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Vanishing theorems on Riemannian manifolds, and geometric applications

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Abstract

A general Liouville-type result and a corresponding vanishing theorem are proved under minimal regularity assumptions. The latter is then applied to conformal deformations of stable minimal hypersurfaces, to the L^2 cohomology of complete manifolds, to harmonic maps under various geometric assumptions, and to the topology of submanifolds of Cartan–Hadamard spaces with controlled extrinsic geometry.

Keywords: Vanishing and Liouville theorems; Harmonic maps; Submanifolds; Isolation results

0. Introduction

The aim of this paper is to present a unified approach to treat different geometrical questions such as the study of the constancy of harmonic maps, the topology of submanifolds, and the L^2-cohomology.

The common feature of the problems we treat lies in the fact that one identifies a suitable function ψ whose vanishing or, more generally, constancy, is the analytic

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counterpart of the desired geometric conclusion, and, using the peculiarities of the geometric data, one shows that the function ψ satisfies a differential inequality of the form

$$\psi\Delta\psi + a(x)\psi^2 + A|\nabla\psi|^2 \geqslant 0 \tag{0.1}$$

weakly on M, as well as some suitable non-integrability condition.

This is reminiscent of Bochner's original method: in the compact case, and under appropriate assumptions on the sign of the function ψ and of the coefficient $a(x)$, one concludes with the aid of the standard maximum principle.

In the non-compact case, one could conclude using a form of the maximum principle at infinity; see for instance [CY] and the very recent [PRS1], where, in some cases, one can also relax the boundedness conditions on ψ.

In the general case, however, where no sign condition is imposed on $a(x)$ and/or the function ψ is not bounded, this method is not feasible.

The novelty of our approach is that the compactness of the ambient manifold is now replaced by the assumption that there exists a positive solution φ of a differential inequality suitably related to (0.1)

$$\Delta\varphi + Ha(x)\varphi \leqslant 0 \tag{0.2}$$

Combining the two inequalities enables us to rephrase the vanishing of ψ into an appropriate Liouville-type result.

We note that the existence of a positive solution to (0.2) is equivalent to the non-negativity of the bottom of the spectrum of the Schrödinger operator $-\Delta - Ha(x)$, and one could interpret the condition on its spectrum as a sign condition on $a(x)$ in a suitably integrated sense. We also remark that a somewhat related approach has been used by other authors, see e.g. Berard [B1]. However, he uses the condition on the spectral radius directly, and is therefore forced to restrict the consideration to the L^2 case.

To exemplify our approach we consider the case of harmonic maps, where the setting is particularly transparent.

We recall that a harmonic map f between two Riemannian manifolds M and N is a stationary point of the (local) energy integral, and it is characterized by the vanishing of the tension field $\tau(f) = \operatorname{trace} Ddf$. Combining the Weitzenböck–Bochner formula for harmonic maps, with a refined Kato inequality in the spirit of Schoen and Yau [SY1], one shows that if N has a non-positive sectional curvature, then the (square root of the) energy density $|df|$ satisfies the inequality

$$|df|\,\Delta(|df|) + a(x)|df|^2 \geqslant \frac{1}{n(m-1)}|\nabla(|df|)|^2,$$

where m and n are the dimensions of M and N, respectively, and $-a(x)$ is a lower bound for the Ricci curvature of M (see Section 2).

Thus, if $a(x)$ is non-positive, $|df|$ is subharmonic, and if M is compact, $|df|$ is constant and in fact f is constant if $a(x)$ is somewhere strictly negative.

In the non-compact case, Schoen and Yau prove that if the Ricci curvature of M is non-negative, i.e., $a(x)$ vanishes identically, $|df|$ is again constant. Thus, since the volume of M is infinite by the assumption on the Ricci curvature, if f has finite energy, it is necessarily constant. The same conclusion is obtained in the case where M is a minimal stable immersion of a hypersurface in N. In this situation, by the Gauss equations, $a(x)$ is equal to the square of the length of the second fundamental tensor $|\mathrm{II}|^2$ of the immersion. Thus, stability is equivalent to the non-negativity of the spectral radius of the operator $\mathcal{L} = -\Delta - a(x)$.

In Section 2, we show that if the operator $\mathcal{L}_H = -\Delta - Ha(x)$ has non-negative spectral radius for some $H \geqslant 1$, then any harmonic map f is constant provided $|df|^2$ is in L^γ for some $1 \leqslant \gamma \leqslant H$. It is immediate to see that this contains both Schoen and Yau's results. Moreover, our approach allows not only to relax the condition on $a(x)$ but establishes a link between the integrability condition imposed on the energy density and the coefficient H in $\mathcal{L}_H$. In particular, the topological consequences deduced in Schoen and Yau hold in this more general context.

The main analytical tool used in proving our geometric results is a Liouville-type theorem for locally Lipschitz solutions of differential inequalities of the type

$$u \,\mathrm{div}\,(\varphi u) \geqslant 0,$$

on M satisfying suitable non-integrability conditions (see Theorem 1.4 below).

Applying this result to a function u constructed in terms of ψ and φ yields the vanishing result for solutions of (0.1) alluded to above. The example just described shows that the fairly weak regularity assumptions imposed on u are indeed necessary in order to treat the geometrical problems at hand.

The paper is organized as follows: Section 1 deals with the analytic results. The main result of this section is the Liouville-type theorem mentioned above, which is then used to prove the vanishing result (see Theorem 1.4) used in the geometric applications presented in Section 2. In Section 2, we first extend a result of D. Fisher-Colbrie and Schoen on the non-existence of complete metrics conformally related to the hyperbolic metric on the disk to the case of stable hypersurfaces of Euclidean space (see Theorem 2.1 and Corollary 2.2). We then consider the case of harmonic maps with finite energy. We continue with the study of the topology at infinity of submanifolds of Cartan–Hadamard spaces. In the last subsection we extend a result of Bourguignon [Bo] on the triviality of the cohomology in the middle dimension, valid in the compact case, to the case of the L^2 reduced cohomology of a complete manifold.

In what follows we let $(M, \langle\,,\,\rangle)$ be a connected Riemannian manifold of dimension m. We fix an origin o, and denote by $r(x)$ the distance function from o, and by B_r and ∂B_r the geodesic ball and sphere of radius r centered at o, and by $\mathrm{vol}\, B_r$ and $\mathrm{vol}\,(\partial B_r)$ the respective Riemannian measures. Finally, we denote by C a positive constant that may vary from line to line.

1. Analytic results

In this section we develop the analytic techniques on which all our geometric applications rest. Our first result is a generalization of a Liouville-type result originally due to by Beresticki et al. [BCN], in Euclidean setting. See also Proposition 2.1 in [AC].

Theorem 1.1. *Let $(M, \langle\, , \rangle)$ be a complete manifold. Assume that $0 < \varphi \in L^2_{\mathrm{loc}}(M)$ and $u \in Lip_{\mathrm{loc}}(M)$ satisfy*

$$u \operatorname{div}(\varphi \nabla u) \geqslant 0, \quad \textit{weakly on } M. \tag{1.1}$$

If, for some $p > 1$,

$$\left(\int_{\partial B_r} |u|^p \varphi\right)^{-1} \notin L^1(+\infty) \tag{1.2}$$

then u is constant.

Proof. We begin observing that assumption (1.1) means that, for every $0 \leqslant \rho \in C^\infty_c(M)$, or equivalently, for every $0 \leqslant \rho \in Lip(M)$ compactly supported in M, we have

$$0 \leqslant -\int \langle \nabla(\rho u), \varphi \nabla u\rangle = -\int \left\{\langle \nabla \rho, \varphi u \nabla u\rangle + \varphi \rho |\nabla u|^2\right\}, \tag{1.3}$$

and it is therefore equivalent to the validity of the differential inequality

$$\operatorname{div}(\varphi u \nabla u) \geqslant \varphi |\nabla u|^2 \tag{1.4}$$

in the weak sense on M.

Next, let $a(t) \in C^1(\mathbb{R})$ and $b(t) \in C^0(\mathbb{R})$ satisfy

$$\text{(i)}\, a(u) \geqslant 0, \quad \text{(ii)}\, a(u) + u a'(u) \geqslant b(u) > 0 \tag{1.5}$$

on M, and, for fixed $\varepsilon, t > 0$, let ψ_ε be the Lipschitz function defined by

$$\psi_\varepsilon(x) = \begin{cases} 1 & \text{if } r(x) \leqslant t, \\ \dfrac{t + \varepsilon - r(x)}{\varepsilon} & \text{if } t < r(x) < t + \varepsilon, \\ 0 & \text{if } r(x) \geqslant t + \varepsilon. \end{cases}$$

The idea of the proof is to apply the divergence theorem to the vector field $a(u)u\varphi\nabla u$. We use an integrated form of this idea in order to deal with the weak regularity of the functions involved.

For every non-negative compactly supported Lipschitz function ρ, we compute

$$\begin{aligned}
&-\int \langle \psi_\varepsilon a(u)\nabla\rho, \varphi u\nabla u\rangle \\
&\quad= -\int \langle \nabla(\rho\psi_\varepsilon a(u)) - \rho\psi_\varepsilon a'(u)\nabla u - \rho a(u)\nabla\psi_\varepsilon, \varphi u\nabla u\rangle \\
&\quad\geqslant \int \rho\psi_\varepsilon\varphi|\nabla u|^2[a(u)+a'(u)u] + \rho a(u)\langle\nabla\psi_\varepsilon, \varphi u\nabla u\rangle \\
&\quad\geqslant \int \rho\psi_\varepsilon\varphi b(u)|\nabla u|^2 - \frac{1}{\varepsilon}\int_{B_{t+\varepsilon}\setminus B_t} \rho a(u)\varphi|u||\nabla u|,
\end{aligned}$$

where the first inequality follows from (1.4) using $\rho\psi_\varepsilon a(u)$ as a test function which is non-negative Lipschitz and compactly supported because of the assumptions imposed on a, u, φ, ψ_ε and ρ (here we need $u \in Lip_{\mathrm{loc}}$), while the second inequality is a consequence of (1.5) (ii), and of the Cauchy–Schwarz inequality.

Choosing ρ in such a way that $\rho \equiv 1$ on $\overline{B}_{t+\varepsilon}$ the integral on the leftmost side vanishes, and, applying the Cauchy–Schwarz inequality to the second integral on the rightmost side and rearranging, we deduce that

$$\begin{aligned}
&\int_{B_t} \varphi b(u)|\nabla u|^2 \\
&\quad\leqslant \left(\frac{1}{\varepsilon}\int_{B_{t+\varepsilon}\setminus B_t} \frac{a(u)^2}{b(u)}\varphi u^2\right)^{1/2} \left(\frac{1}{\varepsilon}\int_{B_{t+\varepsilon}\setminus B_t} b(u)\varphi|\nabla u|^2\right)^{1/2}.
\end{aligned} \tag{1.6}$$

Setting

$$H(t) = \int_{B_t} \varphi b(u)|\nabla u|^2,$$

it follows by the co-area formula (see [F, Theorem 3.2.12]) that

$$H'(t) = \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon}\int_{B_{t+\varepsilon}\setminus B_t} b(u)\varphi|\nabla u|^2 = \int_{\partial B_t} b(u)\varphi|\nabla u|^2\mathcal{H}^{m-1} \quad \text{for a.e. } t.$$

Here $\mathcal{H}^{m-1}$ denotes the $(m-1)$-dimensional Hausdorff measure on ∂B_t, which coincides with the Riemannian measure induced on the regular part of ∂B_t (the intersection of ∂B_t with the complement of the cut locus of o; see [F], 3.2.46, or [Ch], Proposition 3.4).

