Reasoning in Expressive Gödel Description Logics

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Abstract. Fuzzy description logics (FDLs) are knowledge representation formalisms capable of dealing with imprecise knowledge by allowing intermediate membership degrees in the interpretation of concepts and roles. One option for dealing with these intermediate degrees is to use the so-called Gödel semantics, under which conjunction is interpreted by the minimum of the degrees. Despite its apparent simplicity, developing reasoning techniques for expressive FDLs under this semantics is a hard task. In this paper, we illustrate two algorithms for deciding consistency in (sublogics of) $\mathcal{SROIQ}$ under Gödel semantics.

1 Introduction

As it has been widely argued in the literature, one of the important deficits of classical logic is the inability to handle imprecise concepts for which a clear-cut characterization is impossible [15, 21]. To cover this gap, the semantics of DLs has been extended following the ideas of mathematical fuzzy logic [13, 15]. Briefly, fuzzy description logics allow intermediate truth degrees—usually rational numbers between 0 (false) and 1 (true)—to be used in the definition of imprecise knowledge [1]. To interpret these intermediate degrees, the logical connectives need to be extended accordingly. In general, there are many possible extensions that can be used; hence, each DL gives rise to a family of FDLs. However, for most of these extensions, reasoning becomes undecidable, even if the underlying DL is relatively inexpressive [6]. In fact, the only decidable expressive FDLs are those based on the Gödel semantics, and the related Zadeh semantics. Extensions of classical DLs with the Gödel semantics are typically denoted by the prefix "G-".

Developing a reasoning algorithm for classical $\mathcal{SROIQ}$ is far from trivial, as one needs to handle all possible interactions between the constructors, e.g. nominals and number restrictions. This difficulty is accentuated when the Gödel semantics are considered, since this logic does not have the finitely valued model property [5]. This means that there are ontologies whose models must use infinitely many truth degrees. Indeed, this is one of the reasons why the *crispification approach* as described in [4, 7, 23] is only correct under finitely valued semantics.

The study of reasoning algorithms for expressive Gödel FDLs started in [5, 11], where an automata-based approach was developed, showing that the loss of the finitely valued model property does not affect the complexity of reasoning in

G-$\mathcal{ALC}$. Rather than trying to find a model directly, this algorithm produces an abstract representation of a large class of models. In this representation, the actual degrees of truth used in a model are abstracted to consider only the order among them. This abstraction from the actual degrees is also exploited by an extension of the crispification approach [8, 9], which translates a fuzzy ontology into a classical ontology by using concepts that simulate the order between the relevant truth degrees. As an added benefit, considering only the order between concepts allows for a more flexible representation of the domain knowledge in which, for instance, one can express that an individual is *taller* than another, without having to specify explicit degrees of tallness.

Although they provide good theoretical results such as tight complexity bounds for reasoning, these approaches are restricted to sublogics of G-$\mathcal{SROIQ}$ having the forest-model property, and there is no obvious way to extend them to the full expressivity of G-$\mathcal{SROIQ}$. In [10], we have developed a new tableau algorithm to deal with full G-$\mathcal{SROIQ}$. The new algorithm combines the ideas of the classical tableau approach for $\mathcal{SROIQ}$ with the order-based abstraction developed in [5, 9]. It inherits the pay-as-you-go behavior from the classical tableau algorithms, and is the first reasoning algorithm that can handle the full expressivity of fuzzy $\mathcal{SROIQ}$ under Gödel semantics. Due to space restrictions, in this paper we only illustrate the core ideas of the two algorithms from [9, 10] on an example.

2 Preliminaries

The two basic operators of Gödel fuzzy logic are conjunction and implication, interpreted by the *Gödel t-norm* and *residuum*, respectively. The Gödel t-norm of two fuzzy degrees $x, y \in [0, 1]$ is defined as minimum function $\min\{x, y\}$. The residuum $\Rightarrow$ is uniquely defined by the equivalence $\min\{x, y\} \leqslant z$ iff $y \leqslant (x \Rightarrow z)$ for all $x, y, z \in [0, 1]$, and can be computed as

$$x \Rightarrow y = \begin{cases} 1 & \text{if } x \leqslant y, \\ y & \text{otherwise.} \end{cases}$$

We consider both the *residual negation* (that maps each value x to $x \Rightarrow 0$) and the *involutive negation* (that maps x to $1 - x$) in this paper.

An *order structure* is a finite set S containing at least the numbers 0, 0.5, and 1, together with an involutive unary operation $\mathsf{inv}\colon S \to S$ such that $\mathsf{inv}(x) = 1 - x$ for all rational numbers $x \in S \cap [0, 1]$. A *total preorder* (on S) is a transitive and total binary relation $\preccurlyeq_* \subseteq S \times S$. The set $\mathsf{order}(S)$ contains exactly those total preorders $\preccurlyeq_*$ over S which

- have 0 and 1 as least and greatest element, respectively,
- coincide with the order of the rational numbers on $S \cap [0, 1]$, and
- satisfy $\alpha \preccurlyeq_* \beta$ iff $\mathsf{inv}(\beta) \preccurlyeq_* \mathsf{inv}(\alpha)$ for all $\alpha, \beta \in S$.

