ONE-RELATOR MAXIMAL PRO-p GALOIS GROUPS AND THE KOSZULITY CONJECTURES

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To John P. Labute, with respect and admiration.

ABSTRACT. Let p be a prime number and let \mathbb{K} be a field containing a root of 1 of order p. If the absolute Galois group $G_{\mathbb{K}}$ satisfies dim $H^1(G_{\mathbb{K}}, \mathbb{F}_p) < \infty$ and dim $H^2(G_{\mathbb{K}}, \mathbb{F}_p) = 1$, we show that L. Positselski's and T. Weigel's Koszulity conjectures are true for \mathbb{K} . Also, under the above hypothesis we show that the \mathbb{F}_p -cohomology algebra of $G_{\mathbb{K}}$ is the quadratic dual of the graded algebra $\operatorname{gr}_{\bullet}\mathbb{F}_p[G_{\mathbb{K}}]$, induced by the powers of the augmentation ideal of the group algebra $\mathbb{F}_p[G_{\mathbb{K}}]$, and these two algebras decompose as products of elementary quadratic algebras. Finally, we propose a refinement of the Koszulity conjectures, analogous to I. Efrat's Elementary Type Conjecture.

1. INTRODUCTION

Let $A_{\bullet} := \bigoplus_{n \geq 0} A_n$ be a non-negatively graded algebra of finite type over a field k. The algebra A_{\bullet} is called *quadratic* if it is generated in degree 1 (i.e., every element is a combination of products of elements of A_1) and its defining relations are homogeneous relations of degree 2. Namely, one may write $A_{\bullet} \simeq T_{\bullet}(A_1)/(\Omega)$, where $T_{\bullet}(A_1)$ denotes the tensor algebra generated by A_1 and (Ω) is the two-sided ideal generated by a subset $\Omega \subseteq A_1^{\otimes 2}$. A quadratic algebra A_{\bullet} comes equipped with its *quadratic dual* $A_{\bullet}^!$, which is the quadratic algebra generated by the dual A_1^* of A_1 , and with defining relations the orthogonal complement $\Omega^{\perp} \leq (A_1^*)^{\otimes 2}$ of Ω (cf. [18, § 1.2]).

Quadratic algebras gained great importance in Galois theory after the proof of the celebrated Bloch-Kato conjecture by M. Rost and V. Voevodsky (see [27, 29, 31, 32]): from it one deduces that, given a prime number p and a field K containing a root of 1 of order p, the \mathbb{F}_p -cohomology algebra $H^{\bullet}(G_{\mathbb{K}}) = \bigoplus_{n\geq 0} H^n(G_{\mathbb{K}}, \mathbb{F}_p)$ of the absolute Galois group $G_{\mathbb{K}}$ of K, endowed with the (graded-commutative) cup-product

$$\cup : H^{r}(G_{\mathbb{K}}, \mathbb{F}_{p}) \otimes H^{s}(G_{\mathbb{K}}, \mathbb{F}_{p}) \longrightarrow H^{r+s}(G_{\mathbb{K}}, \mathbb{F}_{p}),$$

is a quadratic algebra over the finite field \mathbb{F}_p . This led to the achievement of new results on the structure of maximal pro-*p* Galois groups of fields (see, e.g., [11, 1, 23, 2, 24]). Understanging which pro-*p* groups may occur as maximal pro-*p* Galois groups of fields is one of the main open problems in modern Galois theory.

The class of Koszul algebras is a very peculiar class of quadratic algebras, singled out by S.B. Priddy in [22]. A quadratic algebra A_{\bullet} is called Koszul if \Bbbk ,

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as trivial graded A_{\bullet} module concentrated in degree 0, has a linear resolution (see Definition 2.2 for the formal definition). Koszul algebras arise in various fields of mathematics, and they have an uncommonly nice cohomological behavior — e.g., the cohomology of a Koszul algebra is just its quadratic dual. For the formal definition and properties of Koszul algebras we direct the reader to [18, Ch. 2] and [16, § 2]. Koszul algebras appeared on the Galois stage thanks to the work of L. Positselski and A. Vishik (see, e.g., [21, 19]). In particular, in [20] Positselski conjectured that $H^{\bullet}(G_{\mathbb{K}})$ is Koszul, for \mathbb{K} containing a root of 1 of order p, and he proved that this is the case if \mathbb{K} is a number field. In view of Positselski's conjecture and of the cohomological properties of Koszul algebras, it is natural to ask what the quadratic dual of $H^{\bullet}(G_{\mathbb{K}})$ may look like.

For a profinite group G, let

$$\operatorname{gr}_{\bullet}\mathbb{F}_p[G] := \bigoplus_{n \ge 0} I^n / I^{n+1}$$

denote the graded group algebra of G, with $I \leq \mathbb{F}_p[G]$ the augmentation ideal and $I^0 = \mathbb{F}_p[G]$. If \mathbb{K} contains a root of unity of order p and the maximal pro-p Galois group of \mathbb{K} is finitely generated, $H^{\bullet}(G_{\mathbb{K}})$ and $\operatorname{gr}_{\bullet}\mathbb{F}_p[G_{\mathbb{K}}]$ are expected to be quadratically dual to each other, and Th. Weigel conjectured in [34] that also $\operatorname{gr}_{\bullet}\mathbb{F}_p[G_{\mathbb{K}}]$ is Koszul. Altogether, one has the following three "Koszulity conjectures".

Conjecture 1.1. Let \mathbb{K} be a field containing a root of 1 of order p with $H^1(G_{\mathbb{K}}, \mathbb{F}_p)$ finite.

- (i) The \mathbb{F}_p -cohomology algebra $H^{\bullet}(G_{\mathbb{K}})$ is Koszul (cf. [20]).
- (ii) The graded group algebra $\operatorname{gr}_{\bullet} \mathbb{F}_p[G_{\mathbb{K}}]$ is Koszul (cf. [34]).
- (iii) $H^{\bullet}(G_{\mathbb{K}})^{!} \simeq \operatorname{gr}_{\bullet} \mathbb{F}_{p}[G_{\mathbb{K}}]$ (cf. [16]).

In [16], it is shown that the above conjecture holds true for the class of prop groups of elementary type (introduced by I. Efrat in [4, § 3]), which includes Demushkin pro-p groups, and which is particularly significan from a Galois-theoretic point of view.

Following the trail drawn in [16], in this paper we study the \mathbb{F}_p -cohomology algebra and the graded group algebra for the class of finitely generated one-relator pro-p groups with quadratic \mathbb{F}_p -cohomology. A pro-p group G is said to be one-relator if it has a minimal presentation with a single defining relation. The fundamental example we keep in mind while dealing with one-relator pro-p groups is the class of Demushkin groups — introduced by S.P. Demushkin and calssified completely by J.-P. Serre and J.P. Labute —, which arise as maximal pro-p Galois groups of local fields (see, e.g., [13] and [17, § III.9]). Our investigation on one-relator pro-p groups aims at proving the following.

Theorem 1.2. Let \mathbb{K} be a field containing a root of 1 of order p such that $H^1(G_{\mathbb{K}}, \mathbb{F}_p)$ is finite and dim $H^2(G_{\mathbb{K}}, \mathbb{F}_p) = 1$. Then there exists a closed subgroup S of $G_{\mathbb{K}}$ with $\operatorname{cd}_p(S) = 1$, generating a closed normal subgroup $N \leq G_{\mathbb{K}}$, such that:

- (i) the maximal pro-p quotient of $G_{\mathbb{K}}/N$ is a Demushkin group;
- (ii) $H^{\bullet}(G_{\mathbb{K}}) \simeq H^{\bullet}(S) \sqcap H^{\bullet}(G_{\mathbb{K}}/N);$
- (iii) $\operatorname{gr}_{\bullet}\mathbb{F}_p[G_{\mathbb{K}}] \simeq \operatorname{gr}_{\bullet}\mathbb{F}_p[S] \sqcup \operatorname{gr}_{\bullet}\mathbb{F}_p[G_{\mathbb{K}}/N]$. Moreover, also $\operatorname{cd}_p(N) = 1$, unless p = 2 and $\sqrt{-1} \notin \mathbb{K}$.

Here \sqcap and \sqcup denote the direct product and free product in the category of quadratic algebras (see § 2.2). The assumption on $H^n(G_{\mathbb{K}})$, n = 1, 2, amounts to

3

say that the maximal pro-p quotient of $G_{\mathbb{K}}$ is a finitely generated one-relator pro-p group. Theorem 1.2 may be considered as an "algebras analogue" of a result by T. Würfel on one-relator pro-p groups which occur as absolute Galois groups (cf. [35]).

More importantly, Theorem 1.2 implies that the "Koszulity conjectures" find positive solution under the assumption that $H^1(G_{\mathbb{K}}, \mathbb{F}_p)$ is finite and $H^2(G_{\mathbb{K}}, \mathbb{F}_p)$ has dimension 1.

Corollary 1.3. Let \mathbb{K} be as in Theorem 1.2. Then Conjecture 1.1 holds true for \mathbb{K} .

Finally, in analogy with I. Effrats Elementary Type Conjecture (cf. [4]), we define the classes of Koszul algebras of H-elementary type and of G-elementary type as the minimal classes of Koszul algebras generated by some "basic" quadratic algebras via elementary products (cf. Definition 5.8). In view of the results obtained in [16] and in the present paper, we ask the following.

