Highlights

Complex interplay between monetary and fiscal policies in a real economy model

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- The proposed real economy model grounds on a nonlinear multiplier-accelerator setup
- The interplay between monetary and fiscal policy rules is investigated
- The monetary policy is able to lead the economy toward the targeted level of output
- Policies can be the source of endogenous fluctuations resembling business cycle
- The interplay between policies is responsible for the occurrence of multi-stability

Complex interplay between monetary and fiscal policies in a real economy model

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Abstract

In this paper we consider a nonlinear model for the real economy described by a multiplier-accelerator setup. The model comprises the government sector, which influences the output dynamics by means of the fiscal policy, and the money market, where the money supply depends upon the fluctuations in the economic activity. Through rigorous analytical tools combined with numerical simulations, we investigate the stability conditions of the unique steady state and the emergence of different kinds of endogenous dynamics, which are the results of the action of the fiscal and the monetary policy through their reactivity degrees. Such policies, if properly tuned, can lead the economy toward the desired full employment target but, on the other hand, can also generate endogenous fluctuations in the pace of the economic activity, associated with the occurrence of closed invariant curves and multistability phenomena.

Keywords: Nonlinear dynamics, monetary and fiscal policies, bifurcations, multistability.

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1. Introduction

It is well-known that macroeconomic variables, such as national income, interest rates, inflation rates, money supply, etc. exhibit persistent and irregular fluctuations. The models for explaining the evolutions of these variables are widespread and come from any field of economics, with the aim of detecting the endogenous sources of such fluctuations. In this respect, *nonlinearity* is a key ingredient which is thought to be the source of endogenous cyclic behavior. Accordingly, the literature that stems from this idea is widespread and developed in the last decades. Among the milestones in the macro-dynamics literature we can mention the paper by Kalecki [1], who stressed the crucial role played by investments in a capitalist system and the issue of finding a well-specified investment function to appease many problems within the considered economy. More precisely, the issue of income distribution is one of the pillars of Kalecki's efforts to build a business cycle theory. His theory shows that output, which is of course at the ground of income distribution, is completely determined by investments. He found that the problem of the change in output, and hence the business cycle, is due to changes in the volume of investment, which are the sources of fluctuations of a capitalist economy. During booms, firms are able to produce more cash flow and obtain increases in profits. However, the increase in orders for capital investment increases the stock of capital, until it becomes unprofitable to make more investments. Samuelson [2] built a multiplier-accelerator model to analyze the business cycle, while Kaldor [3] explicitly adopted a nonlinear investment function based on the profit principle. In particular, Samuelson's model analyzes the business cycle, and is based on the Keynesian multiplier (i.e. consumption choices are affected by the level of economic activity) and the accelerator theory of investment (i.e. investments intentions depend on the economic activity growth). In [2], Samuelson proposed a model in which cyclical fluctuations arise as a consequence of the interplay between the accelerator and the multiplier. On the other hand, Kaldor's idea is related to the fact that an expansion in the demand and, consequently, in the production is the source of an increase in induced investments, which in turn bolsters profits and encourages new investments. Finally, Hicks [4] extended the Samuelson's seminal model with the ideas of floor and ceiling to bound the growth of investments. A drawback in the Samuelson model is the possibility of unbounded dynamics. As discussed in [5], if the economy is, for instance, in a depressed phase, in such model we would have a decrease in the capital that could lead not only to negative investments but also to a possible capital erosion. To overcome this phenomenon (and the complementary one related to unbounded expansion), Hicks proposed the introduction of lower and upper bounds (floor and ceiling) on the investments' adjustment mechanism. After these contributions embedded in the general economic thought, the original papers have been extended in several directions thanks to the application of the tools from nonlinear dynamics and bifurcation theory, which allowed reconsidering the various existing economic models. Many interesting improvements of business cycle modeling spread during the recent decades, frequently exhibiting complex dynamics, nonlocal bifurcations and transitions from order to chaos, as in [6]-[10].

The Samuelsonian multiplier-accelerator modelling approach, which is considered a benchmark to explain business cycle fluctuations, has been developed and extended in different ways (see, e.g., [11]-[17]), where the birth of persistent oscillations is discussed through the analysis of the role of different elements, such as the presence of an income ceiling and an investment floor, or the introduction of delays in the consumption or investments function. In particular, the papers in [11]-[13] are developed within a discrete-time framework, while the papers in [14]-[17] are casted into a continuous-time setup, especially to analyze the role of time delays in consumption, saving and investment decisions.

Nonetheless, a usual objection to business cycle models based on the interaction of the multiplier and the accelerator is that they neglect monetary factors and, when the latter are considered, are not an integral part of the model. Only few papers examined the role of the introduction of the monetary policy within a multiplier-accelerator framework, even if the monetary factors may be of relevance for understanding the emergence and evolution of the business cycle. In this regard, we mention the paper by Smith in [18] who introduces money in a simple and linear multiplier-accelerator model, showing that periodic oscillations may arise and increase when monetary factors are considered. In [19], Lovell and Prescott study the role of money supply in determining periods of stability and oscillations in the business cycle within the Samuelsonian multiplier-accelerator setup and investigate the effects of alternative stabilization strategies that might be activated by the monetary authority. The work by Sordi in [20] tackles the issue of business cycle fluctuations in a discrete time multiplier-accelerator model in which a floor and a ceiling are introduced into the accelerator component of the capital stock evolution. Also, the paper by Sordi and Vercelli [21] studies in depth the multiplier-accelerator framework in a context of a monetary economy where the multiplier effects are closely related to the monetary transactions. In fact, as noted in [21], money is an exchange medium that intrinsically introduces a temporal separation between the moment of the expenditure and the realization of the income. This leads to an alternation between these two phenomena, alternation that is further lagged by the effect of money. This in turn affects income dynamics through the multiplier mechanism. Finally, more recently, Karpetis and Varelas in [22] introduced a money market and a balanced government expenditure rule in a linear, discrete-time multiplier-accelerator to study their interaction and how they affect the overall economic stability.

The idea of including the money market in this framework, together with the fiscal policy, fits into the debate on the proper degree of activism in fiscal and monetary policy making. The main question to give an answer is of how much to vary monetary and fiscal instruments to reduce market turmoil and whether a good fiscal policy is more or less effective than a good monetary policy for stabilization purposes. In fact, the economic literature looking at the mix between the two policies mostly dates back to the early 80s and grew during the 90s. Sargent and Wallace ([23]) proved that monetary policy has to lead the fiscal one, in order to guarantee control on inflation. More recently, numerous authors, for instance Schmitt-Grohe and Uribe, ([24]) and Benigno and Woodford ([25]), used DSGE models to investigate the interaction between monetary and fiscal policies. Finally, Eusepi and Preston ([26]) highlighted that the particular choice of monetary policy leads to a constraint on the possible fiscal policies that are compatible with macroeconomic stability requirement. In particular, the Taylor rule can introduce instabilities due to expectation formation mechanisms. The above mentioned contributions mainly focus on the activism of the two policies in order to control inflation and obtain a stable macroeconomic environment.

In the present paper we enrich the discrete time multiplier-accelerator setting considered in [27] by taking into account a money market. The aim is to examine the effects of making monetary factors a relevant part of the economic setup under investigation in order to understand the interplay between these factors and the fiscal policy instruments may give rise, reduce or foster the oscillations in the business cycle. The multiplier-accelerator model encompasses a nonlinear investment function which takes into account the presence of the monetary sector through the interest rate which, in turn, is determined by the equilibrium condition on the money market. The money supply is influenced by the discrepancy between the full employment national income level and the more recent output realizations. Moreover, the public sector may influence the possibility of the economy to reach a full employment output level through a level-adjusting rule. Therefore, it turns out to be relevant to study whether the interplay between these two policies can render the overall system stable or, instead, endogenous fluctuations arise at the ground of the business cycle.

To this end, we analytically obtain the local asymptotic stability conditions of the unique steady state, and, with the help of numerical investigations, we investigate the possible kinds of bifurcations, showing the consequent emergence of periodic, quasi-periodic and chaotic dynamics.

We find that the introduction of the monetary policy is capable of leading the economy toward the targeted level of output.

In general, there is not an unambiguous role played by the two policies since both

can be either the source of endogenous fluctuations, arising when they induce overreaction to real economy signals, or lead to a stabilization of the dynamics. Indeed, there is a certain role played by both policies for the stabilization purposes. For example, instabilities spreading from the money market can be deadened by the activation of a convenient fiscal policy, as well as stability goals can be achieved through an appropriate cautionary monetary policy, grounded on a suitably inertial response to fluctuations in target variables. However, overreactive policies are not beneficial and may generate complex dynamics in the evolution of the national income in the long run. From the mathematical viewpoint, this is described by endogenous fluctuations that arise due to Neimark-Sacker or period doubling bifurcations. In particular, the interplay between fiscal and monetary policies may be the source of quasi-periodic oscillations, resembling the emergence of business cycle, which are not possible, as shown in [27], when the money market is not considered and only a level adjusting fiscal policy is taken into account. Moreover, the effect of endogenizing the money market may imply multistability. Results, containing a variety of dynamic features, are discussed through the analysis of local bifurcations and through numerical examples that give insights about global dynamics.
The rest of the paper is organized as follows: Section 2 introduces the model, Section 3 presents analytical results on the stability of the unique steady state and the conditions for its asymptotic stability, Section 4 reports the numerical simulations showing how the relevant parameters of the two policies may give rise to complex dynamics. Finally, Section 5 collects conclusions and some possible future research perspectives.

2. The baseline model

We present a closed economy model consisting of a real sector, described by a multiplier-accelerator setup, and of a monetary sector. The macroeconomic equilibrium condition, at any time $t \in \mathbb{N}$, is given by

$$Y_t = C_t + I_t + G_t, \tag{1}$$

where Y_t represents the national income, C_t denotes consumption, I_t investments and G_t public expenditures. Consumption linearly depends on the last realization of national income, i.e.

$$C_t = \bar{C} + cY_{t-1}, \tag{2}$$

where $\bar{C}$ is the autonomous consumption and $c \in (0, 1)$ is the marginal propensity to consume.

We assume that government intervenes into the real market to stabilize the economy by means of a level-adjusting rule. In other words, the government has to establish a full employment income Y^F and modifies its expenditures according to the gap between the full employment income and the national income, namely

$$G_t = \bar{G} + g(Y^F - Y_{t-1}). \tag{3}$$

Government expenditures depends on an autonomous component $\bar{G} > 0$ and on a discretionary expenditure $g(Y^F - Y_{t-1})$ where $g > 0$ measures the reactivity of the fiscal policy with respect to deviations from the target Y^F.

