Sharp Convergence Rate of Eigenvalues in a Domain with a Shrinking Tube

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Abstract

In this paper we consider a class of singularly perturbed domains, obtained by attaching a cylindrical tube to a fixed bounded region and letting its section shrink to zero. We use an Almgren-type monotonicity formula to evaluate the sharp convergence rate of perturbed simple eigenvalues, via Courant-Fischer Min-Max characterization and blow-up analysis for scaled eigenfunctions.

Keywords. Singularly perturbed domains, asymptotics of eigenvalues, monotonicity formula.

MSC classification. Primary: 35P20; Secondary: 35P15, 35J25.

1 Introduction and Main Results

The purpose of this work is to investigate the behaviour of the eigenvalues of the Dirichlet-Laplacian in a class of singularly perturbed domains: in particular we are interested in the sharp convergence rate of the eigenvalue variation, i.e. in the evaluation of the leading term in its asymptotic expansion. The perturbation consists in attaching a cylindrical tube to a fixed domain and letting the section of the tube shrink.

Let $N \geq 2$ and $\Omega \subseteq \mathbb{R}^N$ be open, bounded and connected. Suppose that $0 \in \partial\Omega$ and that $\partial\Omega$ is flat in a neighbourhood of the origin, namely

$$\exists\, R_{\max} > 1 \quad \text{such that} \quad M := \{(x_1, \dots, x_N) \in \mathbb{R}^N \colon x_1 = 0,\ |x| \leq R_{\max}\} \subseteq \partial\Omega. \tag{1}$$

Using the following notation for the positive half-space, half-balls and half-spheres

$$\mathbb{R}^N_+ := \{(x_1, \dots, x_N) \in \mathbb{R}^N \colon x_1 > 0\},$$
$$B_r^+ := \{x \in \mathbb{R}^N_+ \colon |x| < r\}, \qquad S_r^+ := \{x \in \mathbb{R}^N_+ \colon |x| = r\},$$

we can suppose, without losing generality, that

$$B^+_{R_{\max}} \subseteq \Omega \subseteq \mathbb{R}^N_+.$$

Let $\Sigma \subset\subset M$ be open, connected and containing the origin 0. For simplicity of exposition we assume that $\partial\Sigma$ is of class C^2; although this regularity assumption can be relaxed, see Remark 3.7. Moreover we assume, for sake of simplicity, that the radius of Σ in $\mathbb{R}^{N-1}$ is 1, i.e.

$$\max_{x \in \partial\Sigma} |x| = 1.$$

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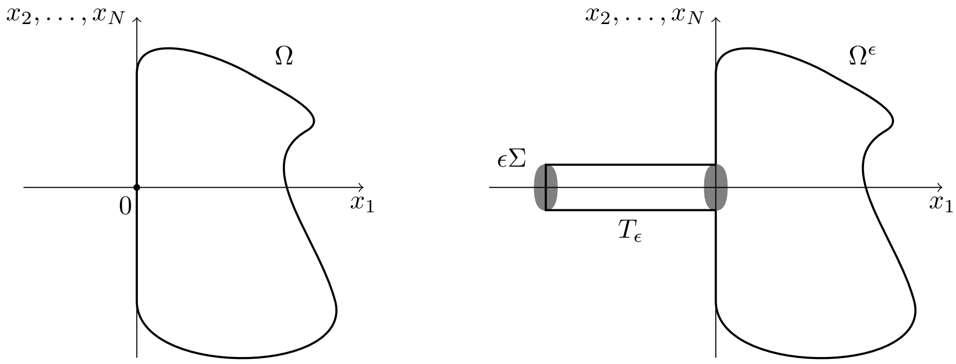


(a) *Unperturbed Domain* (b) *Perturbed Domain*

Figure 1

Finally we assume that Σ is **starshaped** with respect to 0, i.e.

$$x \cdot \boldsymbol{\nu} \geq 0 \text{ for all } x \in \partial\Sigma, \tag{2}$$

where $\boldsymbol{\nu}$ denotes the exterior unit normal vector to $\partial\Sigma$. Let $\epsilon \in \mathbb{R}$, with $0 < \epsilon \leq 1$, and let $T_\epsilon := (-1, 0] \times \epsilon\Sigma$ be a cylindrical tube with section $\epsilon\Sigma = \{\epsilon x \colon x \in \Sigma\}$. Let us denote by

$$\Omega^\epsilon = \Omega \cup T_\epsilon \tag{3}$$

the *perturbed* domain (see Figure 1).

Let $p \in L^\infty(\mathbb{R}^N)$ be a weight function such that $p \geq 0$ a.e. and $p \not\equiv 0$ in Ω. For any open, bounded set $\omega \subseteq \mathbb{R}^N$, we consider the weighted Dirichlet eigenvalue problem for the Laplacian on ω

$$\begin{cases} -\Delta\varphi = \lambda p \varphi, & \text{in } \omega, \\ \varphi = 0, & \text{on } \partial\omega. \end{cases} \tag{E_ω}$$

By classical spectral theory we have that, if $p \not\equiv 0$ in ω, there exists a sequence of positive eigenvalues of (E_ω)

$$0 < \lambda_1(\omega) < \lambda_2(\omega) \leq \lambda_3(\omega) \leq \dots$$

repeated according to their multiplicity. We denote by $(\lambda_n)_n := (\lambda_n(\Omega))_n$ the sequence of eigenvalues of the unperturbed problem (E_Ω), and by $(\varphi_n)_n$ a corresponding sequence of eigenfunctions such that $\int_\Omega p|\varphi_n|^2 \, dx = 1$ and $\int_\Omega p\varphi_n\varphi_m \, dx = 0$ if $n \neq m$. Similarly, we denote by $(\lambda_n^\epsilon)_n := (\lambda_n(\Omega^\epsilon))_n$ and $(\varphi_n^\epsilon)_n$ the sequences of eigenvalues and eigenfunctions of the perturbed problem (E_{Ω^ϵ}), such that $\int_\Omega p|\varphi_n^\epsilon|^2 \, dx = 1$ and $\int_\Omega p\varphi_n^\epsilon\varphi_m^\epsilon \, dx = 0$ if $n \neq m$.

Let $j \in \mathbb{N}$ be such that

$$\lambda_j \quad \text{is } \textbf{simple}. \tag{4}$$

Assumption (4) is not so restrictive: indeed, the simplicity of all eigenvalues is a generic property with respect to perturbations of the domain , see [33, 37].

Classical results (see for instance [13, 19]) ensure the continuity with respect to our domain perturbation, i.e. λ_j^ϵ is simple for ϵ small and

$$\lambda_j^\epsilon \longrightarrow \lambda_j, \qquad \text{as } \epsilon \to 0. \tag{5}$$

Furthermore, for every ϵ we can choose the eigenfunction φ_j^ϵ in such a way that

$$\varphi_j^\epsilon \longrightarrow \varphi_j, \qquad \text{in } H_0^1(\Omega^1), \quad \text{as } \epsilon \to 0, \tag{6}$$

where the functions are trivially extended in Ω^1 outside their domains.

The main goal of this paper is to find the exact asymptotics of the difference $\lambda_j - \lambda_j^\epsilon$ as $\epsilon \to 0$.

The problem of convergence of eigenvalues and eigenfunctions of the Dirichlet-Laplacian with respect to perturbations of the domain has been widely studied in the past. For instance, for general perturbations that cover the shrinking tube, in [9] the authors investigated the stability of the spectrum with respect to general scalar products, while [12] dealt with the convergence of solutions of a nonlinear eigenvalue problem (see also [17, 18]). Within an extensive literature, we mention [26], [27] and [15] as detailed surveys. In [20, 14] bounds for the rate of convergence have been found; furthermore, in [36] the framework is pretty similar to ours and the author proved an estimate of the type $\lambda_j - \lambda_j^\epsilon = O(\epsilon^a)$, where a depends only on the distance between λ_j and its neighbours. We mention also that asymptotic expansions of the eigenvalues of the Dirichlet Laplacian in domains with a thin shrinking cylindrical protrusion of finite length were obtained in [23], see also [6] for a related problem in a two-dimensional domain with thin shoots; we notice that Theorem 4.1 in [23] provides the exact vanishing rate of the eigenvalue variation $\lambda_j - \lambda_j^\epsilon$ only when $\nabla\varphi_j(0) \neq 0$, but it does not say what is the leading term in the expansion when $\nabla\varphi_j(0) = 0$. For what concerns Neumann boundary conditions, among many others, we cite [7, 8, 28, 29, 35], which take into account singular perturbations like the shrinking tube.

A motivation for the interest in studying the spectral behaviour of the Laplacian on thin branching domains comes from physics: for instance, it occurs in the theory of quantum graphs, which models the propagation of waves in quasi one-dimensional structures, like quantum wires, narrow waveguides, photonic crystals, blood vessels, etc. (see e.g. [11, 31] and reference therein). Moreover, this topic is also related with engineering problems, such as elasticity and multi-structure problems, as well explained in surveys [16, 34].

The starting points of this work are [23] and [2, 3, 22]. On the one hand, the present paper aims at providing a criterion for selecting the leading term in the asymptotic expansion given in [23], based on the vanishing order of the limit eigenfunction at the junction; on the other hand, it improves and generalizes some results of [2]. We note that [2] (as well as many of the aforementioned articles) deals with dumbbell domains in which the tubular handle is vanishing. However, from the point of view of both the expected results and the technical approach, our method does not require substantial adaptions to treat also the dumbbell case; hence for the sake of simplicity of exposition, in the present paper we consider only perturbations of type (3).

In order to state our main results, we first need to recall some known facts. Let us consider the eigenvalue problem for the standard Laplacian on the $(N-1)$-dimensional unit sphere

$$-\Delta_{\mathbb{S}^{N-1}}\psi = \mu\psi \qquad \text{in } \mathbb{S}^{N-1}. \tag{7}$$

It is well known that the eigenvalues of (7) are $\mu_k = k(k+N-2)$, for $k = 0, 1, \dots$ and that their multiplicities are (see [10])

$$m_k = \binom{k+N-2}{k} + \binom{k+N-3}{k-1}.$$

If E_k denotes the eigenspace of the eigenvalue μ_k, then $\bigoplus_{k\geq 0} E_k = L^2(\mathbb{S}^{N-1})$. Furthermore it is known that the elements of E_k are spherical harmonics, i.e. homogeneous polynomials (of N variables) of degree k. We are interested in eigenfunctions of (7) that vanish on $\{x_1 = 0\}$, so let us call

$$E_k^0 := \{\psi \in E_k \colon \psi(0, \theta_2, \dots, \theta_N) = 0\}.$$

It is well known (see e.g. [21, Th. 1.3]) that the local behaviour at $0 \in \partial\Omega$ of eigenfunctions of (E_Ω) can be described in term of spherical harmonics vanishing on $\{x_1 = 0\}$. In particular there exist $k \in \mathbb{N}$, $k \geq 1$, and $\Psi \in E_k^0$, $\Psi \neq 0$, such that

$$r^{-k}\varphi_j(r\theta) \to \Psi \qquad \text{in } C^{1,\tau}(S_1^+), \quad \text{as } r \to 0^+, \tag{8}$$

$$r^{1-k}\nabla\varphi_j(r\theta) \to \nabla\Psi \qquad \text{in } C^{0,\tau}(S_1^+, \mathbb{R}^N), \quad \text{as } r \to 0^+, \tag{9}$$

for all $\tau \in (0,1)$. Furthermore the asymptotic homogeneity order k can be characterized as the limit of an Almgren frequency function (see [5]), i.e.

$$\lim_{r\to 0^+} \frac{r\int_{B_r^+}\left(|\nabla\varphi_j|^2 - \lambda_j|\varphi_j|^2\right)\mathrm{d}x}{\int_{S_r^+}|\varphi_j|^2\,\mathrm{d}S} = k.$$

Hereafter we will denote

$$\psi_k(r\theta) := r^k \Psi(\theta), \quad r \geq 0,\ \theta \in S_1^+. \tag{10}$$

The exact asymptotic estimate of the eigenvalue variation we are going to prove involves a nonzero constant $m_k(\Sigma)$ which admits the following variational characterization. Let us consider the functional

$$J\colon \mathcal{D}^{1,2}(\Pi) \longrightarrow \mathbb{R},$$

$$J(u) := \frac{1}{2}\int_{\Pi} |\nabla u|^2 \,\mathrm{d}x - \int_{\Sigma} u \frac{\partial \psi_k}{\partial x_1} \,\mathrm{d}S,$$

where

$$T_1^- := (-\infty, 0] \times \Sigma, \qquad \Pi := T_1^- \cup \mathbb{R}^N_+,$$

and, for any open set $\omega \subseteq \mathbb{R}^N$, $\mathcal{D}^{1,2}(\omega)$ denotes the completion of the space $C_c^\infty(\omega)$ with respect to the L^2 norm of the gradient (see Section 2 for further details). In dimension 2, we will always deal with spaces $\mathcal{D}^{1,2}(\omega)$ with ω such that $\mathbb{R}^N \setminus \omega$ contains a half-line; in this case $\mathcal{D}^{1,2}(\omega)$ can be characterized as a concrete functional space thanks to the validity of a Hardy inequality also in dimension 2, see Theorem 9.1.

By standard minimization methods, one can prove that J is bounded from below and that the infimum

$$m_k(\Sigma) := \inf_{u \in \mathcal{D}^{1,2}(\Pi)} J(u) \tag{11}$$

is attained by some w_k. Moreover

$$m_k(\Sigma) = -\frac{1}{2}\int_{\Pi} |\nabla w_k|^2 \,\mathrm{d}x = -\frac{1}{2}\int_{\Sigma} \frac{\partial \psi_k}{\partial \boldsymbol{\nu}} w_k \,\mathrm{d}S < 0, \tag{12}$$

see [22]. With this framework in mind we are able to state our first (and main) result.

Theorem 1.1. *Under assumptions* (1), (2) *and* (4), *let k denote the vanishing order of the unperturbed eigenfunction φ_j as in* (8)–(9). *Then*

$$\lim_{\epsilon \to 0} \frac{\lambda_j - \lambda_j^\epsilon}{\epsilon^{N+2k-2}} = C_k(\Sigma),$$

where

$$C_k(\Sigma) = -2 m_k(\Sigma) > 0 \tag{13}$$

and $m_k(\Sigma)$ is defined in (11).

We recall that, for $N \geq 3$, an asymptotic expansion for the eigenvalue variation is constructed using the *concordance method* in [23, Theorem 4.1], but explicit formulas are given only for the first perturbed coefficient, which turns out to be a multiple of $|\nabla \varphi_j(0)|^2$; in dimension $N = 2$, [23, Theorem 10.1] performs a more detailed asymptotic analysis with the computation of all the coefficients. Hence, for $N \geq 3$, [23] finds outs what is the leading term in the asymptotic expansion only when $\nabla \varphi_j(0) \neq 0$. We emphasize that, differently from [23], Theorem 1.1 detects the exact vanishing rate of $\lambda_j - \lambda_j^\epsilon$ also when $\nabla \varphi_j(0) = 0$ and $N \geq 3$; more precisely it establishes a direct correspondence between the order of the infinitesimal $\lambda_j - \lambda_j^\epsilon$ and the number k, which is the order of vanishing of φ_j at the junction point 0.

The proof of Theorem 1.1 is based on lower and upper bounds for the difference $\lambda_j - \lambda_j^\epsilon$ carried out using the Min-Max Courant-Fischer characterization of the eigenvalues, see Section 6. To obtain the exact asymptotics for the eigenvalue variation it is crucial to sharply control the energy of perturbed eigenfunctions in neighbourhoods of the junction with radius of order ϵ. The sharpness of our energy estimates is related to the identification of a nontrivial limit profile for blow-up of scaled eigenfunctions, as stated in the following theorem.

Theorem 1.2. *Under the same assumptions of Theorem 1.1, let φ_j^ϵ be chosen as in* (6). *Then*

$$\epsilon^{-k} \varphi_j^\epsilon(\epsilon x) \to \Phi(x) \qquad \text{as } \epsilon \to 0,$$

in $H^1(T_1^- \cup B_R^+)$ for all $R > 1$, where $\Phi := w_k + \psi_k$, being w_k the minimizer for (11) *and ψ_k the homogeneous function defined in* (10).

As mentioned before, Theorem 1.1 generalizes and improves [2, Th. 1.1]: indeed, in [2] the weight p was assumed to vanish in a neighbourhood of the junction Σ and only the case of vanishing order $k = 1$ for the unperturbed eigenfunction φ_j was considered. Furthermore the dimension $N = 2$ was not included in [2]. As in [2], a fundamental tool for the proof of the energy estimates needed to study the local behaviour of eigenfunctions is an Almgren-type monotonicity formula, which was first introduced by Almgren [5] and then used by Garofalo and Lin [24] to study unique continuation properties for elliptic partial differential equations.

In the particular case treated in [2, 22], precise pointwise estimates from above and from below for the perturbed eigenfunction and its gradient were directly obtained via comparison and maximum principles: indeed, if the limit eigenfunction has minimal vanishing order at the origin and the weight vanishes around the junction, then such eigenfunction has a fixed sign and is harmonic in a neighbourhood of 0. These estimates were used in [22] to get rid of a remainder term in the derivative of the Almgren quotient for the perturbed problem, however they are not available in the more general framework of the present paper. Nevertheless, under the geometric assumption (2) on the tube section we succeed in proving that the remainder term has a positive sign, thus obtaining the monotonicity formula, see Proposition 5.9. We also point out that the 2-dimensional case requires the proof of an ad hoc Hardy type inequality for functions vanishing on a fixed half-line, see (117).

We observe that in [2, 22] the limit of the blow-up family $\epsilon^{-k}\varphi_j^\epsilon(\epsilon x)$ was recognized by its frequency at infinity, which must be necessarily equal to the minimal one, i.e. 1, in the particular case $k = 1$. In the general case $k \geq 1$, the monotonicity argument implies that the frequency of the limit profile is less than or equal to k, and this seems to be not enough for a univocal identification. To overcome this difficulty, we use here an argument inspired by [1] and based on a local inversion result giving an energy control for the difference between the blow-up eigenfunction and a k-homogeneous profile, see Corollary 7.4.

The paper is organized as follows. After some preliminary results in Section 2, in Section 3 we prove a Pohozaev-type identity, which is combined with the Poincaré inequalities of Section 4 to develop a monotonicity argument in Section 5. From the monotonicity formula established in Corollary 5.10, we derive some local energy estimates which allow us to deduce sharp upper and lower bounds for the eigenvalue variation in Section 6. In Section 7 we perform a blow-up analysis for scaled eigenfunctions from which we deduce first Theorem 1.2 and then, in Section 8, our main result Theorem 1.1. Finally, in the appendix we recall an Hardy type inequality in dimension 2 for functions vanishing on half-lines and an abstract lemma on maxima of quadratic forms.

2 Preliminaries and Notation

In this section we introduce some basic definitions and notation which will be useful in the rest of the paper. We start fixing some notation:

$$\begin{aligned}
&\Omega_r^\epsilon := T_\epsilon \cup B_r^+, \quad \epsilon \in (0,1),\ r \in (\epsilon, R_{\max}),\\
&\mathcal{C}_r := \partial B_r^+ \setminus S_r^+, \quad r \in (0, R_{\max}),\\
&\Pi_r := T_1^- \cup B_r^+, \quad r > 1.
\end{aligned}$$

For any measurable set $\omega \subseteq \mathbb{R}^N$, we denote as $|\omega|$ its N-dimensional Lebesgue measure.

For any $R \geq 2$, we will denote as η_R a cut-off function satisfying

$$\begin{gathered}
\eta_R \in C^\infty(\overline{\Pi}), \quad \eta_R(x) = \begin{cases} 1, & \text{for } x \in \Pi \setminus \Pi_R, \\ 0, & \text{for } x \in \Pi_{R/2}, \end{cases} \\
|\eta_R(x)| \leq 1, \quad |\nabla \eta_R(x)| \leq 4/R \qquad \text{for all } x \in \Pi.
\end{gathered} \tag{14}$$

We now recall a well known quantitative result about the first eigenvalue of the Dirichlet-Laplacian on bounded domains.

Theorem 2.1 (Faber-Krahn Inequality)**.** *Let $\omega \subseteq \mathbb{R}^N$ be open and bounded and let $\lambda_1^D(\omega)$ denote* *the first eigenvalue of the Dirichlet-Laplacian on ω. Then*

$$\lambda_1^D(\omega) \geq \frac{\lambda_1^D(B_1)|B_1|^{2/N}}{|\omega|^{2/N}},$$

where B_1 denotes the N-dimensional ball centered at the origin and with radius 1.