Given $\preccurlyeq_* \in \mathsf{order}(S)$, the following functions extend the operators of Gödel fuzzy logic from $S \cap [0,1]$ to S:

$$\min_*\{\alpha,\beta\} := \begin{cases}\alpha & \text{if } \alpha \preccurlyeq_* \beta\\ \beta & \text{otherwise,}\end{cases} \qquad \alpha \Rightarrow_* \beta := \begin{cases}1 & \text{if } \alpha \preccurlyeq_* \beta\\ \beta & \text{otherwise.}\end{cases}$$

An *order assertion* (over S) is an expression of the form $\alpha \bowtie \beta$, where $\alpha, \beta \in S$ and $\bowtie \in \{<, \leqslant, =, \geqslant, >\}$. An *order formula* is a Boolean combination of order assertions. The satisfaction of an order formula by an element $\preccurlyeq_* \in \mathsf{order}(S)$ is defined in the obvious way, and can be extended to more complex expressions like $\alpha \geqslant \min\{\beta,\gamma\}$ or $\alpha = (\beta \Rightarrow \gamma)$ using the operators $\min_*$ and $\Rightarrow_*$. A set of order assertions Φ is *satisfiable* if it has a model $\preccurlyeq_* \in \mathsf{order}(S)$. Satisfiability of order assertions can be decided in polynomial time by checking for cycles involving strict order assertions.

2.1 G-$\mathcal{SROIQ}$

The syntax of concepts and roles in G-$\mathcal{SROIQ}$ extends that of classical $\mathcal{SROIQ}$, based on the sets $\mathsf{N_I}$, $\mathsf{N_C}$, and $\mathsf{N_R}$ of *individual names*, *concept names*, and *role names*, respectively. The set $\mathsf{N_R}$ includes the *universal role* r_u, and $\mathsf{N_R^-}$ denotes the set of all (atomic and inverse) roles. Additionally, we allow *truth constants* $\overline{p}$ with $p \in [0,1]$ and *implication* $C \to D$ as concept constructors. The semantics is based on G-*interpretations* $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ over a non-empty *domain* $\Delta^\mathcal{I}$, which assign to each individual name $a \in \mathsf{N_I}$ an element $a^\mathcal{I} \in \Delta^\mathcal{I}$, to each concept name $A \in \mathsf{N_C}$ a fuzzy set $A^\mathcal{I}\colon \Delta^\mathcal{I} \to [0,1]$, and to each role name $r \in \mathsf{N_R}$ a fuzzy binary relation $r^\mathcal{I}\colon \Delta^\mathcal{I} \times \Delta^\mathcal{I} \to [0,1]$. This G-interpretation is extended to complex concepts and roles as defined in the last column of Table 1, for all $d, e \in \Delta^\mathcal{I}$.

We can express the common DL constructors $\top := \overline{1}$ (top), $\bot := \overline{0}$ (bottom), $C \sqcup D := \neg(\neg C \sqcap \neg D)$ (disjunction), and ${\leqslant}n\, s.C := \neg({\geqslant}(n+1)\, s.C)$ (at-most restriction). In some previous work on fuzzy extensions of $\mathcal{SROIQ}$ (e.g. [4]), the latter are defined using the residual negation; that is ${\leqslant}n\, s.C := ({\geqslant}(n+1)\, s.C) \to \bot$. This has the effect that the value of ${\leqslant}n\, r.C$ is always either 0 or 1. However, this discrepancy in definitions is not an issue since our algorithms can handle both the involutive and the residual negation. The use of truth constants $\overline{p}$ for $p \in [0,1]$ is not standard in FDLs, but it allows us to simulate *fuzzy nominals* [2] of the form $\{p_1/a_1, \dots, p_n/a_n\}$ with $p_i \in [0,1]$ and $a_i \in \mathsf{N_I}$, $1 \leqslant i \leqslant n$, using the concept $(\{a_1\} \sqcap \overline{p_1}) \sqcup \dots \sqcup (\{a_n\} \sqcap \overline{p_n})$. Recall that we use only rational numbers.

A *(classical) assertion* is either a *concept assertion* of the form $C(a)$ or a *role assertion* of the form $r(a,b)$ for $a, b \in \mathsf{N_I}$, a concept C, and a role r. A *(fuzzy) assertion* is of the form $\alpha \bowtie p$ or $\alpha \bowtie \beta$, where α, β are classical assertions, $\bowtie \in \{<, \leqslant, =, \geqslant, >\}$, and $p \in [0,1]$. An *ABox* is a finite set of fuzzy assertions and *individual (in)equality assertions* of the form $a \approx b$ ($a \not\approx b$) for $a, b \in \mathsf{N_I}$. A *TBox* is a finite set of *general concept inclusions (GCIs)* of the form $C \sqsubseteq D \geqslant p$ for concepts C, D and $p \in (0,1]$. A *role hierarchy* $\mathcal{R}_h$ is a finite set of *(complex) role inclusions* of the form $w \sqsubseteq r \geqslant p$, where r is a role name different from the universal role, $w \in (\mathsf{N_R^-})^+$ is a non-empty *role chain* not including the universal