Since the same conclusion holds for the first integral on the right-hand side of (1.6), letting $\varepsilon \to 0+$ in (1.6) and squaring, we conclude that

$$H(t)^2 \leqslant \left(\int_{\partial B_t} \frac{a(u)^2}{b(u)} \varphi u^2 \right) H'(t) \quad \text{for a.e.} \quad t. \tag{1.7}$$

At this point the proof follows the lines of that of Lemma 1.1 in [RS2]: assume by contradiction that u is non-constant, so that there exists $R_0 > 0$ such that $|\nabla u|$ does not vanish a.e. in B_{R_o}. Then for each $t > R_0$, $H(t) > 0$, and therefore the RHS of (1.7) is also positive. Integrating the inequality between R and r $(R_0 \leqslant R < r)$ we obtain

$$H(R)^{-1} \geqslant H(R)^{-1} - H(r)^{-1} \geqslant \int_R^r \left(\int_{\partial B_t} \varphi \frac{a(u)^2}{b(u)} u^2 \right)^{-1}. \tag{1.8}$$

Now, we consider the sequence of functions defined by

$$a_n(t) = \left(t^2 + \frac{1}{n} \right)^{\frac{p-2}{2}}, \quad b_n(t) = \min\{p-1, 1\}\, a_n(t), \quad \forall n \in \mathbb{N}.$$

Since condition (1.5) holds for every n, so does (1.8), whence, letting $n \to +\infty$ and using the Lebesgue-dominated convergence theorem and Fatou's lemma we deduce that there exists $C > 0$ which depends only on p such that

$$\left(\int_{B_R} \varphi |u|^{p-2} |\nabla u|^2 \right)^{-1} \geqslant C \int_R^r \left(\int_{\partial B_t} \varphi |u|^p \right)^{-1} dt.$$

The required contradiction is now attained by letting $r \to +\infty$ and using assumption (1.2). □

As the above proof shows, the conclusion of the Theorem holds if one assumes that $\varphi \in L^\infty_{\mathrm{loc}}(M)$ and $u \in H^1_{\mathrm{loc}}(M)$.

We observe that condition (1.2) is implied by $u\varphi^{1/p} \in L^p(M)$. Indeed, if this is the case and we set $f = \int_{\partial B_r} |u|^p \varphi$, then the assumption and the co-area formula show that $f \in L^1(+\infty)$, and by Hölder inequality

$$\int_{r_0}^r f^{-1} \geqslant (r - r_0)^2 \left(\int_{r_0}^r f \right)^{-1} \to +\infty \quad \text{as } r \to +\infty.$$

We also note that the conclusion of Theorem 1.1 fails if we assume that $p = 1$ in (1.2). Indeed, taking $\varphi \equiv 1$, (1.1) reduces to $u\Delta u \geqslant 0$, and Li and Schoen have constructed in [LS] an example of a non-constant, L^1, harmonic, function on a complete manifold.

Finally, we remark that Theorem 1.1 generalizes [BCN] (see the proof of Proposition 2.1 therein) in two directions, even in the case where $M = \mathbb{R}^m$. Firstly, in their case $p = 2$; secondly they replace (1.2) by the more stringent condition

$$\int_{B_r} u^2 \varphi \leqslant C r^2$$

for some constant $C > 0$. To see that the latter implies (1.2) simply note that its validity forces

$$\frac{r}{\int_{B_r} u^2 \varphi} \notin L^1(+\infty)$$

which in turn implies (1.2) (see, e.g., [RS2, Proposition 1.3]). Furthermore, although the approach used in [BCN] is also applicable in the case of Riemannian manifolds, in this general context, it does not yield a sharp result.

For some of our applications of Theorem 1.1 we shall need the following lemma, which is a slightly improved version of a result of Moss and Pieprbrink, [MP], and Fisher-Colbrie and Schoen [FCS].

Lemma 1.2. *Let $(M, \langle\, ,\, \rangle)$ be a Riemannian manifold, and $\Omega \subset M$ be a domain in M and let $a(x) \in L^\infty_{\mathrm{loc}}(\Omega)$. The following facts are equivalent*:

(i) *There exists $w \in C^1(\Omega)$, $w > 0$, a weak solution of*

$$\Delta w + a(x) w = 0 \quad \textit{on } \Omega;$$

(ii) *There exists $\varphi \in H^1_{\mathrm{loc}}(\Omega)$, $\varphi > 0$, a weak solution of*

$$\Delta \varphi + a(x)\varphi \leqslant 0 \quad \textit{on } \Omega; \ \textit{and}$$

(iii) *If $\lambda_1^{\mathcal{L}}(\Omega)$ denotes the bottom of the spectrum of the Schrödinger operator $\mathcal{L} = -\Delta - a(x)$ with Dirichlet boundary conditions, then*

$$\lambda_1^{\mathcal{L}}(\Omega) := \inf_{0 \not\equiv v \in C_c^\infty(\Omega)} \frac{\int |\nabla v|^2 - a(x) v^2}{\int v^2} \geqslant 0.$$

Proof. We sketch the proof, which is a modification of the original proof in [FCS].

It is trivial that (i) implies (ii). To prove that (ii) implies (iii), observe that, by assumptions, φ satisfies

$$\int \langle \nabla \varphi, \nabla \rho \rangle - a(x)\varphi\rho \geqslant 0, \quad \forall 0 \leqslant \rho \in H^1_c(\Omega). \tag{1.9}$$

For every $\varepsilon > 0$, set $\psi_\varepsilon = \log(\varphi + \varepsilon) \in H^1_{\text{loc}}$. Then, given $f \in C^\infty_c(\Omega)$ we compute

$$\begin{aligned}\int \langle \nabla\psi_\varepsilon, \nabla(f^2)\rangle &= \int \left\langle \frac{\nabla\varphi}{\varphi+\varepsilon}, \nabla(f^2)\right\rangle \\ &= \int \left\langle \nabla\varphi, \nabla\left(\frac{f^2}{\varphi+\varepsilon}\right) + \frac{f^2}{(\varphi+\varepsilon)^2}\nabla\varphi\right\rangle \\ &\geqslant \int a\varphi\frac{f^2}{\varphi+\varepsilon} + \int \frac{|\nabla\varphi|^2}{(\varphi+\varepsilon)^2}f^2 = \int a\frac{\varphi}{\varphi+\varepsilon}f^2 + \int |\nabla\psi_\varepsilon|^2 f^2,\end{aligned}$$

where we have used inequality (1.9) applied to the non-negative compactly supported H^1 function $f^2/(\varphi+\varepsilon)$. We use the inequality

$$\langle \nabla\psi_\varepsilon, \nabla(f^2)\rangle \leqslant 2|f||\nabla f||\nabla\psi_\varepsilon| \leqslant f^2|\nabla\psi_\varepsilon|^2 + |\nabla f|^2$$

to estimate the left-hand side, and simplify to obtain

$$\int |\nabla f|^2 \geqslant \int \frac{\varphi}{\varphi+\varepsilon}af^2,$$

and since $a \in L^\infty_{\text{loc}}$, $f^2 \in C^\infty_c$ and $|\varphi/(\varphi+\varepsilon)| \leqslant 1$, we may let $\varepsilon \to 0+$ and apply the dominated convergence theorem to conclude that

$$\int |\nabla f|^2 \geqslant \int af^2,$$

as required.

We now come to the implication (iii)$\Rightarrow$ (i). Let $\{D_n\}$ be an exhaustion of Ω by an increasing sequence of relatively compact domains with smooth boundary. Since $a(x) \in L^\infty_{\text{loc}}(\Omega)$, and $\lambda_1^{\mathcal{L}}(D_n) > 0$, by domain monotonicity, the Dirichlet problem

$$\begin{cases} \mathcal{L}v = 0 & \text{in } D_n, \\ v = 1 & \text{on } \partial D_n, \end{cases} \tag{1.10}$$

has a solution v_n which belongs to $C^{0,\alpha}(\overline{D}_n) \cap H^2(D_n)$ for some $0 < \alpha < 1$ (see, e.g. [GT, Theorems 8.6, 8.12 and 8.29]). Moreover, it follows from Theorem 1.1 in [T] (see also [G, Chapter VII, Theorem 1.2]) that $v_n \in C^{1,\beta}_{\text{loc}}(D_n)$ for some $0 < \beta < 1$ independent of n.

We claim that $v_n > 0$ in D_n. By the maximum principle (see [GT, p. 35], and note that the result extends to functions in C^1 using a comparison argument modelled, e.g., on [PRS1, Proposition 6.1]) it suffices to show that $v_n \geqslant 0$. Assume by contradiction that

$B_n = \{x \in D_n : v_n(x) < 0\} \neq \emptyset$. Then, by the boundary condition, $B_n \Subset D_n$ and $v_n^- = \min\{v_n, 0\} \in H_c^1(D_n)$ is a weak solution of the differential inequality $(-\Delta - a)v_n^- \geqslant 0$. Using the non-zero function $-v_n^-$ as a test function, we obtain

$$\int |\nabla v_n^-|^2 - a(x)(v_n^-)^2 \leqslant 0,$$

contradicting the positivity of $\lambda^{\mathcal{L}}(D_n)$.

Now fix $x_o \in D_0$, and let

$$w_n(x) = \frac{v_n(x)}{v_n(x_o)},$$

so that $w_n \in C^{0,\alpha}(\overline{D}_n) \cap C^{1,\beta}_{\mathrm{loc}}(D_n)$. Furthermore, according to Theorem 8.20 in [GT] and Theorem 1.1 in [T], for every n there exists a constant C_n such that for every $k > n$,

$$\begin{aligned} &C_n^{-1} \leqslant w_k(x) \leqslant C_n \\ &|\nabla w_n| \leqslant C_n \\ &|\nabla w_k(x) - \nabla w_k(y)| \leqslant C_n d(x, y)^\beta \end{aligned}$$

for every $x, y \in \overline{D}_n$.

The Ascoli–Arzelá theorem and a diagonal argument yield a subsequence $\{w_{n_j}\}$ which converges in $C^1_{\mathrm{loc}}(\Omega)$ to a C^1 function w which is a weak solution of $\mathcal{L}w = 0$. Since $w_n(x_o) = 1$ for every n, $w(x_o) = 1$, and, again by the maximum principle, $w > 0$ on Ω. $\square$

Remark 1.3. As the proof shows, the function w belongs in fact to the space $C^{1,\beta}_{\mathrm{loc}}(\Omega)$ for some $0 \leqslant \beta < 1$. Further, if we assume that $a(x) \in C^{0,\alpha}_{\mathrm{loc}}(\Omega)$ for some $0 < \alpha < 1$, then $w \in C^{2,\alpha}_{\mathrm{loc}}(\Omega)$ (see, e.g., [Au, Theorem 3.55]) and it is therefore a classical solution of $\Delta w + a(x)w = 0$ on Ω. Finally, it is easy to see that the equivalences (i)–(iii) extend to the case where Ω is replaced by the exterior $M \setminus K$ of a compact set K, thus extending Proposition 1 in [FC].