Moreover, if we denote

$$C_N := \frac{1}{|B_1|^{2/N}\lambda_1^D(B_1)}, \tag{15}$$

and we combine the previous Theorem with the usual Poincaré Inequality, we have that

$$\int_\omega |u|^2 \,\mathrm{d}x \leq C_N |\omega|^{2/N} \int_\omega |\nabla u|^2 \,\mathrm{d}x \quad \text{for all } u \in H_0^1(\omega). \tag{16}$$

2.1 The Space $\mathcal{H}_R$

For $R > 1$ let us define the function space $\mathcal{H}_R$ as the completion of $C_c^\infty(\Pi_R \cup S_R^+)$ with respect to the norm induced by the scalar product

$$(u, v)_{\mathcal{H}_R} := \int_{\Pi_R} \nabla u \cdot \nabla v \,\mathrm{d}x.$$

Since Π_R is bounded in at least 1 direction, the Poincaré Inequality holds. Hence $\mathcal{H}_R \hookrightarrow H^1(\Pi_R)$ continuously and we have the following characterization

$$\mathcal{H}_R := \left\{u \in H^1(\Pi_R) \colon u = 0 \text{ on } \partial\Pi_R \setminus S_R^+\right\}.$$

Moreover, when $N \geq 3$, the classical Sobolev inequality implies that $\mathcal{H}_R \hookrightarrow L^{2^*}(\Pi_R)$ continuously, where $2^* = \frac{2N}{N-2}$.

2.2 Limit Profiles

In this section we introduce some limit profiles that will appear in the blow-up analysis of scaled eigenfunctions. We recall the following result from [22, Lemma 2.4].

Proposition 2.2. *For every $\psi \in C^2(\mathbb{R}^N_+) \cap C^1(\overline{\mathbb{R}^N_+})$ such that*

$$\begin{cases} -\Delta\psi = 0, & \text{in } \mathbb{R}^N_+, \\ \psi = 0, & \text{on } \partial\mathbb{R}^N_+, \end{cases}$$

there exists a unique $\Phi = \Phi(\psi) : \Pi \to \mathbb{R}$ such that

$$\Phi \in \mathcal{H}_R, \qquad \text{for all } R > 1, \tag{17}$$

$$\begin{cases} -\Delta\Phi = 0, & \text{in } \Pi, \\ \Phi = 0, & \text{on } \partial\Pi, \end{cases} \tag{18}$$

$$\int_\Pi |\nabla(\psi - \Phi)|^2 \,\mathrm{d}x < +\infty. \tag{19}$$

Hereafter we will denote

$$\Phi := \Phi(\psi_k) \tag{20}$$

where ψ_k is the function defined in (10). As observed in [22] we have that

$$\Phi = \begin{cases} \psi_k + w_k, & \text{in } \mathbb{R}^N_+, \\ w_k, & \text{in } \Pi \setminus \mathbb{R}^N_+, \end{cases} \tag{21}$$

where w_k is the function realizing the minimum $m_k(\Sigma)$ in (11). We observe that, in the particular case $N = 2$, the function Φ corresponds to the function X_k introduced in [23, Sections 10-11]. By a classical Dirichlet principle, one can easily obtain the following result.

Lemma 2.3. *For every $R > 1$ there exists a unique function $U_R \in \mathcal{H}_R$ solution to the following minimization problem*

$$\min_{u\in\mathcal{H}_R}\left\{\int_{\Pi_R}|\nabla u|^2\,\mathrm{d}x\colon u=\psi_k \text{ on } S_R^+\right\}.$$

Moreover it weakly solves

$$\begin{cases} -\Delta U_R = 0, & \text{in } \Pi_R,\\ U_R = 0, & \text{on } \partial\Pi_R\setminus S_R^+,\\ U_R=\psi_k, & \text{on } S_R^+.\end{cases}$$

Lemma 2.4. *For every $r > 1$ we have*

$$U_R \longrightarrow \Phi \qquad \text{in } \mathcal{H}_r, \qquad \text{as } R\to+\infty.$$

Proof. We can assume $R > \max\{r, 2\}$. The function $U_R - \Phi$ satisfies, in a weak sense,

$$\begin{cases} -\Delta(U_R-\Phi) = 0, & \text{in } \Pi_R,\\ U_R-\Phi = 0, & \text{on } \partial\Pi_R\setminus S_R^+,\\ U_R-\Phi=\psi_k-\Phi, & \text{on } S_R^+,\end{cases}$$

and then it is the least energy function among those having these boundary conditions. Let $\eta = \eta_R \in C^\infty(\overline{\Pi})$ be the cut-off function defined in (14). Then

$$\begin{aligned}\int_{\Pi_r}|\nabla(U_R-\Phi)|^2\,\mathrm{d}x &\le \int_{\Pi_R}|\nabla(U_R-\Phi)|^2\,\mathrm{d}x \le \int_{\Pi_R}|\nabla(\eta(\psi_k-\Phi))|^2\,\mathrm{d}x \le\\ &\le 2\int_{\Pi_R}|\nabla\eta|^2|\psi_k-\Phi|^2\,\mathrm{d}x + 2\int_{\Pi}|\eta|^2|\nabla(\psi_k-\Phi)|^2\,\mathrm{d}x\le\\ &\le \frac{32}{R^2}\int_{\Pi_R\setminus\Pi_{R/2}}|\psi_k-\Phi|^2\,\mathrm{d}x + 2\int_{\Pi-\Pi_{R/2}}|\nabla(\psi_k-\Phi)|^2\,\mathrm{d}x\le\\ &\le 32\int_{\Pi\setminus\Pi_{R/2}}\frac{|\psi_k-\Phi|^2}{|x|^2}\,\mathrm{d}x + 2\int_{\Pi-\Pi_{R/2}}|\nabla(\psi_k-\Phi)|^2\,\mathrm{d}x\longrightarrow 0\end{aligned}$$

thanks to (19) and Hardy's Inequality. In the case $N = 2$ we use the fact that $1 + |x|^2 \le 2|x|^2$ for $|x| \ge 1$ and the 2-dimensional Hardy's Inequality (117). $\square$

Using again the Dirichlet principle, we construct also the limit profile Z_R as follows.

Lemma 2.5. *For every $R > 1$ there exists a unique function $Z_R \in H^1(B_R^+)$ solution to the following minimization problem*

$$\min_{u\in H^1(B_R^+)}\left\{\int_{B_R^+}|\nabla u|^2\,\mathrm{d}x\colon u=0 \text{ on } \mathcal{C}_R,\ u=\Phi \text{ on } S_R^+\right\}.$$

Moreover it weakly solves

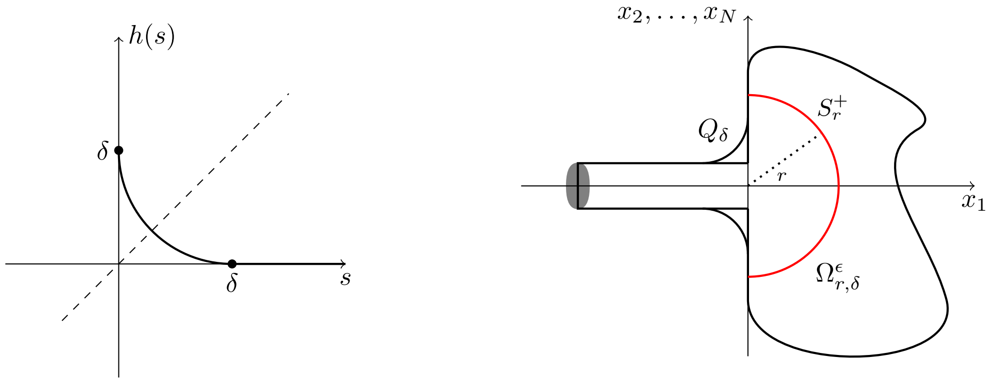
$$\begin{cases} -\Delta Z_R = 0, & \text{in } B_R^+,\\ Z_R = 0, & \text{on } \mathcal{C}_R,\\ Z_R=\Phi, & \text{on } S_R^+.\end{cases}$$

3 A Pohozaev-Type Inequality

The purpose of this section is to prove the following inequality.

Proposition 3.1. *There exists $\tilde\epsilon, \tilde r > 0$, with $0 < \tilde\epsilon < \tilde r \le R_{\max}$, such that, for $\epsilon \in (0, \tilde\epsilon]$, $r \in (\epsilon, \tilde r]$ and $i \in \{1, \dots, j\}$, we have*

$$\int_{S_r^+}|\nabla\varphi_i^\epsilon|^2\,\mathrm{d}S - \frac{N-2}{r}\int_{\Omega_r^\epsilon}|\nabla\varphi_i^\epsilon|^2\,\mathrm{d}x \ge 2\int_{S_r^+}\left(\frac{\partial\varphi_i^\epsilon}{\partial\boldsymbol{\nu}}\right)^2\mathrm{d}S + \frac{2\lambda_i^\epsilon}{r}\int_{\Omega_r^\epsilon}p\varphi_i^\epsilon\nabla\varphi_i^\epsilon\cdot x\,\mathrm{d}x. \tag{22}$$



(a) *The function h* (b) *The Approximating Domain*

Figure 2

We observe that solutions to problems of type

$$\begin{cases} -\Delta u = f, & \text{in } \Omega_r^\epsilon, \\ u = 0, & \text{on } \partial\Omega_r^\epsilon, \end{cases}$$

with $f \in L^2(\Omega_r^\epsilon)$, in general do not belong to $H^2(\Omega_r^\epsilon)$ since $\partial\Omega_r^\epsilon$ is only Lipschitz continuous and doesn't verify a uniform exterior ball condition (which ensures L^2-integrability of second order derivatives, see [4]). But, along a proof of a Pohozaev Identity, one tests the equation with the function $\nabla\varphi_i^\epsilon \cdot x$, which could fail to be H^1 in our case. To overcome this difficulty we implement an approximation process.

3.1 Approximating Domains

Let $\epsilon \in (0,1]$ and $r \in (\epsilon, R_{\max}]$: in order to remove the concave edge $\Gamma_\epsilon := \partial(\epsilon\Sigma)$, we will approximate the domain Ω_r^ϵ with a family of starshaped domains $\Omega_{r,\delta}^\epsilon$ (with $0 \le \delta < r - \epsilon$) such that

$$\Omega_{r,0}^\epsilon = \Omega_r^\epsilon, \quad \Omega_{r,\delta_1}^\epsilon \subset \Omega_{r,\delta_2}^\epsilon \quad \text{for all } 0 \le \delta_1 \le \delta_2 < r - \epsilon,$$

and such that every $\Omega_{r,\delta}^\epsilon$ verifies the uniform exterior ball condition. In particular we will define Q_δ such that

$$\Omega_{r,\delta}^\epsilon = \Omega_r^\epsilon \cup Q_\delta.$$

For $\delta > 0$ small we define a "δ-enlargement" of $\epsilon\Sigma$:

$$\epsilon\Sigma_\delta := \left\{ x \in \mathbb{R}^{N-1} \setminus \epsilon\Sigma \colon \operatorname{dist}(x, \Gamma_\epsilon) < \delta \right\}.$$

Let $h \in C^\infty((0,+\infty)) \cap C([0,+\infty))$ such that $h(0) = \delta$, $h(s) = 0$ for $s \in [\delta, +\infty)$, $h'(s) < 0$ for all $s \in (0,\delta)$ and $h^{-1}(s) = h(s)$ for $s \in (0,\delta)$. We define $G \colon \epsilon\Sigma_\delta \subset \mathbb{R}^{N-1} \to \mathbb{R}$ as

$$G(z) := -h(d(z)),$$

where $d(z) = \operatorname{dist}(z, \Gamma_\epsilon)$. Now, let

$$Q_\delta = \left\{ x \in \mathbb{R}^N \colon (0, x_2, \dots, x_N) \in \epsilon\Sigma_\delta \text{ and } G(x_2, \dots, x_N) < x_1 \le 0 \right\},$$

see Figure 2.

For what concerns the regularity we observe that, since Γ_ϵ is of class C^2, then also d and the graph of G are of class C^2 (see [30]). Moreover it is easy to verify that the approximating domain $\Omega_{r,\delta}^\epsilon$ satisfies the uniform exterior ball condition.

Remark 3.2. We point out that $\Omega^\epsilon_{r,\delta}$ is starshaped. Indeed, first, if $x \in \mathcal{C}_r \setminus (\epsilon\Sigma_\delta \cup \epsilon\Sigma)$, trivially $x \cdot \boldsymbol{\nu}(x) = 0$. If $x \in \{x_1 = -\epsilon\} \cap \partial\Omega^\epsilon_{r,\delta}$, then $x \cdot \boldsymbol{\nu}(x) = -x_1 = \epsilon > 0$. Third, if $x \in (-\epsilon, -\delta) \times \Gamma_\epsilon$, then $\boldsymbol{\nu}(x) = \boldsymbol{\nu}_0(x')$, where $x' = (x_2, \dots, x_N)$ and $\boldsymbol{\nu}_0(x')$ is the exterior unit normal of Γ_ϵ; thus $x \cdot \boldsymbol{\nu}(x) = x' \cdot \boldsymbol{\nu}_0(x') \geq 0$ by (2). Finally, let $x \in \{(G(x'), x') \colon x' \in \epsilon\Sigma_\delta\}$: in this case we have that

$$\boldsymbol{\nu}(x) = \frac{(-1, -h'(d(x'))\nabla d(x'))}{\|(-1, -h'(d(x'))\nabla d(x'))\|},$$

and so

$$x \cdot \boldsymbol{\nu}(x) = \frac{-G(x') - h'(d(x'))\nabla d(x') \cdot x'}{\|(-1, -h'(d(x'))\nabla d(x'))\|} \geq 0,$$

where we used the properties of the functions G and h and the fact that, since $0 \in \Sigma$ and Σ is starshaped, $\frac{\partial d}{\partial x_i} x_i \geq 0$ for any $i = 2, \dots, N$, see [25, Proof of Lemma 14.16].

3.2 Approximating Problems

For $\alpha \in (0,1)$, let us fix $\tilde{r}, \tilde{\epsilon} > 0$, with $0 < \tilde{\epsilon} < \tilde{r} \leq R_{\max}$, such that

$$\left|\Omega^\epsilon_{r,\delta}\right| \leq \left(\frac{1-\alpha}{C_N \lambda_j \|p\|_\infty}\right)^{\frac{N}{2}} \quad \text{for all } \epsilon \in (0, \tilde{\epsilon}),\ r \in (\epsilon, \tilde{r}),\ \delta \in (0, r-\epsilon), \tag{23}$$

where C_N has been defined in (15).

For fixed $i \in \{1, \dots, j\}$, $\epsilon \in (0, \tilde{\epsilon}]$, $r \in (\epsilon, \tilde{r}]$ and for all $\delta \in (0, r-\epsilon)$, let us consider the problem

$$\begin{cases} -\Delta u = \lambda^\epsilon_i p u, & \text{in } D_\delta, \\ u = 0, & \text{on } \partial D_\delta \setminus S^+_r, \\ u = \varphi^\epsilon_i, & \text{on } S^+_r, \end{cases} \tag{24}$$

where, for simplicity of notation, in this section we call $D_\delta := \Omega^\epsilon_{r,\delta}$; we also denote $\delta_0 := r - \epsilon$.

Theorem 3.3. *There exists a unique $u_\delta \in H^1(D_\delta)$ solution to problem (24). Moreover the family $\{u_\delta\}_{\delta \in (0,\delta_0)}$ is bounded in $H^1(D_{\delta_0})$ with respect to δ.*

Proof. We observe that, if we extend φ^ϵ_i to zero in $D_{\delta_0} \setminus \Omega^\epsilon_r$ and we let $v = u - \varphi^\epsilon_i$, then problem (24) is equivalent to

$$\begin{cases} -\Delta v = \lambda^\epsilon_i p v + F, & \text{in } D_\delta, \\ v = 0, & \text{on } \partial D_\delta, \end{cases} \tag{25}$$

where $F := \lambda^\epsilon_i p \varphi^\epsilon_i + \Delta\varphi^\epsilon_i \in H^{-1}(D_\delta)$. Existence and uniqueness of a solution $v_\delta \in H^1_0(D_\delta)$ to (25) easily comes from Lax-Milgram Theorem. Indeed, the bilinear form

$$a(v,w) := \int_{D_\delta} (\nabla v \cdot \nabla w - \lambda^\epsilon_i p v w)\, \mathrm{d}x, \quad v, w \in H^1_0(D_\delta)$$

is coercive, since, by (16) and (23), we have

$$\begin{aligned} a(v,v) &= \int_{D_\delta} \left(|\nabla v|^2 - \lambda^\epsilon_i p |v|^2\right) \mathrm{d}x \\ &\geq \left(1 - C_N \lambda_j \|p\|_\infty |D_\delta|^{\frac{2}{N}}\right) \int_{D_\delta} |\nabla v|^2\, \mathrm{d}x \geq \alpha \int_{D_\delta} |\nabla v|^2\, \mathrm{d}x. \end{aligned} \tag{26}$$

From Lax-Milgram Theorem we also know that

$$\|\nabla v_\delta\|_{L^2(D_{\delta_0})} = \|\nabla v_\delta\|_{L^2(D_\delta)} \leq \frac{\|F\|_{H^{-1}(D_\delta)}}{\alpha},$$

where v_δ has been trivially extended in $D_{\delta_0} \setminus D_\delta$. One can easily prove that $\|F\|_{H^{-1}(D_\delta)} = O(1)$ as $\delta \to 0$. Then $u_\delta := v_\delta + \varphi^\epsilon_i$ is the unique solution to (24) and $\{u_\delta\}_{\delta \in (0,\delta_0)}$ is bounded in $H^1(D_{\delta_0})$. ☐

Theorem 3.4. *If $u_\delta \in H^1(D_\delta)$ is the unique solution to* (24)*, then*

$$u_\delta \longrightarrow \varphi_i^\epsilon \qquad \text{in } H^1(D_{\delta_0}), \quad \text{as } \delta \to 0.$$

Proof. Since $\{u_\delta\}_{\delta\in(0,\delta_0)}$ is bounded in $H^1(D_{\delta_0})$, then there exists $U \in H^1(D_{\delta_0})$ such that, up to a subsequence,

$$u_\delta \rightharpoonup U \qquad \text{weakly in } H^1(D_{\delta_0}), \quad \text{as } \delta \to 0.$$

Actually $U \in H^1(\Omega_r^\epsilon)$ and moreover it weakly solves

$$\begin{cases} -\Delta U = \lambda_i^\epsilon p U, & \text{in } \Omega_r^\epsilon, \\ \quad U = 0, & \text{on } \partial\Omega_r^\epsilon \setminus S_r^+, \\ \quad U = \varphi_i^\epsilon, & \text{on } S_r^+. \end{cases}$$

From the coercivity obtained in (26) we deduce that $U = \varphi_i^\epsilon$.

