \int_{\mathbb{H}} n_t^h x_t^h dh = K_t$$

4. Law of motion of aggregate capital

$$\frac{dK_t}{dt} = \frac{1}{q_t}\int_{f\in\mathbb{F}^n}(\Phi(\iota_t^f)-\delta^f)n_t^f b_t^f df + \frac{1}{q_t}\int_{f\in\mathbb{F}^z}(\Phi(\iota_t^f)-\delta^f)n_t^f l_t^f - \phi\pi n_t^f l_t^f df +$$
$$\frac{1}{q_t}\int_{\mathbb{B}}(\Phi(\iota_t^b)-\delta^b)n_t^b x_t^b dj + \frac{1}{q_t}\int_{\mathbb{H}}(\Phi(\iota_t^h)-\delta^h)n_t^h x_t^h dh$$

5. Credit market

$$\int_{f\in\mathbb{F}^z} n_t^f l_t^f df = \int_{\mathbb{B}} n_t^b x_t^b db$$

6. The market for risk-free debt at rate r_t automatically clears by Walras' Law.

2.2.4 Timing

Taking the description and interaction of the agents above and looking at Figure (2.1) the timing of the economy is as follows:

1. At the beginning of the period all agents start with some endowment of capital that ensures that their net worth is positive. Firms are split in non-zombie and zombie firms with the probability φ and $(1-\varphi)$ respectively.

2. During the period, households can hold capital and short-term risk-free debt in non-zombie firms and deposit their savings in to banks and earn the risk-free rate r_t. Banks can hold capital and channel the deposits from households to zombie firms in the form of risky loans. Non-zombie and zombie firms can hold capital but earn a different returns since zombie firms a subject to default. Non-zombie firms are financed by households and banks, while zombie firms are financed exclusively by banks.

3. During the period zombie firms default with probability p_z and an endogenous idiosyncratic shock $0 < \phi < 1$ hits the zombie firms collateral causing it to reduce. Banks perform a partial liquidation and $(1-\phi)$ of the collateral remains with zombie firms with hopes of recovery back to market-to-market value. The fraction ϕ of collateral is seized and resold (below market value) by banks. Banks also absorb $\phi(\sigma+\sigma_t^q)dZ_t$ of the total risk.

2.2.5 Discussion of assumptions

There are three major departures from Brunnermeier and Sannikov (2014) assumptions:

1. Heterogeneity between firms to study the effect of zombie firms. Firms are modelled following He and Krishnamurthy (2013) using a fraction $0 < \varphi < 1$ for non-zombie firms and $(1-\varphi)$ for zombie firms.

2. Introduction of an explicit banking sector that is subject to regulation. The banks net worth is also important in determining the dynamics of the economy (Holmstrom and Tirole, 1997; Rampini and Viswanathan, 2019).

3. The credit model presented features credit in the form of short-term contingent collateralized debt (Bernanke et al., 1998; Carlstrom and Fuerst, 1997). Credit is backed by capital, therefore it is denominated in capital. Default on credit and collateral shock on capital backing the credit is also modelled.

2.2.6 Recursive equilibrium solution

A. Aggregate state

The two state variables of the economy are the wealth share of the firms and banks:

$$\eta_t \equiv \frac{N_t}{\underline{N}_t + \hat{N}_t + N_t} \equiv \frac{N_t}{q_t K_t} \qquad \text{and}$$

$$\bar{\eta}_t \equiv \frac{\bar{N}_t}{\underline{N}_t + \bar{N}_t + N_t} \equiv \frac{\bar{N}_t}{q_t K_t}$$

where $q_t K_t$ is the aggregate net worth of the economy, i.e. total world wealth. Moving forward, I implement recursive notation where the time sub indexes are suppressed. All equilibrium objects are functions of the two state variables of the economy η_t and $\bar{\eta}_t$, but this dependence is left implicit.

B. Risks

The economy is subject to only one type of aggregate risk. It relates to the exogenous real shock and its amplification mechanism. The total risk or total volatility arises from the exogenous σ and endogenous σ^q component. The exogenous risk or fundamental risk refers to the direct impact of the shock while the behaviour of the agent remains static. They are small and occur frequently. The endogenous risk is the the additional sensitivity generated by agents' endogenous responses, which create amplification. The risks are defined with reference to a variable of interest for example, returns on capital or output growth. The economy is also subject to an idiosyncratic default risk that affects only zombie firms. This default risk transfers risk from zombie firms to banks.

C. Value functions

Since agents have logarithmic utility, I guess and verify that the value function of each agent has the following form:

$$V(n,\zeta) = \frac{\log(n)}{\rho} + \zeta, \qquad V(\underline{n},\underline{\zeta}) = \frac{\log(\underline{n})}{\rho} + \underline{\zeta}, \qquad V(\bar{n},\bar{\zeta}) = \frac{\log(\bar{n})}{\rho} + \bar{\zeta}$$

for some stochastic processes $\{\zeta, \underline{\zeta}, \bar{\zeta}\}$ representing time variations in the set of investment opportunities for firms, households and banks respectively. The law of motion of for the wealth multipliers is postulated as the following Ito process:

$$\frac{d\zeta}{\zeta} = \mu^{\zeta} dt + \sigma^{\zeta} dZ_t, \qquad \frac{d\underline{\zeta}}{\underline{\zeta}} = \mu^{\underline{\zeta}} dt + \sigma^{\underline{\zeta}} dZ_t, \qquad \frac{d\bar{\zeta}}{\bar{\zeta}} = \mu^{\bar{\zeta}} dt + \sigma^{\bar{\zeta}} dZ_t$$

D. Markov Equilibrium

Market clearing. The market clearing for the **final goods** (scaled by total capital) can be written as:

$$\rho q = \varphi\eta(a-\iota)d + (1-\varphi)\eta(a-\iota)l + \bar{\eta}(\bar{a}-\iota)\bar{x} + (1-\eta-\bar{\eta})(\underline{a}-\iota)\underline{x} \tag{2.19}$$

where $\psi = (\varphi d + (1-\varphi)l)\eta$ is the capital share managed by total firms. The left-hand side of the equation(2.15) represents the aggregate demand, and the right-hand side is the aggregate supply. The term on the left is the sum of the aggregate consumption of all agents. The aggregate supply on the right-hand side is given by the firms, households and banks net output weighted by the share of capital that they manage.

The market clearing of **physical capital** (scaled by total wealth) is:

$$\varphi\eta b + (1-\varphi)\eta l + \bar{\eta}\bar{x} + (1-\eta-\bar{\eta})\underline{x} = 1 \tag{2.20}$$

The market clearing for the **credit market** is:

$$x\bar{N} = (1-\varphi)Nl \tag{2.21}$$

where the left-hand side is the total bank loan supply, and the right-hand side is the total bank loan supply in the economy. $\bar{N}$ and N represent the aggregate wealth of banks and firms respectively.

Consistency. Capital price dynamics need to be consistent with the dynamics of the aggregate state, i.e.

$$\begin{aligned} q\mu^q &= q_\eta\mu^\eta\eta + q_{\bar{\eta}}\mu^{\bar{\eta}}\bar{\eta} + \frac{1}{2}q_{\eta\eta}(\sigma^\eta\eta)^2 + \frac{1}{2}q_{\bar{\eta}\bar{\eta}}(\sigma^{\bar{\eta}}\bar{\eta})^2 + q_{\eta\bar{\eta}}\eta\sigma^\eta\bar{\eta}\sigma^{\bar{\eta}} \\ q\sigma^q &= q_\eta\sigma^\eta\eta + q_{\bar{\eta}}\sigma^{\bar{\eta}}\bar{\eta} \end{aligned} \tag{2.22}$$

Investment opportunities for households, firms and banks need to be consistent with the dynamics of the aggregate state, i.e.

$$\begin{aligned} \zeta\mu^\zeta &= \zeta_\eta\mu^\eta\eta + \zeta_{\bar{\eta}}\mu^{\bar{\eta}}\bar{\eta} + \frac{1}{2}\zeta_{\eta\eta}(\sigma^\eta\eta)^2 + \frac{1}{2}\zeta_{\bar{\eta}\bar{\eta}}(\sigma^{\bar{\eta}}\bar{\eta})^2 + \zeta_{\eta\bar{\eta}}\eta\sigma^\eta\bar{\eta}\sigma^{\bar{\eta}} \\ \zeta\sigma^\zeta &= \zeta_\eta\sigma^\eta\eta + \zeta_{\bar{\eta}}\sigma^{\bar{\eta}}\bar{\eta} \end{aligned} \tag{2.23}$$

Markov equilibrium definition. A Markov equilibrium in η and $\bar{\eta}$ is a set of functions $f = f(\eta, \bar{\eta})$ for (i) prices $\{q, r, r^l\}$, (ii) individual controls for non-zombie firms $\{\hat{c}, d, \iota\}$, zombie firms $\{\hat{c}, l, \iota\}$, banks $\{\hat{\bar{c}}, \bar{x}, x, \iota\}$ and households $\{\underline{\hat{c}}, \underline{x}, \underline{\iota}\}$, and the dynamics of firms' wealth share $\{\mu^\eta, \sigma^\eta\}$ and banks wealth share $\mu^{\bar{\eta}}, \sigma^{\bar{\eta}}$ such that:

1. Wealth multipliers $\{\zeta, \bar{\zeta}, \underline{\zeta}\}$ solve agents respective Hamilton-Jacobi Bellman equations with optimal controls (ii) and given prices (i).
2. Markets for output good, capital, credit and risk-free bond clears:
   (a) Goods equation(2.19)
   (b) Capital equation(2.20)
   (c) Credit equation (2.2.6)
   (d) Risk-free bond by Walras' Law.
3. The laws of motion for the state variable η and $\bar{\eta}$ are consistent with equilibrium functions.
4. Capital price dynamics are consistent with the dynamics of the aggregate state equation(2.22)
5. Investment opportunities are consistent with the dynamics of the aggregate state equation(2.23)

E. Real investment decision

The return on capital for firms, banks and households is maximized by choosing the investment rate that solves

$$\max_{\iota} q\Phi(\iota) - \iota.$$

The first-order condition $q\Phi'(\iota) = 1$ also known as the marginal Tobin's Q equates the marginal benefit of investment, i.e., $q\Phi'(\iota)$ to its marginal cost, i.e., a unit of final goods. This implies that the optimal investment rate is a function of the price q_t, i.e.,

$$\iota = \underline{\iota} = \iota(q).$$

The investment decision is a completely static problem in that it only depends on the current capital price as the investment process has no delays.

