The Complexity of Fuzzy $\mathcal{EL}$ under the Łukasiewicz T-norm

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Abstract

Fuzzy Description Logics (DLs) are are a family of knowledge representation formalisms designed to represent and reason about vague and imprecise knowledge that is inherent to many application domains. Previous work has shown that the complexity of reasoning in a fuzzy DL using finitely many truth degrees is usually not higher than that of the underlying classical DL. We show that this does not hold for fuzzy extensions of the light-weight DL $\mathcal{EL}$, which is used in many biomedical ontologies, under the finitely valued Łukasiewicz semantics. More precisely, the complexity of reasoning increases from P to ExpTime, even if only one additional truth value is introduced. When adding complex role inclusions and inverse roles, the logic even becomes undecidable. Even more surprisingly, when considering the infinitely valued Łukasiewicz semantics, reasoning in fuzzy $\mathcal{EL}$ is undecidable.

Keywords: Fuzzy Description Logics, Mathematical Fuzzy Logic, Łukasiewicz T-norm, Fuzzy OWL 2 EL

1. Introduction

Description Logics (DLs) are a family of knowledge representation formalisms that have been successfully applied in many application domains [1]. In particular, these formalisms provide the logical foundation for the Direct Semantics of the standard web ontology language OWL 2 and its profiles.[1] The light-weight DL $\mathcal{EL}$, underlying the OWL 2 EL profile, is of particular interest. The expressive power of this DL suffices for representing many prominent biomedical ontologies like SNOMED CT[2] and the Gene Ontology,[3] while allowing tractable reasoning problems. Knowledge is represented in these logics through a set of *general concept inclusions (GCIs)* like

$$\exists \mathsf{hasDisease.Flu} \sqsubseteq \exists \mathsf{hasSymptom.Headache} \sqcap \exists \mathsf{hasSymptom.Fever} \tag{1}$$

which formally states that every patient with a flu must also show headache and fever as symptoms. Reasoning corresponds to the process of inferring knowledge that is implicitly encoded in the GCIs. It is well-known that this task is polynomial in $\mathcal{EL}$ [2].

An important problem for practical AI applications is to represent and reason with vague or imprecise knowledge in a formal way. Fuzzy Description Logics (FDLs) [3–5] extend classical DLs with the ideas of fuzzy logic to try to achieve this goal. The main premise of fuzzy logics is the use of more than two truth degrees to allow a more fine-grained analysis of dependencies between concepts. Usually, these degrees are arranged in a total order, or *chain*, in the interval $[0, 1]$. For instance, a patient having a body temperature of

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[1] http://www.w3.org/TR/owl2-overview/

[2] http://snomed.org/

[3] http://geneontology.org/

37.5 °C can have a degree of fever of 0.5, whereas a temperature of 39.2 °C may be interpreted as a fever with degree of 0.9. Considering the GCI (1), the severity of the symptoms certainly influences the severity of the disease, and thus truth degrees can be transferred between concepts. Depending on the desired granularity, one can choose to allow a finite number of truth degrees—e.g. 10 or 100—or even admit the whole (infinite) interval $[0, 1]$. Another degree of freedom in FDLs comes from the choice of possible semantics for the logical constructors. The most general semantics are based on *triangular norms (t-norms)* that are used to interpret conjunctions. Among these, the most prominent ones are the *Gödel*, *Łukasiewicz*, and *product* t-norms. All (continuous) t-norms over chains can be expressed as combinations of these three basic ones. Unfortunately, reasoning in many *infinitely* valued FDLs becomes undecidable [6, 7]; for a systematic study on this topic, see [8]. On the other hand, most of the *finitely* valued FDLs that have been studied recently have not only been proved to be decidable, but even to belong to the same complexity class as the corresponding classical DLs [9–13]. Although FDLs are not a new topic, and several monographies have been devoted to them [5, 14–16], there are still several open questions that may lead to surprising results, whose significance goes beyond the immediate application scope and sheds light on the computational behavior of the underlying logical formalisms and semantics.

A question that naturally arose is whether the finitely valued fuzzy framework always yields the same computational complexity as the corresponding classical formalisms. A common opinion was that everything that can be expressed in finitely valued FDLs can be reduced to the corresponding classical DLs without any serious loss of efficiency. Indeed, although some known translations of finitely valued FDLs into classical DLs are exponential [17], more efficient reasoning can be achieved through direct algorithms [10], yielding the same complexity as that of the underlying classical DLs. More recent work has focused on providing practical optimizations for reasoning [18]. However, these results refer to languages that are quite expressive already in their classical versions, and hence it is possible that the added complexity of finitely valued fuzzy semantics is masked by the high complexity of the classical DL reasoning. Hence, it is more fruitful to look for a complexity gap between classical and finitely valued semantics in less expressive DLs like $\mathcal{EL}$. This approach has been first pursued in [19], where different constructors that could cause an increase in the complexity are analyzed, but no specific answer is found. In [20], we have shown that the Łukasiewicz t-norm is a source of nondeterminism able to cause a significant increase in expressivity in very simple propositional languages. The work in [21] built on the methods devised in [20] to show even more dramatic increases in complexity for finitely valued extensions of $\mathcal{EL}$. Hence, the question whether reasoning in finitely valued FDLs has always the same complexity as in the corresponding classical DLs was finally answered. This answer was negative.

The question about the computational complexity of $\mathcal{EL}$ under infinitely valued semantics has also been considered previously. In [22], reasoning in $\mathcal{EL}$ under semantics including the Łukasiewicz t-norm was proven to be coNP-hard, but the proof did not apply to the finitely valued case. In contrast, infinitely valued Gödel semantics do not increase the complexity of reasoning [23]. It is important to notice that more expressive FDLs under infinitely valued Łukasiewicz semantics have been shown to be undecidable. The results in [8] suggest the presence of a negation operator as a likely culprit for undecidability. However, the minimum expressivity necessary to trigger undecidability was still unknown.

In this paper, we extend the preliminary work from [21] and present new results that shed light on the complexity of finitely and infinitely valued fuzzy extensions of $\mathcal{EL}$. In particular, we add two new undecidability proofs for infinitely valued FDLs, and provide further lower bounds for finitely valued extension of $\mathcal{EL}$. Our results can be summarized as follows:

- We prove that $\mathcal{EL}$ under finitely valued semantics is ExpTime-complete whenever the Łukasiewicz t-norm is included in the semantics. This proves a dichotomy similar to one that exists for infinitely valued FDLs [8] since, for all other finitely valued chains of truth values, reasoning in fuzzy $\mathcal{EL}$ can be shown to be in P using the methods from [23].

- We analyze the complexity of adding various means of expressivity that have been considered for classical $\mathcal{EL}$. We show that complex role inclusions, which yield the extension $\mathcal{EL}^+$ of $\mathcal{EL}$, add another exponential blowup, bringing the complexity of reasoning up to 2-ExpTime (as opposed to P in the classical case). We also show that most other extensions of $\mathcal{EL}$ that increase the complexity under the classical semantics have no effect on the baseline complexity of ExpTime for finitely valued fuzzy $\mathcal{EL}$ with the Łukasiewicz t-norm.

- Regarding infinitely valued fuzzy $\mathcal{EL}$, we strengthen previous lower bounds [22] by showing that reasoning under the Łukasiewicz t-norm actually becomes *undecidable*. This is surprising since previous undecidability results for FDLs [6–8] have relied on some kind of negation present in the logical syntax, which is not the case in $\mathcal{EL}$. We also show a partial undecidability result regarding the product t-norm, which extends results from [8]. However, a more complete picture of the complexity of fuzzy $\mathcal{EL}$ under the product t-norm remains open.

2. The Fuzzy $\mathcal{EL}$ Family

Fuzzy Description Logics extend classical DLs by allowing more than two truth degrees in the semantics of concepts and axioms. In this section, we introduce the classes of truth degrees relevant for this paper, recall the description logic $\mathcal{EL}$ and its various extensions, and consider their fuzzy variants.

2.1. Chains of Truth Values

The structures of truth degrees that we consider are called *chains*. Formally, these are algebras of the form $(\mathsf{L}, *_\mathsf{L}, \Rightarrow_\mathsf{L})$, where

- L is a subset of the interval $[0, 1]$ of rational numbers, such that L contains the extreme elements 0 and 1. The elements of L are called *truth degrees*.
- The *t-norm* $*_\mathsf{L}$ is a binary operator on L that is associative, commutative, monotone in both components, and has 1 as unit element [24]. This operator is used as the semantics of logical conjunction.
- The *residuum* $\Rightarrow_\mathsf{L}$ of $*_\mathsf{L}$ is a binary operator on L that satisfies the following condition for all $x, y, z \in \mathsf{L}$: $(x *_\mathsf{L} y) \leqslant z$ iff $y \leqslant (x \Rightarrow_\mathsf{L} z)$. This operator expresses logical implication between truth degrees in the chain.

For ease of presentation, we will often use L to denote the whole structure $(\mathsf{L}, *_\mathsf{L}, \Rightarrow_\mathsf{L})$, and omit the subscript L from the operators if the chain we use is clear from the context. An *interval* in L is a subset of the form $[a, b] \coloneqq \{x \in \mathsf{L} \mid a \leqslant x \leqslant b\}$ with $a, b \in \mathsf{L}$. An *idempotent element* in L is an element x such that $x *_\mathsf{L} x = x$.

We consider in particular the two cases where

(i) L consists of the whole (*infinite*) interval $[0, 1]$, or

(ii) L is a *finite* chain.

In the former case, we always assume that the operator $*_\mathsf{L}$ is *continuous* as a function from $[0, 1] \times [0, 1]$ to $[0, 1]$. One reason for this assumption is that it guarantees that the residuum is uniquely determined by the t-norm [24]. In case (ii), we similarly assume that $*_\mathsf{L}$ is *smooth*, i.e. for every $x, y, z \in \mathsf{L}$, whenever x and y are direct neighbors in L with $x < y$, then there is no $w \in \mathsf{L}$ such that $x *_\mathsf{L} z < w < y *_\mathsf{L} z$ [25]. If $*_\mathsf{L}$ is continuous (smooth), then we call L *continuous (smooth)*.

We emphasize that the restriction of truth values to the interval $[0, 1]$ is not essential, as any finite total order is order isomorphic to a finite subset of $[0, 1]$ containing 0 and 1, and similarly any dense, countable, total order with two extrema is order isomorphic to $[0, 1]$ (see [24] for details). The smooth and continuous chains we use form the basis of the so-called *standard semantics* of Mathematical Fuzzy Logic [26].

Example 2.1. By restricting the algebra of truth values to two elements, the classical Boolean algebra of truth and falsity is obtained: $\mathsf{B} = (\{0, 1\}, *_\mathsf{B}, \Rightarrow_\mathsf{B})$. In this case, $*_\mathsf{B}$ and $\Rightarrow_\mathsf{B}$ are the classical conjunction and the material implication, respectively. It is easy to see that material implication is indeed the residuum of the classical conjunction.

The most studied chains with continuous or smooth t-norms are the ones defined by the Gödel (G), Łukasiewicz ($\mathsf{Ł}$), and product ($\mathsf{\Pi}$) t-norms. The finitely valued versions of the former two, denoted by $\mathsf{Ł}_n$ and G_n for $n \geqslant 2$, are defined over the standard set of n truth values, $\{0, \frac{1}{n-1}, \dots, \frac{n-2}{n-1}, 1\}$. These are formally defined as follows:

- The *(finite) Gödel t-norm* (or *minimum t-norm*) $x *_{\mathsf{G}_n} y \coloneqq x *_{\mathsf{G}} y \coloneqq \min\{x, y\}$ and its residuum

$$x \Rightarrow_{\mathsf{G}_n} y \coloneqq x \Rightarrow_{\mathsf{G}} y \coloneqq \begin{cases} 1 & \text{if } x \leqslant y, \\ y & \text{otherwise.} \end{cases}$$

- The *(finite) Łukasiewicz t-norm* $x *_{\mathsf{Ł}_n} y \coloneqq x *_{\mathsf{Ł}} y \coloneqq \max\{0, x + y - 1\}$ and its residuum

$$x \Rightarrow_{\mathsf{Ł}_n} y \coloneqq x \Rightarrow_{\mathsf{Ł}} y \coloneqq \min\{1, 1 - x + y\}.$$

- The *product t-norm* $x *_{\mathsf{\Pi}} y \coloneqq x \cdot y$ and its residuum

$$x \Rightarrow_{\mathsf{\Pi}} y \coloneqq \begin{cases} 1 & \text{if } x \leqslant y, \\ \frac{y}{x} & \text{otherwise.} \end{cases}$$

A finitely valued version of the product t-norm does not exist (except for the trivial case that $n = 2$) since the chain L needs to be closed under the t-norm, but for any $x \in (0, 1)$, the set $\{x^m \mid m \geqslant 0\}$ is infinite.

The following easy observations about the introduced operators will be useful in the proofs throughout this paper. For details, we refer the interested reader to [24].

Proposition 2.2. *For all $x, y \in \mathsf{L}$, it holds that*

(a) $x *_{\mathsf{L}} y = 1$ *iff both* $x = 1$ *and* $y = 1$*;*

(b) *if* $x \leqslant x'$ *and* $y \leqslant y'$*, then* $x * y \leqslant x' * y'$*;*

(c) $x \Rightarrow_{\mathsf{L}} y = 1$ *iff* $x \leqslant y$*;*

(d) $x \Rightarrow_{\mathsf{L}} y \geqslant y$*;*

(e) $1 \Rightarrow_{\mathsf{L}} y = y$*;*

(f) *if* $\mathsf{L} = \mathsf{Ł}$*, then* $x \Rightarrow_{\mathsf{Ł}} 0 = 1 - x$*;*

(g) *if* $\mathsf{L} = \mathsf{Ł}$ *and* $x *_{\mathsf{Ł}} x \leqslant x *_{\mathsf{Ł}} x *_{\mathsf{Ł}} x$*, then either* $x \leqslant \frac{1}{2}$ *or* $x = 1$*;*

(h) *if* $\mathsf{L} = \mathsf{Ł}_n$ *and* $x *_{\mathsf{Ł}_n} y \geqslant \frac{n-2}{n-1}$*, then* $x = 1$ *or* $y = 1$*;*

(i) *if* $\mathsf{L} = \mathsf{Ł}_n$ *and* $x < 1$*, then* $x *_{\mathsf{Ł}_n} \overset{m}{\cdots} *_{\mathsf{Ł}_n} x = 0$ *for all* $m \geqslant n - 1$*;*

(j) *if* $\mathsf{L} = \mathsf{G}_n$*, then* $x *_{\mathsf{G}_n} \overset{m}{\cdots} *_{\mathsf{G}_n} x = x$ *for all* $m \geqslant 1$*.*

The t-norms defined above can be used to build all other continuous or smooth chains via the following construction.