The principle of *acceleration* determines investments. Precisely, we assume the investments function is made up by three components. Besides an autonomous component, a second component is increasing in the national income variation between period $t-1$ and $t-2$ and is described by a bounded S-shaped function according to the hypothesis that investments can not reach too high or too low values (see [27]). Such a function is continuous and differentiable at each point and it makes this second component constant when the national income does not change for two consecutive periods. A third component is also included to highlight the negative dependence of investments to the interest rate. Thus, the investments function can be summarized as

$$I_t = \bar{I} + \gamma a_2 \left(\frac{a_1 + a_2}{a_1 e^{-(Y_{t-1} - Y_{t-2})} + a_2} - 1 \right) + \varphi R_t, \tag{4}$$

where $\bar{I}$ is the autonomous component of investments, $\gamma > 0$ relates to the accelerator component, a_1 and a_2 are positive parameters that determine the investment function variation range and $\varphi \leq 0$. It is worth noting that a functional form as the one proposed here for the investments function is in line with the classic macroeconomic literature of the 1930s-1950s (see, e.g., [1], [3], [28]). Moreover, the motivation for considering a sigmoid function to model a component of the investments comes from the Hicks' idea of embodying a floor and a ceiling in the evolution of investments, in order to take into account the impossibility of an indefinite growth and disinvestment due to resource and physical constraints (see [5, 13, 29] and [30]). Finally, expression (4) also states that investments negatively depend on the interest rate, being $\varphi \leq 0$. In fact, the interest rate reflects the cost of borrowing in order to finance investment projects and, other things being equal, as the interest rates rise, financing new investment projects becomes more expensive.

Let us now introduce the money market, for which we the equilibrium condition reads as

$$\frac{M_t^S}{\bar{P}} = \frac{M_t^D}{\bar{P}}, \tag{5}$$

that is, the real money demand $M_t^D/\bar{P}$ equals the real money supply $M_t^S/\bar{P}$. In general, see e.g. [31], the equilibrium on the money market implies that the supply equals the real demand of money balances $M/\bar{P}$, where $\bar{P}$ represents is the constant price level, normally set equal to one, acting as a numéraire. In the short run, prices are assumed to be constant, while they are supposed to adjust in the medium or in the long run. For this reason, in what follows we set $\bar{P} = 1$ and we consider the real money demand function

$$M_t^D = d_1 Y_{t-1} + d_2 R_t, \tag{6}$$

which is determined, in accordance with the liquidity preference[1] theory (see e.g.

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[1] Keynes proposed the notion of liquidity preference (i.e. of demand of money, which is the most liquid asset) in [32], to provide explanation of interest rates, in terms of "a reward for parting with liquidity for a specified period". The two terms in the right-hand side of money demand (6) encompass the three motives identified by Keynes for determining the demand of money: the transactive and precautionary motives are represented by $d_1 Y_{t-1}$, while the speculative one by $d_2 R_t$, with the sign of the two coefficients d_1 and d_2 that accounts for the effects respectively played by income and interest rate.

[33] and [19]), by the national income and the interest rate R_t, where $d_1 > 0$ and $d_2 < 0$ accounts for the income and the interest rate effects on real money demand, respectively. As concerns the money supply, the total money quantity at time t is defined by the money creation policy of the monetary authority. For instance, one can suppose that the authority focuses on the money variation with the aim of moderating fluctuations in the economic activity with respect to the benchmark of full employment income Y^F. Accordingly, we shall assume the following target adjusting monetary policy

$$M_t^S = M_{t-1}^S + \mu(Y^F - (1-\theta)Y_{t-1} - \theta Y_{t-2}), \tag{7}$$

where $\mu > 0$ represents the reaction of the monetary policy with respect to deviation of the full employment output to the last observed income variation, while $\theta \in [0, 1)$ weights the past realization of national income. We stress that the supply rule in (7) is in line with similar rules that respond both to the existing money stock and the last levels of output (see e.g. [40]). In (7) the stock money variation depends on the difference between a target income level and a weighted average of the two last income levels. The monetary authority focuses on the money variation with the aim of moderating fluctuations in the economic activity with respect to the benchmark of a full employment income, by responding through some form of inertia to fluctuations in the target variables, so that the policy will continue for some time to depend upon past variable realizations.

When $\theta = 0$, only the last output realization is considered by the money supply rule in reacting to the deviations from the full employment income, while when $0 < \theta < 1/2$ the two most recent output observations are taken into account, giving more relevance to the closest one. For $\theta = 1/2$, both Y_t and Y_{t+1} are equally weighted while for $\theta > 1/2$ a greater importance is assigned to farther output realizations, accounting for a higher level of inertia or cautionary response[2]. Thus, according to (7), the money supply will move

[2]We would like to remark that the fiscal policy, and thus the changes in the government expenditures, affects income more rapidly than the monetary policy and it is reasonable to

cyclically or counter-cyclically in order to moderate adjustments in interest rates as output changes. In particular, when the economic activity is low (namely when $Y^F > (1-\theta)Y_{t-1} + \theta Y_{t-2}$), the aim of the monetary policy is to stimulate the economy through an increase of the money supply. This has the effect of reducing the interest rate, as also evident by making explicit R_t through (5), (6) and (7), which provides

$$R_t = \frac{1}{d_2}(M^S_{t-1} + \mu(Y^F - (1-\theta)Y_{t-1} - \theta Y_{t-2}) - d_1 Y_{t-1}). \tag{8}$$

As a consequence, reduced interest rates have a positive effect on investments and, therefore, on the level of the national income.

Plugging (8) in (4) we obtain the expression of investments

$$I_t = \bar{I} + \gamma_t a_2 \left(\frac{a_1 + a_2}{a_1 e^{-(Y_{t-1} - Y_{t-2})} + a_2} - 1 \right) + \frac{\varphi}{d_2}(M^S_{t-1} + \mu(Y^F - (1-\theta)Y_{t-1} - \theta Y_{t-2}) - d_1 Y_{t-1}),$$

which, inserted in (1) together with (2) and (3) and coupled with the monetary policy equation (7), leads to the model

$$\begin{cases} Y_t &= A + Y_{t-1}c - g(Y_{t-1} - Y^F) + a_2\gamma \left(\frac{a_1 + a_2}{a_2 + a_1 \mathrm{e}^{-(Y_{t-1} - Y_{t-2})}} - 1 \right) \\ & \quad + \frac{\varphi}{d_2}(M^S_{t-1} + \mu(Y^F - (1-\theta)Y_{t-1} - \theta Y_{t-2}) - d_1 Y_{t-1}) \\ M_t &= M^S_{t-1} + \mu(Y^F - (1-\theta)Y_{t-1} - \theta Y_{t-2}). \end{cases}$$

Then, by introducing $Z_t \equiv Y_{t-1}$ in view of the subsequent analysis, we can define the function $T = (T_1, T_2, T_3) : \mathbb{R}^3_+ \to \mathbb{R}^3$, $(Y_t, M_t, Z_t) \mapsto (T_1(Y_t, M_t, Z_t), T_2(Y_t, M_t, Z_t), T_3(Y_t, M_t, Z_t))$, which describes the functioning of the whole economy[3] as a the three-dimensional

assume only one lag in its response to the deviation of the full employment income from the last realization. On the contrary, as argued by Woodford in [35], it is generally optimal for the monetary policy to respond inertially to fluctuations in the target variables and/or their determinants, so that the policy will continue for some time to depend upon past variable realizations, even when these are irrelevant to the determination of the future values of the target variables. See also Gramlich ([36]) for a discussion of monetary versus fiscal policy and the role of time lags in shaping stabilization strategies.

[3]From now on, we drop the superscript S in variable M.

system

$$
T : \begin{cases} Y_t &= T_1(Y_{t-1}, M_{t-1}, Z_{t-1}) = A + Y_{t-1}c - g(Y_{t-1} - Y^F) + a_2\gamma \left(\frac{a_1+a_2}{a_2+a_1 e^{-(Y_{t-1}-Z_{t-1})}} - 1 \right) \\ & \quad + \frac{\varphi}{d_2}(M_{t-1} + \mu(Y^F - (1-\theta)Y_{t-1} - \theta Z_{t-1}) - d_1 Y_{t-1}), \\ M_t &= T_2(Y_{t-1}, M_{t-1}, Z_{t-1}) = M_{t-1} + \mu(Y^F - (1-\theta)Y_{t-1} - \theta Z_{t-1}), \\ Z_t &= T_3(Y_{t-1}, M_{t-1}, Z_{t-1}) = Y_{t-1}. \end{cases} \tag{9}
$$

We stress that economically meaningful values for variables (Y_t, M_t, Z_t) are those belonging to $[0, +\infty)^3$. However, not all the possible parameter configurations and initial conditions give rise to significant economic scenarios. In the following analysis we implicitly limit to the feasible configurations and initial data for which $(Y_t, M_t, Z_t) \in [0, +\infty)^3$, for any $t \geq 0$. Accordingly, the simulations reported in Section 4 are consistent with this restriction.

3. Analytical results on the existence of the steady state and local stability properties

In this section we shall investigate the existence of steady states for the system in (9) and we shall analyze how the relevant monetary and fiscal policy parameters affect stability. Firstly, we study the number and the analytical expression of possible steady states for the map in (9), ending up with the following:

Proposition 1. *The system in* (9) *has a unique steady state given by*

$$
(Y^*, M^*, Z^*) = \left(Y^F, \frac{d_2 \left(Y^F(1-c) - A + \frac{Y^F d_1 \varphi}{d_2} \right)}{\varphi}, Y^F \right)
$$

to which corresponds the interest rate

$$
R^* = \frac{Y^F(1-c) - A}{\varphi}.
$$

Moreover, the values of (Y^*, M^*, Z^*) *are positive provided that*

$$
\frac{A}{(1-c) + d_1\varphi/d_2} < Y^F < \frac{A}{(1-c)}.
$$

The previous Proposition states that the equilibrium level of the national income coincides with the targeted full employment level of output. We observe that the steady state values of the money quantity positively depends on the full employment output level Y^F in accordance to the money supply rule, and on the parameter d_1, signaling a positive relationship with the increase of the output level; moreover, it negatively depends on the parameter $|\varphi|$ since, when it grows, it makes the investments less attractive and, accordingly, the national income grows less with a consequent contraction in the money demand. On the other hand, M^* positively depends on $|d_2|$, since it reduces the speculative component of the money demand (the interest rate) and thus stimulates the economic activity through investments and national income, with a consequent high level of money in equilibrium. We also highlight that the steady state output $Y^* = Y^F$ lies between a value associated with the absence of money $A/((1-c)+d_1\varphi/d_2)$, corresponding to a situation in which no Keynesian functions of money would be considered, and the Samuelsonian steady state $A/(1-c)$, where no policy interventions are taken into account. Finally, we stress that in [27], where only the fiscal policy was considered, the national income steady state does not coincide with the targeted national income level. This means that the combined action of the two policies can be able to drive the economy toward the full employment national income. In what follows, the analysis as well as the numerical simulations will be performed assuming parameters' settings that fulfill the positivity condition reported in Proposition 1.