F. Households' consumption and portfolio decision

The optimal portfolio share of capital for households $\underline{x}$ satisfies

$$\underline{\mu}^R - r \leq \underbrace{\underline{x}(\sigma + \sigma^q)^2}_{\text{risk compensation}} \tag{2.24}$$

The left-hand side is the market excess return of capital over the risk-free rate, and the right-hand side is the compensation of risk required by the households to hold capital which stems from the wealth and investment opportunity risk. If $\underline{x} > 0$, the condition holds with equality and the excess return needs to compensation for the risk.
The following lemma characterizes the optimal consumption and portfolio decision.

Lemma 1 *The optimal consumption for households is given by:*

$$\underline{c} = \rho \underline{n} \tag{2.25}$$

The optimal portfolio share of capital for households is given by:

$$\underline{x} = \frac{\underline{\mu}^R - r}{(\sigma + \sigma^q)^2} \tag{2.26}$$

See proof in Appendix B

G. Firms' consumption and portfolio decision

Non-zombie Firms

The optimal portfolio share of capital financed by short term debt d satisfies

$$\mu_n^R - r = \underbrace{d\varphi(\sigma + \sigma^q)^2}_{\text{risk compensation}} \tag{2.27}$$

The left-hand side is the market excess return of capital over the risk-free rate, and the right-hand side is the compensation of risk required by the non-zombie firms to hold capital which stems from the wealth and investment opportunity risk.

Zombie Firms
The optimal portfolio share of capital financed by bank loans l satisfies

$$\mu_z^R - \phi\pi - r^l = \underbrace{l(1-\varphi)(1-\phi)^2(\sigma+\sigma^q)^2}_{\text{risk compensation}} \tag{2.28}$$

The left-hand side is the market excess return of capital over the intermediation cost and loan rate, and the right-hand side is the compensation of risk required by the zombie firms to hold capital which stems from the wealth and investment opportunity risk.
The following lemma characterizes the optimal consumption and portfolio decisions.

Lemma 2 *The optimal consumption for firms is given by:*

$$c = \rho n \tag{2.29}$$

The optimal portfolio share of capital financed by short term debt d and loans l for non-zombie and zombie firms respectively is given by:

$$\textit{Non-zombie Firms:} \quad d = \frac{\mu_n^R - r}{\varphi(\sigma+\sigma^q)^2} \tag{2.30}$$

$$\textit{Zombie Firms:} \quad l = \frac{\mu_z^R - \phi\pi - r^l}{(1-\varphi)(1-\phi)^2(\sigma+\sigma^q)^2} \tag{2.31}$$

See proof in Appendix B

H. Banks' consumption and portfolio decision

The optimal portfolio share of capital $\bar{x}$ and loan leverage x for banks satisfies:

$$\bar{\mu}^R - r = \underbrace{(\bar{x}+\phi x)(\sigma+\sigma^q)^2}_{\text{risk compensation}} \tag{2.32}$$

where the return on capital is dependent on both the capital and bank loan leverage.

$$r^l - r = \underbrace{(\bar{x}+\phi x)\phi(\sigma+\sigma^q)^2}_{\text{risk compensation}} \tag{2.33}$$

where the loan rate is dependent on both the capital and bank loan leverage and $x \leq x_{\text{reg}}$

Lemma 3 *The optimal consumption for banks is given by:*

$$\bar{c} = \rho\bar{n} \tag{2.34}$$

The optimal portfolio share of capital and lending for banks is given by:

$$\bar{x} + \phi x = \frac{\bar{\mu}^R - r}{(\sigma+\sigma^q)^2} \tag{2.35}$$

$$\bar{x} + \phi x = \frac{r^l - r}{(\sigma+\sigma^q)^2} \tag{2.36}$$

See proof in Appendix B

I. Evolution of the aggregate states

The dynamics of firms and banks net worths are:

$$\frac{dN}{N} = \frac{dn}{n} - \tau \qquad \text{and} \qquad \frac{d\bar{N}}{\bar{N}} = \frac{d\bar{n}}{\bar{n}} - \bar{\tau}.$$

The following lemma characterizes the evolution of the aggregate state variables $\eta = N/qK$ and $\bar{\eta} = \bar{N}/qK$ in the economy.

Lemma 4 *he firm and bank sector's wealth share dynamics are:*

$$\frac{d\eta}{\eta} = \mu^{\eta}dt + \sigma^{\eta}dZ_t \qquad \text{and} \qquad \frac{d\bar{\eta}}{\bar{\eta}} = \bar{\mu}^{\bar{\eta}}dt + \sigma^{\bar{\eta}}dZ_t$$

where

$$\begin{aligned} \eta\mu^{\eta} = \eta[\varphi d(\mu_n^R - r) + (1-\varphi)l(\mu_z^R - \phi\pi - r^l) - \rho - \tau - \\ \mu^K - \mu^q - \sigma\sigma^q + (\sigma+\sigma)^2 - (\varphi d + (1-\varphi)(1-\phi)l)(\sigma+\sigma)^2] \\ \eta\sigma^{\eta} = \eta[(\varphi d + (1-\varphi)(1-\phi)l - 1)(\sigma+\sigma)] \end{aligned} \tag{2.37}$$

$$\begin{aligned} \bar{\eta}\mu^{\bar{\eta}} = \bar{\eta}[r + (\bar{x}-1)(\bar{x}+\phi x)(\sigma+\sigma^q)^2 - x(r^l - r) - \rho - \bar{\tau} - \\ \mu^K - \mu^q - \sigma\sigma^q + (\sigma+\sigma)^2] \\ \bar{\eta}\sigma^{\bar{\eta}} = \bar{\eta}[(\bar{x} + \phi x - 1)(\sigma+\sigma^q)] \end{aligned} \tag{2.38}$$

See proof in appendix B

J. Numerical Approach

Similar to Brunnermeier and Sannikov (2014), this model also has the property of scale-invariance with respect to total physical capital K_t. I look for an equilibrium that is Markov in the state variables ω_t and η_t. In a Markov equilibrium all processes are functions of ω_t and η_t. I use Ito's lemma with respect to the volatility of the capital price to derive a partial differential equation (PDE) with respect to $q(\omega, \eta)$. I solve the PDE by decomposing the problem into a sequence of ordinary differential equations (ODEs). I then solve for the price of capital as the solution of the ODE using the finite difference method. The PDE and its boundary conditions are obtained from equilibrium conditions and Ito's lemma.

K. Baseline parameters

I calibrate the model using the parameters in Brunnermeier and Sannikov (2016) where applicable, i.e., for parameters $a, \underline{a}, \delta, \rho$ and κ. The rest are determined as follows. Since banks are less productive than households, I set their productivity parameter $\bar{a}$ at 2% and their depreciation rate $\bar{\delta}$ at 30%. In this model I study a negative exogenous shock σ on the economy which I set at -3%. Non-zombie firms do not default so their probability p_n of default is set at 0% while zombie do default p_z and is set at 4%. Bank regulation x_{reg} follows the Basel specification and is set at 12.5. I set firms change rate τ higher than the banks change rate $\bar{\tau}$ at 15% and 10% following the standard literature. I set the distance costs π at 25% and I set the non-zombie rate at 88% following **?**. Table 2.1 summarises the chosen parameter values in the baseline calibration.

Table 2.1: Parameter values

Parametes	**Description**	**Baseline**
Production		
a	Firms productivity constant	11%
$\underline{a}$	Households productivity constant	3%
$\bar{a}$	Banks productivity constant	2%
σ	Capital efficiency shock	-3%
δ	Firms depreciation rate	5%
$\underline{\delta}$	Households depreciation rate	10%
$\bar{\delta}$	Banks depreciation rate	30%
κ	Investment adjustment cost	10
p_n	Non-zombie probability of default	0%
p_z	Zombie probability of default	4%
Intermediation		
x_{reg}	Bank regulation	12.5
$\hat{\tau}$	Banks change rate	10%
π	Distance costs	30%
Firms		
φ	Non-zombie rate	88%
τ	Firms change rate	15%
Firms, Banks, Households		
ρ	Discount rate	6%

2.2.7 Economic insights

In this section I present three dimensional graphs displayed in a two-dimensional view. To understand how to interpret the results, I will first explain how to interpret three-dimensional graphs.