Question 1.4. Let \mathbb{K} be a field containing a root of unity of order p such that $H^1(G_{\mathbb{K}}, \mathbb{F}_p)$ is finite.

- (i) Is the \mathbb{F}_p -cohomology $H^{\bullet}(G_{\mathbb{K}})$ a Koszul algebra of H-elementary type?
- (ii) Is the graded group algebra gr_• 𝔽_p[G_𝔅] a Koszul algebra of G-elementary type?

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2. Quadratic Algebras

Throughout the paper every graded algebra $A_{\bullet} = \bigoplus_{n \in \mathbb{Z}} A_n$ is assumed to be unitary associative over the finite field \mathbb{F}_p , and non-negatively graded of finite-type, i.e., $A_0 = \mathbb{F}_p$, $A_n = 0$ for n < 0 and $\dim(A_n) < \infty$ for $n \ge 1$.

In this section we give only the most basic definitions and results on quadratic algebras and Koszul algebras, and some examples, which will be sufficient for our investigation. For a complete account on cohomology of graded algebras and Koszul algebras, we refer to [16, § 2], and also to the first chapters of the books [18] and [14].

2.1. Quadratic algebras and quadratic duals. Given a vector space V of finite dimension, let $T_{\bullet}(V)$ denote the graded tensor algebra generated by V, endowed with the multiplication induced by the tensor product. Moreover, let $V^* = \operatorname{Hom}_{\mathbb{F}_p}(V, \mathbb{F}_p)$ denote the dual space of V. Since dim $V < \infty$, one may identify $(V \otimes V)^* = V^* \otimes V^*$.

Definition 2.1. A graded algebra $A_{\bullet} = \bigoplus_n A_n$ is said to be quadratic if A_{\bullet} is isomorphic to the quotient $T_{\bullet}(V)/(\Omega)$ for some finitely generated vector space V isomorphic to A_1 , and some subset $\Omega \subseteq V \otimes V$, with $(\Omega) \leq T_{\bullet}(V)$ the two-sided ideal generated by Ω . We write $A_{\bullet} = Q(V, \Omega)$.

For a quadratic algebra $A_{\bullet} = Q(V, \Omega)$, let $\Omega^{\perp} \subseteq (V \otimes V)^*$ denote the orthogonal space of Ω , i.e., $\Omega^{\perp} = \{\alpha \in (V \otimes V)^* \mid \alpha(w) = 0 \text{ for all } w \in \Omega\}$. The quadratic dual of A_{\bullet} , denoted by $A_{\bullet}^!$, is the quadratic algebra $Q(V^*, \Omega^{\perp})$.

Note that for a quadratic algebra A_{\bullet} one has $(A_{\bullet}^!)^! = A_{\bullet}$.

Definition 2.2. For a graded algebra $A_{\bullet} = \bigoplus_{n \geq 0} A_n$, the \mathbb{F}_p -cohomology is defined as the direct sum of the derived functors if the functor $\operatorname{Hom}_{A_{\bullet}}(\underline{\ },\mathbb{F}_p)$ evaluated on \mathbb{F}_p . Namely, it is the *bigraded* algebra

$$\bigoplus_{i,j} \operatorname{Ext}_{A_{\bullet}}^{ij}(\mathbb{F}_p, \mathbb{F}_p), \qquad i, j \ge 0,$$

where one grading is induced by the grading of A_{\bullet} , and the other grading is the cohomological grading (cf. [16, § 2.2]). The algebra A_{\bullet} is Koszul if the cohomology is concentrated on the diagonal, i.e., if $\operatorname{Ext}_{A_{\bullet}}^{ij}(\mathbb{F}_{p},\mathbb{F}_{p})=0$ for $i \neq j$.

Every Koszul algebra is quadratic. Moreover, a quadratic algebra A_{\bullet} is Koszul if, and only if, the dual algebra $A_{\bullet}^{!}$ is Koszul (cf. [16, § 2.3]).

2.2. Examples and constructions. Henceforth V denotes a vector space of finite dimension d.

Example 2.3. The tensor algebra $T_{\bullet}(V)$ and the quadratic algebra $Q(V, V^{\otimes 2})$, called the *trivial* quadratic algebra, are Koszul, and $Q(V, V^{\otimes 2})! = T_{\bullet}(V^*)$, and conversely (cf. [14, Examples 3.2.5]).

Let $X = \{X_1, \ldots, X_d\}$ be a set of indeterminates. The free associative algebra $\mathbb{F}_p\langle X \rangle$ — i.e., the algebra of polynomials on the non commutative indeterminates X — comes endowed with the grading induced by the subspaces of homogeneous polynomials. We may identify X with a fixed basis of V, and such identification induces an isomorphism of quadratic algebras $T_{\bullet}(V) \simeq \mathbb{F}_p\langle X \rangle$.

Example 2.4. The symmetric algebra $S_{\bullet}(V) = Q(V, \Omega_S)$ and the exterior algebra $\Lambda_{\bullet}(V) = Q(V, \Omega_{\Lambda})$, where

$$\Omega_S = \{vw - wv \mid v, w \in V\} \quad \text{and} \quad \Omega_\Lambda = \{vw + wv \mid v, w \in V\},\$$

are Koszul, and $\Lambda_{\bullet}(V)^{!} = S_{\bullet}(V^{*})$, and conversely (cf. [14, Examples 3.4.12]).

Given two quadratic algebras $A_{\bullet} = Q(A_1, \Omega_A)$ and $B_{\bullet} = Q(B_1, \Omega_B)$, one has the following contractions (cf. [16, Example 2.5]).

(a) The direct product of A_{\bullet} and B_{\bullet} is the quadratic algebra $A_{\bullet} \sqcap B_{\bullet} = Q(A_1 \oplus B_1, \Omega)$, with

$$\Omega = \Omega_A \cup \Omega_B \cup (A_1 \otimes B_1) \cup (B_1 \otimes A_1).$$

- (b) The free product of A_{\bullet} and B_{\bullet} is the quadratic algebra $A_{\bullet} \sqcup B_{\bullet} = Q(A_1 \oplus B_1, \Omega_A \cup \Omega_B).$
- (c) The symmetric tensor product of A_{\bullet} and B_{\bullet} is the quadratic algebra $A_{\bullet} \otimes B_{\bullet} = Q(A_1 \oplus B_1, \Omega)$, with $\Omega = \Omega_A \cup \Omega_B \cup \Omega_S$, where

$$\Omega_S = \{ab - ba, a \in A_1, b \in B_1\}.$$

(d) The skew-symmetric tensor product of A_{\bullet} and B_{\bullet} is the quadratic algebra $A_{\bullet} \wedge B_{\bullet} = Q(A_1 \oplus B_1, \Omega)$, with $\Omega = \Omega_A \cup \Omega_B \cup \Omega_{\wedge}$, where

$$\Omega_{\wedge} = \{ab + ba, a \in A_1, b \in B_1\}.$$

5

One has the following (cf. $[18, \S 3.1]$).

Proposition 2.5. If both A_{\bullet} and B_{\bullet} are Koszul, then also the algebras $A_{\bullet} \sqcap B_{\bullet}$, $A_{\bullet} \sqcup B_{\bullet}, A_{\bullet} \otimes B_{\bullet}$ and $A_{\bullet} \wedge B_{\bullet}$ are Koszul. Moreover, one has

- (i) $(A_{\bullet} \sqcap B_{\bullet})^{!} \simeq A^{!} \sqcup B^{!}_{\bullet}$ and $(A_{\bullet} \sqcup B_{\bullet})^{!} \simeq A^{!}_{\bullet} \sqcap B^{!}_{\bullet}$, (ii) $(A_{\bullet} \otimes B_{\bullet})^{!} \simeq A^{!}_{\bullet} \land B^{!}_{\bullet}$ and $(A_{\bullet} \land B_{\bullet})^{!} \simeq A^{!}_{\bullet} \otimes B^{!}_{\bullet}$.

3. Pro-*p* groups and graded algebras

The aim of this section is to provide an effective way to describe the graded group algebra of the pro-p groups we are studying.

For a pro-p group G and $n \geq 1$, G^n denotes the closed subgroup of G generated by *n*th powers, and [G, G] denotes the closed commutator subgroup of G.

3.1. The *p*-Zassenhaus filtration. The following is an extract from $[9, \S 2]$.

Let F be a free finitely generated pro-p group with basis $\mathcal{X} = \{x_1, \ldots, x_d\}$. The complete group algebra $\mathbb{F}_p[\![F]\!]$ is defined as $\mathbb{F}_p[\![F]\!] = \lim_{U \to U} \mathbb{F}_p[F/U]$, where U runs through the open normal subgroups of F. The assignment $x \mapsto x - 1$ induces an embedding of sets $F \hookrightarrow \mathbb{F}_p[\![F]\!]$ (this map is not a morphism of groups).

Let $\mathbb{F}_p\langle\!\langle X \rangle\!\rangle$ denote the algebra of formal power series in the non-commuting indeterminates $X = \{X_1, \ldots, X_d\}$. Then there is an isomorphism of topological algebras $\phi \colon \mathbb{F}_p[\![F]\!]$, given by $x_i \mapsto 1 + X_i$. The composition of the embedding $F \hookrightarrow \mathbb{F}_p[\![F]\!]$ with the isomorphism ϕ is the Magnus embedding $\psi \colon F \hookrightarrow \mathbb{F}_p(\langle X \rangle)$.