Table 1. Syntax and semantics of G-$\mathcal{SROIQ}$

Name	Syntax	Semantics ($C^{\mathcal{I}}(d)$ / $r^{\mathcal{I}}(d,e)$)
concept name	A	$A^{\mathcal{I}}(d) \in [0,1]$
truth constant	$\overline{p}$	p
conjunction	$C \sqcap D$	$\min\{C^{\mathcal{I}}(d), D^{\mathcal{I}}(d)\}$
implication	$C \to D$	$C^{\mathcal{I}}(d) \Rightarrow D^{\mathcal{I}}(d)$
negation	$\neg C$	$1 - C^{\mathcal{I}}(d)$
existential restriction	$\exists r.C$	$\sup_{e \in \Delta^{\mathcal{I}}} \min\{r^{\mathcal{I}}(d,e), C^{\mathcal{I}}(e)\}$
value restriction	$\forall r.C$	$\inf_{e \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(d,e) \Rightarrow C^{\mathcal{I}}(e)$
nominal	$\{a\}$	$\begin{cases} 1 & \text{if } d = a^{\mathcal{I}} \\ 0 & \text{otherwise} \end{cases}$
at-least restriction	$\geqslant n\, s.C$	$\sup_{\substack{e_1,\dots,e_n \in \Delta^{\mathcal{I}} \\ \text{pairwise different}}} \min_{i=1}^{n} \min\{s^{\mathcal{I}}(d,e_i), C^{\mathcal{I}}(e_i)\}$
local reflexivity	$\exists s.\mathsf{Self}$	$r^{\mathcal{I}}(d,d)$
role name	r	$r^{\mathcal{I}}(d,e) \in [0,1]$
inverse role	r^-	$r^{\mathcal{I}}(e,d)$
universal role	r_u	1

role, and $p \in (0,1]$. The notions of *regularity* and *simple roles* are defined w.r.t. a given role hierarchy $\mathcal{R}_h$ as for classical $\mathcal{SROIQ}$ [3,18,19], and we adopt the same syntactical restrictions, e.g. that number restrictions can only contain simple roles, to avoid undecidability. An *RBox* $\mathcal{R} = \mathcal{R}_h \cup \mathcal{R}_a$ consists of a regular role hierarchy $\mathcal{R}_h$ and a finite set $\mathcal{R}_a$ of *disjoint role axioms* $\mathsf{dis}(s_1, s_2) \geqslant p$ and *reflexivity axioms* $\mathsf{ref}(r) \geqslant p$, where r is a role, s_1, s_2 are simple roles, and $p \in (0,1]$. An *ontology* $\mathcal{O} = (\mathcal{A}, \mathcal{T}, \mathcal{R})$ consists of an ABox $\mathcal{A}$, a TBox $\mathcal{T}$, and an RBox $\mathcal{R}$.

A G-interpretation $\mathcal{I}$ *satisfies* (or is a *model* of)

- the fuzzy assertion $\alpha \bowtie \beta$ if $\alpha^{\mathcal{I}} \bowtie \beta^{\mathcal{I}}$, where we set $(C(a))^{\mathcal{I}} := C^{\mathcal{I}}(a^{\mathcal{I}})$, $(r(a,b))^{\mathcal{I}} := r^{\mathcal{I}}(a^{\mathcal{I}}, b^{\mathcal{I}})$, and $p^{\mathcal{I}} := p$ for all $p \in [0,1]$;
- the (in)equality assertion $a \approx b$ ($a \not\approx b$) if $a^{\mathcal{I}} = b^{\mathcal{I}}$ ($a^{\mathcal{I}} \neq b^{\mathcal{I}}$);
- the GCI $C \sqsubseteq D \geqslant p$ iff $C^{\mathcal{I}}(d) \Rightarrow D^{\mathcal{I}}(d) \geqslant p$ for all $d \in \Delta^{\mathcal{I}}$;
- the role inclusion $r_1 \dots r_n \sqsubseteq r \geqslant p$ iff $(r_1 \dots r_n)^{\mathcal{I}}(d_0, d_n) \Rightarrow r^{\mathcal{I}}(d_0, d_n) \geqslant p$ for all $d_0, d_n \in \Delta^{\mathcal{I}}$, where
$$(r_1 \dots r_n)^{\mathcal{I}}(d_0, d_n) := \sup_{d_1,\dots,d_{n-1} \in \Delta^{\mathcal{I}}} \min_{i=1}^{n} r_i^{\mathcal{I}}(d_{i-1}, d_i);$$
- the disjoint role axiom $\mathsf{dis}(s_1, s_2) \geqslant p$ iff $\min\{s_1^{\mathcal{I}}(d,e), s_2^{\mathcal{I}}(d,e)\} \leqslant 1 - p$ for all $d, e \in \Delta^{\mathcal{I}}$;

- the reflexivity axiom $\mathsf{ref}(r) \geqslant p$ iff $r^{\mathcal{I}}(d,d) \geqslant p$ for all $d \in \Delta^{\mathcal{I}}$;
- an ontology if it satisfies all its axioms.