A. Interpreting Three-dimensional Graphs

Remember that a three-dimensional graph is a way to represent a multivariable function that has a two-dimensional input and a one-dimensional output. One of the inputs is plotted on the x-axis, while the other input is plotted on the y-axis. The output is plotted on the z-axis and represents the height of the graph. A three-dimensional graph plots the surface of the output of infinitely many points that looks like the function. For example, the surface of a quadratic function $y = x^2$ will look like a parabola.

Figure 2.2 displays the capital price volatility of the economy. The input on the x-axis is the banks' wealth share η_t and input on the y-axis is the firms' wealth share ω_t. The output capital price volatility is plotted on the z-axis. The colors of the graph correspond to the values of the output. They are also presented on the colormap next to the graph. Warmer colors like yellow and green represent high values, whilst cooler colors like blue correspond to low values. A color that deviates from zero represents the height of the output variable. Note that the interesting dynamics of the economy occur where there is a buildup of different colors with one predominant color.

Since I am applying the Brunnermeier and Sannikov (2016) framework, I use the tools learned to analyse graphs and apply them to these results to determine what is happening. The functions in this model are similar to those in their framework only extended to include the net worth of banks. For Figure 2.2, I know from the model characteristics that there are two regions dependent on the capital holdings of firms ψ. When $\psi = 1$ the capital price volatility is at its lowest and firms hold all the capital in the economy. When $\psi < 1$ the fire-sale zone capital price volatility is at its highest and all agents are willing to hold capital.

Equipped with this knowledge, I can look at the graph and see that endogenous risk is at its lowest when firms hold all the capital in the economy indicated the light blue color which corresponds to a low value range on the colormap. In contrast endogenous risk is high in the fire-sale zone. That is, banks and firms hold capital in the economy. Therefore, the economy is very unstable when the total wealth of the economy is concentrated in the hands of banks (seen by the fact that endogenous risk is increasing as banks net worth is increasing). Furthermore, firms have a small fraction of the total wealth in the economy (indicated by their low wealth share). The remaining points on the graph that are flat and correspond to the darkest color on the color map represent zero. They indicate the model is in the region where $\psi = 1$, firms own all of the capital, $\sigma_t^q = 0$ and the price of capital is at its maximum level $q_{max} = 1.2288$.

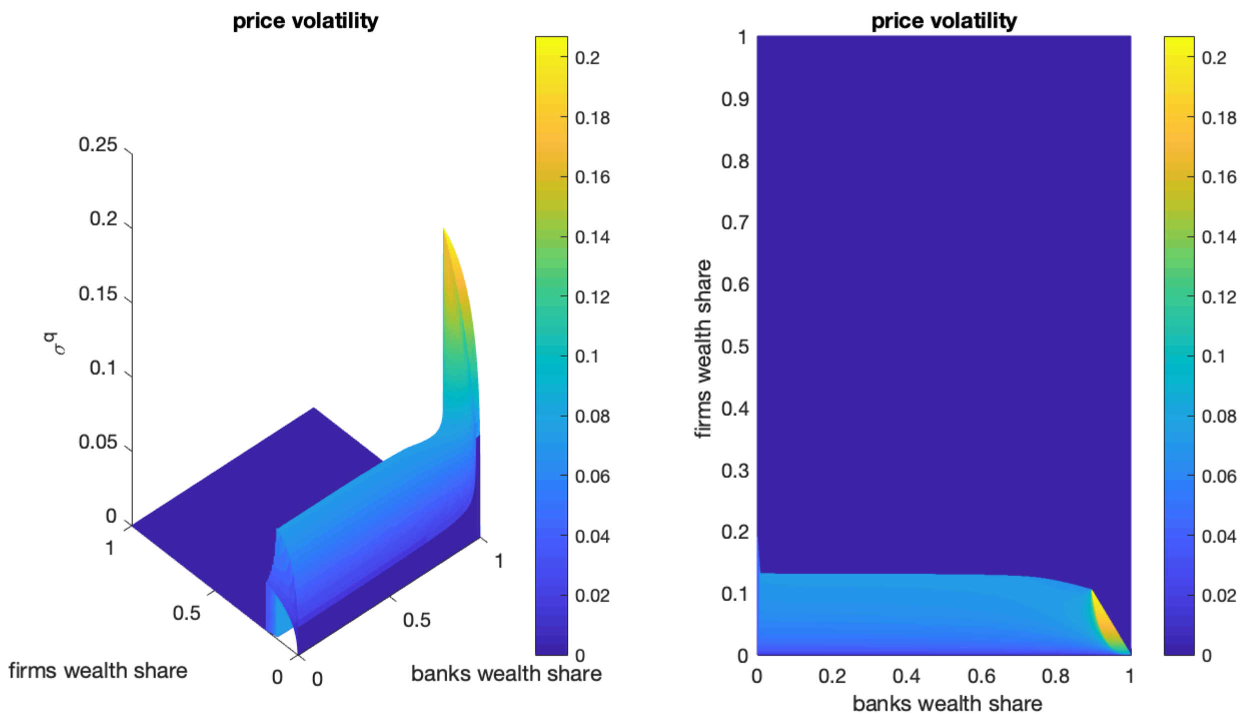


Figure 2.2: Interpreting three-dimensional graphs

The right-hand plot on Figure 2.2 represents the two-dimensional view of the three-dimensional plot on the left-hand. I can generalize interpreting the graph as follows:

- The x-axis represents the banks wealth share in the economy. An increase from zero to one represents the fraction of total wealth they hold in the economy. If the number is small, they hold a small fraction of total wealth in the economy. If the number is large, they hold a large fraction of wealth in the economy.

- The y-axis represents the firms wealth share in the economy. It can be interpreted as above

- The colormap represents the value of the outcome variable. So anywhere where the color differs from the color that represents zero means the outcome variable has impact.

- The model has two regions based on the capital allocation parameter of firms (the most productive agents in the economy). The first region is when $\psi = 1$. Here, firms own all the capital and the economy is stable. The second region is when $\psi < 1$. This is the fire-sale zone when the model goes into crisis.

Using the above one can quickly see that the endogenous risk is high when banks hold a large fraction of total wealth in the economy.

B. General Results

Now that banks and zombie firms are present in the economy the transmission mechanism of the model is extended to incorporate banks' net worth. The exogenous shock now impacts both the firms' and banks' net worth. Recall that banks are exposed to aggregate risk from the collateral that back their loans. When banks liquidate zombie firms' physical capital, the exogenous shock affects the efficient units of physical capital seized by banks. Banks can now only sell the physical capital in the secondary market. The dynamic budget constraint of banks equation 2.13 shows that banks assume high leverage leading to a high-risk exposure to the exogenous shock. The decline in the net worth of both firms and banks has persistent effects on the productivity, investment, asset prices, and financing in the economy.

C. Physical Capital Price and Misallocation

Figure 2.3 depicts the price of physical capital q and the fraction of the physical capital ψ_t and the investment rate ι_t of firms in the economy. Capital is misallocated because productive firms cannot issue outside equity and rely on leverage. When leverage is too high the risk that a firm's net worth will reduce to zero increases. The middle graph shows that when firms' wealth share is close to zero, they hold a small fraction of physical capital. When a firm's net worth falls below 0.2, the fraction of capital held by firms ψ_t is less than one and the economy is in the fire-sale region. Here, capital is also misallocated since less productive banks and households end up managing capital.

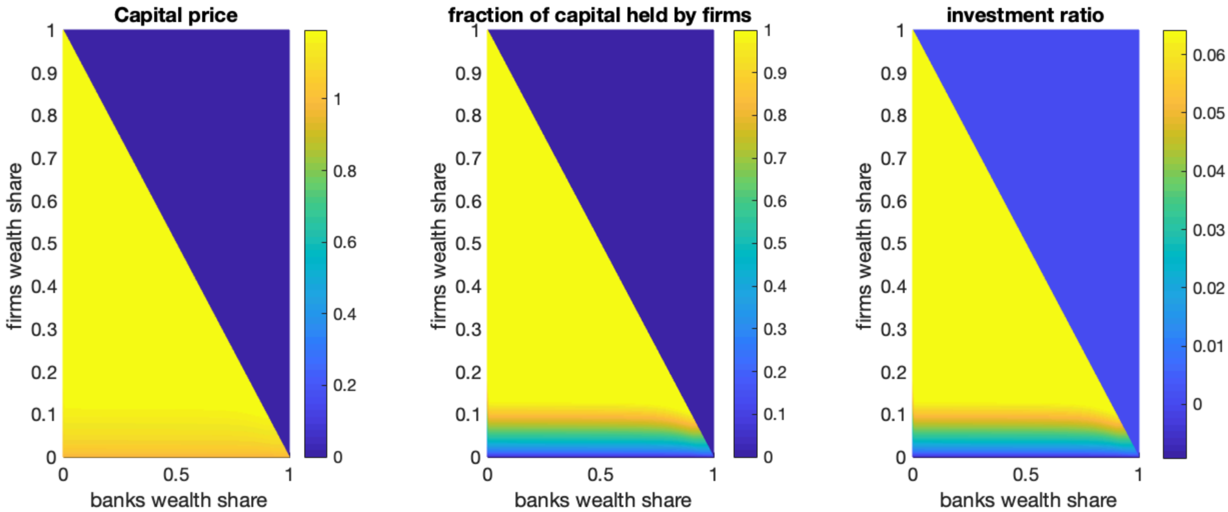


Figure 2.3: Capital price, fraction of capital and investment rate of the model

At the same time the price of physical capital first the plot on the left converges to its lower bound $q_{min} = 0.9725$. Given the same level of firms' net worth, the price of physical capital decreases as banks' net worth declines. The financing in the economy reduces i.e., the supply of risky bank loans becomes smaller when the banking sector is less capitalized. This further amplifies the misallocation of physical capital.

