# UNIQUE CONTINUATION AND CLASSIFICATION OF BLOW-UP PROFILES FOR ELLIPTIC SYSTEMS WITH NEUMANN BOUNDARY COUPLING AND APPLICATIONS TO HIGHER ORDER FRACTIONAL EQUATIONS

VERONICA FELLI AND ALBERTO FERRERO

Abstract. In this paper we develop a monotonicity formula for elliptic systems with Neumann boundary coupling, proving unique continuation and classification of blow-up profiles. As an application, we obtain strong unique continuation for some fourth order equations and higher order fractional problems.

## 1. Introduction and statement of the main results

The present paper is devoted to the study of unique continuation from a boundary point and classification of blow-up profiles for elliptic systems with Neumann boundary coupling. Systems of such a kind arise from higher order extensions of the fractional Laplacian, as first observed in [25], where the well known Caffarelli-Silvestre extension procedure characterizing the fractional Laplacian as the Dirichlet-to-Neumann map in one extra spatial dimension was generalized to higher powers of the Laplacian. More precisely in [25] (see also [7]) it is proved that, if $s \in (1,2)$ and $u \in H^s(\mathbb{R}^N)$, then

$$(-\Delta)^s u = K_s \lim_{t\to 0^+} t^b \frac{\partial(\Delta_b U)}{\partial t} \tag{1}$$

where $b = 3-2s$, $K_s$ is a constant depending only on $s$, $\Delta_b U = \Delta U + \frac{b}{t}\frac{\partial U}{\partial t}$ and $U$ is the unique solution to the problem

$$\begin{cases} \Delta_b^2 U = 0, & \text{in } \mathbb{R}^{N+1}_+ = \mathbb{R}^N \times (0,+\infty),\\ U(x,0) = u(x), & \text{in } \mathbb{R}^N,\\ \frac{\partial U}{\partial \nu^b} = 0, & \text{in } \mathbb{R}^N. \end{cases}$$

where $\frac{\partial U}{\partial \nu^b} := -\lim_{t\to 0^+} t^b \frac{\partial U}{\partial t}$ denotes the conormal exterior derivative.

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Setting $V=\Delta_b U$ and taking into account (1), the above fourth order problem can be rewritten as the system

$$\begin{cases} \Delta_b U = V, & \text{in } \mathbb{R}^{N+1}_+,\\ \Delta_b V = 0, & \text{in } \mathbb{R}^{N+1}_+,\\ U(x,0)=u(x), & \text{in } \mathbb{R}^N,\\ \frac{\partial U}{\partial \nu^b}=0, & \text{in } \mathbb{R}^N,\\ -K_s\frac{\partial V}{\partial \nu^b}=(-\Delta)^s u, & \text{in } \mathbb{R}^N. \end{cases}$$

For further references on higher order fractional powers of the Laplacian see [6, 8, 9].

In [25] an Almgren's frequency formula in the spirit of [3] is derived for solutions to the higher order system

$$\begin{cases} \Delta_b U = V, & \text{in } \mathbb{R}^{N+1}_+,\\ \Delta_b V = 0, & \text{in } \mathbb{R}^{N+1}_+,\\ \frac{\partial U}{\partial \nu^b}=0, & \text{in } \mathbb{R}^N,\\ \frac{\partial V}{\partial \nu^b}=0, & \text{in } \mathbb{R}^N, \end{cases} \tag{2}$$

obtained by extending $s$-harmonic functions; in the spirit of Garofalo and Lin [16], such monotonicity formula allows proving a unique continuation property for solutions to system (2). In [25] a strong unique continuation property is also stated for $s$-harmonic functions.

The main goal of the present paper is to extend, in the case $s=\frac{3}{2}$, the monotonicity formula developed in [25] for the homogeneous case (2) to systems with a Neumann boundary coupling of the type

$$\begin{cases} \Delta U = V, & \text{in } \mathbb{R}^{N+1}_+,\\ \Delta V = 0, & \text{in } \mathbb{R}^{N+1}_+,\\ \frac{\partial U}{\partial \nu}=0, & \text{in } \mathbb{R}^N,\\ \frac{\partial V}{\partial \nu}+2a(x)\,\mathrm{Tr}\,U=0, & \text{in } \mathbb{R}^N, \end{cases} \tag{3}$$

which arise naturally as extension of fractional equations of the form

$$(-\Delta)^{3/2}u=a(x)u$$

once we put $\mathrm{Tr}\,U(x):=U(x,0)$ in (3) for any $x\in\mathbb{R}^N$. Indeed, by [15, Proof of Lemma 3.2, Step 6] we deduce that the constant $C_b$ defined there equals $\sqrt{2}$ when $b=0$ and, since it can be shown that $K_s=C_b^{-2}$ with $b=3-2s$, we deduce that $K_{3/2}=\frac{1}{2}$.

For a closer look at the operator $(-\Delta)^{3/2}$ and its geometric interpretation see [9].

We point out that the proof of a monotonicity formula in the case of a fractional equation of the type

$$(-\Delta)^s u = a(x)u \tag{4}$$

with $s\in(1,2)$ is not straightforward. A first contribution in this direction was given in [15] for equation (4) with $a\equiv 0$ and $s\in(1,2)$. The present paper represents a first step in the study of (4) with a nonzero potential $a$; the difficulties due to the singularity/degeneracy of the operator $\Delta_b$, $b=3-2s$, allow us to treat for the moment just the case $s=\frac{3}{2}$ in which $b=0$ and the extension operator $\Delta_b$ reduces to the standard Laplacian. The general case with $a \not\equiv 0$ and $s \neq \frac{3}{2}$ presents additional technical difficulties and this is the object of current investigation.

In the proofs of our main results we exploit the validity of an Almgren's type monotonicity formula applied to a suitable frequency function $\mathcal{N}$ associated to problem (3). The precise definition of $\mathcal{N}$ can be found in (40).

The classical approach developed by Garofalo and Lin [16] to prove unique continuation through Almgren's monotonicity formula is based on the validity of doubling type conditions, obtained as a consequence of boundedness of the quotient $\mathcal{N}$. We refer to [1, 2, 12, 14, 17, 23, 24] for unique continuation from the boundary established via Almgren monotonicity formula.

While in the local case doubling conditions are enough to establish unique continuation, in the fractional case they provide unique continuation only for the extended local problem and not for the fractional one. Such difficulty was overcome in [11] for the fractional Laplacian $(-\Delta)^s$ with $s \in (0,1)$, by a fine blow-up analysis and a precise classification of the possible blow-up limit profiles in terms of a Neumann eigenvalue problem on the half-sphere.

The problem of unique continuation for fractional laplacians with power $s \in (0,1)$ was also studied in [18] in presence of rough potentials using Carleman estimates and in [26] for fractional operators with variable coefficients using an Almgren type monotonicity formula. As far as higher fractional powers of the laplacian, the main contribution to the problem of unique continuation is due to Seo in papers [20, 21, 22], through Carleman inequalities; in particular papers [20, 21, 22] consider fractional Schrödinger operators with potentials in Morrey spaces and prove a *weak* unique continuation result, i.e. vanishing of solutions which are zero on an open set; we recall that the *strong* unique continuation property instead requires the weaker assumption of infinite vanishing order at some point.

We observe that the presence of a coupling Neumann term in system (3) produces substancial additional difficulties with respect to the extension problem corresponding to the lower order fractional case $s \in (0,1)$ and consisting in a single equation associated with a Neumann boundary condition. In particular the proof of a monotonicity formula for (3) is made quite delicate by the appearance in the derivative of the frequency $\mathcal{N}$ of a term of the type

$$-r \int_{\partial B'_r} auv\, dS' + 2 \int_{B'_r} au\, x \cdot \nabla_x v\, dx, \tag{5}$$

see Lemma 2.11. Throughout the paper we use the notation

$$\begin{aligned} &B_r = \{z \in \mathbb{R}^{N+1} : |z| < r\}, \quad B_r^+ = \{(x,t) \in B_r : t > 0\}, \\ &B'_r = \{x \in \mathbb{R}^N : |x| < r\} = B_r \cap (\mathbb{R}^N \times \{(x,0) : x \in \mathbb{R}^N\}), \\ &S_r^+ = \{(x,t) \in \partial B_r : t > 0\}. \end{aligned} \tag{6}$$

While in the lower order case we have only one component $u = v$ so that an integration by parts allows rewriting the above sum as an integral over $B'_r$, in the case of two components $u, v$ this is no more possible and an estimate of the integral over "the boundary of the boundary" $\int_{\partial B'_r} auv\, dS'$ is required. The method developed here to overcome this difficulty is based on estimates in terms of boundary integrals (see Lemma 2.12) and represents one of the main technical novelty of the present paper in the context of monotonicity formulas; we think that this procedure could have future applications in the extension of some of the results of [11] to rough potentials, since it could avoid the integration by parts needed to write the above sum as an integral over $B'_r$, which requires differentiability of the potential $h$.

Let $N \geqslant 2$, $R > 0$, and $(U, V) \in H^1(B_R^+) \times H^1(B_R^+)$ be a weak solution to the system

$$
\begin{cases}
\Delta U = V, & \text{in } B_R^+, \\
\Delta V = 0, & \text{in } B_R^+, \\
\frac{\partial U}{\partial \nu} = 0, & \text{in } B'_R, \\
\frac{\partial V}{\partial \nu} = hu, & \text{in } B'_R,
\end{cases}
\tag{7}
$$

where $u = \operatorname{Tr} U$ (trace of $U$ on $B'_R$) and $h \in C^1(B'_R)$. We also denote $v = \operatorname{Tr} V$ (trace of $V$ on $B'_R$). By a weak solution to the system (7) we mean a couple $(U, V) \in H^1(B_R^+) \times H^1(B_R^+)$ such that, for every $\varphi \in H^1(B_R^+)$ having zero trace on $S_R^+$,

$$
\begin{cases}
\int_{B_R^+} \nabla U(z) \cdot \nabla \varphi(z) \, dz = -\int_{B_R^+} V(z) \varphi(z) \, dz, \\
\int_{B_R^+} \nabla V(z) \cdot \nabla \varphi(z) \, dz = \int_{B'_R} h(x) u(x) \operatorname{Tr} \varphi(x) \, dx,
\end{cases}
$$

where $\operatorname{Tr} \varphi$ is the trace of $\varphi$ on $B'_R$.

Our first result is an asymptotic expansion of nontrivial solutions to (7); more precisely we prove that blow-up profiles can be described as combinations of spherical harmonics symmetric with respect to the equator $t = 0$.

Let $-\Delta_{\mathbb{S}^N}$ denote the Laplace Beltrami operator on the $N$-dimensional unit sphere $\mathbb{S}^N$. It is well known that the eigenvalues of $-\Delta_{\mathbb{S}^N}$ are given by

$$
\lambda_\ell = (N - 1 + \ell)\ell, \quad \ell = 0, 1, 2, \dots .
$$

For every $\ell \in \mathbb{N}$, it is easy to verify that there exists a spherical harmonic on $\mathbb{S}^N$ of degree $\ell$ which is symmetric with respect to the equator $t = 0$[1]. Therefore the eigenvalues of the problem

$$
\begin{cases}
-\Delta_{\mathbb{S}^N} \psi = \lambda \, \psi, & \text{in } \mathbb{S}^N_+, \\[1ex]
\nabla_{\mathbb{S}^N} \psi \cdot \mathbf{e} = 0, & \text{on } \partial \mathbb{S}^N_+,
\end{cases}
\tag{8}
$$

with

$$
\mathbb{S}^N_+ = \{(\theta_1, \theta_2, \dots, \theta_{N+1}) \in \mathbb{S}^N : \theta_{N+1} > 0\}, \quad \mathbf{e} = (0, 0, \dots, 0, 1),
$$

are given by the sequence $\{\lambda_\ell : \ell = 0, 1, 2, \dots\}$; for every $\ell$, $\lambda_\ell$ has finite multiplicity $M_\ell$ as an eigenvalue of (8). For every $\ell \geqslant 0$, let $\{Y_{\ell,m}\}_{m=1,2,\dots,M_\ell}$ be a $L^2(\mathbb{S}^N_+)$-orthonormal basis of the eigenspace of (8) associated to $\lambda_\ell$ with $Y_{\ell,m}$ being spherical harmonics of degree $\ell$.

We note that, if $\Psi$ is an eigenfunction of (8), then $\Psi \not\equiv 0$ on $\partial \mathbb{S}^N_+ = \mathbb{S}^{N-1}$; indeed, by unique continuation, $\Psi$ and $\nabla_{\mathbb{S}^N} \Psi \cdot \mathbf{e}$ can not both vanish on $\partial \mathbb{S}^N_+$. In particular $Y_{\ell,m} \not\equiv 0$ on $\partial \mathbb{S}^N_+ = \mathbb{S}^{N-1}$ for all $\ell \in \mathbb{N}$ and $1 \leqslant m \leqslant M_\ell$.

**Theorem 1.1.** *Let $N \geqslant 2$ and let $(U, V) \in H^1(B_R^+) \times H^1(B_R^+)$ be a weak solution to (7) such that $(U, V) \neq (0, 0)$. Then there exists $\ell \in \mathbb{N}$ such that*

$$
\lambda^{-\ell} U(\lambda z) \to \widehat{U}(z), \quad \lambda^{-\ell} V(\lambda z) \to \widehat{V}(z), \qquad \textit{strongly in } H^1(B_1^+)
$$

[1] It is enough to take a homogeneous harmonic polynomial $P = P(x_1, x_2, \dots, x_N)$ in $N$ variables of degree $\ell$ and consider the homogeneous harmonic polynomial in $N + 1$ variables $P'(x_1, x_2, \dots, x_N, x_{N+1}) = P(x_1, x_2, \dots, x_N)$, whose restriction to $\mathbb{S}^N$ satisfies the required properties.

*as* $\lambda \to 0^+$ , *where*

$$\widehat{U}(z) = |z|^\ell \sum_{m=1}^{M_\ell} \alpha_{\ell,m} Y_{\ell,m}\Big(\frac{z}{|z|}\Big), \quad \widehat{V}(z) = |z|^\ell \sum_{m=1}^{M_\ell} \alpha'_{\ell,m} Y_{\ell,m}\Big(\frac{z}{|z|}\Big),$$

$$\alpha_{\ell,m} = R^{-\ell}\int_{\mathbb{S}^N_+} U(R\,\theta) Y_{\ell,m}(\theta)\, dS - \frac{R^{-N-2\ell+1}}{N+2\ell-1}\int_0^R t^{N+\ell}\left(\int_{\mathbb{S}^N_+} V(t\,\theta) Y_{\ell,m}(\theta)\, dS\right) dt$$
$$+ \int_0^R \frac{t^{-\ell+1}}{2\ell+N-1}\left(\int_{\mathbb{S}^N_+} V(t\,\theta) Y_{\ell,m}(\theta)\, dS\right) dt, \tag{9}$$

$$\alpha'_{\ell,m} = R^{-\ell}\int_{\mathbb{S}^N_+} V(R\,\theta) Y_{\ell,m}(\theta)\, dS$$
$$- \frac{R^{-N-2\ell+1}}{N+2\ell-1}\int_0^R t^{N+\ell-1}\left(\int_{\mathbb{S}^{N-1}} h(t\theta')U(t\theta',0) Y_{\ell,m}(\theta',0)\, dS'\right) dt$$
$$+ \int_0^R \frac{t^{-\ell}}{2\ell+N-1}\left(\int_{\mathbb{S}^{N-1}} h(t\theta')U(t\theta',0) Y_{\ell,m}(\theta',0)\, dS'\right) dt, \tag{10}$$

*and*

$$\sum_{m=1}^{M_\ell} ((\alpha_{\ell,m})^2 + (\alpha'_{\ell,m})^2) \neq 0.$$

A first remarkable consequence of Theorem 1.1 is the validity of a *strong unique continuation property* (from the boundary point 0) for solutions to (7).