Definition 2.3. Let L be a chain, $(\mathsf{L}_i)_{i \in I}$ be a family of chains, and $(\lambda_i)_{i \in I}$ be order isomorphisms between intervals $[a_i, b_i] \subseteq \mathsf{L}$ and L_i such that the intersection of any two intervals contains at most one element. Then, L is the *ordinal sum* of the family $(\mathsf{L}_i, \lambda_i)_{i \in I}$ if, for all $x, y \in \mathsf{L}$,

$$x *_{\mathsf{L}} y = \begin{cases} \lambda_i^{-1}\big(\lambda_i(x) *_{\mathsf{L}_i} \lambda_i(y)\big) & \text{if } x, y \in (a_i, b_i), \\ \min\{x, y\} & \text{otherwise.} \end{cases}$$

Every continuous chain over $[0,1]$ is isomorphic to an ordinal sum of infinitely valued Łukasiewicz and product chains [26, 27]. Similarly, every smooth finite chain is an ordinal sum of chains of the form $\mathsf{Ł}_n$ with $n \geqslant 3$ [28]. Moreover, these representations are unique, i.e. there cannot exist two non-isomorphic families of Łukasiewicz (and product) chains that produce the same chain as their ordinal sum. All elements that are not contained strictly within one such Łukasiewicz or product component are idempotent and can be thought of as belonging to a (finite) Gödel chain (see Proposition 2.2(j)). We say that a (finite or infinite) chain *contains* the Łukasiewicz t-norm if its ordinal sum representation contains at least one Łukasiewicz component; similarly, it *starts with* the Łukasiewicz t-norm if it contains a Łukasiewicz component in an interval $[0,b]$. Note that every chain that contains the Łukasiewicz t-norm can be represented as the ordinal sum of an arbitrary chain L_1 and another chain L_2 that starts with the Łukasiewicz t-norm.

Another way to view these characterizations is to observe that every smooth finite chain is either a Gödel chain or contains at least one finite Łukasiewicz component, and every continuous infinite chain is either a Gödel chain or contains at least one Łukasiewicz or product component. This is a key insight for our hardness proofs, since reasoning over pure Gödel chains is typically easier than for the other t-norms.

2.2. Fuzzy Description Logics

We now introduce the syntax and semantics of the basic FDL L-$\mathcal{EL}$, and then describe its various extensions. For this purpose, we fix an arbitrary chain $\mathsf{L} = (\mathsf{L}, *, \Rightarrow)$. When L is one of the specific chains introduced in the previous section, e.g. $\mathsf{Ł}_n$, we denote the resulting logic by $\mathsf{Ł}_n$-$\mathcal{EL}$ instead of L-$\mathcal{EL}$.

2.2.1. Fuzzy $\mathcal{EL}$

A *(DL) signature* is a tuple $(\mathsf{N_C}, \mathsf{N_R})$, where $\mathsf{N_C} = \{A, B, \dots\}$ is a countable set of *atomic concepts* (also called *concept names*) and $\mathsf{N_R} = \{r, s, \dots\}$ is a countable set of *atomic roles* (or *role names*). *Complex concepts* in L-$\mathcal{EL}$ are built inductively from atomic concepts and roles by means of the following concept constructors, where $A \in \mathsf{N_C}$ and $r \in \mathsf{N_R}$:

$$
\begin{array}{lll}
C, D \quad \longrightarrow & \top & \textit{top} \\
& A & \textit{atomic concept} \\
& C \sqcap D & \textit{conjunction} \\
& \exists r.C & \textit{existential restriction}
\end{array}
$$

We use the abbreviation C^m, $m \geqslant 1$, for the m-ary conjunction; i.e. $C^1 \coloneqq C$ and $C^{m+1} \coloneqq C^m \sqcap C$.

While the syntax of concepts in L-$\mathcal{EL}$ is the same as in classical $\mathcal{EL}$, the differences between both logics begin in their semantics. An L-*interpretation* is a pair $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ consisting of:

- a nonempty set $\Delta^{\mathcal{I}}$ (called *domain*), and
- a *fuzzy interpretation function* $\cdot^{\mathcal{I}}$ that assigns
  - to each concept name $A \in \mathsf{N_C}$ a fuzzy set $A^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \longrightarrow \mathsf{L}$, and
  - to each role name $r \in \mathsf{N_R}$ a fuzzy relation $r^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \times \Delta^{\mathcal{I}} \longrightarrow \mathsf{L}$.

Similarly, the semantics of a complex concept C is given by a function $C^{\mathcal{I}} \colon \Delta^{\mathcal{I}} \longrightarrow \mathsf{L}$, which is inductively defined as follows:

$$
\begin{aligned}
\top^{\mathcal{I}}(x) &\coloneqq 1, \\
(C \sqcap D)^{\mathcal{I}}(x) &\coloneqq C^{\mathcal{I}}(x) * D^{\mathcal{I}}(x), \\
(\exists r.C)^{\mathcal{I}}(x) &\coloneqq \sup_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x,y) * C^{\mathcal{I}}(y).
\end{aligned}
$$

In infinite chains, L-interpretations are often restricted to be *witnessed* [4], which means that for every existential restriction $\exists r.C$ and $x \in \Delta^{\mathcal{I}}$ there is an element $y \in \Delta^{\mathcal{I}}$ that realizes the supremum in the semantics of $\exists r.C$ at x, i.e. it holds that $(\exists r.C)^{\mathcal{I}}(x) = r^{\mathcal{I}}(x,y) * C^{\mathcal{I}}(y)$. This is a manifestation of the intuition that an existential restriction actually forces the existence of a single individual that satisfies it, instead of infinitely many that only satisfy the restriction in the limit. Under finitely valued semantics (and thus also under classical semantics), this property is always satisfied. We also adopt this restriction in the following, i.e. whenever we speak of an interpretation we implicitly assume that it is witnessed. Likewise, all reasoning problems we investigate are restricted to the class of witnessed interpretations.

In DLs, the domain knowledge about concepts and roles is expressed by *axioms*. In the fuzzy framework, these axioms are usually assigned a minimum degree of truth to which they must be satisfied. The most important kind of axioms that we consider are *general concept inclusions (GCIs)*, which are expressions of the form $\langle C \sqsubseteq D \geqslant \ell\rangle$, where $\ell \in \mathsf{L}$. The L-interpretation $\mathcal{I}$ *satisfies* such an axiom if $C^{\mathcal{I}}(x) \Rightarrow D^{\mathcal{I}}(x) \geqslant \ell$ holds for all domain elements $x \in \Delta^{\mathcal{I}}$. As usual, a TBox is a finite set of GCIs, and an L-interpretation $\mathcal{I}$ *satisfies* a TBox if it satisfies every axiom in it. A *crisp* GCI is of the form $\langle C \sqsubseteq D \geqslant 1\rangle$, and we usually abbreviate such an axiom by $\langle C \sqsubseteq D\rangle$, which has the semantics that $C^{\mathcal{I}}(x) \leqslant D^{\mathcal{I}}(x)$ holds for all $x \in \Delta^{\mathcal{I}}$ (see Proposition 2.2(c)). We also use $\langle C \equiv D\rangle$ as a short-hand for the two axioms $\langle C \sqsubseteq D\rangle$ and $\langle D \sqsubseteq C\rangle$.

Our goal in this paper is to analyze the computational complexity of reasoning in L-$\mathcal{EL}$. In particular, we consider the reasoning problem of deciding whether a concept C is *ℓ-subsumed* by another concept D with respect to a TBox $\mathcal{T}$, where $\ell \in \mathsf{L} \setminus \{0\}$; that is, whether every L-interpretation $\mathcal{I}$ that satisfies $\mathcal{T}$ also satisfies the GCI $\langle C \sqsubseteq D \geqslant \ell\rangle$. To solve this problem, it suffices to consider subsumption problems between *concept names* since a concept C is ℓ-subsumed by another concept D w.r.t. a TBox $\mathcal{T}$ iff the fresh concept name A is ℓ-subsumed by the fresh concept name B w.r.t. $\mathcal{T} \cup \{\langle A \sqsubseteq C\rangle, \langle D \sqsubseteq B\rangle\}$ (see [2]).

2.2.2. Additional Expressivity

Mimicking the study of classical $\mathcal{EL}$, we also consider several extensions of the basic logic L-$\mathcal{EL}$, which are defined by including either new concept constructors or other kinds of axioms. The first such extension is L-$\mathcal{EL}^+$ [29], which allows for *(complex) role inclusions* of the form $\langle r_1 \circ \cdots \circ r_n \sqsubseteq r \geqslant \ell\rangle$, where $r_1, \ldots, r_n, r$ are roles and $\ell \in \mathsf{L} \setminus \{0\}$. TBoxes can then contain also this new kind of axioms, with the semantics that an L-interpretation $\mathcal{I}$ satisfies them if $(r_1 \circ \cdots \circ r_n)^{\mathcal{I}}(x, y) \Rightarrow r^{\mathcal{I}}(x, y) \geqslant \ell$ holds for all $x, y \in \Delta^{\mathcal{I}}$. Here, the *role composition* $r \circ s$ has the following semantics, which is a more general form of the usual composition of classical binary relations:

$$(r \circ s)^{\mathcal{I}}(x, y) := \sup_{z \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, z) * s^{\mathcal{I}}(z, y).$$

Usually, this supremum is also required to be witnessed (like for existential restrictions), i.e. for all $x, y \in \Delta^{\mathcal{I}}$ there must exist an element $z \in \Delta^{\mathcal{I}}$ such that $(r \circ s)^{\mathcal{I}}(x, y) = r^{\mathcal{I}}(x, z) * s^{\mathcal{I}}(z, y)$. However, since we use complex role inclusions only in the context of finitely valued FDLs, where this property always holds, we do not need this restriction here.

There are various other concept constructors that have been considered in combination with $\mathcal{EL}$ before [2]. The names of the extended logics are usually built by appending a designated symbol (e.g., L-$\mathcal{EL}$ with bottom concept and disjunctions is denoted by L-$\mathcal{ELU}_\bot$):

- *Bottom* $\bot$ (designated by the subscript $\bot$), interpreted as $\bot^{\mathcal{I}}(x) = 0$.
- *Nominals* (letter $\mathcal{O}$) of the form $\{a\}$ for an *individual name* a, with the semantics that $\{a\}^{\mathcal{I}}(x) = 1$ if x is equal to a designated domain element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$, and $\{a\}^{\mathcal{I}}(x) = 0$ for all other x.
- *Disjunction* $C \sqcup D$ (letter $\mathcal{U}$) with the semantics $(C \sqcup D)^{\mathcal{I}} := 1 - \big((1 - C^{\mathcal{I}}(x)) * (1 - D^{\mathcal{I}}(x))\big)$.
- *Inverse roles* r^- (letter $\mathcal{I}$) with the semantics $(r^-)^{\mathcal{I}}(x, y) := r^{\mathcal{I}}(y, x)$, which can be used in all places where usually roles would be used, i.e. in existential restrictions or role inclusions.

Hence, the most expressive logic we consider in this paper is L-$\mathcal{ELUOI}^+_\bot$. In contrast to [2], we do not consider *number restrictions*, nor $\mathcal{EL}^{++}$, which extends $\mathcal{ELO}^+_\bot$ with *p-admissible concrete domains* [2], since all of our complexity lower bounds already hold without them.

In more expressive logics than $\mathcal{EL}$, sets of role inclusions are usually restricted to be *regular* in order to ensure decidability [17, 30, 31]. We recall this property here since it is relevant for our results. Let $\lessdot$ be a strict partial order on the set of all role names and inverse roles such that $r \lessdot s$ iff $r^- \lessdot s$. A role inclusion $\langle w \sqsubseteq r \geqslant p \rangle$ is *$\lessdot$-regular* if

- w is of the form $r \circ r$ or r^-, or
- w is of the form $r_1 \circ \cdots \circ r_n$, $r \circ r_1 \circ \cdots \circ r_n$, or $r_1 \circ \cdots \circ r_n \circ r$, and for all $1 \leqslant i \leqslant n$ it holds that $r_i \lessdot r$.

A TBox $\mathcal{T}$ is *regular* if there is a strict partial order $\lessdot$ as above such that each role inclusion in $\mathcal{T}$ is $\lessdot$-regular.

2.2.3. Classical Extensions of $\mathcal{EL}$

Even though, as illustrated in Example 2.1, it is enough to restrict the semantics to the two-element chain B to obtain the classical semantics, we prefer to define both kinds of semantics to aid understanding (and indeed, writing down) the proofs. The resulting logics are denoted simply by $\mathcal{EL}$ and $\mathcal{EL}^+$ instead of B-$\mathcal{EL}$ or B-$\mathcal{EL}^+$.

In the classical view, an interpretation is a pair $\mathcal{I} = (\Delta^\mathcal{I}, \cdot^\mathcal{I})$ consisting of:

- a *domain* $\Delta^\mathcal{I}$, and
- an *interpretation function* $\cdot^\mathcal{I}$ that assigns:
  - to each concept name $A \in \mathsf{N_C}$ a crisp set $A^\mathcal{I} \subseteq \Delta^\mathcal{I}$, and
  - to each role name $r \in \mathsf{N_R}$ a crisp relation $R^\mathcal{I} \subseteq \Delta^\mathcal{I} \times \Delta^\mathcal{I}$.

This function is extended to concepts by setting

$$
\begin{aligned}
\top^\mathcal{I} &:= \Delta^\mathcal{I}, \\
\bot^\mathcal{I} &:= \emptyset, \\
(C \sqcap D)^\mathcal{I} &:= C^\mathcal{I} \cap D^\mathcal{I}, \\
(C \sqcup D)^\mathcal{I} &:= C^\mathcal{I} \cup D^\mathcal{I}, \\
(\exists r.C)^\mathcal{I} &:= \{x \in \Delta^\mathcal{I} \mid \exists y \in \Delta^\mathcal{I} \colon (x,y) \in r^\mathcal{I} \text{ and } y \in C^\mathcal{I}\},
\end{aligned}
$$

and similarly for the other constructors. It is easy to see that, by replacing the sets $C^\mathcal{I}$ by their characteristic functions $\chi_{C^\mathcal{I}} \colon \Delta^\mathcal{I} \to \{0,1\}$, we obtain the fuzzy semantics over B. To distinguish them from their fuzzy counterparts, classical GCIs and role inclusions are written without brackets as $C \sqsubseteq D$ and $r_1 \circ \cdots \circ r_n \sqsubseteq r$, respectively, and are satisfied by $\mathcal{I}$ if $C^\mathcal{I} \subseteq D^\mathcal{I}$ and $r_1^\mathcal{I} \circ \cdots \circ r_n^\mathcal{I} \subseteq r^\mathcal{I}$ hold, respectively.

In this setting, we talk simply about *subsumption*, since for $\ell = 1$ the ℓ-subsumption problem simplifies to the question whether $C^\mathcal{I} \subseteq D^\mathcal{I}$ holds in all interpretations $\mathcal{I}$ that satisfy a given TBox $\mathcal{T}$. Again, it suffices to decide subsumptions between concept names. It is well-known that subsumption in $\mathcal{EL}$ can be decided in polynomial time, and that this upper bound holds also in the more expressive logics $\mathcal{EL}^+$ and $\mathcal{EL}^{++}$ [2, 32]. Although the original algorithm for $\mathcal{EL}^{++}$ presented in [2] is incomplete in the management of nominals, a complete algorithm for $\mathcal{ELO}^+_\bot$ that still runs in polynomial time was presented in [32, 33], and can easily be combined with the rules for p-admissible concrete domains from [2]. On the other hand, if either disjunction or inverse roles are added to the logic, then the complexity increases to ExpTime [2, 34].