We can simulate other axioms of $\mathcal{SROIQ}$ [4, 17] as follows:

- transitivity axioms $\mathsf{tra}(r) \geqslant p$ by $rr \sqsubseteq r \geqslant p$;
- symmetry axioms $\mathsf{sym}(r) \geqslant p$ by $r^- \sqsubseteq r \geqslant p$;
- asymmetry axioms $\mathsf{asy}(s) \geqslant p$ by $\mathsf{dis}(s, s^-) \geqslant p$;
- irreflexivity axioms $\mathsf{irr}(s) \geqslant p$ by $\exists s.\mathsf{Self} \sqsubseteq \neg \overline{p} \geqslant 1$; and
- negated role assertions $\neg r(a,b) \geqslant p$ by $r(a,b) \leqslant 1 - p$.

As usual for FDLs, we consider only *witnessed* G-interpretations [16]. Intuitively, this ensures that the suprema and infima in the semantics of the concept constructors are in fact maxima and minima, respectively. In other words, the degrees of these constructors are witnessed by elements of the domain. Note that this restriction is not without loss of generality. Formally, a G-interpretation $\mathcal{I}$ is *witnessed* if, for every $d \in \Delta^{\mathcal{I}}$, $n \geqslant 0$, $r \in \mathsf{N}_\mathsf{R}^-$, simple $s \in \mathsf{N}_\mathsf{R}^-$, and concept C, there are $e, e', e_1, \ldots, e_n \in \Delta^{\mathcal{I}}$ such that $e_1, \ldots, e_n$ are pairwise different,

$$
\begin{aligned}
(\exists r.C)^{\mathcal{I}}(d) &= \min\{r^{\mathcal{I}}(d,e), C^{\mathcal{I}}(e)\}, \\
(\forall r.C)^{\mathcal{I}}(d) &= r^{\mathcal{I}}(d,e') \Rightarrow C^{\mathcal{I}}(e'), \qquad \text{and} \\
({\geqslant} n\, s.C)^{\mathcal{I}}(d) &= \min_{i=1}^{n} \min\{s^{\mathcal{I}}(d,e_i), C^{\mathcal{I}}(e_i)\}.
\end{aligned}
$$

We could also require witnesses for role chains in complex role inclusions, but this is not usually done, and not necessary for our algorithms. A G-$\mathcal{SROIQ}$ ontology is *consistent* if it has a witnessed G-model.

Other common reasoning problems for FDLs, such as concept satisfiability and subsumption can be reduced to consistency in linear time [11]. For instance, the subsumption between C and D to degree p w.r.t. a TBox $\mathcal{T}$ and an RBox $\mathcal{R}$ is equivalent to the inconsistency of $(\{(C \to D)(a) < p\}, \mathcal{T}, \mathcal{R})$, where a is a fresh individual name. Likewise, the satisfiability of C to degree p w.r.t. $\mathcal{T}$ and $\mathcal{R}$ is equivalent to the consistency of $(\{C(a) \geqslant p\}, \mathcal{T}, \mathcal{R})$. One can even show that the *best* satisfiability and subsumption degrees are always values occurring in the input ontology, and can be computed using linearly many consistency tests [11]. Hence, we restrict our attention to the problem of deciding consistency of fuzzy ontologies.

3 The Algorithms

The main idea for both our algorithms is that, instead of explicitly defining the degrees of all concepts and roles for all domain elements, we only represent the *order* between different values. For example, to satisfy the semantics of $\to$, i.e. $(C \to D)^{\mathcal{I}}(x) = C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x)$, it suffices to consider the two cases

- $(C \to D)^{\mathcal{I}}(x) = 1$ and $C^{\mathcal{I}}(x) \leqslant D^{\mathcal{I}}(x)$; or
- $(C \to D)^{\mathcal{I}}(x) = D^{\mathcal{I}}(x)$ and $C^{\mathcal{I}}(x) > D^{\mathcal{I}}(x)$.

In both cases, it is irrelevant what the actual values of $C^{\mathcal{I}}(x)$ and $D^{\mathcal{I}}(x)$ are, as long as they satisfy a certain order relationship. We exploit this property of the Gödel operators in the following constructions, by using order structures and order assertions to represent the semantics of concepts. This idea has also been used for other reasoning problems based on the Gödel semantics [14].

For the nonce, we use the small example ontology $\mathcal{O} := (\mathcal{A}, \mathcal{T}, \emptyset)$, where

$$\mathcal{A} := \{(\exists r.A)(a) \geqslant p_A,\ (\exists r.B)(a) \geqslant p_B,\ (\leqslant 1\, r.C)(a) \geqslant p_C\},$$
$$\mathcal{T} := \{A \sqsubseteq C \geqslant 1,\ B \sqsubseteq C \geqslant 1\},$$

and p_A, p_B, p_C are arbitrary values, to illustrate the algorithms.