Let I(X) denote the augmentation ideal of $\mathbb{F}_p(\langle X \rangle)$, i.e., I(X) is the two-sided ideal (X_1, \ldots, X_d) . The p-Zassenhaus filtration of F is the filtration given by the subgroups

$$F_{(n)} = \{ x \in F \mid \phi(x-1) \in I(X)^n \} = \{ x \in F \mid \psi(x) \in I(X)^n \}, \qquad n \ge 1$$

In particular, one has $F_{(1)} = G$, $F_{(2)} = F^p[F,F]$ (namely, $F_{(2)}$ is the Frattini subgroup of F), and

(3.1)
$$F_{(3)} = \begin{cases} F^p[[F,F],F] & \text{if } p \neq 2\\ F^4[F,F]^2[[F,F],F] & \text{if } p = 2. \end{cases}$$

Moreover, the p-Zassenhaus filtration of F is the fastest descending series of Fstarting at F and such that

(3.2)
$$[F_{(n)}, F_{(m)}] \subseteq F_{(n+m)} \quad \text{and} \quad F_{(n)}^p \subseteq F_{(np)}$$

for every $n, m \ge 1$ (cf. [3, § 11.1]), and the quotient $F_{(n)}/F_{(n+1)}$ is finite. Moreover, one has a canonical isomorphism of graded algebras

$$\operatorname{gr}\mathbb{F}_p\langle\!\langle X \rangle\!\rangle := \bigoplus_{n \ge 0} I(X)^n / I(X)^{n+1} \xrightarrow{\sim} \mathbb{F}_p\langle X \rangle,$$

so that for a series $f \in \mathbb{F}_p(\langle X \rangle)$ such that $f \in I(X)^n \setminus I(X)^{n+1}$, one may consider the class $f + I(X)^{n+1}$ as a homogeneous polynomial of $\mathbb{F}_p(X)$ of degree n.

For an element $x \in F_{(n)} \setminus F_{(n+1)}$, the class $\psi(x) + I(X)^{n+1}$, considered as a homogeneous polynomial of $\mathbb{F}_p(X)$ of degree n, is called the *initial form* of x in $\mathbb{F}_p(X)$. Thus, one may consider the quotient $F_{(n)}/F_{(n+1)}$ as a subspace of the space of homogeneous polynomials on X of degree n. In particular, the initial form of x_i is X_i , and the initial form of the commutator $[x_i, x_j] = x_i^{-1} x_j^{-1} x_i x_j$ is the algebra commutator $-[X_i, X_j] = X_i X_j + X_j X_i$, for every $i, j \in \{1, \dots, d\}$.

3.2. Restricted lie algebras. In this subsection we give a description of the graded group algebra of a finitely generated pro-*p* group as quotient of the algebra $\mathbb{F}_p\langle X \rangle$.

For $X = \{X_1, \ldots, X_d\}$, one can make the algebra $\mathbb{F}_p \langle X \rangle$ into a Lie algebra, by setting Lie brackets $[f_1, f_2] = f_1 f_2 - f_2 f_1$ for $f_1, f_2 \in \mathbb{F}_p \langle X \rangle$.

Definition 3.1. Let L be a Lie subalgebra of $\mathbb{F}_p\langle X \rangle$. Then L is said to be restricted if $f^p \in L$ for each element $f \in L$. In particular, the restricted Lie algebra L(X)is the restricted Lie subalgebra of $\mathbb{F}_p\langle X \rangle$ generated by X, it is a free restricted Lie algebra, and $\mathbb{F}_p\langle X \rangle$ is its universal envelope.

(For the general definition of restricted Lie algebra see [10, § V.7] and [3, § 12.1].) Let F be a free pro-p group with basis $\mathcal{X} = \{x_1, \ldots, x_d\}$. For every $n \ge 1$, the subspace of L(X) of the homogeneous elements of degree n is the image of the quotient $F_{(n)}/F_{(n+1)}$ via ψ , and we may identify L(X) with $\bigoplus_{n\ge 0} F_{(n)}/F_{(n+1)}$ (cf. [9, Rem. 2.3]).

Let G be a pro-p group with presentation

$$(3.3) 1 \longrightarrow R \longrightarrow F \longrightarrow G \longrightarrow 1.$$

Recall that a presentation (3.3) is minimal when $R \subseteq F_{(2)}$ (cf., e.g., [2, p. 215]). A subset $\{r_1, \ldots, r_m\}$ of F is said to be a set of defining relations for G if it generates R as closed normal subgroup of F.

 Set

$$I(R) = \bigoplus_{n>2} \frac{(R \cap F_{(n)})F_{(n+1)}}{F_{(n+1)}}$$

Since R is a normal subgroup of F, I(R) is a restricted ideal of L(X), i.e., it is an ideal in the sense of a Lie algebra, with the further condition that $f^p \in I(R)$ for each $f \in I(R)$ (cf. [16, § 7.2]). We set L(G) = L(X)/I(R), and $G_{(n)} = F_{(n)}/(R \cap F_{(n)})$ for $n \ge 1$, i.e.,

$$L(G) = \bigoplus_{n \ge 1} G_{(n)} / G_{(n+1)} = L(X) / I(R)$$

(for the original definition of the *p*-Zassenhaus filtration $\{G_{(n)}\}_{n\geq 1}$ of a pro-*p* group G see [3, § 11.1]). The graded group algebra $\operatorname{gr}_{\bullet} \mathbb{F}_p[G]$ and the restricted Lie algebra L(G) related by Jennings' theorem (cf. [3, § 12.2], see also [16, Thm. 3.9])

Proposition 3.2. Let G be as above. Then $\operatorname{gr}_{\bullet}\mathbb{F}_p[G]$ is isomorphic to the universal envelope of the restricted Lie algebra L(G).

Proposition 3.2 and [9, Prop. 2.1] yield the following.

Proposition 3.3. Let G be a finitely generated pro-p group with presentation (3.3). Then one has a short exact sequence of graded \mathbb{F}_p -algebras

$$0 \longrightarrow \mathcal{I} \longrightarrow \mathbb{F}_p\langle X \rangle \longrightarrow \operatorname{gr}_{\bullet} \mathbb{F}_p[G] \longrightarrow 0$$

where $\mathcal{I} \leq \mathbb{F}_p \langle X \rangle$ is the two-sided ideal (as ideal of an associative algebra) generated by I(R) as subset of $\mathbb{F}_p \langle X \rangle$.

7

3.3. Mild pro-*p* groups. Let $X = \{X_1, \ldots, X_d\}$ be a set of non-commuting indeterminates. For $m \ge 1$, let $\rho_1, \ldots, \rho_m \in \mathbb{F}_p\langle X \rangle$ be homogeneous polynomials of degree deg $(\rho_i) = s_i \ge 2$ for each *i*. Let $\mathcal{I} \trianglelefteq \mathbb{F}_p\langle X \rangle$ denote the two-sided ideal generated by ρ_1, \ldots, ρ_m , and set $A_{\bullet} = \mathbb{F}_p\langle X \rangle / \mathcal{I}$. The sequence $\{\rho_1, \ldots, \rho_m\}$ is called strongly free if one has an identity of formal power series

$$\sum_{n \ge 0} \dim(A_n) \cdot t^n = \frac{1}{1 - dt + (t^{s_1} + \ldots + t^{s_m})}$$

(cf. [9, Def. 2.6]).

Definition 3.4. Let G be a finitely generated pro-p group with minimal presentation (3.3), where F has a basis $\mathcal{X} = \{x_1, \ldots, x_d\}$. The group G is said to be mild if there exists a finite set of defining relations $\{r_1, \ldots, r_m\}$ such that $\{\rho_1, \ldots, \rho_m\}$ is a strongly free sequence in $\mathbb{F}_p\langle X \rangle$, with ρ_i the initial form of r_i for each *i*. Such a presentation is called mild.

The following result is fundamental for dealing with the algebras arising from mild pro-p groups — in particular, it provides an effective description for the ideal \mathcal{I} as in Proposition 3.3 (cf. [9, Thm. 2.12]).

Proposition 3.5. Let G be a mild pro-p group, with mild presentation (3.3).

- (i) G has cohomological dimension 2, i.e., $H^n(G, \mathbb{F}_p) = 0$ for $n \geq 3$.
- (ii) One has an isomorphism of graded algebras $\operatorname{gr}_{\bullet}\mathbb{F}_p[G] \simeq \mathbb{F}_p\langle X \rangle / \mathcal{I}$, where $X = \{X_1, \ldots, X_d\}$ and \mathcal{I} is the two-sided ideal generated by the initial forms of the defining relations of G.

Note that for a generic pro-p group G the dieal \mathcal{I} as in Proposition 3.3 may not be generated only by the initial forms of a set of defining relations.