We are now ready for the following consequence of Theorem 1.1, which will be the main component in the geometric applications of Section 2.

Theorem 1.4. *Let $(M, \langle\, , \rangle)$ be a complete manifold, $a(x) \in L^\infty_{\mathrm{loc}}(M)$ and let $\varphi \in Lip_{\mathrm{loc}}(M)$ be a positive solution of*

$$\Delta\varphi + Ha(x)\varphi \leqslant 0 \quad \textit{weakly on } M, \tag{1.11}$$

for some $H \geqslant 1 + \max\{A, 0\}$. *Let* $\psi \in Lip_{\mathrm{loc}}(M)$ *satisfy the differential inequality*

$$\psi\Delta\psi + a(x)\psi^2 + A|\nabla\psi|^2 \geqslant 0 \quad \textit{weakly on } M. \tag{1.12}$$

If

$$\left(\int_{\partial B_r} |\psi|^{2(\beta+1)}\right)^{-1} \notin L^1(+\infty) \tag{1.13}$$

for some β *such that* $\max\{A, 0\} \leqslant \beta \leqslant H - 1$, *then there exists a constant* $C \in \mathbb{R}$ *such that*

$$C\varphi = |\psi|^H \operatorname{sgn}\psi. \tag{1.14}$$

Further,

(i) *If* $H - 1 > A$, *then* ψ *is constant on* M, *and if in addition*, $a(x)$ *does not vanish identically*, *then* ψ *is identically zero*;
(ii) *If* $H - 1 = A$, *and* ψ *does not vanish identically*, *then* φ *and therefore* $|\psi|^H$ *satisfy* (1.11) *with equality sign.*

Proof. Set, for ease of notation, $\alpha = \frac{\beta+1}{H}$, and let u be the locally Lipschitz function defined by

$$u = \varphi^{-\alpha}|\psi|^\beta\psi,$$

so that the first assertion in the statement is that u is constant on M.

Noting that

$$\int \varphi^{2\alpha}|u|^2 = \int |\psi|^{2(\beta+1)},$$

so that (1.13) implies that (1.2) holds with $\varphi^{2\alpha}$ in place of φ, and $p = 2$, the constancy of u follows from Theorem 1.1 once we show that the differential inequality

$$u \operatorname{div}\left(\varphi^{2\alpha}\nabla u\right) \geqslant 0 \tag{1.15}$$

holds weakly on M, i.e. (see the beginning of the proof of Theorem 1.1), that for every non-negative, compactly supported Lipschitz function ρ on M, we have

$$I = \int \left[\langle \varphi^{2\alpha} u\nabla u, \nabla\rho\rangle + \varphi^{2\alpha}|\nabla u|^2\rho\right] \leqslant 0.$$

From the definition of u we compute

$$\nabla u = -\alpha\varphi^{-\alpha-1}|\psi|^{\beta}\psi\nabla\varphi + (\beta+1)\varphi^{-\alpha}|\psi|^{\beta}\nabla\psi$$

whence

$$\begin{aligned} I = (\beta+1)\int \langle\nabla\psi, \psi|\psi|^{2\beta}\nabla\rho\rangle - \alpha\int \varphi^{-1}|\psi|^{2\beta+2}\langle\nabla\varphi,\nabla\rho\rangle \\ + \int\left[(\beta+1)^2|\psi|^{2\beta}|\nabla\psi|^2\rho + \alpha^2|\psi|^{2\beta+2}\frac{|\nabla\varphi|^2}{\varphi^2}\rho\right] \\ -2\alpha(\beta+1)\int |\psi|^{2\beta}\psi\left\langle\frac{\nabla\varphi}{\varphi},\nabla\psi\right\rangle. \end{aligned} \tag{1.16}$$

We first consider the first integral on the right-hand side

$$\begin{aligned} \int |\psi|^{2\beta}\psi\langle\nabla\psi,\nabla\rho\rangle &= \lim_{\varepsilon\to 0+}\int (\psi^2+\varepsilon)^{\beta}\psi\langle\nabla\psi,\nabla\rho\rangle \\ &= \lim_{\varepsilon\to 0+}\left\{\int \langle\nabla\psi, \nabla\big[\psi(\psi^2+\varepsilon)^{\beta}\rho\big]\rangle\right. \\ &\qquad\left. -(\psi^2+\varepsilon)^{\beta}\frac{(2\beta+1)\psi^2+\varepsilon}{\psi^2+\varepsilon}|\nabla\psi|^2\rho\right\}. \end{aligned} \tag{1.17}$$

According to (1.12), for every non-negative, compactly supported Lipschitz function σ,

$$\int \langle\nabla\psi,\nabla(\sigma\psi)\rangle \leqslant \int \big(a(x)\psi^2 + A|\nabla\psi|^2\big)\sigma.$$

Applying the above inequality with $\sigma = \rho(\psi^2+\varepsilon)^{\beta}$, and applying the dominated convergence theorem, we deduce that

$$\lim_{\varepsilon\to 0+}\int (\psi^2+\varepsilon)^{\beta}\frac{(2\beta+1)\psi^2+\varepsilon}{\psi^2+\varepsilon}|\nabla\psi|^2\rho = (2\beta+1)\int |\psi|^{2\beta}|\nabla\psi|^2\rho,$$

and

$$\begin{aligned} \lim_{\varepsilon\to 0+}\int \langle\nabla\psi,\nabla\big[\psi(\psi^2+\varepsilon)^{\beta}\rho\big]\rangle &\leqslant \lim_{\varepsilon\to 0+}\int \big[a(x)\psi^2 + A|\nabla\psi|^2\big](\psi^2+\varepsilon)^{\beta}\rho \\ &= \int \big[a(x)|\psi|^{2\beta+2} + A|\psi|^{2\beta}|\nabla\psi|^2\big]\rho. \end{aligned}$$

Inserting into (1.17) we conclude that

$$\int |\psi|^{2\beta}\psi\langle\nabla\psi, \nabla\rho\rangle \leqslant \int \big[a(x)|\psi|^{2\beta+2} + (A-2\beta-1)|\psi|^{2\beta}|\nabla\psi|^2\big]\rho. \tag{1.18}$$

In a similar, but easier way, using (1.11) one verifies that

$$-\int \varphi^{-1}|\psi|^{2\beta+2}\langle\nabla\varphi, \nabla\rho\rangle \leqslant \int \left[-Ha(x)|\psi|^{2\beta+2} - |\psi|^{2\beta+2}\frac{|\nabla\varphi|^2}{\varphi^2}\right.$$
$$\left. +2(\beta+1)|\psi|^{2\beta}\psi\left\langle\frac{\nabla\varphi}{\varphi}, \nabla\psi\right\rangle\right]\rho. \tag{1.19}$$

Substituting (1.18) and (1.19) into (1.16), recalling the value of α, and the condition satisfied by β, we conclude that

$$I \leqslant (\beta+1)\int\left[(A-\beta)|\psi|^{2\beta}|\nabla\psi|^2\rho + \frac{\beta+1-H}{H^2}|\psi|^{2\beta+2}\frac{|\nabla\varphi|^2}{\varphi^2}\rho\right] \leqslant 0,$$

as required to show that (1.15) holds.

In particular, ψ has a constant sign, and if we assume that $\psi \not\equiv 0$, multiplying ψ by a suitable constant we may assume that ψ is strictly positive, and

$$\varphi = \psi^H.$$

Inserting this equality into (1.11) we have

$$H\psi^{H-2}\big[\psi\Delta\psi + (H-1)|\nabla\psi|^2 + a(x)\psi^2\big] \leqslant 0, \tag{1.20}$$

whence, multiplying (1.12) by $H\psi^{H-2}$, and subtracting the resulting inequality from (1.20) we obtain

$$H\big[(H-1)-A\big]\psi^{H-2}|\nabla\psi|^2 \leqslant 0. \tag{1.21}$$

Thus, if $H-1 > A$, $|\nabla\psi|^2 \equiv 0$, and ψ and therefore φ are constant. It follows from (1.11) that

$$\Delta\varphi + Ha(x)\varphi = Ha(x)\varphi \leqslant 0 \quad \text{ so that } a(x) \leqslant 0,$$

while (1.12) implies that

$$\psi\Delta\psi + a(x)\psi^2 + A|\nabla\psi|^2 = a(x)\psi^2 \geqslant 0 \quad \text{ so that } a(x) \geqslant 0,$$

and we conclude that $a(x) \equiv 0$. In particular, if $a(x) \not\equiv 0$, then ψ must vanish identically.

Finally, assume that $A = H - 1$, and that ψ does not vanish identically, so that, as noted above, we may assume that ψ is strictly positive, and that $\varphi = \psi^H$. On the other hand, it follows from (1.11) and Lemma 1.2 that there exists a positive C^1 function v satisfying

$$\Delta v + Ha(x)v = 0 \quad \text{weakly on } M. \tag{1.22}$$

Repeating the argument with v in place of φ, we deduce that there exists $\tilde{c} \neq 0$, such that

$$\tilde{c}v = \psi^H = \varphi.$$

Thus φ is a positive multiple of v and we conclude that it also satisfies (1.22). $\square$

We remark that Theorem 1.4 fails if the exponent $2(\beta + 1)$ in the integrability condition (1.14) is replaced by $p(\beta+1)$ for some $p > 2$. Indeed, it was shown in [BR] that if $a(x)$ and $b(x)$ are non-negative continuous functions on $\mathbb{R}^m$ satisfying

$$a(x) \leqslant \frac{(m-2)^2}{4}|x|^{-2}, \quad a(x) = \frac{(m-2)^2}{4}|x|^{-2} \quad \text{if} \quad |x| \gg 1$$

and

$$b(x) = \frac{|x|^{(m-2)(\sigma-1)/2}}{(\log|x|)^{\sigma+1}(\log\log|x|)(\log\log\log|x|)^2} \quad \text{if } |x| \gg 1$$

for some $\sigma > 1$, then the equation

$$\Delta u + a(x)u - b(x)u^\sigma = 0 \tag{1.23}$$

has a family of positive solutions u_α $(\alpha > 0)$ satisfying

$$u_\alpha(0) = \alpha \quad \text{and} \quad u_\alpha(x) \sim |x|^{-(m-2)/2}\log|x| \quad \text{as} \quad |x| \to +\infty.$$

In particular, u_α is a solution of (1.12) with $A = 0$, and

$$\int_{\partial B_r} |u_\alpha|^q \asymp r^{1+(m-2)(2-q)/2}(\log r)^q,$$

so that

$$\left(\int_{\partial B_r} |u_\alpha|^q\right)^{-1} \notin L^1(+\infty)$$

for every $q > 2$.

On the other hand, it is well known that in this case $\lambda_1^{-\Delta - a(x)}(\mathbb{R}^m) = 0$, so there exists a positive solution φ of

$$\Delta\varphi + a(x)\varphi = 0 \quad \text{on } \mathbb{R}^m \tag{1.24}$$

(see, e.g., [BRS1, Lemma 3 and subsequent Remark 4]).