In order to prove strong convergence in $H^1(D_{\delta_0})$, we notice that

$$\int_{D_{\delta_0}} |\nabla u_\delta|^2 \,\mathrm{d}x = \lambda_i^\epsilon \int_{D_{\delta_0}} p u_\delta^2 \,\mathrm{d}x + \int_{S_r^+} \varphi_i^\epsilon \frac{\partial u_\delta}{\partial \boldsymbol{\nu}} \,\mathrm{d}S. \tag{27}$$

From the compactness of the embedding $H^1(D_{\delta_0}) \hookrightarrow L^2(D_{\delta_0})$ we have that $u_\delta \to \varphi_i^\epsilon$ in $L^2(D_{\delta_0})$ and so $\int_{D_{\delta_0}} p|u_\delta|^2 \,\mathrm{d}x \to \int_{D_{\delta_0}} p|\varphi_i^\epsilon|^2 \,\mathrm{d}x$. Moreover, from the equation (24), we have that $\nabla u_\delta \rightharpoonup \nabla\varphi_i^\epsilon$ weakly in $H(\mathrm{div}, D_{\delta_0})$ as $\delta \to 0$. Hence classical trace theorems for vector functions yield

$$\int_{S_r^+} \frac{\partial u_\delta}{\partial \boldsymbol{\nu}} \varphi_i^\epsilon \,\mathrm{d}S \to \int_{S_r^+} \frac{\partial \varphi_i^\epsilon}{\partial \boldsymbol{\nu}} \varphi_i^\epsilon \,\mathrm{d}S, \quad \text{as } \delta \to 0.$$

Therefore, from (27) and from the equation satisfied by φ_i^ϵ, we conclude that, along a subsequence,

$$\lim_{\delta\to 0} \int_{D_{\delta_0}} |\nabla u_\delta|^2 \,\mathrm{d}x = \lambda_i^\epsilon \int_{D_{\delta_0}} p|\varphi_i^\epsilon|^2 \,\mathrm{d}x + \int_{S_r^+} \frac{\partial \varphi_i^\epsilon}{\partial \boldsymbol{\nu}} \varphi_i^\epsilon \,\mathrm{d}S = \int_{\Omega_r^\epsilon} |\nabla\varphi_i^\epsilon|^2 \,\mathrm{d}x = \int_{D_{\delta_0}} |\nabla\varphi_i^\epsilon|^2 \,\mathrm{d}x.$$

Thanks to *Urysohn's Subsequence Principle* the proof is thereby complete. □

Theorem 3.5. *Let $u_\delta \in H^1(D_\delta)$ be the unique solution to* (24)*. Then*

$$\nabla u_\delta \longrightarrow \nabla\varphi_i^\epsilon \qquad \text{in } L^2(S_r^+), \quad \text{as } \delta \to 0.$$

Proof. By classical elliptic regularity theory, it is easy to prove that an odd reflection of u_δ through the hyperplane $\{x_1 = 0\}$ in a neighbourhood of $\{x\colon |x| = r\}$ converges to φ_i^ϵ in H^2, as $\delta \to 0$. Hence the conclusion follows by trace embeddings. □

Theorem 3.6. *Let u_δ be the unique solution to* (24)*. Then the following identity holds*

$$\begin{aligned} &\int_{S_r^+} |\nabla u_\delta|^2 \,\mathrm{d}S - \frac{N-2}{r} \int_{D_\delta} |\nabla u_\delta|^2 \,\mathrm{d}x \\ &\qquad = \frac{1}{r} \int_{\partial D_\delta \setminus S_r^+} \left(\frac{\partial u_\delta}{\partial \boldsymbol{\nu}}\right)^2 x \cdot \boldsymbol{\nu} \,\mathrm{d}S + 2 \int_{S_r^+} \left(\frac{\partial u_\delta}{\partial \boldsymbol{\nu}}\right)^2 \mathrm{d}S + \frac{2\lambda_i^\epsilon}{r} \int_{D_\delta} p\, u_\delta \nabla u_\delta \cdot x \,\mathrm{d}x. \end{aligned}$$

Proof. We first observe that, by classical regularity theory, $u_\delta \in H^2(D_\delta)$ since D_δ verifies an exterior ball condition. Let us now test equation (24) with the function $\nabla u_\delta \cdot x \in H^1(D_\delta)$. Integrating by parts and using the following identity

$$\nabla u_\delta \cdot \nabla(\nabla u_\delta \cdot x) = \frac{1}{2} \mathrm{div}\left(|\nabla u_\delta|^2 x\right) - \frac{N-2}{2} |\nabla u_\delta|^2$$

we obtain the conclusion. □

Proof of Proposition 3.1. Let u_δ be the unique solution to (24). From Theorem 3.6 and Remark 3.2 we know that

$$\int_{S_r^+} |\nabla u_\delta|^2 \,\mathrm{d}S - \frac{N-2}{r} \int_{D_\delta} |\nabla u_\delta|^2 \,\mathrm{d}x \geq 2 \int_{S_r^+} \left(\frac{\partial u_\delta}{\partial \boldsymbol{\nu}} \right)^2 \mathrm{d}S + \frac{2\lambda_i^\epsilon}{r} \int_{D_\delta} p\, u_\delta \, \nabla u_\delta \cdot x \,\mathrm{d}x.$$

Now, thanks to Theorems 3.4 and 3.5, we can pass to the limit as $\delta \to 0$ in the above inequality, thus obtaining (22). □

Remark 3.7. We observe that the assumption of C^2-regularity for $\partial\Sigma$ can be relaxed; indeed, if Σ is less regular (e.g. if $\partial\Sigma$ is Lipschitz continuous), we can approximate $\epsilon\Sigma$ with a class of C^2-regular domains $(\epsilon\Sigma)_\beta$ and start the procedure of Section 3 from the domain $(\epsilon\Sigma)_\beta$.

4 Poincaré-Type Inequalities

In this section we consider the following spaces for $\epsilon \in (0,1]$ and $r > \epsilon$:

$$V_\epsilon(B_r^+) := \left\{ u \in H^1(B_r^+) \colon u = 0 \text{ on } \mathcal{C}_r \setminus \epsilon\Sigma \right\}, \quad V_0(B_r^+) := \{ u \in H^1(B_r^+) \colon u = 0 \text{ on } \mathcal{C}_r \}.$$

We point out that, for $0 \leq \epsilon_1 \leq \epsilon_2 < r$,

$$H_0^1(B_r^+) \subseteq V_0(B_r^+) \subseteq V_{\epsilon_1}(B_r^+) \subseteq V_{\epsilon_2}(B_r^+) \subseteq H^1(B_r^+). \tag{28}$$

Lemma 4.1 (Poincaré-Type Inequality)**.** *Let $r > 0$. Then, for every $u \in H^1(B_r^+)$, the following inequality holds*

$$\frac{N-1}{r^2} \int_{B_r^+} |u|^2 \,\mathrm{d}x \leq \int_{B_r^+} |\nabla u|^2 \,\mathrm{d}x + \frac{1}{r} \int_{S_r^+} |u|^2 \,\mathrm{d}S. \tag{29}$$

Proof. Integrating the equality $\operatorname{div}(u^2 x) = 2u\nabla u \cdot x + N u^2$ over B_r^+ and recalling the elementary inequality $0 \leq (u + \nabla u \cdot x)^2 = |u|^2 + |\nabla u \cdot x|^2 + 2u\nabla u \cdot x$, we obtain that

$$\int_{\partial B_r^+} |u|^2 x \cdot \boldsymbol{\nu} \,\mathrm{d}S = \int_{B_r^+} \left(2u\nabla u \cdot x + N|u|^2 \right) \mathrm{d}x \geq - \int_{B_r^+} \left(|u|^2 + |\nabla u \cdot x|^2 \right) \mathrm{d}x + N \int_{B_r^+} |u|^2 \,\mathrm{d}x.$$

Since $x \cdot \boldsymbol{\nu} = 0$ on $\mathcal{C}_r$ and $|x| \leq r$, then

$$r \int_{S_r^+} |u|^2 \,\mathrm{d}S \geq -r^2 \int_{B_r^+} |\nabla u|^2 \,\mathrm{d}x + (N-1) \int_{B_r^+} |u|^2 \,\mathrm{d}x.$$

Reorganizing the terms and dividing by r^2 yields the thesis. □

Lemma 4.2. *For $0 \leq \sigma < 1$ the infimum*

$$m_\sigma = \inf_{\substack{u \in V_\sigma(B_1^+) \\ u \neq 0}} \frac{\int_{B_1^+} |\nabla u|^2 \,\mathrm{d}x}{\int_{S_1^+} |u|^2 \,\mathrm{d}S}$$

is achieved. Moreover $m_\sigma > 0$, the map $\sigma \mapsto m_\sigma$ is non-increasing in $[0,1)$ and continuous in 0 and $m_0 = 1$.

Proof. For $u \in V_\sigma(B_1^+)$, let us denote

$$F(u) := \frac{\int_{B_1^+} |\nabla u|^2 \,\mathrm{d}x}{\int_{S_1^+} |u|^2 \,\mathrm{d}S}.$$

Let $\{u_n\}_n \subseteq V_\sigma(B_1^+)$ be a minimizing sequence such that $\int_{S_1^+} |u_n|^2 \,\mathrm{d}S = 1$. From (29) it follows that $\{u_n\}_n$ is bounded in $H^1(B_1^+)$ and so there exists $\tilde{u} \in H^1(B_1^+)$ such that, up to a subsequence, $u_n \rightharpoonup \tilde{u}$ in $H^1(B_1^+)$. Taking into account the compact embedding $H^1(B_1^+) \hookrightarrow L^2(\partial B_1^+)$, we have that $\int_{S_1^+}|\tilde u|^2\,\mathrm{d}S = 1$ and then $\tilde u \neq 0$. Moreover $\tilde u = 0$ on $\mathcal{C}_r \setminus \sigma\Sigma$, since $\{u_n\}$ do; in particular $\tilde u \in V_\sigma(B_1^+)$. By weak lower semicontinuity, we have that

$$\int_{B_1^+}|\nabla \tilde u|^2\,\mathrm{d}x \le \liminf_{n\to+\infty}\int_{B_1^+}|\nabla u_n|^2\,\mathrm{d}x = m_\sigma.$$

Then $m_\sigma = F(\tilde u)$, i.e. $\tilde u$ attains the infimum m_σ. Trivially $m_\sigma > 0$, due to the null boundary conditions on $\mathcal{C}_r \setminus \sigma\Sigma$. The monotonicity of the map $\sigma \mapsto m_\sigma$ follows from (28).

Now we have to prove continuity. Let $\sigma_n \to 0^+$. For every n there exists $\tilde u_n \in V_{\sigma_n}(B_1^+)$ such that

$$\int_{S_1^+}|\tilde u_n|^2\,\mathrm{d}S = 1, \qquad \int_{B_1^+}|\nabla \tilde u_n|^2\,\mathrm{d}x = m_{\sigma_n}.$$

Then, since $m_{\sigma_n} \le m_0$ for all n, we have that $\{\tilde u_n\}_n$ is bounded in $H^1(B_1^+)$. Thus there exists $u_0 \in H^1(B_1^+)$ sucht that, up to a subsequence, $\tilde u_n \rightharpoonup u_0$ weakly in $H^1(B_1^+)$. So, first

$$\int_{S_1^+}|u_0|^2\,\mathrm{d}S = 1 \qquad \text{and} \qquad u_0 = 0 \quad \text{on} \quad \mathcal{C}_r.$$

Furthermore

$$m_0 \le \int_{B_1^+}|\nabla u_0|^2\,\mathrm{d}x \le \liminf m_{\sigma_n} \le \limsup m_{\sigma_n} \le m_0.$$

Then, along the subsequence, $m_0 = \lim_{n\to+\infty} m_{\sigma_n}$. Thanks to *Urysohn's Subsequence Principle* we may conclude that $m_0 = \lim_{\sigma\to 0^+} m_\sigma$.

Finally we prove that $m_0 = 1$. Since the function u_0 achieving m_0 is harmonic in B_1^+, thanks to classical monotonicity arguments (we refer to [21] for further details) we can say that the function

$$r \longmapsto M(r) := \frac{r\int_{B_r^+}|\nabla u_0|^2\,\mathrm{d}x}{\int_{S_r^+}|u_0|^2\,\mathrm{d}S}$$

is non-decreasing and that there exists $k \in \mathbb{N}$, $k \ge 1$, such that $\lim_{r\to 0^+} M(r) = k$. Hence $M(r) \ge 1$ for every $r \ge 0$. Then

$$m_0 = \frac{\int_{B_1^+}|\nabla u_0|^2\,\mathrm{d}x}{\int_{S_1^+}|u_0|^2\,\mathrm{d}S} = M(1) \ge 1.$$

Furthermore the function $v(x) = x_1$ belongs to $V_0(B_1^+)$ and $F(v) = 1$; hence $m_0 = 1$. $\square$

Corollary 4.3. *Let $\epsilon \in (0,1]$ and $r > \epsilon$. Then*

$$\frac{m_{\epsilon/r}}{r}\int_{S_r^+}|u|^2\,\mathrm{d}S \le \int_{B_r^+}|\nabla u|^2\,\mathrm{d}x \qquad \textit{for all } u \in V_\epsilon(B_r^+).$$

Moreover, for every $\rho \in (0,1)$ there exists $\mu_\rho > 1$ such that, if $\epsilon < \frac{r}{\mu_\rho}$, then

$$\frac{1-\rho}{r}\int_{S_r^+}|u|^2\,\mathrm{d}S \le \int_{B_r^+}|\nabla u|^2\,\mathrm{d}x \qquad \textit{for all } u \in V_\epsilon(B_r^+). \tag{30}$$

Proof. If we let $\sigma = \epsilon/r$ in the previous Lemma, we have that

$$m_{\epsilon/r}\int_{S_1^+}|u|^2\,\mathrm{d}S \le \int_{B_1^+}|\nabla u|^2\,\mathrm{d}x \qquad \text{for all } u \in V_{\epsilon/r}(B_1^+).$$

The first inequality follows by the change of variables $y = rx$, while the second one trivially comes from the continuity of m_σ in 0. $\square$

From (29) and Corollary 4.3 one can easily prove the following corollary.

Corollary 4.4. *For every $\rho \in (0,1)$ there exists $\mu_\rho > 1$ such that, for every $r > 0$ and $\epsilon < \frac{r}{\mu_\rho}$,*

$$\frac{N-1}{r^2}\int_{B_r^+}|v|^2\,\mathrm{d}x \le \left(1 + \frac{1}{1-\rho}\right)\int_{B_r^+}|\nabla v|^2\,\mathrm{d}x \quad \textit{for all } v \in V_\epsilon(B_r^+). \tag{31}$$

5 Monotonicity Formula

For any $0 < \epsilon < r \le R_{\max}$, $\lambda > 0$, $\varphi \in H^1(\Omega_r^\epsilon)$, let us introduce the functions

$$E(\varphi, r, \lambda, \epsilon) := \frac{1}{r^{N-2}} \int_{\Omega_r^\epsilon} \left(|\nabla \varphi|^2 - \lambda p |\varphi|^2 \right) \mathrm{d}x$$

and

$$H(\varphi, r) := \frac{1}{r^{N-1}} \int_{S_r^+} |\varphi|^2 \, \mathrm{d}S.$$

Moreover we define the Almgren type frequency function

$$\mathcal{N}(\varphi, r, \lambda, \epsilon) := \frac{E(\varphi, r, \lambda, \epsilon)}{H(\varphi, r)}.$$

Lemma 5.1 (Integration on the Tube)**.** *There exists a constant $\kappa = \kappa(N, \Sigma) > 0$, depending only on N and $|\Sigma|$, such that, for every $\epsilon \in (0, 1]$ and for every $u \in H^1(T_\epsilon)$ such that $u = 0$ on $\partial T_\epsilon \setminus \epsilon\Sigma$,*

$$\int_{T_\epsilon} |u|^2 \, \mathrm{d}x \le \kappa \epsilon^{2(N-1)/N} \int_{T_\epsilon} |\nabla u|^2 \, \mathrm{d}x.$$

Proof. Let $\tilde{T}_\epsilon = T_\epsilon \cup \varsigma(T_\epsilon)$, where ς is the reflection through the hyperplane $\{x_1 = 0\}$, and let $\tilde{u}$ be the even extension of u on $\tilde{T}_\epsilon$. Since $\tilde{u} \in H_0^1(\tilde{T}_\epsilon)$, thanks to (16) we have that

$$\int_{T_\epsilon} |u|^2 \, \mathrm{d}x = \frac{1}{2} \int_{\tilde{T}_\epsilon} |\tilde{u}|^2 \, \mathrm{d}x \le \frac{C_N}{2} |\tilde{T}_\epsilon|^{2/N} \int_{\tilde{T}_\epsilon} |\nabla \tilde{u}|^2 \, \mathrm{d}x = C_N 2^{2/N} |\Sigma|^{2/N} \epsilon^{2(N-1)/N} \int_{T_\epsilon} |\nabla u|^2 \, \mathrm{d}x.$$

Hence we can conclude the proof letting $\kappa = C_N 2^{2/N} |\Sigma|^{2/N}$. □

Lemma 5.2. *There exists $\epsilon_1 \in (0, 1]$, $R_1 > 0$, with $0 < \epsilon_1 < R_1 \le R_{\max}$, such that*

$$H(\varphi_i^\epsilon, r) > 0 \qquad \text{for all } \epsilon \in (0, \epsilon_1], \quad \text{for all } r \in (\epsilon, R_1], \quad \text{for all } i = 1, \dots, j.$$

Proof. Suppose by contradiction that for every n there exists $\epsilon_n \in (0, 1]$, $r_n \in (\epsilon_n, R_{\max}]$ and $i_n \in \{1, \dots, j\}$ such that $r_n \to 0$ and $H(\varphi_{i_n}^{\epsilon_n}, r_n) = 0$. Let us denote $\nu_n := \lambda_{i_n}^{\epsilon_n}$, $\xi_n := \varphi_{i_n}^{\epsilon_n}$ and $\Omega_n := \Omega_{r_n}^{\epsilon_n}$. From this it follows that $\xi_n = 0$ on $S_{r_n}^+$ and that

$$\int_{\Omega_n} |\nabla \xi_n|^2 \, \mathrm{d}x = \nu_n \int_{\Omega_n} p |\xi_n|^2 \, \mathrm{d}x \le \lambda_j \|p\|_\infty \int_{\Omega_n} |\xi_n|^2 \, \mathrm{d}x.$$

Using Lemma 5.1 when integrating on the tube we obtain

$$\int_{T_{\epsilon_n}} |\xi_n|^2 \, \mathrm{d}x \le \kappa \epsilon_n^{2(N-1)/N} \int_{\Omega_n} |\nabla \xi_n|^2 \, \mathrm{d}x.$$

Moreover (29) says that

$$\int_{B_{r_n}^+} |\xi_n|^2 \, \mathrm{d}x \le \frac{r_n^2}{N-1} \int_{\Omega_n} |\nabla \xi_n|^2 \, \mathrm{d}x.$$

Then we have that

$$\int_{\Omega_n} |\nabla \xi_n|^2 \, \mathrm{d}x \le \lambda_j \|p\|_\infty \left(\kappa \epsilon_n^{2(N-1)/N} + \frac{r_n^2}{N-1} \right) \int_{\Omega_n} |\nabla \xi_n|^2 \, \mathrm{d}x.$$

Thus $\xi_n \equiv 0$ in Ω_n, provided n is sufficiently large. Thanks to classical unique continuation properties for elliptic equations it follows that $\xi_n = 0$ in Ω^{ϵ_n}, which is a contradiction. □

Lemma 5.3. *Let*

$$R_2 = \min \left\{ \left(\frac{N-1}{\lambda_j \|p\|_\infty} \right)^{1/2}, R_{\max} \right\}.$$

For every $r \in (0, R_2]$ there exist $c_r > 0$ and $\epsilon_r \in (0, 1]$, with $\epsilon_r < r$, such that

$$H(\varphi_i^\epsilon, r) \ge c_r \qquad \text{for all } \epsilon \in (0, \epsilon_r), \quad \text{for all } i = 1, \dots, j.$$

Proof. We will prove the lemma for a fixed $i \in \{1, \dots, j\}$ and take as c_r the minimum among the constants found for each i. Suppose by contradiction that for a certain $r \in (0, R_2]$ and for every n (large enough) there exists $\epsilon_n \in (0, 1/n)$ such that

$$H(\varphi_i^{\epsilon_n}, r) < \frac{1}{n}. \tag{32}$$

We first note that, since $\epsilon_n \to 0$, then $\lambda_i^{\epsilon_n} \to \lambda_i$ (see [19]). Moreover

$$\int_{\Omega^1} |\nabla \varphi_i^{\epsilon_n}| \, \mathrm{d}x = \int_{\Omega^{\epsilon_n}} |\nabla \varphi_i^{\epsilon_n}| \, \mathrm{d}x = \lambda_i^{\epsilon_n} \int_{\Omega^{\epsilon_n}} p |\varphi_i^{\epsilon_n}|^2 = \lambda_i^{\epsilon_n} \le \lambda_j.$$

Hence there exists $v \in H_0^1(\Omega^1)$ such that $\varphi_i^{\epsilon_n} \rightharpoonup v$ weakly in $H_0^1(\Omega^1)$ along a subsequence. Note that actually $v \in H_0^1(\Omega)$ and $v \not\equiv 0$ in Ω, since $\int_\Omega p|v|^2 \, \mathrm{d}x = 1$. Moreover $\varphi_i^{\epsilon_n} \to v$ strongly in $L^2(\Omega^1)$ and in $L^2(S_r^+)$, so that (32) implies that $v = 0$ on S_r^+. This tells us that v weakly solves

$$\begin{cases} -\Delta v = \lambda_i p v, & \text{in } B_r^+, \\ \quad v = 0, & \text{on } \partial B_r^+. \end{cases}$$

Testing the above equation with v we obtain that $\int_{B_r^+} |\nabla v|^2 \, \mathrm{d}x = \lambda_i \int_{B_r^+} p|v|^2 \, \mathrm{d}x$ and then, thanks to (29) and the fact that $\lambda_i \le \lambda_j$ and $r \le R_2$,

$$\begin{aligned} 0 = \int_{B_r^+} \left(|\nabla v|^2 - \lambda_i p |v|^2 \right) \mathrm{d}x &\ge \left(\frac{N-1}{r^2} - \lambda_j \|p\|_\infty \right) \int_{B_r^+} |v|^2 \, \mathrm{d}x \\ &\ge \left(\frac{N-1}{R_2^2} - \lambda_j \|p\|_\infty \right) \int_{B_r^+} |v|^2 \, \mathrm{d}x. \end{aligned}$$

Due to the initial choice of R_2 we have that this last factor is positive; then $v = 0$ in B_r^+. This, together with classical unique continuation principles, implies that $v = 0$ in Ω, which is a contradiction. □

Let $\tilde{\epsilon}$ and $\tilde{r}$ be the constants found in Proposition 3.1.