3. Non-Idempotent Finite Chains Make $\mathcal{EL}$ Harder

In this section, we exhibit a surprising reversal of the above-mentioned classical results for the finite Łukasiewicz t-norm. First, reasoning in $\mathsf{Ł}_n$-$\mathcal{EL}$, for any $n \geqslant 3$, is already exponentially harder than in $\mathcal{EL}$; more precisely, it is ExpTime-complete. Moreover, in this setting complex role inclusions actually make it worse, resulting in 2-ExpTime-completeness for $\mathsf{Ł}_n$-$\mathcal{EL}^+$ and $\mathsf{Ł}_n$-$\mathcal{EL}^{++}$ with regular role inclusions, and even undecidability for $\mathsf{Ł}_n$-$\mathcal{ELI}^+$ with unrestricted role inclusions. In contrast, the other constructors identified as

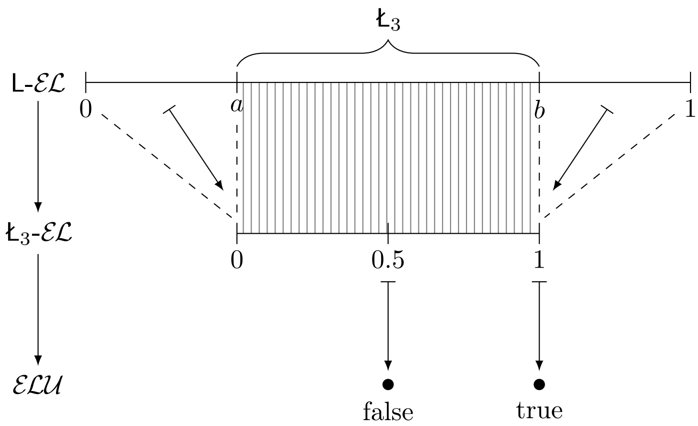

Figure 1: Illustration of the reductions of Section 3

harmful in the classical setting do *not* raise the baseline complexity of ExpTime: reasoning in $Ł_n$-$\mathcal{ELUOI}_\bot$ (without $^+$) is still ExpTime-complete. Moreover, all these results hold for any finite chain *containing* the Łukasiewicz t-norm. We focus on showing the lower bounds for these complexity results. Matching upper bounds have been shown for more expressive fuzzy DLs [10, 17, 35]. Recall that every finite chain that does not contain a Łukasiewicz component must be a finite Gödel chain, which has only idempotent elements; for these structures, the complexity of reasoning in $\mathcal{EL}^{++}$ is known to remain the same as in the classical case; that is, subsumption can be decided in polynomial time [31, 35].

The key insight for the hardness proofs is that the Łukasiewicz t-norm is powerful enough to simulate the (two-valued) disjunction constructor. Hence, we can reduce concept subsumption in $\mathcal{ELU}$ to concept subsumption in L-$\mathcal{EL}$, which yields ExpTime-hardness [2].[4] The structure of our proofs is illustrated in Figure 1 for the special case of a finite chain L containing an $Ł_3$-component. To obtain the semantics of $\mathcal{ELU}$, the values 0.5 and 1 in $Ł_3$-$\mathcal{EL}$ are used to simulate the classical truth values *false* and *true*, respectively. The chain $Ł_3$ can then be embedded into L as depicted. For a finite chain containing an $Ł_n$-component with $n > 3$, the only difference is that the value $\frac{n-2}{n-1}$ is used instead of 0.5.

3.1. Finite Łukasiewicz Chains

We first consider the case of a finite Łukasiewicz chain $Ł_n$ with $n \geqslant 3$. For ease of presentation, we omit the subscript $Ł_n$ from $*$ and $\Rightarrow$ in the following. In our reduction from subsumption in $\mathcal{ELU}$ to subsumption in $Ł_n$-$\mathcal{EL}$, we can restrict our considerations to $\mathcal{ELU}$ TBoxes that are in *normal form*; that is, TBoxes consisting only of axioms of the following forms:

$$A_1 \sqcap A_2 \sqsubseteq B$$
$$\exists r.A \sqsubseteq B$$
$$A \sqsubseteq \exists r.B$$
$$A \sqsubseteq B_1 \sqcup B_2$$

where A, A_1, A_2, B, B_1 and B_2 are concept names or $\top$. It is easy to see that every $\mathcal{ELU}$ TBox can be polynomially reduced to an equivalent one in normal form (see [2] for details).

In the reduction, we will simulate a classical concept name in $Ł_n$-$\mathcal{EL}$ by considering all values below $\frac{n-2}{n-1}$ to represent *false*, and thus only the value 1 can be used to express that a domain element belongs to the concept name. We can then express a classical disjunction of the form $B_1 \sqcup B_2$ by restricting the value of the fuzzy conjunction $B_1 \sqcap B_2$ to be $\geqslant \frac{n-2}{n-1}$ since the latter implies that B_1 *or* B_2 has value 1 (see

[4] This is closely related to the argument that usually non-convex extensions of classical $\mathcal{EL}$, i.e. those that can express disjunctions, are ExpTime-hard [2].

Proposition 2.2(h)). Furthermore, we reformulate classical subsumption between C and D as 1-subsumption between C^{n-1} and D^{n-1} since the latter two concepts can take only the values 0 or 1 (see Proposition 2.2(i)).

More formally, let $n \geqslant 3$, $\mathcal{T}$ be an $\mathcal{ELU}$ TBox in normal form, and C, D be two atomic concepts. We construct an $Ł_n$-$\mathcal{EL}$ TBox $\rho_n(\mathcal{T})$ such that C is subsumed by D w.r.t. $\mathcal{T}$ if and only if C^{n-1} is subsumed by D^{n-1} w.r.t. $\rho_n(\mathcal{T})$. Since $\mathcal{T}$ is in normal form, we can define the reduction ρ_n individually for each of the four kinds of axioms listed above:

$$\begin{aligned}
\rho_n(A_1 \sqcap A_2 \sqsubseteq B) &:= \langle A_1 \sqcap A_2 \sqsubseteq B \geqslant 1\rangle \\
\rho_n(\exists r.A \sqsubseteq B) &:= \langle \exists r.A \sqsubseteq B \geqslant 1\rangle \\
\rho_n(A \sqsubseteq \exists r.B) &:= \langle A \sqsubseteq (\exists r.B)^{n-1} \geqslant \tfrac{1}{n-1}\rangle \\
\rho_n(A \sqsubseteq B_1 \sqcup B_2) &:= \langle A \sqsubseteq B_1 \sqcap B_2 \geqslant \tfrac{n-2}{n-1}\rangle
\end{aligned}$$

Finally, we set $\rho_n(\mathcal{T}) := \{\rho_n(\alpha) \mid \alpha \in \mathcal{T}\}$.

Notice that $\rho_n(\mathcal{T})$ has as many axioms as $\mathcal{T}$, and the size of each axiom is increased by a factor of at most n. Hence, the translation $\rho_n(\mathcal{T})$ can be performed in polynomial time. We show that this translation satisfies the properties described above.

3.2. Soundness

In this subsection we prove that if C is subsumed by D with respect to $\mathcal{T}$, then C^{n-1} is 1-subsumed by D^{n-1} with respect to the $Ł_n$-$\mathcal{EL}$ TBox $\rho_n(\mathcal{T})$. In order to achieve this result, for any $Ł_n$-interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ we define the crisp interpretation $\mathcal{I}_{cr} = (\Delta^{\mathcal{I}_{cr}}, \cdot^{\mathcal{I}_{cr}})$, where:

- $\Delta^{\mathcal{I}_{cr}} := \Delta^{\mathcal{I}}$;
- for every concept name $A \in \mathsf{N_C}$ and $x \in \Delta^{\mathcal{I}}$, $x \in A^{\mathcal{I}_{cr}}$ iff $A^{\mathcal{I}}(x) = 1$; and
- for every $r \in \mathsf{N_R}$ and $x, y \in \Delta^{\mathcal{I}}$, $(x, y) \in r^{\mathcal{I}_{cr}}$ iff $r^{\mathcal{I}}(x, y) = 1$.

Note that it also holds for all $x \in \Delta^{\mathcal{I}}$ that $x \in \top^{\mathcal{I}_{cr}}$ iff $\top^{\mathcal{I}}(x) = 1$. Thus, to increase the readability of the following proofs we can treat $\top$ as an ordinary concept name. Before proving soundness of $\rho_n(\cdot)$ we prove that the translation $\cdot_{cr}$ of interpretations preserves satisfaction of the TBoxes.

Lemma 3.1. *Let $\mathcal{I}$ be an $Ł_n$-interpretation that satisfies $\rho_n(\mathcal{T})$. Then $\mathcal{I}_{cr}$ satisfies $\mathcal{T}$.*

Proof. We make a case distinction on the type of axiom in $\mathcal{T}$.

- Consider an axiom of the form $A_1 \sqcap A_2 \sqsubseteq B \in \mathcal{T}$ and $x \in A_1^{\mathcal{I}_{cr}} \cap A_2^{\mathcal{I}_{cr}}$. By the definition of $\mathcal{I}_{cr}$, we have that $A_1^{\mathcal{I}}(x) = 1$ and $A_2^{\mathcal{I}}(x) = 1$. Hence $(A_1 \sqcap A_2)^{\mathcal{I}}(x) = 1$. Since $\mathcal{I}$ satisfies $\rho_n(\mathcal{T})$, this implies that $B^{\mathcal{I}}(x) = 1$ by Proposition 2.2(c). Again by the definition of $\mathcal{I}_{cr}$, we get $x \in B^{\mathcal{I}_{cr}}$.
- Consider an axiom of the form $\exists r.A \sqsubseteq B \in \mathcal{T}$ and $x \in (\exists r.A)^{\mathcal{I}_{cr}}$. Hence there exists an element $y \in \Delta^{\mathcal{I}_{cr}}$ such that $(x, y) \in r^{\mathcal{I}_{cr}}$ and $y \in A^{\mathcal{I}_{cr}}$. By the definition of $\mathcal{I}_{cr}$, we have that $r^{\mathcal{I}}(x, y) = 1$ and $A^{\mathcal{I}}(y) = 1$. Hence $\sup_{z \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, z) * A^{\mathcal{I}}(z) = r^{\mathcal{I}}(x, y) * A^{\mathcal{I}}(y) = 1$. Since $\mathcal{I}$ satisfies $\rho_n(\mathcal{T})$, we get $B^{\mathcal{I}}(x) = 1$. Again by the definition of $\mathcal{I}_{cr}$, we conclude that $x \in B^{\mathcal{I}_{cr}}$.
- Consider an axiom of the form $A \sqsubseteq \exists r.B \in \mathcal{T}$ and $x \in A^{\mathcal{I}_{cr}}$. By the definition of $\mathcal{I}_{cr}$, we have $A^{\mathcal{I}}(x) = 1$. Since $\mathcal{I}$ satisfies $\rho_n(\mathcal{T})$, Proposition 2.2(e) implies that $((\exists r.B)^{n-1})^{\mathcal{I}}(x) \geqslant \frac{1}{n-1}$. However, by Proposition 2.2(i) we have $((\exists r.B)^{n-1})^{\mathcal{I}}(x) \in \{0, 1\}$, and hence it must be that case that $((\exists r.B)^{n-1})^{\mathcal{I}}(x) = 1$. Thus,

  $$1 = (\exists r.B)^{\mathcal{I}}(x) = \sup_{z \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, z) * B^{\mathcal{I}}(z).$$

  Since every $Ł_n$-interpretation is witnessed, there exists $y \in \Delta^{\mathcal{I}}$ such that $r^{\mathcal{I}}(x, y) = 1$ and $B^{\mathcal{I}}(y) = 1$. Again by the definition of $\mathcal{I}_{cr}$, we have $(x, y) \in r^{\mathcal{I}_{cr}}$ and $y \in B^{\mathcal{I}_{cr}}$, and hence $x \in (\exists r.B)^{\mathcal{I}_{cr}}$.

- Consider an axiom of the form $A \sqsubseteq B_1 \sqcup B_2 \in \mathcal{T}$ and $x \in A^{\mathcal{I}_{cr}}$. By the definition of $\mathcal{I}_{cr}$, we have that $A^{\mathcal{I}}(x) = 1$. Since $\mathcal{I}$ satisfies $\rho_n(\mathcal{T})$, this implies that $(B_1 \sqcap B_2)^{\mathcal{I}}(x) \geqslant \frac{n-2}{n-1}$. By Proposition 2.2(h), either $B_1^{\mathcal{I}}(x) = 1$ or $B_2^{\mathcal{I}}(x) = 1$. Again by the definition of $\mathcal{I}_{cr}$, we have that $x \in B_1^{\mathcal{I}_{cr}}$ or $x \in B_2^{\mathcal{I}_{cr}}$. $\square$

Now we are ready to prove the following proposition.

Proposition 3.2. *If C is subsumed by D w.r.t. $\mathcal{T}$, then C^{n-1} is 1-subsumed by D^{n-1} w.r.t. $\rho_n(\mathcal{T})$.*

Proof. Let $\mathcal{I}$ be an Ł$_n$-interpretation satisfying $\rho_n(\mathcal{T})$ and $x \in \Delta^{\mathcal{I}}$ such that $(C^{n-1})^{\mathcal{I}}(x) > 0$. Hence $(C^{n-1})^{\mathcal{I}}(x) = 1$ and thus $C^{\mathcal{I}}(x) = 1$. By the definition of $\mathcal{I}_{cr}$, we have $x \in C^{\mathcal{I}_{cr}}$. By Lemma 3.1 we know that $\mathcal{I}_{cr}$ satisfies $\mathcal{T}$, and thus we get $x \in D^{\mathcal{I}_{cr}}$ by assumption. Again by the definition of $\mathcal{I}_{cr}$, we obtain $D^{\mathcal{I}}(x) = 1$ and therefore $(D^{n-1})^{\mathcal{I}}(x) = 1$. Hence $(C^{n-1})^{\mathcal{I}}(x) \Rightarrow (D^{n-1})^{\mathcal{I}}(x) = 1$, that is, C^{n-1} is 1-subsumed by D^{n-1} with respect to $\rho_n(\mathcal{T})$. $\square$

3.3. Completeness

We now prove the converse direction; that is, that the translation does not introduce new subsumption relations of the specified form. Similarly to before, we define for any classical interpretation $\mathcal{I} = (\Delta^{\mathcal{I}}, \cdot^{\mathcal{I}})$ an Ł$_n$-interpretation $\mathcal{I}_n = (\Delta^{\mathcal{I}_n}, \cdot^{\mathcal{I}_n})$, where:

- $\Delta^{\mathcal{I}_n} := \Delta^{\mathcal{I}}$,
- $A^{\mathcal{I}_n}(x) := 1$ if $x \in A^{\mathcal{I}}$ and $A^{\mathcal{I}_n}(x) := \frac{n-2}{n-1}$ otherwise, for every $A \in \mathsf{N_C}$ and $x \in \Delta^{\mathcal{I}}$,
- $r^{\mathcal{I}_n}(x,y) := 1$ if $(x,y) \in r^{\mathcal{I}}$ and $r^{\mathcal{I}_n}(x,y) := \frac{n-2}{n-1}$ otherwise, for every $r \in \mathsf{N_R}$ and $x, y \in \Delta^{\mathcal{I}}$.