We recall that the existence of a unique steady state has been widely observed even when nonlinear elements have been introduced to produce more realistic dynamics for the investments. In particular, if we compare our model those considering the role of money within a multiplier-accelerator framework, they exhibit a unique equilibrium (see e.g. [22]). Conversely, the emergence of multiple equilibria has been observed when the role of individual expectations has been considered within the multiplier-accelerator framework, see e.g. [37]-[38], or when the role of the debt cycle is coupled with the multiplier-accelerator setup, see [39].

In the next propositions we study the stability of the equilibrium with respect to the reactivity of the monetary policy μ. To this end, we say that we are in an unconditionally unstable, mixed or destabilizing scenario if (Y^*, M^*, Z^*) is respectively unstable for any $\mu > 0$, locally asymptotically stable for $\mu \in (\bar{\mu}_1, \bar{\mu}_2)$ or locally asymptotically stable for $\mu \in (0, \bar{\mu}_2)$, for some $0 < \bar{\mu}_1 < \bar{\mu}_2$. Finally, we do not discuss cases in which stability occurs for single values of the parameters. Before studying the model in (9), it is worth recalling the stability condition of the steady state for the model studied in [27, Prop 3.], in which only the fiscal policy is considered and money is not present[4] In this regard, henceforth, we shall make use of substitution $\tilde{\gamma} \equiv \gamma(a_1 + a_2)/(a_1 a_2)$.

Proposition 2. *When no monetary policy is considered, the steady state* (Y^*, Z^*) *is locally asymptotically stable provided that* $\tilde{\gamma} < 1$ *and*

$$g < c + 2\tilde{\gamma} + 1. \tag{10}$$

When the money market is not considered, the fiscal policy should be set in a way such that it does not react too aggressively[5] to deviations of output realizations with respect to the full employment income target, otherwise periodic fluctuations arise in the business cycle.

Now we move the analysis to the study of the monetary policy parameters on the steady state stability. The following Proposition, in which we make use of

[4]The corresponding system is actually a two-dimensional reduction of system (9), when the second equation is neglected as well as any influence of the monetary sector on the investments ($\varphi = 0$.) The resulting stability conditions can be studied in terms of the trace and determinant of the two-dimensional map.

[5]Through the fiscal policy in (3) the government acts to compensate the difference between the effective demand and some desired level (i.e. the full employment). In this respect, it turns out to be relevant the magnitude through which such a policy is designed, since it can both mitigate and exacerbate economic fluctuations. The fiscal policy is addressed as "aggressive" when the government firmly and strongly react to these discrepancies in order, for instance, to fight a recession.

$\tilde{\varphi} \equiv \varphi/d_2$ to collect the joint effect of the interest rate on investments and money demand, reports the stability conditions for the steady state (Y^*, M^*, Z^*).

Proposition 3. *The local stability conditions for* (Y^*, M^*, Z^*) *are given by:*

$$\begin{cases} -\tilde{\varphi}(1-2\theta)\mu + 4\tilde{\gamma} - 2(g + d_1\tilde{\varphi} - c) + 2 > 0, \\ -\tilde{\varphi}(\tilde{\gamma}(1-\theta)+\theta)\mu + (g + d_1\tilde{\varphi} - c) - \tilde{\gamma} - (g + d_1\tilde{\varphi} - c)\tilde{\gamma} + 1 > 0, \\ -\tilde{\varphi}\theta\mu + g + d_1\tilde{\varphi} - c - 2\tilde{\gamma} + 3 > 0. \end{cases} \tag{11}$$

Then, since we are interested in the role that the monetary policy parameter μ plays on the steady state stability, in the next Proposition we make explicit the role of such a parameter. Thus the following result holds:

Proposition 4. *The steady state* (Y^*, M^*, Z^*) *is locally asymptotically stable provided that* $\tilde{\gamma} < 1$ *and*

- *when* $0 \leq \theta < \dfrac{1}{2}$ *if*

$$0 \leq \mu < s_2 \quad \textit{provided that} \quad g < c + 4\tilde{\gamma} - 1 + 4\theta(1-\tilde{\gamma}) - d_1\tilde{\varphi} \tag{12a}$$

$$0 \leq \mu < s_1 \quad \textit{provided that} \quad c + 4\tilde{\gamma} - 1 + 4\theta(1-\tilde{\gamma}) - d_1\tilde{\varphi} \leq g < 2\tilde{\gamma} + 1 + c - d_1\tilde{\varphi} \tag{12b}$$

- *when* $\dfrac{1}{2} \leq \theta \leq 1$ *if*

$$0 \leq \mu < s_2 \quad \textit{provided that} \quad g < c + 2\tilde{\gamma} + 1 - d_1\tilde{\varphi} \tag{13a}$$

$$s_1 < \mu < s_2 \quad \textit{provided that} \quad 2\tilde{\gamma} + 1 + c - d_1\tilde{\varphi} < g < c + 4\tilde{\gamma} - 1 + 4\theta(1-\tilde{\gamma}) - d_1\tilde{\varphi} \tag{13b}$$

where

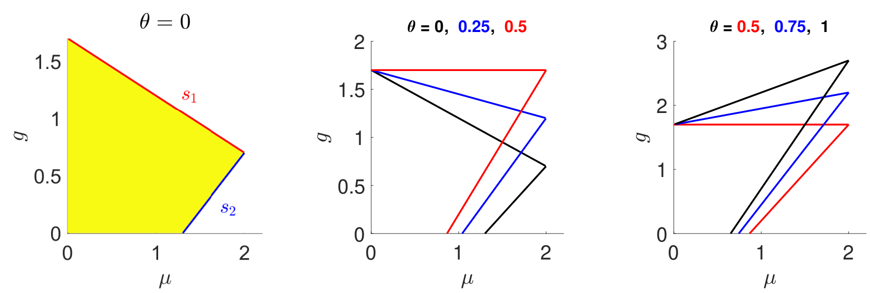
$$s_1 = \frac{2\tilde{\varphi}(2\tilde{\gamma} - (g - c + d_1\tilde{\varphi}) + 1)}{1-2\theta} \;\; \textit{and} \;\; s_2 = \frac{\tilde{\varphi}(g - c + d_1\tilde{\varphi} + 1)(1-\tilde{\gamma})}{(\tilde{\gamma}(1-\theta)+\theta)}. \tag{14}$$

Conditions (12) *and* (13a) *give rise to destabilizing scenarios with respect to* μ*, while condition* (13b) *gives rise to a mixed scenario. For all the remaining parameters' configurations we have unconditionally unstable scenarios.*

Proposition 4 allows us to understand the effect of the accelerator mechanism and of the policies on the stability of (Y^*, M^*, Z^*). The first comment is about the role of the accelerator parameter. If the reactivity of investments to income variations is too strong ($\tilde{\gamma} > 1$), neither fiscal nor monetary policy intervention is able to stabilize the system, and the source of endogenous fluctuations is ascribed to the accelerator mechanism, in agreement with the setting studied in Proposition 2. Conversely, when $\tilde{\gamma} < 1$ and the money supply is adjusted giving more relevance to the most recent real economy signals, the money market can be the source of instability. This can be understood by comparing the stability condition (10) with those in (12). If (10) holds true, conditions in (12) can be violated either due to an overreaction of the monetary policy (when $\mu > s_i$) or when the parameters characterizing the money market (d_1 and d_2) and its effects on investments (ϕ) are too large, so that $g < 2\tilde{\gamma} + 1 + c - d_1\tilde{\varphi}$. This may happen also when more relevance is given to the least recent real economy signal, but in this case there is some space for a stabilizing role of the monetary policy, the steady state can be stable even if (10) is violated (as evident from (13b)) and hence the fiscal policy is the source of endogenous fluctuations. This is a scenario in which the monetary authority, by sufficiently adjusting the level of money supply, is able to stabilize an otherwise unstable scenario up to a certain level, beyond which the steady state becomes unstable again. To summarize, the reaction of the monetary authority in changing the money supply is not always stabilizing since when such a reactivity is sufficiently large, the steady state may turn unstable. Thus, from the stability analysis it is clear that if the monetary policy is not considered, the stability condition of the equilibrium reduces to the one in [27]. Additionally, comparing the previous results with the paper by Lovell and Prescott ([19]), they provided a double stability threshold for the monetary policy parameter in order to get stability of the unique equilibrium, but their model does not consider the presence of a fiscal policy that acts at the same time. For this reason, our stability conditions can be seen an improvement of the latter, as we jointly consider the reactivity of the two policies together with the accelerator parameter.

We now comment more in detail the joint role of μ, g and θ on the stability of (Y^*, M^*, Z^*) with the help of Figures 1 and 2. In Figure 1 we report the stability regions of the steady state in the (μ, g) plane, obtained by setting $c = 0.7$, $\tilde{\gamma} = 0.5$, $d_1 = 1$ and $\tilde{\varphi} = 1$, for different values of θ. In Figure 1 (a) the stability region is represented using yellow color and shows, in accordance with the result of Proposition 4, how the steady state become unstable as μ sufficiently increases. Moreover, for small values of μ, increasing g has a destabilizing effect, as in [27] while if μ increases, we find that introducing a fiscal policy has an initial stabilizing effect, which is thwarted by increasing the degree of its reactivity g. In other words, there exists a double stability threshold, which arises due to the intervention of the monetary policy and makes the fiscal policy able to stabilize the economy if its reactivity is not too strong. Figures 1 (b-c) highlight the evolution of the stability region as long as an additional weight is assigned to the past level of national income on determining the money supply. If we compare the stability regions when $0 \leq \theta \leq 1/2$ and when $1/2 \leq \theta \leq 1$, we note that the two stability thresholds, s_1 (lower lines) and s_2 (upper lines), move upward, signaling that, in order to get stability when a higher weight is assigned to the past levels of national income, a stronger reaction of the fiscal policy is required. However, when $0 \leq \theta \leq 1/2$ the unique possible scenarios on increasing μ are those destabilizing (for small enough reactivity of the fiscal policy) and unconditionally unstable, while when $1/2 < \theta \leq 1$ a sufficiently reactive monetary policy can counterbalance the destabilizing effects of the fiscal policy, provided that it is not too overreactive. In this case we have a mixed scenario for μ.