Proposition 5.4. *Let $i \in \{1, \dots, j\}$, $\epsilon \in (0, \tilde{\epsilon}]$ and $r \in (\epsilon, \tilde{r}]$. Then*

$$\begin{aligned} \frac{\mathrm{d}E}{\mathrm{d}r}(\varphi_i^\epsilon, r, \lambda_i^\epsilon, \epsilon) \ge \frac{1}{r^{N-2}} \Bigg[2 \int_{S_r^+} \left(\frac{\partial \varphi_i^\epsilon}{\partial \boldsymbol{\nu}} \right)^2 \mathrm{d}S + \\ + \frac{2\lambda_i^\epsilon}{r} \int_{\Omega_r^\epsilon} p \, \varphi_i^\epsilon \, \nabla \varphi_i^\epsilon \cdot x \, \mathrm{d}x - \lambda_i^\epsilon \int_{S_r^+} p |\varphi_i^\epsilon|^2 \, \mathrm{d}S + \frac{N-2}{r} \lambda_i^\epsilon \int_{\Omega_r^\epsilon} p |\varphi_i^\epsilon|^2 \, \mathrm{d}x \Bigg] \end{aligned} \tag{33}$$

and

$$\frac{\mathrm{d}H}{\mathrm{d}r}(\varphi_i^\epsilon, r) = \frac{2}{r^{N-1}} \int_{S_r^+} \varphi_i^\epsilon \frac{\partial \varphi_i^\epsilon}{\partial \boldsymbol{\nu}} \, \mathrm{d}S = \frac{2}{r} E(\varphi_i^\epsilon, r, \lambda_i^\epsilon, \epsilon). \tag{34}$$

Proof. We compute the derivative

$$\frac{\mathrm{d}E}{\mathrm{d}r} = \frac{2-N}{r^{N-1}} \int_{\Omega_r^\epsilon} \left(|\nabla \varphi_i^\epsilon|^2 - \lambda_i^\epsilon p |\varphi_i^\epsilon|^2 \right) \mathrm{d}x + \frac{1}{r^{N-2}} \int_{S_r^+} \left(|\nabla \varphi_i^\epsilon|^2 - \lambda_i^\epsilon p |\varphi_i^\epsilon|^2 \right) \mathrm{d}S.$$

Then, thanks to (22), we obtain (33). The proof of (34) follows from direct computations, the equation satisfied by φ_i^ϵ and integration by parts. □

Lemma 5.5. *Let $\rho \in (0, 1/2]$, μ_ρ be as in Corollary 4.3, $\epsilon \in (0, 1]$ and $r \in (\epsilon, R_{\max}]$. If $\epsilon \mu_\rho < r$, then*

$$\int_{\Omega_r^\epsilon} |u|^2 \, \mathrm{d}x \le K_{\epsilon,r}^1 \int_{\Omega_r^\epsilon} |\nabla u|^2 \, \mathrm{d}x$$

for any $u \in H^1(\Omega_r^\epsilon)$ such that $u = 0$ on $\partial \Omega_r^\epsilon \setminus S_r^+$, where

$$K_{\epsilon,r}^1 = \kappa \epsilon^{2(N-1)/N} + \frac{3r^2}{N-1}$$

and κ is as in Lemma 5.1.

Proof. Thanks to Lemma 5.1 we have an estimate about the integral on the tube, i.e.

$$\int_{T_\epsilon} |u|^2 \,\mathrm{d}x \le \kappa\epsilon^{2(N-1)/N} \int_{T_\epsilon} |\nabla u|^2 \,\mathrm{d}x \le \kappa\epsilon^{2(N-1)/N} \int_{\Omega_r^\epsilon} |\nabla u|^2 \,\mathrm{d}x.$$

On the other hand, by (31) we have that

$$\int_{B_r^+} |u|^2 \,\mathrm{d}x \le \frac{r^2}{N-1}\left(1+\frac{1}{1-\rho}\right) \int_{B_r^+} |\nabla u|^2 \,\mathrm{d}x \le \frac{3r^2}{N-1} \int_{\Omega_r^\epsilon} |\nabla u|^2 \,\mathrm{d}x.$$

The conclusion follows by adding the two parts. □

Lemma 5.6. *Let $\rho \in (0, 1/2)$, μ_ρ be as in Corollary 4.3, $\epsilon \in (0, 1]$ and $r \in (\epsilon, R_{\max}]$. If $\epsilon\mu_\rho < r$, then*

$$\int_{\Omega_r^\epsilon} |u\nabla u \cdot x| \,\mathrm{d}x \le K_{\epsilon,r}^2 \int_{\Omega_r^\epsilon} |\nabla u|^2 \,\mathrm{d}x$$

for any $u \in H^1(\Omega_r^\epsilon)$ such that $u = 0$ on $\partial\Omega_r^\epsilon \setminus S_r^+$, where

$$K_{\epsilon,r}^2 = \sqrt{2\kappa}\epsilon^{(N-1)/N} + \sqrt{\frac{3}{N-1}}r^2$$

and κ is as in Lemma 5.1.

Proof. First we consider the integral over T_ϵ: thanks to Cauchy-Schwarz Inequality and Lemma 5.1 we know that

$$\int_{T_\epsilon} |u\nabla u \cdot x| \,\mathrm{d}x \,\mathrm{d}x \le \sqrt{2\kappa}\epsilon^{(N-1)/N} \int_{\Omega_r^\epsilon} |\nabla u|^2 \,\mathrm{d}x.$$

From the Cauchy-Schwarz Inequality and (31) it follows that

$$\int_{B_r^+} |u\nabla u \cdot x| \,\mathrm{d}x \le \sqrt{\frac{3}{N-1}}r^2 \int_{\Omega_r^\epsilon} |\nabla u|^2 \,\mathrm{d}x.$$

Adding the two parts we conclude the proof. □

Corollary 5.7. *Let $\rho \in (0, 1/2)$, μ_ρ be as in Corollary 4.3, $\epsilon \in (0, 1]$, $r \in (\epsilon, R_{\max}]$. If $\epsilon\mu_\rho < r$ then*

$$\int_{\Omega_r^\epsilon} \left(|\nabla u|^2 - \lambda_i^\epsilon p|u|^2\right) \mathrm{d}x \ge \left(1 - \lambda_i^\epsilon \|p\|_\infty K_{\epsilon,r}^1\right) \int_{\Omega_r^\epsilon} |\nabla u|^2 \,\mathrm{d}x \tag{35}$$

for any $u \in H^1(\Omega_r^\epsilon)$ such that $u = 0$ on $\partial\Omega_r^\epsilon \setminus S_r^+$ and for all $i \in \{1, \ldots, j\}$. Furthermore there exists $r_0 \le R_{\max}$ such that, for every r, ϵ satisfying $\epsilon\mu_\rho < r \le r_0$, we have

$$\int_{\Omega_r^\epsilon} |\nabla u|^2 \,\mathrm{d}x \le 2\int_{\Omega_r^\epsilon} \left(|\nabla u|^2 - \lambda_i^\epsilon p|u|^2\right) \mathrm{d}x$$

for any $u \in H^1(\Omega_r^\epsilon)$ such that $u = 0$ on $\partial\Omega_r^\epsilon \setminus S_r^+$ and for all $i \in \{1, \ldots, j\}$.

Proof. The first statement (35) easily comes from Lemma 5.5. Besides, if we choose $r_0 \le R_{\max}$ such that

$$K_{\epsilon,r}^1 \le K_{r_0,r_0}^1 \le \frac{1}{2\lambda_j \|p\|_\infty},$$

from (35), we can conclude the proof. □

Lemma 5.8. *Let $\rho \in (0, 1/2)$ and μ_ρ be as in Corollary 4.3. Let R_1 and ϵ_1 be as in Lemma 5.2. Then there exists $\tau > 0$ depending only on N, λ_j, $\|p\|_\infty$ and $|\Sigma|$ such that, for every $\epsilon \in (0, \epsilon_1]$, r_1, r_2, with $0 < \mu_\rho\epsilon < r_1 \le r_2 \le \min\{1, R_1\}$, we have that*

$$\frac{H(\varphi_i^\epsilon, r_2)}{H(\varphi_i^\epsilon, r_1)} \ge \exp\left(-\tau R_1^{2(N-1)/N}\right)\left(\frac{r_2}{r_1}\right)^{2(1-\rho)} \quad \textit{for all } i \in \{1, \ldots, j\}.$$

Proof. With the notation $E(r) = E(\varphi_i^\epsilon, r, \lambda_i^\epsilon, \epsilon)$, $H(r) = H(\varphi_i^\epsilon, r)$ and $\frac{\mathrm{d}H}{\mathrm{d}r}(\varphi_i^\epsilon, r) = H'(r)$, we have that, from Proposition 5.4 and Corollary 5.7

$$H'(r) = \frac{2}{r}E(r) = \frac{2}{r^{N-1}} \int_{\Omega_r^\epsilon} \left(|\nabla \varphi_i^\epsilon|^2 - \lambda_i^\epsilon p |\varphi_i^\epsilon|^2 \right) \mathrm{d}x \geq \frac{2}{r^{N-1}} (1 - \lambda_i^\epsilon \|p\|_\infty K_{\epsilon,r}^1) \int_{\Omega_r^\epsilon} |\nabla \varphi_i^\epsilon|^2 \, \mathrm{d}x$$

for all $\epsilon\mu_\rho < r \leq \min\{1, R_1\}$. Hence, since $\lambda_i^\epsilon \leq \lambda_j$, thanks to (30)

$$H'(r) \geq \frac{2}{r^{N-1}} (1 - \lambda_j \|p\|_\infty K_{\epsilon,r}^1) \frac{1-\rho}{r} \int_{S_r^+} |\varphi_i^\epsilon|^2 \, \mathrm{d}S = \frac{2(1-\rho)}{r} (1 - \lambda_j \|p\|_\infty K_{\epsilon,r}^1) H(r).$$

So we have

$$\frac{H'(r)}{H(r)} \geq \frac{2(1-\rho)}{r} \left[1 - \tau_1 \epsilon^{2(N-1)/N} - \tau_2 r^2 \right]$$

where $\tau_1 = \lambda_j \|p\|_\infty \kappa$ and $\tau_2 = \lambda_j \|p\|_\infty \frac{3}{N-1}$. Since $\epsilon < r$ and $r \leq 1$, if $\tau_0 = \tau_1 + \tau_2$, then

$$(\log H(r))' = \frac{H'(r)}{H(r)} \geq \frac{2(1-\rho)}{r} \left[1 - \tau_0 r^{2(N-1)/N} \right] \geq \frac{2(1-\rho)}{r} - 2\tau_0 r^{1-\frac{2}{N}}.$$

Integrating from r_1 to r_2 and letting $\tau := \tau_0 N/(N-1)$, we obtain

$$\log \frac{H(\varphi_i^\epsilon, r_2)}{H(\varphi_i^\epsilon, r_1)} \geq 2(1-\rho) \log \frac{r_2}{r_1} - \tau (r_2^{2(N-1)/N} - r_1^{2(N-2)/N}) \geq 2(1-\rho) \log \frac{r_2}{r_1} - \tau R_1^{2(N-1)/N}.$$

Taking the exponentials yields the thesis. □

Hereafter let $R_0 := \min\{1, R_1, R_2, r_0\}$ where R_1, R_2, r_0 are defined in Lemma 5.2, Lemma 5.3 and Corollary 5.7 respectively. Moreover let $\epsilon_0 = \min\{1, \tilde{\epsilon}, \epsilon_1\}$ where $\tilde{\epsilon}, \epsilon_1$ are defined in Proposition 3.1 and Lemma 5.2 respectively.

Proposition 5.9. *Let $\rho \in (0, 1/2)$ and μ_ρ be as in Corollary 4.3. Then, for every $r \in (0, R_0]$, $\epsilon \in (0, \epsilon_0]$ such that $0 < \epsilon\mu_\rho < r \leq R_0$*

$$\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}r}(\varphi_i^\epsilon, r, \lambda_i^\epsilon, \epsilon) \geq -f(r)\mathcal{N}(\varphi_i^\epsilon, r, \lambda_i^\epsilon, \epsilon) \quad \text{for all } i \in \{1, \ldots, j\},$$

where

$$f(r) = c_1 r + c_2 r^{(N-2)/N} + c_3 r^{-1/N}$$

and c_n's are positive constants depending only on ρ, $\|p\|_\infty$, λ_j, the dimension N and the geometry of the problem (in particular on Ω and on $|\Sigma|_{N-1}$).

Proof. With the usual notation

$$\frac{\mathrm{d}\mathcal{N}}{\mathrm{d}r}(\varphi_i^\epsilon, r, \lambda_i^\epsilon, \epsilon) =: \mathcal{N}'(r), \quad \frac{\mathrm{d}E}{\mathrm{d}r}(\varphi_i^\epsilon, r, \lambda_i^\epsilon, \epsilon) := E'(r), \quad \frac{\mathrm{d}H}{\mathrm{d}r}(\varphi_i^\epsilon, r) := H'(r),$$

from Proposition 5.4 we have that

$$\begin{aligned} \mathcal{N}'(r) \geq \frac{1}{H^2} \frac{2}{r^{2N-3}} \Bigg\{ & \left[\left(\int_{S_r^+} \left(\frac{\partial \varphi_i^\epsilon}{\partial \boldsymbol{\nu}} \right)^2 \mathrm{d}S \right) \left(\int_{S_r^+} |\varphi_i^\epsilon|^2 \, \mathrm{d}S \right) - \left(\int_{S_r^+} \varphi_i^\epsilon \frac{\partial \varphi_i^\epsilon}{\partial \boldsymbol{\nu}} \, \mathrm{d}S \right)^2 \right] \\ & + \left[\frac{\lambda_i^\epsilon}{r} \int_{\Omega_r^\epsilon} p \, \varphi_i^\epsilon \, \nabla \varphi_i^\epsilon \cdot x \, \mathrm{d}x - \frac{\lambda_i^\epsilon}{2} \int_{S_r^+} p |\varphi_i^\epsilon|^2 \, \mathrm{d}S + \frac{N-2}{2r} \lambda_i^\epsilon \int_{\Omega_r^\epsilon} p |\varphi_i^\epsilon|^2 \, \mathrm{d}x \right] \int_{S_r^+} |\varphi_i^\epsilon|^2 \, \mathrm{d}S \Bigg\}. \end{aligned}$$

By Cauchy-Schwarz Inequality we have that

$$\mathcal{N}'(r) \geq \frac{2\lambda_i^\epsilon}{\int_{S_r^+} |\varphi_i^\epsilon|^2} \left[\int_{\Omega_r^\epsilon} p \, \varphi_i^\epsilon \, \nabla \varphi_i^\epsilon \cdot x \, \mathrm{d}x + \frac{N-2}{2} \int_{\Omega_r^\epsilon} p |\varphi_i^\epsilon|^2 \, \mathrm{d}x - \frac{r}{2} \int_{S_r^+} p |\varphi_i^\epsilon|^2 \, \mathrm{d}S \right].$$

Thanks to Lemmas 5.5, 5.6, Corollary 4.3 and Corollary 5.7 we can say that

$$\begin{aligned}\mathcal{N}'(r) &\geq -\frac{2\lambda_i^\epsilon \|p\|_\infty}{\int_{S_r^+} |\varphi_i^\epsilon|^2} \left[K_{\epsilon,r}^2 + \frac{(N-2)}{2} K_{\epsilon,r}^1 + \frac{r^2}{2(1-\rho)}\right] \int_{\Omega_r^\epsilon} |\nabla \varphi_i^\epsilon|^2 \, \mathrm{d}x \\ &\geq -\frac{4\lambda_i^\epsilon \|p\|_\infty}{r^{N-1} H(r)} r^{N-2} E(r) \left[K_{\epsilon,r}^2 + \frac{(N-2)}{2} K_{\epsilon,r}^1 + r^2\right].\end{aligned}$$

Taking into account that $K_{\epsilon,r}^n < K_{r,r}^n$, $n = 1, 2$, we have

$$\mathcal{N}'(r) \geq -\left(c_1 r + c_2 r^{(N-2)/N} + c_3 r^{-1/N}\right) \mathcal{N}(r) = -f(r)\mathcal{N}(r)$$

by some constants $c_1, c_2, c_3 > 0$ independent of r and ϵ. □

Corollary 5.10. *Let $\rho \in (0, \frac{1}{2})$ and μ_ρ be as in Corollary 4.3. Then, for every $\mu > \mu_\rho$, $r \in (0, R_0]$, and $\epsilon \in (0, \epsilon_0]$ such that $\epsilon\mu \leq r \leq R_0$, we have that*

$$\mathcal{N}(\varphi_i^\epsilon, r, \lambda_i^\epsilon, \epsilon) \leq e^{\int_r^{R_0} f(t)\, dt} \mathcal{N}(\varphi_i^\epsilon, R_0, \lambda_i^\epsilon, \epsilon).$$

Proof. Form Proposition 5.9 it follows that $\left(e^{-\int_r^{R_0} f(t)\, dt} \mathcal{N}(r)\right)' \geq 0$, which, by integration over (r, R_0), yields the conclusion. □

5.1 Energy Estimates

Proposition 5.11. *Let $\rho \in (0, 1/2)$. Then there exists $K_\rho > 0$ such that, for every $R \geq K_\rho$ and for every $i \in \{1, \ldots, j\}$, we have*

$$\int_{\Omega_{R\epsilon}^\epsilon} |\nabla \varphi_i^\epsilon|^2 \, \mathrm{d}x = O(\epsilon^{N-2} H(\varphi_i^\epsilon, K_\rho \epsilon)) \quad \text{as } \epsilon \to 0^+, \tag{36}$$

$$\int_{\Omega_{R\epsilon}^\epsilon} |\varphi_i^\epsilon|^2 \, \mathrm{d}x = O(\epsilon^{N-\frac{2}{N}} H(\varphi_i^\epsilon, K_\rho \epsilon)) \quad \text{as } \epsilon \to 0^+, \tag{37}$$

$$\int_{S_{R\epsilon}^+} |\varphi_i^\epsilon|^2 \, \mathrm{d}S = O(\epsilon^{N-1} H(\varphi_i^\epsilon, K_\rho \epsilon)) \quad \text{as } \epsilon \to 0^+. \tag{38}$$

Proof. For $\rho \in (0, 1/2)$ let us consider μ_ρ as in Corollary 4.3, $\epsilon_0 = \min\{1, \tilde{\epsilon}, \epsilon_1\}$, ϵ_{R_0} as in Lemma 5.3 and let $K_\rho > \max\{\mu_\rho, R_0/\epsilon_0, R_0/\epsilon_{R_0}\}$. From Corollary 5.10 we deduce that, if $R \geq K_\rho$ and $R\epsilon < R_0$

$$\mathcal{N}(\varphi_i^\epsilon, R\epsilon, \lambda_i^\epsilon, \epsilon) \leq e^{\int_{R\epsilon}^{R_0} f(t)\, dt} \mathcal{N}(\varphi_i^\epsilon, R_0, \lambda_i^\epsilon, \epsilon). \tag{39}$$

Now let us analyze the frequency function $\mathcal{N}$ at radius R_0:

$$E(\varphi_i^\epsilon, R_0, \lambda_i^\epsilon, \epsilon) = \frac{1}{R_0^{N-2}} \int_{\Omega_{R_0}^\epsilon} \left(|\nabla \varphi_i^\epsilon|^2 - \lambda_i^\epsilon |\varphi_i^\epsilon|^2\right) \mathrm{d}x \leq \frac{1}{R_0^{N-2}} \int_{\Omega^\epsilon} |\nabla \varphi_i^\epsilon|^2 \, \mathrm{d}x \leq \frac{\lambda_j}{R_0^{N-2}}.$$

Moreover, thanks to Lemma 5.3

$$H(\varphi_i^\epsilon, R_0) \geq c_{R_0}.$$

Thus we have that

$$\mathcal{N}(\varphi_i^\epsilon, R_0, \lambda_i^\epsilon, \epsilon) \leq \frac{\lambda_j}{c_{R_0} R_0^{N-2}}. \tag{40}$$

Then, from (39)

$$\int_{\Omega_{R\epsilon}^\epsilon} \left(|\nabla \varphi_i^\epsilon|^2 - \lambda_i^\epsilon |\varphi_i^\epsilon|^2\right) \mathrm{d}x \leq \mathrm{const}\, H(\varphi_i^\epsilon, R\epsilon)(R\epsilon)^{N-2}. \tag{41}$$

From the second statement of Corollary 5.7 we have that

$$\int_{\Omega_{R\epsilon}^\epsilon} |\nabla \varphi_i^\epsilon|^2 \, \mathrm{d}x \leq 2\, \mathrm{const}\, H(\varphi_i^\epsilon, R\epsilon)(R\epsilon)^{N-2}. \tag{42}$$

Now let $K_\rho\epsilon \le r \le R_0$. Then, from Proposition 5.4

$$\frac{H'(\varphi_i^\epsilon, r)}{H(\varphi_i^\epsilon, r)} = \frac{2}{r}\mathcal{N}(\varphi_i^\epsilon, r, \lambda_i^\epsilon, \epsilon)$$

and from Corollary 5.10 and (40)

$$\frac{H'(\varphi_i^\epsilon, r)}{H(\varphi_i^\epsilon, r)} \le \frac{C}{r}. \tag{43}$$

Now, integrating the previous inequality from $K_\rho\epsilon$ to $R\epsilon$, we obtain

$$\log\frac{H(\varphi_i^\epsilon, R\epsilon)}{H(\varphi_i^\epsilon, K_\rho\epsilon)} \le C\log\frac{R\epsilon}{K_\rho\epsilon},$$

hence $H(\varphi_i^\epsilon, R\epsilon) \le \mathrm{const}_{\rho,R}\, H(\varphi_i^\epsilon, K_\rho\epsilon)$, i.e. $H(\varphi_i^\epsilon, R\epsilon) = O(H(\varphi_i^\epsilon, K_\rho\epsilon))$ as $\epsilon \to 0$. Then (36) follows from (42), whereas (38) is a direct consequence of the previous estimate and definition of H. Finally, thanks to Lemma 5.5 and (36), we have

$$\int_{\Omega_{R\epsilon}^\epsilon} |\varphi_i^\epsilon|^2\,\mathrm{d}x \le (c_1\epsilon^{2(N-1)/N} + c_2(R\epsilon)^2)\int_{\Omega_{R\epsilon}^\epsilon}|\nabla\varphi_i^\epsilon|\,\mathrm{d}x = O(\epsilon^{N-\frac{2}{N}}H(\varphi_i^\epsilon, K_\rho\epsilon)),$$

as $\epsilon \to 0$, thus proving (37). □

Proposition 5.12. *Let $\rho \in (0,1/2)$ and K_ρ be as in Proposition 5.11. Then there exists $C_\rho > 0$ such that, for every $R \ge K_\rho$, for every $\epsilon \in (0,\epsilon_0]$ such that $R\epsilon \le R_0$, and for every $i \in \{1,\dots,j\}$ we have*

$$\int_{\Omega_{R\epsilon}^\epsilon}|\nabla\varphi_i^\epsilon|^2\,\mathrm{d}x \le C_\rho(R\epsilon)^{N-2\rho},$$
$$\int_{\Omega_{R\epsilon}^\epsilon}|\varphi_i^\epsilon|^2\,\mathrm{d}x \le C_\rho(R\epsilon)^{N+2-2\rho-2/N},$$
$$\int_{S_{R\epsilon}^+}|\varphi_i^\epsilon|^2\,\mathrm{d}x \le C_\rho(R\epsilon)^{N+1-2\rho}.$$

Proof. From Lemma 5.8 we know that

$$H(\varphi_i^\epsilon, R\epsilon) \le \exp\left(\tau R_1^{2(N-1)/N}\right)\left(\frac{R\epsilon}{R_0}\right)^{2(1-\rho)} H(\varphi_i^\epsilon, R_0) \tag{44}$$

and, from (30), we have

$$H(\varphi_i^\epsilon, R_0) = \frac{1}{R_0^{N-1}}\int_{S_{R_0}^+}|\varphi_i^\epsilon|^2\,\mathrm{d}S \le \frac{1}{R_0^{N-2}(1-\rho)}\int_{\Omega_{R_0}^\epsilon}|\nabla\varphi_i^\epsilon|^2\,\mathrm{d}x \le \frac{\lambda_j}{R_0^{N-2}(1-\rho)}. \tag{45}$$

Combining (44) and (45) with Proposition 5.11 (in particular estimate (42) in the proof) we can deduce all the claims. □

As a consequence of Proposition 5.12 we can say that

$$H(\varphi_i^\epsilon, K_\rho\epsilon) = O(\epsilon^{2-2\rho}) \quad \text{as } \epsilon \to 0. \tag{46}$$

As a byproduct of the proof of Proposition 5.11 we obtain the following result.