Example 3.6. For p odd let G be the pro-p group generated by $\{x_1, x_2, x_3, x_4\}$ and subject to the defining relations

$$[x_1, x_2]x_1^{\alpha_{12}}x_2^{\beta_{12}} = [x_2, x_3]x_2^{\alpha_{23}}x_3^{\beta_{23}} = [x_3, x_4]x_3^{\alpha_{34}}x_4^{\beta_{34}} = [x_4, x_1]x_4^{\alpha_{41}}x_1^{\beta_{41}} = 1,$$

for some $\alpha_{ij}, \beta_{ij} \in p\mathbb{Z}_p$. Tge group may be reslized as the generalised right-angled Artin pro-p group associated to the square graph with vertices $\{x_1, x_2, x_3, x_4\}$ (cf. [25, Def. 5.3]). By [25, Thm. F], G is mild. Since the initial form of the defining relations above are the commutators $[X_1, X_2], \ldots, [X_4, X_1]$, by Proposition 3.5 one has

$$\operatorname{gr}_{\bullet}\mathbb{F}_{p}[G] \simeq \frac{\mathbb{F}_{p}\langle X \rangle}{([X_{1}, X_{2}], [X_{2}, X_{3}], [X_{3}, X_{4}], [X_{4}, X_{1}])},$$

where $X = \{X_1, \ldots, X_4\}$. Consequently, by [33, § 4.2.2] the algebra $\operatorname{gr}_{\bullet} \mathbb{F}_p[G]$ is Koszul.

4. One-relator pro-p groups

For a pro-*p* group *G* we shall denote the \mathbb{F}_p -cohomology groups simply by $H^n(G)$ for every $n \geq 0$. Thus $H^0(G) = \mathbb{F}_p$, and given a presentation (3.3), one has the following isomorphisms of vector spaces:

(4.1)
$$H^{1}(G) \simeq H^{1}(F) \simeq (G/G_{(2)})^{*}, H^{2}(G) \simeq H^{1}(R)^{G} \simeq (R/R^{p}[R,F])^{*}$$

(cf. [17, Prop. 3.9.1 and Prop. 3.5.9]). The \mathbb{F}_p -cohomology of a pro-*p* group comes endowed with the *cup*-product

$$H^i(G) \times H^j(G) \xrightarrow{\cup} H^{i+j}(G)$$

which is graded-commutative, i.e., $\alpha \cup \beta = (-1)^{ij}\beta \cup \alpha$ for $\alpha \in H^i(G)$ and $\beta \in H^j(G)$. For further facts on cohomology of pro-*p* groups we refer to [17, § III.9].

Henceforth G is assumed to be a finitely generated one-relator pro-p group. Thus, if (3.3) is a minimal presentation, then R is generated by a single defining relation $r \in F_{(2)}$. Given a fixed basis $\mathcal{X} = \{x_1, \ldots, x_d\}$ of F, and a basis $\mathcal{B} = \{\chi_1, \ldots, \chi_d\}$ of $H^1(G)$ dual to \mathcal{X} (i.e., $\chi_i(x_j) = \delta_{ij}$), one has the following (cf. [30, Prop. 1.3.2]).

Proposition 4.1. One may write

(4.2)
$$r = \begin{cases} \prod_{i < j} [x_i, x_j]^{a_{ij}} \cdot r' & \text{if } p \neq 2\\ \prod_{i=1}^n x_i^{2a_{ii}} \cdot \prod_{i < j} [x_i, x_j]^{a_{ij}} \cdot r' & \text{if } p = 2 \end{cases} \quad r' \in F_{(3)},$$

with $0 \leq a_{ij} < p$, and these numbers are uniquely determined by r. Moreover, one has an isomorphism $\operatorname{tr}: H^2(G) \to \mathbb{F}_p$ such that $\operatorname{tr}(\chi_i \cup \chi_j) = -a_{ij}$, with a_{ij} as in (4.2).

In particular, from Proposition 4.1 one deduces that if $\chi_h \cup \chi_k \neq 0$ for some $1 \leq h \leq k \leq d$, then

(4.3)
$$\chi_i \cup \chi_j = \frac{a_{ij}}{a_{hk}} \chi_h \cup \chi_k, \quad \text{for all } 1 \le i \le j \le d.$$

Let $G^{ab} = G/[G, G]$ be the abelianization of G. Then $F^{ab} \simeq \mathbb{Z}_p^d$, and one may choose a basis \mathcal{X} of F such that $r \equiv x_1^q \mod [F, F]$, with q a power of p or q = 0 if $r \in [F, F]$ — i.e., one has an isomorphism of abelian pro-p groups

(4.4)
$$G^{ab} \simeq \mathbb{Z}_p^d$$
 or $G^{ab} \simeq \mathbb{Z}_p/q\mathbb{Z}_p \times \mathbb{Z}_p^{d-1}$

In particular, with such basis one has $a_{ii} = 0$ for $i \ge 2$, if p = 2. Henceforth, \mathcal{X} will always denote a fixed basis of F satisfying this condition.

The following two propositions are crucial for applying the theory exposed in Section 3.

Proposition 4.2. Let G be a finitely generated one-relator pro-p group, with minimal presentation (3.3), basis \mathcal{X} , and defining relation r, such that $q \neq 2$. Then the following are equivalent:

- (i) $H^{\bullet}(G)$ is quadratic;
- (ii) $r \notin F_{(3)}$, *i.e.*, not all a_{ij} are equal to θ .

Moreover, if the above conditions are satisfied then G is mild.

Proof. Since $q \neq 2$, one has $a_{ii} = 0$ for all $i \in \{1, \ldots, d\}$. Thus, by Proposition 4.1 one has $\chi_i \cup \chi_i = 0$ for all i, and moreover the initial form of r is a Lie polynomial, namely, it is a combination of commutators $[X_i, X_j]$.

Suppose first that $H^{\bullet}(G)$ is quadratic, and let $\mathcal{B} = \{\chi_1, \ldots, \chi_d\}$ be a basis of $H^1(G)$ dual to \mathcal{X} . If p is odd, then $H^2(G)$ is a quotient of $\Lambda_2(H^1(G))$ of dimension 1, so that there are i, j, with $1 \leq i < j \leq d$, such that $\chi_i \cup \chi_j \neq 0$. If p = 2, then $H^2(G)$ is a quotient of $S_2(H^1(G))$ of dimension 1, so that there are i, j, with $1 \leq i < j \leq d$, such that $\chi_i \cup \chi_j \neq 0$. If p = 2, then $H^2(G)$ is a quotient of $S_2(H^1(G))$ of dimension 1, so that there are i, j, with $1 \leq i < j \leq d$, such that $\chi_i \cup \chi_j \neq 0$. In both cases, Proposition 4.1 implies that $a_{ij} \neq 0$ as trcolon $H^2(G) \to \mathbb{F}_p$ is an isomorphism, and this yields (ii).

Suppose now that $a_{ij} \neq 0$ for some $1 \leq i \leq j \leq d$. Then [9, Thm. 5.9–(i)] implies that G is mild. In particular, $H^n(G) = 0$ for $n \geq 3$ by Proposition 3.5. We claim that $H^{\bullet}(G)$ is quadratic. Since $H^2(G)$ is the 1-dimensional vector space generated by $\chi_i \cup \chi_j$, the algebra $H^{\bullet}(G)$ is 1-generated. Moreover, the fact that $H^3(G)$ is trivial is a consequence of the relations which hold in $H^2(G)$. Indeed, let $1 \leq h < k < l \leq d$ be any triplet of indices. Then either $\chi_h \cup \chi_k = 0$, or $\chi_h \cup \chi_k = b(\chi_i \cup \chi_j)$ for some $b \in \mathbb{F}_p^{\times}$. In the former case one has $\chi_h \cup \chi_k \cup \chi_l = 0$, whereas in the latter case one has

$$\chi_h \cup \chi_k \cup \chi_l = b(\chi_i \cup \chi_j) \cup \chi_l.$$

Then again either $\chi_j \cup \chi_l = 0$, or $\chi_j \cup \chi_l = b'(\chi_i \cup \chi_j)$ for some $b' \in \mathbb{F}_p^{\times}$, so that

$$\chi_h \cup \chi_k \cup \chi_l = bb'(\chi_i \cup \chi_i \cup \chi_j) = 0,$$

as $\chi_i \cup \chi_i = 0$. Therefore, the relations which hold in $H^2(G)$ imply that $H^3(G) = 0$.

Proposition 4.3. Let G be a finitely generated one-relator pro-2 group, with minimal presentation (3.3), basis \mathcal{X} , and defining relation r, such that q = 2.

- (i) If $H^{\bullet}(G)$ is quadratic, then $r \notin F_{(3)}$, i.e., not all a_{ij} are equal to 0.
- (ii) If $a_{ij} \neq 0$ for some $1 \leq i < j \leq d$, then G is mild and $H^{\bullet}(G)$ is quadratic.

Proof. The proof of statement (i) is the same as the proof of the implication (i) \Rightarrow (ii) in Proposition 4.2.

If $a_{ij} \neq 0$ for some $1 \leq i < j \leq d$, then the initial form of r is the polynomial

$$\rho = X_1^2 + \sum_{1 \le h < k \le d} a_{hk} [X_h, X_k] \in \mathbb{F}_2 \langle X \rangle, \qquad a_{hk} \in \{0, 1\},$$

with $a_{ij} = 1$. Therefore, one may choose an order on the set $X = \{X_1, \ldots, X_d\}$ such that the leading monomial of the homogeneous polynomial ρ is $X_i X_j$. Since $i \neq j$, the sequence $\{X_i X_j\}$ is a combinatorially free sequence of monomials of degree 2 (cf. [9, Def. 3.1]), and therefore the sequence $\{\rho\}$ is strongly free by [9, Thm. 3.5], and r yields a mild presentation of G. In particular, Proposition 3.5 yields $H^n(G) = 0$ for all $n \geq 3$.