The plot on the right is the investment rate in the economy. When the firms' wealth share increases firms hold more physical capital and the investment in physical capital and the capital price increase. When the firms' wealth share is low there is under-investment since $\iota(q_t) < \iota(\bar{q})$. Since, banks raise the financing costs in the economy the price of physical capital and the investment to capital decreases as the banks' wealth share declines.

Overall Figure 2.3, show that when the firms' and banks' wealth share increase, firms hold more physical capital as they have access to affordable financing. This increases the price of physical capital and the investment in physical capital.

D. Endogenous Risk and Amplification

Endogenous risk refers to changes in asset prices that arise not due to changes in fundamentals, but rather due to adjustments that agents make in response to shocks, which may be driven by constraints or simply the precautionary motive. While exogenous fundamental shocks σ cause initial losses, endogenous risk σ_t^q is created through feedback loops that arise when agents react to losses. In the model, exogenous risk σ is assumed to be constant, but endogenous risk σ_t^q varies with the state of the system. The total volatility is the sum of exogenous and endogenous risk, $\sigma + \sigma_t^q$.

The amplification of shocks that creates endogenous risk depends on (i) firm and bank leverage and (ii) feedback loops that arise as prices react to changes in firm and bank net worth and affect firm and bank net worth further.

The exogenous Brownian shock affects both the firms' and banks' net worth in the economy. The impact of the exogenous shock is amplified through the following two inter-connected vicious spirals. First, the decline in firms' net worth lowers their holdings of physical capital, which depresses the capital price and reduces the firms' net worth. Second, the decline in banks' net worth raises the cost of obtaining bank loans. This also lowers the aggregate productivity and pushes down the price of physical capital, which in turn impairs the net worth of firms and banks further. Figure **??** illustrates the feedback mechanism of amplification.

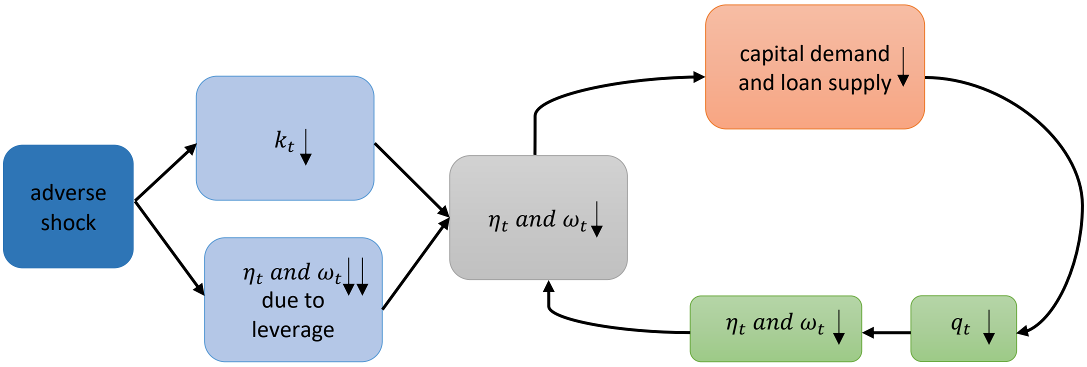


Figure 2.4: Adverse feed back loop

Rewriting equation 2.33, the amplification mechanism is given by:

Proposition 1 *Amplification: the endogenous volatility of the state variables η_t and $\bar{\eta}_t$ are given by:*

$$\sigma^{\eta}\eta = \frac{(\varphi d + (1-\varphi)(1-\phi)l - 1)\sigma}{1 - \frac{q_\eta}{q}(\varphi d + (1-\varphi)(1-\phi)l - 1) - \frac{q_{\bar{\eta}}}{q}(\bar{x} + \phi x - 1)}$$
$$\sigma^{\bar{\eta}}\bar{\eta} = \frac{(\bar{x} + \phi x - 1)\sigma}{1 - \frac{q_\eta}{q}(\varphi d + (1-\varphi)(1-\phi)l - 1) - \frac{q_{\bar{\eta}}}{q}(\bar{x} + \phi x - 1)} \tag{2.39}$$

where

$$q_\eta = \frac{\partial q(\eta, \bar{\eta})}{\partial \eta} \qquad \text{and} \qquad q_{\bar{\eta}} = \frac{\partial q(\eta, \bar{\eta})}{\partial \bar{\eta}}.$$

As in Brunnermeier and Sannikov (2014), an amplification spiral arises because of a feed back loop between decreasing wealth of firms and banks and higher endogenous volatility (see Figure). This can be seen from the denominator of this equation that corresponds to the sum of two geometric series. The size of this amplification factor depends on the derivatives of the price function with respect to the two state variables η and $\bar{\eta}$.
FIX

Equation (2.34) shows that the size of endogenous risk depends on the sensitivity of the price of physical capital to the change of wealth shares of the firms and banks q_η and $q_{\hat{\eta}}$ and the exposure of their wealth shares to the aggregate risk. As prices react to shocks, fundamental risk becomes amplified. Equation 2.34 shows that this amplification is nonlinear since q_η and $q_{\hat{\eta}}$ enters not only the numerator, but also the denominator. This happens due to the adverse feedback effect. An initial shock causes η and $\hat{\eta}$ to drop, which leads to a drop in q, which hurts firms who are holding

capital and banks that are lending. This leads to a further decrease in η and $\hat{\eta}$, and so on.
Figure ?? illustrates the risk in the economy. The price volatility plot is the endogenous risk which is determined by the price volatility. When the firms' wealth share, the fraction of capital held by firms is less than one. The economy is in the fire-sale region. Here, all agents (households, banks and firms) are willing to hold capital. Therefore, the endogenous risk is high. The price volatility plot shows that while the wealth share for firms is low and the wealth share of banks is increasing the endogenous risk in the economy also increases. The endogenous risk is at its highest when the wealth share in the economy is concentrated in the hands of banks (lower right region on the price volatility plot). This is because endogenous risk originates from the risk of asset fire-sales, which depends on the net worth of the firms and not banks.

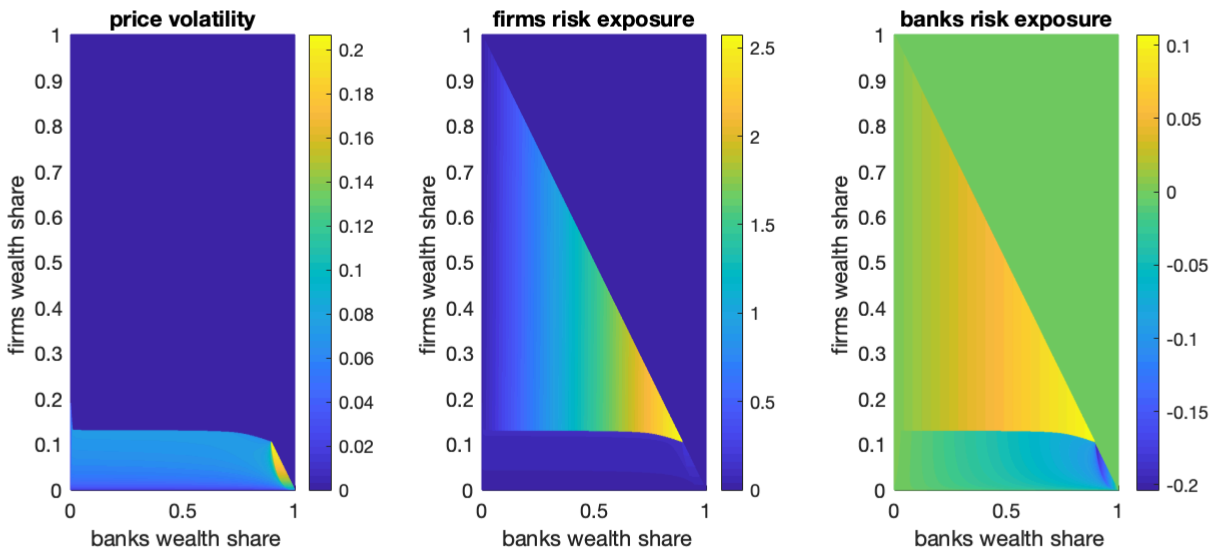


Figure 2.5: Capital price volatility of the model with zombie firms, non-zombie firms and banks

The middle plot shows the firms' risk exposure $\eta_t \sigma^\eta$. Firms risk exposure is the highest where the fire-sales start (lower right region of the firms' risk exposure plot). It will then decline to zero and remain as the wealth share of firms increase to one (top left region of the firms' risk exposure plot). Endogenous risk originates from the firms' risk exposure so a firm's risk exposure is higher when there wealth share is lower.
The right-hand plot shows the banks' risk exposure $\hat{\eta}\sigma^{\hat{\eta}}$. A banks' risk exposure is the amount an bank stands to lose if investments should fail i.e. zombie firms default on their loans. The banks' risk exposure is the highest when fire-sales begin (lower right region of the banks' risk exposure plot). Here, firms have low net worth and are at a higher risk to default on their loans. The banks' risk exposure declines as they become better capitalized i.e. as their wealth share increases to one (bottom right region of the banks' risk exposure plot). Therefore, when the firms' wealth share is low and banks' wealth share is high there is excess supply of loans in the economy. Zombie firms then continue to increase their leverage which amplifies asset

fire-sales.
Overall, Figure **??** show that endogenous risk and firms' risk exposure is the highest when fire-sales occur and firms hold a small wealth share of the total wealth in the economy. Banks contribute to increasing endogenous risk when they hold capital in the economy (resource misallocallotion) and when their risk exposure is high due to zombie firms' risk of default on the bank loans. When banks' risk exposure is low there is excess supply of loans in the economy and zombie firms who have low wealth share continue to increase their leverage. This amplifies asset fire-sales further.