Corollary 5.13. *Let $\rho \in (0,1/2)$ and K_ρ be as in Proposition 5.11. Then there exist $\bar{C}, q > 0$ such that, if $\epsilon \in (0,\epsilon_0]$ and $K_\rho\epsilon < R_0$,*

$$H(\varphi_j^\epsilon, K_\rho\epsilon) \ge \bar{C}\epsilon^q. \tag{47}$$

Proof. If we integrate (43) over $(K_\rho\epsilon, R_0)$ and take the exponentials, we obtain

$$\frac{H(\varphi_j^\epsilon, R_0)}{H(\varphi_j^\epsilon, K_\rho\epsilon)} \leq \left(\frac{R_0}{K_\rho\epsilon}\right)^q,$$

denoting by q the constant C in (43). Then Lemma 5.3 implies that

$$H(\varphi_j^\epsilon, K_\rho\epsilon) \geq c_{R_0}\left(\frac{K_\rho\epsilon}{R_0}\right)^q.$$

Hence the claim is proved with $\bar{C} := c_{R_0}(K_\rho/R_0)^q$. □

6 Estimates on the Difference $\lambda_j - \lambda_j^\epsilon$

6.1 Upper Bound

For any $i \in \{0,\ldots,j\}$, $R > 1$ and $\epsilon \in (0,1]$, with $R\epsilon \leq R_{\max}$, let us consider the following minimization problem

$$\min\left\{\int_{B_{R\epsilon}^+} |\nabla u|^2\,\mathrm{d}x \colon u \in H^1(B_{R\epsilon}^+),\ u = 0 \text{ on } \mathcal{C}_{R\epsilon},\ u = \varphi_i^\epsilon \text{ on } S_{R\epsilon}^+\right\}.$$

One can prove that this problem has a unique solution $v_{i,R,\epsilon}^{\mathrm{int}}$, which weakly solves

$$\begin{cases} -\Delta v_{i,R,\epsilon}^{\mathrm{int}} = 0, & \text{in } B_{R\epsilon}^+, \\ v_{i,R,\epsilon}^{\mathrm{int}} = 0, & \text{on } \mathcal{C}_{R\epsilon}, \\ v_{i,R,\epsilon}^{\mathrm{int}} = \varphi_i^\epsilon, & \text{on } S_{R\epsilon}^+. \end{cases}$$

Proposition 6.1. *Let $\rho \in (0,1/2)$ and K_ρ be as in Proposition 5.11. Then*

$$\int_{B_{R\epsilon}^+} \left|\nabla v_{i,R,\epsilon}^{\mathrm{int}}\right|^2\mathrm{d}x = O(\epsilon^{N-2}H(\varphi_i^\epsilon, K_\rho\epsilon)) \quad \text{as } \epsilon \to 0^+, \tag{48}$$

$$\int_{B_{R\epsilon}^+} \left|v_{i,R,\epsilon}^{\mathrm{int}}\right|^2\mathrm{d}x = O(\epsilon^{N}H(\varphi_i^\epsilon, K_\rho\epsilon)) \quad \text{as } \epsilon \to 0^+, \tag{49}$$

$$\int_{S_{R\epsilon}^+} \left|v_{i,R,\epsilon}^{\mathrm{int}}\right|^2\mathrm{d}x = O(\epsilon^{N-1}H(\varphi_i^\epsilon, K_\rho\epsilon)) \quad \text{as } \epsilon \to 0^+. \tag{50}$$

for all $R \geq 2$ and for any $i = 1,\ldots,j$. Moreover there exists $\hat{C}_\rho$ such that, if $R \geq \max\{2, K_\rho\}$ and $\epsilon < R_0/R$,

$$\int_{B_{R\epsilon}^+} \left|\nabla v_{i,R,\epsilon}^{\mathrm{int}}\right|^2\mathrm{d}x \leq \hat{C}_\rho(R\epsilon)^{N-2\rho}, \tag{51}$$

$$\int_{B_{R\epsilon}^+} \left|v_{i,R,\epsilon}^{\mathrm{int}}\right|^2\mathrm{d}x \leq \hat{C}_\rho(R\epsilon)^{N+2-2\rho}, \tag{52}$$

$$\int_{S_{R\epsilon}^+} \left|v_{i,R,\epsilon}^{\mathrm{int}}\right|^2\mathrm{d}x \leq \hat{C}_\rho(R\epsilon)^{N+1-2\rho}. \tag{53}$$

Proof. Proving (50) is trivial due to (38), since $v_{i,R,\epsilon}^{\mathrm{int}} = \varphi_i^\epsilon$ on $S_{R\epsilon}^+$. Let $\eta = \eta_R(\frac{\cdot}{\epsilon})$, with η_R defined in (14); then

$$\int_{B_{R\epsilon}^+} \left|\nabla v_{i,R,\epsilon}^{\mathrm{int}}\right|^2\mathrm{d}x \leq \int_{B_{R\epsilon}^+} |\nabla(\eta\varphi_i^\epsilon)|^2\,\mathrm{d}x \leq$$

$$\leq 2\left(\int_{B_{R\epsilon}^+} |\nabla\varphi_i^\epsilon|^2 + \frac{16}{(R\epsilon)^2}\int_{B_{R\epsilon}^+} |\varphi_i^\epsilon|^2\,\mathrm{d}x\right) \leq \mathrm{const}_\rho \int_{\Omega_{R\epsilon}^\epsilon} |\nabla\varphi_i^\epsilon|^2\,\mathrm{d}x,$$

where the last step comes from (31). Combining this inequality with (36) we obtain (48). Moreover (31) and (48) yield (49). Finally estimates (51)–(53) follow from the above argument and Proposition 5.12. $\square$

Now let us define, for all $i \in \{1,\dots,j\}$, for all $R>1$ and $\epsilon \in (0,1]$ such that $R\epsilon \le R_{\max}$,

$$v_{i,R,\epsilon} := \begin{cases} v^{\rm int}_{i,R,\epsilon}, & \text{in } B^+_{R\epsilon}, \\ \varphi^\epsilon_i, & \text{in } \Omega \setminus B^+_{R\epsilon}, \end{cases} \tag{54}$$

and

$$Z^\epsilon_R(x) := \frac{v^{\rm int}_{j,R,\epsilon}(\epsilon x)}{\sqrt{H(\varphi^\epsilon_j, K_\rho \epsilon)}}, \qquad \tilde\varphi^\epsilon(x) := \frac{\varphi^\epsilon_j(\epsilon x)}{\sqrt{H(\varphi^\epsilon_j, K_\rho \epsilon)}}. \tag{55}$$

It is easy to prove that the family of functions $\{v_{1,R,\epsilon}, \dots, v_{j,R,\epsilon}\}$ is linearly independent in $H^1_0(\Omega)$.

Lemma 6.2. *For all $R \ge \max\{2, K_\rho\}$, we have that, as $\epsilon \to 0^+$,*

$$\int_\Omega |\nabla v_{j,R,\epsilon}|^2 \,\mathrm{d}x = \lambda^\epsilon_j + \epsilon^{N-2} H(\varphi^\epsilon_j, K_\rho\epsilon)\left(\int_{B^+_R} |\nabla Z^\epsilon_R|^2 \,\mathrm{d}x - \int_{\Pi_R} |\nabla \tilde\varphi^\epsilon|^2 \,\mathrm{d}x\right), \tag{56}$$

$$\int_\Omega |\nabla v_{i,R,\epsilon}|^2 \,\mathrm{d}x = \lambda^\epsilon_i + O(\epsilon^{N-2\rho}) \quad \textit{for all } i \in \{1,\dots,j\}, \tag{57}$$

$$\int_\Omega \nabla v_{i,R,\epsilon}\cdot\nabla v_{j,R,\epsilon} \,\mathrm{d}x = O\left(\epsilon^{N-1-\rho}\sqrt{H(\varphi^\epsilon_j, K_\rho\epsilon)}\right) \quad \textit{for all } i \in \{1,\dots,j-1\}, \tag{58}$$

$$\int_\Omega \nabla v_{i,R,\epsilon}\cdot\nabla v_{n,R,\epsilon} \,\mathrm{d}x = O(\epsilon^{N-2\rho}) \quad \textit{for all } i,n \in \{1,\dots,j\},\ i \ne n, \tag{59}$$

$$\int_\Omega p|v_{j,R,\epsilon}|^2 \,\mathrm{d}x = 1 + O(\epsilon^{N-2/N} H(\varphi^\epsilon_j, K_\rho\epsilon)), \tag{60}$$

$$\int_\Omega p|v_{i,R,\epsilon}|^2 \,\mathrm{d}x = 1 + O(\epsilon^{N+2-2\rho-2/N}) \quad \textit{for all } i \in \{1,\dots,j\}, \tag{61}$$

$$\int_\Omega p v_{i,R,\epsilon} v_{j,R,\epsilon} \,\mathrm{d}x = O\left(\epsilon^{N+1-\rho-2/N}\sqrt{H(\varphi^\epsilon_j, K_\rho\epsilon)}\right) \quad \textit{for all } i \in \{1,\dots,j-1\}, \tag{62}$$

$$\int_\Omega p v_{i,R,\epsilon} v_{n,R,\epsilon} \,\mathrm{d}x = O(\epsilon^{N+2-2\rho-2/N}) \quad \textit{for all } i,n \in \{1,\dots,j\},\ i \ne n, \tag{63}$$

where, in (56), *$\tilde\varphi^\epsilon$ has been trivially extended in Π_R outside its domain.*

Proof. We will only prove the first part of the estimates, i.e. (56), (57), (58), (59), since the second part is completely analogous. To prove (56) we observe that, by scaling,

$$\begin{aligned}\int_\Omega |\nabla v_{j,R,\epsilon}|^2 \,\mathrm{d}x &= \int_{B^+_{R\epsilon}} \left|\nabla v^{\rm int}_{j,R,\epsilon}\right|^2 \mathrm{d}x + \int_{\Omega^\epsilon} \left|\nabla \varphi^\epsilon_j\right|^2 \mathrm{d}x - \int_{\Omega^\epsilon_{R\epsilon}} \left|\nabla \varphi^\epsilon_j\right|^2 \mathrm{d}x \\ &= \lambda^\epsilon_j + \epsilon^{N-2} H(\varphi^\epsilon_j, K_\rho\epsilon)\left(\int_{B^+_R} |\nabla Z^\epsilon_R|^2 \,\mathrm{d}x - \int_{\Pi_R} |\nabla \tilde\varphi^\epsilon|^2 \,\mathrm{d}x\right).\end{aligned}$$

Thanks to Propositions 5.12 and 6.1 we have that

$$\begin{aligned}\int_\Omega |\nabla v_{i,R,\epsilon}|^2 \,\mathrm{d}x &= \int_{B^+_{R\epsilon}} \left|\nabla v^{\rm int}_{i,R,\epsilon}\right|^2 \mathrm{d}x + \int_{\Omega^\epsilon} |\nabla \varphi^\epsilon_i|^2 \,\mathrm{d}x - \int_{\Omega^\epsilon_{R\epsilon}} |\nabla \varphi^\epsilon_i|^2 \,\mathrm{d}x \\ &= \lambda^\epsilon_i + O(\epsilon^{N-2\rho}),\end{aligned}$$

as $\epsilon \to 0^+$, thus proving (57) and, by Cauchy-Schwarz Inequality, for $i<j$

$$\begin{aligned}\int_\Omega \nabla v_{i,R,\epsilon}\cdot\nabla v_{j,R,\epsilon} &= \int_{B^+_{R\epsilon}} \nabla v^{\rm int}_{i,R,\epsilon} \nabla v^{\rm int}_{j,R,\epsilon} \,\mathrm{d}x + \int_{\Omega^\epsilon} \nabla\varphi^\epsilon_i\cdot\nabla\varphi^\epsilon_j \,\mathrm{d}x - \int_{\Omega^\epsilon_{R\epsilon}} \nabla\varphi^\epsilon_i\cdot\nabla\varphi^\epsilon_j \,\mathrm{d}x \\ &= O(\epsilon^{\frac{N-2\rho}{2}}) O\left(\epsilon^{\frac{N-2}{2}}\sqrt{H(\varphi^\epsilon_j, K_\rho\epsilon)}\right) = O\left(\epsilon^{N-1-\rho}\sqrt{H(\varphi^\epsilon_j, K_\rho\epsilon)}\right),\end{aligned}$$

as $\epsilon \to 0^+$, thus proving (58). Similarly, for $i \neq n$

$$\int_\Omega \nabla v_{i,R,\epsilon} \cdot \nabla v_{n,R,\epsilon} = \int_{B^+_{R\epsilon}} \nabla v^{\text{int}}_{i,R,\epsilon} \nabla v^{\text{int}}_{n,R,\epsilon} \, \mathrm{d}x + \int_{\Omega^\epsilon} \nabla \varphi^\epsilon_i \cdot \nabla \varphi^\epsilon_n \, \mathrm{d}x - \int_{\Omega^\epsilon_{R\epsilon}} \nabla \varphi^\epsilon_i \cdot \nabla \varphi^\epsilon_n \, \mathrm{d}x$$
$$= O(\epsilon^{N-2\rho}),$$

as $\epsilon \to 0^+$, which provides (59). $\square$

We construct a basis $\{\hat{v}_{1,R,\epsilon}, \dots, \hat{v}_{j,R,\epsilon}\}$ of the space span $\{v_{1,R,\epsilon}, \dots, v_{j,R,\epsilon}\}$ such that

$$\int_\Omega p \, \hat{v}_{n,R,\epsilon} \hat{v}_{m,R,\epsilon} \, dx = 0 \quad \text{for } n \neq m,$$

by defining

$$\hat{v}_{j,R,\epsilon} = v_{j,R,\epsilon}, \quad \hat{v}_{i,R,\epsilon} = v_{i,R,\epsilon} - \sum_{n=i+1}^{j} d^\epsilon_{i,n} \hat{v}_{n,R,\epsilon}, \quad \text{for all } i = 1, \dots, j-1 \ ,$$

where

$$d^\epsilon_{i,n} = \frac{\int_\Omega p \, v_{i,R,\epsilon} \, \hat{v}_{n,R,\epsilon} \, \mathrm{d}x}{\int_\Omega p |\hat{v}_{n,R,\epsilon}|^2 \, \mathrm{d}x}.$$

Using the estimates established in Lemma 6.2, one can prove the following

$$\int_\Omega |\nabla \hat{v}_{j,R,\epsilon}|^2 \, \mathrm{d}x = \lambda^\epsilon_j + \epsilon^{N-2} H(\varphi^\epsilon_j, K_\rho \epsilon) \left(\int_{B^+_R} |\nabla Z^\epsilon_R|^2 \, \mathrm{d}x - \int_{\Pi_R} |\nabla \tilde{\varphi}^\epsilon|^2 \, \mathrm{d}x \right), \tag{64}$$

$$\int_\Omega |\nabla \hat{v}_{i,R,\epsilon}|^2 \, \mathrm{d}x = \lambda^\epsilon_i + O(\epsilon^{N-2\rho}) \quad \text{for all } i \in \{1, \dots, j\}, \tag{65}$$

$$\int_\Omega \nabla \hat{v}_{j,R,\epsilon} \cdot \nabla \hat{v}_{i,R,\epsilon} \, \mathrm{d}x = O(\epsilon^{N-1-\rho} \sqrt{H(\varphi^\epsilon_j, K_\rho \epsilon)}) \quad \text{for all } i \in \{1, \dots, j-1\}, \tag{66}$$

$$\int_\Omega \nabla \hat{v}_{i,R,\epsilon} \cdot \nabla \hat{v}_{m,R,\epsilon} \, \mathrm{d}x = O(\epsilon^{N-2\rho}) \quad \text{for all } i, m \in \{1, \dots, j\}, \quad i \neq m, \tag{67}$$

$$\int_\Omega p |\hat{v}_{j,R,\epsilon}|^2 \, \mathrm{d}x = 1 + O(\epsilon^{N-2/N} H(\varphi^\epsilon_j, K_\rho \epsilon)), \tag{68}$$

$$\int_\Omega p |\hat{v}_{i,R,\epsilon}|^2 \, \mathrm{d}x = 1 + O(\epsilon^{N+2-2\rho-2/N}) \quad \text{for all } i \in \{1, \dots, j\}, \tag{69}$$

as $\epsilon \to 0$.

Proposition 6.3. *Let $\rho \in (0, 1/2)$, K_ρ as defined in Proposition 5.11 and $R \geq K_\rho$. For $\epsilon < R_0/R$ there exists $f_R(\epsilon)$ such that*

$$\lambda_j - \lambda^\epsilon_j \leq \epsilon^{N-2} H(\varphi^\epsilon_j, K_\rho \epsilon)(f_R(\epsilon) + o(1)) \quad \text{as } \epsilon \to 0$$

and

$$f_R(\epsilon) = \int_{B^+_R} |\nabla Z^\epsilon_R|^2 \, \mathrm{d}x - \int_{\Pi_R} |\nabla \tilde{\varphi}^\epsilon|^2 \, \mathrm{d}x,$$

where $\tilde{\varphi}^\epsilon$ has been trivially extended in Π_R outside its domain.