**Theorem 1.2.** *Let* $N \geqslant 2$ *and let* $(U,V) \in H^1(B_R^+) \times H^1(B_R^+)$ *be a weak solution to* (7)*. If*

$$U(z) = o(|z|^n) \quad \text{as } |z| \to 0 \text{ for all } n \in \mathbb{N} \tag{11}$$

*then* $U \equiv V \equiv 0$ *in* $B_R^+$*.*

We observe that in the case of a single equation a blow-up result as the one stated in Theorem 1.1 directly yields the strong unique continuation: indeed, if the solution has a precise vanishing order it cannot vanish of any order. On the other hand, in the case of a system of type (7), the blow-up Theorem 1.1 ensures that the couple of the limit profiles $(\widehat{U}, \widehat{V})$ is not trivial, i.e. at least one of the two components $U, V$ has a precise vanishing order; hence some further analysis is needed to deduce strong unique continuation from Theorem 1.1.

When $N > 3$, system (7) is related to fourth order elliptic equations arising in Caffarelli-Silvestre type extensions for higher order fractional laplacians in the spirit of [25]. Let us define $\mathcal{D}$ as the completion of

$$\mathcal{T} := \Big\{ U \in C_c^\infty(\overline{\mathbb{R}^{N+1}_+}) : \tfrac{\partial U}{\partial \nu} = 0 \text{ in } \mathbb{R}^N \Big\} \tag{12}$$

with respect to the norm

$$\|U\|_{\mathcal{D}} = \left(\int_{\mathbb{R}^{N+1}_+} |\Delta U(x,t)|^2\, dx\, dt\right)^{1/2}.$$

By [15] there exists a well defined continuous trace map

$$\mathrm{Tr} : \mathcal{D} \to \mathcal{D}^{3/2}(\mathbb{R}^N),$$

where for any $s > 0$ the space $\mathcal{D}^s(\mathbb{R}^N)$ is defined as the completion of $C_c^\infty(\mathbb{R}^N)$ with respect to the scalar product

$$(u,v)_{\mathcal{D}^s(\mathbb{R}^N)} := \int_{\mathbb{R}^N} |\xi|^{2s}\, \widehat{u}(\xi)\overline{\widehat{v}(\xi)}\, d\xi. \tag{13}$$

In (13) $\widehat{u}$ denotes the Fourier transform of $u$ in $\mathbb{R}^N$:

$$\widehat{u}(\xi) = \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i\xi x} u(x)\, dx\,.$$

Moreover in (13) we denoted by $\overline{\widehat{v}(\xi)}$ the complex conjugate of $\widehat{v}(\xi)$.

We observe that, since $u$ and $v$ are real functions, (13) is really a scalar product although their respective Fourier transforms are complex functions.

As a corollary of Theorem 1.1 we derive sharp asymptotic estimates and a strong unique continuation principle for weak $\mathcal{D}$-solutions to the fourth order elliptic problem

$$\begin{cases} \Delta^2 U = 0, & \text{in } \mathbb{R}^{N+1}_+, \\ \frac{\partial U}{\partial \nu} = 0, & \text{in } \mathbb{R}^N\,, \\ \frac{\partial(\Delta U)}{\partial \nu} = h \text{ Tr } U, & \text{in } \Omega\,. \end{cases} \tag{14}$$

By a weak $\mathcal{D}$-solution to (14) we mean some $U \in \mathcal{D}$ such that

$$\int_{\mathbb{R}^{N+1}_+} \Delta U(x,t) \Delta\varphi(x,t)\, dx\, dt = - \int_\Omega h(x) \operatorname{Tr} U(x) \operatorname{Tr}\varphi(x)\, dx$$

for all $\varphi \in \mathcal{D}$ such that $\operatorname{supp}(\operatorname{Tr}\varphi) \subset \Omega$.

**Theorem 1.3.** *Suppose that $N > 3$.*

(i) *Let $U \in \mathcal{D}$, $U \not\equiv 0$, be a nontrivial weak solution to* (14) *for some $h \in C^1(\Omega)$, with $\Omega$ being an open bounded set in $\mathbb{R}^N$ such that $0 \in \Omega$. Then there exists $\ell \in \mathbb{N}$ such that*

$$\lambda^{-\ell} U(\lambda z) \to |z|^\ell \sum_{m=1}^{M_\ell} \alpha_{\ell,m} Y_{\ell,m}\Big(\frac{z}{|z|}\Big), \quad \lambda^{-\ell} \Delta U(\lambda z) \to |z|^\ell \sum_{m=1}^{M_\ell} \alpha'_{\ell,m} Y_{\ell,m}\Big(\frac{z}{|z|}\Big),$$

*strongly in $H^1(B_1^+)$, where $\sum_{m=1}^{M_\ell}((\alpha_{\ell,m})^2 + (\alpha'_{\ell,m})^2) \neq 0$ and $\alpha_{\ell,m}, \alpha'_{\ell,m}$ are given in* (9)–(10) *with $V = \Delta U$.*

(ii) *If $U \in \mathcal{D}$ is a weak solution to* (14) *such that*

$$U(z) = o(|z|^n) \quad \text{as } |z| \to 0 \text{ for all } n \in \mathbb{N},$$

*then $U \equiv 0$ in $B_R^+$.*

As mentioned above, a motivation for the study of higher order equations of type (14) and consequently of systems (7) comes from the interest in higher order fractional laplacians and their characterization as a Dirichlet-to-Neumann map in the spirit of [5].

Let us consider the fractional laplacian $(-\Delta)^{3/2}$ defined as

$$\widehat{(-\Delta)^{\frac{3}{2}} u}(\xi) = |\xi|^3 \widehat{u}(\xi)\,.$$

If $u \in \mathcal{D}^{3/2}(\mathbb{R}^N)$ then $(-\Delta)^{3/2}u$ may be interpreted as a distribution which acts on test functions as follows:

$$\langle(-\Delta)^{3/2}u, \varphi\rangle = \int_{\mathbb{R}^N} |\xi|^3 \widehat{u}(\xi)\overline{\widehat{\varphi}(\xi)}\, d\xi \qquad \text{for any } \varphi \in C_c^\infty(\mathbb{R}^N).$$

**Theorem 1.4.** *For $N > 3$, let $\Omega \subseteq \mathbb{R}^N$ be open, $a \in C^1(\Omega)$, and $u \in \mathcal{D}^{3/2}(\mathbb{R}^N)$ be a weak solution to the problem*

$$(-\Delta)^{3/2}u = a\, u, \quad \text{in } \Omega, \tag{15}$$

*i.e.*

$$(u, \varphi)_{\mathcal{D}^{3/2}(\mathbb{R}^N)} = \int_\Omega a\, u\, \varphi\, dx \quad \text{for all } \varphi \in C_c^\infty(\Omega).$$

*Let us also assume that*

$$(-\Delta)^{3/2}u \in (\mathcal{D}^{1/2}(\mathbb{R}^N))^\star, \tag{16}$$

*where $(\mathcal{D}^{1/2}(\mathbb{R}^N))^\star$ is the topological dual space of $\mathcal{D}^{1/2}(\mathbb{R}^N)$.*

(i) *If $u$ vanishes at some point $x_0 \in \Omega$ of infinite order, i.e. if*

$$u(x) = o(|x - x_0|^n) \quad \text{as } x \to x_0 \text{ for every } n \in \mathbb{N}, \tag{17}$$

*then $u \equiv 0$ in $\Omega$.*

(ii) *If $u$ vanishes on a set $E \subset \Omega$ of positive Lebesgue measure, then $u \equiv 0$ in $\Omega$.*

**Remark 1.5.** We observe that assumption (16) is satisfied in each of the following cases:

(i) $u \in \mathcal{D}^{5/2}(\mathbb{R}^N)$;

(ii) $u \in \mathcal{D}^{3/2}(\mathbb{R}^N)$ solves (15) with $\Omega = \mathbb{R}^N$ and $a \in L^{N/2}(\mathbb{R}^N) \cap C^1(\mathbb{R}^N)$; in this case the validity of (16) follows by (15) because of Sobolev embeddings.

The proof of Theorem 1.4 is based on Theorem 1.3 and the generalization of the Caffarelli-Silvestre extension to higher order fractional laplacians given in [25], see also [15]. Indeed, according to [25], we have that if $u$ solves (15), then $u$ is the trace on $\mathbb{R}^N \times \{0\}$ of some $U \in \mathcal{D}$ solving (14) with $h = -2a$.

We observe that the unique continuation result stated in Theorem 1.4 does not overlap with the results in [20, 21, 22]. Indeed, from one hand [20, 21, 22] consider more general potentials; on the other hand we obtain here a *strong* unique continuation and a unique continuation from sets of positive measure, which are stronger results than the weak unique continuation obtained in [20, 21, 22]. We also observe that we assume that equation (15) is satisfied only on the set $\Omega$ and not in the whole $\mathbb{R}^N$.

The paper is organized as follows. In section 2 we develop the monotonicity argument, proving in particular the existence of a finite limit for the frequency function $\mathcal{N} = \mathcal{N}(r)$ as $r \to 0^+$. In section 3 we carry out a careful blow-up analysis for scaled solutions, which allows proving Theorem 1.1 and, as a consequence, Theorem 1.2. Finally section 4 is devoted to applications of Theorem 1.1 to fourth order problems (14) and higher order fractional problems (15), with the proofs of Theorems 1.3 and 1.4.

**List of notations.**

- For the definitions of $B_r$, $B_r^+$, $B'_r$ and $S_r^+$ see (6).

- $z = (x,t) \in \mathbb{R}^{N+1}_+$ with $x \in \mathbb{R}^N$ and $t > 0$.
- $dz = dx, dt$: element of volume in $\mathbb{R}^{N+1}$.
- $\frac{\partial U}{\partial \nu}$ or $U_\nu$ in $\mathbb{R}^N$: exterior normal derivative of a function $U$ defined in $\overline{\mathbb{R}^{N+1}_+}$, i.e.
$$\tfrac{\partial U}{\partial \nu}(x) := -\tfrac{\partial U}{\partial t}(x,0) \qquad \text{for any } x \in \mathbb{R}^N .$$
- $\mathcal{D}$: completion of the space of smooth functions $U \in C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ with $U_\nu = 0$ in $\mathbb{R}^N$, with respect to the norm $\|U\|_{\mathcal{D}} := \left(\int_{\mathbb{R}^{N+1}_+} |\Delta U(z)|^2 dz\right)^{\frac{1}{2}}$.
- $\widehat{u}$: Fourier transform of a function defined in $\mathbb{R}^N$, i.e. $\widehat{u}(\xi) := \frac{1}{(2\pi)^{N/2}} \int_{\mathbb{R}^N} e^{-i\xi x} u(x)\, dx$.
- $\mathcal{D}^s(\mathbb{R}^N)$: completion of the space $C^\infty_c(\mathbb{R}^N)$ with respect to the norm
$$\|u\|_{\mathcal{D}^s(\mathbb{R}^N)} := \left(\int_{\mathbb{R}^N} |\xi|^{2s} |\widehat{u}(\xi)|^2 d\xi\right)^{\frac{1}{2}} .$$
- Fractional Laplacian $(-\Delta)^{\frac{3}{2}}$: it is defined implicitly by $\widehat{(-\Delta)^{\frac{3}{2}} u}(\xi) := |\xi|^3 \widehat{u}(\xi)$. If $u \in \mathcal{D}^{3/2}(\mathbb{R}^N)$ then $(-\Delta)^{3/2} u$ is the distribution defined by $\langle (-\Delta)^{3/2} u, \varphi\rangle := \int_{\mathbb{R}^N} |\xi|^3 \widehat{u}(\xi) \overline{\widehat{\varphi}(\xi)}\, d\xi$ for any $\varphi \in C^\infty_c(\mathbb{R}^N)$.
- $\mathrm{Tr}\, U$: it is the trace operator which maps any function $U$ defined over $\overline{\mathbb{R}^{N+1}_+}$ into the function $x \mapsto U(x,0)$ defined for any $x \in \mathbb{R}^N$.

## 2. The monotonicity argument

Suppose that $N \geqslant 2$. For all $r \in (0,R)$ we define the functions

$$D(r) = r^{-N+1} \left[ \int_{B^+_r} \left(|\nabla U|^2 + |\nabla V|^2 + UV\right) dz - \int_{B'_r} h(x) u(x) v(x)\, dx \right] \tag{18}$$

and

$$H(r) = r^{-N} \int_{S^+_r} (U^2 + V^2)\, dS. \tag{19}$$

We define the space $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+)$ as the completion of the space $C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$ with respect to the norm

$$\|U\|_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+)} := \left(\int_{\mathbb{R}^{N+1}_+} |\nabla U|^2 dz\right)^{1/2} \qquad \text{for any } U \in C^\infty_c(\overline{\mathbb{R}^{N+1}_+}) .$$

From [4], we have that there exists a constant $K > 0$ such that

$$K \|\mathrm{Tr}\, U\|_{\mathcal{D}^{\frac{1}{2}}(\mathbb{R}^N)} \leqslant \|U\|_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+)} \qquad \text{for any } U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+). \tag{20}$$

Here we are denoting as Tr the trace operator $\mathrm{Tr} : \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+) \to \mathcal{D}^{\frac{1}{2}}(\mathbb{R}^N)$. We recall that, for all $\gamma < \frac{N}{2}$ the following Sobolev embedding holds: there exists a positive constant $S(N,\gamma)$ depending only on $N$ and $\gamma$ such that

$$S(N,\gamma) \|u\|^2_{L^{2^*(N,\gamma)}(\mathbb{R}^N)} \leqslant \|u\|^2_{\mathcal{D}^\gamma(\mathbb{R}^N)} \qquad \text{for any } u \in C^\infty_c(\mathbb{R}^N) \tag{21}$$

where $2^*(N,\gamma) = 2N/(N-2\gamma)$. Moreover the following Hardy type inequality due to Herbst [10] holds: there exists $\Lambda > 0$

$$\Lambda \int_{\mathbb{R}^N} \frac{\varphi^2(x)}{|x|}\, dx \leqslant \|\varphi\|^2_{\mathcal{D}^{1/2}(\mathbb{R}^N)}, \quad \text{for all } \varphi \in \mathcal{D}^{1/2}(\mathbb{R}^N). \tag{22}$$

Combining (20) and (21) we obtain that

$$S\big(N, \tfrac{1}{2}\big) K^2 \,\|\mathrm{Tr}\, U\|^2_{L^{\frac{2N}{N-1}}(\mathbb{R}^N)} \leqslant \|U\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+)} \qquad \text{for any } U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+). \tag{23}$$

Similarly, combining (20) with (22), we infer

$$\Lambda K^2 \int_{\mathbb{R}^N} \frac{|\mathrm{Tr}\, U|^2}{|x|}\, dx \leqslant \|U\|^2_{\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+)} \qquad \text{for any } U \in \mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+). \tag{24}$$

We recall the following lemmas from [11], which provide Sobolev and Hardy type trace inequalities with boundary terms in $N+1$-dimensional half-balls.