In Figure 2 we report the stability region of the steady state in the (θ, μ) parameter plane for different values of g, where the red and the blue curves are related to the first two stability conditions of Proposition 3. The parameters are the same used for Figure 1, with the exception of $d_1 = 0.1$. These panels allow us to appraise the role that the two parameters of the monetary policy exert on the stability of (Y^*, M^*, Z^*). In the left panel the case of a weakly reactive fiscal policy is depicted and the region of parameters that guarantees



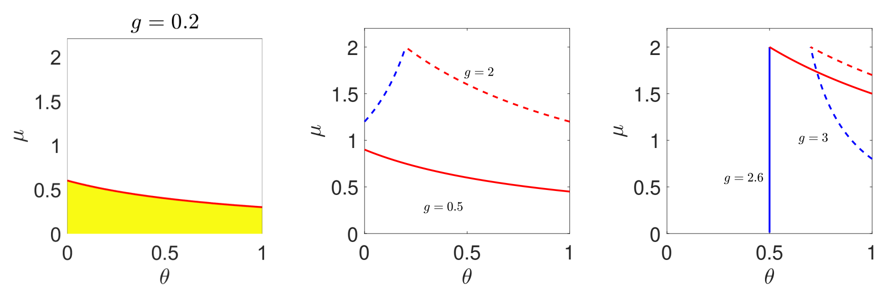
(a) (b) (c)

Figure 1: (a) Stability region (yellow color) of (Y^*, M^*, Z^*) for $\theta = 0$, on varying μ and g. The stability thresholds s_1 and s_2, defined in (14), are represented by a red and blue line, respectively. (b-c) Evolution of the stability region, bounded by the vertical axis and by the lines with the same color, for increasing values of θ.

the stability of (Y^*, M^*, Z^*) is represented by yellow color. In this case, when μ is sufficiently small, the steady state is always stable while it can be destabilized for increasing values of θ as long as the monetary authority reacts slightly more to the deviations of the income realizations to its full employment level. On the other hand, if the reaction parameter μ grows more, there are no means of stabilization even if a growing weight is assigned to the past income levels. Such behavior occurs for all values of g that preserve the stability of the economy when no monetary policy is used (i.e. when $\mu = 0$), which, in the present setting, corresponds[6] to $g < 2.6$, as noticeable also from Figure 2 (b), where, for $g = 0.5$ and $g = 2$, stability is achieved in the regions below the solid and the dashed lines, respectively. When the degree of the fiscal policy reactivity increases, if μ is not too large, the steady state is stable for any value of θ, while there exists a double stability threshold on increasing the reactivity of the monetary policy μ. Moreover, also recalling Figure 2 (a), we can note that the stability region increases in size due to the impact played by the fiscal policy.

Conversely, if the reactivity of the fiscal policy is too large, (i.e. $g \geq 2.6$),

[6]The threshold is obtained from condition (12a) when $\theta = 0$.



(a) (b) (c)

Figure 2: (a) Stability region (yellow color) of (Y^*, M^*, Z^*) for $g = 0.2$, on varying θ and μ. Middle and right panels show the evolution of stability regions for couples of increasing values of g. The red and blue colors are related to the thresholds defined by the first and the second condition in Proposition 3, respectively. Stability is guaranteed for θ and μ belonging to the regions lying below (b) or between (c) stability thresholds.

as in Figure 2 (c), it turns out to be useful to introduce a form of inertia in the response of the monetary policy ($\theta > 1/2$) in order to gain stability. In this case, the size of the stability region diminishes as g increases and a certain degree of reactivity is needed to preserve stability, even if we note that increasing values of μ are not always beneficial since the stability region may shrink as long as the reactivity of the monetary policy gets more aggressive.

Finally, before investigating the behavior of model (9) through numerical simulations, we analytically study the kind of local bifurcations occurring when the steady state loses its stability on increasing μ. To do this, by standard homeomorphic change of variable $f(Y, M, Z) = (Y - Y^*, M - M^*, Z - Z^*)$, we rewrite the original model (9) into the topologically conjugate System T_0 : $(Y_{t-1}, M_{t-1}, Z_{t-1}) \mapsto (Y_t, M_t, Z_t)$ defined by

$$\begin{cases} Y_t &= Y_{t-1}c - Y_{t-1}g - a_2\gamma + \dfrac{a_2^2\gamma + a_1 a_2 \gamma}{a_2 + a_1 e^{Z_{t-1} - Y_{t-1}}} + \varphi \dfrac{M_{t-1} - Y_{t-1}(d_1 + \mu - \mu\theta) - Z_{t-1}\mu\theta}{d_2}, \\ M_t &= M_t - \mu(Y_t(1-\theta) + Z_t\theta), \\ Z_t &= Y_{t-1}, \end{cases} \tag{15}$$

for which the origin is the unique steady state. The next two propositions allow connecting critical values s_1 and s_2 provided by Proposition 4 to the corresponding kind of bifurcation. We start studying the emergence of a flip bifurcation.

Proposition 5. *Let*

$$\frac{g - c + d_1\tilde{\varphi} + 1 - 4\tilde{\gamma} - 4\theta + 4\tilde{\gamma}\theta}{1 - 2\theta} > 0. \tag{16}$$

For values of the parameter μ belonging to a suitable neighborhood of s_1 (defined in (14)*), there exists a one-dimensional invariant manifold W_μ such that W_{s_1} is the center manifold at the bifurcation value. The restriction of System* (15) *to W_{s_1} around the origin $(0,0,0)$ is locally topologically conjugate (in suitable coordinates) to the map*

$$\xi \mapsto -\xi + \chi\xi^3 + O(\xi^4), \tag{17}$$

where

$$\chi = \frac{8\tilde{\gamma}(1-2\theta)(a_1^2 - 4a_1a_2 + a_2^2)}{3(a_1+a_2)^2(g - c + d_1\tilde{\varphi} + 1 - 4\tilde{\gamma} - 4\theta + 4\tilde{\gamma}\theta)}. \tag{18}$$

Proposition 5 shows that at $\mu = s_1$ a flip bifurcation occurs. Since condition (16) guarantees that the sign of (18) is uniquely determined by the factor $a_1^2 - 4a_1a_2 + a_2^2$, we can conclude that the flip bifurcation is supercritical when $0 < a_1/a_2 < 2 - \sqrt{3} \vee a_1/a_2 > 2 + \sqrt{3}$ and a subcritical when $2 - \sqrt{3} < a_1/a_2 < 2 + \sqrt{3}$. Moreover, recalling Proposition 4, we have that for $\theta < 1/2$ we have a period-doubling bifurcation, while for $\theta > 1/2$ a period-halving bifurcation occurs.

We stress that the subcritical (respectively, supercritical) case occurs when the bounds of the investment variation range are suitably close (respectively, far) to a symmetric configuration with $a_1 = a_2$. To understand the economic rationale underlying these behaviors we assume that we have $\theta < 1/2$, so that a flip bifurcation results in period-doubling one and we set $\mu \gtrapprox s_1$. In this case, instabilities arising due to the reactive monetary policy lead to oscillating dynamics around M^*, which, in turn, affect the trajectories of Y. When responses of investments to positive and negative variations of the national income are not too different (i.e. when $a_1/a_2 \in (2-\sqrt{3}, 2+\sqrt{3})$), both upward and downward oscillations around Y^F induced by instabilities in the money market are reinforced to a similar extent by the investment function, leading to divergence. We extensively checked through simulations that when $a_1/a_2 \in (2-\sqrt{3}, 2+\sqrt{3})$ and a subcritical flip bifurcation occurs, this leads to diverging trajectories (and no coexisting attractor has been identified). Since this is indeed not interesting from an economical viewpoint, in the next Section we will only focus on the supercritical case, in which, conversely, strongly asymmetric responses in investments partially counterbalance monetary oscillations, giving rise to a stable period-2 cycle.

In the next proposition we study what happens when $\mu = s_2$.

Proposition 6. *Let*

$$g - c + d_1\tilde{\varphi} + 1 - 4\tilde{\gamma} - 4\theta + 4\tilde{\gamma}\theta < 0 \tag{19}$$

and that nonresonance conditions

$$g - c - d_1\tilde{\varphi} + 1 \neq k(\tilde{\gamma} + \theta - \tilde{\gamma}\theta), \quad k = 2, 3, \tag{20}$$

hold true. For values of the parameter μ belonging to a suitable neighborhood of s_2 (defined in (14)*) there exists a two-dimensional invariant manifold W_μ such that W_{s_2} is the center manifold at the bifurcation value. The restriction of System* (15) *to W_{s_2} around the origin* $(0, 0, 0)$ *is locally topologically conjugate (in suitable complex coordinates) to the map*

$$z \mapsto e^{i\omega_0} z(1+\delta)|z|^2 + O(|z|^4), \tag{21}$$

in which $e^{i\omega_0}$ is the complex eigenvalue of J^* with $0 < \omega_0 < \pi$ and

$$\mathrm{Re}(\delta) = b_2(g - c - d_1\tilde{\varphi} + 1)^2 + b_1(g - c - d_1\tilde{\varphi} + 1) + b_0, \tag{22}$$

where

$$b_2 = \frac{2a_1^2 a_2^2 \gamma(\theta - 1)^2}{a_1 + a_2},$$

$$b_1 = -\frac{a_1 a_2 \gamma (1-\theta)^2 (a_1\theta + a_2\theta + a_1 a_2 \gamma - a_1 a_2 \gamma \theta)}{(a_1+a_2)^3} \cdot (-\gamma a_1^3 a_2 + a_1^3 + 2\gamma a_1^2 a_2^2 + 7a_1^2 a_2 - \gamma a_1 a_2^3 + 7a_1 a_2^2 + a_2^3),$$

and

$$b_0 = \frac{(1-\theta)^2 (a_1\theta + a_2\theta + a_1 a_2\gamma - a_1 a_2 \gamma\theta)^2}{(a_1+a_2)^4} \cdot [-3a_1^4 a_2^2 \gamma^2 + 2a_1^4 a_2 \gamma + a_1^4 + 4a_1^3 a_2^3 \gamma^2 + 2a_1^3 a_2^2 \gamma - 2a_1^3 a_2 - 3a_1^2 a_2^4 \gamma^2 + 2a_1^2 a_2^3 \gamma - 6a_1^2 a_2^2 + 2a_1 a_2^4 \gamma - 2a_1 a_2^3 + a_2^4].$$

Proposition 6 shows that at $\mu = s_2$ a Neimark-Sacker bifurcation can take place, which is supercritical (resp. subcritical) if the sign of (22) is negative (resp. positive). We shall deepen the investigation of the possible unstable dynamics through numerical simulations, focusing on their economic relevance. The main focus is to catch the economic insight about the interplay between policies. We remark that due to the peculiarity of the parameter configurations for which flip and Neimark-Sacker bifurcations degenerate, we avoid deepening the investigation of such scenarios, as this would not add significant additional information.