Since in this case $H = 1$, applying Theorem 1.4 we would conclude that

$$c\varphi = u_\alpha$$

for some constant c which is necessarily positive, since both u, $\varphi > 0$. But then u_α would be a solution of (1.24) and this is impossible since it satisfies (1.23) and b is non-zero.

As an immediate corollary of Theorem 1.4 we have

Corollary 1.5. *Let $a(x) \in L^\infty_{\text{loc}}(M)$ and $H \geqslant 1 + \max\{A, 0\}$, and set $\mathcal{L} = -\Delta - Ha(x)$. Assume that $\psi \in Lip_{\text{loc}}(M)$ is a changing sign solution of* (1.12) *satisfying* (1.13) *for some β such that* $\max\{A, 0\} \leqslant \beta \leqslant H - 1$. *Then $\lambda_1^{\mathcal{L}}(M) < 0$.*

Proof. Assume by contradiction that $\lambda_1^{\mathcal{L}}(M) \geqslant 0$. By Lemma 1.2 there exists $0 < \varphi \in C^1(M)$ satisfying $\Delta\varphi + Ha(x)\varphi = 0$ on M. By Theorem 1.4, there exists a constant c such that $c\varphi|\psi|^{H-1}\psi$, and since ψ changes sign, while φ is strictly positive, this yields the required contradiction. $\square$

In the case of Euclidean space, the integrability condition (1.13) follows assuming a suitable upper estimate for ψ, and yields the following (slight) improvement of [BCN] Theorem 1.7:

Corollary 1.6. *Let $a(x) \in L^\infty_{\text{loc}}(\mathbb{R}^m)$, and let $\psi \in Lip_{\text{loc}}(\mathbb{R}^m)$ be a changing sign solution of*

$$\psi\Delta\psi + a(x)\psi^2 \geqslant 0 \quad \textit{on } \mathbb{R}^m,$$

such that, for some $H \geqslant 1$,

$$|\psi(x)| = \mathcal{O}\Big(r(x)^{-(m-2)/2H}(\log r(x))^{1/2H}\Big) \quad \textit{as } r(x) \to +\infty.$$

Then $\lambda_1^{-\Delta - Ha(x)}(\mathbb{R}^m) < 0$.

Similar results can be obtained on Riemannian manifolds where vol ∂B_r satisfies a suitable upper bound. This in turn follows, by the volume comparison theorem, from appropriate lower bounds on the Ricci curvature (see, e.g.,[BRS2, Appendix]). We leave the details to the interested reader.

Theorem 1.4 also yields the following generalization of Theorem 2 (and Corollary 2) in [FCS].

Corollary 1.7. *Let* $(M, \langle\, ,\, \rangle)$ *be a complete manifold, and let* $a(x) \in L^{\infty}_{\mathrm{loc}}(M)$ *and* $A < 0$. *Suppose that* $\psi \in Lip_{\mathrm{loc}}$ *is a non-constant weak solution of the differential inequality*

$$\psi\Delta\psi + a(x)\psi^2 + A|\nabla\psi|^2 \geqslant 0,$$

satisfying

$$\left(\int_{\partial B_r} \psi^2\right)^{-1} \notin L^1(+\infty). \tag{1.25}$$

Then, there exists $H_0 \in [0, 1)$ *such that, for every* $H > H_0$, *the differential inequality*

$$\Delta\varphi + Ha(x)\varphi \leqslant 0 \tag{1.26}$$

has no positive, locally Lipschitz weak solution on M, *while if* $0 \leqslant H \leqslant H_0$, *such a solution of* (1.26) *exists.*

Proof. Recall that, according to Lemma 1.2, the existence of a positive, locally Lipschitz weak solution of (1.26) is equivalent to

$$\lambda_1^{-\Delta - Ha(x)} \geqslant 0. \tag{1.27}$$

Observe next that if $0 < H_1 \leqslant H_2$, then, by the variational characterization of $\lambda_1^{-\Delta - Ha(x)}$, we have

$$\lambda_1^{-\Delta - H_1 a(x)} \geqslant \frac{H_1}{H_2} \lambda_1^{-\Delta - H_2 a(x)}. \tag{1.28}$$

(see the argument in the proof of Theorem 2 in [FCS]). Thus, if we denote by S the set of $H \geqslant 0$ such that (1.26) holds, S is not empty, since $\lambda_1^{-\Delta} \geqslant 0$, and if H_2 is in S then so is H_1.

An application of Theorem 1.4 with $A < \max\{A, 0\} = 0 = \beta = H - 1$ implies that if $H = 1$, then (1.26) has no positive locally Lipschitz solution, for otherwise ψ would necessarily be constant, against the assumption. Thus $1 \notin S$, and $H_0 = \sup S \leqslant 1$

Now one concludes as in Corollary 2 in [FCS] showing, by an approximation argument, that S is closed, so that $1 > H_0 \in S$. $\square$

To see that Corollary 1.7 implies Theorem 2 and Corollary 2 in [FCS], it suffices to observe that if $ds^2 = \mu(z)|dz|^2$ is a complete metric on the unit disk D, with Gaussian curvature K, then $\psi = \mu^{-1/2}$ is a non-constant solution of

$$\psi\Delta\psi - K\psi^2 = |\nabla\psi|^2$$

and

$$\int \psi^2 dV_{ds^2} = \int \mu^{-1}\mu\, dx\, dy = \mathrm{vol}_{Eucl}(D) < +\infty.$$

According to the remark after the proof of Theorem 1.1, condition (1.25) holds, and Corollary 1.7 implies that there exists $H_0 \in [0, 1)$ such that equation

$$\Delta\varphi - HK(x)\varphi = 0$$

has no positive solution if $H > H_0$ and has a positive solution if $0 \leqslant H \leqslant H_0$.

2. Geometric consequences

2.1. Conformal metrics on stable minimal hypersurfaces

Our main analytic result, Theorem 1.4 above, generalizes Theorem 2 of Fischer-Colbrie and Schoen [FCS]. Following their line of investigation, we are naturally led to the following result, which extends some known facts in minimal surfaces theory to (higher dimensional) minimal hypersurfaces of Euclidean space; see Corollary 2.2 below.

We recall that a minimal hypersurface $f : (M^m, \langle\, ,\, \rangle) \to \mathbb{R}^{m+1}$ is stable if it (locally) minimizes area up to second order or, equivalently, if the spectral radius $\lambda_1^{\mathcal{L}}(M)$ of the operator $\mathcal{L} = -\Delta - |\mathrm{II}|^2$ is non-negative. Here $|\mathrm{II}|$ denotes the length of the second fundamental tensor of the immersion.

We also recall that a Riemannian metric $\widetilde{\langle\, ,\, \rangle}$ on a (generic) manifold M is said to be a pointwise conformal deformation of a metric $\langle\, ,\, \rangle$ if there exists a positive function $\rho \in C^\infty(M)$ such that $\widetilde{\langle\, ,\, \rangle}_x (v, w) = \rho^2(x) \langle\, ,\, \rangle_x (v, w)$, for every $x \in M$ and $v, w \in T_x M$.

Theorem 2.1. *Let $f : (M^m, \langle\, ,\, \rangle) \to \mathbb{R}^{m+1}$ be a complete, stable, minimal hypersurface of dimension $m \geqslant 2$. Then $\langle\, ,\, \rangle$ cannot be pointwise conformally deformed to any Riemannian metric $\widetilde{\langle\, ,\, \rangle}$ of scalar curvature $\tilde{S}(x) \leqslant 0$ and finite volume $\widetilde{\mathrm{vol}}\,(M) < +\infty$.*

Proof. We first consider the case $m \geqslant 3$. By contradiction, we assume that there exists a conformal metric $\widetilde{\langle\, ,\, \rangle}$ on M with scalar curvature $\tilde{S}(x) \leqslant 0$ and finite volume $\widetilde{vol}(M) < +\infty$. Denoting by $S(x)$ the scalar curvature of the original metric, minimality and the Gauss equations imply

$$S(x) = -\left|\mathrm{II}\,(x)\right|^2. \tag{2.1}$$

The stability of f is then equivalent to the existence of a positive solution $\varphi \in C^\infty(M)$ of

$$\Delta\varphi - S(x)\varphi = 0 \quad \text{on} \quad M. \tag{2.2}$$

Setting

$$H = \frac{4\,(m-1)}{m-2} > 1; \quad a(x) = -\frac{1}{H}S(x),$$

we can rewrite (2.2) in the form

$$\Delta\varphi + Ha(x)\varphi = 0 \quad \text{on} \quad M.$$

Now, let

$$\widetilde{\langle\, ,\, \rangle} = \psi^{\frac{4}{m-2}}\langle\, ,\, \rangle. \tag{2.3}$$

By the scalar curvature equation and the assumption that $\tilde{S}\,(x) \leqslant 0$, the smooth positive function ψ satisfies

$$\Delta\psi + a(x)\psi = -\frac{1}{H}\tilde{S}(x)\psi^{\frac{m+2}{m-2}} \geqslant 0, \qquad \text{on} \quad M. \tag{2.4}$$

Since

$$\int_M \psi^{\frac{2m}{m-2}} dvol = \widetilde{\mathrm{vol}}(M) < +\infty$$

we have

$$\frac{1}{\int_{\partial B_r(o)} \psi^{2(\beta+1)}} \notin L^1(+\infty),$$

where

$$\beta = \frac{2}{m-2}$$

satisfies

$$0 < \beta < H - 1.$$

Applying Theorem 1.4, case 1, with $A = 0$ we therefore conclude that ψ, and therefore φ, is a positive constant and $S(x) \equiv 0$. According to (2.1) and (2.3) we deduce that $f(M)$ is an affine plane and hence $\left(M, \widetilde{\langle\, ,\, \rangle}\right)$ is homothetic to $\left(\mathbb{R}^m, can\right)$. But this clearly contradicts the assumption $\widetilde{\mathrm{vol}}(M) < +\infty$.

The case $m = 2$ is completely similar. This time, we replace (2.3) with

$$\widetilde{\langle\, ,\, \rangle} = \psi^2 \langle\, ,\, \rangle$$

and, instead of (2.4), we use the corresponding Yamabe equation

$$\psi \Delta \psi - S(x) \psi^2 = -\tilde{S}(x) \psi^4 + |\nabla \psi|^2 .$$

Thus, ψ satisfies

$$\psi \Delta \psi - S(x) \psi^2 \geqslant |\nabla \psi|^2 .$$

Since

$$\int_M \psi^2 d\mathrm{vol} = \widetilde{\mathrm{vol}}(M) < +\infty$$

we have

$$\frac{1}{\int_{\partial B_r(o)} \psi^2} \notin L^1(+\infty) .$$

On the other hand, the stability assumption implies the existence of a positive, smooth solution φ of (2.2). Therefore we can apply Theorem 1.4, case 1, with the choices $\beta = 0$, $a(x) = -S(x)$, $H = 1$, $A = -1$. Reasoning as above, we reach the desired contradiction. $\square$

Using a classical universal covering argument, together with the Riemann–Köbe uniformization theorem, we easily recover Corollary 4 in [FCS]:

Corollary 2.2. *A two-dimensional, complete, stable, minimal surface* $f : (M, \langle\, ,\, \rangle) \to \mathbb{R}^3$ *is parabolic, and hence it is an affine plane.*

Indeed let $\pi : \left(\bar{M}, \bar{\langle\, ,\, \rangle}\right) \to (M, \langle\, ,\, \rangle)$ be the Riemannian universal covering of M. Then, $\bar{f} = f \circ \pi : \left(\bar{M}, \bar{\langle\, ,\, \rangle}\right) \to \mathbb{R}^3$ defines a complete, minimal surface. Moreover, $\bar{f}$ is stable because any positive solution φ of (2.2) on M lifts to a positive solution $\bar{\varphi} = \varphi \circ \pi$ of $\bar{\Delta}\bar{\varphi} - \bar{S}(y)\bar{\varphi} = 0$ on $\bar{M}$. Here the bar-quantities refer to the covering metric $\bar{\langle\, ,\, \rangle}$. Since there are no compact minimal surfaces in the Euclidean space, the uniformization theorem implies that $\left(\bar{M}, \bar{\langle\, ,\, \rangle}\right)$ is conformally diffeomorphic to either $\mathbb{R}^2$ or the open unit disk $D_1 \subset \mathbb{R}^2$. In view of Theorem 2.1 the second possibility cannot occur so that M must be parabolic. To conclude that f is totally geodesic, simply note that, by (2.2), φ is a positive superharmonic function. Therefore φ must be constant and $S(x) = -\,|\mathrm{II}|^2 \equiv 0$.