Again, $\top$ behaves exactly like the concept names since $\top^{\mathcal{I}_n}(x)$ is always 1. Before proving completeness of $\rho_n(\cdot)$ we need to prove that the translation $\cdot_n$ preserves satisfaction of all axioms.

Lemma 3.3. *If a classical interpretation $\mathcal{I}$ satisfies $\mathcal{T}$, then $\mathcal{I}_n$ satisfies $\rho_n(\mathcal{T})$.*

Proof. We prove case by case that $\mathcal{I}_n$ satisfies $\rho_n(\mathcal{T})$.

- Consider an axiom of the form $\langle A_1 \sqcap A_2 \sqsubseteq B \geqslant 1\rangle \in \rho_n(\mathcal{T})$ and any $x \in \Delta^{\mathcal{I}_n}$. If $(A_1 \sqcap A_2)^{\mathcal{I}_n}(x) = 1$, then both $A_1^{\mathcal{I}_n}(x) = 1$ and $A_2^{\mathcal{I}_n}(x) = 1$. By the definition of $\mathcal{I}_n$, we have that $x \in A_1^{\mathcal{I}} \cap A_2^{\mathcal{I}}$. Since $\mathcal{I}$ satisfies $\mathcal{T}$, this yields $x \in B^{\mathcal{I}}$. Again by the definition of $\mathcal{I}_n$, we get $B^{\mathcal{I}_n}(x) = 1$, and hence by Proposition 2.2(c) the axiom is satisfied.
  In the case that $(A_1 \sqcap A_2)^{\mathcal{I}_n}(x) < 1$, we have $(A_1 \sqcap A_2)^{\mathcal{I}_n}(x) \leqslant \frac{n-2}{n-1} \leqslant B^{\mathcal{I}_n}(x)$ by the definition of $\mathcal{I}_n$, and thus also $(A_1 \sqcap A_2)^{\mathcal{I}_n}(x) \Rightarrow B^{\mathcal{I}_n}(x) = 1$.
- Consider an axiom of the form $\langle \exists r.A \sqsubseteq B \geqslant 1\rangle \in \rho_n(\mathcal{T})$ and any $x \in \Delta^{\mathcal{I}_n}$. If $(\exists r.A)^{\mathcal{I}_n}(x) = 1$, then $\sup_{z \in \Delta^{\mathcal{I}_n}} r^{\mathcal{I}_n}(x,z) * A^{\mathcal{I}_n}(z) = 1$. By Proposition 2.2(a), this means that there exists $y \in \Delta^{\mathcal{I}_n}$ such that $r^{\mathcal{I}_n}(x,y) = 1$ and $A^{\mathcal{I}_n}(y) = 1$. By the definition of $\mathcal{I}_n$, we know that $(x,y) \in r^{\mathcal{I}}$ and $y \in A^{\mathcal{I}}$. Hence $x \in (\exists r.A)^{\mathcal{I}}$. Since $\mathcal{I}$ satisfies $\mathcal{T}$, we get $x \in B^{\mathcal{I}}$. Again by the definition of $\mathcal{I}_n$, we have that $B^{\mathcal{I}_n} = 1$.
  Otherwise, we have $(\exists r.A)^{\mathcal{I}_n}(x) \Rightarrow B^{\mathcal{I}_n}(x) = 1$ as in the previous case.
- Consider an axiom of the form $\langle A \sqsubseteq (\exists r.B)^{n-1} \geqslant \frac{1}{n-1}\rangle \in \rho_n(\mathcal{T})$ and any $x \in \Delta^{\mathcal{I}_n}$. If we have $((\exists r.B)^{n-1})^{\mathcal{I}_n}(x) = 0$, then

$$1 > (\exists r.B)^{\mathcal{I}_n}(x) = \sup_{z \in \Delta^{\mathcal{I}_n}} r^{\mathcal{I}_n}(x,z) * B^{\mathcal{I}_n}(z).$$

  By Proposition 2.2(a), every $y \in \Delta^{\mathcal{I}_n}$ must satisfy either $r^{\mathcal{I}_n}(x,y) < 1$ or $B^{\mathcal{I}_n}(y) < 1$. By the definition of $\mathcal{I}_n$, for all $y \in \Delta^{\mathcal{I}}$ we have either $(x,y) \notin r^{\mathcal{I}}$ or $y \notin B^{\mathcal{I}}$, and hence $x \notin (\exists r.B)^{\mathcal{I}}$. Since $\mathcal{I}$ satisfies $\mathcal{T}$, we get $x \notin A^{\mathcal{I}}$. Again by the definition of $\mathcal{I}_n$, we have $A^{\mathcal{I}_n}(x) = \frac{n-2}{n-1}$. Hence

$A^{\mathcal{I}_n}(x) \Rightarrow ((\exists r.B)^{n-1})^{\mathcal{I}_n}(x) = \frac{1}{n-1}$.
In the case that $((\exists r.B)^{n-1})^{\mathcal{I}_n}(x) > 0$, Proposition 2.2(d) yields

$$A^{\mathcal{I}_n}(x) \Rightarrow ((\exists r.B)^{n-1})^{\mathcal{I}_n}(x) \geqslant ((\exists r.B)^{n-1})^{\mathcal{I}_n}(x) \geqslant \tfrac{1}{n-1}$$

since $\frac{1}{n-1}$ is the smallest non-zero truth degree.

- Consider an axiom of the form $\langle A \sqsubseteq B_1 \sqcap B_2 \geqslant \frac{n-2}{n-1}\rangle \in \rho_n(\mathcal{T})$ and any $x \in \Delta^{\mathcal{I}_n}$. In the case that $(B_1 \sqcap B_2)^{\mathcal{I}_n}(x) < \frac{n-2}{n-1}$, we must have $B_1^{\mathcal{I}_n}(x) = B_2^{\mathcal{I}_n}(x) = \frac{n-2}{n-1}$. By the definition of $\mathcal{I}_n$, it follows that $x \notin B_1^{\mathcal{I}} \cup B_2^{\mathcal{I}}$. Since $\mathcal{I}$ satisfies $\mathcal{T}$, this implies that $x \notin A^{\mathcal{I}}$. Again by the definition of $\mathcal{I}_n$, we get $A^{\mathcal{I}_n}(x) = \frac{n-2}{n-1}$. Since by the definition of $\mathcal{I}_n$ and supposition we have $(B_1 \sqcap B_2)^{\mathcal{I}_n}(x) = \frac{n-3}{n-1}$, we can conclude that $A^{\mathcal{I}_n}(x) \Rightarrow (B_1 \sqcap B_2)^{\mathcal{I}_n}(x) = \frac{n-2}{n-1}$.
In the case that $(B_1 \sqcap B_2)^{\mathcal{I}_n}(x) \geqslant \frac{n-2}{n-1}$, Proposition 2.2(d) again yields

$$A^{\mathcal{I}_n}(x) \Rightarrow (B_1 \sqcap B_2)^{\mathcal{I}_n}(x) \geqslant (B_1 \sqcap B_2)^{\mathcal{I}_n}(x) \geqslant \tfrac{n-2}{n-1},$$

i.e. the axiom is satisfied by $\mathcal{I}_n$. □

From this lemma, we immediately obtain the desired result.

Proposition 3.4. *If C is not subsumed by D w.r.t. $\mathcal{T}$, then C^{n-1} is not 1-subsumed by D^{n-1} w.r.t. $\rho_n(\mathcal{T})$.*

Proof. Let $\mathcal{I}$ be a crisp interpretation satisfying $\mathcal{T}$ and $x \in \Delta^{\mathcal{I}}$ such that $x \in C^{\mathcal{I}} \setminus D^{\mathcal{I}}$. By Lemma 3.3, we know that $\mathcal{I}_n$ satisfies $\rho_n(\mathcal{T})$. Moreover, by the definition of $\mathcal{I}_n$, we have $C^{\mathcal{I}_n}(x) = 1$ and $D^{\mathcal{I}_n}(x) = \frac{n-2}{n-1}$. Hence $(C^{n-1})^{\mathcal{I}_n}(x) = 1$ and $(D^{n-1})^{\mathcal{I}_n}(x) = 0$, and therefore $(C^{n-1})^{\mathcal{I}_n}(x) \Rightarrow (D^{n-1})^{\mathcal{I}_n}(x) = 0 < 1$. □

Taking together the results of Propositions 3.2 and 3.4 and the complexity bounds derived in [2, 10], we have thus found the precise complexity of subsumption in Ł$_n$-$\mathcal{EL}$.

Theorem 3.5. *Let $n \geqslant 3$. Deciding ℓ-subsumption with respect to a TBox in Ł$_n$-$\mathcal{EL}$ is* ExpTime*-complete.*

Proof. The result follows from the above reduction and the fact that the subsumption problem with respect to a TBox for the language $\mathcal{ELU}$ is ExpTime-hard [2]. The ExpTime upper bound was shown in [10] for the more expressive language Ł$_n$-$\mathcal{ALC}$. □

3.4. Extensions of $\mathcal{EL}$

The goal of this section is to investigate the complexity of extensions of Ł$_n$-$\mathcal{EL}$, following the analysis of [2]. We start by considering complex role inclusions; that is, Ł$_n$-$\mathcal{EL}^+$. The following result is not surprising in the light of the previous reduction, which shows that Ł$_n$-$\mathcal{EL}$ can simulate classical disjunction, and the fact that reasoning with complex role inclusions in classical DLs with disjunction is 2-ExpTime-hard [36].

Theorem 3.6. *For any $n \geqslant 3$, deciding ℓ-subsumption with respect to a TBox in Ł$_n$-$\mathcal{EL}^+$ is 2-*ExpTime*-hard.*

Proof. In [2], a series of polynomial reductions from satisfiability in the classical DL $\mathcal{ALC}$ to subsumption in $\mathcal{ELU}$ are provided. It is easy to see that the same reductions can be used in the presence of complex role inclusions. Since satisfiability in $\mathcal{ALC}$ with complex role inclusions is 2-ExpTime-hard [36], it thus suffices to extend the reduction of the previous section to complex role inclusions.

Using the same notation as before, we extend the translation ρ_n to complex role inclusions:

$$\rho_n(r_1 \circ \cdots \circ r_n \sqsubseteq r) \coloneqq \langle r_1 \circ \cdots \circ r_n \sqsubseteq r \geqslant 1\rangle.$$

The constructions used to show soundness and completeness remain the same as before; the only thing left to prove are the missing cases of Lemmas 3.1 and 3.3 for role inclusions:

- We need to show that, if an Ł_n-interpretation $\mathcal{I}$ satisfies the fuzzy role inclusion above, then $\mathcal{I}_{cr}$ satisfies the original axiom. Hence, assume that $(x,y) \in r_1^{\mathcal{I}_{cr}} \circ \cdots \circ r_n^{\mathcal{I}_{cr}}$, i.e. there are domain elements $x_1, \dots, x_{n-1} \in \Delta^{\mathcal{I}}$ with $(x, x_1) \in r_1^{\mathcal{I}_{cr}}, \dots, (x_{n-1}, y) \in r_n^{\mathcal{I}_{cr}}$. By the definition of $\mathcal{I}_{cr}$, we know that $r_1^{\mathcal{I}}(x,x_1) = \cdots = r_n^{\mathcal{I}}(x_{n-1},y) = 1$, and hence $r^{\mathcal{I}}(x,y)$ since $\mathcal{I}$ satisfies the fuzzy axiom. The definition of $\mathcal{I}_{cr}$ thus yields $(x,y) \in r^{\mathcal{I}_{cr}}$, which shows that $\mathcal{I}_{cr}$ satisfies the classical role inclusion.
- Assume now that a classical interpretation $\mathcal{I}$ satisfies $r_1 \circ \cdots \circ r_n \sqsubseteq r$. We show that $\mathcal{I}_n$ satisfies its translation under ρ_n. If $(r_1 \circ \cdots \circ r_n)^{\mathcal{I}_n}(x,y) = 1$, then there must exist elements $x_1, \dots, x_{n-1}$ such that $r_1^{\mathcal{I}_n}(x,x_1) = 1, \dots, r_n^{\mathcal{I}_n}(x_{n-1},y) = 1$. By the definition of $\mathcal{I}_n$, we get $(x,x_1) \in r_1^{\mathcal{I}}, \dots, (x_{n-1},y) \in r_n^{\mathcal{I}}$, and hence $(x,y) \in r^{\mathcal{I}}$ by our assumption. This shows that $r^{\mathcal{I}_n}(x,y) = 1$, as required.
  In the case that $(r_1 \circ \cdots \circ r_n)^{\mathcal{I}_n}(x,y) < 1$, we obtain $(r_1 \circ \cdots \circ r_n)^{\mathcal{I}_n}(x,y) \leqslant \frac{n-2}{n-1} \leqslant r^{\mathcal{I}_n}(x,y)$ by the definition of $\mathcal{I}_n$. Hence, the translated axiom is also satisfied in this case. □

For classical DLs with disjunction, a matching upper bound is only known for *regular* sets of complex role inclusions. Under this restriction, reasoning in even very expressive DLs, in particular $\mathcal{EL}$ extended with complex role inclusions, disjunction, inverse roles, nominals, and bottom is in 2-ExpTime [37]. This upper bound can be transferred to the corresponding finitely valued FDLs using generic (polynomial) reductions from fuzzy to classical TBoxes, such as the one described in [35], which is based on the ideas first presented in [17]. Since these reductions introduce disjunctions, they cannot be applied to derive a polynomial upper bound for Ł_n-$\mathcal{EL}^+$ from the known polynomial complexity of classical $\mathcal{EL}^+$.

Without the restriction to regular role inclusions, an upper bound is not known; it is still open whether reasoning in $\mathcal{ELU}^+$ is even decidable. If this logic is further extended with inverse roles, i.e. to $\mathcal{ELUI}^+$, then reasoning even becomes undecidable [30].[5] Hence, also subsumption in Ł_n-$\mathcal{ELI}^+$ (with non-regular role inclusions) is undecidable, which can be shown by a similar reduction as above (the presence of inverse roles does not affect the arguments at all).

Theorem 3.7. *For any $n \geqslant 3$, deciding ℓ-subsumption with respect to a regular TBox in Ł_n-$\mathcal{ELUOI}^+_\bot$ is* 2-ExpTime*-complete. The problem is undecidable in Ł_n-$\mathcal{ELI}^+$ with unrestricted role inclusions.*

On the other hand, if we do not allow complex role inclusions, then the other constructors that make reasoning in classical $\mathcal{EL}$ exponentially harder, e.g. disjunction or inverse roles, do not affect the complexity of Ł_n-$\mathcal{EL}$. In essence, this is because subsumption in Ł_n-$\mathcal{EL}$ is already ExpTime-hard. These results again follow from known upper bounds for more expressive FDLs [38].