**Lemma 2.1** ([11] Lemma 2.6)**.** *Let $N \geqslant 2$. For any $r > 0$ and any $U \in H^1(B_r^+)$ we have*

$$\widetilde{S} \left( \int_{B_r'} |u|^{\frac{2N}{N-1}} dx \right)^{\frac{N-1}{N}} \leqslant \int_{B_r^+} |\nabla U|^2 dz + \frac{N-1}{2r} \int_{S_r^+} U^2 dS$$

*where $u = \mathrm{Tr}\, U$ and $\widetilde{S}$ is a positive constant depending only on $N$.*

**Lemma 2.2** ([11] Lemma 2.5)**.** *Let $N \geqslant 2$. For any $r > 0$ and any $U \in H^1(B_r^+)$ we have that*

$$\widetilde{\Lambda} \int_{B_r'} \frac{u^2}{|x|}\, dx \leqslant \int_{B_r^+} |\nabla U|^2 dz + \frac{N-1}{2r} \int_{S_r^+} U^2 dS$$

*where $u = \mathrm{Tr}\, U$ and $\widetilde{\Lambda}$ is a positive constant depending only on $N$.*

The following Poincaré type inequality on half-balls will be useful in the sequel.

**Lemma 2.3.** *Let $N \geqslant 2$. For every $r > 0$ and $W \in H^1(B_r^+)$ we have that*

$$\frac{N}{r^2} \int_{B_r^+} W^2(z)\, dz \leqslant \frac{1}{r} \int_{S_r^+} W^2(z)\, dS + \int_{B_r^+} |\nabla W(z)|^2\, dz.$$

*Proof.* From the Divergence Theorem we have that

$$\begin{aligned}
(N+1) \int_{B_r^+} W^2(z)\, dz &= \int_{B_r^+} \big(\mathrm{div}(W^2 z) - 2W \nabla W \cdot z\big)\, dz \\
&= r \int_{S_r^+} W^2(z)\, dS - 2 \int_{B_r^+} W \nabla W \cdot z\, dz \\
&\leqslant r \int_{S_r^+} W^2(z)\, dS + \int_{B_r^+} W^2(z)\, dz + r^2 \int_{B_r^+} |\nabla W|^2\, dz
\end{aligned}$$

thus yielding the stated inequality. □

The following lemma contains a Pohozaev type identity for solutions to system (7).

**Lemma 2.4.** *Let $N \geqslant 2$ and let $(U,V) \in H^1(B_R^+) \times H^1(B_R^+)$ be a weak solution to* (7)*. Then for a.e. $r \in (0,R)$*

$$\int_{B_r^+} \left(|\nabla U|^2 + |\nabla V|^2 + UV\right) dz = \int_{S_r^+} \left(\frac{\partial U}{\partial \nu} U + \frac{\partial V}{\partial \nu} V\right) dS + \int_{B_r'} h(x)u(x)v(x)\, dx \tag{25}$$

*and*

$$\begin{aligned} -\frac{N-1}{2} \int_{B_r^+} \left(|\nabla U|^2 + |\nabla V|^2\right) dz + \int_{B_r^+} V(z \cdot \nabla U)\, dz + \frac{r}{2} \int_{S_r^+} \left(|\nabla U|^2 + |\nabla V|^2\right) dS \\ = \int_{B_r'} h(x)u(x)(x \cdot \nabla_x v)\, dx + r \int_{S_r^+} \left( \left|\frac{\partial U}{\partial \nu}\right|^2 + \left|\frac{\partial V}{\partial \nu}\right|^2 \right) dS \end{aligned} \tag{26}$$

*where $u(x) := U(x,0)$ and $v(x) = V(x,0)$.*

*Proof.* Identity (25) follows by testing the equation for $U$ with $U$ and the equation for $V$ with $V$ and by integrating by parts over $B_r^+$.

To prove (26) we first observe that $U, V \in H^2(B_r^+)$ for all $r \in (0,R)$. Indeed, since $\frac{\partial U}{\partial \nu} = 0$ on $B_R'$, the function

$$\widetilde{U}(x,t) = \begin{cases} U(x,t), & \text{if } t > 0, \\ U(x,-t), & \text{if } t < 0, \end{cases}$$

satisfies the equation $\Delta \widetilde{U} = \widetilde{V}$, where $\widetilde{V}(x,t) = V(x,t)$ if $t > 0$ and $\widetilde{V}(x,t) = V(x,-t)$ if $t < 0$. Since $\widetilde{V} \in L^2(B_R)$, by classical elliptic regularity we have that $\widetilde{U} \in H^2(B_r)$ and hence $U \in H^2(B_r^+)$ for all $r \in (0,R)$. By the Gagliardo Trace Theorem we have that $u = \operatorname{Tr} U \in H^{1/2}(B_r')$ for all $r \in (0,R)$ . Since $h \in C^1(B_R')$ we have that $hu \in H^{1/2}(B_r')$ for all $r \in (0,R)$. Therefore, for all $r \in (0,R)$, $V$ satisfies

$$\begin{cases} \Delta V = 0, & \text{in } B_r^+, \\ \frac{\partial V}{\partial \nu} \in H^{1/2}(B_r'). \end{cases}$$

From elliptic regularity under Neumann boundary conditions (see in particular [19, Theorem 8.13]) we conclude that $V \in H^2(B_r^+)$ for all $r \in (0,R)$.

Since, for every $r \in (0,R)$, $U, V \in H^2(B_r^+)$, we can test the equation for $U$ with $\nabla U \cdot z$ (which belongs to $H^1(B_r^+)$) and the equation for $V$ with $\nabla V \cdot z$ (which belongs to $H^1(B_r^+)$), thus obtaining (26). □

**Lemma 2.5.** *Let $N \geqslant 2$ and let $(U,V) \in H^1(B_R^+) \times H^1(B_R^+)$ be a weak solution to* (7) *such that $(U,V) \neq (0,0)$ (i.e. $U$ and $V$ are not both identically null). Let $D = D(r)$ and $H = H(r)$ be the functions defined in* (18) *and* (19)*. Then there exists $r_0 \in (0,R)$ such that $H(r) > 0$ for any $r \in (0,r_0)$.*

*Proof.* Suppose by contradiction that for any $r_0 > 0$ there exists $r \in (0,r_0)$ such that $H(r) = 0$. Then there exists a sequence $r_n \to 0^+$ such that $H(r_n) = 0$, i.e. $U = V = 0$ on $S_{r_n}^+$. From (25) it follows that

$$\int_{B_{r_n}^+} \left(|\nabla U|^2 + |\nabla V|^2 + UV\right) dz = \int_{B_{r_n'}} h(x)u(x)v(x)\, dx. \tag{27}$$

From (27), Lemma 2.3, and Lemma 2.2 it follows that

$$\begin{aligned}\left(1-\frac{r_n^2}{2N}\right)\int_{B^+_{r_n}}\left(|\nabla U|^2+|\nabla V|^2\right)dz &\leqslant \int_{B^+_{r_n}}\left(|\nabla U|^2+|\nabla V|^2+UV\right)dz\\ &=\int_{B_{r'_n}} h(x)u(x)v(x)\,dx \leqslant \mathrm{const}\, r_n\left(\int_{B'_{r_n}}\frac{u^2}{|x|}\,dx+\int_{B'_{r_n}}\frac{v^2}{|x|}\,dx\right)\\ &\leqslant \mathrm{const}\, r_n\int_{B^+_{r_n}}\left(|\nabla U|^2+|\nabla V|^2\right)dz.\end{aligned}$$

Since $r_n \to 0^+$ as $n\to+\infty$, the above inequality implies that $\int_{B^+_{r_n}}\left(|\nabla U|^2+|\nabla V|^2\right)dz=0$ for $n$ sufficiently large. Hence, in view of Lemma 2.3, $U\equiv V\equiv 0$ in $B^+_{r_n}$. Classical unique continuation principles then imply that $U\equiv V\equiv 0$ in $B^+_R$ giving rise to a contradiction. □

**Lemma 2.6.** *Let $N\geqslant 2$. Letting $(U,V)\in H^1(B^+_R)\times H^1(B^+_R)$ be as in Lemma 2.5 and $D,H$ as in* (18)–(19)*, there holds*

$$D(r)\geqslant r^{1-N}\left(\int_{B^+_r}\left(|\nabla U|^2+|\nabla V|^2\right)dz\right)(1+O(r))-H(r)O(r), \tag{28}$$

$$D(r)\geqslant Nr^{-1-N}\left(\int_{B^+_r}(U^2+V^2)dz\right)(1+O(r))-H(r)O(1), \tag{29}$$

*as $r\to 0^+$.*

*Proof.* From Lemma 2.3 we have that

$$\int_{B^+_r}(U^2+V^2)\,dz\leqslant \frac{r^{1+N}}{N}H(r)+\frac{r^2}{N}\int_{B^+_r}\left(|\nabla U|^2+|\nabla V|^2\right)dz. \tag{30}$$

From (30) it follows that

$$\left|\int_{B^+_r}UV\,dz\right|\leqslant \frac{r^{1+N}}{2N}H(r)+\frac{r^2}{2N}\int_{B^+_r}\left(|\nabla U|^2+|\nabla V|^2\right)dz \tag{31}$$

whereas Lemma 2.2 implies that, for all $r\in(0,r_0)$,

$$\begin{aligned}\left|\int_{B'_r}huv\,dx\right| &\leqslant \|h\|_{L^\infty(B'_{r_0})}\frac{r}{2}\int_{B'_r}\frac{u^2+v^2}{|x|}\,dx\\ &\leqslant \|h\|_{L^\infty(B'_{r_0})}\frac{r}{2\widetilde\Lambda}\left(\int_{B^+_r}\left(|\nabla U|^2+|\nabla V|^2\right)dz\right)+\|h\|_{L^\infty(B'_{r_0})}\frac{N-1}{4\widetilde\Lambda}r^N H(r).\end{aligned} \tag{32}$$

From (31) and (32) it follows that

$$\begin{aligned}D(r)\geqslant r^{1-N}&\left(\int_{B^+_r}(|\nabla U|^2+|\nabla V|^2)dz\right)\left(1-\frac{r^2}{2N}-\|h\|_{L^\infty(B'_{r_0})}\frac{r}{2\widetilde\Lambda}\right)\\ &-rH(r)\left(\frac{r}{2N}+\|h\|_{L^\infty(B'_{r_0})}\frac{N-1}{4\widetilde\Lambda}\right).\end{aligned}$$

The proof of (28) is thereby complete. Estimate (29) follows by combination of (28) and (30). □

**Remark 2.7.** We observe that estimates (28) and (29) can be rewritten as

$$\int_{B_r^+}\big(|\nabla U|^2+|\nabla V|^2\big)\,dz \leqslant D(r)r^{N-1}(1+O(r))+H(r)O(r^N), \tag{33}$$

$$\int_{B_r^+}(U^2+V^2)dz \leqslant \frac{1}{N}r^{N+1}D(r)(1+O(r))+H(r)O(r^{N+1}), \tag{34}$$

as $r\to 0^+$.

**Lemma 2.8.** *Let $N\geqslant 2$. We have that $H\in W^{1,1}_{\rm loc}(0,R)$ and*

$$H'(r)=2r^{-N}\int_{S_r^+}\big(U\tfrac{\partial U}{\partial\nu}+V\tfrac{\partial V}{\partial\nu}\big)\,dS, \quad \textit{in a distributional sense and for a.e. } r\in(0,R), \tag{35}$$

$$H'(r)=\frac{2}{r}D(r), \quad \textit{for every } r\in(0,R). \tag{36}$$

*Proof.* See the proof of [11, Lemma 3.8]. □

**Lemma 2.9.** *Let $N\geqslant 2$. The function $D$ defined in (18) belongs to $W^{1,1}_{\rm loc}(0,R)$ and*

$$\begin{aligned} D'(r)=&\frac{2}{r^{N-1}}\int_{S_r^+}\left(\left|\frac{\partial U}{\partial\nu}\right|^2+\left|\frac{\partial V}{\partial\nu}\right|^2\right)dS+\frac{1}{r^{N-1}}\int_{S_r^+}UV\,dS\\ &-\frac{2}{r^N}\int_{B_r^+}V\,\nabla U\cdot z\,dz-\frac{N-1}{r^N}\int_{B_r^+}UV\,dz\\ &+\frac{N-1}{r^N}\int_{B_r'}huv-\frac{1}{r^{N-1}}\int_{\partial B_r'}huv\,dS'+2\frac{1}{r^N}\int_{B_r'}hu(x\cdot\nabla_x v)\,dx \end{aligned} \tag{37}$$

*in a distributional sense and for a.e. $r\in(0,R)$.*

*Proof.* For any $r\in(0,R)$ let

$$I(r)=\int_{B_r^+}\big(|\nabla U|^2+|\nabla V|^2+UV\big)\,dz-\int_{B_r'}h(x)u(x)v(x)\,dx. \tag{38}$$

From the fact that $U,V\in H^1(B_R^+)$ and Lemma 2.2 it follows that $I\in W^{1,1}(0,R)$ and

$$I'(r)=\int_{S_r^+}\big(|\nabla U|^2+|\nabla V|^2+UV\big)\,dS-\int_{\partial B_r'}h(x)u(x)v(x)\,dS' \tag{39}$$

for a.e. $r\in(0,R)$ and in the distributional sense. Therefore $D\in W^{1,1}_{\rm loc}(0,R)$ and, replacing (26), (38), and (39) into $D'(r)=r^{-N}[-(N-1)I(r)+rI'(r)]$, we obtain (37). □

In view of Lemma 2.5, the function

$$\mathcal{N}:(0,r_0)\to\mathbb{R},\quad \mathcal{N}(r)=\frac{D(r)}{H(r)} \tag{40}$$

is well defined. As a consequence of estimate (28) we obtain the following corollary.