2.2. Harmonic maps of finite energy

In this section, we prove a Liouville-type theorem for harmonic maps that generalizes, in some respects, classical work by Schoen and Yau, [SY1]. Compared with [SY1], one realizes that our result, emphasizing the role of a suitable Schrodinger operator related to the Ricci curvature of the domain manifold, unifies in a single statement the situations considered in the paper by Schoen and Yau; see Remark 2.4.

Later, we shall employ our version of this Liouville theorem to study the topology at infinity of submanifolds of a Cartan–Hadamard space; see Theorem 2.10 below.

Recall that a smooth map $f : (M^m, \langle\, ,\, \rangle) \to (N^n, (\, ,\,))$ is said to be harmonic if it is a critical point of the energy functional

$$E_\Omega(f) = \int_\Omega |df|^2 \tag{2.5}$$

for every domain $\Omega \subset\subset M$. Here $|df|^2$, called the energy density of *f*, denotes the square of the Hilbert–Schmidt norm of the differential map $d_x f \in T_x^*M \otimes T_{f(x)}N$. If we consider *df* as a section of the bundle $T^*M \otimes f^{-1}TN$, $f^{-1}TN$ denoting the (Riemannian) pull-back bundle, then the Euler–Lagrange equations corresponding to (2.5) are

$$trace_{\langle\, ,\, \rangle} Ddf = 0, \tag{2.6}$$

where the symbol D stands for the covariant derivative of the $f^{-1}TN$-valued 1-form *df*.

Now, fix local o.n. frames $\{e_i\}_{i=1}^m$ of M and $\{E_A\}_{A=1}^n$ of N, denote by $\left\{\theta^i\right\}_i$ and $\left\{\Theta^A\right\}_A$ the corresponding o.n. co-frames and let $\left\{\theta^i_j\right\}_{i,j}$ and $\left\{\Theta^A_B\right\}_{A,B}$ be the associated connection forms. Then, up to pull-backs,

$$\begin{cases} df = f_i^A \theta^i \otimes E_A, \\ Ddf = f_{ij}^A \theta^i \otimes \theta^j \otimes E_A, \end{cases} \tag{2.7}$$

where

$$\begin{cases} f_{ij}^A = f_{ji}^A, \\ f_{ij}^A \theta^j = df_i^A - f_h^A \theta_i^h + f_i^B \Theta_B^A. \end{cases} \tag{2.8}$$

According to (2.6), f is a harmonic map if and only if

$$\sum_i f_{ii}^A = 0, \quad A = 1, \dots, n. \tag{2.9}$$

Theorem 2.3. *Let $(M, \langle\, , \rangle)$ be a complete, m-dimensional manifold whose Ricci tensor satisfies*

$$^M Ricci \geqslant -a(x), \quad on \;\; M \tag{2.10}$$

for some continuous function $a(x)$. Having fixed $H \geqslant 1$ consider the Schrödinger operator

$$\mathcal{L}_H = -\Delta - Ha(x)$$

and assume that

$$\lambda_1^{\mathcal{L}_H}(M) \geqslant 0. \tag{2.11}$$

Let $(N, (\, ,))$ be an n-dimensional manifold of non-positive sectional curvature

$$^N Sec \leqslant 0. \tag{2.12}$$

Then, any harmonic map $f : M \to N$ whose energy density $|df|^2 \in L^\gamma(M)$ for some $1 \leqslant \gamma \leqslant H$, is constant.

Remark 2.4. If $(M, \langle\, , \rangle)$ has a non-negative Ricci curvature, we can obviously choose $a(x) \equiv 0$ in (2.10) so that condition (2.11) is automatically satisfied.

Similarly, suppose that $(M^m, \langle\, , \rangle)$ is isometrically immersed in the Euclidean space $\mathbb{R}^{m+1}$ as a complete, stable, minimal hypersurface. Then, according to the Gauss equations, $^M Ricci \geqslant -|\mathrm{II}|^2$, where $|\mathrm{II}|$ denotes the length of the second fundamental tensor of the immersion. Moreover, the stability assumption implies that the differential operator $\mathcal{L} = -\Delta - |\mathrm{II}|^2$ satisfies (2.11).

These are the geometric situations considered in [SY1].

Proceeding as in [SY1] (see in particular the Corollary to Theorem 1), one deduces the following topological obstruction for a manifold to have a metric with a suitable control on the Ricci curvature.

Corollary 2.5. *Let $(M, \langle\, ,\, \rangle)$ be a complete, m-dimensional manifold whose Ricci tensor satisfies*

$$ {}^{M}Ricci \geqslant -a(x), \quad on \ M $$

and assume that

$$ \lambda_1^{-\Delta - a(x)}(M) \geqslant 0. $$

If D is any compact domain in M with a smooth, simply connected boundary, then there is no non-trivial homomorphism of $\pi_1(D)$ into the fundamental group of a compact manifold with a non-positive sectional curvature.

Before proving Theorem 2.3, we also point out the following consequence that should be compared with Proposition 0.1 in [RS1].

Corollary 2.6. *Let $(M, \langle\, ,\, \rangle)$ be a complete, simply connected Riemannian manifold of dimension $m \geqslant 3$. Assume that the sectional and Ricci curvatures of M satisfy*

$$ \begin{array}{ll} \text{(i)} & {}^{M}Sec \leqslant -A^2, \\ \text{(ii)} & {}^{M}Ricci \geqslant -(m-1)B^2 \end{array} \tag{2.13} $$

for some constants A, B such that

$$ 0 < A^2 \leqslant B^2 \leqslant \frac{(m-1)}{4} A^2. \tag{2.14} $$

Then there are no, non-constant, harmonic maps with finite energy from M into any Riemannian manifold of non-positive sectional curvature.

Proof. Set $a(x) = (m-1)B^2$, define $\mathcal{L} = -\Delta - a(x)$ and note that

$$ \lambda_1^{\mathcal{L}}(M) = \lambda_1^{-\Delta}(M) - (m-1)B^2. \tag{2.15} $$

Using a comparison estimate for the first Dirichlet eigenvalue due to McKean, see [McK], we see that

$$\lambda_1^{-\Delta}(M) \geqslant \lambda_1^{-\Delta}\left(\mathbb{H}^m_{-A^2}\right) = \frac{A^2(m-1)^2}{4}, \tag{2.16}$$

where $\mathbb{H}^m_{-A^2}$ denotes the m-dimensional space form of constant sectional curvature $-A^2$. Inserting (2.16) into (2.15) and recalling (2.14) we conclude that $\lambda_1^{\mathcal{L}}(M) \geqslant 0$. The result now follows from Theorem 2.3. $\square$

Proof of Theorem 2.3. Assume that f is a non-constant harmonic map. The Weitzenbock–Bochner formula for harmonic maps (see e.g. [EL,W]), (2.10) and (2.12) implies that

$$\Delta |df|^2 \geqslant 2|Ddf|^2 - 2a(x)|df|^2.$$

Therefore, the non-negative function $\psi = |df| \in Lip_{\mathrm{loc}}(M)$ satisfies

$$\psi\Delta\psi + a(x)\psi^2 \geqslant |Ddf|^2 - |\nabla\psi|^2 \tag{2.17}$$

pointwise on $\Omega = \{x \in M : \psi(x) \neq 0\}$ and weakly on all of M. We claim that

$$|Ddf|^2 - |\nabla\psi|^2 \geqslant \frac{1}{(m-1)}|\nabla\psi|^2. \tag{2.18}$$

Assume for the moment the validity of (2.18). Then, from (2.17) we deduce that

$$\psi\Delta\psi + a(x)\psi^2 \geqslant \frac{1}{(m-1)}|\nabla\psi|^2.$$

Moreover, the condition $\psi \in L^{2\gamma}(M)$ implies

$$\frac{1}{\int_{\partial B_r}\psi^{2\gamma}} \notin L^1(+\infty).$$

On the other hand, since $\lambda_1^{\mathcal{L}_H}(M) \geqslant 0$ there exists a positive function $\varphi \in C^1(M)$ satisfying

$$\Delta\varphi + Ha(x)\varphi = 0.$$

Applying Theorem 1.4, case (i), with the choices $A = -1/(m-1)$, $0 \leqslant \beta \leqslant \gamma - 1$ we conclude that ψ is a non-negative constant and $a(x) \equiv 0$. This clearly means that M

has non-negative Ricci curvature so that, in particular, $\mathrm{vol} M = +\infty$. The integrability condition of the constant ψ then forces ψ to be identically zero. Therefore, f is constant.