In order to prove that $H^{\bullet}(G)$ is quadratic if $a_{ij} \neq 0$ for $1 \leq i < j \leq d$, note that by Proposition 4.1 one has $\chi_i \cup \chi_j \neq 0$, and this cup-product generates $H^2(G)$. Hence, if $\chi_h \cup \chi_k \neq 0$ for some $1 \leq h \leq k \leq d$, then Proposition 4.1 implies $\chi_h \cup \chi_k = \chi_i \cup \chi_j$ (in particular, $\chi_1 \cup \chi_1 = \chi_i \cup \chi_j$). Therefore, the same argument as in the proof of implication (ii) \Rightarrow (i) of Proposition 4.2 — taking any triplet of indices $1 \leq h \leq k \leq l \leq d$ (thus allowing equal indices) — shows that $H^3(G) = 0$ because of the relations which hold in $H^2(G)$, and this completes the proof of statement (ii).

4.1. **Demushkin pro-**p **groups.** Here we describe briefly the example we keep in mind while dealing with one-relator pro-p groups: Demushkin groups (see [17, § III.9] and [16, § 5.2] for further details).

A Demushkin group is a finitely generated one-relator pro-p group G such that the cup-product induces a perfect pairing $H^1(G) \times H^1(G) \to \mathbb{F}_p$. Equivalently, a finitely generated one-relator pro-p group G is Demushkin if and only if it has a presentation (3.3) with defining relation r such that one of the following holds:

- (a) *d* is even and $r = x_1^{p^f}[x_1, x_2][x_3, x_4] \cdots [x_{d-1}, x_d]$ for some $f \in \{1, 2, \dots, \infty\}$ such that $p^f \neq 2$;
- (b) d is even, p = 2 and $r = x_1^{2+\alpha}[x_1, x_2]x_3^{2f}[x_3, x_4] \cdots [x_{d-1}, x_d]$ for some $f \in \{2, 3, \ldots, \infty\}$ and $\alpha \in 4\mathbb{Z}_4$;
- (c) *d* is odd, p = 2 and $r = x_1^2 x_2^{2f} [x_2, x_3] [x_4, x_5] \cdots [x_{d-1}, x_d]$ for some $f \in$ $\{2,3,\ldots,\infty\}$

(cf. [13] and [17, Thm. 3.9.11 and Thm. 3,9,19]). In particular, the only finite Demushkin group is the cyclic group of order 2 (case (c) with d=1). In this case, the \mathbb{F}_2 -cohomology algebra $H^{\bullet}(G)$ is the ring of polynomials in one indeterminate with coefficients in \mathbb{F}_2 . Otherwise, $H^n(G) = 0$ for $n \geq 3$. In both cases, $H^{\bullet}(G)$ is quadratic.

Moreover, the graded group algebra $\operatorname{gr}_{\bullet}\mathbb{F}_{p}[G]$ is isomorphic to the polynomial algebra $\mathbb{F}_p(X)/(f)$, with $X = \{X_1, \ldots, X_d\}$, and f a polynomial such that one of the following cases holds:

- (a) d is even and $f = [X_1, X_2] + [X_3, X_4] + \ldots + [X_{d-1}, X_d];$ (b) d is even, p = 2 and $f = X_1^2 + [X_1, X_2] + [X_3, X_4] + \ldots + [X_{d-1}, X_d];$ (c) d is odd, p = 2 and $f = X_1^2 + [X_2, X_3] + [X_4, X_5] + \ldots + [X_{d-1}, X_d].$

We call such a graded algebra $\mathbb{F}_p(X)/(f)$ a Demushkin graded \mathbb{F}_p -algebra.

Remark 4.4. A quadratic algebra A_{\bullet} has a single relation as above if, and only if, $\operatorname{Ext}_{A_{\bullet}}^{2,2}(\mathbb{F}_p,\mathbb{F}_p)$ has dimension 1 and the cup-product induces a non-degenerate alternating pairing $\operatorname{Ext}_{A_{\bullet}}^{1,1}(\mathbb{F}_p,\mathbb{F}_p) \times \operatorname{Ext}_{A_{\bullet}}^{1,1}(\mathbb{F}_p,\mathbb{F}_p) \to \mathbb{F}_p.$

Finally, one has the following (cf. [16, Thm. 5.2]).

Proposition 4.5. If G is a Demushkin group, then $\operatorname{gr}_{\bullet}\mathbb{F}_p[G]$ is isomorphic to the quadratic dual of $H^{\bullet}(G)$, and both algebras are Koszul.

4.2. Cohomology. The isomorphism tr: $H^2(G) \to \mathbb{F}_p$ induces a skewcommutative pairing tr($_ \cup _$): $H^1(G) \times H^1(G) \to \mathbb{F}_p$. If this pairing is perfect, then G is a Demushkin group by definition. Otherwise, let $V_2 = H^1(G)^{\perp}$ be the radical of $H^1(G)$ with respect to the cup-product — i.e.,

$$V_2 = H^1(G)^{\perp} = \{ \chi \in H^1(G) \mid \chi \cup \psi = 0 \text{ for all } \psi \in H^1(G) \}$$

Then $H^1(G) = V_1 \oplus V_2$, so that the cup-product induces a perfect pairing $V_1 \times V_1 \to V_2$ \mathbb{F}_p . Set q and x_1 as in (4.4). Then (4.2) yields

(4.5)
$$r \equiv \begin{cases} \prod_{i < j} [x_i, x_j]^{a_{ij}} \mod F_{(3)}, & \text{if } q \neq 2\\ x_1^2 \cdot \prod_{i < j} [x_i, x_j]^{a_{ij}} \mod F_{(3)}, & \text{if } q = 2. \end{cases}$$

Proposition 4.6. Set V_1 and V_2 as above. Then

(4.6)
$$H^{\bullet}(G) \simeq A_{\bullet} \sqcap Q(V_2, V_2^{\otimes 2}),$$

where $A_1 = V_1$ and $A_2 \simeq H^2(G)$, with the cup-product inducing a perfect pairing $A_1 \times A_1 \to \mathbb{F}_p.$

Proof. Let (3.3) be a minimal presentation of G. If $q \neq 2$, then the pairing induced by tr is alternating, so that $m = \dim(V_1)$ is even. Hence, V_1 decomposes into a direct sum of hyperbolic planes (cf. [17, Prop. 3.9.16]). Therefore, one may find a basis $\mathcal{B}_1 = \{\chi_1, \ldots, \chi_m\}$ of V_1 which completes to a basis \mathcal{B} of $H^1(G)$ such that

(4.7)
$$\chi_1 \cup \chi_2 = \chi_3 \cup \chi_4 = \ldots = \chi_{n-1} \cup \chi_n = 1$$

and $\chi_i \cup \chi_j = 0$ in any other case for $i \leq j$.

If q = 2, let $\mathcal{B}_1 = \{\chi_1, \ldots, \chi_m\}$ be a basis of V_1 with χ_1 dual to x_1 . Then by Proposition 4.1 one has $\chi_1 \cup \chi_1 = 1$, and m can be both odd or even. Thus, by [13, Prop. 4] we may choose the basis \mathcal{B}_1 and complete it to a basis \mathcal{B} of $H^1(G)$ such that

$$\chi_1 \cup \chi_2 = \chi_3 \cup \chi_4 = \dots = \chi_{m-1} \cup \chi_m = 1, \quad \text{if } 2 \mid m, \\ \chi_2 \cup \chi_3 = \chi_4 \cup \chi_5 = \dots = \chi_{m-1} \cup \chi_m = 1, \quad \text{if } 2 \nmid m.$$

and $\chi_i \cup \chi_j = 0$ in any other case for $i \leq j$.

Remark 4.7. If $q \neq 2$, then necessarily $\dim(V_1) \geq 2$. On the other hand, if q = 2 then one may have $\dim(V_1) = 1$. Then A_{\bullet} is isomorphic to the polynomial algebra in one indeterminate $\mathbb{F}_2[\chi_1]$. This is the only case when $H^{\bullet}(G)$ is quadratic and $H^3(G) \neq 0$.

4.3. The graded group algebra. Let G and $V_1, V_2 \subseteq H^1(G)$ be as above. First we deal with the case q = 2 and $\dim(V_1) = 1$, since by Proposition 4.2 and Remark 4.7, this is the only case with G not mild.

Proposition 4.8. Let G be a finitely generated one-relator pro-2 group with $H^{\bullet}(G)$ quadratic and dim $(V_1) = 1$. Then one has an isomorphism of graded \mathbb{F}_2 -algebras gr $_{\bullet}\mathbb{F}_p[G] = \mathbb{F}_2\langle X \rangle / (X_1^2)$, with $X = \{X_1, \ldots, X_d\}, d = d(G)$.

Proof. By (4.5) and Proposition 4.1, one has $r = x_1^2 \cdot t$ with $t \in F_{(3)}$, with $\mathcal{X} = \{x_1, \ldots, x_d\}$ the basis of F. Thus, the initial form of r is $X_1^2 \in F_2\langle X \rangle$, with $X = \{X_1, \ldots, X_d\}$. We claim that I(R) is the restricted ideal of $\mathbb{F}_p\langle X \rangle$ generated by X_1^2 .