Proof. By the *Courant-Fischer Min-Max* characterization of eigenvalues

$$\lambda_j = \min \left\{ \max_{\substack{\alpha_1, \dots, \alpha_j \in \mathbb{R} \\ \sum_{i=1}^j |\alpha_i|^2 = 1}} \frac{\int_\Omega \left| \nabla \left(\sum_{i=1}^j \alpha_i u_i \right) \right|^2 \mathrm{d}x}{\int_\Omega p \left| \sum_{i=1}^j \alpha_i u_i \right|^2 \mathrm{d}x} : \{u_1, \dots, u_j\} \subseteq H^1_0(\Omega) \text{ linearly independent} \right\}.$$

Testing the Rayleigh quotient with the family of functions

$$\tilde{v}_{i,R,\epsilon} = \frac{\hat{v}_{i,R,\epsilon}}{\sqrt{\int_\Omega p |\hat{v}_{i,R,\epsilon}|^2 \, \mathrm{d}x}}$$

we obtain that

$$\lambda_j - \lambda_j^\epsilon \le \max_{\substack{\alpha_1,\dots,\alpha_j \in \mathbb{R} \\ \sum_{i=1}^j |\alpha_i|^2 = 1}} \int_\Omega \left| \nabla \Bigl(\sum_{i=1}^j \alpha_i \tilde{v}_{i,R,\epsilon} \Bigr) \right|^2 \mathrm{d}x - \lambda_j^\epsilon = \max_{\substack{\alpha_1,\dots,\alpha_j \in \mathbb{R} \\ \sum_{i=1}^j |\alpha_i|^2 = 1}} \sum_{i,n=1}^j M_{i,n}^\epsilon \alpha_i \alpha_n$$

where

$$M_{i,n}^\epsilon = \frac{\int_\Omega \nabla \hat{v}_{i,R,\epsilon} \cdot \nabla \hat{v}_{n,R,\epsilon} \, \mathrm{d}x}{\left(\int_\Omega p |\hat{v}_{i,R,\epsilon}|^2 \, \mathrm{d}x \right)^{1/2} \left(\int_\Omega p |\hat{v}_{n,R,\epsilon}|^2 \, \mathrm{d}x \right)^{1/2}} - \lambda_j^\epsilon \delta_i^n,$$

with δ_i^n denoting the usual *Kronecker delta*, i.e. $\delta_i^n = 0$ for $i \neq n$ and $\delta_i^n = 1$ for $i = n$. From estimates (64)–(69) one can derive the following estimates

$$\begin{aligned}
&M_{j,j}(\epsilon) = \epsilon^{N-2} H(\varphi_j^\epsilon, K_\rho \epsilon)(f_R(\epsilon) + O(\epsilon^{2-2/N})), \\
&M_{i,j}(\epsilon) = O\left(\epsilon^{N-1-\rho} \sqrt{H(\varphi_j^\epsilon, K_\rho \epsilon)} \right) \quad \text{and} \quad M_{i,i}(\epsilon) = \lambda_i^\epsilon - \lambda_j^\epsilon + o(1) \quad \text{for all } i < j, \\
&M_{i,n}(\epsilon) = O(\epsilon^{N-2\rho}) \quad \text{for all } i, n < j, \quad i \neq n,
\end{aligned}$$

as $\epsilon \to 0$. Moreover, from Corollary 5.13, we know that $H(\varphi_j^\epsilon, K_\rho \epsilon) \ge \bar{C} \epsilon^q$ for some $\bar{C}, q > 0$. Therefore, taking also into account (46) and the fact that $f_R(\epsilon) = O(1)$ as $\epsilon \to 0$ in view of (36) and (48), the hypotheses of Lemma 9.3 are satisfied with

$$\sigma(\epsilon) = \epsilon^{N-2} H(\varphi_j^\epsilon, K_\rho \epsilon), \quad \mu(\epsilon) = f_R(\epsilon) + o(1), \quad \alpha = \frac{N}{2} - \rho, \quad M > (2\rho - 2 + q) \frac{2}{N - 2\rho}.$$

The proof is thereby complete. □

6.2 Lower Bound

For any $R > 1$ and $\epsilon \in (0, 1]$, with $R\epsilon \le R_{\max}$, let us consider the following minimization problem

$$\min \left\{ \int_{\Omega_{R\epsilon}^\epsilon} |\nabla u|^2 \, \mathrm{d}x \colon u \in H^1(\Omega_{R\epsilon}^\epsilon), \ u = 0 \text{ on } \partial \Omega_{R\epsilon}^\epsilon \setminus S_{R\epsilon}^+, \ u = \varphi_j \text{ on } S_{R\epsilon}^+ \right\}. \tag{70}$$

One can prove that this problem has a unique solution $w_{j,R,\epsilon}^{\mathrm{int}}$, which weakly verifies

$$\begin{cases} -\Delta w_{j,R,\epsilon}^{\mathrm{int}} = 0, & \text{in } \Omega_{R\epsilon}^\epsilon, \\ w_{j,R,\epsilon}^{\mathrm{int}} = 0, & \text{on } \partial \Omega_{R\epsilon}^\epsilon \setminus S_{R\epsilon}^+, \\ w_{j,R,\epsilon}^{\mathrm{int}} = \varphi_j, & \text{on } S_{R\epsilon}^+. \end{cases}$$

Let us define

$$w_{j,R,\epsilon} := \begin{cases} w_{j,R,\epsilon}^{\mathrm{int}}, & \text{in } \Omega_{R\epsilon}^\epsilon, \\ \varphi_j, & \text{in } \Omega \setminus B_{R\epsilon}^+. \end{cases}$$

Lemma 6.4. *There exists $\tilde{C} > 0$ such that, for all $i \in \{1, \dots, j-1\}$, for all $R > 1$ and $\epsilon \in (0, 1]$, with $R\epsilon \le R_{\max}$,*

$$\int_{B_{R\epsilon}^+} |\nabla \varphi_i|^2 \, \mathrm{d}x \le \tilde{C}(R\epsilon)^N, \quad \int_{B_{R\epsilon}^+} |\varphi_i|^2 \, \mathrm{d}x \le \tilde{C}(R\epsilon)^{N+2}, \quad \int_{S_{R\epsilon}^+} |\varphi_i|^2 \, \mathrm{d}x \le \tilde{C}(R\epsilon)^{N+1},$$

and

$$\int_{B_{R\epsilon}^+} |\nabla \varphi_j|^2 \, \mathrm{d}x \le \tilde{C}(R\epsilon)^{N+2k-2}, \quad \int_{B_{R\epsilon}^+} |\varphi_j|^2 \, \mathrm{d}x \le \tilde{C}(R\epsilon)^{N+2k}, \quad \int_{S_{R\epsilon}^+} |\varphi_j|^2 \, \mathrm{d}x \le \tilde{C}(R\epsilon)^{N+2k-1}.$$

Proof. It follows from classical asymptotic estimates at the boundary, see e.g. [21, Th. 1.3] and (8),(9). □

Lemma 6.5. *There exists $\hat{C} > 0$ such that, for all $R > 1$ and $\epsilon \in (0, 1]$, with $R\epsilon \le R_{\max}$,*

$$\int_{\Omega^\epsilon_{R\epsilon}} \left|\nabla w^{\mathrm{int}}_{j,R,\epsilon}\right|^2 \mathrm{d}x \le \hat{C}(R\epsilon)^{N+2k-2}, \tag{71}$$

$$\int_{S^+_{R\epsilon}} \left|w^{\mathrm{int}}_{j,R,\epsilon}\right|^2 \mathrm{d}x \le \hat{C}(R\epsilon)^{N+2k-1}. \tag{72}$$

Furthermore, for all $R > \mu_{1/2}$ and $\epsilon \in (0, 1]$, with $R\epsilon \le R_{\max}$,

$$\int_{\Omega^\epsilon_{R\epsilon}} \left|w^{\mathrm{int}}_{j,R,\epsilon}\right|^2 \mathrm{d}x \le \hat{C}(R\epsilon)^{N+2k-2/N}. \tag{73}$$

Proof. (72) is trivial since $w^{\mathrm{int}}_{j,R,\epsilon} = \varphi_j$ on $S^+_{R\epsilon}$. Also (71) is simple since φ_j is an admissible test function for (70). Finally (73) comes from (71), (72) and Lemma 5.5. □

Now let us define, for all $R > 1$ and $\epsilon \in (0, 1]$ such that $R\epsilon \le R_{\max}$,

$$U^\epsilon_R(x) = \frac{w^{\mathrm{int}}_{j,R,\epsilon}(\epsilon x)}{\epsilon^k}, \qquad W^\epsilon(x) = \frac{\varphi_j(\epsilon x)}{\epsilon^k}. \tag{74}$$

From (8), we easily deduce that

$$W^\epsilon \longrightarrow \psi_k \qquad \text{in } H^1(B^+_R) \quad \text{as } \epsilon \to 0, \quad \text{for all } R > 0,$$

where ψ_k has been defined in (10).

Lemma 6.6. *We have that*

$$U^\epsilon_R \longrightarrow U_R \qquad \text{in } \mathcal{H}_R \quad \text{as } \epsilon \to 0, \quad \text{for all } R > 1,$$

where U_R is defined in Lemma 2.3.

Proof. From Lemma 6.5 and from the definition of U^ϵ_R we know that

$$\int_{\Pi_R} |\nabla U^\epsilon_R|^2 \,\mathrm{d}x = O(1) \qquad \text{as } \epsilon \to 0.$$

where U^ϵ_R has been trivially extended in Π_R outside its domain. So there exists $V = V_R \in \mathcal{H}_R$ such that, along a sequence $\epsilon = \epsilon_n \to 0$,

$$U^\epsilon_R \rightharpoonup V \qquad \text{weakly in } \mathcal{H}_R \quad \text{as } \epsilon = \epsilon_n \to 0.$$

This means that

$$\nabla U^\epsilon_R \rightharpoonup \nabla V \qquad \text{weakly in } L^2(\Pi_R) \quad \text{as } \epsilon = \epsilon_n \to 0.$$

Since $U^\epsilon_R = W^\epsilon$ on S^+_R and $W^\epsilon \to \psi_k$ in $L^2(S^+_R)$, then V satisfies (in a weak sense) the same equation as U_R, defined in Lemma 2.3. So, by uniqueness, $V = U_R$. Since the limit $V = U_R$ is the same for every subsequence, *Urysohn's Subsequence Principle* implies that the convergence $U^\epsilon_R \rightharpoonup U_R$ holds as $\epsilon \to 0$ (not only along subsequences).

To prove strong convergence it is enough to show that $\|U^\epsilon_R\|_{\mathcal{H}_R} \to \|U_R\|_{\mathcal{H}_R}$ as $\epsilon \to 0$. First we notice that, trivially, $-\Delta U^\epsilon_R \rightharpoonup -\Delta U_R$ weakly in $L^2(\Pi_R)$: so, we have that $\nabla U^\epsilon_R \rightharpoonup \nabla U_R$ in $H(\mathrm{div}, \Pi_R)$, thus

$$\frac{\partial U^\epsilon_R}{\partial \boldsymbol{\nu}} \rightharpoonup \frac{\partial U_R}{\partial \boldsymbol{\nu}} \qquad \text{in } \left(H^{1/2}_{00}(S^+_R)\right)^* \quad \text{as } \epsilon \to 0,$$

where $(H^{1/2}_{00}(S^+_R))^*$ is the dual of the Lions-Magenes space $H^{1/2}_{00}(S^+_R)$. Then, since $W^\epsilon \to \psi_k$ in $H^{1/2}_{00}(S^+_R)$ as $\epsilon \to 0$, we obtain that

$$\int_{\Pi_R} |\nabla U^\epsilon_R|^2 \,\mathrm{d}x = \int_{S^+_R} \frac{\partial U^\epsilon_R}{\partial \boldsymbol{\nu}} W^\epsilon \,\mathrm{d}S \to \int_{S^+_R} \frac{\partial U_R}{\partial \boldsymbol{\nu}} \psi_k \,\mathrm{d}S = \int_{\Pi_R} |\nabla U_R|^2 \quad \text{as } \epsilon \to 0,$$

thus completing the proof. □

It is easy to prove that the family of functions $\{\varphi_1, \varphi_2, \dots, \varphi_{j-1}, w_{j,R,\epsilon}\}$ is linearly independent in $H_0^1(\Omega^\epsilon)$. As in the previous section, we construct a new basis of the space

$$\mathrm{span}\{\varphi_1, \varphi_2, \dots, \varphi_{j-1}, w_{j,R,\epsilon}\} \subseteq H_0^1(\Omega^\epsilon)$$

by defining, for all $i = 1, \dots, j-1$

$$\hat{w}_{i,R,\epsilon} = \varphi_i$$

and

$$\hat{w}_{j,R,\epsilon} = w_{j,R,\epsilon} - \sum_{i=1}^{j-1} c_i^\epsilon \varphi_i,$$

where

$$c_i^\epsilon = \int_{\Omega^\epsilon} p w_{j,R,\epsilon} \varphi_i \,\mathrm{d}x.$$

In this way we have that $\int_{\Omega^\epsilon} p\, \hat{w}_{n,R,\epsilon} \hat{w}_{m,R,\epsilon}\, dx = 0$ if $n \neq m$.

Using the estimates established in Lemmas 6.4 and 6.5, one can prove the following

$$\int_{\Omega^\epsilon} |\nabla \hat{w}_{j,R,\epsilon}|^2 \,\mathrm{d}x = \lambda_j + \epsilon^{N+2k-2} \left(\int_{\Pi_R} |\nabla U_R^\epsilon|^2 \,\mathrm{d}x - \int_{B_R^+} |\nabla W^\epsilon|^2 \,\mathrm{d}x + o(1) \right), \tag{75}$$

$$\int_{\Omega^\epsilon} \nabla \hat{w}_{j,R,\epsilon} \cdot \nabla \hat{w}_{i,R,\epsilon} \,\mathrm{d}x = O(\epsilon^{N+k-1}) \quad \text{for all } i \in \{1, \dots, j-1\}, \tag{76}$$

$$\int_{\Omega^\epsilon} p |\hat{w}_{j,R,\epsilon}|^2 \,\mathrm{d}x = 1 + O(\epsilon^{N+2k-2/N}), \tag{77}$$

as $\epsilon \to 0$.

Proposition 6.7. *Let $\rho \in (0, 1/2)$, K_ρ as defined in Proposition 5.11 and $R \geq K_\rho$. For $\epsilon < R_0/R$ there exists $g_R(\epsilon)$ such that*

$$\lambda_j^\epsilon - \lambda_j \leq \epsilon^{N+2k-2}(g_R(\epsilon) + o(1)) \quad \text{as } \epsilon \to 0$$

and

$$g_R(\epsilon) = \int_{\Pi_R} |\nabla U_R^\epsilon|^2 \,\mathrm{d}x - \int_{B_R^+} |\nabla W^\epsilon|^2 \,\mathrm{d}x,$$

where U_R^ϵ has been trivially extended in Π_R outside its domain.

Proof. By the *Courant-Fischer Min-Max* characterization of eigenvalues

$$\lambda_j^\epsilon = \min \left\{ \max_{\substack{\alpha_1, \dots, \alpha_j \in \mathbb{R} \\ \sum_{i=1}^j |\alpha_i|^2 = 1}} \frac{\int_{\Omega^\epsilon} \left| \nabla \left(\sum_{i=1}^j \alpha_i u_i \right) \right|^2 \mathrm{d}x}{\int_{\Omega^\epsilon} p \left| \sum_{i=1}^j \alpha_i u_i \right|^2 \mathrm{d}x} : \{u_1, \dots, u_j\} \subseteq H_0^1(\Omega^\epsilon) \text{ linearly independent} \right\}.$$

Testing the Rayleigh quotient with the family of functions

$$\tilde{w}_{i,R,\epsilon} = \frac{\hat{w}_{i,R,\epsilon}}{\sqrt{\int_{\Omega^\epsilon} p |\hat{w}_{i,R,\epsilon}|^2 \,\mathrm{d}x}}$$

we obtain that

$$\lambda_j^\epsilon - \lambda_j \leq \max_{\substack{\alpha_1, \dots, \alpha_j \in \mathbb{R} \\ \sum_{i=1}^j |\alpha_i|^2 = 1}} \int_{\Omega^\epsilon} \left| \nabla \left(\sum \alpha_i \tilde{w}_{i,R,\epsilon} \right) \right|^2 \mathrm{d}x - \lambda_j = \max_{\substack{\alpha_1, \dots, \alpha_j \in \mathbb{R} \\ \sum_{i=1}^j |\alpha_i|^2 = 1}} \sum_{i,n=1}^j L_{i,n}^\epsilon \alpha_i \alpha_n,$$

where

$$L_{i,n}^\epsilon = \frac{\int_{\Omega^\epsilon} \nabla \hat{w}_{i,R,\epsilon} \cdot \nabla \hat{w}_{n,R,\epsilon} \,\mathrm{d}x}{\left(\int_{\Omega^\epsilon} p |\hat{w}_{i,R,\epsilon}|^2 \,\mathrm{d}x \right)^{1/2} \left(\int_{\Omega^\epsilon} p |\hat{w}_{n,R,\epsilon}|^2 \,\mathrm{d}x \right)^{1/2}} - \lambda_j \delta_i^n,$$

with δ_i^n denoting the usual *Kronecker delta*. From estimates (75)–(77) it follows that

$$L^\epsilon_{j,j} = \epsilon^{N+2k-2}(g_R(\epsilon)+o(1)), \quad L^\epsilon_{i,j} = O(\epsilon^{N+k-1}) \quad \text{for all } i<j,$$
$$L^\epsilon_{i,i} = \lambda_i - \lambda_j \quad \text{for all } i<j, \quad L^\epsilon_{i,n} = 0 \quad \text{for all } i,n<j, \quad i \neq n,$$

as $\epsilon \to 0$. Therefore, taking into account that $g_R(\epsilon) = O(1)$ as $\epsilon \to 0$ in view of Lemma 6.4 and (71), the hypotheses of Lemma 9.3 are satisfied with

$$\sigma(\epsilon) = \epsilon^{N+2k-2}, \quad \mu(\epsilon) = g_R(\epsilon)+o(1), \quad \alpha = \frac{N}{2}, \quad M > \frac{4(k-1)}{N}.$$

The proof is thereby complete. □

From the fact that $W^\epsilon \to \psi_k$ in $H^1(B_R^+)$, as $\epsilon \to 0$, for all $R>0$, and from Lemma 6.6 we can deduce the following result.

Lemma 6.8. *For all $R>1$ we have that*

$$g_R(\epsilon) \longrightarrow g_R \qquad \text{as } \epsilon \to 0,$$

where

$$g_R := \int_{\Pi_R} |\nabla U_R|^2 \,\mathrm{d}x - \int_{B_R^+} |\nabla \psi_k|^2 \,\mathrm{d}x. \tag{78}$$

In order to compute the limit $\lim_{R\to\infty} g_R$, we introduce the functions

$$\zeta(r) := \int_{S_1^+} \Phi(r\theta)\Psi(\theta)\,\mathrm{d}S(\theta) \qquad \text{for } r \geq 1, \tag{79}$$

$$\chi_R(r) := \int_{S_1^+} U_R(r\theta)\Psi(\theta)\,\mathrm{d}S(\theta) \qquad \text{for } 1 \leq r \leq R. \tag{80}$$

Moreover, we denote

$$\gamma_N := \int_{S_1^+} |\Psi(\theta)|^2 \,\mathrm{d}S(\theta). \tag{81}$$

We immediatly notice, thanks to Lemma 2.4 and to the embedding $H^1(B_1^+) \hookrightarrow L^2(S_1^+)$, that

$$\zeta(1) = \lim_{R\to+\infty} \chi_R(1). \tag{82}$$

Lemma 6.9. *Let ζ be the function defined in* (79)*, γ_N the constant defined in* (81) *and $m_k(\Sigma)$ the one defined in* (11)*. Then*

$$\zeta(1) = \gamma_N - \frac{2m_k(\Sigma)}{N+2k-2}. \tag{83}$$

Proof. From the definition of Φ, given in (20), one can easily prove that ζ satisfies the following ODE

$$\left(r^{N+2k-1}(r^{-k}\zeta(r))'\right)' = 0 \qquad \text{in } (1,+\infty).$$

This yields

$$r^{-k}\zeta(r) = \zeta(1) + C\frac{1-r^{-N-2k+2}}{N+2k-2} \tag{84}$$

for some constant $C \in \mathbb{R}$. Now we note that $r^{-k}\zeta(r) \to \gamma_N$ as $r \to +\infty$. Indeed, since $\Phi = w_k + \psi_k$, we can rewrite

$$\zeta(r) = \int_{S_1^+} w_k(r\theta)\Psi(\theta)\,\mathrm{d}S(\theta) + \gamma_N r^k.$$

By evaluating the vanishing order at 0 of the Kelvin transform of the restriction of the function w_k on $\Pi \setminus \Pi_1$, we can prove that

$$|w_k(x)| \leq \text{const}|x|^{1-N} \qquad \text{for } |x|>1.$$

Hence, when $r \to +\infty$

$$\left|r^{-k}\zeta(r) - \gamma_N\right| \leq \int_{S_1^+} \frac{|w_k(r\theta)|}{r^k} |\Psi(\theta)| \,\mathrm{d}S(\theta) \leq \mathrm{const}\, r^{1-N-k} \to 0.$$

Then we can find the constant C in (84), letting $r \to +\infty$; so we can rewrite ζ as

$$\zeta(r) = \gamma_N r^k + (\zeta(1) - \gamma_N) r^{-N-k+2} \qquad \text{in } (1, +\infty). \tag{85}$$

Taking the derivative leads to

$$\begin{aligned} \zeta'(r) &= k\gamma_N r^{k-1} + (N+k-2)(\gamma_N - \zeta(1)) r^{-N-k+1} \\ &= (N+2k-2)\gamma_N r^{k-1} - \frac{(N+k-2)\zeta(r)}{r}. \end{aligned} \tag{86}$$

Hence, taking into account the definition of ζ and evaluating its derivative at $r = 1$, we obtain

$$\int_{S_1^+} \frac{\partial \Phi}{\partial \boldsymbol{\nu}}(\theta) \Psi(\theta) \,\mathrm{d}S(\theta) = (N+2k-2)\gamma_N - (N+k-2)\zeta(1). \tag{87}$$

Since $-\Delta\Phi = 0$ in B_1^+, multiplying this equation by ψ_k and integrating by parts we obtain that

$$\int_{B_1^+} \nabla\Phi \cdot \nabla\psi_k \,\mathrm{d}x = \int_{S_1^+} \frac{\partial \Phi}{\partial \boldsymbol{\nu}} \psi_k \,\mathrm{d}S = \int_{S_1^+} \frac{\partial \Phi}{\partial \boldsymbol{\nu}} \Psi \,\mathrm{d}S. \tag{88}$$

Then, let us test the equation $-\Delta\psi_k = 0$ with Φ. From (12) and (21) it follows that

$$\begin{aligned} \int_{B_1^+} \nabla\psi_k \cdot \nabla\Phi \,\mathrm{d}x = \int_{S_1^+} \frac{\partial \psi_k}{\partial \boldsymbol{\nu}} \Phi \,\mathrm{d}S - \int_\Sigma \frac{\partial \psi_k}{\partial x_1} \Phi \,\mathrm{d}S &= \int_{S_1^+} \frac{\partial \psi_k}{\partial \boldsymbol{\nu}} \Phi \,\mathrm{d}S - \int_\Sigma \frac{\partial \psi_k}{\partial x_1} w_k \,\mathrm{d}S \\ &= \int_{S_1^+} \frac{\partial \psi_k}{\partial \boldsymbol{\nu}} \Phi \,\mathrm{d}S + 2m_k(\Sigma). \end{aligned} \tag{89}$$

Moreover we note that

$$\frac{\partial \psi_k}{\partial \boldsymbol{\nu}}(\theta) = k\Psi(\theta) \qquad \text{on } S_1^+. \tag{90}$$

Then, from (89) and (90) we obtain

$$\int_{B_1^+} \nabla\psi_k \cdot \nabla\Phi \,\mathrm{d}x = k\int_{S_1^+} \Phi\Psi \,\mathrm{d}S + 2m_k(\Sigma) = k\zeta(1) + 2m_k(\Sigma). \tag{91}$$

Finally, combining (87), (88) and (91) leads to the thesis. □

Lemma 6.10. *Let g_R be as defined in* (78) *and $m_k(\Sigma)$ as in* (11). *Then* $\lim_{R\to+\infty} g_R = 2m_k(\Sigma)$.