4. Numerical simulations

In this section we present some numerical simulations in order to complement the previous analysis and check whether complex dynamics may arise when the steady state turns unstable, as a consequence of the joint actions of the two policies. Unless differently stated, we shall make use of these parameter values: $\gamma = 1.11, A = 100, Y^F = 205, \varphi = -20, d_1 = 1, d_2 = -20,\ a_1 = 4.5, a_2 = 0.5$ and $c = 0.7$. The parameters are set according to what done in [27], in order to compare results, and to get reasonable and economically meaningful values for the interest rate in percentage, which does not exceed 5%. We point out that in the two dimensional bifurcation diagrams reported in Figures 3-7 the white color refers to convergence toward the steady state, while other colors are used to represent attractors consisting of more than a single point. Moreover, hatched regions correspond to parameter configurations characterized by divergence or unfeasibility. Finally, the initial datum is chosen in a suitably small neighborhood of the steady state.

Figures 3 (a) -5 (a) report the two-dimensional bifurcation diagrams in the (μ, g) parameter plane for different values of θ.

The first consideration that can be inferred from such diagrams is the different kind of dynamics arising when stability is lost, in accordance with Propositions 5 and 6. In fact, when the white region is crossed through its upper border (corresponding to the line defined by $\mu = s_1$ in Proposition 4, see also Figure 1) convergence to the steady state is replaced by convergence to a period two cycle, as we enter the red region. Conversely, for the parameter configuration used for each simulation in Figures 3-7 (a), when $\mu = s_2$ and condition (19) holds true, the left hand side of (22) is always positive, so a subcritical Neimark-Sacker occurs. However, in such cases, for values of μ suitably close to s_1, trajectories converge toward coexisting stable attractors (see e.g. the leftmost parts of one-dimensional bifurcation diagrams reported in Figures 3-7 (b)), giving rise to economically significant dynamics.

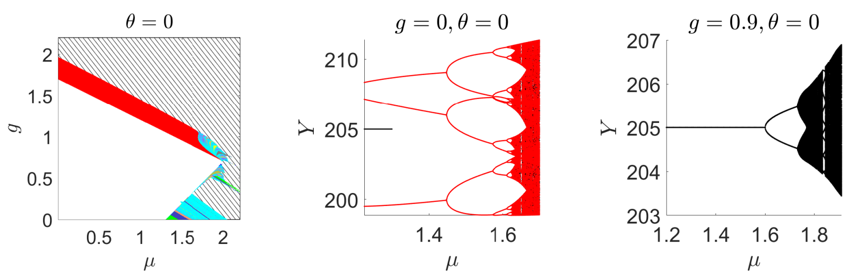
To enter more into details about the role of the reactivity of the monetary policy on overall dynamics, we can observe the two examples of one-dimensional bifurcation diagrams with respect to μ, reported in Figure 3 (b-c), obtained for two different values of g. In Figure 3 (b), there is no intervention of the fiscal policy nor degree of inertia in the monetary policy. Observing the black bifurcation diagram, we can see that when μ increases, the steady state loses stability through a subcritical Neimark-Sacker bifurcation when $\mu = s_2$. However, when the reactivity μ of the monetary policy is not too strong, the steady state also coexists with a cycle of period 3 (red bifurcation diagram[7].). In this case, the actual final outcome of the economy depends not only on its parameters but also

[7] Throughout this section, red bifurcation diagrams are computed following the attractor along a sequence $\{x_i\}$ of parameter values. This means that the initial datum for the simulation related to parameter value x_{i+1} is chosen suitably close to the attractor toward which convergence occurred for parameter value x_i

on the initial level of national income, whose evolution is then significantly path dependent. When the steady state becomes unstable, trajectories converge to the periodic attractor, which then evolves through a cascade of period doubling bifurcations leading to complex dynamics.

The previous phenomena also emerge with the introduction of a small degree of inertia ($\theta \leq 0.5$) in the money supply rule, as exemplified by the two-dimensional bifurcation diagram obtained for $\theta = 0.25$ and reported in Figure 4 (a). It is worth to remark that, when a certain degree of intervention of the government is encompassed, the intervention of the public authority in reaction to the deviation of the income from the full employment level is able to drive the economy toward the objective of the policy, even for large values of the monetary policy reactivity μ. However, we note that the parameter g has an ambiguous effect on the stability of the steady state, for a given value of μ. In fact, an increase of g has the effect of reducing the complexity of the orbits, and lets the steady state gain stability; however, as g keeps increasing, we cross the upper bound of the stability region, associated with the white color, and the steady state undergoes a period doubling bifurcation. With this respect we consider Figure 4 (b), in which the one dimensional bifurcation diagram is computed with respect to g. When the fiscal policy is not sufficiently strong, the dynamics of the output are not convergent toward the steady state, being the latter associated with the occurrence of a Neimark-Sacker bifurcation. On the other hand, when the policy acts sufficiently strong, it is able to stabilize the dynamics, leading the economy to the desired output level or, at least, to a reduction of the qualitative complexity of the trajectories, with the occurrence of a cycle of period two. We stress that when the money market is considered, the interplay of both monetary and fiscal policies gives rise to possibly high levels of complexity, differently from [27], in which it is shown that on changing the reactivity of a level targeting fiscal policy the steady state could lose stability just giving rise to a period 2 cycle.

The same qualitative behaviors are observed also when the parameter θ is further increased (Figure 5 (a)). In particular, comparing the two dimensional

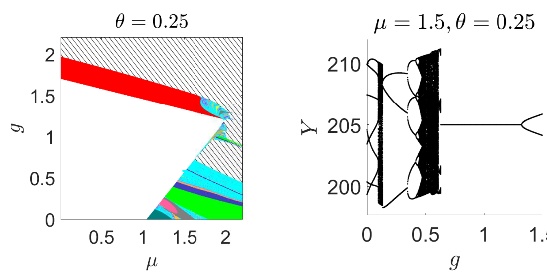


(a) (b) (c)

Figure 3: (a) Two-dimensional bifurcation diagram in the (μ, g) plane for $\theta = 0$. White color refers to convergence toward the steady state, while other colors are used to represent attractors consisting of more than a single point. The hatched region stands for divergence or unfeasible economic values of the variables. (b-c) Bifurcation diagrams with respect to μ for different values of g, showing the possibility of attractors coexistence, even when the steady state is locally stable (b). For the red bifurcation diagram we modified the initial value of variables Y and Z, setting $Y_0 = Z_0 = 210$.

bifurcation diagrams in Figures 3- 5, we can observe that when the two bifurcation curves move upward, the size of the region associated with complex dynamics increases. It is worth to note that, when the inertia in the money supply grows, a sufficiently large degree of the fiscal policy reaction is necessary to get the stability of the steady state if the monetary authority overreacts to the deviations of the full employment income to its recent realizations. Moreover, in the case reported in Figure 5, as predicted by Proposition 4, if we consider a high level of reactivity in the fiscal policy, we have the appearance of a double stability threshold, as already commented before, and the steady state is locally asymptotically stable only for intermediate values of the policy parameter μ. This is also evident from the one dimensional bifurcation diagram reported in Figure 5 (b), from which we can additionally observe a situation of multistability for increasing values of μ, where the steady state coexists with cycles, making again crucial the choice of the national income to consider when setting the policy intervention.

Figures 6 and 7 represent the two-dimensional bifurcation diagram in the

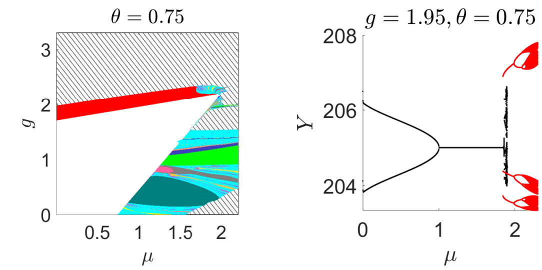


(a) (b)

Figure 4: (a) Two-dimensional bifurcation diagrams in the (μ, g) plane for $\theta = 0.25$. White color refers to convergence toward the steady state, while other colors are used to represent attractors consisting of more than a single point. The hatched region stands for divergence or unfeasible economic values of the variables.(b) Bifurcation diagram on varying g.

(θ, μ) parameter plane for different values of the fiscal policy parameter g. In Figure 6 we consider the situation in which the fiscal policy does not intervene. We observe that the steady state is locally stable if the degree of the monetary policy reaction is not too large. In fact, for increasing values of μ, and for any θ, the steady state turns unstable and complex dynamics arise with consequent endogenous fluctuations that characterize the course of the business cycle. In Figure 6 (b) we report a bifurcation diagram on increasing parameter θ in order to show how the degree of inertia in the money supply rule may generate different dynamic scenarios. In this case, when there is no space for the fiscal policy and the reaction of the monetary authority is set at an intermediate level, the steady state is locally asymptotically stable when more weight is assigned to the most recent output observations while the steady state loses stability via a Neimark-Sacker bifurcation when θ increases, with the consequent emergence of complex dynamics. Hence, in this case, an increase in the degree of inertia in the money supply rule is not necessarily an advantage in terms of reaching the desired output level.

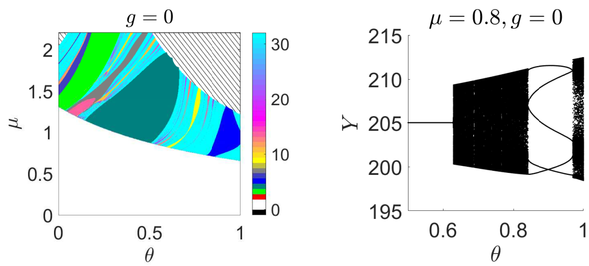
When the economy accounts for a certain level of fiscal policy intervention



(a) (b)

Figure 5: (a) Two-dimensional bifurcation diagram in the (μ, g) plane for $\theta = 0.75$. White color refers to convergence toward the steady state, while other colors are used to represent attractors consisting of more than a single point. The hatched region stands for divergence or unfeasible economic values of the variables. (b) Bifurcation diagram with respect to μ and reveals the possibility of different coexisting business cycles for a sufficiently high level of the reactivity parameter μ. The red bifurcation diagram is computed following the attractor along a decreasing sequence of parameter values ranging from $\mu = 2.3$ to $\mu = 1.84$.