**Corollary 2.10.** *Let $N\geqslant 2$ and let $(U,V)\in H^1(B_R^+)\times H^1(B_R^+)$ be as in Lemma 2.5 and let $D,H,\mathcal{N}$ be defined in (18), (19), and (40) respectively. For every $\varepsilon>0$ there exists $r_\varepsilon>0$ such that*

$$\mathcal{N}(r)+\varepsilon\geqslant 0 \quad \textit{for all } 0<r<r_\varepsilon,$$

*i.e.*

$$\liminf_{r\to 0^+} \mathcal{N}(r) \geqslant 0. \tag{41}$$

**Lemma 2.11.** *Let $N \geqslant 2$. The function $\mathcal{N}$ defined in* (40) *belongs to $W^{1,1}_{\rm loc}(0, r_0)$ and*

$$\mathcal{N}'(r) = \nu_1(r) + \nu_2(r) \tag{42}$$

*in a distributional sense and for a.e. $r \in (0, r_0)$, where*

$$\nu_1(r) = \frac{2r\Big[\Big(\int_{S_r^+}\Big(\big|\frac{\partial U}{\partial\nu}\big|^2 + \big|\frac{\partial V}{\partial\nu}\big|^2\Big)\,dS\Big)\cdot\Big(\int_{S_r^+}(U^2+V^2)\,dS\Big) - \Big(\int_{S_r^+}\big(U\frac{\partial U}{\partial\nu} + V\frac{\partial V}{\partial\nu}\big)\,dS\Big)^2\Big]}{\Big(\int_{S_r^+}(U^2+V^2)\,dS\Big)^2}$$

*and*

$$\begin{aligned}\nu_2(r) = {} & \frac{r\int_{S_r^+} UV\,dS - 2\int_{B_r^+} V\,\nabla U\cdot z\,dz - (N-1)\int_{B_r^+} UV\,dz}{\int_{S_r^+}(U^2+V^2)\,dS}\\ & + \frac{(N-1)\int_{B_r'} huv\,dx - r\int_{\partial B_r'} huv\,dS' + 2\int_{B_r'} hu\,x\cdot\nabla_x v\,dx}{\int_{S_r^+}(U^2+V^2)\,dS}.\end{aligned} \tag{43}$$

*Proof.* It follows directly from the definition of $\mathcal{N}$ and Lemmas 2.8 and 2.9. □

We now estimate the term $\nu_2$ in (43). This is the most delicate point in the development of the monotonicity argument for system (7), due to the presence of the integral over "the boundary of the boundary" $\int_{\partial B_r'} huv\,dS'$ in the term $\nu_2$.

**Lemma 2.12.** *Let $N \geqslant 2$ and let $\nu_2$ be as in* (43)*. Then*

$$\nu_2(r) = O\left(1 + \mathcal{N}(r) + r\sqrt{\frac{B(r)}{r^N H(r)}}\right) \quad \text{as } r \to 0^+,$$

*where*

$$B(r) = \int_{S_r^+} (|\nabla U|^2 + |\nabla V|^2)\,dS. \tag{44}$$

*Proof.* We observe that

$$\frac{r\left|\int_{S_r^+} UV\,dS\right|}{\int_{S_r^+}(U^2+V^2)\,dS} = O(r) \quad \text{as } r \to 0^+. \tag{45}$$

From (33) and (34) we have that

$$\begin{aligned}\frac{\left|\int_{B_r^+} V\,\nabla U\cdot z\,dz\right|}{\int_{S_r^+}(U^2+V^2)\,dS} & \leqslant \frac{1}{2r^N H(r)}\left(\int_{B_r^+} V^2\,dz + r^2\int_{B_r^+}|\nabla U|^2\,dz\right)\\ & \leqslant \frac{N+1}{2N}\mathcal{N}(r) r(1+O(r)) + O(r) \leqslant \mathcal{N}(r) r(1+O(r)) + O(r)\end{aligned} \tag{46}$$

and

$$\frac{\left|\int_{B_r^+} UV\,dz\right|}{\int_{S_r^+}(U^2+V^2)\,dS} \leqslant \frac{r}{2N}\mathcal{N}(r)(1+O(r))+O(r) \tag{47}$$

as $r \to 0^+$. From (32) and (33) we have that

$$\frac{\left|\int_{B_r'} huv\,dx\right|}{\int_{S_r^+}(U^2+V^2)\,dS} \leqslant \frac{\|h\|_{L^\infty(B'_{r_0})}}{2\widetilde{\Lambda}}\mathcal{N}(r)(1+O(r))+O(1) = \mathcal{N}(r)O(1)+O(1) \tag{48}$$

as $r \to 0^+$.

Integration by parts yields

$$\int_{B_r'} hu\,x\cdot\nabla_x v\,dx = r\int_{\partial B_r'} huv\,dS' - \int_{B_r'} v(Nhu + u\nabla h\cdot x + hx\cdot\nabla_x u)\,dx$$

so that

$$\begin{aligned} -r\int_{\partial B_r'} huv\,dS' &+ 2\int_{B_r'} hu\,x\cdot\nabla_x v\,dx \\ &= \int_{B_r'} hu\,x\cdot\nabla_x v\,dx - \int_{B_r'} hv\,x\cdot\nabla_x u\,dx - \int_{B_r'} uv(Nh + x\cdot\nabla h)\,dx. \end{aligned} \tag{49}$$

From Lemma 2.2 and (33) we have that

$$\begin{aligned}\frac{\left|\int_{B_r'} uv(Nh+x\cdot\nabla h)\,dx\right|}{\int_{S_r^+}(U^2+V^2)\,dS} &\leqslant \frac{\|Nh + x\cdot\nabla_x h\|_{L^\infty(B'_{r_0})}}{2\widetilde{\Lambda}}\mathcal{N}(r)(1+O(r))+O(1) \\ &= \mathcal{N}(r)O(1)+O(1)\end{aligned} \tag{50}$$

as $r \to 0^+$.

On the other hand, by the Divergence Theorem we have that

$$\begin{aligned}\int_{B_r'} hu\,x\cdot\nabla_x v\,dx &= -\int_{B_r'} hu\,(x\cdot\nabla_x v)\mathbf{e}_{N+1}\cdot\nu\,dx \\ &= \int_{S_r^+} h(x)U(x,t)\,(z\cdot\nabla V)\mathbf{e}_{N+1}\cdot\nu\,dS - \int_{B_r^+}\frac{\partial}{\partial t}\left[h(x)U(x,t)\,(z\cdot\nabla V)\right]\,dz \\ &= \int_{S_r^+} h(x)U(x,t)\,(z\cdot\nabla V)\mathbf{e}_{N+1}\cdot\nu\,dS - \int_{B_r^+} h(x)U_t\,(z\cdot\nabla V)\,dz \\ &\quad - \int_{B_r^+} h(x)U\,(V_t + z\cdot\nabla V_t)\,dz \\ &= \int_{S_r^+} h(x)U(x,t)\,(z\cdot\nabla V)\mathbf{e}_{N+1}\cdot\nu\,dS - \int_{B_r^+} h(x)U_t\,(z\cdot\nabla V)\,dz \\ &\quad - r\int_{S_r^+} h(x)UV_t\,dS + \int_{B_r^+}(Nh(x)+\nabla h\cdot x)UV_t\,dz + \int_{B_r^+} hV_t(\nabla U\cdot z)\,dz.\end{aligned} \tag{51}$$

Hence, taking into account Lemma 2.3,

$$\left|\int_{B'_r} hu\, x\cdot\nabla_x v\, dx\right| \leqslant \mathrm{const}\left(r\sqrt{r^N H(r)B(r)} + r\int_{B_r^+}(|\nabla U|^2+|\nabla V|^2)\, dz + \int_{S_r^+}(U^2+V^2)\, dS\right) \tag{52}$$

for some const >0 independent of $r$. In a similar way we obtain that

$$\left|\int_{B'_r} hv\, x\cdot\nabla_x u\, dx\right| \leqslant \mathrm{const}\left(r\sqrt{r^N H(r)B(r)} + r\int_{B_r^+}(|\nabla U|^2+|\nabla V|^2)\, dz + \int_{S_r^+}(U^2+V^2)\, dS\right).$$

As a consequence, in view of (33) we conclude that

$$\frac{\left|-r\int_{\partial B'_r} huv\, dS' + 2\int_{B'_r} hu\, x\cdot\nabla_x v\, dx\right|}{\int_{S_r^+}(U^2+V^2)\, dS} \leqslant \mathcal{N}(r)O(1) + \sqrt{\frac{B(r)}{r^N H(r)}}\, O(r) + O(1) \tag{53}$$

as $r\to 0^+$.

Inserting (45)-(53) into (43) the proof of the lemma follows. □

Inspired by [14, Lemma 5.9], in the following lemma we estimate $B$ in terms of the derivative $D'$.

**Lemma 2.13.** *Let $N\geqslant 2$ and let $B$ be defined in (44). Then there exist $C_1, C_2, \bar r > 0$ such that*

$$B(r)\leqslant 2r^{N-1}D'(r) + C_1 r^{N-2}(D(r)+C_2H(r)) \quad \text{and} \quad D(r)+C_2H(r)\geqslant 0 \quad \text{for all } r\in(0,\bar r).$$

*Proof.* From the definition of $D$ (see (18)) we have that

$$D'(r) = r^{1-N}B(r) - (N-1)r^{-1}D(r) + r^{1-N}\int_{S_r^+} UV\, dS - r^{1-N}\int_{\partial B'_r} huv\, dS'. \tag{54}$$

From (49) it follows that

$$\int_{\partial B'_r} huv\, dS' = \frac{1}{r}\int_{B'_r} hu\, x\cdot\nabla_x v\, dx + \frac{1}{r}\int_{B'_r} hv\, x\cdot\nabla_x u\, dx + \frac{1}{r}\int_{B'_r} uv(Nh + x\cdot\nabla h)\, dx.$$

By (52) and (33) we deduce that, for every $\varepsilon>0$, there exists $C_\varepsilon>0$ such that

$$\begin{aligned}\left|\frac{1}{r}\int_{B'_r} hu\, x\cdot\nabla_x v\, dx\right| & \\ &\leqslant \varepsilon B(r) + C_\varepsilon r^N H(r) + O(1)\int_{B_r^+}\left(|\nabla U|^2+|\nabla V|^2\right) dz + O(1)r^{N-1}H(r)\\ &\leqslant \varepsilon B(r) + O(1)r^{N-1}H(r) + O(1)r^{N-1}D(r) \quad \text{as } r\to 0^+.\end{aligned}$$

An analogous estimate holds for the term $\frac{1}{r}\int_{B'_r} hv\, x\cdot\nabla_x u\, dx$, whereas (50) implies that

$$\frac{1}{r}\int_{B'_r} uv(Nh + x\cdot\nabla h)\, dx = O(1)r^{N-1}H(r) + O(1)r^{N-1}D(r) \quad \text{as } r\to 0^+.$$

Therefore we conclude that

$$\left|\int_{\partial B'_r} huv\, dS'\right| \leqslant 2\varepsilon B(r) + O(1)r^{N-1}H(r) + O(1)r^{N-1}D(r) \quad \text{as } r\to 0^+. \tag{55}$$

From Corollary 2.10, (54) and (55), choosing $\varepsilon = \frac{1}{4}$, we deduce that, for some constants $C_1, C_2 > 0$ independent of $r$, $D(r) + C_2 H(r) \geqslant 0$ and

$$D'(r) \geqslant \frac{1}{2} r^{1-N} B(r) - \frac{C_1}{2} r^{-1}(D(r) + C_2 H(r)) \quad \text{for all } r \text{ sufficiently small.}$$

The proof is thereby complete. □

**Lemma 2.14.** *Let $N \geqslant 2$ and let $\mathcal{N} : (0, r_0) \to \mathbb{R}$ be defined in* (40). *Then*

$$\mathcal{N}(r) = O(1) \quad \text{as } r \to 0^+. \tag{56}$$

*Furthermore the limit*

$$\gamma := \lim_{r \to 0^+} \mathcal{N}(r)$$

*exists, is finite and*

$$\gamma \geqslant 0.$$

*Proof.* Let us consider the set

$$\Sigma = \{r \in (0, r_0) : D'(r)H(r) \leqslant H'(r)D(r)\}$$

(which is well-defined up to a zero measure set).

If there exists $r \in (0, r_0]$ such that $|(0, r) \cap \Sigma|_1 = 0$ (where $|\cdot|_1$ stands for the Lebesgue measure in $\mathbb{R}$) we have that $\mathcal{N}' \geqslant 0$ a.e. in $(0, r)$ and hence $\mathcal{N}$ is non-decreasing in $(0, r)$ and admits a limit as $r \to 0^+$ which is necessarily finite and non-negative due to (41).

Let us now assume that, for all $r \in (0, r_0]$, $|(0, r) \cap \Sigma|_1 > 0$. In view of Lemma 2.13 and (36) we have that, a.e. in $(0, r_0) \cap \Sigma$,

$$\begin{aligned} B(r) &\leqslant 2r^{N-1} \frac{H'(r)D(r)}{H(r)} + C_1 r^{N-2}(D(r) + C_2 H(r)) \\ &= 4r^{N-2} \frac{D^2(r)}{H(r)} + C_1 r^{N-2}(D(r) + C_2 H(r)). \end{aligned} \tag{57}$$

Schwarz inequality implies that the function $\nu_1$ appearing in Lemma 2.11 is non-negative, hence (42), Lemma 2.12, and (57) imply that

$$\mathcal{N}'(r) \geqslant O(1) \left(1 + \mathcal{N}(r) + \sqrt{4\mathcal{N}^2(r) + C_1(\mathcal{N}(r) + C_2)}\right)$$

as $r \to 0^+$, $r \in \Sigma$. Hence there exist $\tilde{C}, \tilde{r} > 0$ such that

$$\mathcal{N}'(r) \geqslant -\tilde{C}\,(1 + \mathcal{N}(r)) \quad \text{for a.e. } r \in (0, \tilde{r}) \cap \Sigma.$$

Since the above inequality is obviously true in $(0, \tilde{r}) \setminus \Sigma$ (provided $\tilde{r}$ is sufficiently small), we deduce that

$$\mathcal{N}'(r) \geqslant -\tilde{C}\,(1 + \mathcal{N}(r)) \quad \text{for a.e. } r \in (0, \tilde{r}). \tag{58}$$

Integrating the above inequality in $(r, \tilde{r})$ we obtain that

$$\mathcal{N}(r) + 1 \leqslant e^{\tilde{C}\tilde{r}}(\mathcal{N}(\tilde{r}) + 1) \quad \text{for all } r \in (0, \tilde{r}).$$

The above estimate together with Corollary 2.10 yield (56). Furthermore (58) implies that

$$\left(e^{\tilde{C}r}(1 + \mathcal{N}(r))\right)' \geqslant 0 \quad \text{a.e. in } (0, \tilde{r}),$$

hence the function $r \mapsto e^{\tilde{C}r}(1+\mathcal{N}(r))$ admits a limit as $r \to 0^+$. Therefore also the limit $\gamma := \lim_{r\to 0^+} \mathcal{N}(r)$ exists; furthermore $\gamma$ is finite in view of (56) and $\gamma \geqslant 0$ in view of (41). □

A first consequence of the previous monotonicity argument is the following estimate of the function $H$.