Proof. Integrating by parts we have that

$$g_R = \int_{S_R^+} \left(\frac{\partial U_R}{\partial \boldsymbol{\nu}} - \frac{\partial \psi_k}{\partial \boldsymbol{\nu}} \right) \psi_k \,\mathrm{d}S.$$

If χ_R is the function defined in (80), then

$$\chi_R'(r) = \int_{S_1^+} \frac{\partial U_R}{\partial \boldsymbol{\nu}}(r\theta) \Psi(\theta) \,\mathrm{d}S(\theta)$$

and, by a change of variable,

$$\chi_R'(r) = r^{1-N-k} \int_{S_r^+} \frac{\partial U_R}{\partial \boldsymbol{\nu}} \psi_k \,\mathrm{d}S. \tag{92}$$

By simple computations one can prove that χ_R solves

$$\left(r^{N+2k-1}(r^{-k}\chi_R(r))'\right)' = 0 \qquad \text{in } (1, R).$$

By integration, we arrive at

$$r^{-k}\chi_R(r) = \chi_R(1) + C\frac{1 - r^{-N-2k+2}}{N+2k-2}. \tag{93}$$

From the fact that $U_R = R^K\Psi$ on S_R^+, we have that $\chi_R(R) = R^k\gamma_N$, and this allows us to know the constant C. After some computations, the expression (93) then becomes as follows

$$r^{-k}\chi_R(r) = \chi_R(1) + (\gamma_N - \chi_R(1))\frac{1 - r^{-N-2k+2}}{1 - R^{-N-2k+2}}, \qquad r \in (1, R).$$

From (92), we get

$$\begin{aligned}\int_{S_R^+} \frac{\partial U_R}{\partial \boldsymbol{\nu}}\psi_k \, \mathrm{d}S &= \chi_R'(R)R^{N+k-1} \\ &= \frac{[\gamma_N(N+k-2) - \chi_R(1)(N+2k-2)]R^{-N-k+1} + k\gamma_N R^{k-1}}{1 - R^{-N-2k+2}}R^{N+k-1} \\ &= \frac{\gamma_N(N+k-2) - \chi_R(1)(N+2k-2) + k\gamma_N R^{N+2k-2}}{1 - R^{-N-2k+2}}.\end{aligned} \tag{94}$$

For what concerns the second part of g_R, it is easy to see that

$$\frac{\partial \psi_k}{\partial \boldsymbol{\nu}}(r\theta) = kr^{k-1}\Psi(\theta).$$

Therefore

$$\int_{S_R^+} \frac{\partial \psi_k}{\partial \boldsymbol{\nu}}\psi_k \, \mathrm{d}S = \int_{S_1^+} kR^{N+2k-2}|\Psi|^2 \, \mathrm{d}S(\theta) = kR^{N+2k-2}\gamma_N. \tag{95}$$

Finally, combining (82), (94), (95) and Lemma 6.9 and taking the limit when $R \to +\infty$ we reach the conclusion. □

Combining Propositions 6.3 and 6.7 with Lemmas 6.8 and 6.10 we obtain the following upper-lower estimate for the eigenvalue variation.

Proposition 6.11. *Let $\rho \in (0, 1/2)$, K_ρ as defined in Proposition 5.11 and $m_k(\Sigma)$ as in (11). Then, for all $R \geq K_\rho$, we have that, as $\epsilon \to 0$,*

$$-2m_k(\Sigma) + o(1) \leq \frac{\lambda_j - \lambda_j^\epsilon}{\epsilon^{N+2k-2}} \leq \frac{H(\varphi_j^\epsilon, K_\rho\epsilon)}{\epsilon^{2k}}(f_R(\epsilon) + o(1)).$$

Since $-2m_k(\Sigma) > 0$, as a direct consequence of Proposition 6.11 we obtain the following estimate from below for $H(\varphi_j^\epsilon, K_\rho\epsilon)$.

Corollary 6.12. *We have that*

$$\frac{\epsilon^{2k}}{H(\varphi_j^\epsilon, K_\rho\epsilon)} = O(1) \qquad \text{as } \epsilon \to 0.$$

7 Blow-up Analysis

Let us introduce the functional

$$\begin{aligned} F\colon \mathbb{R} \times H_0^1(\Omega) &\longrightarrow \mathbb{R} \times H^{-1}(\Omega) \\ (\lambda, \varphi) &\longmapsto (\|\varphi\|^2_{H_0^1(\Omega)} - \lambda_j, -\Delta\varphi - \lambda p\varphi)\end{aligned}$$

where $\|\varphi\|^2_{H_0^1(\Omega)} = \int_\Omega |\nabla\varphi|^2 \, dx$ and

$${}_{H^{-1}(\Omega)}\langle -\Delta\varphi - \lambda p\varphi, v\rangle_{H_0^1(\Omega)} = \int_\Omega (\nabla\varphi \cdot \nabla v - \lambda p\varphi v) \, \mathrm{d}x.$$

From the assumptions we know that $F(\lambda_j, \varphi_j) = (0, 0)$. Moreover, from the simplicity assumption (4) and Fredholm Alternative, one can easily prove the following result (see e.g. [1] for details for a similar operator).

Lemma 7.1. *The functional F is differentiable at (λ_j, φ_j) and its differential*

$$\mathrm{d}F(\lambda_j, \varphi_j) \colon \mathbb{R} \times H^1_0(\Omega) \longrightarrow \mathbb{R} \times H^{-1}(\Omega)$$
$$\mathrm{d}F(\lambda_j, \varphi_j)(\lambda, \varphi) = \left(2 \int_\Omega \nabla\varphi_j \cdot \nabla\varphi \,\mathrm{d}x, -\Delta\varphi - \lambda p \varphi_j - \lambda_j p \varphi\right)$$

is invertible.

Lemma 7.2. *Let $\rho \in (0, 1/2)$, K_ρ as defined in Proposition 5.11 and $R \geq K_\rho$. Then, when $\epsilon \to 0$,*

$$v_{j,R,\epsilon} \longrightarrow \varphi_j \qquad \text{in} \quad H^1_0(\Omega),$$

where $v_{j,R,\epsilon}$ is defined in (54).

Proof. First note that

$$\int_\Omega |\nabla(v_{j,R,\epsilon} - \varphi_j)|^2 \,\mathrm{d}x = \int_{\Omega^\epsilon} \left|\nabla(\varphi_j^\epsilon - \varphi_j)\right|^2 \mathrm{d}x - \int_{\Omega^\epsilon_{R\epsilon}} \left|\nabla(\varphi_j^\epsilon - \varphi_j)\right|^2 \mathrm{d}x + \int_{B^+_{R\epsilon}} \left|\nabla(v^{\mathrm{int}}_{j,R,\epsilon} - \varphi_j)\right|^2 \mathrm{d}x.$$

The first term tends to zero because of (6). For the second and the third term we can exploit the energy estimates in Proposition 5.12, Lemma 6.4 and Proposition 6.1 to conclude. □

Lemma 7.3. *Let $\rho \in (0, 1/2)$, K_ρ as defined in Proposition 5.11 and $R \geq K_\rho$. Then*

$$\|v_{j,R,\epsilon} - \varphi_j\|_{H^1_0(\Omega)} = O\left(\epsilon^{N/2-1}\sqrt{H(\varphi_j^\epsilon, K_\rho\epsilon)}\right) \qquad \text{as} \quad \epsilon \to 0$$

and, in particular,

$$\int_{\Omega \setminus B^+_{R\epsilon}} \left|\nabla(\varphi_j^\epsilon - \varphi_j)\right|^2 \mathrm{d}x = O\left(\epsilon^{N-2} H(\varphi_j^\epsilon, K_\rho\epsilon)\right) \qquad \text{as} \quad \epsilon \to 0. \tag{96}$$

Proof. Taking into account Lemma 7.2 and (5), from the differentiability of the functional F it follows that

$$F(\lambda_j^\epsilon, v_{j,R,\epsilon}) = \mathrm{d}F(\lambda_j, \varphi_j)(\lambda_j^\epsilon - \lambda_j, v_{j,R,\epsilon} - \varphi_j) + o\left(\left|\lambda_j^\epsilon - \lambda_j\right| + \|v_{j,R,\epsilon} - \varphi_j\|_{H^1_0(\Omega)}\right)$$

as $\epsilon \to 0$. Now let us apply $\mathrm{d}F(\lambda_j, \varphi_j)^{-1}$ to both members and obtain

$$\begin{aligned}\left|\lambda_j^\epsilon - \lambda_j\right| &+ \|v_{j,R,\epsilon} - \varphi_j\|_{H^1_0(\Omega)} \\ &\leq \left\|\mathrm{d}F(\lambda_j, \varphi_j)^{-1}\right\|_{\mathcal{L}(\mathbb{R}\times H^{-1}(\Omega), \mathbb{R}\times H^1_0(\Omega))} \left\|F(\lambda_j^\epsilon, v_{j,R,\epsilon})\right\|_{\mathbb{R}\times H^{-1}(\Omega)} (1 + o(1))\end{aligned}$$

and so

$$\|v_{j,R,\epsilon} - \varphi_j\|_{H^1_0(\Omega)} \leq C\left(\left|\|v_{j,R,\epsilon}\|^2_{H^1_0(\Omega)} - \lambda_j\right| + \left\|-\Delta v_{j,R,\epsilon} - \lambda_j^\epsilon p v_{j,R,\epsilon}\right\|_{H^{-1}(\Omega)}\right). \tag{97}$$

Thanks to (56), Proposition 6.11, and the fact that $f_R(\epsilon) = O(1)$ as $\epsilon \to 0$ in view of (36) and (48),

$$\left|\|v_{j,R,\epsilon}\|^2_{H^1_0(\Omega)} - \lambda_j\right| \leq \left|\|v_{j,R,\epsilon}\|_{H^1_0(\Omega)} - \lambda_j^\epsilon\right| + \left|\lambda_j^\epsilon - \lambda_j\right| = O(\epsilon^{N-2} H(\varphi_j^\epsilon, K_\rho\epsilon)). \tag{98}$$

Let $u \in H^1_0(\Omega)$ be such that $\|u\|_{H^1_0(\Omega)} \leq 1$. Note that

$$\begin{aligned}\int_\Omega \nabla v_{j,R,\epsilon} \cdot \nabla u \,\mathrm{d}x &= \int_{B^+_{R\epsilon}} \nabla v^{\mathrm{int}}_{j,R,\epsilon} \cdot \nabla u \,\mathrm{d}x + \int_{\Omega^\epsilon} \nabla\varphi_j^\epsilon \cdot \nabla u \,\mathrm{d}x - \int_{\Omega^\epsilon_{R\epsilon}} \nabla\varphi_j^\epsilon \cdot \nabla u \,\mathrm{d}x \leq \\ &\leq \sqrt{\int_{B^+_{R\epsilon}} \left|\nabla v^{\mathrm{int}}_{j,R,\epsilon}\right|^2 \mathrm{d}x} + \lambda_j^\epsilon \int_{\Omega^\epsilon} p\varphi_j^\epsilon u \,\mathrm{d}x + \sqrt{\int_{\Omega^\epsilon_{R\epsilon}} \left|\nabla\varphi_j^\epsilon\right|^2 \mathrm{d}x}.\end{aligned}$$

So we have that

$$\begin{aligned}\int_\Omega \nabla v_{j,R,\epsilon}\cdot\nabla u\,\mathrm{d}x - \lambda_j^\epsilon\int_\Omega p v_{j,R,\epsilon} u\,\mathrm{d}x \le\\ \le \sqrt{\int_{B^+_{R\epsilon}}\left|\nabla v^{\mathrm{int}}_{j,R,\epsilon}\right|^2\mathrm{d}x} + \lambda_j^\epsilon\left(\int_{\Omega^\epsilon} p\varphi_j^\epsilon u\,\mathrm{d}x - \int_\Omega p v_{j,R,\epsilon}u\,\mathrm{d}x\right) + \sqrt{\int_{\Omega^\epsilon_{R\epsilon}}\left|\nabla\varphi_j^\epsilon\right|^2\mathrm{d}x}.\end{aligned}\tag{99}$$

Now let us analyze the middle term

$$\begin{aligned}\int_{\Omega^\epsilon} p\varphi_j^\epsilon u\,\mathrm{d}x - \int_\Omega p v_{j,R,\epsilon}u\,\mathrm{d}x = \int_{B^+_{R\epsilon}} p\varphi_j^\epsilon u\,\mathrm{d}x - \int_{B^+_{R\epsilon}} p v^{\mathrm{int}}_{j,R,\epsilon}u\,\mathrm{d}x \le\\ \le \mathrm{const}\left(\sqrt{\int_{B^+_{R\epsilon}}\left|\varphi_j^\epsilon\right|^2\mathrm{d}x} + \sqrt{\int_{B^+_{R\epsilon}}|v^{\mathrm{int}}_{j,R,\epsilon}|^2\,\mathrm{d}x}\right)\end{aligned}$$

where we implicitly used the Poincaré Inequality. Thanks to inequality (29) and to the energy estimates made in Proposition 5.11

$$\int_{B^+_{R\epsilon}}\left|\varphi_j^\epsilon\right|^2\mathrm{d}x \le \frac{(R\epsilon)^2}{N-1}\int_{B^+_{R\epsilon}}\left|\nabla\varphi_j^\epsilon\right|^2\mathrm{d}x + \frac{R\epsilon}{N-1}\int_{S^+_{R\epsilon}}|\varphi_j^\epsilon|^2\,\mathrm{d}x = O(\epsilon^N H(\varphi_j^\epsilon, K_\rho\epsilon)) \quad \text{as } \epsilon\to 0.$$

Then, from (99), Proposition 5.12 and Proposition 6.1 we obtain that

$$\int_\Omega \nabla v_{j,R,\epsilon}\cdot\nabla u\,\mathrm{d}x - \lambda_j^\epsilon\int_\Omega p v_{j,R,\epsilon}u\,\mathrm{d}x = O\left(\epsilon^{N/2-1}\sqrt{H(\varphi_j^\epsilon, K_\rho\epsilon)}\right) \quad \text{as } \epsilon\to 0$$

uniformly with respect to $u\in H^1_0(\Omega)$ with $\|u\|_{H^1_0(\Omega)}\le 1$ and hence

$$\left\|-\Delta v_{j,R,\epsilon} - \lambda_j^\epsilon p v_{j,R,\epsilon}\right\|_{H^{-1}(\Omega)} = O\left(\epsilon^{N/2-1}\sqrt{H(\varphi_j^\epsilon, K_\rho\epsilon)}\right) \quad \text{as } \epsilon\to 0. \tag{100}$$

The conclusion follows by combining (97), (98), and (100). □

Corollary 7.4. *Let $\rho\in(0,1/2)$, K_ρ as defined in Proposition 5.11 and $R\ge K_\rho$. Then*

$$\int_{\frac{1}{\epsilon}\Omega\setminus B^+_R}\left|\nabla\tilde\varphi^\epsilon - \epsilon^k H(\varphi_j^\epsilon, K_\rho\epsilon)^{-1/2}\nabla W^\epsilon\right|^2\mathrm{d}x = O(1), \qquad \text{as } \epsilon\to 0,$$

where $\tilde\varphi^\epsilon$ is defined in (55) and W^ϵ in (74), while $\frac{1}{\epsilon}\Omega = \{\frac{1}{\epsilon}x\colon x\in\Omega\}$.

Proof. It directly follows from a change of variables in (96). □

The following Theorem provides a blow-up analysis for scaled eigenfunctions, which contains Theorem 1.2.

Theorem 7.5. *Let $\rho\in(0,1/2)$ and K_ρ as defined in Proposition 5.11. Then*

$$\tilde\varphi^\epsilon \longrightarrow \frac{1}{\sqrt{\Lambda_\rho}}\Phi \qquad \text{in } \mathcal{H}_R \quad \text{for all } R>2, \tag{101}$$

$$\frac{H(\varphi_j^\epsilon, K_\rho\epsilon)}{\epsilon^{2k}} \longrightarrow \Lambda_\rho, \tag{102}$$

$$\frac{\varphi_j^\epsilon(\epsilon x)}{\epsilon^k} \longrightarrow \Phi(x) \qquad \text{in } \mathcal{H}_R \quad \text{for all } R>2, \tag{103}$$

as $\epsilon\to 0$, where

$$\Lambda_\rho := \frac{1}{K_\rho^{N-1}}\int_{S^+_{K_\rho}}|\Phi|^2\,\mathrm{d}S.$$

Proof. Let $\epsilon_n \to 0$. From Corollary 6.12 we deduce that, up to a subsequence,

$$\frac{(\epsilon_n)^k}{\sqrt{H(\varphi_j^{\epsilon_n}, K_\rho \epsilon_n)}} \longrightarrow c \geq 0.$$

Since, in view of Proposition 5.11, $\{\tilde{\varphi}^{\epsilon_n}\}$ is bounded in $\mathcal{H}_R$, by a diagonal process there exists $\tilde{\Phi}$, with $\tilde{\Phi} \in \mathcal{H}_R$ for all $R > 2$, and a subsequence (still denoted by ϵ_n) such that

$$\tilde{\varphi}^{\epsilon_n} \rightharpoonup \tilde{\Phi} \qquad \text{weakly in } \mathcal{H}_R \quad \text{for all } R > 2. \tag{104}$$

Moreover $\int_{S^+_{K_\rho}} |\tilde{\varphi}^{\epsilon_n}|^2 \, \mathrm{d}S = K_\rho^{N-1}$, hence, by compactness of trace embeddings,

$$\int_{S^+_{K_\rho}} |\tilde{\Phi}|^2 \, \mathrm{d}S = K_\rho^{N-1}, \tag{105}$$

thus implying that $\tilde{\Phi} \not\equiv 0$.