3.1 Reduction to Classical DLs

Our first algorithm is based on a reduction of the fuzzy ontology $\mathcal{O}$ to a classical ontology $\mathsf{red}(\mathcal{O})$. We use special concept names to express order assertions over a specific order structure $\mathcal{U}$. This order structure contains all values occurring in $\mathcal{O}$, all relevant subconcepts and roles, e.g. $\exists r.A_1$ and r, relevant assertions over known individuals, such as $(\leqslant 1\, r.C)(a)$, and special role assertions of the form $r(*, a)$, as explained below. For example, the concept name $\boxed{C > (\exists r.A)(a)}$ expresses that the value of C at the current domain element should exceed the value of $\exists r.A$ at a. We call these concept names *order concepts* and, to improve readability, will denote them always with a surrounding box. This approach can be seen as an extension of previous algorithms for reasoning in fuzzy DLs based on reductions to classical DLs [2, 4, 7], which use *cut-concepts* of the form $\boxed{A \geqslant p}$, but are applicable only for fuzzy semantics based on finitely many values.

To achieve the correct behavior, our reduction explicitly specifies the semantics of the order structure and the concept constructors. For example, we use the classical axioms $\top \sqsubseteq \boxed{\alpha \leqslant \beta} \sqcup \boxed{\beta \leqslant \alpha}$, for all $\alpha, \beta \in \mathcal{U}$ to express that $\leqslant$ should be total. The assertions in our ABox $\mathcal{A}$ are translated into classical assertions, e.g. $\boxed{(\exists r.A)(a) \geqslant p_A}(a)$. To ensure that $(\exists r.A)(a)$ actually represents the value of the existential restriction $\exists r.A$ at the individual a, we use the additional assertion $\boxed{(\exists r.A)(a) = (\exists r.A)}(a)$. The GCIs from our example ontology have the straightforward translations

$$\top \sqsubseteq \boxed{A \Rightarrow C \geqslant 1} \text{ and } \top \sqsubseteq \boxed{B \Rightarrow C \geqslant 1},$$

which require that they are satisfied in every element of the domain.

In the reduction, domain elements are connected via only one special role, denoted by $\mathfrak{r}$. This role is used to transfer information between domain elements. The goal is that, except for the named individuals, the role $\mathfrak{r}$ will generate a forest-shaped structure in the classical interpretation; hence this approach is restricted to logics having the forest-model property, i.e. $\mathcal{SRIQ}$, $\mathcal{SROQ}$, and $\mathcal{SROI}$ [12].

Information about the named individuals is transferred to all $\mathfrak{r}$-connected domain elements using GCIs like $\boxed{(\exists r.A)(a) \geqslant (\exists r.B)(a)} \sqsubseteq \forall \mathfrak{r}.\boxed{(\exists r.A)(a) \geqslant (\exists r.B)(a)}$, i.e. whenever a domain element x "knows" something about the behavior of a, then all $\mathfrak{r}$-successors of x share that knowledge. Special elements of $\mathcal{U}$ of the form $\langle C\rangle_{\uparrow}$ are used to refer to the value of a concept C at the parent node in the tree. These elements are restricted by axioms like $\boxed{(\exists r.B) \leqslant C} \sqsubseteq \forall\mathfrak{r}.\big(\mathsf{AN} \rightarrow \boxed{\langle\exists r.B\rangle_{\uparrow} \leqslant \langle C\rangle_{\uparrow}}\big)$, which express that order relations between concepts of the parent are known by all child nodes, i.e. $\mathfrak{r}$-successors. The special concept name AN is used to distinguish anonymous domain elements from those that are designated by an individual name (and are hence not part of the forest).

In our example, to generate a witness for the existential restriction $\exists r.A$ at a (and all other domain elements), we introduce the axiom

$$\top \sqsubseteq \exists\mathfrak{r}.\big(\mathsf{AN} \sqcap \boxed{\langle\exists r.A\rangle_{\uparrow} \leqslant \min\{r, A\}}\big) \sqcup \big(\exists\mathfrak{r}.\{a\} \sqcap \boxed{(\exists r.A) \leqslant \min\{r(*, a), A(a)\}}\big).$$

That is, either a has an anonymous (AN) $\mathfrak{r}$-successor at which the value of $\exists r.A$ at the parent node ($\langle\exists r.A\rangle_{\uparrow}$), in this case a, is bounded by the minimum between the r-connection to the parent node (r) and the value of A at the current node (A); or there is an $\mathfrak{r}$-successor that satisfies $\{a\}$, i.e. a itself, and the value of $\exists r.A$ at a is bounded by the minimum between the value of the role connection from the current node (represented by $*$) to a and the value of A at a ($A(a)$). In general, the second part has to consider all named domain elements as possible successors; in our example we have only a.

On the other hand, all $\mathfrak{r}$-successors have to be restricted to not exceed the value of $\exists r.A$ using the similar axioms

$$\top \sqsubseteq \forall\mathfrak{r}.\big(\mathsf{AN} \rightarrow \boxed{\langle\exists r.A\rangle_{\uparrow} \geqslant \min\{r, A\}}\big) \text{ and } \exists\mathfrak{r}.\{a\} \sqsubseteq \boxed{(\exists r.A) \geqslant \min\{r(*, a), A(a)\}}.$$

Similar axioms are introduced to express the semantics of $\exists r.B$.