Theorem 3.8. *For any $n \geqslant 3$, deciding ℓ-subsumption with respect to a TBox in Ł_n-$\mathcal{ELUOI}_\bot$ is* ExpTime-*complete.*

3.5. Arbitrary Finite Chains

The hardness results presented in the previous subsection can in fact be extended to cover almost all logics of the form L-$\mathcal{EL}$ (or L-$\mathcal{EL}^+$) where L is a finite chain. The only exceptions are the finite chains using the minimum as t-norm—this case can be shown to be tractable following the arguments from [23]. As detailed in Section 2, any chain L that is not of this special form must contain a finite Łukasiewicz chain in an interval $[a,b]$ with at least three elements. This is the starting point of our reduction to the results from the previous section (see Figure 1). More formally, we reduce the subsumption problem in Ł_n-$\mathcal{EL}$ (resp., Ł_n-$\mathcal{EL}^+$), where $n \geqslant 3$ is the cardinality of $[a,b]$, to the subsumption problem in L-$\mathcal{EL}$ (resp., L-$\mathcal{EL}^+$).

In the following, let $\mathcal{T}$ be a TBox in Ł_n-$\mathcal{EL}^+$, $\ell \in \text{Ł}_n \setminus \{0\}$, and A, B two concept names for which we want to check whether A is ℓ-subsumed by B w.r.t. $\mathcal{T}$. We consider the inverse λ^{-1} of the bijection $\lambda\colon [a,b] \to \text{Ł}_n$ that exists due to the ordinal sum representation of L (see Definition 2.3). We define the new L-$\mathcal{EL}$ TBox $\mathcal{T}'$ based on $\mathcal{T}$ as follows:

$$\mathcal{T}' := \{\langle \top \sqsubseteq B \geqslant a\rangle\} \cup \{\langle \alpha \sqsubseteq \beta \geqslant \lambda^{-1}(p)\rangle \mid \langle \alpha \sqsubseteq \beta \geqslant p\rangle \in \mathcal{T}\}.$$

[5] The proof in [30] also uses functionality restrictions on roles, which are, however, not necessary to show undecidability.

Recall that B is one of the concept names for which we want to check subsumption in $Ł_n$-$\mathcal{EL}$, and a is the lower bound of the interval $[a, b]$ in L. The full proof of the following lemma is very technical, and can be found in Appendix A. Here we present only a brief sketch.

Lemma 3.9. *A is ℓ-subsumed by B w.r.t. $\mathcal{T}$ iff A is $\lambda^{-1}(\ell)$-subsumed by B w.r.t. $\mathcal{T}'$.*

Proof (sketch). The main insight is that the bijections λ and λ^{-1} are compatible with the t-norm and residuum of the two chains L and $Ł_n$ (under suitable restrictions). Based on this, one can show that they are also compatible with the semantics of all concepts (top, conjunctions, existential restrictions) and axioms (concept and role inclusions). In this way, λ and λ^{-1} can be used to show completeness and soundness, respectively, of our reduction. □

We can now generalize Theorems 3.5, 3.7, and 3.8, where the upper bounds again follow from known results [10, 17, 35, 37, 38]. Observe that, if $\mathcal{T}$ is an $Ł_n$-$\mathcal{EL}$ TBox, then $\mathcal{T}'$ is an L-$\mathcal{EL}$ TBox, i.e. it also does not contain role inclusions.

Theorem 3.10. *Let L be a finite chain that is not of the form G_n. Then deciding ℓ-subsumption with respect to a TBox in any logic between L-$\mathcal{EL}$ and L-$\mathcal{ELUOI}_\perp$ is* ExpTime*-complete. The problem is* 2-ExpTime*-complete in any logic between L-$\mathcal{EL}^+$ and L-$\mathcal{ELUOI}_\perp^+$ with regular role inclusions, and undecidable in L-$\mathcal{ELI}^+$ with unrestricted role inclusions.*

In contrast, subsumption for fuzzy extensions of $\mathcal{EL}$ based on G_n, $n \geqslant 2$, can be shown to be in the same complexity classes as for the underlying classical DLs, using the polynomial reductions of fuzzy TBoxes to classical TBoxes described in [31, 35]. In particular, subsumption in G_n-$\mathcal{ELO}_\perp^+$ remains polynomial [2, 32, 33], for G_n-$\mathcal{ELU}$ and G_n-$\mathcal{ELI}$ it becomes ExpTime-complete [2, 34], and in G_n-$\mathcal{ELU}^+$ with regular role inclusions the complexity increases to 2-ExpTime [36, 37].

4. The Final Nail in the Coffin of the Infinite Łukasiewicz T-norm

In this section we consider the case where L is defined over the interval $[0, 1]$. We show that in this case, subsumption in the very inexpressive L-$\mathcal{EL}$ becomes undecidable whenever L *contains* the Łukasiewicz t-norm. Interestingly, this is the first instance of undecidability for a fuzzy description logic that does not allow for any negation constructor. Indeed, the required expressivity is a consequence of the properties of the Łukasiewicz t-norm itself. We emphasize once again that this result covers a large class of cases, as it holds for any continuous t-norm containing the Łukasiewicz t-norm.

Rather than proving undecidability of this problem directly, we take advantage of the general framework recently developed for FDLs [8, 39]. This framework, as all previously existing undecidability proofs for FDLs [6, 7], considers a different but related decision problem; namely, *ontology consistency* in the presence of concept assertions. Thus, we prove first that ontology consistency in the logic L-$\mathcal{EL}_=$ (which extends L-$\mathcal{EL}$ with so-called *equality assertions*) is undecidable if L *starts with* the Łukasiewicz t-norm, and even if all GCIs are restricted to be crisp. Afterwards, we adapt this result to cover also subsumption in L-$\mathcal{EL}$ (without equality assertions, but with non-crisp GCIs), and use a generic result from [22] to remove the restriction requiring that the Łukasiewicz component has to be the first component of L.

4.1. Undecidability of Consistency in L-$\mathcal{EL}_=$

For this section, we consider an arbitrary but fixed infinite chain L that starts with the Łukasiewicz t-norm. That is, there is an interval of the form $[0, b]$ over which the t-norm is isomorphic to $*_Ł$. The logic L-$\mathcal{EL}_=$ is an extension of L-$\mathcal{EL}$ where an ontology, in addition to a TBox, may also contain finitely many *equality assertions* of the form $\langle a{:}C = \ell\rangle$, where C is a concept, $\ell \in \mathsf{L}$, and a is an *individual name* from a countable set $\mathsf{N_I}$ disjoint from $\mathsf{N_C}$ and $\mathsf{N_R}$.[6] The definition of an L-interpretation $\mathcal{I}$ is extended to map

[6]In general, also assertions about role connections between individual names are allowed (see [5]). We do not consider them here since they are not necessary to show undecidability.

each individual name $a \in \mathsf{N_I}$ to a domain element $a^{\mathcal{I}} \in \Delta^{\mathcal{I}}$. Such an L-interpretation *satisfies* the equality assertion from above if $C^{\mathcal{I}}(a^{\mathcal{I}}) = \ell$. An ontology (a finite set of GCIs and assertions) is *consistent* if there exists an L-interpretation that satisfies all its axioms. In this section, we use only crisp GCIs of the form $\langle C \sqsubseteq D \rangle$.

We prove that consistency of L-$\mathcal{EL}_=$ ontologies is undecidable using the framework from [8]. We now briefly describe the notions of this framework required for the present paper. For the full details, we refer the reader to the original work [8]. According to this framework, undecidability can be shown by proving that a given logic satisfies several properties, which together allow for the construction of an ontology simulating instances of the undecidable Post Correspondence Problem (PCP) [40]. Hence, we consider an arbitrary but fixed instance $\mathcal{P}$ of the PCP, which consists of pairs $(v_1, w_1), \dots, (v_n, w_n)$ of words over an alphabet Σ of the form $\{1, \dots, s\}$ for a natural number $s > 1$. The problem is to find a *solution* of $\mathcal{P}$, which is a finite sequence of the form $i_1 \dots i_k \in \{1, \dots, n\}^*$ such that $v_1 v_{i_1} \dots v_{i_k} = w_1 w_{i_1} \dots w_{i_k}$.[7] For a candidate solution $\nu \in \{1, \dots, n\}^*$, we denote these two words by v_ν and w_ν, respectively.

The first requirement of the framework is to provide a *valid encoding function* $\mathsf{enc}\colon \Sigma_0^* \to 2^{[0,1]}$ that allows us to represent words over $\Sigma_0 := \Sigma \cup \{0\}$ as (sets of) truth degrees. For L-$\mathcal{EL}_=$, recall that L contains the Łukasiewicz t-norm over the interval $[0, b]$. We define the encoding function enc as follows:

$$\mathsf{enc}(u) := \begin{cases} [b, 1] & \text{if } u \in \{0\}^* \\ \left\{ b\left(1 - \frac{1}{2} 0.\overleftarrow{u}\right) \right\} & \text{otherwise,} \end{cases}$$

where $\overleftarrow{u}$ is the word u written in reverse and interpreted as a sequence of digits in base $s+1$. In the latter case—when $u \notin \{0\}^*$—we sometimes treat $\mathsf{enc}(u)$ as a single value in $[0, 1]$. For example, if $s = 9$, then we have $\mathsf{enc}(1) = 0.95b$ and $\mathsf{enc}(81) = 0.91b$. It is an important property of this function that the encoding of every word is always strictly greater than $\frac{b}{2}$.

For this encoding function to be *valid*, it must satisfy the following condition: There must exist two words $u_\varepsilon, u_+ \in \Sigma_0^*$ such that for every candidate solution $\nu \in \{1, \dots, n\}^*$ and all encodings $p \in \mathsf{enc}(v_\nu)$, $q \in \mathsf{enc}(w_\nu)$ and $m \in \mathsf{enc}(u_\varepsilon \cdot u_+^{|\nu|})$ it holds that $u_\varepsilon \cdot u_+^{|\nu|} \in \{\varepsilon\} \cup \Sigma\Sigma_0^*$ (that is, it is either the empty word or a sequence not starting with 0), and $v_\nu \neq w_\nu$ iff $\min\{p \Rightarrow q, q \Rightarrow p\} \leqslant m$. That is, we can use an encoding of $u_\varepsilon \cdot u_+^c$ to check the equality of any two words v_ν, w_ν belonging to a candidate solution ν of length c.

Lemma 4.1. *The function* enc *is a valid encoding function (according to [8, Definition 11]).*

Proof. Given an instance $\mathcal{P}$ of the PCP, let k be the maximal length of any word v_i, w_i appearing in $\mathcal{P}$. Choose $u_\varepsilon := 1 \cdot 0^k$—that is, the word consisting of the digit 1 followed by k zeros—and $u_+ := 0^k$. It can be verified as in [8] that the two required conditions hold over these words. In particular, if $v_\nu \neq w_\nu$, then these words must differ in one of the first $K := (|\nu| + 1)k$ letters. Thus, either $\mathsf{enc}(v_\nu) > \mathsf{enc}(w_\nu)$, and hence

$$\mathsf{enc}(v_\nu) \Rightarrow \mathsf{enc}(w_\nu) = \min\left\{b, b(1 + \tfrac{1}{2} 0.\overleftarrow{v_\nu} - \tfrac{1}{2} 0.\overleftarrow{w_\nu})\right\} \leqslant b\left(1 - \tfrac{1}{2} 0.\overleftarrow{1 \cdot 0^K}\right) = \mathsf{enc}\left(u_\varepsilon \cdot u_+^{|\nu|}\right),$$

or else $\mathsf{enc}(v_\nu) < \mathsf{enc}(w_\nu)$ and $\mathsf{enc}(w_\nu) \Rightarrow \mathsf{enc}(v_\nu) \leqslant \mathsf{enc}\left(u_\varepsilon \cdot u_+^{|\nu|}\right)$. □

Based on this encoding function, the following *canonical model* $\mathcal{I}_\mathcal{P}$ of $\mathcal{P}$ is used to encode the *search tree* for a solution of $\mathcal{P}$, as illustrated in Figure 2:

- the domain $\Delta^{\mathcal{I}_\mathcal{P}} := \{1, \dots, n\}^*$ contains all candidate solutions for $\mathcal{P}$;
- we set $a^{\mathcal{I}_\mathcal{P}} := \varepsilon$ for a certain individual name a that denotes the root node of the search tree;
- $V^{\mathcal{I}_\mathcal{P}}(\nu) := \mathsf{enc}(v_\nu)$ and $W^{\mathcal{I}_\mathcal{P}}(\nu) := w_\nu$ represent the words v_ν and w_ν, respectively, of the candidate solution at a node $\nu \in \{1, \dots, n\}^*$;

[7] Without loss of generality we can restrict the search to solutions that start with v_1 and w_1.

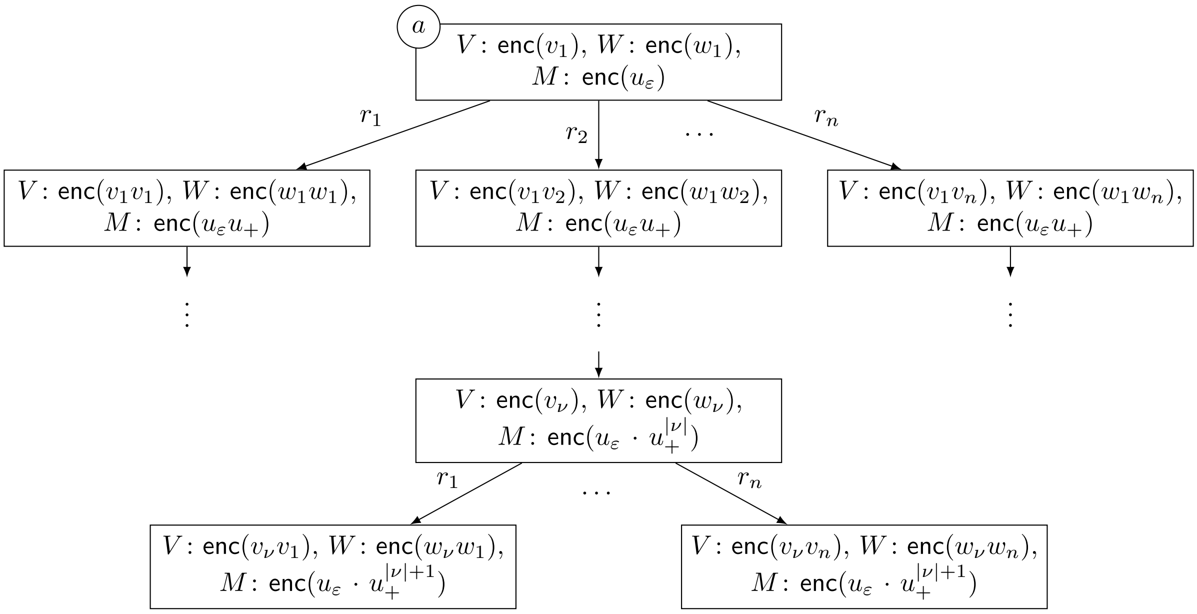

Figure 2: The canonical model $\mathcal{I}_\mathcal{P}$ for an instance $\mathcal{P}$ of the PCP (taken from [8]).