E. Bank Lending

Raising financing from banks involves compensating banks for their exposure to both the exogenous riskσ and the endogenous risk $\phi x(\sigma + \sigma^q)$

$$r^l \leq (>) r + (\hat{x} + \phi x)\phi(\sigma + \sigma^q)^2 \quad \text{with equality if } 0 < x < x_{\text{reg}} \text{ (if } \bar{x} = 0).$$

The cost of bank loans consists of the cost of liquidation and the interest rate charged by banks. Banks channel money from households to firms, and can only issue risk-free debt to households in return. The costs fluctuate endogenously in the dynamics of the economy.

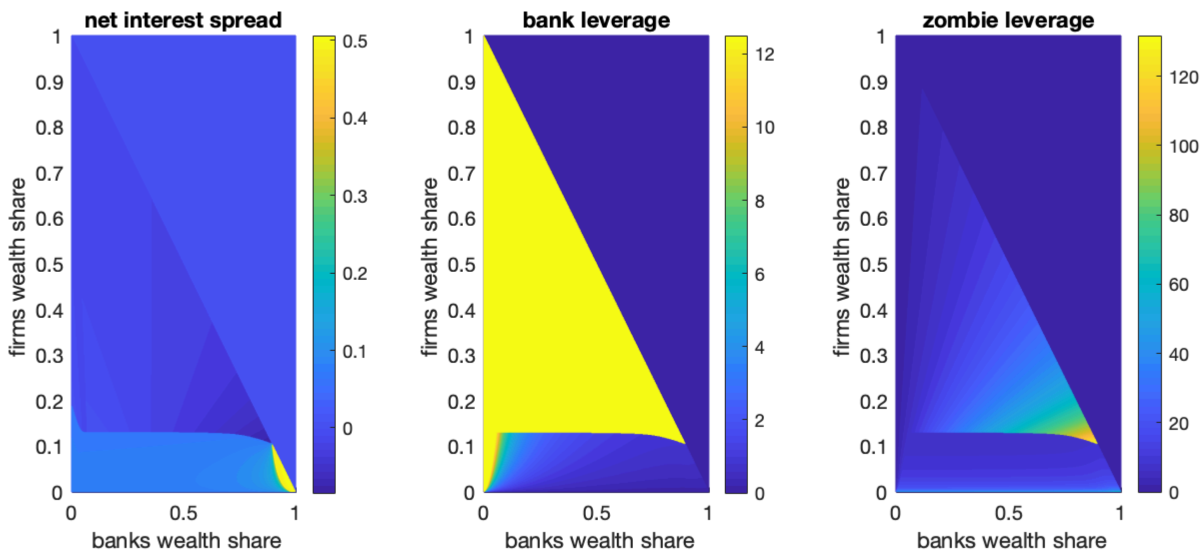


Figure 2.6: Net interest spread, bank leverage and zombie leverage of the model

The net interest spread $r^l - r$ that banks earn from loans made to zombie firms depend on the bank's leverage x, the exposure to zombie firms default risk ϕ, and the size of endogenous risk σ^q. When banks have a high net worth they can withstand adverse exogenous shocks. Therefore, the banks' leverage x and endogenous risk σ^q are small during economic upturns and the financing costs are also low. Conversely, during economic downturns banks have less net worth and become less tolerant of taking risks, also endogenous risk goes up.
Figure **??** shows the net interest spread, the banks' leverage and the zombie firms' leverage. The net interest spread is the difference between the interest rate on loans and the interest rate paid on borrowed funds. Looking at the net interest spread

plot, when the wealth share of banks is low the net interest spread is low. When the economy is in distress and fire-sales occur firms have a small wealth share of the total wealth share in the economy. In this region when banks have a high wealth share they have the highest net interest spread (lower right region of the net interest spread plot). This is because during the crisis period banks increase their financing costs. Zombie firms leverage is at its highest (lower right region of zombie leverage plot) despite the high financing costs. In this economy the probability of firm default ϕ is high which hurts the aggregate productivity because inefficient zombie firms are left to operate.
Looking at the bank leverage plot, when the economy is in a boom i.e. where all the firms hold the physical capital in the economy i.e. $\psi = 1$ the bank leverage is high. When the economy is in a bust i.e. where $\psi < 1$, fire-sales occur and there is capital misallocation because unproductive agents can hold capital bank leverage is low. This shows that bank leverage is procyclical.
The supply of credit by banks is dependent on the portfolio of the banks which is their loan assets. Therefore, procyclical leverage has a direct bearing on the supply of credit through fluctuations in lending.

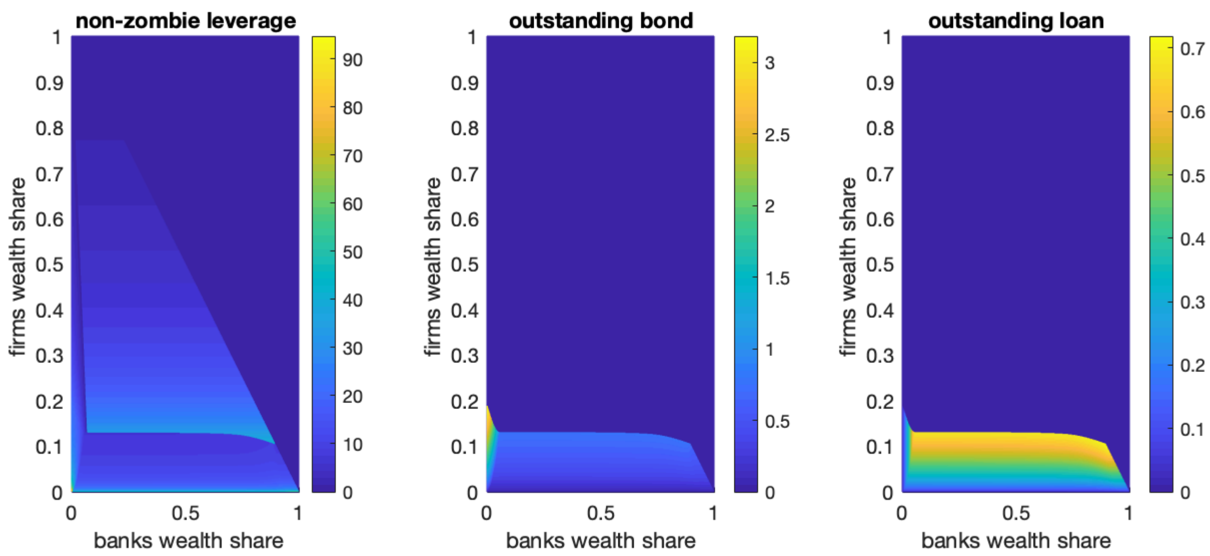


Figure 2.7: Non-zombie leverage, outstanding bond, outstanding loan of the model

Figure 2.7 depicts the leverage of non-zombie firms, the outstanding bond and the outstanding loan. Looking at the zombie and non-zombie leverage plots, it is evident that zombie firms in the economy have higher leverage than non-zombie firms. Zombie firms take on more debt and have higher cumulative default probability rates than non-zombie firms who lower debt and no risk of default.
Looking at the leverage of non-zombie firms and outstanding bond plots, when the economy is in crisis and the wealth share of firms is low and the leverage of non-zombie firms' is high, and the outstanding bonds in total wealth is increasing. During an economic boom the share of outstanding bonds in total wealth goes down and firms' wealth share increases and their leverage decreases.
Looking at the leverage of banks and zombie firms and the outstanding loan plots, it shows that bank loans are procyclical. Zombie firms are highly leveraged during

economic downturns when endogenous risk is high. In particular, as the bank sector becomes more and more financially healthy, firms are more levered in order to absorb all of the risk. Total outstanding loans is high.