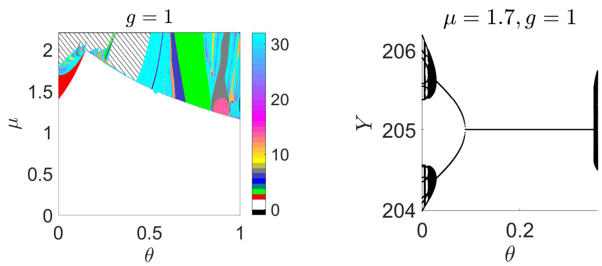
(Figure 7 (a)), we note a general increase in the white region, which is associated with the stability of the steady state. Moreover, there exists a double stability threshold when the monetary policy aggressively reacts. In fact, on one hand, when the degree of inertia θ is very low, the dynamics are periodic while a sufficient degree of inertia in the response of the monetary authority is able to stabilize the economy; on the other hand, when the policy assigns more weights to the past output realization, the coupling with the reaction to the deviation of the output from its full employment level renders the steady state unstable and complex dynamics arise again. Looking at the bifurcation diagram in Figure 7 (b), we can see that θ is able to stabilize the dynamics by reducing the complexity of the orbits through a reverted Neimark-Sacker bifurcation followed by a period halving bifurcation. Nonetheless, the range of parameters θ for which the steady state remains locally stable is quite narrow and a Neimark-Sacker bifurcation of the steady state occurs, with dynamics that start oscillating in a complex and intricate manner.



(a) (b)

Figure 6: (a) Two-dimensional bifurcation diagram in the (θ, μ) plane for $g = 0$. White color refers to convergence toward the steady state, while other colors are used to represent attractors consisting of more than a single point. The hatched region stands for divergence or unfeasible economic values of variables. (b) Bifurcation diagram with respect to θ in the absence of fiscal policy.

Finally, in Figure 8 (a) we show a bifurcation diagram with respect to the parameter μ when the fiscal policy is not considered ($g = 0$) and there is no inertia in the money supply rule ($\theta = 0$). As it is clearly visible from the picture, when the reactivity of the monetary policy increases, the steady state loses stability confirming all the previous analytical results; moreover, when it is still locally stable, it may coexist with a closed invariant curve associated with orbits that largely oscillates above and below the steady state output level. It is worth stressing the evidence that the introduction of the monetary policy in this simple multiplier-accelerator setting is the responsible for generating the business cycle with fluctuations in the national income dynamics, in a context in which the same qualitative dynamics would not occur if only the fiscal policy were considered (see [27]). This may be due to the fact that an excessive tightening of monetary policy may lead to instability in other sectors of the economy, with a negative effect on economic actors' behavior by weakening their assessment of the future state of the economy. Moreover, as evident from the bifurcation diagram and the times series reported in Figure 8 (b), there can be the possibility of coexisting business cycles in which, being the national income different

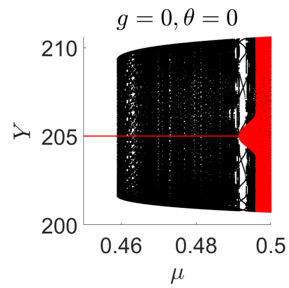


(a) (b)

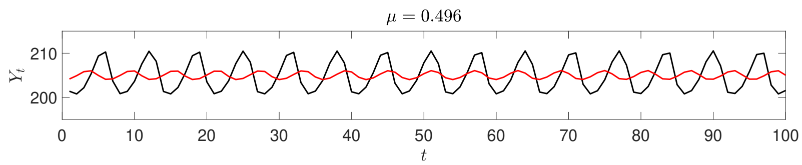
Figure 7: (a) Two-dimensional bifurcation diagram in the (θ, μ) plane for $g = 1$. White color refers to convergence toward the steady state, while other colors are used to represent attractors consisting of more than a single point. The hatched region stands for divergence or unfeasible economic values of variables. (b) Bifurcation diagram with respect to θ when also the fiscal policy is present, and shows the double stability threshold for the steady state.

from its desired level, the final state of the economy can be characterized by persistent higher or lower level of output.

The corresponding basins of attraction of this multistability situation is depicted in Figure 9 where we highlight the evolution of the two coexisting attractors as long as μ increases. In particular, the basin of the steady state is colored in blue while the basin of the closed curve is represented in yellow. In the first row, moving from left to right, which corresponds to an increase of the value of the reactivity parameter μ, we observe a quite sharp shrink in the basin of the steady state. Such a shrinking is even more evident in the bottom-left panel where a higher value of μ is considered. For this value, the steady state is unstable and another closed curve coexists with the previously existing one. The enlargement of Figure 8 (d) allows us to appraise the smallest of the two coexisting curves and the corresponding basins. This situation, in which one basin is extremely small, is associated with the consequent unpredictability of the asymptotic state of the economy in a wide region of the state variables.



(a)



(b)

Figure 8: (a) Bifurcation diagram on varying μ showing the coexistence of different attractors. (b) Times series of Y_t showing coexisting business cycles. Parameters are set as declared at the beginning of the section except for $\gamma = 1.7$ and $d_2 = -200.$. The red diagram is obtained with initial datum suitably close to the steady state while the black diagram is obtained with $Y_0 = Z_0 = 220, M_0 = 88$.

5. Concluding remarks

In this paper we have shown how a rich variety of dynamical outcomes may arise in a real economy described by a nonlinear multiplier-accelerator model when the public authority influences the dynamics of the national income either via fiscal or monetary policy. The consideration of the two policy instruments fits into the debate on which of the two instruments is better able to purse the stabilization objective. Most of the times the issue is studied having in mind the problem of controlling inflation and regulatory programs (see [26]-[40]). The investigation of the effects of these policies is pertinent not only for deepening the discussion on the potential implications of the different fiscal and monetary programs, but also with respect to the dynamical outcomes that they can give

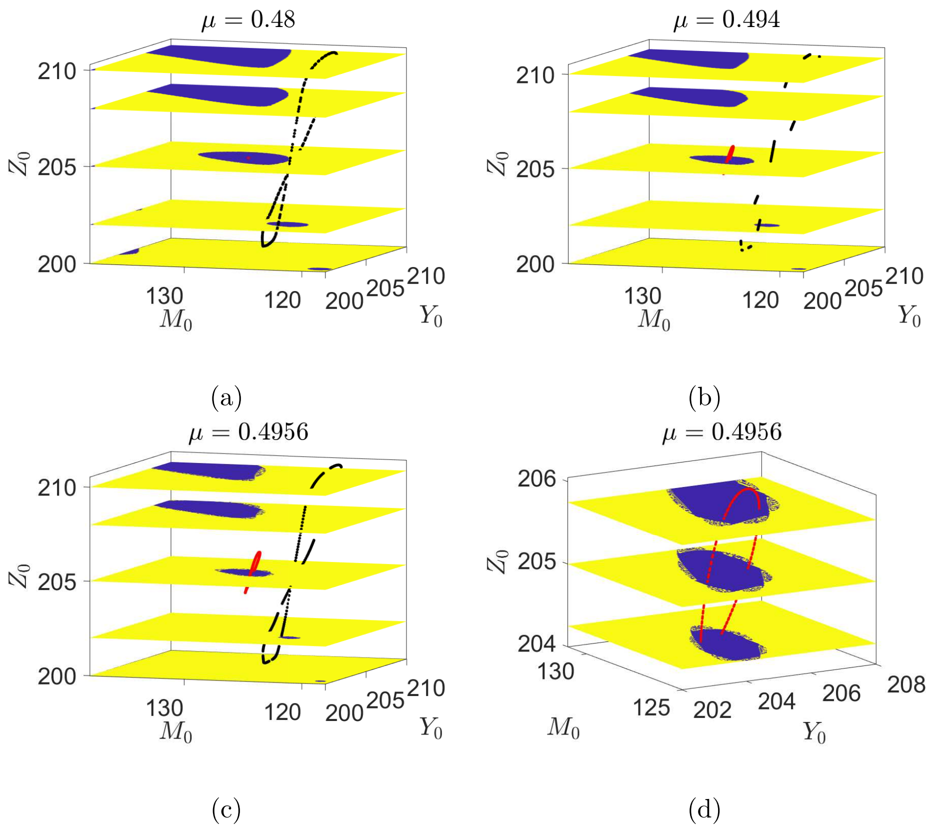


Figure 9: Basins of attraction for different values of the reactivity parameter μ. The yellow color refers to the basin of the closed invariant curve surrounding the steady state (a-b), whose basin is depicted in blue, and the other closed curve (c-d) arising when the steady state loses stability, whose basin is still represented in blue.

rise. In this respect, one of the papers that bear resemblance with the present one is the work in [22]. Both setups are grounded on a multiplier-accelerator framework, but the couples of policy rules are different. In particular, in the present work we study the capability of policies to drive the economy toward the full employment level. Moreover, since the model considered in [22] is essentially linear, the possible endogenous dynamics are just convergent or divergent, and thus stable cyclical fluctuations around the steady state (business cycle) are not possible. Conversely, the present model is nonlinear and hence complex dynamics can arise. In particular, the dynamics basically depend on two parameters, one related to the reactivity of the fiscal policy in adjusting the deviation of the output realizations from its full employment level, and another linked to the responsiveness of the money supply with respect to the variation between the full employment income and the two last output realizations.

It is firstly shown that, from a static viewpoint, the introduction of the monetary policy is able to lead the economy to the desired output equilibrium level. Secondly, from a dynamic point of view, the interaction of the two policy instruments causes a variety of local bifurcation scenarios (Neimark-Sacker and period doubling), multistability, as well as complex dynamics that are not possible for the model considered in [22]. In particular, the introduction of the monetary policy can have beneficial effects in reducing the complexity of the orbits and in leading the economy toward the full employment income, if it is not too aggressive. Otherwise endogenous oscillations in the national income dynamics can take place along an attractive closed invariant curve, which is interpreted as a business cycle in economics. Both the government and the monetary authority are able to influence the size and the persistence of the oscillations by properly tuning their policy instruments. Finally, from a global analysis perspective, we have shown the coexistence of different attractors, occurring even when the national income steady state is locally stable, and thus making the choices of policy makers crucial to shift the output in the desired direction. Within the present framework, the introduction of the monetary policy allowed us to thoroughly investigate the dynamics of the economic activity, which can exhibit interesting dynamic features that would not be present in a context where only the fiscal policy is present. In particular, the role of the monetary policy has to be read in terms of the possibility of stabilizing the national income dynamics but, at the same time, it can also be responsible for the generation of the business cycle. This confirms the importance of the role played by the monetary sector and the relevance of studying how manipulating and targeting the money supply (and, ultimately, the interest rate) influences the real economic variables. The present setting can be extended in several directions that take into account the role of the monetary policy: firstly, the introduction of the financial sector would add realism in order to contribute to the debate on whether the monetary authority should respond to financial factors, such as asset prices, in monetary policy rules; secondly, the introduction of an asset market would allow to account for different agents' expectations, whose interaction would affect the overall economic stability; thirdly, the role of expectation can also be considered on the real side, as the consumption choices can also be affected by individual perceptions on the output level which, in turn, may contribute to the amplification of business cycle fluctuations.