**Lemma 2.15.** *Let $N \geqslant 2$. Letting $\gamma$ be as in Lemma 2.14, we have that*

$$H(r) = O(r^{2\gamma}) \quad \text{as } r \to 0^+. \tag{59}$$

*Furthermore, for any $\sigma > 0$ there exist $K(\sigma) > 0$ depending on $\sigma$ such that*

$$H(r) \geqslant K(\sigma)\, r^{2\gamma+\sigma} \quad \text{for all } r \in (0, r_0). \tag{60}$$

*Proof.* See the proof of [11, Lemma 3.16]. □

## 3. Blow-up analysis

**Lemma 3.1.** *Let $N \geqslant 2$ and let $(U,V) \in H^1(B_R^+) \times H^1(B_R^+)$ be a weak solution to (7) such that $(U,V) \neq (0,0)$, let $\mathcal{N}$ be defined in (40), and let $\gamma := \lim_{r\to 0^+} \mathcal{N}(r)$ be as in Lemma 2.14. Then*

(i) *there exists $\ell \in \mathbb{N}$ such that $\gamma = \ell$;*

(ii) *for every sequence $\lambda_n \to 0^+$, there exist a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and $2M_\ell$ real constants $\beta_{\ell,m}, \beta'_{\ell,m}$, $m = 1, 2, \dots, M_\ell$, such that $\sum_{m=1}^{M_\ell}((\beta_{\ell,m})^2 + (\beta'_{\ell,m})^2) = 1$ and*

$$\frac{U(\lambda_{n_k} z)}{\sqrt{H(\lambda_{n_k})}} \to |z|^\ell \sum_{m=1}^{M_\ell} \beta_{\ell,m} Y_{\ell,m}\Big(\frac{z}{|z|}\Big), \qquad \frac{V(\lambda_{n_k} z)}{\sqrt{H(\lambda_{n_k})}} \to |z|^\ell \sum_{m=1}^{M_\ell} \beta'_{\ell,m} Y_{\ell,m}\Big(\frac{z}{|z|}\Big),$$

*weakly in $H^1(B_1^+)$ and strongly in $H^1(B_r^+)$ for all $r \in (0,1)$. See Section 1 for the definition of $M_\ell$ and $Y_{\ell,m}$.*

*Proof.* Let us define

$$U_\lambda(z) = \frac{U(\lambda z)}{\sqrt{H(\lambda)}}, \quad V_\lambda(z) = \frac{V(\lambda z)}{\sqrt{H(\lambda)}}. \tag{61}$$

We notice that

$$\Delta U_\lambda = \lambda^2 V_\lambda \quad \text{and} \quad \int_{S_1^+} (U_\lambda^2 + V_\lambda^2) dS = 1. \tag{62}$$

By scaling and (56) we have

$$\int_{B_1^+} \big(|\nabla U_\lambda(z)|^2 + |\nabla V_\lambda(z)|^2 + \lambda^2 U_\lambda(z) V_\lambda(z)\big)\, dz - \lambda \int_{B'_1} h(\lambda x) U_\lambda(x,0) V_\lambda(x,0)\, dx = \mathcal{N}(\lambda) = O(1) \tag{63}$$

as $\lambda \to 0^+$. On the other hand, Lemmas 2.2 and 2.3 imply

$$\mathcal{N}(\lambda) \geqslant \left(\int_{B_1^+} \big(|\nabla U_\lambda(z)|^2 + |\nabla V_\lambda(z)|^2\big)\, dz\right)\left(1 - \frac{\lambda^2}{2N} - \frac{\lambda \|h\|_{L^\infty(B'_{r_0})}}{2\widetilde{\Lambda}}\right) - \frac{\lambda^2}{2N} - \frac{\lambda \|h\|_{L^\infty(B'_{r_0})}(N-1)}{4\widetilde{\Lambda}}$$

so that (63) and Lemma 2.3 imply that

$$\{U_\lambda\}_{\lambda\in(0,\tilde\lambda)} \text{ and } \{V_\lambda\}_{\lambda\in(0,\tilde\lambda)} \text{ are bounded in } H^1(B_1^+)$$

for some $\tilde\lambda > 0$.

Therefore, for any given sequence $\lambda_n \to 0^+$, there exists a subsequence $\lambda_{n_k} \to 0^+$ such that $U_{\lambda_{n_k}} \rightharpoonup \widetilde U$ and $V_{\lambda_{n_k}} \rightharpoonup \widetilde V$ weakly in $H^1(B_1^+)$ for some $\widetilde U, \widetilde V \in H^1(B_1^+)$. From compactness of the trace embedding $H^1(B_1^+) \hookrightarrow L^2(S_1^+)$ and from (62) we deduce that

$$\int_{S_1^+} (\widetilde U^2 + \widetilde V^2)\,dS = 1, \tag{64}$$

hence $(\widetilde U, \widetilde V) \neq (0,0)$, i.e. $\widetilde U$ and $\widetilde V$ can not both vanish identically. For every $\lambda \in (0,\tilde\lambda)$, the couple $(U_\lambda, V_\lambda)$ satisfies

$$\begin{cases} \Delta U_\lambda = \lambda^2 V_\lambda, & \text{in } B_1^+,\\ \Delta V_\lambda = 0, & \text{in } B_1^+,\\ \partial_\nu U_\lambda = 0, & \text{on } B_1',\\ \partial_\nu V_\lambda = \lambda h(\lambda x) u_\lambda, & \text{on } B_1', \end{cases} \tag{65}$$

in a weak sense, i.e.

$$\begin{cases} \int_{B_1^+} \nabla U_\lambda \cdot \nabla\varphi\,dz = -\lambda^2 \int_{B_1^+} V_\lambda \varphi\,dz\\ \int_{B_1^+} \nabla V_\lambda \cdot \nabla\varphi\,dz = \lambda \int_{B_1'} h(\lambda x) u_\lambda(x)\,\mathrm{Tr}\,\varphi(x)\,dx, \end{cases} \tag{66}$$

for all $\varphi \in H^1(B_1^+)$ such that $\varphi = 0$ on $S_1^+$, where $u_\lambda = \mathrm{Tr}\,U_\lambda$. From the weak convergences $U_{\lambda_{n_k}} \rightharpoonup \widetilde U$ and $V_{\lambda_{n_k}} \rightharpoonup \widetilde V$ in $H^1(B_1^+)$, we can pass to the limit in (66) to obtain

$$\begin{cases} \int_{B_1^+} \nabla \widetilde U \cdot \nabla\varphi\,dz = 0,\\ \int_{B_1^+} \nabla \widetilde V \cdot \nabla\varphi\,dz = 0, \end{cases} \qquad \text{for all } \varphi \in H^1(B_1^+) \text{ such that } \varphi = 0 \text{ on } S^1_+,$$

i.e. $(\widetilde U, \widetilde V)$ weakly solves

$$\begin{cases} \Delta \widetilde U = 0, & \text{in } B_1^+,\\ \Delta \widetilde V = 0, & \text{in } B_1^+,\\ \partial_\nu \widetilde U = 0, & \text{on } B_1',\\ \partial_\nu \widetilde V = 0, & \text{on } B_1'. \end{cases} \tag{67}$$

From elliptic regularity under Neumann boundary conditions (see in particular [19, Theorem 8.13]) we conclude that

$$\{U_\lambda\}_{\lambda\in(0,\tilde\lambda)} \text{ and } \{V_\lambda\}_{\lambda\in(0,\tilde\lambda)} \text{ are bounded in } H^2(B_r^+) \text{ for all } r \in (0,1), \tag{68}$$

hence, by compactness, up to passing to a subsequence,

$$U_{\lambda_{n_k}} \to \widetilde U \text{ and } V_{\lambda_{n_k}} \to \widetilde V \ \text{ weakly in } H^2(B_r^+) \text{ and strongly in } H^1(B_r^+) \text{ for all } r \in (0,1). \tag{69}$$

For any $r \in (0,1)$ and $k \in \mathbb{N}$, let us define the functions

$$D_k(r) = r^{-N+1}\bigg[\int_{B_r^+}\Big(|\nabla U_{\lambda_{n_k}}|^2 + |\nabla V_{\lambda_{n_k}}|^2 + \lambda_{n_k}^2 U_{\lambda_{n_k}} V_{\lambda_{n_k}}\Big)\,dz \\ - \lambda_{n_k}\int_{B_r'} h(\lambda_{n_k}x)u_{\lambda_{n_k}}(x)v_{\lambda_{n_k}}(x)\,dx\bigg],$$

$$H_k(r) = r^{-N}\int_{S_r^+}(U_{\lambda_{n_k}}^2 + V_{\lambda_{n_k}}^2)\,dS,$$

where we have set $v_\lambda = \operatorname{Tr} V_\lambda$. By direct calculations we have

$$\mathcal{N}_k(r) := \frac{D_k(r)}{H_k(r)} = \frac{D(\lambda_{n_k}r)}{H(\lambda_{n_k}r)} = \mathcal{N}(\lambda_{n_k}r) \quad \text{for all } r \in (0,1). \tag{70}$$

From (69) it follows that, for any fixed $r \in (0,1)$,

$$D_k(r) \to \widetilde{D}(r) \quad \text{and} \quad H_k(r) \to \widetilde{H}(r) \quad \text{as } k \to +\infty \tag{71}$$

where

$$\widetilde{D}(r) = r^{-N+1}\int_{B_r^+}\Big(|\nabla\widetilde{U}|^2 + |\nabla\widetilde{V}|^2\Big)\,dz \quad \text{and} \quad \widetilde{H}(r) = r^{-N}\int_{S_r^+}(\widetilde{U}^2 + \widetilde{V}^2)\,dS \tag{72}$$

for all $r \in (0,1)$. We observe that $\widetilde{H}(r) > 0$ for all $r \in (0,1)$; indeed, if $\widetilde{H}(\bar r) = 0$ for some $\bar r \in (0,1)$, the fact that $\widetilde{U}, \widetilde{V}$ (and their even extension for $t < 0$) are harmonic would imply that $\widetilde{U} \equiv \widetilde{V} \equiv 0$ in $B_{\bar r}^+$, thus contradicting the classical unique continuation principle. Therefore the function

$$\widetilde{\mathcal{N}}(r) := \frac{\widetilde{D}(r)}{\widetilde{H}(r)}$$

is well defined for $r \in (0,1)$. From (70), (71), and Lemma 2.14, we deduce that

$$\widetilde{\mathcal{N}}(r) = \lim_{k\to\infty}\mathcal{N}(\lambda_{n_k}r) = \gamma \tag{73}$$

for all $r \in (0,1)$. Therefore $\widetilde{\mathcal{N}}$ is constant in $(0,1)$ and hence $\widetilde{\mathcal{N}}'(r) = 0$ for any $r \in (0,1)$. Arguing as in the proof of Lemma 2.11 we can prove that

$$\widetilde{\mathcal{N}}'(r) = \frac{2r\Big[\Big(\int_{S_r^+}\Big(\Big|\frac{\partial\widetilde{U}}{\partial\nu}\Big|^2 + \Big|\frac{\partial\widetilde{V}}{\partial\nu}\Big|^2\Big)dS\Big)\cdot\Big(\int_{S_r^+}(\widetilde{U}^2 + \widetilde{V}^2)\,dS\Big) - \Big(\int_{S_r^+}\Big(\widetilde{U}\frac{\partial\widetilde{U}}{\partial\nu} + \widetilde{V}\frac{\partial\widetilde{V}}{\partial\nu}\Big)\,dS\Big)^2\Big]}{\Big(\int_{S_r^+}(\widetilde{U}^2 + \widetilde{V}^2)\,dS\Big)^2}$$

for all $r \in (0,1)$. Therefore for all $r \in (0,1)$

$$\Big(\int_{S_r^+}\Big(\Big|\frac{\partial\widetilde{U}}{\partial\nu}\Big|^2 + \Big|\frac{\partial\widetilde{V}}{\partial\nu}\Big|^2\Big)dS\Big)\cdot\Big(\int_{S_r^+}(\widetilde{U}^2 + \widetilde{V}^2)\,dS\Big) - \Big(\int_{S_r^+}\Big(\widetilde{U}\frac{\partial\widetilde{U}}{\partial\nu} + \widetilde{V}\frac{\partial\widetilde{V}}{\partial\nu}\Big)\,dS\Big)^2 = 0$$

which implies that $(\widetilde{U}, \widetilde{V})$ and $(\frac{\partial\widetilde{U}}{\partial\nu}, \frac{\partial\widetilde{V}}{\partial\nu})$ have the same direction as vectors in $L^2(S_r^+) \times L^2(S_r^+)$. Hence there exists a function $\eta = \eta(r)$ such that $\Big(\frac{\partial\widetilde{U}}{\partial\nu}(r\theta), \frac{\partial\widetilde{V}}{\partial\nu}(r\theta)\Big) = \eta(r)(\widetilde{U}(r\theta), \widetilde{V}(r\theta))$ for all

$r \in (0,1)$ and $\theta \in \mathbb{S}^N_+$. By integration we obtain

$$\widetilde{U}(r\theta) = e^{\int_1^r \eta(s)ds}\widetilde{U}(\theta) = \varphi(r)\Psi_1(\theta), \quad r \in (0,1),\ \theta \in \mathbb{S}^N_+, \tag{74}$$

$$\widetilde{V}(r\theta) = e^{\int_1^r \eta(s)ds}\widetilde{V}(\theta) = \varphi(r)\Psi_2(\theta), \quad r \in (0,1),\ \theta \in \mathbb{S}^N_+, \tag{75}$$

where $\varphi(r) = e^{\int_1^r \eta(s)ds}$ and $\Psi_1 = \widetilde{U}\big|_{\mathbb{S}^N_+}$, $\Psi_2(\theta) = \widetilde{V}\big|_{\mathbb{S}^N_+}$. From (67), (74), and (75), it follows that

$$\begin{cases} r^{-N}\big(r^N\varphi'\big)'\Psi_i(\theta) + r^{-2}\varphi(r)\Delta_{\mathbb{S}^N_+}\Psi_i(\theta) = 0, & \text{on } \mathbb{S}^N_+, \\ \partial_\nu \Psi_i = 0, & \text{on } \partial\mathbb{S}^N_+, \end{cases} \quad i = 1,2. \tag{76}$$

Taking $r$ fixed we deduce that $\Psi_1, \Psi_2$ are either zero or restrictions to $\mathbb{S}^N_+$ of eigenfunctions of $-\Delta_{\mathbb{S}^N}$ associated to the same eigenvalue and symmetric with respect to the equator $\partial\mathbb{S}^N_+$. Therefore there exist $\ell \in \mathbb{N}$, $\{\beta_{\ell,m}, \beta'_{\ell,m}\}_{m=1}^{M_\ell} \subset \mathbb{R}$ such that

$$\begin{cases} -\Delta_{\mathbb{S}^N_+}\Psi_1 = \lambda_\ell \Psi_1, & \text{on } \mathbb{S}^N_+, \\ \partial_\nu \Psi_1 = 0, & \text{on } \partial\mathbb{S}^N_+, \end{cases} \qquad \begin{cases} -\Delta_{\mathbb{S}^N_+}\Psi_2 = \lambda_\ell \Psi_2, & \text{on } \mathbb{S}^N_+, \\ \partial_\nu \Psi_2 = 0, & \text{on } \partial\mathbb{S}^N_+, \end{cases}$$

and

$$\Psi_1 = \sum_{m=1}^{M_\ell} \beta_{\ell,m} Y_{\ell,m}, \quad \Psi_2 = \sum_{m=1}^{M_\ell} \beta'_{\ell,m} Y_{\ell,m}.$$