It remains to prove (the Kato-type) inequality (2.18). A version of such an inequality, with a slightly worse constant, is proved in [SY1]. We present a proof of the improved version for completeness and for the convenience of the reader. To this end, it suffices to consider the pointwise inequality on Ω. Let $\left\{f_i^A\right\}, \left\{f_{ij}^A\right\}$ be as in (2.7) and (2.8), i.e. the coefficients of the (local expressions of the) differential and of the Hessian of f. Then $\psi = \sqrt{\sum_{A,i} \left(f_i^A\right)^2}$ so that

$$\nabla\psi = \frac{\sum_i \left\{\sum_{A,j} f_{ij}^A f_j^A\right\} e_i}{\sqrt{\sum_{A,i} \left(f_i^A\right)^2}}$$

and we have

$$|Ddf|^2 - |\nabla\psi|^2 = \sum_{A,i,j} \left(f_{ij}^A\right)^2 - \frac{\sum_i \left\{\sum_{A,j} f_{ij}^A f_j^A\right\}^2}{\sum_{A,i} \left(f_i^A\right)^2}. \tag{2.19}$$

For $A = 1, \ldots, n$, define

$$M^A = \left(f_{ij}^A\right) \in M_m(\mathbb{R}), \quad y^A = \left(f_i^A\right)^t \in \mathbb{R}^m.$$

Then (2.19) reads

$$|Ddf|^2 - |\nabla\psi|^2 = \sum_A \left\|M^A\right\|^2 - \frac{\left|\sum_A M^A y^A\right|^2}{\sum_A \left|y^A\right|^2},$$

where $\|M\|^2 = trace\left(MM^t\right)$ and $|y|$ denotes the $\mathbb{R}^m$-norm of y. We have to show that

$$\sum_A \left\|M^A\right\|^2 - \frac{\left|\sum_A M^A y^A\right|^2}{\sum_A \left|y^A\right|^2} \geqslant \frac{1}{(m-1)} \frac{\left|\sum_A M^A y^A\right|^2}{\sum_A \left|y^A\right|^2}.$$

Since, by (2.8) and (2.9), each matrix M^A is traceless and symmetric this inequality is an immediate consequence of the next simple algebraic lemma. $\square$

Lemma 2.7. *For $A = 1, \dots, n$, let $M^A \in M_m(\mathbb{R})$ be a symmetric matrix satisfying trace $\left(M^A\right) = 0$. Then, for every $y^1, \dots, y^n \in \mathbb{R}^m$ with $\sum_A \left|y^A\right|^2 \neq 0$,*

$$\sum_A \left\|M^A\right\|^2 - \frac{\left|\sum_A M^A y^A\right|^2}{\sum_A \left|y^A\right|^2} \geqslant \frac{1}{(m-1)\,n} \frac{\left|\sum_A M^A y^A\right|^2}{\sum_A \left|y^A\right|^2}. \tag{2.20}$$

Proof. First, we consider the case $A = 1$. Let $\lambda_1 \leqslant \cdots \leqslant \lambda_s \leqslant 0 \leqslant \lambda_{s+1} \leqslant \cdots \leqslant \lambda_m$ be the eigenvalues of M. Without loss of generality we may assume that $\lambda_m \geqslant |\lambda_1|$. We are thus reduced to proving that

$$\sum_{i=1}^m \lambda_i^2 \geqslant \left(1 + \frac{1}{m-1}\right) \lambda_m^2.$$

To this end we note that, since M is traceless,

$$-\sum_{j=1}^{m-1} \lambda_j = \lambda_m$$

and therefore, from Schwarz inequality,

$$\lambda_m^2 \leqslant (m-1) \sum_{j=1}^{m-1} \lambda_j^2.$$

This implies

$$\sum_{i=1}^m \lambda_i^2 = \lambda_m^2 + \sum_{j=1}^{m-1} \lambda_j^2 \geqslant \left(1 + \frac{1}{m-1}\right) \lambda_m^2,$$

as desired.

Now we let A be any positive integer. We note that

$$\sum_A \left\|M^A\right\|^2 - \frac{\left|\sum_A M^A y^A\right|^2}{\sum_A \left|y^A\right|^2} \geqslant \sum_A \left\|M^A\right\|^2 - \frac{\left(\sum_A \left|M^A y^A\right|\right)^2}{\sum_A \left|y^A\right|^2}.$$

Applying the first part of the proof we obtain, for every $A = 1, \ldots, n$,

$$\left|M^A y^A\right| \leqslant \sqrt{\frac{m-1}{m}} \left\|M^A\right\| \left|y^A\right|$$

which in turn, used in the above, yields

$$\begin{aligned}
&\sum_A \left\|M^A\right\|^2 - \frac{\left|\sum_A M^A y^A\right|^2}{\sum_A \left|y^A\right|^2} \\
&\geqslant \sum_A \left\|M^A\right\|^2 - \frac{\left(\sum_A \sqrt{\frac{m-1}{m}} \left\|M^A\right\| \left|y^A\right|\right)^2}{\sum_A \left|y^A\right|^2} \\
&\geqslant \sum_A \left\|M^A\right\|^2 - \frac{m-1}{m} \frac{\sum_A \left\|M^A\right\|^2 \sum_A \left|y^A\right|^2}{\sum_A \left|y^A\right|^2} \\
&= \frac{1}{m} \sum_A \left\|M^A\right\|^2,
\end{aligned}$$

whence, rearranging and simplifying we obtain (2.20). □

We end this section with the following straightforward application of Theorem 1.1 to the problem of uniqueness of harmonic maps. To this aim, we recall that a ball $B_R(q)$ in a Riemannian manifold $(N, (\ ,\ ))$ is said to be regular if it does not intersect the cut locus of q, and, having denoted by $B \geqslant 0$ an upper bound for the sectional curvature of N on $B_R(q)$, one has $\sqrt{B}R < \pi/2$. Define the function

$$q_B(t) = \begin{cases} \frac{1}{2} t^2 & \text{if } B = 0, \\ \frac{1}{B}(1 - \cos\sqrt{B}t) & \text{if } B > 0. \end{cases}$$

Assume that $f, g : M \to B_R(q) \subset N$ are harmonic maps taking values in the regular ball $B_R(q)$ and define functions $\Phi, \psi, \varphi, u : M \to \mathbb{R}$ by setting

$$\Phi(x) = -\log(\cos(\sqrt{B}\mathrm{dist}_N(q, f(x))(\cos(\sqrt{B}\mathrm{dist}_N(q, g(x)))$$
$$\varphi(x) = e^{-\Phi(x)} \quad \text{and} \quad u = \varphi(x)^{-1} q_B(\mathrm{dist}_N(f(x), g(x))).$$

Clearly, $u \geqslant 0$ and, since f and g take values in the regular ball $B_R(q)$, there exists a constant $C \geqslant 1$ such that

$$C^{-1} \leqslant \varphi \leqslant 1 \quad \text{and} \quad C^{-1} \mathrm{dist}_N(f(x), g(x))^2 \leqslant u(x) \leqslant C \mathrm{dist}_N(f(x), g(x))^2 \tag{2.21}$$

on M. Further, a result of Jäger and Kaul [JK] shows that

$$\mathrm{div}\big(\varphi \nabla u\big) \geqslant 0 \quad \text{on } M,$$

and therefore (1.1) is "a fortiori" satisfied. With this preparation we have:

Theorem 2.8. *Maintaining the notation introduced above, let $f, g : M \to N$ be harmonic maps adopting values in the regular ball $B_R(q) \subset N$, and assume that, for some $p > 1$*

$$\mathrm{dist}_N\,(f, g)^{2p} \in L^1(M). \tag{2.22}$$

If $\mathrm{vol}\,(M) = +\infty$, *then* $f \equiv g$.

Proof. As noted above, the functions φ and u satisfy (1.1), and, according to (2.21), the integrability condition (2.22) implies that $\varphi u^p = \varphi^{1-p} q_B\big(\mathrm{dist}_N(f(x), g(x))\big)^p \in L^1(M)$, and therefore (1.2) holds. According to Theorem 1.1, it follows that u is constant, and there exists a constant $C_1 \geqslant 0$, such that

$$q_B\big(\mathrm{dist}_N(f(x), g(x))\big) = C_1 \varphi(x).$$

Since $\mathrm{vol}\, M = +\infty$ and φ is bounded away from zero, the integrability condition forces $C_1 = 0$ and therefore $\mathrm{dist}_N(f(x), g(x)) \equiv 0$, as required. $\square$

2.3. Topology at infinity of submanifolds of C–H spaces

In order to put the next geometric result into the appropriate perspective, we recall that the topology at infinity of a submanifold M^m of $\mathbb{R}^n$ is influenced and, in some cases determined, by the size of its second fundamental tensor II.

Given a compact set $K \subset M$, an end E of M corresponding to K is an unbounded, connected component of the set $M \backslash K$. Clearly, if we take two compact sets $K_1 \subset\subset K_2$, then the number of ends corresponding to K_1 is less than or equal to the number of ends corresponding to K_2. Hence, we say that M has finitely many ends if there exists a positive integer b such that, for any compact set $K \subset M$, the number of ends corresponding to K is bounded by b. In this case we can obviously find an integer $b_0 \leqslant b$ and a compact set K_0 in such a way that M has precisely b_0 ends for every compact set containing K_0. We say that b_0 is the number of ends of M.

In the setting of complete, minimal hypersurfaces of the Euclidean space, Tysk [Ty] has shown that the L^m-integrability of $|\mathrm{II}|$ forces the submanifold to possess only a finite number of ends (see the more recent [N] for a different proof and related results). If we also add the stability assumption, by a result of Shen and Zhu, [SZ], the immersion is totally geodesic. On the other hand, Cao et al. have shown in [CSZ] that stability alone implies that the hypersurface has a simple topology at infinity, i.e., it has only one end.

We note that a suitable control of the L^m size of $|\mathrm{II}|$ implies stability; see [S] and Lemma 2.12 below. According to an isolation phenomenon pointed out by Anderson, [A] and quantified by Berard [B2] if we allow the codimension of the minimal immersion to be greater than 1 and the L^m size of $|\mathrm{II}|$ is sufficiently small, then the submanifold is again an affine space; see also Theorem 2.14 below.

Finally, we know from a work of Ni [N] that if we relax the bound on $|\mathrm{II}|$ the minimal submanifold still has only one end.

Our main purpose is to extend both the results in [CSZ] and in [N] by showing that small perturbations of the minimal immersion (so that minimality is lost) do not modify the topology at infinity of the submanifold. In fact, we are able to quantify the amount of such perturbation and to replace the Euclidean ambient space with a Cartan–Hadamard manifold, i.e., a complete, simply connected Riemannian manifold of non-positive sectional curvature; see Theorems 2.10 and 2.13 below. We should remark that G. Carron has a similar result obtained in [C1], with a different method and a less precise condition on the second fundamental form of the immersion.

We recall that, according to Hoffman and Spruck [HS] if $f : (M, \langle\ ,\ \rangle) \to (N, (\ ,\ ))$ is an isometric immersion of a complete manifold M of dimension $m \geqslant 2$ into a Cartan–Hadamard manifold N, denoted by H the mean curvature vector field of f, then the following L^1-Sobolev inequality holds:

$$S_1(m)^{-1} \left(\int_M |u|^{\frac{m}{m-1}} \right)^{\frac{m-1}{m}} \leqslant \int_M (|\nabla u| + |H|\,|u|), \quad \forall u \in W_0^{1,1}(M) \tag{2.23}$$

with

$$S_1(m) = \frac{\pi 2^{m-1}}{\omega_m^{\frac{1}{m}}} \frac{(m+1)^{1+\frac{1}{m}}}{m-1} \tag{2.24}$$

ω_m being the volume of the unit ball of $\mathbb{R}^m$. In particular, if we assume that $H \in L^m(M)$, so that, for a suitable compact K,

$$\|H\|_{L^m(M \setminus K)} < S_1(m)^{-1},$$

then, applying Holder inequality, the term involving the mean curvature can be absorbed in the left-hand side, showing that the standard L^1 Sobolev inequality

$$C \left(\int |u|^{\frac{m}{m-1}} \right)^{\frac{m-1}{m}} \leqslant \int |\nabla u| \tag{2.25}$$

holds for every $u \in W_0^{1,1}(M)$ supported in $M \setminus K$. This in turn implies that for every geodesic ball $B_r(p)$ of $M \setminus K$

$$C\,(\mathrm{vol}\, B_r(p))^{\frac{m-1}{m}} \leqslant \mathrm{vol}\partial B_r(p),$$

for some constant $C > 0$ independent of p, whence, integrating,

$$\mathrm{vol}B_r(p) \geqslant Cr^m. \tag{2.26}$$

On the other hand, if $m \geqslant 3$, applying (2.25) to $u = |v|^{\frac{2(m-1)}{m-2}}$ with $v \in C_c^\infty(M \setminus K)$, one deduces, after some manipulations, that the L^2 Sobolev inequality

$$C\left(\int |v|^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leqslant \left(\int |\nabla v|^2\right)^{\frac{1}{2}} \tag{2.27}$$

holds in $M \setminus K$. By Proposition 2.5 in [C2], (2.27) holds on the whole of M (with a different, and non-explicitly computable constant C).