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Appendix

Proof of Proposition 1. The assert follows by setting $Y_{t-1} = Y_t = Y^*$, $Z_t = Z_{t-1} = Z^*$ and $M_t = M_{t-1} = M^*$. From the second equation in (9) we immediately obtain $Y^* = Y^F$, which indeed provides $Z^* = Y^F$ from the third equation in (9). Setting $Y^* = Z^* = Y^F$ in the first equation in (9) we find M^*.
Then, setting $M_{t-1} = M^*$ and $Y_{t-1} = Y^*$ in (8) we obtain R^*. Recalling that $d_2 < 0$ and $\varphi < 0$, imposing $M^* > 0$ and $R^* > 0$ and solving with respect to Y^F we easily find the positivity conditions. □

Proof of Proposition 2. See [27]. □

Proof of Propositions 3-4. The Jacobian matrix of the system is given by

$$J = \begin{pmatrix} c - g - \tilde{\varphi}(d_1 - \mu(\theta - 1)) + \frac{\tilde{\gamma} e^{Z-Y}(a_1+a_2)^2}{(a_2+a_1 e^{Z-Y})^2} & \tilde{\varphi} & -\tilde{\varphi}\mu\theta - \frac{\tilde{\gamma} e^{Z-Y}(a_1+a_2)^2}{(a_2+a_1 e^{Z-Y})^2} \\ \mu(\theta-1) & 1 & -\mu\theta \\ 1 & 0 & 0 \end{pmatrix},$$

which, evaluated at the steady state (Y^*, M^*, Z^*), becomes

$$J^* = \begin{pmatrix} c - g + \tilde{\gamma} - \tilde{\varphi}(d_1 + \mu(1-\theta)) & \tilde{\varphi} & -\tilde{\gamma} - \tilde{\varphi}\mu\theta \\ -\mu(1-\theta) & 1 & -\mu\theta \\ 1 & 0 & 0 \end{pmatrix}.$$

The characteristic polynomial $P(\lambda) = \lambda^3 + C_1\lambda^2 + C_2\lambda + C_3$ associated to matrix J^* is defined by

$$C_1 = g - c - \tilde{\gamma} + d_1\tilde{\varphi} + \tilde{\varphi}\mu - \tilde{\varphi}\mu\theta - 1,\ C_2 = c - g + 2\tilde{\gamma} - d_1\tilde{\varphi} + \tilde{\varphi}\mu\theta,\ C_3 = -\tilde{\gamma}.$$

The stability conditions (see [41]) require

$$\begin{cases} 1 + C_1 + C_2 + C_3 > 0, \\ 1 - C_1 + C_2 - C_3 > 0, \\ 1 - C_2 + C_1C_3 - (C_3)^2 > 0, \\ C_2 < 3, \end{cases}$$

namely

$$\begin{cases} \tilde{\varphi}\mu > 0, \\ 2c - 2g + 4\tilde{\gamma} - 2d_1\tilde{\varphi} - \tilde{\varphi}\mu + 2\tilde{\varphi}\mu\theta + 2 > 0, \\ g - c - \tilde{\gamma} + c\tilde{\gamma} + d_1\tilde{\varphi} - g\tilde{\gamma} - d_1\tilde{\varphi}\tilde{\gamma} - \tilde{\varphi}\tilde{\gamma}\mu - \tilde{\varphi}\mu\theta + \tilde{\varphi}\tilde{\gamma}\mu\theta + 1 > 0, \\ g - c - 2\tilde{\gamma} + d_1\tilde{\varphi} - \tilde{\varphi}\mu\theta + 3 > 0. \end{cases}$$

The first condition is always true. Introducing $\alpha = g - c + d_1\tilde{\varphi}$ we can rewrite the last three condition of the previous system as

$$\begin{cases} -\tilde{\varphi}(1 - 2\theta)\mu + 4\tilde{\gamma} - 2\alpha + 2 > 0, \\ -\tilde{\varphi}(\tilde{\gamma}(1-\theta) + \theta)\mu + \alpha - \tilde{\gamma} - \alpha\tilde{\gamma} + 1 > 0, \\ -\tilde{\varphi}\theta\mu + \alpha - 2\tilde{\gamma} + 3 > 0. \end{cases} \tag{23}$$

Before making the stability conditions explicit from System (23), we collect some identities and inequalities that will be used in the rest of the proof. Let us define

$$s_3 = \frac{\alpha - 2\tilde{\gamma} + 3}{\tilde{\varphi}\theta}.$$

and note that since $c < 1$, we have $\alpha > -1$. We have

$$
\begin{aligned}
& s_1 > 0 \Leftrightarrow \begin{cases} 0 \leq \theta < 1/2, \\ \alpha < 2\tilde{\gamma} + 1, \end{cases} \cup \begin{cases} 1/2 < \theta \leq 1, \\ \alpha > 2\tilde{\gamma} + 1, \end{cases} \\
& s_2 > 0 \Leftrightarrow \tilde{\gamma} < 1, \\
& \theta \neq 0 \text{ and } s_2 > 0 \Rightarrow s_3 > 0,
\end{aligned} \tag{24}
$$

where the last implication is due to the fact that, when $\theta \neq 0$, $s_3 > 0$ is equivalent to $\alpha > 2\tilde{\gamma} - 3$, but $2\tilde{\gamma} - 3 < -1$ when $\tilde{\gamma} < 1$.

Moreover, we have

$$
s_1 - s_2 = \frac{(\tilde{\gamma} + 1)(-\alpha + 4\tilde{\gamma} + 4\theta - 4\tilde{\gamma}\theta - 1)}{\tilde{\varphi}(1 - 2\theta)(\tilde{\gamma}(1 - \theta) + \theta)}, \tag{25}
$$

and

$$
\theta \neq 0 \text{ and } s_2 > 0 \Rightarrow s_2 - s_3 < 0 \tag{26}
$$

since

$$
s_2 - s_3 = \frac{-3\tilde{\gamma} - 2\theta - \alpha\tilde{\gamma} + 4\tilde{\gamma}\theta - 2\tilde{\gamma}^2\theta + 2\tilde{\gamma}^2}{\tilde{\varphi}\theta(\tilde{\gamma}(1 - \theta) + \theta)} < 0
$$

is equivalent to

$$
\alpha > \frac{-3\tilde{\gamma} - 2\theta + 4\tilde{\gamma}\theta - 2\tilde{\gamma}^2\theta + 2\tilde{\gamma}^2}{\tilde{\gamma}} = \frac{2\gamma^2 - 3\tilde{\gamma} - 2\theta(1 - \tilde{\gamma})^2}{\tilde{\gamma}},
$$

in which the rightmost term is smaller than -1, since when $s_2 > 0$ we have $\tilde{\gamma} < 1$ and consequently

$$
\frac{2\gamma^2 - 3\tilde{\gamma} - 2\theta(1 - \tilde{\gamma})^2}{\tilde{\gamma}} < -1 \Leftrightarrow -2\tilde{\gamma}(1 - \tilde{\gamma}) - 2\theta(1 - \tilde{\gamma})^2 < 0.
$$

We can now make (23) explicit with respect to μ. In particular, we distinguish different cases depending on the values of μ.

- $\theta = 0$

System (23) becomes

$$
\begin{cases} -\tilde{\varphi}\mu + 4\tilde{\gamma} - 2\alpha + 2 > 0, \\ -\tilde{\varphi}\tilde{\gamma}\mu + \alpha - \tilde{\gamma} - \alpha\tilde{\gamma} + 1 > 0, \\ \alpha - 2\tilde{\gamma} + 3 > 0, \end{cases} \tag{27}
$$

in which the last condition requires $\alpha > 2\tilde{\gamma} - 3$, while the first and the second one respectively become $\mu < s_1$ and $\mu < s_2$. Hence, we can write

$$\begin{cases} \theta = 0, \\ \alpha > 2\tilde{\gamma} - 3, \\ 0 \leq \mu < \min\{s_1, s_2\}. \end{cases}$$

To have a non-empty stability interval we need $s_1 > 0$ and $s_2 > 0$, which, setting $\theta = 0$ in (24), respectively provide $\alpha < 2\tilde{\gamma} + 1$ and $\tilde{\gamma} < 1$ (so $\alpha > 2\tilde{\gamma} - 3$ holds true). Moreover, setting $\theta = 0$ in (25) we have that $s_1 < s_2$ when $\alpha > 4\tilde{\gamma} - 1$. System (27) then becomes

$$0 \leq \mu < s_2 \text{when} \begin{cases} \theta = 0, \\ -1 < \alpha < 4\tilde{\gamma} - 1, \\ \tilde{\gamma} < 1, \end{cases} \tag{28}$$

and

$$0 \leq \mu < s_1 \text{when} \begin{cases} \theta = 0, \\ 4\tilde{\gamma} - 1 \leq \alpha < 2\tilde{\gamma} + 1, \\ \tilde{\gamma} < 1. \end{cases} \tag{29}$$

• $0 < \theta < 1/2$

Conditions in (23) respectively become $\mu < s_1, \mu < s_2$ and $\mu < s_3$. Thus, to have a non-empty stability interval, we need $s_1 > 0$, i.e. $\alpha < 2\tilde{\gamma} + 1$ and $s_2 > 0$, i.e. $\tilde{\gamma} < 1$. Recalling (24) and (26), this last condition guarantees that $s_3 > 0$ and $s_2 < s_3$. Hence, we can write

$$\begin{cases} 0 < \theta < 1/2, \\ -1 < \alpha < 2\tilde{\gamma} + 1, \\ \tilde{\gamma} < 1, \\ 0 \leq \mu < \min\{s_1, s_2\}. \end{cases}$$

Since from (25) we have $s_1 < s_2$ when $\alpha > 4\tilde{\gamma} - 1 + 4\theta(1 - \tilde{\gamma})$ and noting that

$$4\tilde{\gamma} - 1 + 4\theta(1 - \tilde{\gamma}) < 2\tilde{\gamma} + 1 \Leftrightarrow 2(1 - \tilde{\gamma})(2\theta - 1) \tag{30}$$

is negative for $\tilde{\gamma} < 1$ and $0 < \theta < 1/2$, we can conclude

$$0 \le \mu < s_2 \text{ when } \begin{cases} 0 < \theta < 1/2, \\ -1 < \alpha < 4\tilde{\gamma} - 1 + 4\theta(1-\tilde{\gamma}), \\ \tilde{\gamma} < 1, \end{cases} \tag{31}$$

and

$$0 \le \mu < s_1 \text{ when } \begin{cases} 0 < \theta < 1/2, \\ 4\tilde{\gamma} - 1 + 4\theta(1-\tilde{\gamma}) \le \alpha < 2\tilde{\gamma} + 1, \\ \tilde{\gamma} < 1. \end{cases} \tag{32}$$

Combining (28) and (31) we obtain condition (12a), while (29) and (32) together provides (12b).