In view of (64) we have that $\int_{\mathbb{S}^N_+}(\Psi_1^2 + \Psi_2^2)\, dS = 1$ and hence

$$\sum_{m=1}^{M_\ell}((\beta_{\ell,m})^2 + (\beta'_{\ell,m})^2) = 1.$$

Since $\Psi_1$ and $\Psi_2$ are not both identically zero, from (76) it follows that $\varphi(r)$ solves the equation

$$\varphi''(r) + \frac{N}{r}\varphi'(r) - \frac{\lambda_\ell}{r^2}\varphi(r) = 0$$

and hence $\varphi(r)$ is of the form

$$\varphi(r) = c_1 r^\ell + c_2 r^{-(N-1)-\ell}$$

for some $c_1, c_2 \in \mathbb{R}$. Since either $|z|^{-(N-1)-\ell}\Psi_1(\frac{z}{|z|}) \notin H^1(B_1^+)$ or $|z|^{-(N-1)-\ell}\Psi_2(\frac{z}{|z|}) \notin H^1(B_1^+)$ (being $(\Psi_1, \Psi_2) \not\equiv (0,0)$), we have that $c_2 = 0$ and $\varphi(r) = c_1 r^\ell$. Moreover, from $\varphi(1) = 1$ we deduce that $c_1 = 1$. Then

$$\widetilde{U}(r\theta) = r^\ell \Psi_1(\theta), \quad \widetilde{V}(r\theta) = r^\ell \Psi_2(\theta), \quad \text{for all } r \in (0,1) \text{ and } \theta \in \mathbb{S}^N_+. \tag{77}$$

From (77) and the fact that

$$\int_{\mathbb{S}^N_+}(\Psi_1^2 + \Psi_2^2)\, dS = 1 \quad \text{and} \quad \int_{\mathbb{S}^N_+}(|\nabla_{\mathbb{S}^N}\Psi_1|^2 + |\nabla_{\mathbb{S}^N}\Psi_2|^2)\, dS = \lambda_\ell$$

it follows that

$$\begin{aligned} \widetilde{D}(r) &= \frac{1}{r^{N-1}}\int_{B_r^+}(|\nabla \widetilde{U}|^2 + |\nabla \widetilde{V}|^2)\, dt\, dx \\ &= r^{1-N}\ell^2 \int_0^r t^{N+2(\ell-1)}dt + r^{1-N}\lambda_\ell \int_0^r t^{N+2(\ell-1)}dt = \frac{\ell^2 + \ell(N-1+\ell)}{N+2\ell-1}\, r^{2\ell} = \ell\, r^{2\ell} \end{aligned}$$

and

$$\widetilde{H}(r) = \int_{\mathbb{S}^N_+} \left(\widetilde{U}^2(r\theta) + \widetilde{V}^2(r\theta)\right) dS = r^{2\ell}.$$

Hence from (73) it follows that $\gamma = \widetilde{\mathcal{N}}(r) = \frac{\widetilde{D}(r)}{\widetilde{H}(r)} = \ell$. The proof of the lemma is complete. $\square$

**Lemma 3.2.** *Let $N \geqslant 2$ and let $(U,V) \in H^1(B_R^+) \times H^1(B_R^+)$ be a weak solution to (7) such that $(U,V) \neq (0,0)$, let $H$ be defined in (19), and let $\ell$ be as in Lemma 3.1. Then the limit*

$$\lim_{r\to 0^+} r^{-2\ell} H(r)$$

*exists and it is finite.*

*Proof.* We recall from Lemma 3.1 that $\ell = \lim_{r\to 0^+} \mathcal{N}(r)$ with $\mathcal{N}$ as in (40).

In view of (59) it is sufficient to prove that the limit exists. By (36) and Lemma 2.14 we have

$$\begin{aligned}(78)\qquad \frac{d}{dr}\frac{H(r)}{r^{2\ell}} &= -2\ell r^{-2\ell-1}H(r) + r^{-2\ell}H'(r) = 2r^{-2\ell-1}(D(r) - \ell H(r))\\ &= 2r^{-2\ell-1}H(r)\int_0^r \mathcal{N}'(\rho)d\rho.\end{aligned}$$

From (58) and (56) it follows that there exists some $c > 0$ such that $\mathcal{N}'(r) \geqslant -c$ for all $r \in (0,\tilde{r})$. Then we can write $\mathcal{N}'(r) = -c + f(r)$ for some function $f \in L^1_{\rm loc}(0,r_0)$ such that $f(r) \geqslant 0$ a.e. in $(0,\tilde{r})$. Since $\mathcal{N}$ has a finite limit as $r \to 0^+$ we have that necessarily $f \in L^1(0,r_0)$.

Then integration of (78) over $(r,\tilde{r})$ yields

$$(79)\qquad \frac{H(\tilde{r})}{\tilde{r}^{2\ell}} - \frac{H(r)}{r^{2\ell}} = 2\int_r^{\tilde{r}} \rho^{-2\ell-1}H(\rho)\left(\int_0^\rho f(t)dt\right)d\rho - 2c\int_r^{\tilde{r}} \rho^{-2\ell}H(\rho)d\rho.$$

Since $f \geqslant 0$, we have that $\lim_{r\to 0^+}\int_r^{\tilde{r}} \rho^{-2\ell-1}H(\rho)\left(\int_0^\rho f(t)dt\right)d\rho$ exists. On the other hand, (59) implies that $\rho^{-2\ell}H(\rho) \in L^1(0,\widetilde{r})$ and hence the second term in the right hand side of (79) has a finite limit. Therefore both terms at the right hand side of (79) admit a limit as $r \to 0^+$ the second of which is finite and hence their sum has a limit. $\square$

Let $(U,V) \in H^1(B_R^+) \times H^1(B_R^+)$ be a weak solution to (7) such that $(U,V) \neq (0,0)$. Let us expand $U$ and $V$ as

$$U(z) = U(\lambda\theta) = \sum_{k=0}^{\infty}\sum_{m=1}^{M_k} \varphi_{k,m}(\lambda)Y_{k,m}(\theta), \quad V(z) = V(\lambda\theta) = \sum_{k=0}^{\infty}\sum_{m=1}^{M_k} \widetilde{\varphi}_{k,m}(\lambda)Y_{k,m}(\theta)$$

where $\lambda = |z| \in (0,R]$, $\theta = z/|z| \in \mathbb{S}^N_+$, and

$$(80)\qquad \varphi_{k,m}(\lambda) = \int_{\mathbb{S}^N_+} U(\lambda\,\theta)Y_{k,m}(\theta)\,dS, \quad \widetilde{\varphi}_{k,m}(\lambda) = \int_{\mathbb{S}^N_+} V(\lambda\,\theta)Y_{k,m}(\theta)\,dS$$

**Lemma 3.3.** *Let $N \geqslant 2$ and let $(U,V) \in H^1(B_R^+) \times H^1(B_R^+)$ be a weak solution to (7) such that $(U,V) \neq (0,0)$, let $\ell$ be as in Lemma 3.1, and let $\widetilde{\varphi}_{\ell,m}, \varphi_{\ell,m}$ be as in (80). Then, for all*

$1 \leqslant m \leqslant M_\ell$,

$$\varphi_{\ell,m}(\lambda) = \lambda^\ell \bigg( c_1^{\ell,m} + \int_\lambda^R \frac{t^{-\ell+1}}{2\ell+N-1} \widetilde{\varphi}_{\ell,m}(t)\,dt \bigg) + \lambda^{-(N-1)-\ell} \int_0^\lambda \frac{t^{N+\ell}}{N+2\ell-1} \widetilde{\varphi}_{\ell,m}(t)\,dt \tag{81}$$
$$= \lambda^\ell \bigg( c_1^{\ell,m} + \int_\lambda^R \frac{t^{-\ell+1}}{2\ell+N-1} \widetilde{\varphi}_{\ell,m}(t)\,dt + O(\lambda^2) \bigg), \quad \textit{as } \lambda \to 0^+,$$

$$\widetilde{\varphi}_{\ell,m}(\lambda) = \lambda^\ell \bigg( d_1^{\ell,m} + \int_\lambda^R \frac{t^{-\ell+1}}{2\ell+N-1} \zeta_{\ell,m}(t)\,dt \bigg) + \lambda^{-(N-1)-\ell} \int_0^\lambda \frac{t^{N+\ell}}{N+2\ell-1} \zeta_{\ell,m}(t)\,dt \tag{82}$$
$$= \lambda^\ell \bigg( d_1^{\ell,m} + \int_\lambda^R \frac{t^{-\ell+1}}{2\ell+N-1} \zeta_{\ell,m}(t)\,dt + O(\lambda) \bigg), \quad \textit{as } \lambda \to 0^+,$$

*where*

$$\zeta_{\ell,m}(\lambda) = \frac{1}{\lambda} \int_{\mathbb{S}^{N-1}} h(\lambda\theta') U(\lambda\theta',0) Y_{\ell,m}(\theta',0)\,dS', \tag{83}$$

*and*

$$c_1^{\ell,m} = R^{-\ell} \int_{\mathbb{S}^N_+} U(R\,\theta) Y_{\ell,m}(\theta)\,dS - \frac{R^{-N-2\ell+1}}{N+2\ell-1} \int_0^R t^{N+\ell} \left( \int_{\mathbb{S}^N_+} V(t\,\theta) Y_{\ell,m}(\theta)\,dS \right) dt, \tag{84}$$

$$d_1^{\ell,m} = R^{-\ell} \int_{\mathbb{S}^N_+} V(R\,\theta) Y_{\ell,m}(\theta)\,dS \tag{85}$$
$$- \frac{R^{-N-2\ell+1}}{N+2\ell-1} \int_0^R t^{N+\ell-1} \left( \int_{\mathbb{S}^{N-1}} h(t\theta') U(t\theta',0) Y_{k,m}(\theta',0)\,dS' \right) dt.$$

*Proof.* From the Parseval identity it follows that

$$H(\lambda) = \int_{\mathbb{S}^N_+} \big( U^2(\lambda\theta) + V^2(\lambda\theta) \big)\,dS = \sum_{k=0}^{\infty} \sum_{m=1}^{M_k} \big( \varphi_{k,m}^2(\lambda) + \widetilde{\varphi}_{k,m}^2(\lambda) \big), \quad \text{for all } 0 < \lambda \leqslant R. \tag{86}$$

In particular (59) and (86) yield, for all $k \geqslant 0$ and $1 \leqslant m \leqslant M_k$,

$$\varphi_{k,m}(\lambda) = O(\lambda^\ell) \quad \text{and} \quad \widetilde{\varphi}_{k,m}(\lambda) = O(\lambda^\ell) \quad \text{as } \lambda \to 0^+. \tag{87}$$

Equations (7) and (8) imply that, for every $k \geqslant 0$ and $1 \leqslant m \leqslant M_k$,

$$\begin{cases} -\varphi''_{k,m}(\lambda) - \frac{N}{\lambda}\varphi'_{k,m}(\lambda) + \frac{k(N-1+k)}{\lambda^2}\,\varphi_{k,m}(\lambda) = \widetilde{\varphi}_{k,m}(\lambda), & \text{in } (0,R), \\ -\widetilde{\varphi}''_{k,m}(\lambda) - \frac{N}{\lambda}\widetilde{\varphi}'_{k,m}(\lambda) + \frac{k(N-1+k)}{\lambda^2}\,\widetilde{\varphi}_{k,m}(\lambda) = \zeta_{k,m}(\lambda), & \text{in } (0,R), \end{cases}$$

where

$$\zeta_{k,m}(\lambda) = \frac{1}{\lambda} \int_{\mathbb{S}^{N-1}} h(\lambda\theta') U(\lambda\theta',0) Y_{k,m}(\theta',0)\,dS'. \tag{88}$$

By direct calculations we have, for some $c_1^{k,m}, c_2^{k,m}, d_1^{k,m}, d_2^{k,m} \in \mathbb{R}$,

$$\begin{aligned}\varphi_{k,m}(\lambda) = \lambda^k\biggl(c_1^{k,m} + \int_\lambda^R \frac{t^{-k+1}}{2k+N-1}\widetilde{\varphi}_{k,m}(t)\,dt\biggr)\\ + \lambda^{-(N-1)-k}\biggl(c_2^{k,m} + \int_\lambda^R \frac{t^{N+k}}{1-N-2k}\widetilde{\varphi}_{k,m}(t)\,dt\biggr),\end{aligned}\tag{89}$$

$$\begin{aligned}\widetilde{\varphi}_{k,m}(\lambda) = \lambda^k\biggl(d_1^{k,m} + \int_\lambda^R \frac{t^{-k+1}}{2k+N-1}\zeta_{k,m}(t)\,dt\biggr)\\ + \lambda^{-(N-1)-k}\biggl(d_2^{k,m} + \int_\lambda^R \frac{t^{N+k}}{1-N-2k}\zeta_{k,m}(t)\,dt\biggr).\end{aligned}\tag{90}$$

We observe that

$$\zeta_{k,m}(\lambda) = \frac{2^{N-1}\sqrt{H(2\lambda)}}{\lambda}\int_{\partial B'_{1/2}} h(2\lambda x)U_{2\lambda}(x,0)Y_{k,m}\bigl(\tfrac{x}{|x|},0\bigr)\,dS'\tag{91}$$

with $U_\lambda$ as in (61). Since $\{U_\lambda\}_\lambda$ is bounded in $H^2(B^+_{1/2})$ in view of (68), from continuity of the trace embedding $H^2(B^+_{1/2}) \hookrightarrow H^{3/2}(B'_{1/2})$ we deduce that $\{\operatorname{Tr} U_\lambda\}_\lambda$ is bounded in $H^1(B'_{1/2})$ and its trace on $\partial B'_{1/2}$ is bounded in $L^2(\partial B'_{1/2})$. Hence from (91) and (59) we conclude that, for all $k \geqslant 0$ and $1 \leqslant m \leqslant M_k$,

$$\zeta_{k,m}(\lambda) = O(\lambda^{\ell-1}) \quad \text{as } \lambda \to 0^+.\tag{92}$$

From (87) and (92) it follows that, for all $1 \leqslant m \leqslant M_\ell$, the functions

$$t \mapsto t^{-\ell+1}\widetilde{\varphi}_{\ell,m}(t), \quad t \mapsto t^{N+\ell}\widetilde{\varphi}_{\ell,m}(t), \quad t \mapsto t^{-\ell+1}\zeta_{\ell,m}(t), \quad t \mapsto t^{N+\ell}\zeta_{\ell,m}(t),$$

belong to $L^1(0,R)$. Hence

$$\lambda^\ell\biggl(c_1^{\ell,m} + \int_\lambda^R \frac{t^{-\ell+1}}{2\ell+N-1}\widetilde{\varphi}_{\ell,m}(t)\,dt\biggr) = o(\lambda^{-(N-1)-\ell}), \quad \text{as } \lambda \to 0^+,$$

$$\lambda^\ell\biggl(d_1^{\ell,m} + \int_\lambda^R \frac{t^{-\ell+1}}{2\ell+N-1}\zeta_{\ell,m}(t)\,dt\biggr) = o(\lambda^{-(N-1)-\ell}), \quad \text{as } \lambda \to 0^+,$$

and consequently, by (87), there must be

$$c_2^{\ell,m} = -\int_0^R \frac{t^{N+\ell}}{1-N-2\ell}\widetilde{\varphi}_{\ell,m}(t)\,dt \quad \text{and} \quad d_2^{\ell,m} = -\int_0^R \frac{t^{N+\ell}}{1-N-2\ell}\zeta_{\ell,m}(t)\,dt.$$

Using (87) and (92), we then deduce that

$$\lambda^{-(N-1)-\ell}\biggl(c_2^{\ell,m} + \int_\lambda^R \tfrac{t^{N+\ell}}{1-N-2\ell}\widetilde{\varphi}_{\ell,m}(t)\,dt\biggr) = \lambda^{-(N-1)-\ell}\int_0^\lambda \tfrac{t^{N+\ell}}{N+2\ell-1}\widetilde{\varphi}_{\ell,m}(t)\,dt = O(\lambda^{\ell+2}),\tag{93}$$

$$\lambda^{-(N-1)-\ell}\biggl(d_2^{\ell,m} + \int_\lambda^R \tfrac{t^{N+\ell}}{1-N-2\ell}\zeta_{\ell,m}(t)\,dt\biggr) = \lambda^{-(N-1)-\ell}\int_0^\lambda \tfrac{t^{N+\ell}}{N+2\ell-1}\zeta_{\ell,m}(t)\,dt = O(\lambda^{\ell+1}),\tag{94}$$

as $\lambda \to 0^+$. From (89), (90), (93), and(94) we deduce (81) and (82). Finally, (84) and (85) follow by computing (81) and (82) for $\lambda = R$ and recalling (80). □

We now prove that $\lim_{r\to 0^+} r^{-2\ell}H(r)$ is strictly positive.