F. Drifts of the State Variables

The map from the history of aggregate shocks dZ_t to the state variables η_t and $\bar{\eta}$ is captured by the drift $\eta\mu^e ta_t$ and $\bar{\eta}\mu^{\bar{\eta}}$ and the firm's risk exposure $\eta\sigma^{\eta}$ and $\bar{\eta}\sigma^{\bar{\eta}}$.
In Figure **??** depicts the drifts of firms and banks. Looking at both plots, where the drift of firms η and banks $\bar{\eta}$ becomes zero plays the role of the steady state of the system. In the absence of shocks, the system stays still at the steady state. When small shocks occur the respective drifts push the system back to the steady state.
Looking at the drift of firms plot, it is negative at the point where the fire-sales begin (blue region in the right side) then jumps when the risk premia jump at the boundary of the crisis region (yellow region of the plot where the firms' wealth share is low) and the drift is positive here.
Similarly, the drift of banks plot is negative at the point where fire-sales begin (dark aqua region in the right side) then jump when the risk premia jump at the boundary of the crisis region (yellow region where the firms' wealth share is low) and the drift is positive here.
Overall, when the drift of firms η_t and banks $\bar{\eta}$ becomes zero the system is at steady state. When there are no shocks, the system stays still at the steady state and when small shocks occur the respective drifts push the system back to the steady state.

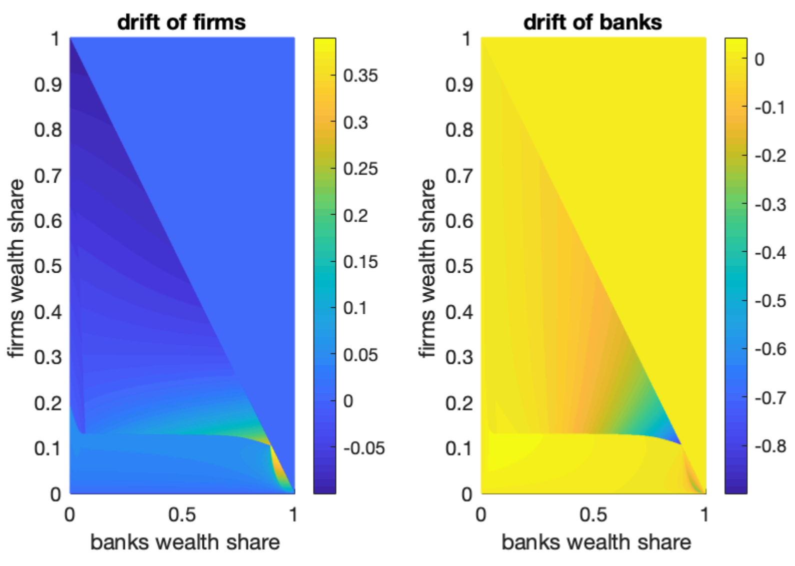
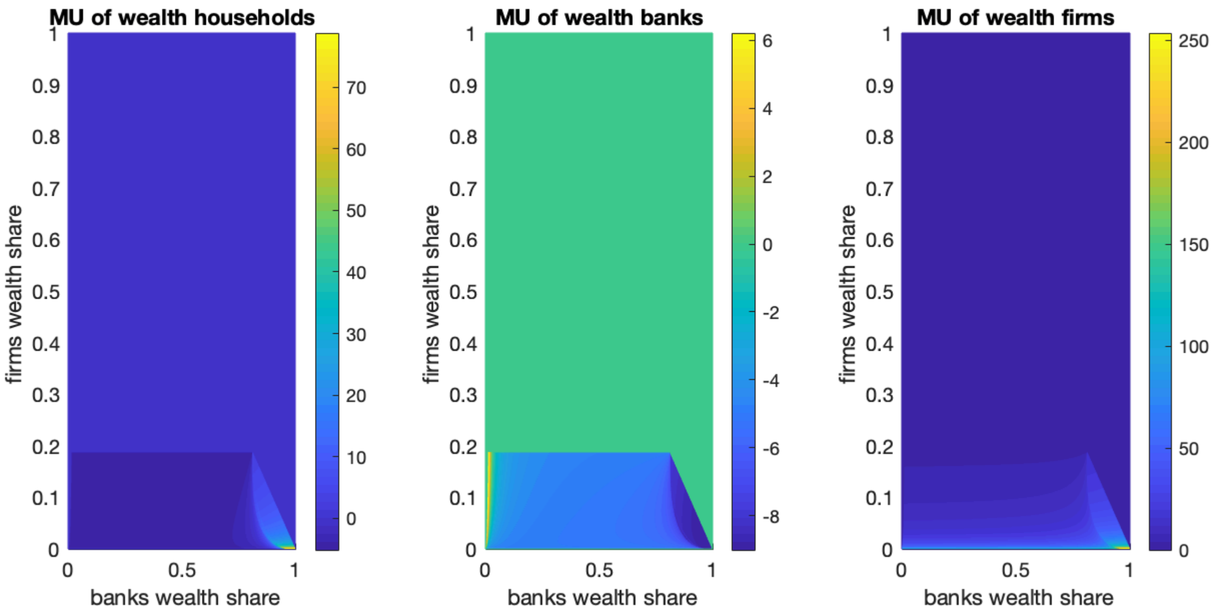


Figure 2.8: Drift of firms $\omega_t\mu_t^{\omega}$ of the model with zombie firms, non-zombie firms and banks

2.2.8 Welfare



2.3 Conclusion

In this chapter, I present a dynamic general framework to study the impact of zombie firms in the economy. Zombie firms in the economy increases endogenous risk and worsens the effect of shocks that leads to inefficiencies in the economy. During an economic downturn the presence of zombie firms slow down the recovery of the banks and firms.

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Appendix A

Analytical Results

Derivation 1
Households' problem. The Hamilton Jacobi Bellman (HJB) equation for the households' problem is:

$$\rho \underline{V}(\underline{n}, \underline{\zeta}) = \max_{\{\underline{c}\}} \log(\underline{c})\, dt + E_t[d\underline{V}(\underline{n}, \underline{\zeta})]$$

subject to the law of motion of wealth:

$$\frac{d\underline{n}}{\underline{n}} = \left(r - \frac{\underline{c}}{\underline{n}}\right) dt$$

and the solvency constraint:

$$\underline{n} \geq 0$$

Solution. Take as given $V(\underline{n}, \underline{\zeta}) = \log(\underline{n})/\rho + \underline{\zeta}$ with $d\underline{\zeta} = \underline{\mu}^{\zeta}\underline{\zeta} dt + \underline{\sigma}^{\zeta}\underline{\zeta} dZ_t$. Given the conjecture and using Ito's lemma, the evolution of $\underline{V}$ can be written as:

$$E(d\underline{V}) = \underline{V}_n \underline{\mu}^n \underline{n} + \underline{V}_{\zeta} \underline{\mu}^{\zeta} \underline{\zeta} + \frac{1}{2}[\underline{V}_{nn}(\underline{\sigma}^n \underline{n})^2 + \underline{V}_{\zeta\zeta}(\underline{\sigma}^{\zeta}\underline{\zeta})^2 + 2\underline{V}_{n\zeta}\underline{\sigma}^n \underline{n} \underline{\sigma}^{\zeta}\underline{\zeta}]$$

Then the HJB reduces to

$$\log(\underline{n}) + \rho \underline{\zeta}(\eta) = \max_{\{\underline{c}\}} \log(\underline{c}) + \frac{1}{\rho}\left(r - \frac{\underline{c}}{\underline{n}}\right) + \underline{\mu}^{\zeta}\zeta.$$

The first-order condition delivers:

$$\text{consumption} \qquad \underline{c} = \rho \underline{n}$$

Replacing the optimal consumption decision the HJB equation becomes:

$$\rho \underline{\zeta} = \log(\rho) + \frac{1}{\rho}(r - \rho) + \underline{\mu}^{\zeta}\underline{\zeta} \tag{A.1}$$

Derivation 2
Experts' problem. The Hamilton Jacobi Bellman (HJB) equation for the experts' problem is:

$$\rho V(n, \zeta) = \max_{\{c, x \geq 0\}} \log(c)\, dt + E_t[dV(n, \zeta)]$$

subject to the law of motion of wealth

$$\frac{dn}{n} = \left(r + x(\mu^R - r) - \frac{c}{n}\right) dt + x\sigma^R dZ_t$$

and the solvency constraint:

$$n \geq 0$$

Solution. Take as given $V(n,\zeta) = \log(n)/\rho + \zeta$ [1] with $d\zeta = \mu^\zeta \zeta dt + \sigma^\zeta \zeta dZ_t$. Given the conjecture and using Ito's lemma, the evolution of V can be written as:

$$E(dV) = V_n \mu^n n + V_\zeta \mu^\zeta \zeta + \frac{1}{2}[V_{nn}(\sigma^n n)^2 + V_{\zeta\zeta}(\sigma^\zeta \zeta)^2 + 2V_{n\zeta}\sigma^n n \sigma^\zeta \zeta]$$

Then the HJB reduces to

$$\log(n) + \rho\zeta(\eta) = \max_{\{c,x\geq 0\}} \log(c) + \frac{1}{\rho}\left(r + x(\mu^R - r) - \frac{c}{n}\right) + \mu^\zeta \zeta - \frac{1}{2\rho}(x\sigma^R)^2$$

The first-order conditions delivers:

$$\text{consumption} \qquad c = \rho n$$

$$\text{portfolio weight} \qquad x = \frac{\mu^R - r}{(\sigma + \sigma^q)^2}$$

Replacing the optimal consumption decision the HJB equation becomes:

$$\rho\zeta(\eta) = \log(\rho) + \frac{1}{\rho}(r + (x(\sigma + \sigma^q))^2 - \rho) + \mu^\zeta \zeta - \frac{1}{2\rho}(x(\sigma + \sigma^q))^2 \tag{A.2}$$