The subgroup $R \subseteq F$ is the (pro-2 closure of the) normal closure of the pro-2cyclic group generated by r. For $n \geq 2$, the non trivial elements of the subspace

$$\frac{(R \cap F_{(n)})F_{(n+1)}}{F_{(n+1)}} \le \frac{F_{(n)}}{F_{(n+1)}}$$

are the initial forms of all the elements of R of degree n. Such elements of R are products of elements of the form $[y, r^{2^{m-1}}]$, with $m, s \ge 0$ such that $n = 2^m + s$, and $y \in F_{(s)}$; and also $y^{-1}r^{2^{m-1}}y$ in case n is a power of 2, with $2^m = n$ and $y \in F$. Commutator calculus and the properties (3.2) yield

$$r^{2^{m-1}} = (x_1^2 \cdot t)^{2^{m-1}} \equiv x_1^{2^m} \mod F_{(2^m+1)}$$

for all $m \ge 1$, and consequently

(4.8)
$$\begin{bmatrix} y, r^{2^{m-1}} \end{bmatrix} = \begin{bmatrix} y, x^{2^m} t_m \end{bmatrix} \equiv \begin{bmatrix} y, x_1^{2^m} \end{bmatrix} \mod F_{(2^m+1+s)}$$
$$y^{-1} \cdot r^{2^{m-1}} \cdot y = \begin{pmatrix} y^{-1} x_1^{2^m} y \end{pmatrix} \cdot \begin{pmatrix} y^{-1} t_m y \end{pmatrix} \equiv x_1^{2^m} \mod F_{(2^m+1)},$$

with $t_m \in F_{2^m+1}$, for $y \in F$ as above. Therefore, the space $(R \cap F_{(n)})F_{(n+1)}/F_{(n+1)}$ — viewed as subspace of the space of homogeneous polynomials of degree n in $\mathbb{F}_2\langle X \rangle$ — is generated by the polynomials $[\wp(X), X_1^{2^m}]$, with $\wp(X)$ running through all Lie polynomials in $F_2\langle X \rangle$ of degree s, with $m, s \geq 0$ such that $n = 2^m + s$; together with the monomial X_1^n in case n is a power of 2.

Hence, I(R) is generated by the monomial X_1^2 as restricted ideal of the restricted \mathbb{F}_2 -Lie algebra L(X), and Proposition 3.3 yields $\operatorname{gr}_{\bullet} \mathbb{F}_p[G] \simeq \operatorname{gr}_{\bullet} \mathbb{F}_p[F]/(X_1^2)$. \Box

On the other hand, if $\dim(V_1) \ge 2$ one has the following.

Proposition 4.9. Set V_1 and V_2 as above, and assume that $\dim(V_1) \ge 2$. Then the graded group \mathbb{F}_p -algebra of G decomposes as free product

(4.9)
$$\operatorname{gr}_{\bullet}\mathbb{F}_{p}[G] \simeq A_{\bullet} \sqcup T^{\bullet}(V_{2}^{*}),$$

where A_{\bullet} is a Demushkin quadratic \mathbb{F}_p -algebra with $A_1 = V_1^*$.

Proof. Set $m = \dim(V_1)$, and let (3.3) be a minimal presentation of G. Also, let \mathcal{B} be a basis of $H^1(G)$ as in the proof of Proposition 4.6.

Let $S = \{x_1, \ldots, x_m, y_1, \ldots, y_{d-m}\} \subseteq F$ be the basis dual to \mathcal{B} . Then by (4.3) from (4.5) one obtains

(4.10)
$$r \equiv [x_1, x_2]^a [x_3, x_4]^a \cdots [x_{m-1}, x_m]^a \mod F_{(3)}$$

for some $a \in \{1, \ldots, p-1\}$, if $q \neq 2$, and

(4.11)
$$r \equiv \begin{cases} x_1^2[x_1, x_2][x_3, x_4] \cdots [x_{m-1}, x_m] \mod F_{(3)} & \text{if } 2 \mid m, \\ x_1^2[x_2, x_3][x_4, x_5] \cdots [x_{m-1}, x_m] \mod F_{(3)} & \text{if } 2 \nmid m, \end{cases}$$

if q = 2. Since $[x, y]^a \equiv [x^a, y] \equiv [x, y^a] \mod F_{(3)}$, after a suitable change of basis S we may assume that a = 1 in (4.10). Therefore, after identifying $\operatorname{gr}_{\bullet} \mathbb{F}_p[G] = \mathbb{F}_p\langle X \rangle$, whith $X = \{X_1, \ldots, X_d\}$, the initial form of r in $F_{(2)}/F_{(3)}$ is the homogeneous polynomial

(4.12)
$$\rho = [X_1, X_2] + [X_3, X_4] + \dots + [X_{m-1}, X_m] \in \mathbb{F}_p\langle X \rangle,$$

if $q \neq 2$, and

(4.13)
$$\rho = \begin{cases} X_1^2 + [X_1, X_2] + [X_3, X_4] + \dots + [X_{m-1}, X_m] & \text{if } 2 \mid m, \\ X_1^2 + [X_2, X_3] + [X_4, X_5] + \dots + [X_{m-1}, X_m] & \text{if } 2 \nmid m, \end{cases}$$

if q = 2. Since G is mild, Proposition 3.5 yields the claim.

4.4. **Demushkin groups as quotients.** In [35], T. Würfel proved that if a field
$$\mathbb{K}$$
 contains all roots of 1 of order a power of p and its absolute Galois group $G_{\mathbb{K}}$ is a finitely generated one-relator pro- p group, then one has a short exact sequence of pro- p groups

$$(4.14) 1 \longrightarrow N \longrightarrow G_{\mathbb{K}} \longrightarrow G_{\mathbb{K}}/N \longrightarrow 1$$

where N is free and $G_{\mathbb{K}}/N$ is a Demushkin group, and moreover the inflation map $\inf_{U,N}^2: H^2(U/N, \mathbb{Z}/p^s) \to H^2(U, \mathbb{Z}/p^s)$ is an isomorphism for every open subgroup $U \subseteq G_{\mathbb{K}}$ containing N and every $s \geq 1$.

If one considers finitely generated pro-p groups whose closed subgroups have quadratic \mathbb{F}_p -cohomology (see Remark 4.11 below), one obtains the following.

Proposition 4.10. Let G be a finitely generated one-relator pro-p group such that every closed subgroup of G has quadratic \mathbb{F}_p -cohomology. Then there exists a free closed subgroup $S \leq G$ and a short exact sequence of pro-p groups

$$(4.15) 1 \longrightarrow N \longrightarrow G \longrightarrow G/N \longrightarrow 1$$

where N is the normal closure of S in G, G/N is a Demushkin group, and N is free if $H^3(G) = 0$. Moreover, one has the isomorphisms of quadratic algebras

(4.16)
$$H^{\bullet}(G) \simeq H^{\bullet}(S) \sqcap H^{\bullet}(G/N),$$
$$\operatorname{gr}_{\bullet}\mathbb{F}_{p}[G] \simeq \operatorname{gr}_{\bullet}\mathbb{F}_{p}[S] \sqcup \operatorname{gr}_{\bullet}\mathbb{F}_{p}[G/N].$$

Remark 4.11. The assumption that every closed subgroup of G has quadratic \mathbb{F}_p -cohomology might seem quite restrictive and unnatural. Still, by the Rost-Voevodsky Theorem every closed subgroup of the (maximal pro-p quotient of the) absolute Galois group of a field containing a root of 1 of order p has quadratic \mathbb{F}_p -cohomology (cf. Remark 5.2 below). Thus, this assumption is natural in view of the application of Proposition 4.10 to the Galois-theoretic case (cf. Theorem 5.4).

Proof of Proposition 4.10. Let S be the closed subgroup of G such that the restriction morphism $\operatorname{res}_{G,S}^1 \colon H^1(G) \to H^2(S)$ induces an isomorphism $H^1(G)^{\perp} \simeq H^1(S)$. In particular, one has $\operatorname{ker}(\operatorname{res}_{G,S}^1) = V_1$. Therefore, the commutative diagram

$$\begin{array}{c|c} H^1(G) \times H^1(G) & \longrightarrow & H^2(G) \\ \hline res^1_{G,S} & res^1_{G,S} & & res^2_{G,S} \\ H^1(S) \times H^1(S) & \longrightarrow & H^2(S) \end{array}$$

implies that the lower horizontal arrow is trivial and thus $H^2(S) = 0$, as $H^{\bullet}(S)$ is quadratic. Consequently, S is a free pro-p group (cf. [17, Prop. 3.5.17]).