**Lemma 3.4.** *Under the same assumption as in Lemmas 3.2, we have*

$$\lim_{r\to 0^+} r^{-2\ell}H(r) > 0.$$

*Proof.* Let us assume by contradiction that $\lim_{\lambda\to 0^+}\lambda^{-2\ell}H(\lambda)=0$. Then, for all $1\leqslant m\leqslant M_\ell$, (86) would imply that

$$\lim_{\lambda\to 0^+}\lambda^{-\ell}\varphi_{\ell,m}(\lambda)=\lim_{\lambda\to 0^+}\lambda^{-\ell}\widetilde{\varphi}_{\ell,m}(\lambda)=0.$$

Hence, in view of (81) and (82),

$$c_1^{\ell,m}+\int_0^R\frac{t^{-\ell+1}}{2\ell+N-1}\widetilde{\varphi}_{\ell,m}(t)\,dt=0\quad\text{and}\quad d_1^{\ell,m}+\int_0^R\frac{t^{-\ell+1}}{2\ell+N-1}\zeta_{\ell,m}(t)\,dt=0$$

which, in view of (92) and (87), yields

$$\lambda^\ell\bigg(c_1^{\ell,m}+\int_\lambda^R\frac{t^{-\ell+1}}{2\ell+N-1}\widetilde{\varphi}_{\ell,m}(t)\,dt\bigg)=\lambda^\ell\int_0^\lambda\frac{t^{-\ell+1}}{1-2\ell-N}\widetilde{\varphi}_{\ell,m}(t)\,dt=O(\lambda^{\ell+2}) \tag{95}$$

$$\lambda^\ell\bigg(d_1^{\ell,m}+\int_\lambda^R\frac{t^{-\ell+1}}{2\ell+N-1}\zeta_{\ell,m}(t)\,dt\bigg)=\lambda^\ell\int_0^\lambda\frac{t^{-\ell+1}}{1-2\ell-N}\zeta_{\ell,m}(t)\,dt=O(\lambda^{\ell+1}) \tag{96}$$

as $\lambda\to 0^+$. Estimates (81), (82), (95), and (96) imply that

$$\varphi_{\ell,m}(\lambda)=O(\lambda^{\ell+2})\quad\text{and}\quad\widetilde{\varphi}_{\ell,m}(\lambda)=O(\lambda^{\ell+1})\quad\text{as }\lambda\to 0^+\quad\text{for every }1\leqslant m\leqslant M_\ell,$$

namely,

$$\sqrt{H(\lambda)}\,(U_\lambda,Y_{\ell,m})_{L^2(\mathbb{S}^N_+)}=O(\lambda^{\ell+2})\quad\text{and}\quad\sqrt{H(\lambda)}\,(V_\lambda,Y_{\ell,m})_{L^2(\mathbb{S}^N_+)}=O(\lambda^{\ell+1})\quad\text{as }\lambda\to 0^+,$$

for every $1\leqslant m\leqslant M_\ell$. From (60), there exists $K>0$ such that $\sqrt{H(\lambda)}\geqslant K\lambda^{\ell+\frac12}$ for $\lambda$ sufficiently small. Therefore

$$(U_\lambda,Y_{\ell,m})_{L^2(\mathbb{S}^N_+)}=O(\lambda^{\frac32})\quad\text{and}\quad(V_\lambda,Y_{\ell,m})_{L^2(\mathbb{S}^N_+)}=O(\lambda^{\frac12})\quad\text{as }\lambda\to 0^+, \tag{97}$$

for every $1\leqslant m\leqslant M_\ell$. From Lemma 3.1, for every sequence $\lambda_n\to 0^+$, there exist a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and $2M_\ell$ real constants $\beta_{\ell,m},\beta'_{\ell,m}$, $m=1,2,\dots,M_\ell$, such that

$$\sum_{m=1}^{M_\ell}((\beta_{\ell,m})^2+(\beta'_{\ell,m})^2)=1 \tag{98}$$

and

$$U_{\lambda_{n_k}}\to|z|^\ell\sum_{m=1}^{M_\ell}\beta_{\ell,m}Y_{\ell,m}\Big(\frac{z}{|z|}\Big),\quad V_{\lambda_{n_k}}\to|z|^\ell\sum_{m=1}^{M_\ell}\beta'_{\ell,m}Y_{\ell,m}\Big(\frac{z}{|z|}\Big),\quad\text{as }k\to+\infty,$$

weakly in $H^1(B_1^+)$ and hence strongly in $L^2(S_1^+)$. It follows that, for all $m=1,2,\dots,M_\ell$,

$$\beta_{\ell,m}=\lim_{k\to+\infty}(U_{\lambda_{n_k}},Y_{\ell,m})_{L^2(\mathbb{S}^N_+)}\quad\text{and}\quad\beta'_{\ell,m}=\lim_{k\to+\infty}(V_{\lambda_{n_k}},Y_{\ell,m})_{L^2(\mathbb{S}^N_+)}$$

and hence, in view of (97),

$$\beta_{\ell,m}=0\quad\text{and}\quad\beta'_{\ell,m}=0\quad\text{for every }m=1,2,\dots,M_\ell,$$

thus contradicting (98). □

*Proof of Theorem 1.1.* From Lemmas 3.1 and 3.4 there exist $\ell \in \mathbb{N}$ such that, for every sequence $\lambda_n \to 0^+$, there exist a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and $2M_\ell$ real constants $\alpha_{\ell,m}, \alpha'_{\ell,m}$, $m = 1, 2, \ldots, M_\ell$, such that $\sum_{m=1}^{M_\ell}((\alpha_{\ell,m})^2 + (\alpha'_{\ell,m})^2) \neq 0$ and

$$\lambda_{n_k}^{-\ell} U(\lambda_{n_k} z) \to |z|^\ell \sum_{m=1}^{M_\ell} \alpha_{\ell,m} Y_{\ell,m}\Big(\frac{z}{|z|}\Big), \quad \lambda_{n_k}^{-\ell} V(\lambda_{n_k} z) \to |z|^\ell \sum_{m=1}^{M_\ell} \alpha'_{\ell,m} Y_{\ell,m}\Big(\frac{z}{|z|}\Big), \tag{99}$$

strongly in $H^1(B_r^+)$ for all $r \in (0,1)$, and then, by homogeneity, strongly in $H^1(B_1^+)$.

From above, (80), (81), (82), (83), (84), and (85), we deduce that

$$\begin{aligned}
\alpha_{\ell,m} &= \lim_{k\to\infty} \lambda_{n_k}^{-\ell} \int_{\mathbb{S}^N_+} U(\lambda_{n_k}\theta) Y_{\ell,m}(\theta)\, dS \\
&= \lim_{k\to\infty} \lambda_{n_k}^{-\ell} \varphi_{\ell,m}(\lambda_{n_k}) = c_1^{\ell,m} + \int_0^R \frac{t^{-\ell+1}}{2\ell+N-1} \widetilde{\varphi}_{\ell,m}(t)\, dt \\
&= R^{-\ell} \int_{\mathbb{S}^N_+} U(R\theta) Y_{\ell,m}(\theta)\, dS - \frac{R^{-N-2\ell+1}}{N+2\ell-1} \int_0^R t^{N+\ell} \left( \int_{\mathbb{S}^N_+} V(t\theta) Y_{\ell,m}(\theta)\, dS \right) dt \\
&\qquad + \int_0^R \frac{t^{-\ell+1}}{2\ell+N-1} \left( \int_{\mathbb{S}^N_+} V(t\theta) Y_{\ell,m}(\theta)\, dS \right) dt
\end{aligned}$$

and

$$\begin{aligned}
\alpha'_{\ell,m} &= \lim_{k\to\infty} \lambda_{n_k}^{-\ell} \int_{\mathbb{S}^N_+} V(\lambda_{n_k}\theta) Y_{\ell,m}(\theta)\, dS \\
&= \lim_{k\to\infty} \lambda_{n_k}^{-\ell} \widetilde{\varphi}_{\ell,m}(\lambda_{n_k}) = d_1^{\ell,m} + \int_0^R \frac{t^{-\ell+1}}{2\ell+N-1} \zeta_{\ell,m}(t)\, dt \\
&= R^{-\ell} \int_{\mathbb{S}^N_+} V(R\theta) Y_{\ell,m}(\theta)\, dS \\
&\qquad - \frac{R^{-N-2\ell+1}}{N+2\ell-1} \int_0^R t^{N+\ell-1} \left( \int_{\mathbb{S}^{N-1}} h(t\theta') U(t\theta', 0) Y_{k,m}(\theta', 0)\, dS' \right) dt \\
&\qquad + \int_0^R \frac{t^{-\ell}}{2\ell+N-1} \left( \int_{\mathbb{S}^{N-1}} h(t\theta') U(t\theta', 0) Y_{\ell,m}(\theta', 0)\, dS' \right) dt.
\end{aligned}$$

We observe that the coefficients $\alpha_{\ell,m}, \alpha'_{\ell,m}$ depend neither on the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$. Hence the convergences in (99) hold as $\lambda \to 0^+$ and the theorem is proved. □

*Proof of Theorem 1.2.* Let us assume by contradiction that $(U, V) \neq (0, 0)$. Then Theorem 1.1 implies that there exist $\ell \in \mathbb{N}$ such that

$$\lambda^{-\ell} U(\lambda z) \to \widehat{U}(\theta), \quad \lambda^{-\ell} V(\lambda z) \to \widehat{V}(\theta), \tag{100}$$

strongly in $H^1(B_1^+)$, where $(\widehat{U}, \widehat{V}) \neq (0, 0)$.

Assumption (11) implies that $\widehat{U} \equiv 0$. Hence $\widehat{V} \not\equiv 0$. Let us denote $\widetilde{U}_\lambda(z) = \lambda^{-\ell-2} U(\lambda z)$. Then $\widetilde{U}_\lambda$ satisfies

$$-\Delta \widetilde{U}_\lambda(z) = \lambda^{-\ell} V(\lambda z).$$

We have that, for all $\varphi \in C^\infty_{\rm c}(B_1^+)$,

$$\lim_{\lambda\to 0^+}\int_{B_1^+}\nabla\widetilde U_\lambda(z)\cdot\nabla\varphi(z)\,dz=\lim_{\lambda\to 0^+}\int_{B_1^+}\lambda^{-\ell}V(\lambda z)\varphi(z)\,dz=\int_{B_1^+}\widehat V(z)\varphi(z)\,dz.$$

On the other, by assumption (11) we have that

$$\begin{aligned}\lim_{\lambda\to 0^+}\int_{B_1^+}\nabla\widetilde U_\lambda(z)\cdot\nabla\varphi(z)\,dz&=-\lim_{\lambda\to 0^+}\int_{B_1^+}\widetilde U_\lambda(z)\Delta\varphi(z)\,dz\\&=-\lim_{\lambda\to 0^+}\lambda^{-\ell-2}\int_{B_1^+}U(\lambda z)\Delta\varphi(z)\,dz=0.\end{aligned}$$

Therefore we obtain that

$$\int_{B_1^+}\widehat V(z)\varphi(z)\,dz=0\quad\text{for all }\varphi\in C^\infty_{\rm c}(B_1^+)$$

which implies that $\widehat V\equiv 0$ in $B_1^+$, a contradiction. □

## 4. Applications to fourth order problems and higher order fractional equations

In this section we discuss applications of Theorem 1.1 to fourth order problems and higher order fractional equations, by proving Theorems 1.3 and 1.4.

*Proof of Theorem 1.3.* From [15, Proposition 7.2] we have that, if $U\in\mathcal D$, then $U\in H^1(B_R^+)$. Furthermore, [15, Proposition 2.4] implies that, if $U\in\mathcal D$ is a nontrivial weak solution to (14) for some $h\in C^1(\Omega)$, then $V:=\Delta U$ belongs to $H^1(B_R^+)$ for some $R>0$ so that the couple $(U,V)\in H^1(B_R^+)\times H^1(B_R^+)$ is a weak solution to (7) such that $(U,V)\neq(0,0)$. Then statement (i) follows from Theorem 1.1 while (ii) comes from Theorem 1.2. □

*Proof of Theorem 1.4.* In view of [25] (see also [15]), we have that, if $u\in\mathcal D^{3/2}(\mathbb R^N)$, then there exists a unique $U\in\mathcal D$ such that $\Delta^2U=0$ in $\mathbb R^{N+1}_+$ and $\operatorname{Tr}U=u$ on $\mathbb R^{N+1}_+$. Moreover

$$\int_{\mathbb R^{N+1}_+}\Delta U(x,t)\Delta\varphi(x,t)\,dx\,dt=2\,(u,\operatorname{Tr}\varphi)_{\mathcal D^{3/2}(\mathbb R^N)}\tag{101}$$

for all $\varphi\in\mathcal D$. In particular, if $u$ solves (15), we have that $U$ is a weak solution to (14). Let $V=\Delta U$. Since $(-\Delta)^{3/2}u\in(\mathcal D^{1/2}(\mathbb R^N))^*$, by (101) we have that

$$\int_{\mathbb R^{N+1}_+}V(x,t)\Delta\varphi(x,t)\,dx\,dt=2\ {}_{(\mathcal D^{1/2}(\mathbb R^N))^\star}\Big\langle(-\Delta)^{3/2}u,\operatorname{Tr}\varphi\Big\rangle_{\mathcal D^{1/2}(\mathbb R^N)}\tag{102}$$

for all $\varphi\in\mathcal T$ with $\mathcal T$ as in (12). Applying [15, Proposition 2.4] to $V$ we deduce that $V\in H^1(B_r^+)$ for all $r>0$ and hence by (102) and integration by parts we obtain

$$-\int_{\mathbb R^{N+1}_+}\nabla V(x,t)\cdot\nabla\varphi(x,t)\,dx\,dt=2\ {}_{(\mathcal D^{1/2}(\mathbb R^N))^\star}\Big\langle(-\Delta)^{3/2}u,\operatorname{Tr}\varphi\Big\rangle_{\mathcal D^{1/2}(\mathbb R^N)}\tag{103}$$

for all $\varphi\in\mathcal T$.