Derivation 3

Evolution of the state variable η. The experts wealth share N/qK is the state variable in the economy. From equation(1.13):

$$x(\sigma + \sigma^q) = \mu^R - r$$

Substitute in $\sigma^q = 0$ and $x = 1/\eta$

$$r = \mu^R - \frac{\sigma^2}{\eta}$$

$$r = \underbrace{\frac{a - \iota}{q}}_{\rho} + \Phi(\iota) - \delta - \frac{\sigma^2}{\eta}$$

The dividend is equal to the preference rate from the goods market clearing condition. Therefore, the risk-free rate is:

$$r = \rho + \Phi(\iota) - \delta - \frac{\sigma^2}{\eta} \tag{A.3}$$

The aggregate wealth share of the expert sector is:

$$\frac{dN}{N} = (r + \underbrace{x}_{\frac{1}{\eta}} \underbrace{(\mu^R - r)}_{\frac{\sigma^2}{\eta}} - \underbrace{\frac{C}{N}}_{\rho})dt + \underbrace{x}_{\frac{1}{\eta}} \underbrace{\sigma^R}_{\sigma} dZ_t$$

[1] Given the guess the derivatives are:

$$V_n = \frac{1}{\rho n}; \qquad V_{nn} = -\frac{1}{\rho n^2}; \qquad V_\zeta = 1; \qquad V_{\zeta\zeta} = 0; \qquad V_{n\zeta} = 0$$

from the first-order conditions of experts and the market clearing conditions. So:

$$\frac{dN}{N} = (r + \frac{\sigma^2}{\eta^2} - \rho)dt + \frac{\sigma}{\eta}dZ_t \tag{A.4}$$

The total wealth $d(qK)/qK$ in the economy can be found by applying Ito's product rule for two stochastic processes to the postulated price of capital equation:

$$\frac{dq}{q} = \mu^q dt + \sigma^q dZ_t$$

and the aggregate law of motion of capital:

$$\frac{dK}{K} = (\Phi(\iota) - \delta)dt + \sigma dZ_t.$$

The total wealth is:

$$\frac{d(qK)}{qK} = (\Phi(\iota) - \delta + \mu^q + \sigma\sigma^q)dt + (\sigma + \sigma^q)dZ_t$$

Since q is constant $\mu^q = \sigma^q = 0$. The law of motion of total wealth simplifies to:

$$\frac{d(qK)}{qK} = (\Phi(\iota) - \delta)dt + \sigma dZ_t. \tag{A.5}$$

Now apply Ito's ratio rule for two stochastic processes equations(A.4) and (A.5) to obtain the law of motion of the state variable η:

$$\frac{d\eta}{\eta} = (r + \frac{\sigma^2}{\eta^2} - \rho - \Phi(\iota) + \delta + \sigma^2 - \frac{\sigma^2}{\eta})dt + (\frac{\sigma}{\eta} - \sigma)dZ_t$$

substitute in the value of r_t given by equation(A.3) and simplify to obtain:

$$\frac{d\eta}{\eta} = \underbrace{\frac{(\eta-1)^2}{\eta^2}\sigma^2}_{\mu^\eta} dt + \underbrace{\frac{(1-\eta)}{\eta}\sigma}_{\sigma^\eta} dZ_t \tag{A.6}$$

where the absolute values $\eta\mu^\eta = \eta\frac{(\eta-1)^2}{\eta^2}\sigma^2$ and $\eta\sigma^\eta = (1-\eta)\sigma$.

Derivation 4
Households' problem. The Hamilton Jacobi Bellman (HJB) equation for the households' problem is:

$$\underline{\rho}\underline{V}(\underline{n}, \zeta) = \max_{\{\underline{c}, \underline{x}, \geq 0\}} \frac{\underline{c}^{1-\gamma}}{1-\gamma} + E_t[d\underline{V}(\underline{n}, \zeta)]$$

subject to the law of motion of wealth:

$$\frac{d\underline{n}}{\underline{n}} = \left(r + \underline{x}(\underline{\mu}^R - r) - \frac{\underline{c}}{\underline{n}}\right) dt + \underline{x}\underline{\sigma}^R dZ_t$$

and the solvency constraint:

$$\underline{n} \geq 0$$

Solution. Take as given $\underline{V}(\underline{n},\underline{\zeta}) = \frac{1}{\underline{\rho}}\frac{(\underline{n\zeta})^{1-\gamma}}{1-\gamma}$ with $d\underline{\zeta} = \underline{\mu}^{\zeta}\underline{\zeta}dt + \underline{\sigma}^{\zeta}\underline{\zeta}dZ_t$. Given the conjecture and using Ito's lemma, the evolution of $\underline{V}$ can be written as:

$$E(d\underline{V}) = \underline{V}_n\underline{\mu}^n\underline{n} + \underline{V}_{\zeta}\underline{\mu}^{\zeta}\underline{\zeta} + \frac{1}{2}[\underline{V}_{nn}(\underline{\sigma}^n\underline{n})^2 + \underline{V}_{\zeta\zeta}(\underline{\sigma}^{\zeta}\underline{\zeta})^2 + 2\underline{V}_{n\zeta}\underline{\sigma}^n\underline{n}\underline{\sigma}^{\zeta}\underline{\zeta}]$$

Then the HJB reduces to

$$\frac{(\underline{\zeta n})^{1-\gamma}}{1-\gamma} = \max_{\{\underline{c},\underline{x},\geq 0\}} \frac{\underline{c}^{1-\gamma}}{1-\gamma} + \frac{(\underline{\zeta n})^{1-\gamma}}{\underline{\rho}}\left(r + \underline{x}(\underline{\mu}^R - r) - \frac{\underline{c}}{\underline{n}}\right) + \frac{(\underline{\zeta n})^{1-\gamma}}{\underline{\rho}}\underline{\mu}^{\zeta} -$$
$$\frac{\gamma}{2}\frac{(\underline{\zeta n})^{1-\gamma}}{\underline{\rho}}(\underline{x}\underline{\sigma}^R)^2 - \frac{\gamma}{2}\frac{(\underline{\zeta n})^{1-\gamma}}{\underline{\rho}}(\underline{\sigma}^{\zeta})^2 + (1-\gamma)\frac{(\underline{\zeta n})^{1-\gamma}}{\underline{\rho}}\underline{x}\underline{\sigma}^R\underline{\sigma}^{\zeta}$$

The first-order condition delivers:

$$\text{consumption} \qquad \underline{c} = \underline{\zeta}^{1-\frac{1}{\gamma}}\underline{\rho}^{\frac{1}{\gamma}}n$$
$$\text{portfolio weight} \qquad \underline{x} = \frac{\underline{\mu}^R + (1-\gamma)\underline{\sigma}^R\underline{\sigma}^{\zeta}}{\gamma(\underline{\sigma}^R)^2}$$

Replacing the optimal consumption decision the HJB equation becomes:

$$\frac{\underline{\zeta}^{1-\gamma}}{1-\gamma} = (\underline{\zeta}^{1-\frac{1}{\gamma}}\underline{\rho}^{\frac{1}{\gamma}})^{1-\gamma} + \frac{\underline{\zeta}^{1-\gamma}}{\underline{\rho}}(r + \gamma(\underline{x}(\sigma+\sigma^q))^2 - (1-\gamma)\underline{x}(\sigma+\sigma^q)\underline{\sigma}^{\zeta} - \underline{\zeta}^{1-\frac{1}{\gamma}}\underline{\rho}^{\frac{1}{\gamma}})$$
$$\frac{\underline{\zeta}^{1-\gamma}}{\underline{\rho}}\underline{\mu}^{\zeta} - \frac{\gamma}{2}\frac{\underline{\zeta}^{1-\gamma}}{\underline{\rho}}(\underline{x}(\sigma+\sigma^q))^2 - \frac{\gamma}{2}\frac{\underline{\zeta}^{1-\gamma}}{\underline{\rho}}(\underline{\sigma}^{\zeta})^2 + (1-\gamma)\frac{\underline{\zeta}^{1-\gamma}}{\underline{\rho}}\underline{x}(\sigma+\sigma^q)\underline{\sigma}^{\zeta} \qquad \text{(A.7)}$$

where $\underline{x}$ is given by the first-order condition to obtain the portfolio weight. Since equation (A.7) does not depend on the individual state $\underline{n}$, function $\underline{\zeta}(\eta)$ can be chosen to ensure that it is always satisfied. This verifies the conjecture.