- $V_i^{\mathcal{I}_\mathcal{P}}(\nu) \coloneqq \mathsf{enc}(v_i)$ and $W_i^{\mathcal{I}_\mathcal{P}}(\nu) \coloneqq \mathsf{enc}(w_i)$ for $i \in \{1, \dots, n\}$ encode the words v_i and w_i, respectively, at every node of the search tree;
- $M^{\mathcal{I}_\mathcal{P}}(\nu) \coloneqq \mathsf{enc}(u_\varepsilon \cdot u_+^{|\nu|})$ and $M_+^{\mathcal{I}_\mathcal{P}}(\nu) \coloneqq \mathsf{enc}(u_+)$ encode the words used to compare v_ν and w_ν;
- $r_i^{\mathcal{I}_\mathcal{P}}(\nu, \nu i) \coloneqq 1$ and $r_i^{\mathcal{I}_\mathcal{P}}(\nu, \nu') \coloneqq 0$ for $\nu' \neq \nu i$ are used to distinguish the successors in the search tree;
- $H^{\mathcal{I}_\mathcal{P}}(\nu) \coloneqq h$ is an auxiliary concept name that has a constant value $h \in [0, 1]$ everywhere.

Strictly speaking, this construction is slightly different from the one described in [8], since the original construction does not contain the concept name H. It is easy to show, however, that this change does not affect the correctness of the approach.

In the following, let $p \sim q$ denote the fact that both p and q belong to the same set $\mathsf{enc}(u)$ for some word $u \in \Sigma_0^*$. We first want to show that $\mathsf{L}\text{-}\mathcal{EL}_=$ is capable of constructing the canonical model, as expressed by the following property.

The Canonical Model Property:

There is an ontology $\mathcal{O}_\mathcal{P}$ such that every model $\mathcal{I}$ of $\mathcal{O}_\mathcal{P}$ admits a mapping $g\colon \Delta^{\mathcal{I}_\mathcal{P}} \to \Delta^\mathcal{I}$ that satisfies

$$A^{\mathcal{I}_\mathcal{P}}(\nu) \sim A^\mathcal{I}(g(\nu)) \quad \text{and} \quad H^\mathcal{I}(g(\nu)) = h$$

for every concept name $A \in \{V, W, M, M_+\} \cup \bigcup_{i=1}^n \{V_i, W_i\}$ and $\nu \in \{1, \dots, n\}^*$.

In other words, the ontology $\mathcal{O}_\mathcal{P}$ required by this property enforces that the canonical model can be *embedded* into every L-interpretation satisfying it. As shown in [8, Theorem 12], the canonical model property is implied by the following four simpler properties, which are used, in that order, to initialize the values of the concept names at the root node, to enforce the existence of the r_i-successors, to construct the encodings of the next candidate solutions $(v_{\nu i}, w_{\nu i})$ by concatenation, and to transfer these encodings along the r_i-connections to the successors.

The Initialization Property:

Let C be a concept, $a \in \mathsf{N_I}$, and $u \in \Sigma_0^*$. There is an ontology $\mathcal{O}$ such that for every model $\mathcal{I}$ of $\mathcal{O}$ it holds that $C^{\mathcal{I}}(a^{\mathcal{I}}) \in \mathsf{enc}(u)$.

The Successor Property:

Let $r \in \mathsf{N_R}$. There is an ontology $\mathcal{O}$ such that for every model $\mathcal{I}$ of $\mathcal{O}$ and every $x \in \Delta^{\mathcal{I}}$ with $H^{\mathcal{I}}(x) = h$ there is a $y \in \Delta^{\mathcal{I}}$ with $r^{\mathcal{I}}(x,y) \geqslant b$ and $H^{\mathcal{I}}(y) = h$.

The Concatenation Property:

Let $u \in \Sigma_0^*$, and C and C_u be concepts. There is an ontology $\mathcal{O}$ and a concept name D such that for every model $\mathcal{I}$ of $\mathcal{O}$ and every $x \in \Delta^{\mathcal{I}}$, if $C_u^{\mathcal{I}}(x) \in \mathsf{enc}(u)$ and $C^{\mathcal{I}}(x) \in \mathsf{enc}(u')$ for some $u' \in \{\varepsilon\} \cup \Sigma\Sigma_0^*$, then $D^{\mathcal{I}}(x) \in \mathsf{enc}(u'u)$.

The Transfer Property:

Let C, D be concepts and $r \in \mathsf{N_R}$. There is an ontology $\mathcal{O}$ such that for every model $\mathcal{I}$ of $\mathcal{O}$ and every $x, y \in \Delta^{\mathcal{I}}$, if $H^{\mathcal{I}}(x) = h$, $r^{\mathcal{I}}(x,y) \geqslant b$, $H^{\mathcal{I}}(y) = h$, and $C^{\mathcal{I}}(x) \in \mathsf{enc}(u)$ for some $u \in \Sigma_0^*$, then also $D^{\mathcal{I}}(y) \in \mathsf{enc}(u)$.

We show that $\mathsf{L}\text{-}\mathcal{EL}_=$ satisfies all these properties.

Lemma 4.2. *$\mathsf{L}\text{-}\mathcal{EL}_=$ has the initialization, successor, concatenation, and transfer properties, and hence the canonical model property.*

Proof. The initialization property follows trivially from the availability of equality assertions: given $u \in \Sigma_0^*$, choose any $\ell \in \mathsf{enc}(u)$ and set $\mathcal{O} := \{\langle a{:}C = \ell\rangle\}$.

For successor property, we choose $h := \frac{b}{2}$ as the constant for the concept name H, and consider the TBox $\{\langle H \equiv G^2\rangle, \langle G \sqsubseteq \exists r.G\rangle, \langle \exists r.H \sqsubseteq H\rangle\}$. Since we assume that $H^{\mathcal{I}}(x) = \frac{b}{2}$, the first axiom yields that $G^{\mathcal{I}}(x) = \frac{3b}{4}$. Then, by the second axiom and the assumption that our interpretations are witnessed, we find an element $y \in \Delta^{\mathcal{I}}$ such that $\frac{3b}{4} \leqslant r^{\mathcal{I}}(x,y) *_{\L} G^{\mathcal{I}}(y)$. By the third axiom, $r^{\mathcal{I}}(x,y) *_{\L} (G^2)^{\mathcal{I}}(y) \leqslant (\exists r.H)^{\mathcal{I}}(x) \leqslant H^{\mathcal{I}}(x) = \frac{b}{2}$. Since $*_{\L}$ is monotone (Proposition 2.2(b)), we get

$$r^{\mathcal{I}}(x,y) *_{\L} (G^2)^{\mathcal{I}}(y) \leqslant \tfrac{b}{2} = \tfrac{3b}{4} *_{\L} \tfrac{3b}{4} \leqslant r^{\mathcal{I}}(x,y) *_{\L} r^{\mathcal{I}}(x,y) *_{\L} (G^2)^{\mathcal{I}}(y) \tag{2}$$

This implies that $r^{\mathcal{I}}(x,y) \geqslant b$, since otherwise we would have

$$r^{\mathcal{I}}(x,y) *_{\L} \big(r^{\mathcal{I}}(x,y) *_{\L} (G^2)^{\mathcal{I}}(y)\big) \leqslant r^{\mathcal{I}}(x,y) *_{\L} \tfrac{b}{2} = \max\{0, r^{\mathcal{I}}(x,y) - \tfrac{b}{2}\} < \tfrac{b}{2},$$

in contradiction to (2). From this, we obtain $H^{\mathcal{I}}(y) = (G^2)^{\mathcal{I}}(y) = \frac{b}{2}$, as required.

For the concatenation property, consider the TBox $\{\langle C'^{(s+1)^{|u|}} \equiv C\rangle, \langle D \equiv C' \sqcap C_u\rangle\}$, where C' is a fresh auxiliary concept name. Let $\mathcal{I}$ be a model of these axioms and $x \in \Delta^{\mathcal{I}}$ such that $C_u^{\mathcal{I}}(x) \in \mathsf{enc}(u)$ and $C^{\mathcal{I}}(x) \in \mathsf{enc}(u')$ for some $u' \in \{\varepsilon\} \cup \Sigma\Sigma_0^*$. Suppose that $u' \neq \varepsilon$. Then from the first axiom it follows that both $C^{\mathcal{I}}(x)$ and $C'^{\mathcal{I}}(x)$ belong to the interval $(0,b)$, and hence $C'^{\mathcal{I}}(x) = b\Big(1 - \frac{(s+1)^{-|u|}}{2} 0.\overleftarrow{u'}\Big)$. If $u \notin \{0\}^*$, then

$$D^{\mathcal{I}}(x) = b\Big(1 - \tfrac{1}{2}0.\overleftarrow{u} - \tfrac{(s+1)^{-|u|}}{2}0.\overleftarrow{u'}\Big) = \mathsf{enc}(u'u).$$

Otherwise, $C_u^{\mathcal{I}}(x) \in [b,1]$ and thus $D^{\mathcal{I}}(x) = C'^{\mathcal{I}}(x) = \mathsf{enc}(u'u)$. In case that $u' = \varepsilon$, then $C^{\mathcal{I}}(x) \in [b,1]$ which implies that $C'^{\mathcal{I}}(x) \in [b,1]$, and by the second axiom $D^{\mathcal{I}}(x) \in \mathsf{enc}(u) = \mathsf{enc}(\varepsilon u)$.

We now look at the transfer property. Consider concepts C, D and a role name r, and let $\overline{C}$ be a fresh concept name. For every model $\mathcal{I}$ of $\langle H \equiv C \sqcap \overline{C}\rangle$ and every $x \in \Delta^{\mathcal{I}}$ with $H^{\mathcal{I}}(x) = \frac{b}{2}$ and $C^{\mathcal{I}}(x) < b$ we have

$C^{\mathcal{I}}(x) + \overline{C}^{\mathcal{I}}(x) - b = H^{\mathcal{I}}(x) = \frac{b}{2} > 0$, and hence $\overline{C}^{\mathcal{I}}(x) = \frac{3b}{2} - C^{\mathcal{I}}(x)$. That is, $\overline{C}$ simulates an involutive negation of C over the interval $(\frac{b}{2}, b)$. In the case that $C^{\mathcal{I}}(x) = [b, 1]$, the above axiom implies $\overline{C}^{\mathcal{I}}(x) = \frac{b}{2}$. We can hence summarize the effect of this axiom as $\overline{C}^{\mathcal{I}}(x) = \frac{3b}{2} - \min\{b, C^{\mathcal{I}}(x)\}$. Since $H^{\mathcal{I}}(y) = \frac{b}{2}$, we can use a similar construction as above to obtain a concept $\overline{D}$ that simulates a kind of involutive negation of D at y; as a side effect, the axiom $\langle H \equiv D \sqcap \overline{D}\rangle$ already enforces that $D^{\mathcal{I}}(y) \geqslant \frac{b}{2}$.

Consider now the axioms $\langle \exists r.D \sqsubseteq C\rangle$ and $\langle \exists r.\overline{D} \sqsubseteq \overline{C}\rangle$. The first axiom implies that $D^{\mathcal{I}}(y) \leqslant C^{\mathcal{I}}(x)$. From the second axiom, we similarly get that

$$\tfrac{3b}{2} - \min\{b, D^{\mathcal{I}}(y)\} = \overline{D}^{\mathcal{I}}(y) \leqslant \overline{C}^{\mathcal{I}}(x) = \tfrac{3b}{2} - \min\{b, C^{\mathcal{I}}(x)\}.$$

If $C^{\mathcal{I}}(x) = \mathsf{enc}(u) < b$, then $C^{\mathcal{I}}(x) \leqslant D^{\mathcal{I}}(y)$, and hence $D^{\mathcal{I}}(y) = C^{\mathcal{I}}(x) = \mathsf{enc}(u)$. Otherwise, it must hold that $C^{\mathcal{I}}(x) \in [b, 1] = \mathsf{enc}(u)$, and thus $\overline{C}^{\mathcal{I}}(x) = \frac{b}{2}$, which implies that $\overline{D}^{\mathcal{I}}(x) = \frac{b}{2}$, and hence also $D^{\mathcal{I}}(x) \in [b, 1] = \mathsf{enc}(u)$.

Overall, the TBox $\{\langle H \equiv C \sqcap \overline{C}\rangle, \langle H \equiv D \sqcap \overline{D}\rangle, \langle \exists r.D \sqsubseteq C\rangle, \langle \exists r.\overline{D} \sqsubseteq \overline{C}\rangle\}$ has the desired properties. $\square$

A logic satisfying the canonical model property means that it is possible to create an ontology $\mathcal{O}_{\mathcal{P}}$ whose models all embed the search tree for a solution of a PCP instance $\mathcal{P}$. To obtain undecidability, one needs to further guarantee that the existence of such a solution can be decided. We achieve this through the following property, which intuitively states that no node of the search tree is a solution; thus, the ontology is inconsistent if and only if $\mathcal{P}$ has a solution.

The Solution Property:

$\mathcal{I}_{\mathcal{P}}$ can be extended to a model of $\mathcal{O}_{\mathcal{P}}$ and there is an ontology $\mathcal{O}$ such that:

1. For every model $\mathcal{I}$ of $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}$ and every $\nu \in \{1, \dots, n\}^*$,

$$\min\left\{V^{\mathcal{I}}(g(\nu)) \Rightarrow W^{\mathcal{I}}(g(\nu)),\ W^{\mathcal{I}}(g(\nu)) \Rightarrow V^{\mathcal{I}}(g(\nu))\right\} \leqslant M^{\mathcal{I}}(g(\nu)).$$

2. If for every $\nu \in \{1, \dots, n\}^*$ we have

$$\min\left\{V^{\mathcal{I}_{\mathcal{P}}}(\nu) \Rightarrow W^{\mathcal{I}_{\mathcal{P}}}(\nu),\ W^{\mathcal{I}_{\mathcal{P}}}(\nu) \Rightarrow V^{\mathcal{I}_{\mathcal{P}}}(\nu)\right\} \leqslant M^{\mathcal{I}_{\mathcal{P}}}(\nu),$$

then $\mathcal{I}_{\mathcal{P}}$ can be extended to a model of $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}$.

It is usually easy to show the first part, that $\mathcal{I}_{\mathcal{P}}$ can be extended to a model of $\mathcal{O}_{\mathcal{P}}$: one just has to be careful that none of the auxiliary concepts introduced in the constructions are unsatisfiable or contradict the shape of $\mathcal{I}_{\mathcal{P}}$ in any way (e.g. by restricting the value of V to be below $\frac{b}{2}$, which cannot be a valid encoding of a word). The remaining conditions again require a more intricate proof.