For the number restriction $\leqslant 1\, r.C = \neg{\geqslant 2}\, r.C$, we first create witnesses as for the existential restrictions above:

$$\begin{aligned}\top \sqsubseteq {\geqslant 2}\,\mathfrak{r}.\big(\mathsf{AN} \sqcap \boxed{\langle{\geqslant 2}\, r.C\rangle_{\uparrow} \leqslant \min\{r, C\}}\big) \sqcup\\ \big({\geqslant 1}\,\mathfrak{r}.\big(\mathsf{AN} \sqcap \boxed{\langle{\geqslant 2}\, r.C\rangle_{\uparrow} \leqslant \min\{r, C\}}\big) \sqcap \boxed{({\geqslant 2}\, r.C) \leqslant \min\{r(*, a), C(a)\}}\big)\end{aligned}$$

That is, either there exist two anonymous witnesses for the value of ${\geqslant 2}\, r.C$, or one anonymous witness and a serves as another witness. In general, the reduction needs to consider all possible (exponentially many) combinations of named and unnamed domain elements as witnesses for number restrictions; in this example there are only 2 cases. Dually, there can be at most one r-successor that exceeds the value given by ${\geqslant 2}\, r.C$ at a, which is encoded in the assertion

$${\leqslant 1}\,\mathfrak{r}.\big(\big(\mathsf{AN} \sqcap \boxed{\langle{\geqslant 2}\, r.C\rangle_{\uparrow} < \min\{r, C\}}\big) \sqcup \big(\neg\mathsf{AN} \sqcap \boxed{({\geqslant 2}\, r.C)(a) < \min\{r(a, *), C\}}\big)\big)(a).$$

All the axioms listed above are collected into a classical ontology $\mathsf{red}(\mathcal{O})$, and any classical model of this ontology obtained by a classical reasoner can be used to construct a G-model of $\mathcal{O}$. Hence, while this reduction incurs an exponential blow-up in the size of the ontology due to the interaction of nominals and number restrictions, it enables us to use existing optimized reasoners to decide consistency of G-$\mathcal{SROIQ}$ ontologies.

3.2 The Tableau Algorithm

In contrast, our tableau algorithm explicitly creates a G-model of $\mathcal{O}$ by introducing new domain elements, which we call *nodes*. It uses an order structure that is similar to the one used for the reduction described above. The main difference is that the order structure now also contains concept and role assertions of the form $B(x)$ and $r(x, y)$, where x and y are nodes. In this way, we can express the semantics directly using order assertions, e.g. $(\exists r.A)(x) \geqslant \min\{r(x, y), A(y)\}$ for all nodes x and y. However, the latter expression is not fully determined: that is, we do not know whether $(\exists r.A)(x) \geqslant r(x, y)$, or $(\exists r.A)(x) \geqslant A(y)$ holds. In our tableau algorithm, we resolve this nondeterminism by considering only *atomic* order assertions, i.e. without using the abbreviations min and $\Rightarrow$. In order to guarantee that these sets can be used to construct a G-model of $\mathcal{O}$, we need to ensure that they remain satisfiable.

In our example, the tableau algorithm is initialized with one node a representing the individual of the same name, and the order assertions from $\mathcal{A}$, where the at-most assertion is equivalent to an upper bound on the corresponding at-least-restriction: $({\geqslant}2\, r.C)(a) \leqslant 1 - p_C$. Afterwards, (nondeterministic) tableau rules are applied exhaustively to create new nodes and order assertions; we only present a few selected nondeterministic choices here. Similar to classical tableau algorithms, first the $(\exists)$-rule creates two witnesses x and y for the existential restrictions $\exists r.A$ and $\exists r.B$, respectively, at a. For example, we need to ensure that $(\exists r.A)(a) = \min\{r(a, x), A(x)\}$ is satisfied. One possibility is to introduce the order assertions

$$(\exists r.A)(a) = r(a, x) \text{ and } (\exists r.A)(a) \leqslant A(x),$$

expressing that the above minimum is realized by the value of the role connection from a to x. Although it does not seem necessary, we need to have equality here in order to prove completeness of the algorithm. Likewise, for y we assert that

$$(\exists r.B)(a) \leqslant r(a, y) \text{ and } (\exists r.B)(a) = B(y).$$

Moreover, the supremum-based semantics of existential restrictions also imposes an upper bound on all other r-successors, similar to the behavior of classical value restrictions. Hence, we also assert that

$$(\exists r.B)(a) \geqslant B(x) \text{ and } (\exists r.A)(a) \geqslant r(a, y).$$