Since the trace map Tr is continuous from $\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+)$ into $\mathcal{D}^{1/2}(\mathbb{R}^N)$, in view of assumption (16) we have that $W \mapsto {}_{(\mathcal{D}^{1/2}(\mathbb{R}^N))^\star}\big\langle(-\Delta)^{3/2}u, \operatorname{Tr} W\big\rangle_{\mathcal{D}^{1/2}(\mathbb{R}^N)}$ belongs to $(\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+))^\star$. Then, by classical minimization methods, we have that the minimum

$$\min_{W\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+)}\left[\frac12\int_{\mathbb{R}^{N+1}_+}|\nabla W(x,t)|^2\,dx\,dt + 2\ {}_{(\mathcal{D}^{1/2}(\mathbb{R}^N))^\star}\Big\langle(-\Delta)^{3/2}u, \operatorname{Tr} W\Big\rangle_{\mathcal{D}^{1/2}(\mathbb{R}^N)}\right]$$

is attained by some $\widetilde V\in\mathcal{D}^{1,2}(\mathbb{R}^{N+1}_+)$ weakly solving

$$\begin{aligned}-\int_{\mathbb{R}^{N+1}_+}\nabla\widetilde V(x,t)\cdot\nabla\varphi(x,t)\,dx\,dt &= 2\ {}_{(\mathcal{D}^{1/2}(\mathbb{R}^N))^\star}\Big\langle(-\Delta)^{3/2}u, \operatorname{Tr}\varphi\Big\rangle_{\mathcal{D}^{1/2}(\mathbb{R}^N)}\\ &= 2\int_{\mathbb{R}^N}|\xi|^3\widehat u\,\overline{\widehat{\operatorname{Tr}\varphi}}\,d\xi\end{aligned}\tag{104}$$

for all $\varphi\in C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$. Combining (103) and (104) we infer that

$$\int_{\mathbb{R}^{N+1}_+}\nabla(V(x,t)-\widetilde V(x,t))\cdot\nabla\varphi(x,t)\,dxdt = 0\qquad\text{for all }\varphi\in\mathcal{T}.\tag{105}$$

Actually (105) still holds true for any $\varphi\in C^\infty_c(\overline{\mathbb{R}^{N+1}_+})$. Indeed, for any $\varphi\in C^\infty_c(\overline{\mathbb{R}^{N+1}})$, one can test (105) with $\varphi_k(x,t)=\varphi(x,t)-\varphi_t(x,0)\,t\,\eta(kt)$, $k\in\mathbb{N}$, where $\eta\in C^\infty_c(\mathbb{R})$, $0\leqslant\eta\leqslant 1$, $\eta(t)=1$ for any $t\in[-1,1]$ and $\eta(t)=0$ for any $t\in(-\infty,-2]\cup[2,+\infty)$, and pass to the limit as $k\to+\infty$. Therefore, if we define

$$\widetilde{W}=\begin{cases}V(x,t)-\widetilde V(x,t), & \text{if } t\geqslant 0,\\ V(x,-t)-\widetilde V(x,-t), & \text{if } t<0,\end{cases}$$

we easily deduce that $\int_{\mathbb{R}^{N+1}}\nabla\widetilde{W}\cdot\nabla\varphi\,dz=0$ for all $\varphi\in C^\infty_c(\mathbb{R}^{N+1})$. In particular $\widetilde{W}$ is harmonic in $\mathbb{R}^{N+1}$. Furthermore, since $V\in L^2(\mathbb{R}^{N+1}_+)$ and $\widetilde V\in\mathcal{D}^{1,2}(\overline{\mathbb{R}^{N+1}_+})$, we have that $\widetilde{W}=W_1+W_2$ for some $W_1\in L^2(\mathbb{R}^{N+1})$ and $W_2\in L^{\frac{2(N+1)}{N-1}}(\mathbb{R}^{N+1})$. The mean value property for harmonic functions ensures that, for every $z\in\mathbb{R}^{N+1}$ and $R>0$,

$$\begin{aligned}|\widetilde{W}(z)| &= \frac{1}{|B(z,R)|_{N+1}}\left|\int_{B(z,R)}\widetilde{W}(y)\,dy\right|\leqslant\frac{\text{const}}{R^{N+1}}\left(\int_{B(z,R)}|W_1(y)|\,dy+\int_{B(z,R)}|W_2(y)|\,dy\right)\\ &\leqslant\frac{\text{const}}{R^{N+1}}\left(\|W_1\|_{L^2(\mathbb{R}^{N+1})}R^{\frac{N+1}{2}}+\|W_2\|_{L^{\frac{2(N+1)}{N-1}}(\mathbb{R}^{N+1})}R^{\frac{N+3}{2}}\right)\end{aligned}$$

where $|\cdot|_{N+1}$ stands for the Lebesgue measure in $\mathbb{R}^{N+1}$ and const is a positive constant independent of $z$ and $R$ which could vary from line to line. Since the right hand side of the previous inequality tends to 0 as $R\to+\infty$, we deduce that $\widetilde{W}\equiv 0$, and then $\widetilde V=V$. In particular, in view of [5] and (104), this implies that

$$(v,\varphi)_{\mathcal{D}^{1/2}(\mathbb{R}^N)}=-2\,(u,\varphi)_{\mathcal{D}^{3/2}(\mathbb{R}^N)}\quad\text{for all }\varphi\in C^\infty_c(\mathbb{R}^N),$$

where we put $v=\operatorname{Tr}V$. This implies that $-2\,|\xi|^3\widehat u=|\xi|\widehat v$ and hence $v=2\,\Delta u$ in $\mathbb{R}^N$.

To prove (i), it is not restrictive to assume $x_0=0$. Let us assume, by contradiction, that $u\not\equiv 0$. Then the couple $(U,V)\neq(0,0)$ is a weak solution to (7) in $H^1(B^+_R)\times H^1(B^+_R)$ for some $R>0$ with $h=-2a$.

From Theorem 1.1 it follows that either $u$ or $v$ (which are the traces of $U$ and $V$ respectively) have vanishing order $\ell \in \mathbb{N}$ at 0. In view of assumption (17) we have that necessarily $V$ vanishes of order $\ell$, i.e. there exists $\Psi : \mathbb{S}^N_+ \to \mathbb{R}$, a nontrivial linear combination of spherical harmonics symmetric with respect to the equator $t = 0$, such that $\Psi \not\equiv 0$ on $\partial\mathbb{S}^N_+$,

$$\lambda^{-\ell} V(\lambda z) \to |z|^\ell \Psi\Big(\frac{z}{|z|}\Big) \qquad \text{strongly in } H^1(B_1^+) \text{ as } \lambda \to 0^+$$

and consequently

$$\lambda^{-\ell} v(\lambda x) \to |x|^\ell \Psi\Big(\frac{x}{|x|}, 0\Big) \qquad \text{strongly in } H^{1/2}(B_1') \text{ as } \lambda \to 0^+.$$

Let us denote

$$v_\lambda(x) = \lambda^{-\ell} v(\lambda x) \quad \text{and} \quad \widetilde{u}_\lambda(x) = \lambda^{-2-\ell} u(\lambda x),$$

so that

$$v_\lambda \to |x|^\ell \Psi\Big(\frac{x}{|x|}, 0\Big) \qquad \text{strongly in } H^{1/2}(B_1') \text{ as } \lambda \to 0^+ \tag{106}$$

and

$$2\Delta \widetilde{u}_\lambda = v_\lambda \text{ in } \mathbb{R}^N.$$

For every $\varphi \in C^\infty_{\rm c}(B_1')$ we have that

$$-2\int_{\mathbb{R}^N} \widetilde{u}_\lambda(-\Delta\varphi)\,dx = -2\int_{\mathbb{R}^N} \varphi(-\Delta\widetilde{u}_\lambda)\,dx = \int_{\mathbb{R}^N} \varphi v_\lambda\,dx. \tag{107}$$

From one hand, assumption (17) implies that

$$\lim_{\lambda\to 0^+} \int_{\mathbb{R}^N} \widetilde{u}_\lambda(-\Delta\varphi)\,dx = 0$$

whereas convergence (106) yields

$$\lim_{\lambda\to 0^+} \int_{\mathbb{R}^N} \varphi v_\lambda\,dx = \int_{\mathbb{R}^N} |x|^\ell \Psi\Big(\frac{x}{|x|}, 0\Big)\varphi(x)\,dx.$$

Hence passing to the limit in (107) we obtain that

$$\int_{\mathbb{R}^N} |x|^\ell \Psi\Big(\frac{x}{|x|}, 0\Big)\varphi(x)\,dx = 0 \quad \text{for every } \varphi \in C^\infty_{\rm c}(B_1'),$$

thus contradicting the fact that $|x|^\ell \Psi\big(\frac{x}{|x|}, 0\big) \not\equiv 0$.

To prove (ii), let us assume by contradiction, that $u \not\equiv 0$ in $\Omega$ and $u(x) = 0$ a.e. in a set $E \subset \Omega$ with $|E|_N > 0$, where $|\cdot|_N$ denotes the $N$-dimensional Lebesgue measure. Since $2\Delta u = v$ and $v \in \mathcal{D}^{1/2}(\mathbb{R}^N) \subset L^2_{\rm loc}(\mathbb{R}^N)$, by classical regularity theory we have that $u \in H^2_{\rm loc}(\Omega)$. Since $u(x) = 0$ for a.e. $x \in E$, we have that $\nabla u(x) = 0$ for a.e. $x \in E$ and hence, since $\frac{\partial u}{\partial x_i} \in H^1_{\rm loc}(\Omega)$ for every $i$, $\Delta u = 0$ a.e. in $E$. In particular there exists a set $E' \subset E \subset \Omega$ with $|E'|_N > 0$ such that $u(x) = \Delta u(x) = 0$ a.e. in $E'$. In particular $v(x) = 0$ a.e. in $E'$.

By Lebesgue's density Theorem, a.e. point of $E'$ is a density point of $E'$. Let $x_0$ be a density point of $E'$. Hence, for all $\varepsilon > 0$ there exists $r_0 = r_0(\varepsilon) \in (0,1)$ such that, for all $r \in (0, r_0)$,

$$\frac{|(\mathbb{R}^N \setminus E') \cap B_r'(x_0)|_N}{|B_r'(x_0)|_N} < \varepsilon, \tag{108}$$

where $B'_r(x_0) = \{x \in \mathbb{R}^N : |x - x_0| < r\}$. Theorem 1.1 implies that there exist $\Psi_1, \Psi_2 : \mathbb{S}^N_+ \to \mathbb{R}$ linear combination of spherical harmonics such that either $\Psi_1 \not\equiv 0$ or $\Psi_2 \not\equiv 0$ and

$$\lambda^{-\ell} u(x_0 + \lambda(x - x_0)) \to |x - x_0|^{\ell} \Psi_1\Big(\frac{x - x_0}{|x - x_0|}, 0\Big) \tag{109}$$

and

$$\lambda^{-\ell} v(x_0 + \lambda(x - x_0)) \to |x - x_0|^{\ell} \Psi_2\Big(\frac{x - x_0}{|x - x_0|}, 0\Big) \tag{110}$$

strongly in $H^{1/2}(B'_1(x_0))$ as $\lambda \to 0^+$ and then strongly in $L^{\frac{2N}{N-1}}(B'_1(x_0))$ thanks to the Sobolev embedding $H^{\frac{1}{2}}(B'_1(x_0)) \hookrightarrow L^{\frac{2N}{N-1}}(B'_1(x_0))$. Since $u \equiv v \equiv 0$ in $E'$, by (108) we have

$$\begin{aligned}
\int_{B'_r(x_0)} u^2(x)\,dx &= \int_{(\mathbb{R}^N \setminus E') \cap B'_r(x_0)} u^2(x)\,dx \\
&\leqslant \Big( \int_{(\mathbb{R}^N \setminus E') \cap B'_r(x_0)} |u(x)|^{2N/(N-1)} dx \Big)^{\frac{N-1}{N}} |(\mathbb{R}^N \setminus E') \cap B'_r(x_0)|_N^{1/N} \\
&< \varepsilon^{1/N} |B'_r(x_0)|_N^{1/N} \Big( \int_{(\mathbb{R}^N \setminus E') \cap B'_r(x_0)} |u(x)|^{2N/(N-1)} dx \Big)^{\frac{N-1}{N}}
\end{aligned}$$

and similarly

$$\int_{B'_r(x_0)} v^2(x)\,dx < \varepsilon^{1/N} |B'_r(x_0)|_N^{1/N} \Big( \int_{(\mathbb{R}^N \setminus E') \cap B'_r(x_0)} |v(x)|^{2N/(N-1)} dx \Big)^{\frac{N-1}{N}}$$

for all $r \in (0, r_0)$. Then, letting $u^r(x) := r^{-\ell} u(x_0 + r(x - x_0))$ and $v^r(x) := r^{-\ell} v(x_0 + r(x - x_0))$,

$$\int_{B'_1(x_0)} |u^r(x)|^2 dx < \Big(\frac{\omega_{N-1}}{N}\Big)^{\frac{1}{N}} \varepsilon^{\frac{1}{N}} \Big( \int_{B'_1(x_0)} |u^r(x)|^{\frac{2N}{N-1}} dx \Big)^{\frac{N-1}{N}},$$

$$\int_{B'_1(x_0)} |v^r(x)|^2 dx < \Big(\frac{\omega_{N-1}}{N}\Big)^{\frac{1}{N}} \varepsilon^{\frac{1}{N}} \Big( \int_{B'_1(x_0)} |v^r(x)|^{\frac{2N}{N-1}} dx \Big)^{\frac{N-1}{N}},$$

for all $r \in (0, r_0)$, where $\omega_{N-1} = \int_{\mathbb{S}^{N-1}} 1\, dS'$. Letting $r \to 0^+$, from (109) and (110) we have that

$$\begin{aligned}
&\int_{B'_1(x_0)} |x - x_0|^{2\ell} \Psi_i^2\Big(\tfrac{x - x_0}{|x - x_0|}, 0\Big)\,dx \\
&\qquad \leqslant \Big(\frac{\omega_{N-1}}{N}\Big)^{\frac{1}{N}} \varepsilon^{\frac{1}{N}} \Big( \int_{B'_1(x_0)} |x - x_0|^{\frac{2N\ell}{N-1}} \Big|\Psi_i\Big(\tfrac{x - x_0}{|x - x_0|}, 0\Big)\Big|^{\frac{2N}{N-1}} dx \Big)^{\frac{N-1}{N}} \quad \text{for } i = 1, 2,
\end{aligned}$$

which yields a contradiction as $\varepsilon \to 0^+$, since either $\Psi_1 \not\equiv 0$ or $\Psi_2 \not\equiv 0$. □

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Veronica Felli,
Università degli Studi di Milano – Bicocca,
Dipartimento di Matematica e Applicazioni,
Via Cozzi 55, 20125 Milano, Italy.

Alberto Ferrero,
Università del Piemonte Orientale,
Dipartimento di Scienze e Innovazione Tecnologica,
Viale Teresa Michel 11, 15121 Alessandria, Italy.

*E-mail addresses:* veronica.felli@unimib.it, alberto.ferrero@uniupo.it.