Lemma 4.3. *L-$\mathcal{EL}_{=}$ has the solution property.*

Proof. Consider the ontology

$$\begin{aligned} \mathcal{O} := \{&\langle X^2 \sqsubseteq X^3\rangle,\ \langle H \equiv X \sqcap \overline{X}\rangle, && (3)\\ &\langle X^2 \sqcap V \sqsubseteq W \sqcap M\rangle, && (4)\\ &\langle \overline{X}^2 \sqcap W \sqsubseteq V \sqcap M\rangle\}. && (5)\end{aligned}$$

Since $H^{\mathcal{I}}(g(\nu)) = \frac{b}{2}$, the canonical model property implies that for every model $\mathcal{I}$ of (3) it holds that $X^{\mathcal{I}}(g(\nu)) \in \{\frac{b}{2}\} \cup [b, 1]$ and $\overline{X}^{\mathcal{I}}(g(\nu)) \in \{\frac{b}{2}\} \cup [b, 1]$ (see Proposition 2.2(g)). Furthermore, X and $\overline{X}$ complement each other in the sense that $X^{\mathcal{I}}(g(\nu)) = \frac{b}{2}$ iff $\overline{X}^{\mathcal{I}}(g(\nu)) \in [b, 1]$.

Let now $\mathcal{I}$ be a model of $\mathcal{O}_{\mathcal{P}} \cup \mathcal{O}$ and $\nu \in \{1, \dots, n\}^*$. If $X^{\mathcal{I}}(g(\nu)) \geqslant b$, then axiom (4) requires that $V^{\mathcal{I}}(g(\nu)) \leqslant W^{\mathcal{I}}(g(\nu)) *_{\mathsf{L}} M^{\mathcal{I}}(g(\nu))$, while axiom (5) is trivially satisfied. Let $K := (|\nu| + 1)k$, where k is as in the proof of Lemma 4.1. If $w_\nu \neq \varepsilon$, then since $|w_\nu| \leqslant K$ it follows that

$$W^{\mathcal{I}}(g(\nu)) *_{\mathsf{L}} M^{\mathcal{I}}(g(\nu)) = b(1 - \tfrac{1}{2}0.\overleftarrow{w_\nu} - \tfrac{1}{2}0.\overleftarrow{1 \cdot 0^K}) \in (\tfrac{b}{2}, b).$$

In the case that $w_\nu = \varepsilon$, the term $\frac{1}{2}0.\overleftarrow{w_\nu}$ disappears, but we still have

$$W^{\mathcal{I}}(g(\nu)) *_{\mathsf{L}} M^{\mathcal{I}}(g(\nu)) = M^{\mathcal{I}}(g(\nu)) \in (\tfrac{b}{2}, b).$$

Thus, for any $z > M^{\mathcal{I}}(g(\nu))$ it holds that $W^{\mathcal{I}}(g(\nu)) *_{\mathsf{L}} z > W^{\mathcal{I}}(g(\nu)) *_{\mathsf{L}} M^{\mathcal{I}}(g(\nu)) \geqslant V^{\mathcal{I}}(g(\nu))$. Hence,

$$\begin{aligned} W^{\mathcal{I}}(g(\nu)) \Rightarrow V^{\mathcal{I}}(g(\nu)) &= \sup\{z \in [0,1] \mid W^{\mathcal{I}}(g(\nu)) *_{\mathsf{L}} z \leqslant V^{\mathcal{I}}(g(\nu))\} \\ &\leqslant \inf\{z \in [0,1] \mid z > M^{\mathcal{I}}(g(\nu))\} \\ &= M^{\mathcal{I}}(g(\nu)). \end{aligned}$$

Dually, if $X^{\mathcal{I}}(g(\nu)) = \frac{b}{2}$, then $V^{\mathcal{I}}(g(\nu)) \Rightarrow W^{\mathcal{I}}(g(\nu)) \leqslant M^{\mathcal{I}}(g(\nu))$. Thus,

$$\min\{V^{\mathcal{I}}(g(\nu)) \Rightarrow W^{\mathcal{I}}(g(\nu)),\ W^{\mathcal{I}}(g(\nu)) \Rightarrow V^{\mathcal{I}}(g(\nu))\} \leqslant M^{\mathcal{I}}(g(\nu)).$$

For the last part of the property, let $\min\{V^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow W^{\mathcal{I}_\mathcal{P}}(\nu), W^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow V^{\mathcal{I}_\mathcal{P}}(\nu)\} \leqslant M^{\mathcal{I}_\mathcal{P}}(\nu) < 1$ and consider an extension $\mathcal{I}$ of $\mathcal{I}_\mathcal{P}$ that satisfies $\mathcal{O}_\mathcal{P}$. We show that $\mathcal{I}$ can be extended to satisfy $\mathcal{O}$. The only required extension is to provide the adequate interpretation of the concept names $X, \overline{X}$ on elements $\nu \in \{1, \dots, n\}^*$. If $V^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow W^{\mathcal{I}_\mathcal{P}}(\nu) = 1$, set $X^{\mathcal{I}}(\nu) := b$, which requires $\overline{X}^{\mathcal{I}}(\nu) := \frac{b}{2}$ and trivially satisfies axiom (5). We must then have $W^{\mathcal{I}_\mathcal{P}}(\nu) \Rightarrow V^{\mathcal{I}_\mathcal{P}}(\nu) \leqslant M^{\mathcal{I}_\mathcal{P}}(\nu)$, which shows that axiom (4) is also satisfied. Otherwise, $X^{\mathcal{I}}(\nu) := \frac{b}{2}$ provides the desired result. □

As a consequence of Lemmas 4.2 and 4.3, the undecidability framework guarantees that consistency in L-$\mathcal{EL}_=$ is undecidable (see [8, Theorem 13]).

Theorem 4.4. *Ontology consistency in L-$\mathcal{EL}_=$ is undecidable if L starts with the Łukasiewicz t-norm. This result holds even even if all GCIs are crisp.*

We now turn our attention to the problem of subsumption.

4.2. Undecidability of Subsumption in L-$\mathcal{EL}$

We now use the result from Theorem 4.4 to show that subsumption in L-$\mathcal{EL}$ is undecidable if L starts with the Łukasiewicz t-norm. By known results [22], this immediately implies that undecidability holds for any chain that *contains* the Łukasiewicz t-norm.

Notice first that in the construction from Section 4.1, the equality assertions are only used to satisfy the initialization property. In the general proof of undecidability from [8], this property is used to ensure that a can serve as the root of the search tree for the PCP instance, thus requiring the initialization of the interpretation of several concept names (V, W, M, etc.) at a (see Figure 2). Using this insight, we show that undecidability arises already if *only one* equality assertion is allowed in the ontology. To prove this, it suffices to show that the initialization property can also be obtained using one fixed equality assertion. However, in the following we also use a single non-crisp GCI.

Lemma 4.5. *Given a concept C and $u \in \Sigma_0^*$, there exists a TBox $\mathcal{T}$ such that for every model $\mathcal{I}$ of $\mathcal{T} \cup \{\langle a{:}Y = \frac{b}{2}\rangle\}$ it holds that $C^{\mathcal{I}}(a^{\mathcal{I}}) \in \mathsf{enc}(u)$.*

Proof. For any model $\mathcal{I}$ of the TBox $\mathcal{T}_0 := \{\langle Y^2 \sqsubseteq Y^3\rangle, \langle \top \sqsubseteq H \geqslant \frac{b}{2}\rangle, \langle H \equiv Y \sqcap \overline{Y}\rangle, \langle a{:}Y = \frac{b}{2}\rangle\}$ it holds that

$$\tfrac{b}{2} \leqslant H^{\mathcal{I}}(a^{\mathcal{I}}) \leqslant (Y \sqcap \overline{Y})^{\mathcal{I}}(a^{\mathcal{I}}) \leqslant Y^{\mathcal{I}}(a^{\mathcal{I}}) = \tfrac{b}{2}.$$

In particular, this already initializes the concept name H as desired. In addition, we have that $\overline{Y}^{\mathcal{I}}(a^{\mathcal{I}}) \geqslant b$.

To ensure that $C^{\mathcal{I}}(a^{\mathcal{I}}) \in \mathsf{enc}(u)$, let $\mathcal{T}_1 := \{\langle H \equiv A^{(s+1)^{|u|}}\rangle, \langle \overline{Y}^2 \sqcap C \equiv \overline{Y}^2 \sqcap A^{\overleftarrow{u}}\rangle\}$, where A is an auxiliary concept name. The first axiom implies that $(s+1)^{|u|}(A^{\mathcal{I}}(a^{\mathcal{I}}) - b) + b = \frac{b}{2}$, and thus $A^{\mathcal{I}}(a^{\mathcal{I}}) = b\big(1 - \frac{1}{2(s+1)^{|u|}}\big)$. Since $(\overline{Y}^2)^{\mathcal{I}}(a^{\mathcal{I}}) \geqslant b$, the second axiom entails that either $C^{\mathcal{I}}(a^{\mathcal{I}})$ and $(A^{\overleftarrow{u}})^{\mathcal{I}}(a^{\mathcal{I}})$ are both in the interval $[b, 1]$, or $C^{\mathcal{I}}(a^{\mathcal{I}}) = (A^{\overleftarrow{u}})^{\mathcal{I}}(a^{\mathcal{I}}) < b$. If $u \in \{0\}^*$, then $A^{\overleftarrow{u}}$ is equivalent to $\top$, and hence we get $C^{\mathcal{I}}(a^{\mathcal{I}}) \in [b, 1] = \mathsf{enc}(u)$. Otherwise, we obtain $C^{\mathcal{I}}(a^{\mathcal{I}}) = (A^{\overleftarrow{u}})^{\mathcal{I}}(a^{\mathcal{I}}) = b(1 - \frac{1}{2}0.\overleftarrow{u}) = \mathsf{enc}(u)$. □

Moreover, we can easily extend $\mathcal{I}_\mathcal{P}$ to a model of these axioms by setting $Y^{\mathcal{I}_\mathcal{P}}(\nu) := 1$, $\overline{Y}^{\mathcal{I}_\mathcal{P}}(\nu) := \frac{b}{2}$, and $A^{\mathcal{I}_\mathcal{P}}(\nu) := b\big(1 - \frac{1}{2(s+1)^{|u|}}\big)$ for all other domain elements $\nu \neq a^{\mathcal{I}_\mathcal{P}} = \varepsilon$. We have hence re-proven the canonical model property and the solution property, and obtain undecidability of consistency in L-$\mathcal{EL}_=$ even with only one equality assertion (namely $\langle a{:}Y = \frac{b}{2}\rangle$).

We now use this result to prove undecidability of subsumption in L-$\mathcal{EL}$, i.e. without equality assertions. Consider the ontology $\mathcal{O}_\mathcal{P}$ used in the new proof of undecidability of L-$\mathcal{EL}_=$, and let $\mathcal{T}_\mathcal{P}$ be the TBox $\mathcal{T}_\mathcal{P} := \mathcal{O}_\mathcal{P} \setminus \{\langle a{:}Y = \frac{b}{2}\rangle\}$, which contains only GCIs. Due to the axioms $\langle Y^2 \sqsubseteq Y^3\rangle$, $\langle H \equiv Y \sqcap \overline{Y}\rangle$, and $\langle \top \sqsubseteq H \geqslant \frac{b}{2}\rangle$, the interpretation of Y in a model of $\mathcal{T}_\mathcal{P}$ is always in $\{\frac{b}{2}\} \cup [b, 1]$. Hence, $\mathcal{O}_\mathcal{P}$ is consistent iff $\top$ is not b-subsumed by Y w.r.t. $\mathcal{T}_\mathcal{P}$ (since in the latter case there must be a model $\mathcal{I}$ of $\mathcal{T}_\mathcal{P}$ such that $Y^\mathcal{I}(x) = \frac{b}{2}$ for some domain element $x \in \Delta^\mathcal{I}$, i.e. x can serve the function of $a^\mathcal{I}$). This shows that subsumption in L-$\mathcal{EL}$ is undecidable if L starts with the Łukasiewicz t-norm.

To generalize this result to chains that *contain* the Łukasiewicz t-norm, we recall a result from [22, Theorem 13]: If L is the ordinal sum of two continuous chains L_1 and L_2, then subsumption in L-$\mathcal{EL}$ is at least as hard as subsumption in L_2-$\mathcal{EL}$. Since any continuous chain L that contains the Łukasiewicz t-norm can be represented in the above form such that L_2 *starts* with the Łukasiewicz t-norm, this shows undecidability of L-$\mathcal{EL}$ whenever at least one Łukasiewicz component is present in any subinterval of L.

Theorem 4.6. *Deciding ℓ-subsumption with respect to a TBox in L-$\mathcal{EL}$ is undecidable if L contains the Łukasiewicz t-norm.*

In particular, from the results presented in [22] we obtain that the following statements are equivalent (where b is determined as before, λ_2 is the order isomorphism between the subinterval of L and L_2 given by Definition 2.3, and $\lambda_2^{-1}(\mathcal{T}_\mathcal{P})$ is obtained from $\mathcal{T}_\mathcal{P}$ by replacing all values according to λ_2^{-1}):

(a) $\top$ is b-subsumed by Y w.r.t. $\mathcal{T}_\mathcal{P}$ in L_2.

(b) $\top$ is $\lambda_2^{-1}(b)$-subsumed by Y w.r.t. $\lambda_2^{-1}(\mathcal{T}_\mathcal{P})$ in L.

As above, the axioms $\langle Y^2 \sqsubseteq Y^3\rangle$, $\langle H \equiv Y \sqcap \overline{Y}\rangle$, and $\langle \top \sqsubseteq H \geqslant \lambda_2^{-1}(\frac{b}{2})\rangle$ in $\lambda_2^{-1}(\mathcal{T}_\mathcal{P})$ ensure that the value of Y is always in $\{\lambda_2^{-1}(\frac{b}{2})\} \cup [\lambda_2^{-1}(b), 1]$. Hence, the problem (b) is equivalent to the inconsistency of $\lambda_2^{-1}(\mathcal{T}_\mathcal{P}) \cup \{\langle a{:}Y = \lambda_2^{-1}(\frac{b}{2})\rangle\}$. Since $\lambda_2^{-1}(1) = 1$ and the axiom $\langle \top \sqsubseteq H \geqslant \lambda_2^{-1}(\frac{b}{2})\rangle$ is needed only for initializing certain concepts at a, which can equivalently be achieved by using several equality assertions, we do not even need non-crisp GCIs in the result.

Corollary 4.7. *Ontology consistency in L-$\mathcal{EL}_=$ is undecidable if L contains the Łukasiewicz t-norm. This result holds even if all GCIs are crisp.*

These results further extend the undecidability analysis from [8]. In previous work it was shown that the extension L-$\mathfrak{N}\mathcal{EL}$ of L-$\mathcal{EL}$ with a negation constructor, but only *crisp* assertions of the form $\langle a{:}C \geqslant 1\rangle$, is undecidable whenever L starts with the Łukasiewicz t-norm (and is decidable otherwise). We have now shown that the negation constructor can be omitted if we instead allow to state *equality* in assertions (which one might argue represents a weak kind of negation). Furthermore, undecidability holds even if the Łukasiewicz t-norm does not occur in an initial interval $[0, b]$.