Let $N \subseteq G$ be the normal closure of S in G, and set $\overline{G} = G/N$. Since $H^1(N)^G \simeq H^1(S)$, the exact sequence

$$0 \longrightarrow H^1(\bar{G}) \xrightarrow{\inf_{G,N}^1} H^1(G) \xrightarrow{\operatorname{res}_{G,N}^1} H^1(N)^{\bar{G}} \longrightarrow H^2(\bar{G}) \xrightarrow{\inf_{G,N}^2} H^2(G)$$

induced by the quotient G/N implies that $H^1(\bar{G}) \simeq V_1$ and that the inflation map $\inf_{G,N}^2: H^2(\bar{G}) \to H^2(G)$ is a monomorphism (cf. [17, Prop. 1.6.7]). Thus, in the commutative diagram

$$\begin{array}{c|c} H^1(\bar{G}) \times H^1(\bar{G}) & \longrightarrow & H^2(\bar{G}) \\ & & & & \\ \inf^1_{G,N} & & & & \inf^1_{G,N} \\ H^1(G) \times H^1(G) & \longrightarrow & H^2(G) \end{array}$$

the upper line is a non-degenerate pairing — in particular, \overline{G} is a one relator pro-p group too. Therefore, \overline{G} is a Demushkin group (cf. § 4.1). Moreover, $\inf_{G,N}^2$ is an isomorphism, so that if $H^3(G) = 0$ then [36, Prop. 1] implies that N is free — recall that $H^3(G) = 0$ if, and only if, $\dim(V_1) = 1$ (cf. Remark 4.7).

Finally, (4.16) follows form Proposition 4.6 and Proposition 4.9, since $V_1 \simeq H^1(G/N)$ and $V_2 \simeq H^1(S)$.

5. Absolute Galois groups of fields

Hereinafter \mathbb{K} will denote a field containing a root of 1 of order p. Moreover, $G_{\mathbb{K}}(p)$ will denote the maximal pro-p quotient of the absolute Galois group of \mathbb{K} — namely, $G_{\mathbb{K}}(p)$ is the maximal pro-p Galois group (i.e., the Galois group of the maximal pro-p extension) of \mathbb{K} .

5.1. Maximal pro-*p* Galois groups. Let \mathbb{K}^{\times} denote the multiplicative group of \mathbb{K} . By Kummer theory one has an isomorphism $\mathbb{K}^{\times}/(\mathbb{K}^{\times})^p \simeq H^1(G_{\mathbb{K}})$. Moreover, note that if p = 2 then q = 2 (where q is defined for $G_{\mathbb{K}}(2)$ as in (4.4)) only if $\sqrt{-1} \notin \mathbb{K}$.

The Rost-Voevodsky theorem has the following fundamental consequence (see, e.g., [7, p. 222]).

Proposition 5.1. The \mathbb{F}_p -cohomology ring $H^{\bullet}(G_{\mathbb{K}})$ of the absolute Galois group of \mathbb{K} is quadratic. In particular, the epimorphism $G_{\mathbb{K}} \to G_{\mathbb{K}}(p)$ induces an isomorphism of graded algebras $H^{\bullet}(G_{\mathbb{K}}(p)) \simeq H^{\bullet}(G_{\mathbb{K}})$.

Remark 5.2. If S is a closed subgroup of $G_{\mathbb{K}}$, respectively of $G_{\mathbb{K}}(p)$, then S is the absolute Galois group $G_{\mathbb{L}}$, representively the maximal pro-p Galois group $G_{\mathbb{L}}(p)$, of a suitable extension \mathbb{L}/\mathbb{K} , and obviously \mathbb{L} contains a root of 1 of oder p as well. Then by Proposition 5.1, the \mathbb{F}_p -cohomology algebra $H^{\bullet}(S)$ is again quadratic.

By Proposition 5.1 and (4.1), $G_{\mathbb{K}}(p)$ is one-relator if, and only if, $H^2(G_{\mathbb{K}})$ has dimension 1. Recall that the cohomological *p*-dimension of a profinite group G is the non-negative integer $\operatorname{cd}_p(G)$ defined by

 $\operatorname{cd}_p(G) = \max\{n \ge 0 \mid H^n(G, M) \ne 0 \text{ for all } p\text{-torsion } G\text{-modules } M\}$

(cf. [17, Def. 3.3.1]). If G is a pro-p group, then $\operatorname{cd}_p(G) = \operatorname{cd}(G)$. Let N denote the kernel of the epimorphism $G_{\mathbb{K}} \to G_{\mathbb{K}}(p)$. Since $H^1(G_{\mathbb{K}}(p)) \simeq H^1(G_{\mathbb{K}})$, the group N is p-perfect, i.e., $H^1(N, \mathbb{F}_p) = 0$, and hence $\operatorname{cd}_p(N) = 0$. Moreover, if $\operatorname{cd}(G_{\mathbb{K}}(p)) < \infty$, then [17, Prop. 3.3.8] implies that $\operatorname{cd}_p(G_{\mathbb{K}}) = \operatorname{cd}(G_{\mathbb{K}}(p))$. Furthermore, if $\dim(H^1(G_{\mathbb{K}})) < \infty$, then $\operatorname{cd}(G_{\mathbb{K}}(p))$ (and hence $\operatorname{cd}_p(G_{\mathbb{K}})$) is finite (cf. [23, Prop. 4.1]).

On the other hand, the group algebras $\operatorname{gr}_{\bullet}\mathbb{F}_p[G_{\mathbb{K}}]$ and $\operatorname{gr}_{\bullet}\mathbb{F}_p[G_{\mathbb{K}}(p)]$ are related as follows (cf. [16, Rem 1.4] and [28]).

Proposition 5.3. If $H^1(G_{\mathbb{K}})$ if finite the epimorphism $G_{\mathbb{K}} \to G_{\mathbb{K}}(p)$ induces an isomorphism of graded algebras $\operatorname{gr}_{\bullet} \mathbb{F}_p[G_{\mathbb{K}}] \simeq \operatorname{gr}_{\bullet} \mathbb{F}_p[G_{\mathbb{K}}(p)].$

From the results of Section 4, we may prove Theorem 1.2.

Theorem 5.4. Suppose that $H^1(G_{\mathbb{K}})$ is finite and dim $H^2(G_{\mathbb{K}}) = 1$. Then one has isomorphisms of quadratic algebras

(5.1)
$$H^{\bullet}(G_{\mathbb{K}}) \simeq A_{\bullet} \sqcap Q(V_2, V_2^{\otimes 2}) \quad and \quad \operatorname{gr}_{\bullet} \mathbb{F}_p[G_{\mathbb{K}}] \simeq B_{\bullet} \sqcup T_{\bullet}(V_2^*),$$

where $V_2 = H^1(G_{\mathbb{K}})^{\perp}$ (with respect to the pairing induced by the cup-product), $A_1 \simeq H^1(G_{\mathbb{K}})/V_2$ and B_{\bullet} a Demushkin algebra (cf. § 4.1).

Moreover, there exists a closed subgroup $\tilde{S} \leq G_{\mathbb{K}}$ with $\operatorname{cd}_{p}(\tilde{S}) = 1$ such that

$$A_{\bullet} \simeq H^{\bullet}(G_{\mathbb{K}}/N_{\tilde{S}}) \quad and \quad B_{\bullet} \simeq \operatorname{gr}_{\bullet} \mathbb{F}_{p}[G_{\mathbb{K}}/N_{\tilde{S}}]$$
$$Q(V_{2}, V_{2}^{\otimes 2}) \simeq H^{\bullet}(\tilde{S}) \quad and \quad T_{\bullet}(V_{2}^{*}) \simeq \operatorname{gr}_{\bullet} \mathbb{F}_{p}[\tilde{S}].$$

— here $N_{\tilde{S}}$ denotes the closed normal subgroup generated by \tilde{S} , and $\operatorname{cd}_p(N_{\tilde{\xi}}) = 1$ as well, unless p = 2 and $\sqrt{-1} \notin \mathbb{K}$.

Proof. By Theorem 5.1 and Proposition 5.3, it is enough to show (5.1) for $H^{\bullet}(G_{\mathbb{K}}(p))$ and $\operatorname{gr}_{\bullet}\mathbb{F}_{p}[G_{\mathbb{K}}(p)]$. By hypothesis, $G_{\mathbb{K}}(p)$ is a finitely generated one-relator pro-p group, thus (5.1) follows from Theorem 4.10.

Now let $\tilde{S} \subseteq G_{\mathbb{K}}$ be the lift of S, with S and N as in Proposition 4.10 for $G = G_{\mathbb{K}}(p)$. I.e., for $\pi \colon G_{\mathbb{K}} \to G_{\mathbb{K}}(p)$ the canonical projection, one has $\ker(\pi|_{\tilde{S}}) = \ker(\pi)$. By Remark 5.2, both S and \tilde{S} have quadratic \mathbb{F}_p -cohomology. In fact, S is the maximal pro-p quotient of \tilde{S} , and thus $\operatorname{cd}_p(\tilde{S}) = \operatorname{cd}(S)$ (cf. [17, Prop. 3.3.8]).

Moreover, one has an isomorphism $G_{\mathbb{K}}/N_{\tilde{S}} \simeq G_{\mathbb{K}}(p)/N$, as $N_{\tilde{S}}$ is the lift of N and thus $N_{\tilde{S}}/\ker(\pi) \simeq N$. Hence, $G_{\mathbb{K}}/N_{\tilde{S}}$ is a Demushkin group, and $\operatorname{cd}_p(N_{\tilde{S}}) = \operatorname{cd}(N)$. Finally, $\operatorname{cd}(N) = 1$ unless $\dim(H^1(G_{\mathbb{K}}(p))/N) = 1$, and this case occurs inly if p = q = 2, namely, only if $\sqrt{-1} \notin \mathbb{K}$.