According to a nice argument of Cao, Shen and Zhu, see Lemma 2 in [CSZ], the validity of (2.27) together with a "uniform" volume growth condition (like that expressed in (2.26)) can be used to relate the number of ends of M to the presence of non-constant, bounded harmonic functions with finite energy.

Lemma 2.9. *Let $f : (M, \langle\, , \rangle) \to (N, (\, ,))$ be an isometric immersion of a complete manifold M of dimension $m \geqslant 3$ into a Cartan–Hadamard manifold N. Denote by H the mean curvature vector field of f and assume that $H \in L^m(M)$. Then, each end of M has infinite volume. Moreover, in case there are at least two ends, then M supports a non-constant, bounded, harmonic function with finite energy.*

We are now in a position to prove our first result on the topology at infinity of immersed submanifold of a Cartan–Hadamard space with controlled extrinsic geometry, which generalizes [CSZ], Theorem 1. We note that assumption (2.30) below is the counterpart of the stability condition assumed there.

Theorem 2.10. *Let $f : (M, \langle\, , \rangle) \to (N, (\, ,))$ be an isometric immersion of a complete manifold M of dimension $m \geqslant 3$ into a Cartan–Hadamard manifold N whose sectional curvature (along f) satisfies*

$$(0 \geqslant)\, {}^N Sec_{f(x)} \geqslant -\ {}^N R(x) \tag{2.28}$$

for some non-negative function ${}^N R \in C^0(M)$. Denote by H and II the mean curvature vector field and the second fundamental tensor of f, respectively, and let $a(x) \in C^0(M)$

be the function defined by

$$a\,(x) = (m-1)\ \ {}^{N}R\,(x) + |\mathrm{II}|\,(|\mathrm{II}| + m\,|H|)\,(x)\,. \tag{2.29}$$

If $H \in L^m(M)$ *and*

$$\lambda_1^{-\Delta - a(x)}(M) \geqslant 0, \tag{2.30}$$

then either f *is totally geodesic or* M *has only one end.*

Proof. Assume that f is not totally geodesic. Then $a(x)$ does not vanish identically.

According to Lemma 2.9 we have to show that M does not support any non-constant, bounded, harmonic function with a finite Dirichlet integral. But this follows immediately from Theorem 2.3 above. Indeed, the Gauss equations imply that the Ricci tensor of M satisfies

$$^{M}Ricci\,(x) \geqslant -\,a\,(x)\,. \qquad \square$$

Our next task is to quantify the heuristic idea according to which the bottom of the spectrum of $-\Delta - a(x)$ is non-negative provided the norm of the function $a(x)$ in (2.29) is small.

It is well known that if an L^2 Sobolev inequality holds on M, then $\lambda_1^{-\Delta - a(x)}(M) \geqslant 0$ provided a suitable L^p-norm of a is strictly less than the Sobolev constant (see, e.g., [S]).

In the next lemma, we obtain the same conclusion in terms of an L^2-Sobolev inequality with a potential like (2.31) below.

Lemma 2.11. *Suppose that the following Sobolev-type inequality*:

$$S(\alpha)^{-1}\left(\int_M v^{\frac{2}{1-\alpha}}\right)^{1-\alpha} \leqslant \int_M \left(|\nabla v|^2 + h(x)v^2\right), \quad \forall v \in C_c^{\infty}\,(M) \tag{2.31}$$

holds on M, *where* $0 < \alpha < 1$, $S\,(\alpha) > 0$ *is a constant*, *and* $h\,(x) \in C^0\,(M)$ *is a non-negative function. Consider the Schrodinger operator*

$$\mathcal{L} = -\Delta - a(x)$$

with $a(x) \in C^0\,(M)$. *Set* $a_+\,(x) = \max\{a(x), 0\}$ *and assume that*

$$\|h\,(x) + a_+\,(x)\|_{L^{\frac{1}{\alpha}}(M)} \leqslant S\,(\alpha)^{-1}\,. \tag{2.32}$$

Then

$$\lambda_1^{\mathcal{L}}(M) \geqslant 0.$$

Proof. We let $\tilde{\mathcal{L}} = \Delta + a_+(x)$ and we note that, for any domain $\Omega \subset\subset M$, $\lambda_1^{\mathcal{L}}(\Omega) \geqslant \lambda_1^{\tilde{\mathcal{L}}}(\Omega)$. Therefore we can limit ourselves to proving that

$$\lambda_1^{\tilde{\mathcal{L}}}(\Omega) \geqslant 0, \quad \forall \Omega \subset\subset M.$$

By contradiction, suppose that, for some $\Omega \subset\subset M$,

$$\lambda_1^{\tilde{\mathcal{L}}}(\Omega) < 0.$$

Then we can find $v \in C_c^\infty(\Omega) \setminus \{0\}$ such that

$$\int_M \left(|\nabla v|^2 - a_+(x)\, v^2\right) < 0.$$

Using (2.31) in this latter and applying Hölder inequality, we obtain

$$S(\alpha)^{-1}\left(\int_M v^{\frac{2}{1-\alpha}}\right)^{1-\alpha} < \left(\int_M (a_+(x) + h(x))^{\frac{1}{\alpha}}\right)^{\alpha}\left(\int_M v^{\frac{2}{1-\alpha}}\right)^{1-\alpha}.$$

This contradicts (2.32). $\square$

There are a number of geometric situations where the Sobolev inequality (2.31) is satisfied for some choices of α, $S(\alpha)$, $h(x)$. The interested reader can consult e.g. [He].

Assume now that (M, g) is a submanifold of a Cartan–Hadamard manifold, so that the L^1-type Sobolev inequality (2.23) holds. As above, we apply this inequality to the function $u = |v|^{\frac{2(m-1)}{m-2}}$ with $v \in C_c^\infty(M)$ to obtain

$$\begin{aligned} S_1(m)^{-2}&\left(\int_M |v|^{\frac{2(m-)}{m-2}}\right)^{\frac{m-2}{m}} \\ &\leqslant \left\{\frac{2(m-1)}{m-2}\left(\int_M |\nabla v|^2\right)^{\frac{1}{2}} + \left(\int_M |H|^2\,|v|^2\right)^{\frac{1}{2}}\right\}^2. \end{aligned} \tag{2.33}$$

Expanding the square on the right-hand side and applying the inequality $2ab \leqslant \varepsilon^2 a^2 + \varepsilon^{-2} b^2$ with $\varepsilon > 0$, we finally obtain the L^2-Sobolev inequality

$$S_2(m,\varepsilon)^{-1}\left(\int_M |v|^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}} \leqslant \int_M |\nabla v|^2 + \varepsilon^2 \int_M |H|^2\, v^2, \tag{2.34}$$

where

$$S_2(m,\varepsilon) = \left(\frac{2\sqrt{1+\varepsilon^{-2}}\,(m-1)}{(m-2)}\right)^2 S_1(m)^2 . \tag{2.35}$$

We observe that in case f is a minimal immersion, the best L^2-Sobolev constant in (2.35) is achieved by choosing $\varepsilon = +\infty$. In this situation, we set

$$S_2(m) = S_2(m, +\infty).$$

In particular, from Lemma 2.11 we immediately conclude

Lemma 2.12. *Let $f : (M, \langle\, ,\, \rangle) \to (N, (\, ,\,))$ be an isometric immersion of a complete manifold M of dimension $m \geqslant 3$ into a Cartan–Hadamard manifold N. Denote by H the mean curvature vector field of f. Consider the Schrodinger operator*

$$\mathcal{L} = -\Delta - a(x) \tag{2.36}$$

with $a \in C^0(M)$. If, for some $\varepsilon > 0$,

$$\left\| \varepsilon^2 |H|^2 + a(x) \right\|_{L^{\frac{m}{2}}} \leqslant S_2(m,\varepsilon)^{-1} \tag{2.37}$$

then

$$\lambda_1^{\mathcal{L}}(M) \geqslant 0.$$

Lemma 2.12 allows to obtain a version of Theorem 2.10 above, where the assumption on the bottom of the spectrum of $-\Delta - a(x)$ is replaced by a suitable control on the second fundamental tensor of the immersion. We note that our result extends the above-mentioned result of Ni [N] valid for minimal submanifold of $\mathbb{R}^n$, and yields a qualitative improvement on the result of Carron [C1].

We also note that since $S_2(m,\varepsilon) > \varepsilon^2 S_1(m)$, assumption (2.38) below implies that $\|H\|_{L^m} < S_1(m)^{-1}$. Thus in this case, the standard L^1-Sobolev inequality (2.25) holds on M and we may conclude that L^2-Sobolev inequality (2.27) is valid on M, without having to appeal to Proposition 2.5 in [C2].

Theorem 2.13. *Maintaining the notation of Theorem* 2.10, *assume that the sectional curvature* (*along* f) *of N satisfies* (2.28) *and that*

$$\left\| \varepsilon^2 |H|^2 + (m-1)\ {}^N R(x) + |\mathrm{II}|\left(|\mathrm{II}| + m|H|\right)(x) \right\|_{L^{\frac{m}{2}}} \leqslant S_2(m,\varepsilon)^{-1} \tag{2.38}$$

for some $\varepsilon > 0$, *where* $S_2(m,\varepsilon)$ *is the* L^2*-Sobolev constant defined in* (2.35). *Then either* f *is totally geodesic or* M *has only one end.*

Finally, combining Lemma 2.12 with some careful computations of Berard [B2] we obtain the following isolation phenomena for minimal submanifolds of the Euclidean space. Note that our constant improves on that of Proposition II.1 in [B2].

Theorem 2.14. *Let* $f : (M, \langle\, , \rangle) \to \mathbb{R}^n$ *be a complete, minimal, immersed submanifold of dimension* $m \geqslant 3$ *whose second fundamental tensor* II *satisfies*

$$\left(\int_M |\mathrm{II}|^m\right)^{\frac{2}{m}} \leqslant \frac{2}{m\left(2-\frac{1}{n-m}\right)} S_2(m)^{-1}. \tag{2.39}$$

Then f *is totally geodesic.*

Proof. From Proposition I.2 of [B2] we know that the function $\psi = |\mathrm{II}|$ is a (weak) solution of

$$\psi\Delta\psi + \left(2-\frac{1}{n-m}\right)|\mathrm{II}|^2\psi^2 \geqslant \frac{2}{(m+2)(n-m)-2}|\nabla\psi|^2.$$

Moreover, by (2.39), $\psi \in L^m(M)$ so that

$$\frac{1}{\int_{\partial B_r}\psi^m} \notin L^1(+\infty).$$

We define the differential operator

$$\mathcal{L} = -\Delta - \frac{m}{2}\left(2-\frac{1}{n-m}\right)|\mathrm{II}|^2.$$

and we note that, according to (2.39), we can apply Lemma 2.12 to obtain $\lambda_1^{\mathcal{L}}(M) \geqslant 0$. This means that there exists a positive solution $\varphi \in C^\infty(M)$ of the equation

$$\Delta\varphi + \frac{m}{2}\left(2-\frac{1}{n-m}\right)|\mathrm{II}|^2\varphi = 0.$$

Applying Theorem 1.4, case 1, with the choices $a(x) = \left(2-\frac{1}{n-m}\right)|\mathrm{II}|^2$, $H = \frac{m}{2}$, $\beta = \frac{m}{2}-1$, $A = -\frac{2}{(m+2)(n-m)-2}$ we therefore conclude that ψ is constant and $a(x) \equiv 0$, i.e., $|\mathrm{II}| \equiv 0$. $\square$

2.4. Topology of locally conformally flat manifolds

Our last geometric result relies on the topology of compact, locally conformally flat manifolds; see Corollary 2.20 below.