Actually we can prove that the convergence in (104) is strong. Indeed, consider the equation solved by $\tilde{\varphi}^{\epsilon_n}$:

$$\begin{cases} -\Delta \tilde{\varphi}^{\epsilon_n} = (\epsilon_n)^2 \lambda_j^{\epsilon_n} p \, \tilde{\varphi}^{\epsilon_n}, & \text{in } \Big(\big(-\frac{1}{\epsilon_n}, 0 \big] \times \Sigma \Big) \cup B_R^+, \\ \qquad \tilde{\varphi}^{\epsilon_n} = 0, & \text{on } \partial \left(\Big(\big(-\frac{1}{\epsilon_n}, 0 \big] \times \Sigma \Big) \cup B_R^+ \right) \setminus S_R^+, \\ \tilde{\varphi}^{\epsilon_n}(x) = \frac{\varphi_j^{\epsilon_n}(\epsilon_n x)}{\sqrt{H(\varphi_j^{\epsilon_n}, K_\rho \epsilon_n)}}, & \text{on } S_R^+. \end{cases}$$

If we consider the restriction to $B_R^+ \setminus B_{R/2}^+$ and the odd reflection through the hyperplane $x_1 = 0$, we have that $\{\tilde{\varphi}^{\epsilon_n}\}$ is bounded in $H^2(B_R \setminus B_{R/2})$, where $B_R = \{x \in \mathbb{R}^N : |x| < R\}$. Hence, up to a subsequence, $\frac{\partial \tilde{\varphi}^{\epsilon_n}}{\partial \boldsymbol{\nu}} \to \frac{\partial \tilde{\Phi}}{\partial \boldsymbol{\nu}}$ in $L^2(S_R^+)$ and therefore

$$\int_{\Pi_R} |\nabla \tilde{\varphi}^{\epsilon_n}|^2 \, \mathrm{d}x = (\epsilon_n)^2 \lambda_j^{\epsilon_n} \int_{\Pi_R} p |\tilde{\varphi}^{\epsilon_n}|^2 \, \mathrm{d}x + \int_{S_R^+} \frac{\partial \tilde{\varphi}^{\epsilon_n}}{\partial \boldsymbol{\nu}} \tilde{\varphi}^{\epsilon_n} \, \mathrm{d}S \to \int_{S_R^+} \frac{\partial \tilde{\Phi}}{\partial \boldsymbol{\nu}} \tilde{\Phi} \, \mathrm{d}S = \int_{\Pi_R} |\nabla \tilde{\Phi}|^2 \, \mathrm{d}x.$$

Then we conclude that $\tilde{\varphi}^{\epsilon_n} \to \tilde{\Phi}$ strongly in $\mathcal{H}_R$ for all $R > 2$.

From Corollary 7.4 it follows that there exist $c' > 0$ and $n_0 \in \mathbb{N}$ such that, for all $n \geq n_0$ and $\tilde{R} > R$,

$$\int_{B^+_{\tilde{R}} \setminus B^+_R} \left| \nabla \tilde{\varphi}^{\epsilon_n} - (\epsilon_n)^k H(\varphi_j^{\epsilon_n}, K_\rho \epsilon_n)^{-1/2} \nabla W^{\epsilon_n} \right|^2 \mathrm{d}x \leq c'.$$

Let us recall that $W^{\epsilon_n} \to \psi_k$ in $H^1(B^+_{\tilde{R}})$ and (since the norms are equivalent) also in $\mathcal{H}_{\tilde{R}}$: so, passing to the limit as $n \to \infty$ in the above estimate, we obtain that

$$\int_{B^+_{\tilde{R}} \setminus B^+_R} |\nabla \tilde{\Phi} - c \nabla \psi_k|^2 \, \mathrm{d}x \leq c'.$$

Since the constant c' is independent on $\tilde{R}$, we deduce that

$$\int_{\Pi} |\nabla \tilde{\Phi} - c \nabla \psi_k|^2 \, \mathrm{d}x < +\infty. \tag{106}$$

Moreover, the function $\tilde{\Phi}$ satisfies the following equation

$$\begin{cases} -\Delta \tilde{\Phi} = 0, & \text{in } \Pi, \\ \quad \tilde{\Phi} = 0, & \text{on } \partial \Pi. \end{cases} \tag{107}$$

We claim that $c > 0$. Otherwise, if $c = 0$ then, by (106) and (107), we could say that $\tilde{\Phi} = 0$, which would contradict (105).

From Proposition 2.2 we conclude that $\tilde{\Phi} = c \, \Phi$ and hence, in view of (105), $c = \Lambda_\rho^{-1/2}$. Since the limit of the sequence $\{\tilde{\varphi}^{\epsilon_n}\}$ is the same for any choice of the subsequence, we conclude the proof by invoking the *Urysohn's Subsequence Principle.* ☐

Corollary 7.6. *For all $R > 2$ we have that*

$$Z_R^\epsilon \longrightarrow \frac{1}{\sqrt{\Lambda_\rho}} Z_R \qquad \text{in } H^1(B_R^+) \quad \text{as } \epsilon \to 0.$$

Proof. From the definitions of the functions Z_R^ϵ and Z_R (in (55) and Lemma 2.5 respectively)

$$\begin{cases} -\Delta(\sqrt{\Lambda_\rho} Z_R^\epsilon - Z_R) = 0, & \text{in } B_R^+, \\ \sqrt{\Lambda_\rho} Z_R^\epsilon - Z_R = 0, & \text{on } \mathcal{C}_R, \\ \sqrt{\Lambda_\rho} Z_R^\epsilon - Z_R = \sqrt{\Lambda_\rho}\tilde{\varphi}^\epsilon - \Phi, & \text{on } S_R^+. \end{cases}$$

So $Z_R^\epsilon - Z_R$ is the unique, least energy solution with these prescribed boundary conditions. Now, let $\eta = \eta_R$ be as defined in (14). We have that

$$\begin{aligned} &\int_{B_R^+} \left|\nabla(\sqrt{\Lambda_\rho} Z_R^\epsilon - Z_R)\right|^2 \mathrm{d}x \le \int_{B_R^+} \left|\nabla(\eta(\sqrt{\Lambda_\rho}\tilde{\varphi}^\epsilon - \Phi))\right|^2 \mathrm{d}x \le \\ &\le 2\int_{B_R^+} |\nabla\eta|^2 \left|\sqrt{\Lambda_\rho}\tilde{\varphi}^\epsilon - \Phi\right|^2 \mathrm{d}x + 2\int_{B_R^+} \eta^2 \left|\nabla(\sqrt{\Lambda_\rho}\tilde{\varphi}^\epsilon - \Phi)\right|^2 \mathrm{d}x \le \\ &\le \frac{32}{R^2}\int_{B_R^+} \left|\sqrt{\Lambda_\rho}\tilde{\varphi}^\epsilon - \Phi\right|^2 \mathrm{d}x + 2\int_{B_R^+} \left|\nabla(\sqrt{\Lambda_\rho}\tilde{\varphi}^\epsilon - \Phi)\right|^2 \mathrm{d}x \to 0 \end{aligned}$$

as $\epsilon \to 0$, thanks to (101) and to the embedding $\mathcal{H}_R \subset L^2(\Pi_R)$. The conclusion follows taking into account Poincaré Inequality for functions vanishing on a portion of the boundary. □

8 Proof of Theorem 1.1

Thanks to Theorem 7.5 and Corollary 7.6, we know that

$$f_R := \lim_{\epsilon\to 0} f_R(\epsilon) = \frac{1}{\Lambda_\rho}\int_{B_R^+} |\nabla Z_R|^2 \,\mathrm{d}x - \frac{1}{\Lambda_\rho}\int_{\Pi_R} |\nabla\Phi|^2 \,\mathrm{d}x.$$

Moreover, in view of Proposition 6.11 and (102), we have that, for any $R > \max\{2, K_\rho\}$

$$C_k(\Sigma) \le \liminf_{\epsilon\to 0} \frac{\lambda_j - \lambda_j^\epsilon}{\epsilon^{N+2k-2}} \le \limsup_{\epsilon\to 0} \frac{\lambda_j - \lambda_j^\epsilon}{\epsilon^{N+2k-2}} \le \Lambda_\rho f_R, \tag{108}$$

where $C_k(\Sigma) = -2m_k(\Sigma) > 0$. To complete the proof of our main result it is then enough to show that

$$\lim_{R\to+\infty} \Lambda_\rho f_R = C_k(\Sigma).$$

For every $R > 2$ let us define

$$\xi_R(r) := \int_{S_1^+} Z_R(r\theta)\Psi(\theta)\,\mathrm{d}S(\theta) \qquad \text{for } 0 \le r \le R. \tag{109}$$

Lemma 8.1. *There holds*

$$\int_{S_R^+} \frac{\partial(Z_R - \psi_k)}{\partial\boldsymbol{\nu}}(\Phi - \psi_k)\,\mathrm{d}S \longrightarrow 0 \qquad \text{as } R \to +\infty, \tag{110}$$

$$\int_{S_R^+} \frac{\partial(\psi_k - \Phi)}{\partial\boldsymbol{\nu}}(\Phi - \psi_k)\,\mathrm{d}S \longrightarrow 0 \qquad \text{as } R \to +\infty. \tag{111}$$

Proof. In order to prove (110), we first take into account the equation solved by $Z_R - \psi_k$, i.e.

$$\begin{cases} -\Delta(Z_R - \psi_k) = 0, & \text{in } B_R^+, \\ Z_R - \psi_k = 0, & \text{on } \mathcal{C}_R, \\ Z_R - \psi_k = \Phi - \psi_k, & \text{on } S_R^+. \end{cases} \tag{112}$$

Let $\eta = \eta_R$ as defined in (14). Testing (112) with $\eta(\Phi - \psi_k)$, we obtain that

$$\int_{B_R^+} \nabla(Z_R - \psi_k) \cdot \nabla(\eta(\Phi - \psi_k)) \, \mathrm{d}x = \int_{S_R^+} \frac{\partial(Z_R - \psi_k)}{\partial \boldsymbol{\nu}} (\Phi - \psi_k) \, \mathrm{d}S.$$

Then, by the Dirichlet principle,

$$\int_{S_R^+} \frac{\partial(Z_R - \psi_k)}{\partial \boldsymbol{\nu}} (\Phi - \psi_k) \, \mathrm{d}S \leq \sqrt{\int_{B_R^+} |\nabla(Z_R - \psi_k)|^2 \, \mathrm{d}x} \sqrt{\int_{B_R^+} |\nabla(\eta(\Phi - \psi_k))|^2 \, \mathrm{d}x} \leq$$

$$\leq \int_{B_R^+} |\nabla(\eta(\Phi - \psi_k))|^2 \, \mathrm{d}x \leq \frac{32}{R^2} \int_{B_R^+ \setminus B_{R/2}^+} |\Phi - \psi_k|^2 \, \mathrm{d}x + 2 \int_{B_R^+ \setminus B_{R/2}^+} |\nabla(\Phi - \psi_k)|^2 \, \mathrm{d}x \to 0$$

as $R \to +\infty$, thanks to the fact that $\Phi - \psi_k \in \mathcal{D}^{1,2}(\Pi)$ and to Hardy's inequality (reasoning as in Lemma 2.4).

For the second part, since $-\Delta(\Phi - \psi_k) = 0$ in $\Pi \setminus \Pi_R$ and $\Phi - \psi_k = 0$ on $\{x_1 = 0\} \setminus \Sigma$, then

$$\int_{S_R^+} \frac{\partial(\psi_k - \Phi)}{\partial \boldsymbol{\nu}} (\Phi - \psi_k) \, \mathrm{d}S = \int_{\Pi \setminus \Pi_R} |\nabla(\Phi - \psi_k)|^2 \, \mathrm{d}x \to 0$$

as $R \to +\infty$. ☐

Lemma 8.2. *We have that* $\lim_{R \to +\infty} \Lambda_\rho f_R = -2m_k(\Sigma)$.

Proof. Thanks to Lemma 8.1 we know that

$$\lim_{R \to +\infty} \Lambda_\rho f_R = \lim_{R \to +\infty} \int_{S_R^+} \left(\frac{\partial Z_R}{\partial \boldsymbol{\nu}} - \frac{\partial \Phi}{\partial \boldsymbol{\nu}} \right) \psi_k \, \mathrm{d}S. \tag{113}$$

From the definition of ζ (79) and from (86) we deduce that

$$\int_{S_R^+} \frac{\partial \Phi}{\partial \boldsymbol{\nu}} \psi_k \, \mathrm{d}S = R^{N+k-1} \zeta'(R) = k \gamma_N R^{N+2k-2} + (N + k - 2)(\gamma_N - \zeta(1)). \tag{114}$$

It's easy to verify that the function ξ_R defined in (109) satisfies the following ODE

$$\left(r^{N+2k-1} (r^{-k} \xi_R(r))' \right)' = 0 \qquad \text{in } (0, R).$$

By integration, we obtain

$$r^{N+k-2} \xi_R(r) = r^{N+2k-2} R^{-k} \xi_R(R) - \frac{C}{N + 2k - 2} + \frac{C}{N + 2k - 2} r^{N+2k-2} R^{-N-2k+2}.$$

Since Z_R is regular at 0, we have necessarily that $C = 0$; hence

$$\xi_R(r) = \left(\frac{r}{R} \right)^k \xi_R(R).$$

From the definition of ξ_R (109) we have

$$\int_{S_R^+} \frac{\partial Z_R}{\partial \boldsymbol{\nu}} \psi_k \, \mathrm{d}S = R^{N+k-1} \xi_R'(R) = k R^{N+k-2} \xi_R(R) = k R^{N+k-2} \zeta(R). \tag{115}$$

Then, from (113), (114), (115), (85) and (83)

$$\begin{aligned} \lim_{R \to +\infty} \Lambda_\rho f_R &= \lim_{R \to +\infty} \left(k R^{N+k-2} \zeta(R) - k \gamma_N R^{N+2k-2} - (N + k - 2)(\gamma_N - \zeta(1)) \right) \\ &= \lim_{R \to +\infty} R^{N+k-2} (N + 2k - 2)(\zeta(R) - \gamma_N R^k) \\ &= (N + 2k - 2)(\zeta(1) - \gamma_N) = -2m_k(\Sigma), \end{aligned}$$

thus concluding the proof. ☐

We are now able to prove our main result.

Proof of Theorem 1.1. From (108) and Lemma 8.2 we conclude that

$$\liminf_{\epsilon \to 0} \frac{\lambda_j - \lambda_j^\epsilon}{\epsilon^{N+2k-2}} = \limsup_{\epsilon \to 0} \frac{\lambda_j - \lambda_j^\epsilon}{\epsilon^{N+2k-2}} = C_k(\Sigma),$$

thus completing the proof. ☐

9 Appendix

It is well known that the classical Hardy's Inequality

$$\left(\frac{N-2}{2}\right)^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\,\mathrm{d}x \le \int_{\mathbb{R}^N} |\nabla u|^2\,\mathrm{d}x, \quad u \in C_c^\infty(\mathbb{R}^N), \quad N \ge 3,$$

fails in dimension 2. However we observe, in the following theorem, that, under a vanishing condition on part of the domain (at least on a half-line), it is possible to recover a Hardy-type Inequality even in dimension 2.

Let $\mathbf{p} = (x_{\mathbf{p}}, 0) \in \mathbb{R}^2$ with $x_{\mathbf{p}} > 0$ and let $s_{\mathbf{p}} := \{(x,0) \colon x \ge x_{\mathbf{p}}\}$. Let $\mathcal{D}_{\mathbf{p}}$ denote the completion of the space $C_c^\infty(\mathbb{R}^2 \setminus s_{\mathbf{p}})$ with respect to the norm

$$\|u\|_{\mathcal{D}_{\mathbf{p}}} := \left(\int_{\mathbb{R}^2} |\nabla u|^2\,\mathrm{d}x\right)^{1/2}.$$

Let us consider the function

$$\theta_{\mathbf{p}} \colon \mathbb{R}^2 \setminus s_{\mathbf{p}} \longrightarrow (0, 2\pi), \quad \theta_{\mathbf{p}}(x_{\mathbf{p}} + r\cos t, r\sin t) = t.$$

We have that $\theta_{\mathbf{p}} \in C^\infty(\mathbb{R}^2 \setminus s_{\mathbf{p}})$.

Theorem 9.1. *For all $\varphi \in C_c^\infty(\mathbb{R}^2 \setminus s_{\mathbf{p}})$*

$$\frac{1}{4} \int_{\mathbb{R}^2} \frac{|\varphi(z)|^2}{|z-\mathbf{p}|^2}\,\mathrm{d}z \le \int_{\mathbb{R}^2} |\nabla \varphi(z)|^2\,\mathrm{d}z. \tag{116}$$

Moreover the space $\mathcal{D}_{\mathbf{p}}$ can be characterized as

$$\mathcal{D}_{\mathbf{p}} = \left\{u \in L^1_{\mathrm{loc}}(\mathbb{R}^2) : \nabla u \in L^2(\mathbb{R}^2),\ \tfrac{u}{|z-\mathbf{p}|} \in L^2(\mathbb{R}^2),\ \text{and } u = 0 \text{ on } s_{\mathbf{p}}\right\}$$

and inequality (116) *holds for every $\varphi \in \mathcal{D}_{\mathbf{p}}$.*

Proof. Let $\varphi \in C_c^\infty(\mathbb{R}^2 \setminus s_{\mathbf{p}})$ and let $\tilde{\varphi}(z) := \varphi(z) e^{i\frac{\theta_{\mathbf{p}}(z)}{2}} \in C_c^\infty(\mathbb{R}^2 \setminus s_{\mathbf{p}}, \mathbb{C})$. By direct calculations, we have that

$$i\nabla\varphi(z) = e^{-i\frac{\theta_p(z)}{2}}(i\nabla + \mathrm{A}_{\mathbf{p}})\tilde{\varphi}(z),$$

where

$$\mathrm{A}_{\mathbf{p}}(x,y) := \frac{1}{2}\left(\frac{-y}{(x-x_{\mathbf{p}})^2 + y^2}, \frac{x - x_{\mathbf{p}}}{(x-x_{\mathbf{p}})^2 + y^2}\right)$$

is the Aharonov-Bohm vector potential with pole $\mathbf{p}$ and circulation $1/2$. Now let us compute the L^2-norm of $|i\nabla\varphi|$ and use the Hardy's Inequality for Aharonov-Bohm operators (see [32]):

$$\int_{\mathbb{R}^2} |\nabla\varphi|^2\,\mathrm{d}z = \int_{\mathbb{R}^2} |(i\nabla + \mathrm{A}_{\mathbf{p}})\tilde{\varphi}|^2\,\mathrm{d}z \ge \frac{1}{4}\int_{\mathbb{R}^2} \frac{|\tilde{\varphi}|^2}{|z-\mathbf{p}|^2}\,\mathrm{d}z = \frac{1}{4}\int_{\mathbb{R}^2} \frac{|\varphi|^2}{|z-\mathbf{p}|^2}\,\mathrm{d}z.$$

The second part of the statement follows from (116) by classical completion and density arguments. □

Corollary 9.2. *There exists $C = C(\mathbf{p}) > 0$ such that*

$$\int_{\mathbb{R}^2} \frac{|\varphi(z)|^2}{1+|z|^2}\,\mathrm{d}z \le C \int_{\mathbb{R}^2} |\nabla\varphi(z)|^2\,\mathrm{d}z \quad \textit{for all } \varphi \in \mathcal{D}_{\mathbf{p}}. \tag{117}$$

Proof. We observe that there exists $K = K(|\mathbf{p}|) > 0$ such that

$$|z-\mathbf{p}|^2 \le K(|\mathbf{p}|)(1+|z|^2) \quad \text{for all } z \in \mathbb{R}^2.$$

Therefore the claim easily follows from Theorem 9.1 with $C(\mathbf{p}) = 4K(|\mathbf{p}|)$. □

We conclude this appendix by recalling from [1] the following lemma about maxima of quadratic forms depending on a parameter, which we used in Section 6.

Lemma 9.3. *For every $\epsilon > 0$ let us consider a quadratic form*

$$Q_\epsilon \colon \mathbb{R}^j \longrightarrow \mathbb{R},$$
$$Q_\epsilon(\xi_1, \dots, \xi_j) = \sum_{i,n=1}^{j} M_{i,n}(\epsilon)\xi_i\xi_n,$$

with real coefficients $M_{i,n}(\epsilon)$ such that $M_{i,n}(\epsilon) = M_{n,i}(\epsilon)$. Let us assume that there exist $\alpha > 0$, $\epsilon \mapsto \sigma(\epsilon) \in \mathbb{R}$ with $\sigma(\epsilon) \geq 0$ and $\sigma(\epsilon) = O(\epsilon^{2\alpha})$ as $\epsilon \to 0$, and $\epsilon \mapsto \mu(\epsilon) \in \mathbb{R}$ with $\mu(\epsilon) = O(1)$ as $\epsilon \to 0$, such that the coefficients $M_{i,n}(\epsilon)$ satisfy the following conditions:

$$\begin{aligned}
&M_{j,j}(\epsilon) = \sigma(\epsilon)\mu(\epsilon),\\
&\text{for all } i < j \ M_{i,i}(\epsilon) \to M_i < 0, \ \text{ as } \epsilon \to 0,\\
&\text{for all } i < j \ M_{i,j}(\epsilon) = O(\epsilon^{\alpha}\sqrt{\sigma(\epsilon)}) \ \text{ as } \epsilon \to 0,\\
&\text{for all } i, n < j \text{ with } i \neq n \ M_{i,n} = O(\epsilon^{2\alpha}) \ \text{ as } \epsilon \to 0,\\
&\text{there exists } M \in \mathbb{N} \text{ such that } \epsilon^{(2+M)\alpha} = o(\sigma(\epsilon)) \ \text{ as } \epsilon \to 0.
\end{aligned}$$

Then

$$\max_{\substack{\xi \in \mathbb{R}^j \\ \|\xi\|=1}} Q_\epsilon(\xi) = \sigma(\epsilon)(\mu(\epsilon) + o(1)) \quad \text{as } \epsilon \to 0,$$

where $\|\xi\| = \|(\xi_1, \dots, \xi_j)\| = \left(\sum_{i=1}^{j} \xi_i^2\right)^{1/2}$.

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