Therefore, one may deduce all the claims of the statement from Proposition 4.10. $\hfill \square$

Note that Theorem 5.4 (in fact already Proposition 4.10) implies the statement of Würfel's result, but for the bijectivity of the maps $\inf_{U,N}^2$. Corollary 1.3 follows from Theorem 5.4 together with Example 2.3 and Proposition 2.5, as Demushkin algebras are Koszul (cf. § 4.1).

Example 5.5. Let $G = G_{\circ} *_{\hat{p}} S$ be the free product (in the category of pro-*p* groups) of a Demushkin group G_{\circ} with a finitely generated free pro-*p* group *S*. Then $G/N \simeq G_{\circ}$ (with *N* the normal closure of *S*), and one has

$$H^{\bullet}(G) \simeq H^{\bullet}(G_{\circ}) \sqcap H^{\bullet}(S) \qquad \text{and} \qquad \operatorname{gr}_{\bullet} \mathbb{F}_p[G] \simeq \operatorname{gr}_{\bullet} \mathbb{F}_p[G_{\circ}] \sqcup \operatorname{gr}_{\bullet} \mathbb{F}_p[S].$$

Such group is a pro-p groups of elementary type (see next subsection).

Example 5.6. For p odd let G be the pro-p group with minimal presentation

$$G = \langle x_1, x_2, x_3 \mid [x_1, x_2] = x_3^q \rangle$$

with q > 1 a power of p. Such pro-p group satisfies all the conditions in Würfel's theorem, as stated in [12, Thm.2], and by Proposition 4.2 its \mathbb{F}_p -cohomology algebra is quadratic. In particular, one has $H^{\bullet}(G) \simeq H^{\bullet}(S) \sqcap H^{\bullet}(\overline{G})$, with $S = \langle x_3 \rangle$ and $\overline{G} = G/N \simeq \mathbb{Z}_p^2$, with N the normal closure of S, and

$$\operatorname{gr}_{\bullet} \mathbb{F}_p[G] \simeq \mathbb{F}_p[X_1, X_2] \sqcup \mathbb{F}_p[X_3].$$

Yet, the group G is not realizable as maximal pro-p Galois group of any \mathbb{K} , by [8, Thm. 4.2 and Thm. 8.1].

5.2. Koszul algebras of elementary type. Let $\mu_{p^{\infty}}$ denote the group of roots of unity of order a power of p contained in the maximal pro-p extension of \mathbb{K} . The maximal pro-p Galois group $G_{\mathbb{K}}(p)$ acts on $\mu_{p^{\infty}}$, fixing the roots of order p (as they lie in \mathbb{K}). Since the subgroup of $\operatorname{Aut}(\mu_{p^{\infty}})$ which fixes the roots of order p is isomorphic to the (multiplicative) group $1 + p\mathbb{Z}_p = \{1 + p\lambda \mid \lambda \in \mathbb{Z}_p\}$, one has a homomorphism of pro-p groups

$$\theta_{\mathbb{K}} \colon G_{\mathbb{K}}(p) \longrightarrow 1 + p\mathbb{Z}_p,$$

called the cyclotomic character.

Following [6, § 3], we call a cyclotomic pro-p pair a pair (G, θ) consisting of a finitely generated pro-p group G and a homomorphism $\theta: G \to 1 + p\mathbb{Z}_p$ (the homomorphism θ is also called an orientation of G, cf. [23, 26]). A cyclotomic pro-p pair is realizable arithmetically if there exists a field K such that $G \simeq G_{\mathbb{K}}(p)$ and θ coincides with the cyclotomic character. The class of cyclotomic pro-p pairs of elementary type is the class of cyclotomic pro-p pairs containing

- (a) any pair (F, θ) , with F a finitely generated free pro-p group and $\theta: F \to 1 + p\mathbb{Z}_p$ any orientation (including the trivial group with trivial orientation θ);
- (b) any pair (G, θ), with G a Demushkin group and θ: 1 + pZ_p as defined in [13, Thm. 4] (including the cyclic group of order 2 with the non-trivial orientation θ: Z/2 → {±1});

and such that any pair contained in this class which is not of the type (a) or (b) may be obtained iterating the following to operations:

- (c) by taking the free product $(G_1, \theta_1) * (G_2, \theta_2)$ of two cyclotomic pro-*p* pairs of elementary type, given by the free pro-*p* product $G_1 *_{\hat{p}} G_2$;
- (d) by taking the the semi-direct product $(\mathbb{Z}_p \rtimes G, \theta \circ \pi)$ of a cyclotomic pro-p pair (G, θ) , defined by $gzg^{-1} = \theta(g) \cdot z$ for all $z \in \mathbb{Z}_p$ and with $\pi \colon \mathbb{Z}_p \rtimes G \to G$ the canonical projection.

Remark 5.7. (i) The \mathbb{F}_p -cohomology algebra of a pro-*p* group of elementary type is always quadratic.

(ii) A one-relator pro-p group G is of elementary type if, and only if, G is isomorphic to the free pro-p product $G_{\circ} *_{\hat{p}} S$ of a Demushkin group G_{\circ} with a (possibly trivial) free pro-p group S. In particular, by [12, Thm. 12–(f)] the pro-p group G as in Example 5.6 is not of elementary type.

Not all Demushkin groups are known to occur as $G_{\mathbb{K}}(p)$ for some field \mathbb{K} . On the other hand, if one takes only Demushkin groups which occur as $G_{\mathbb{K}}(p)$ for some field \mathbb{K} in item (b) above, then all cyclotomic pro-*p* pairs of elementary type obtained are realizable as maximal pro-*p* Galois groups. I. Efrat's *Elementary type Conjecture* states that if a cyclotomic pro-*p* pair is realizable arithmetically, then it is of elementary type (cf. [4], see also [5, Question 4.8] and [15, § 10]).

In analogy with the class of cyclotomic pro-p pairs of elementary type, we define the classes of Koszul graded algebras of G-elementary type and of H-elementary type.

Definition 5.8. The class of Koszul graded \mathbb{F}_p -algebras of G-elementary type \mathcal{KET}_G is the smallest class of quadratic \mathbb{F}_p -algebras such that, for V be any finite \mathbb{F}_p -vector space,

- (a) the free algebra $T^{\bullet}(V)$ is in \mathcal{KET}_G ;
- (b) any Demushkin algebra A_{\bullet} is in \mathcal{KET}_G (including the trivial \mathbb{F}_2 -algebra on one generator $Q(\mathbb{F}_2, \mathbb{F}_2^{\otimes 2})$);

and such that, for V any finite \mathbb{F}_p -vector space,

- (c) if A_{\bullet}, B_{\bullet} are in \mathcal{KET}_G , then also the free product $A_{\bullet} \sqcup B_{\bullet}$ is in \mathcal{KET}_G ;
- (d) if A_{\bullet} is in \mathcal{KET}_G , then also the symmetric tensor product $A_{\bullet} \otimes S_{\bullet}(V)$ is in \mathcal{KET}_G .

Dually, the class of Koszul graded \mathbb{F}_p -algebras of H-elementary type \mathcal{KET}_H is the smallest class of quadratic \mathbb{F}_p -algebras such that, for V be any finite \mathbb{F}_p -vector space,

- (a') the trivial algebra $A_{\bullet} = Q(V, V^{\otimes 2})$ is in \mathcal{KET}_H (including the case V = 0);
- (b') the quadratic dual $A^!_{\bullet}$ of any Demushkin algebra A_{\bullet} is in \mathcal{KET}_H (including the polynomial algebra on one indeterminate $\mathbb{F}_2[X]$);

and such that, for V be any finite \mathbb{F}_p -vector space,

- (c') if A_{\bullet}, B_{\bullet} are in \mathcal{KET}_H , then also the direct product $A_{\bullet} \sqcap B_{\bullet}$ is in \mathcal{KET}_H ;
- (d') if A_{\bullet} is in \mathcal{KET}_H , then also the skew-symmetric tensor product $A_{\bullet} \wedge \Lambda_{\bullet}(V)$ is in \mathcal{KET}_H .

By Example 2.3, Proposition 2.5 and [16, § 5.2], all algebras of *G*-elementary and *H*-elementary type are in fact Koszul. Combining the restults obtained in [16, § 5] together Proposition 4.10, one deduces the following.

Proposition 5.9. Let \mathbb{K} be a field containing a root of unity of order p, and assume that $H^1(G_{\mathbb{K}})$ is finite.

- (i) If (G_K, θ) is of elementary type, then gr_•F_p[G_K] and H[•](G_K) are Koszul algebras of G-, respectively H-, elementary type.
- (ii) If G_K(p) is one-relator, then gr_●F_p[G_K] and H[●](G_K) are Koszul algebras of G-, respectively H-, elementary type.

Therefore, in analogy with Efrat's conjecture, we formulate the following refinement of Conjecture 1.1.

Conjecture 5.10. Let \mathbb{K} be a field containing a root of unity of order p, such that $H^1(G_{\mathbb{K}})$ is finite.

- (i) The graded group algebra $\operatorname{gr}_{\bullet} \mathbb{F}_p[G_{\mathbb{K}}]$ is a Koszul algebra of G-elementary type.
- (ii) The 𝔽_p-cohomology ring H[●](G_𝔅, 𝔽_p) is a Koszul algebra of H-elementary type.

By Proposition 5.9 a positive solution of the Elementary Type Conjecture would imply a positive answer to Conjecture 5.10.

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