In the next step, the GCIs are applied to all nodes; we ignore a here since it is not relevant for this example. For the node x, we know already that

$$B(x) \leqslant (\exists r.B)(a) \leqslant r(a, y) \leqslant (\exists r.A)(a) \leqslant A(x),$$

and hence it suffices to assert in addition that $A(x) \leqslant C(x)$, which then implies that also $B(x) \leqslant C(x)$ holds. For y, we introduce the order assertions

$$A(y) \leqslant C(y) \text{ and } B(y) \leqslant C(y).$$

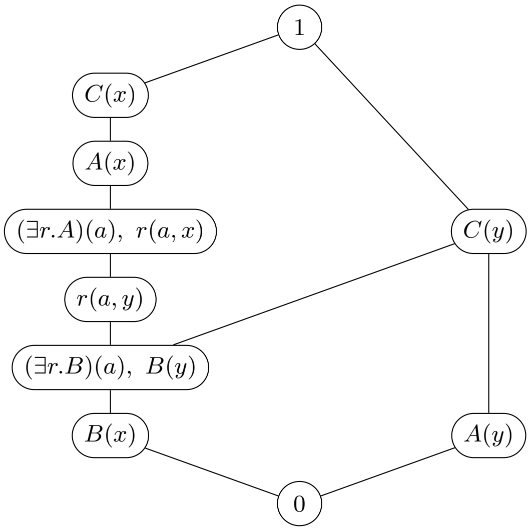

Fig. 1. Order diagram of the preorder induced by the order assertions produced in the example by the first applications of tableau rules.

The resulting set of assertions entails the preorder depicted in Figure 1, where we ignore p_A, p_B, p_C, and all irrelevant elements of the order structure. Note that, although we consider as models only total preorders, the assertions themselves need not define a single total order over all elements of the order structure.

Now we deal with the number restriction $(\leqslant 1\, r.C)(a)$. In the following, we ignore the required witnesses, as they are not essential for the example. As in the classical tableau algorithm, we use a tableau rule that forces each r-successor of a to choose whether it wants to "challenge" the number restriction or not. In the classical setting, this means choosing whether to satisfy C or not. Here, we have to decide whether $\neg(\leqslant 1\, r.C)(a) < \min\{r(a,x), C(x)\}$ holds. If this inequality holds for at least 2 nodes, then the supremum in the semantics of the at-least restriction $(\geqslant 2\, r.C) = \neg(\leqslant 1\, r.C)$ is violated. We analyze several possibilities:

– If it holds that $(\geqslant 2\, r.C)(a) \geqslant r(a,x) = \min\{r(a,x), C(x)\}$ and additionally $(\geqslant 2\, r.C)(a) < \min\{r(a,y), C(y)\}$, then

$$r(a,y) \leqslant (\exists r.A)(a) = r(a,x) \leqslant (\geqslant 2\, r.C)(a) < \min\{r(a,y), C(y)\} \leqslant r(a,y).$$

In this case, the resulting set of order assertions is not satisfiable anymore.

– If $(\geqslant 2\, r.C)(a) < r(a,x)$ and $(\geqslant 2\, r.C)(a) \geqslant \min\{r(a,y), C(y)\}$, then it depends on the values of p_A, p_B, and p_C whether we can build a G-model. If $(\geqslant 2\, r.C)(a) \leqslant 1 - p_C < p_B \leqslant (\exists r.B)(a) \leqslant (\geqslant 2\, r.C)(a)$, then this is obviously not possible. On the other hand, supposing that $p_A = \frac{1}{2}$ and $p_C = p_B = \frac{1}{4}$, we can construct a G-model by assigning the value $\frac{1}{4}$ to $A(y), B(x), B(y), r(a,y), C(y)$ and $\frac{1}{2}$ to $r(a,x), A(x), C(x)$. This means that

$(\exists r.A)(a)$ evaluates to $\frac{1}{2}$, $(\exists r.B)(a)$ to $\frac{1}{4}$, and $(\leqslant 1\, r.C)(a)$ to $\frac{1}{4}$, and hence $\mathcal{O}$ is satisfied.

– If $(\geqslant 2\, r.C)(a) < r(a,x)$ and $(\geqslant 2\, r.C)(a) < \min\{r(a,y), C(y)\}$, then the at-least restriction is violated. Thus, we have to apply another rule to *merge* the node y into x (or vice versa), which essentially amounts to discarding the node y and replacing all occurrences of y in the order assertions by x. Hence, almost all relevant elements of the order structure become equivalent, the only exception being $(\geqslant 2\, r.C)(a)$, which must be strictly smaller than all other elements. A possible resulting G-model could simply assign 1 to $A(x), B(x), C(x), r(a,x)$, which would result in $(\geqslant 2\, r.C)(a)$ being evaluated to 0. Again, all axioms of $\mathcal{O}$ are satisfied.