• $\theta = 1/2$

The first condition in (23) is $\alpha < 2\tilde{\gamma} + 1$, the second one requires $\mu < s_2$ and the third one requires $\mu < s_3$. Thus, to have a non-empty stability interval we need $s_2 > 0$, i.e. $\tilde{\gamma} < 1$, which guarantees $s_3 > 0$ and $s_2 < s_3$. The system (23) then becomes

$$0 \le \mu < s_2 \text{ when } \begin{cases} \theta = 1/2, \\ -1 < \alpha < 2\tilde{\gamma} + 1, \\ \tilde{\gamma} < 1. \end{cases} \tag{33}$$

• $1/2 < \theta \le 1$

The first condition in (23) is $\alpha > s_1$, the second one requires $\mu < s_2$ and the third one requires $\mu < s_3$. Thus, to have a non-empty stability interval we need $s_2 > 0$, i.e. $\tilde{\gamma} < 1$, which again guarantees that $s_3 > 0$ and $s_2 < s_3$. The system (23) becomes

$$\begin{cases} 1/2 < \theta \le 1, \\ \tilde{\gamma} < 1, \\ s_1 < \mu < s_2. \end{cases} \tag{34}$$

To have a non-empty stability interval we need $s_1 < s_2$. Recalling (25), we have

$$\alpha < 4\tilde{\gamma} - 1 + 4\theta(1-\tilde{\gamma}),$$

where it is easy to the see that the right-hand side is greater than -1.

In particular, a destabilizing scenario occurs when $s_1 \leq 0$, i.e. from (24) when $\alpha < 2\tilde{\gamma} + 1$, which allows writing

$$0 \leq \mu < s_2 \text{ when } \begin{cases} 1/2 < \theta \leq 1, \\ -1 < \alpha \leq 2\tilde{\gamma} + 1, \\ \tilde{\gamma} < 1, \end{cases}$$

which combined with (33) provides condition (13a). Finally, we have a mixed scenario when $s_1 > 0$, i.e. from (24) when $\alpha > 2\tilde{\gamma} + 1$, which allows rewriting System (34) as (13b). We stress that, thanks to (30), the second condition in the rightmost system is always fulfilled by some α. □

Proof of Proposition 5. We follow the projection method described in [42, Chapter 5.4]. Since computations are long but standard, we just report a sketch of the main steps of the proof. As in the proof of Proposition 4, we set $\alpha = g - c + d_1\tilde{\varphi} + 1$.

To have an eigenvalue $\lambda_1 = -1$ and the remaining couple of eigenvalues $\lambda_{2,3}$ inside the unitary circle[8] we need

$$\begin{cases} -\tilde{\varphi}(1-2\theta)\mu + 4\tilde{\gamma} - 2\alpha = 0, \\ -\tilde{\varphi}(\tilde{\gamma}(1-\theta) + \theta)\mu + \alpha - \tilde{\gamma} - \alpha\tilde{\gamma} > 0, \end{cases} \Leftrightarrow \begin{cases} \mu = s_1, \\ \dfrac{\alpha - 4\tilde{\gamma} - 4\theta + 4\tilde{\gamma}\theta}{1-2\theta} > 0, \end{cases}$$

from which we have condition (16).

After noting that the Jacobian matrix of System (15) evaluated at the origin actually coincides with J^*, let $q \in \mathbb{R}^3$ and $p \in \mathbb{R}^3$ be two eigenvectors related to eigenvalue λ_1 for J^* and $(J^*)^T$, respectively. We choose p, q so that the normalization condition $\langle p, q \rangle = 1$ is satisfied, where $\langle , \rangle$ is the Euclidean scalar product in $\mathbb{R}^3$. The Taylor series expansion of System (15) can be written as

$$T_0(x) = J^*x + \frac{1}{2}B(x,x) + \frac{1}{6}C(x,x,x) + O\left(\|x\|^4\right), \tag{35}$$

[8]We recall that (proof of Proposition 4) in (11) the left hand side in first condition corresponds to $p(-1)$, where p is the characteristic polynomial of J^*.

where $x = (Y, M, Z)^T$ and the multilinear functions $B(x, y)$ and $C(x, y, z)$ correspond to

$$B_i(x,y) = \sum_{j,k=1}^{3} \frac{\partial T_{0,i}(\xi)}{\partial \xi_j \partial \xi_k}\bigg|_{\xi=0} x_i y_k, \quad C_i(x,y,z) = \sum_{j,k,l=1}^{3} \frac{\partial T_{0,i}(\xi)}{\partial \xi_j \partial \xi_k \delta \xi_l}\bigg|_{\xi=0} x_i y_k z_l,$$

for $i = 1, 2$ and 3. The center manifold W_{s_1} for T_0 can be then represented by a function whose restriction on W_{s_1} is topologically conjugate to the normal form (17) of the flip bifurcation, where

$$\chi = \frac{1}{6}\langle p, C(q,q,q)\rangle - \frac{1}{2}\langle p, B(q,(A-I)^{-1}B(q,q))\rangle$$

and I is the three-dimensional identity matrix. Computing quantities involved in the previous identity leads to (18). $\square$

Proof of Proposition 6. We follow the projection method described in [42, Chapter 5.4]. As before, we only report a sketch of the main steps of the proof.

We note that all the eigenvalues of J^* lie inside (respectively outside) the unitary circle if and only if each condition in (11) holds true (respectively holds with the opposite inequality). When the left hand side in the first condition in (11) becomes null, we know that we have an eigenvalue equal to -1, while in the proof of Proposition 4 we proved that the third condition in (11) is implied by the first two conditions. So, the unique possibility to have a (real) eigenvalue λ_1 inside the unit circle and a couple of complex eigenvalues $\lambda_{2,3}$ on the unitary circumference is

$$\begin{cases} -\tilde{\varphi}(1-2\theta)\mu + 4\tilde{\gamma} - 2\alpha > 0, \\ -\tilde{\varphi}(\tilde{\gamma}(1-\theta)+\theta)\mu + \alpha - \tilde{\gamma} - \alpha\tilde{\gamma} = 0, \end{cases} \Leftrightarrow \begin{cases} -\dfrac{(\tilde{\gamma}+1)(\alpha - 4\tilde{\gamma} - 4\theta + 4\tilde{\gamma}\theta)}{\tilde{\gamma}+\theta-\tilde{\gamma}\theta} > 0, \\ \mu = s_2. \end{cases} \tag{36}$$

It is easy to see that computing the eigenvalues of J^* and evaluating them at $\mu = s_2$, we find $\lambda_1 = \tilde{\gamma} < 1$ and, setting $\beta = \tilde{\gamma} + \theta - \tilde{\gamma}\theta > 0$, the couple of complex conjugated eigenvalues

$$\lambda_{2,3} = \frac{2\beta - \alpha \pm i\sqrt{\alpha(4\beta - \alpha)}}{2\beta}.$$

Note that $|\lambda_{2,3}| = 1$ and, from the first condition in (36), we have $4\beta > \alpha$.

Strong resonances are avoided provided that $(2\beta-\alpha)/2\beta \neq 1$ (i.e. $\alpha \neq 0$, guaranteed by $\alpha > 0$), $(2\beta-\alpha)/2\beta \neq 0$ (i.e. $\alpha \neq 2\beta$), $(2\beta-\alpha)/2\beta \neq -1$ (i.e. $\alpha \neq 4\beta$, guaranteed by $4\beta > \alpha$), $(2\beta-\alpha)/2\beta \neq -1/2$ (i.e. $\alpha \neq 3\beta$)

Let $q \in \mathbb{C}^3$ and $p \in \mathbb{C}^3$ be two eigenvectors related to eigenvalue λ_2 for J^* and $(J^*)^T$, respectively. We choose p, q such that the normalization condition $< p, q >= 1$ is satisfied, where $<,>$ is the Euclidean scalar product in $\mathbb{C}^3$. Recalling the Taylor expansion of T_0 in (35), the center manifold W_{s_2} for T_0 can be then represented by a function whose restriction on W_{s_2} is topologically conjugate to the normal form (21) of the Neimark-Sacker bifurcation, where

$$\mathrm{Re}(\delta) = \frac{1}{2}\mathrm{Re}\{\lambda_3[\langle p, C(q,q,\bar{q})\rangle + 2\langle p, B(q,(A-I)^{-1}B(q,\bar{q}))\rangle + \langle p, B(\bar{q},(\lambda_2^2 I - J^*)^{-1}B(q,q))\rangle]\}$$

in which I is the three-dimensional identity matrix. Computing quantities involved in the previous identity leads to $\mathrm{Re}(\delta) = N/D$ where

$$N = \alpha^2\tilde{\gamma}(1-\theta)^2(\beta-\theta)\{2a_1a_2(\beta-\theta)(1-\theta)\alpha^2 - \beta(\beta-\theta)[(a_1-a_2)^2(1-\beta)$$
$$+ 8a_1a_2(1-\theta)]\alpha + \beta^2(1-\beta)[(1-\beta)(a_1^2 - 4a_1a_2 + a_2^2) + 4(a_1-a_2)^2(\beta-\theta)]\}$$

and

$$D = 2\beta(a_1+a_2)^2((\beta-\theta)(1-\theta)\alpha + \beta(1-\beta)^2)(\alpha(4\beta-\alpha)(\beta-\theta)(1-\theta) + \beta^2(\beta-1)^2)$$

Recalling that $\beta > \theta, 0 < \beta < 1, \alpha > 0$ and $\theta < 1$ we have that $D > 0$, as well as $\alpha^2\tilde{\gamma}(1-\theta)^2(\beta-\theta) > 0$, so non-degeneracy is guaranteed by $b_2\alpha^2 + b_1\alpha + b_0 \neq 0$, where

$$b_2 = 2a_1a_2(\beta-\theta)(1-\theta)\alpha^2 > 0, \quad b_1 = -\beta(\beta-\theta)[(a_1-a_2)^2(1-\beta) + 8a_1a_2(1-\theta)]\alpha < 0$$

while b_0 can be either positive or negative. The parabola $b_2\alpha^2 + b_1\alpha + b_0$ is then convex with vertex having positive abscissa, so both a supercritical and a subcritical bifurcation are possible. Replacing the original expressions of α, β and $\tilde{\gamma}$ allows concluding. $\square$

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