We recall that a Riemannian manifold $(M, \langle\, ,\, \rangle)$ is said to be locally conformally flat if M is covered by a family $\{(U_\alpha, \xi_\alpha)\}$ of smooth charts with the property that, on each coordinate domain U_α, the metric $\left(\xi_\alpha^{-1}\right)^* \langle\, ,\, \rangle$ is a pointwise conformal deformation of the canonical metric of $\mathbb{R}^m$. When the dimension $m \geqslant 4$ this is equivalent to the fact that the Weyl tensor of (M, g) vanishes identically.

We also need to recall the concept of L^2-Betti numbers. Let $L^p H^q(M)$ denote the space of p-integrable, harmonic q-forms on M. By the Hodge–de Rham–Kodaira representation theorem, $L^2 H^q(M)$ is isomorphic to the qth group of reduced L^2-cohomology of M. In the case where $\pi : M \to N = M/\Gamma$ is the Riemannian universal covering of a compact manifold N with deck transformation group Γ then $L^2 H^q(M)$ becomes a Γ-module. Its Von Neumann dimension $L^2 b^q(N) = \dim_\Gamma L^2 H^q(M)$ is finite for every q and is called the qth L^2 Betti number of the base manifold N. It is known from the fundamental work of Dodziuk [D] that $L^2 b^q(N)$ is a homotopy invariant of N.

We begin generalizing to the non-compact case a well-known Theorem of Bourguignon [Bo]. For a generalization in a different direction we refer the reader to [PRS2].

Theorem 2.15. *Let $(M, \langle\, ,\, \rangle)$ be a complete, non-compact, locally conformally flat Riemannian manifold of dimension $m = 2k \geqslant 4$ with scalar curvature $S(x)$. Given $H \geqslant 1$, assume that the differential operator*

$$\mathcal{L}_H = -\Delta + H \frac{k!k}{2(2k-1)} S(x)$$

satisfies

$$\lambda_1^{\mathcal{L}_H}(M) \geqslant 0.$$

If $L^p H^k(M) \neq \{0\}$ for some $2 \leqslant p \leqslant 2H$, then

(a) $S(x) \equiv 0$

(b) $\mathrm{vol} M < +\infty$

so that, in particular,

(c) *M cannot be conformally immersed into the standard sphere $\mathbb{S}^m$.*

Proof. Fix any $0 \not\equiv \omega \in L^p H^k(M) \neq \{0\}$. Since M is locally conformally flat, the Weyl tensor vanishes and the Weitzenbock formula reads

$$\Delta |\omega|^2 - \frac{k!k}{(2k-1)} S(x) |\omega|^2 = 2 |D\omega|^2 \quad \text{pointwise on } M. \tag{2.40}$$

We recall that from the first Kato inequality,

$$|D\omega|^2 - |\nabla |\omega||^2 \geqslant 0 \tag{2.41}$$

pointwise on the open set $\Omega = \{x \in M : \omega(x) \neq 0\}$ and weakly on all of M. Moreover, the equality sign holds if and only if

$$\text{(i) } \omega = |\omega|\, \omega_1 \quad \text{on} \quad \Omega \quad \text{where} \quad \text{(ii) } D\omega_1 = 0, \tag{2.42}$$

and, by the unique continuation property of harmonic forms,

$$\bar{\Omega} = M.$$

From (2.40) and (2.41) we deduce that $\psi = |\omega|$ satisfies

$$\psi\Delta\psi - \frac{k!k}{2(2k-1)} S(x)\psi^2 \geqslant 0 \tag{2.43}$$

pointwise on Ω and weakly on M. Since $\lambda_1^{\mathcal{L}_H}(M) \geqslant 0$ and $\psi \in L^p(M)$, we can apply Theorem 1.4, with $A = 0$, and $\beta = p/2 - 1$, to conclude that if $H > 1$, then, by case (i), ψ is a non-zero constant, and therefore $S(x) \equiv 0$, and since ψ is in L^p, $\operatorname{vol} M < +\infty$.

On the other hand, if $H = 1$, then, by case (ii) in Theorem 1.4, the equality sign holds in (2.43), and therefore also in (2.41). Thus (2.42) is satisfied, and inserting the parallel form ω_1 into (2.40) we again conclude that $S(x) = 0$ on Ω and by continuity, $S(x) \equiv 0$ on M. Further, again by (2.40),

$$\Delta |\omega|^2 = 2\, |D\omega|^2 . \tag{2.44}$$

Since $|\omega| \in L^2(M)$, by a standard cut-off argument, we can integrate (2.44) to obtain

$$\int_M |D\omega|^2 = 0$$

which in turn forces ω to be a parallel form on M. Using again the integrability condition we therefore conclude that $\operatorname{vol} M < +\infty$.

That M cannot be conformally immersed into $\mathbb{S}^m$ now follows readily. Indeed let

$$Q(M) = \inf \left\{ \frac{\int_M |\nabla v|^2 + \frac{(m-2)}{4(m-1)} S(x) v^2}{\left(\int_M v^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}}} : v \in C_0^\infty(M) \setminus \{0\} \right\}$$

be the Yamabe invariant of M. It is known (see, e.g., [SY2, Theorem 2.2, Chapter VI]) that $Q(M) = Q(\mathbb{S}^m) > 0$ provided M has a conformal immersion into the standard sphere. However, since $S(x) \equiv 0$ and $(M, \langle\, ,\, \rangle)$ is a complete, non-compact manifold of finite volume the latter vanishes. To see this, choose a family $\{v_R\}_{R>0}$ of smooth, cut-off functions satisfying $v_R \equiv 1$ on B_R, $v_R \equiv 0$ on $M \backslash B_{2R}$ and $|\nabla v_R| \leqslant 4/R$. Then, for each $R > 0$,

$$Q(M) \leqslant \frac{\int_M |\nabla v_R|^2}{\left(\int_M v_R^{\frac{2m}{m-2}}\right)^{\frac{m-2}{m}}} \leqslant \frac{16 \mathrm{vol} M}{R^2 (\mathrm{vol} B_R)^{\frac{m-2}{2}}}. \qquad \square$$

Remark 2.16. The result is still true when $(M, \langle\, ,\, \rangle)$ is compact and without boundary. Obviously, in this case, $L^2 H^k(M)$ is the whole space of harmonic k-forms which, in turn, is isomorphic to the ordinary k-co-homology group of M.

Remark 2.17. The sign of the first eigenvalue of the differential operator $\mathcal{L}_H$ is not a conformal invariant. Note that we cannot replace $\mathcal{L}_H$ with the conformal Laplacian as the case of standard Hyperbolic space shows.

Remark 2.18. During the proof of Theorem 2.15 we observed that an m-dimensional manifold $(M, \langle\, ,\, \rangle)$ is conformally immersed into the standard sphere, then $Q(M) = Q(\mathbb{S}^m) > 0$. In particular, if we assume that the scalar curvature $S(x)$ of M is non-positive, we have the validity of the L^2-Sobolev inequality

$$Q(\mathbb{S}^m) \left(\int_M v^{\frac{2m}{m-2}} \right)^{\frac{m-2}{m}} \leqslant \int_M |\nabla v|^2 \quad \forall v \in C_c^\infty(M) \setminus \{0\}.$$

Combining this fact with Lemma 2.11 and Theorem 2.15 we therefore conclude that, for such a manifold with $m = 2k$, $L^2 H^k(M) = 0$ provided $\left\| \frac{k!k}{(2k-1)} S(x) \right\|_{L^k(M)} \leqslant Q(\mathbb{S}^m)$. Since we know from a theorem of N. Kuiper (see e.g. [SY2, Theorem 1.2, Chapter VI]) that every simply connected, locally conformally flat manifold can be immersed into the standard sphere we have proved the following vanishing result:

Proposition 2.19. *Let $(M, \langle\, ,\, \rangle)$ be a complete, simply connected, locally conformally flat manifold of dimension $m = 2k \geqslant 4$. Assume that the scalar curvature $S(x)$ of M satisfies*

$$\text{(i)}\ S(x) \leqslant 0; \quad \text{(ii)}\ \|S(x)\|_{L^k(M)} \leqslant \frac{(2k-1)}{k!k} Q(\mathbb{S}^m).$$

Then $L^2 H^k(M) = 0$.

We are now in a position to prove our topological result that should be considered as a new version of the theorem by Bourguignon alluded to above.

Corollary 2.20. *Let $(M, \langle\, ,\, \rangle)$ be a compact, locally conformally flat Riemannian manifold of dimension $m = 2k$ and scalar curvature $S(x)$. Assume that the differential operator*

$$\mathcal{L} = -\Delta + \frac{k!k}{2\,(2k-1)} S(x) \tag{2.45}$$

satisfies

$$\lambda_1^{\mathcal{L}}(M) \geqslant 0. \tag{2.46}$$

Then $L^2 b^k(M) = 0$.

Proof. Consider the Riemannian universal covering $\pi : (\tilde{M}, \widetilde{\langle\, ,\, \rangle}) \to (M, \langle\, ,\, \rangle)$, and denote with a tilde the geometric objects defined on $\tilde{M}$. Then $(\tilde{M}, \widetilde{\langle\, ,\, \rangle})$ is a complete, simply connected, locally conformally flat manifold of scalar curvature $\tilde{S}(y) = S \circ \pi(y)$. By the above-mentioned theorem of Kuiper, $\tilde{M}$ admits a conformal immersion into the standard sphere $\mathbb{S}^m$, and, as noted after Corollary 2.2, the operator $\tilde{\mathcal{L}} = -\tilde{\Delta} + \frac{k!k}{2(2k-1)}\tilde{S}(x)$, satisfies $\lambda_1^{\tilde{\mathcal{L}}}(\tilde{M}) \geqslant 0$. Indeed, any positive solution φ of $\mathcal{L}\varphi \geqslant 0$ on M lifts to a positive solution $\tilde{\varphi} = \varphi \circ \pi$ of $\tilde{L}\tilde{\varphi} \geqslant 0$ on $\tilde{M}$. Applying Theorem 2.15 we conclude that $L^2 H^k(\tilde{M}) = 0$ whence $L^2 b^k(M) = 0$. $\square$

Remark 2.21. It should be pointed out that condition (2.46) does not imply any vanishing result for the ordinary Betti numbers of M. Consider for instance the $2k$-dimensional, flat torus.

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