3.3 Complex Role Inclusions

For the tableau algorithm in [18], certain finite automata are constructed in order to deal with complex role inclusions. For each role r, the automaton $\mathbf{A}_r$ reads role chains, i.e. words over the alphabet of all roles, and recognizes exactly those role chains that imply r. In [18], these automata are then used in concept expressions of the form $\forall\mathbf{A}.C$ with the intuitive semantics that all domain elements connected by a chain of roles recognized by $\mathbf{A}$ to the current domain element should satisfy C. This allows to decompose inferences about complex role inclusions into single steps by enforcing certain connections between $\forall\mathbf{A}^q.C$ and $\forall\mathbf{A}^{q'}.C$, where $\mathbf{A}^q$ denotes the automaton $\mathbf{A}$ with q as initial state, and q' is a successor state of q. For example, if q' is reachable via an r-transition from q and the current domain element "satisfies" $\forall\mathbf{A}^q.C$, then any r-successor has to satisfy $\forall\mathbf{A}^{q'}.C$

For our setting, we generalize this construction to weighted finite automata recognizing the degree to which a given role chain implies a certain role r [9]. We closely follow the ideas from [18], but need to incorporate the degrees to which the role inclusions hold to the transitions of the automata. As in [18], the construction of $\mathbf{A}_r$ causes an exponential blowup in the size of $\mathcal{R}$. However, it is known that such a blowup cannot be avoided [20].

3.4 Results

In addition to the blow-up from this automata construction, our first algorithm, based on the reduction to a classical ontology, produces an exponential blowup in the (binary encoding of) the largest number n involved in a number restriction in $\mathcal{O}$, and in the number of individual names occurring in $\mathcal{O}$. However, we can avoid both if either nominals or number restrictions are disallowed.

In the reduction, from $\mathcal{O}$ we always obtain a classical $\mathcal{ALCOQ}$ ontology $\mathsf{red}(\mathcal{O})$, regardless of whether $\mathcal{O}$ uses inverse roles or nominals. However, if $\mathcal{O}$ does not use number restrictions, then $\mathsf{red}(\mathcal{O})$ is an $\mathcal{ALCO}$ ontology. As mentioned before, the reduction is only correct for logics having the forest-model property, i.e. G-$\mathcal{SRIQ}$, G-$\mathcal{SROQ}$, and G-$\mathcal{SROI}$ and their sublogics [12]. We can thus lift the following complexity results from classical DLs.

Theorem 1. *Deciding consistency is*

- 2-ExpTime-*complete in* G-$\mathcal{SRIQ}$, G-$\mathcal{SROI}$, *and* G-$\mathcal{SROQ}$, *and*
- ExpTime-*complete in all FDLs between* G-$\mathcal{ALC}$ *and* G-$\mathcal{SHOI}$ *or* G-$\mathcal{SHIQ}$.

Proof. The consistency of the $\mathcal{ALCOQ}$ ontology $\mathsf{red}(\mathcal{O})$ is decidable in exponential time in the size of $\mathsf{red}(\mathcal{O})$ [12]. The first upper bound thus follows from the fact that the size of $\mathsf{red}(\mathcal{O})$ is exponential in the size of $\mathcal{O}$. 2-ExpTime-hardness, even without involutive negation and assertions restricted to the form $\alpha \geqslant p$, follows from classical results [20] since in this case reasoning in sublogics of G-$\mathcal{SROIQ}$ is equivalent to reasoning in the underlying classical DLs [6].

Without complex role inclusions, i.e. restricting to simple role inclusions and transitivity axioms, the size of the automata $\mathbf{A}_r$ is polynomial in the size of $\mathcal{R}$ [18]. The other exponential blowup can be avoided by disallowing nominals or number restrictions. Hence, for G-$\mathcal{SHOI}$ and G-$\mathcal{SHIQ}$, the size of $\mathsf{red}(\mathcal{O})$ is polynomial in the size of $\mathcal{O}$, and the lower bound follows again from the reduction in [6] and ExpTime-hardness of consistency in classical $\mathcal{ALC}$ [22]. □

These results hold regardless of whether the numbers in number restrictions are encoded in unary or in binary. We leave open the complexity of consistency in G-$\mathcal{SHOQ}$, which is ExpTime-complete in the classical case [12].

The tableau algorithm does not allow us to derive a tight bound on the complexity of G-$\mathcal{SROIQ}$ since it may create triply exponentially many nodes in the size of $\mathcal{O}$. The resulting worst-case complexity of 3-NExpTime is the same bound that is obtained from the classical tableau algorithm for $\mathcal{SROIQ}$ [17]. This is in contrast to 2-NExpTime-completeness of classical $\mathcal{SROIQ}$ [20], where the upper bound is obtained by a reduction to the two-variable fragment of first-order logic with counting quantifiers. The 2-NExpTime-hardness can again be transferred to our setting via the reduction in [6].

4 Conclusions

We have described two algorithms for deciding consistency of (sublogics of) fuzzy $\mathcal{SROIQ}$ under infinitely valued Gödel semantics. The first approach involves an exponential blowup in the "depth" of the role hierarchy (which cannot be avoided [20]) and in the number of individual names if number restrictions and nominals are combined. However, it allows to directly exploit optimized classical reasoners for FDL reasoning, like the previous crispification algorithms for finitely valued FDLs [2, 4, 7, 23]. The tableau-based algorithm is more goal-oriented, but has not been implemented yet. While it also uses the automata-based encoding of role inclusions, it has the same complexity as the classical tableau algorithm for $\mathcal{SROIQ}$ [17].

In this paper, our goal was to provide a general intuition of the two algorithms via a simple example. For the full details of the algorithms, including proofs of correctness, we refer the interested reader to [8–10].

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