Derivation 5

Experts' problem. The Hamilton Jacobi Bellman (HJB) equation for the experts' problem is:

$$\rho V(n,\zeta) = \max_{\{c,x\geq 0\}} \frac{c^{1-\gamma}}{1-\gamma} + E_t[dV(n,\zeta)]$$

subject to the law of motion of wealth:

$$\frac{dn}{n} = \left(r + x(\mu^R - r) - \frac{c}{n}\right)dt + x\sigma^R dZ_t$$

and the solvency constraint:

$$n \geq 0$$

Solution. Take as given $V(n,\zeta) = \frac{1}{\rho}\frac{(n\zeta)^{1-\gamma}}{1-\gamma}$ [2] with $d\zeta = \mu^{\zeta}\zeta dt + \sigma^{\zeta}\zeta dZ_t$. Given the conjecture and using Ito's lemma, the evolution of V can be written as:

$$E(dV) = V_n\mu^n n + V_{\zeta}\mu^{\zeta}\zeta + \frac{1}{2}[V_{nn}(sigma^n n)^2 + V_{\zeta\zeta}(\sigma^{\zeta}\zeta)^2 + 2V_{n\zeta}\sigma^n n\sigma^{\zeta}\zeta]$$

[2]Given the guess the derivatives are:

$$V_n = \frac{\zeta^{1-\gamma}n^{-\gamma}}{\rho}; \qquad V_{nn} = -\frac{\gamma\zeta^{1-\gamma}n^{-\gamma-1}}{\rho};$$
$$V_{\zeta} = \frac{\zeta^{-\gamma}n^{1-\gamma}}{\rho}; \qquad V_{\zeta\zeta} = -\frac{\gamma\zeta^{-\gamma-1}n^{1-\gamma}}{\rho}; \qquad V_{n\zeta} = \frac{(1-\gamma)\zeta^{-\gamma}n^{-\gamma}}{\rho}$$

Then the HJB reduces to

$$\frac{(\zeta n)^{1-\gamma}}{1-\gamma} = \max_{\{c,x\geq 0\}} \frac{c^{1-\gamma}}{1-\gamma} + \frac{(\zeta n)^{1-\gamma}}{\rho}\left(r + x(\mu^R - r) - \frac{c}{n}\right) + \frac{(\zeta n)^{1-\gamma}}{\rho}\mu^{\zeta} -$$

$$\frac{\gamma}{2}\frac{(\zeta n)^{1-\gamma}}{\rho}(x\sigma^R)^2 - \frac{\gamma}{2}\frac{(\zeta n)^{1-\gamma}}{\rho}(\sigma^{\zeta})^2 + (1-\gamma)\frac{(\zeta n)^{1-\gamma}}{\rho}x\sigma^R\sigma^{\zeta}$$

The first-order condition delivers:

$$\text{consumption} \qquad c = \zeta^{1-\frac{1}{\gamma}}\rho^{\frac{1}{\gamma}}n$$

$$\text{portfolio weight} \qquad x = \frac{\mu^R + (1-\gamma)\sigma^R\sigma^{\zeta}}{\gamma(\sigma^R)^2}$$

Replacing the optimal consumption decision the HJB equation becomes:

$$\frac{\zeta^{1-\gamma}}{1-\gamma} = (\zeta^{1-\frac{1}{\gamma}}\rho^{\frac{1}{\gamma}})^{1-\gamma} + \frac{\zeta^{1-\gamma}}{\rho}(r + \gamma(x(\sigma+\sigma^q))^2 - (1-\gamma)x(\sigma+\sigma^q)\sigma^{\zeta} - \zeta^{1-\frac{1}{\gamma}}\rho^{\frac{1}{\gamma}})$$

$$\frac{\zeta^{1-\gamma}}{\rho}\mu^{\zeta} - \frac{\gamma}{2}\frac{\zeta^{1-\gamma}}{\rho}(x(\sigma+\sigma^q))^2 - \frac{\gamma}{2}\frac{\zeta^{1-\gamma}}{\rho}(\sigma^{\zeta})^2 + (1-\gamma)\frac{\zeta^{1-\gamma}}{\rho}x(\sigma+\sigma^q)\sigma^{\zeta} \tag{A.8}$$

where x is given by the first-order condition to obtain the portfolio weight. Since equation (A.8) does not depend on the individual state n, function $\zeta(\eta)$ can be chosen to ensure that it is always satisfied. This verifies the conjecture.

Derivation 6
Wealth share dynamics. The state variable is $\eta = \frac{N}{qK}$. The numerator is given by equation (1.41):

$$\frac{dN}{N} = (r + x(\mu^R - r) - \hat{C})dt + x(\sigma+\sigma^q)dZ_t$$

and the denominator is found by applying Ito's rule [3] to:

$$\frac{dq}{q} = \mu^q dt + \sigma^q dZ_t \qquad \text{and}$$

$$\frac{dK}{k} = (\Phi(\iota) - \delta)dt + \sigma dZ_t$$

The evolution of total world wealth is:

$$\frac{d(qK)}{qK} = (\Phi(\iota) - \delta + \mu^q + \sigma\sigma^q)dt + (\sigma+\sigma^q)dZ_t$$

[3] Ito's product rule for two stochastic processes is applied, where:

$$\frac{d(X_tY_t)}{X_tY_t} = (\mu_t^X + \mu_t^Y + \sigma_t^X\sigma_t^Y)dt + (\sigma_t^X + \sigma_t^Y)dZ_t$$

The evolution of the state variable can then be found by applying Ito's rule [4] to the numerator and denominator to obtain:

$$\frac{d\eta}{\eta} = \mu^\eta dt + \sigma^\eta dZ_t$$

$$= (r+x(\mu^R-r)-\hat{C}-\Phi(\iota)+\delta-\mu^q-\sigma\sigma^q+(\sigma+\sigma^q)^2-x(\sigma+\sigma^q)^2)dt+(x-1)(\sigma+\sigma^q)^2 dZ_t$$

[4]Ito's ratio rule for two stochastic processes is applied, where:

$$\frac{dX_t/Y_t}{X_t/Y_t} = (\mu_t^X - \mu_t^Y + (\sigma_t^Y)^2 - \sigma_t^X\sigma_t^Y)dt + (\sigma_t^X - \sigma_t^Y)dZ_t$$

Appendix B

Analytical Results

Proof of Lemma 1

Households' problem. The Hamilton Jacobi Bellman (HJB) equation for the households' problem is:

$$\rho V(n, \eta, \bar{\eta}) = \max_{\{\underline{c}, \underline{x} \geq 0\}} \log(\underline{c})\, dt + E_t[dV(n, \eta, \bar{\eta})]$$

subject to the law of motion of wealth

$$\frac{d\underline{n}}{\underline{n}} = (r + \underline{x}(\underline{\mu}^R - r) - \hat{\underline{c}})dt + \underline{x}(\sigma + \sigma^q)dZ_t$$

and the solvency constraint:

$$\underline{n} \geq 0$$

Solution. Take as given $V(\underline{n}, \eta, \bar{\eta}) = \log(\underline{n})/\rho + \underline{\zeta}(\eta, \bar{\eta})$ [1] with $d\underline{\zeta} = \mu^{\underline{\zeta}}\underline{\zeta}dt + \sigma^{\underline{\zeta}}\zeta dZ_t$. Given the conjecture and using Ito's lemma, the evolution of $\underline{V}$ can be written as:

$$E(d\underline{V}) = \underline{V}_{\underline{n}}\mu^{\underline{n}}\underline{n} + \underline{V}_{\eta}\mu^{\eta}\eta + \underline{V}_{\bar{\eta}}\mu^{\bar{\eta}}\bar{\eta} + \frac{1}{2}\underline{V}_{\underline{nn}}(\sigma^{\underline{n}}\underline{n})^2 + \frac{1}{2}\underline{V}_{\eta\eta}(\sigma^{\eta}\eta)^2 + \frac{1}{2}\underline{V}_{\bar{\eta}\bar{\eta}}(\sigma^{\bar{\eta}}\bar{\eta})^2 + \underline{V}_{\eta\bar{\eta}}\sigma^{\eta}\eta\sigma^{\bar{\eta}}\bar{\eta}$$

Then the HJB reduces to

$$\log(\underline{n}) + \rho\underline{\zeta}(\eta, \bar{\eta}) = \max_{\{\underline{c}, \underline{x} \geq 0\}} \log(\underline{c}) + \frac{1}{\rho}\left(r + \underline{x}(\underline{\mu}^R - r) - \frac{\underline{c}}{\underline{n}}\right) + \mu^{\eta}\eta + \mu^{\bar{\eta}}\bar{\eta} - \frac{1}{2\rho}(\underline{x}(\sigma + \sigma^q))^2$$

The first-order conditions delivers:

$$\text{consumption} \qquad \underline{c} = \rho\underline{n}$$

$$\text{portfolio weight} \qquad \underline{x} = \frac{\underline{\mu}^R - r}{(\sigma + \sigma^q)^2}$$

Substituting in the optimal consumption and portfolio decision the HJB equation becomes:

$$\rho\underline{\zeta}(\eta, \bar{\eta}) = \log(\rho) + \frac{1}{\rho}(r + (\underline{x}(\sigma + \sigma^q))^2 - \rho) + \mu^{\eta}\eta + \mu^{\bar{\eta}}\bar{\eta} - \frac{1}{2\rho}(x(\sigma + \sigma^q))^2 \quad \text{(B.1)}$$

where $\underline{x}$ is given by the first-order condition to obtain the portfolio weight. Since equation (B.1) does not depend on the individual state $\underline{n}$, function $\underline{\zeta}(\eta, \bar{\eta})$ can be chosen to ensure that it is always satisfied. This verifies the conjecture.

[1] Given the guess the derivatives are:

$$\underline{V}_{\underline{n}} = \frac{1}{\rho\underline{n}}; \qquad \underline{V}_{\underline{nn}} = -\frac{1}{\rho\underline{n}^2}; \qquad \underline{V}_{\eta} = 1; \qquad \underline{V}_{\bar{\eta}} = 1; \qquad V_{\eta\eta} = 0; \qquad V_{\bar{\eta}\bar{\eta}} = 0; \qquad V_{\eta\bar{\eta}} = 0$$

Micro-foundation