It is still unknown whether a similar result holds for the product t-norm (and, more generally, for continuous t-norms containing several product components). Decidability is known only for the case of the infinite Gödel t-norm, where subsumption is P-complete [23]. In the following section we provide an intermediate result by proving that ontology consistency in Π-$\mathcal{ELU}_=$ is undecidable.

5. A Short Visit to the Product T-norm

For this section we focus on the logic Π-$\mathcal{ELU}_=$.

Remark 5.1. It does not make much sense to consider extensions of $\mathcal{ELU}$ over chains containing Π in an interval other than $[0,1]$, because in such cases the disjunction constructor has no reasonable semantics. Since the semantics of disjunction uses the involutive negation $1-x$, its behavior in the product component $[a,b]$ depends on the behavior of the t-norm in $[1-b,1-a]$, which may be completely arbitrary. In particular, the disjunction of two values from $[a,b]$ may even lie outside of that interval.

As in Section 4, we use the framework for undecidability from [8]. In this case, we encode any word $u \in \Sigma_0^*$ as the number $2^{-(u+1)}$, where again u is viewed as a number represented in base $s+1$. It can be shown, using arguments similar to those in [8] that this is a valid encoding function, if we choose the words $u_\varepsilon := 1$ and $u_+ := \varepsilon$. Moreover, the presence of equality assertions again makes the initialization property easy to show. Hence, it remains to show the concatenation, successor, transfer, and solution properties.

Lemma 5.2. *L-$\mathcal{ELU}_=$ has the concatenation, successor, and transfer properties.*

Proof. For the concatenation property, consider a model $\mathcal{I}$ of the TBox $\{\langle X \sqcap H \equiv C\rangle, \langle D \equiv X^{(s+1)^{|u|}} \sqcap C_u\rangle\}$, and any $x \in \Delta^\mathcal{I}$ such that $H^\mathcal{I}(x) = \frac{1}{2}$ and $C^\mathcal{I}(x) = \mathsf{enc}(u')$. Then, the first axiom ensures that $X^\mathcal{I}(x) = 2^{-u'}$, and by the second axiom it follows that

$$D^\mathcal{I}(x) = 2^{-(u'(s+1)^{|u|}+u+1)} = 2^{-(u'u+1)} = \mathsf{enc}(u'u).$$

We obtain the successor property from the same TBox $\{\langle H \equiv G^2\rangle, \langle G \sqsubseteq \exists r.G\rangle, \langle \exists r.H \sqsubseteq H\rangle\}$ and the same arguments as in Lemma 4.2, except that G now takes the value $\frac{1}{\sqrt{2}}$ instead of $\frac{3}{4}$.

For the transfer property, consider the TBox $\{\langle C \sqcup \overline{C} \equiv H\rangle, \langle D \sqcup \overline{D} \equiv H\rangle, \langle \exists r.D \sqsubseteq C\rangle, \langle \exists r.\overline{D} \sqsubseteq \overline{C}\rangle\}$, where $\overline{C}, \overline{D}$ are two fresh concept names, and let $\mathcal{I}$ be a model of $\mathcal{T}$ and $x, y \in \Delta^\mathcal{I}$ such that $H^\mathcal{I}(x) = \frac{1}{2}$, $r^\mathcal{I}(x,y) = 1$, and $H^\mathcal{I}(y) = \frac{1}{2}$.

If $C^\mathcal{I}(x) \in \mathsf{enc}(u) \leqslant \frac{1}{2}$, the axiom $\langle C \sqcup \overline{C} \equiv H\rangle$ ensures that $\big(1 - C^\mathcal{I}(x)\big)\big(1 - \overline{C}^\mathcal{I}(x)\big) = 1 - \frac{1}{2} = \frac{1}{2}$, and hence $\overline{C}^\mathcal{I}(x) = 1 - \frac{1}{2(1-C^\mathcal{I}(x))}$, which is a strictly decreasing function in $C^\mathcal{I}(x)$. A similar behavior is obtained for $\overline{D}$ and D through the second axiom. Therefore, the last two axioms guarantee that $D^\mathcal{I}(y) \leqslant C^\mathcal{I}(x)$ and $\overline{D}^\mathcal{I}(y) \leqslant \overline{C}^\mathcal{I}(x)$, respectively. The latter also implies $D^\mathcal{I}(y) \geqslant C^\mathcal{I}(x)$, which proves the desired result. ☐

The only remaining step is to prove that this logic also satisfies the solution property. To achieve this, we follow a similar idea as in the previous section, creating new concept names X and $\overline{X}$ that will serve as flags for ensuring that at every element x either $V^\mathcal{I}(x) < W^\mathcal{I}(x)$ or $V^\mathcal{I}(x) > W^\mathcal{I}(x)$.

Lemma 5.3. *L-$\mathcal{ELU}_=$ has the solution property.*

Proof. Consider the ontology

$$\begin{aligned}\mathcal{O} := \{&\langle X \sqsubseteq X^2\rangle, \langle \overline{X} \sqsubseteq \overline{X}^2\rangle, &(6)\\ &\langle X \sqcap \overline{X} \sqsubseteq H\rangle, \langle H \sqsubseteq X \sqcup \overline{X}\rangle, &(7)\\ &\langle X \sqcap V \sqsubseteq W \sqcap H\rangle, &(8)\\ &\langle \overline{X} \sqcap W \sqsubseteq V \sqcap H\rangle\}. &(9)\end{aligned}$$

Every model $\mathcal{I}$ of the first two axioms is such that both X and $\overline{X}$ are interpreted in $\{0,1\}$ at every element of the domain. By the canonical model property, we also know that $H^\mathcal{I}(g(\nu)) = \frac{1}{2}$ and hence the axioms (7) entail that $X^\mathcal{I}(g(\nu)) = 0$ iff $\overline{X}^\mathcal{I}(g(\nu)) = 1$. Based on these arguments, the rest of the proof follows the same steps as the proof of Lemma 4.3. ☐

Using the framework of [8], this implies the claimed result.

Theorem 5.4. *Ontology consistency in Π-$\mathcal{ELU}_=$ is undecidable.*

This extends the results of [8], where it was shown that consistency is undecidable in Π-$\mathcal{ELC}$ with crisp assertions, and hence also in Π-$\mathcal{ELC}_=$. The letter $\mathcal{C}$ denotes the involutive negation constructor, which can be used to simulate concept disjunction. For Π-$\mathcal{ELU}_\geqslant$ (having only assertions of the form $\langle a{:}C \geqslant \ell\rangle$), it is known that ontology consistency is trivially reducible to classical $\mathcal{ELU}$ [8], so Theorem 5.4 is as strong as possible with respect to the kind of assertions that are allowed.

6. Related Work

For a broad overview of the field of FDL research, we refer the reader to [5]. We describe in more detail those publications that are most closely related to the results presented here.

The research on fuzzy extensions of the light-weight DL $\mathcal{EL}$ started in [23], where the authors presented a completion-based reasoning procedure for $\mathsf{G}\text{-}\mathcal{EL}^{++}$ that is based on the original algorithm for $\mathcal{EL}$ [2, 41]. In [32, 33], it was shown that the original algorithm provided for $\mathcal{EL}^{++}$ in [2] is incomplete concerning the treatment of nominals. Since the procedure for $\mathsf{G}\text{-}\mathcal{EL}^{++}$ in [23] is based on [2], it is also incomplete for this case. We conjecture that the correct algorithms from [32, 33] can be adapted to handle $\mathsf{G}\text{-}\mathcal{EL}^{++}$ and $\mathsf{G}_n\text{-}\mathcal{EL}^{++}$ using the techniques in [23], and in particular adopting the restrictions on p-admissible concrete domains introduced there. Developing such an algorithm is out of the scope of the present paper, as it would require completely different techniques from the ones we use here. Nevertheless, for the sublogic $\mathsf{G}\text{-}\mathcal{ELO}^+_\bot$ of $\mathsf{G}\text{-}\mathcal{EL}^{++}$ without concrete domains, a polynomial-time algorithm can alternatively be obtained from known polynomial reductions from finitely valued FDLs to classical DLs [31, 35], and the fact that reasoning in $\mathsf{G}\text{-}\mathcal{ELO}_\bot$ can be restricted to a fixed, finite set of truth values without loss of generality (this property is also exploited by the algorithm in [23]). For t-norms containing the Łukasiewicz t-norm, intractability was first shown in [22], where ℓ-subsumption in $\mathsf{L}\text{-}\mathcal{EL}$ was proven to be coNP-hard. This result was subsequently strengthened to ExpTime-hardness in [21], the precursor to the current paper. There it was also shown that reasoning in the finitely valued $\text{Ł}_n\text{-}\mathcal{EL}$ is ExpTime-complete, using results originally developed for more expressive FDLs.

Regarding finitely valued FDLs in general, the most popular reasoning technique is a reduction of FDL ontologies to classical ontologies, which was originally proposed in [42]. While it is easy to devise a polynomial reduction for the finitely valued Gödel t-norm [31], the first such reductions proposed for arbitrary finitely valued t-norms incurred an exponential blow-up [17]. This problem was solved in [35] by including a polynomial preprocessing step. However, for finitely valued $\mathsf{L}\text{-}\mathcal{EL}$ these reductions introduce disjunctions into the classical ontology; that is, the ontology obtained through them requires a higher expressivity than that provided by $\mathcal{EL}$ (unless L is a Gödel chain). Our results show that this cannot be avoided since the finite Łukasiewicz t-norm can express actual disjunctions. However, the reductions can be used to obtain the upper bounds for Section 3 from classical complexity results [37, 43]. Other work on finitely valued FDLs introduced more direct reasoning algorithms, such as adaptations of tableau- or automata-based procedures for classical DLs [3, 10–12, 38, 44]. In particular, [10, 11, 38] were the first to show the ExpTime-upper bounds we need for Theorems 3.5, 3.8, and 3.10.

For infinitely valued FDLs, several tableau algorithms have been developed for reasoning with so-called *acyclic* TBoxes [45, 46]. In contrast, reasoning with full GCIs was first shown to be undecidable for FDLs under the product t-norm [6], then for the Łukasiewicz t-norm [7]. This initiated a study of the border between decidability and undecidability in infinitely valued FDLs, depending on the constructors of the logics and the precise t-norms used [8, 47]. However, all of the previous undecidability results depend on some kind of negation constructor for concepts. We show here that the negation can also be provided by the Łukasiewicz t-norm itself. Complementary decidability results again make use of reductions to classical DLs [8, 48, 49], or introduce automata- and tableau-based algorithms tailored towards the infinite Gödel t-norm [50, 51].

7. Conclusions

In this paper, we provide a thorough study of the complexity of reasoning in fuzzy extensions of description logics from the $\mathcal{EL}$ family. Our results show that, for finite chains, the presence of at least one non-idempotent truth degree damages the tractability of the underlying classical logics. In the infinite case, the presence of a Łukasiewicz component causes undecidability. These results are summarized in Table 1. In particular, reasoning in finitely valued extensions of fuzzy $\mathcal{EL}$ becomes exponentially harder than in classical $\mathcal{EL}$. If regular role inclusions are allowed in addition, then the problem becomes 2-ExpTime-hard. We stress that these hardness results apply to any finitely valued extension of these logics that contains at least

Table 1: Complexity of subsumption in fuzzy extensions of $\mathcal{EL}$ with regular TBoxes. All results also hold for the extensions with nominals and bottom ($\mathcal{O}_\bot$). The precise complexity of $\mathcal{ELI}^+$ is unknown even under classical semantics.

chain L	$\mathcal{EL}$	$\mathcal{EL}^+$	$\mathcal{ELU}$ / $\mathcal{ELI}$ / $\mathcal{ELUI}$	$\mathcal{ELI}^+$	$\mathcal{ELU}^+$ / $\mathcal{ELUI}^+$
G or G_n, $n \geqslant 2$	P	P	ExpTime	?	2-ExpTime
finite, not G_n	ExpTime	2-ExpTime	ExpTime	2-ExpTime	2-ExpTime
containing $\mathsf{Ł}$	undec.	undec.	undec.	undec.	undec.

one truth degree that is not idempotent. For the case where all degrees are idempotent—i.e., the underlying chain uses the Gödel t-norm—it is already known that tractability is preserved. Interestingly, other constructors (e.g., disjunctions or inverse roles) that had been highlighted as harmful for the tractability of classical $\mathcal{EL}$ do not cause a further increase in complexity in their non-idempotent fuzzy extensions. In the case of L-$\mathcal{ELI}^+$ with arbitrary (non-regular) role inclusions, we even obtain undecidability if L is not of the form G_n; the precise complexity of the underlying classical DL $\mathcal{ELI}^+$ is still unknown.

Although the complexity lower bounds can be matched by the algorithms developed for expressive finitely valued DLs [10, 11], which are out of the scope of the present paper, it is still worth to consider algorithms that exploit the restricted syntax of our logic. To this end, we plan to look at suitable adaptations of consequence-based algorithms for classical DLs [2, 52].

If infinitely many truth degrees are allowed, then the picture is more extreme. Indeed, we have shown that for any t-norm that contains a Łukasiewicz component, subsumption in L-$\mathcal{EL}$ is undecidable. This greatly improves the preliminary work [21], where only an ExpTime lower bound was obtained. For t-norms containing the product t-norm, the question of decidability remains open. However, we show that a related problem—consistency of $\mathsf{\Pi}$-$\mathcal{ELU}_=$ ontologies—is undecidable. Although we still do not provide a full answer for subsumption in L-$\mathcal{EL}$, our results do fill some gaps previously left open [8].

An obvious direction for future work is to find exact complexity bounds for the continuous t-norms that have not yet been solved. The precise complexity of reasoning with acyclic TBoxes in infinitely valued FDLs is also not fully explored yet (although decidability is known [7, 45]). One can also investigate the complexity of fuzzy extensions of other inexpressive DLs like $\mathcal{FL}_0$ [53] or *DL-Lite* [54]. In the former case, it is known that the complexity $\mathcal{FL}_0$ with cyclic TBoxes does not increase under Gödel semantics [55]. The effect of the Łukasiewicz and other semantics on these logics remains to be understood.

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Appendix A. Proof of Lemma 3.9

We split the soundness and completeness proofs for the reduction of Section 3.5 into several lemmas. For the following proofs, we extend the bijection $\lambda\colon [a,b] \to \text{Ł}_n$ as follows to the whole chain L:

- $\lambda(x) \coloneqq 0$ if $x < a$ and

- $\lambda(x) \coloneqq 1$ if $x > b$.

We also make use of the inverse $\lambda^{-1}\colon Ł_n \to 2^{\mathsf{L}}$ of this function, for which we in particular have $\lambda^{-1}(0) = [0, a]$ and $\lambda^{-1}(1) = [b, 1]$ (see Figure 1 for an illustration). When we sometimes treat $\lambda^{-1}(x)$ as a single value, we implicitly refer to the original bijection $\lambda^{-1}\colon Ł_n \to [a, b]$. The two functions λ^{-1} and λ are used to prove soundness and completeness, respectively, of the reduction from the L-$\mathcal{EL}^+$ TBox $\mathcal{T}$ to the $Ł_n$-$\mathcal{EL}^+$ TBox $\mathcal{T}'$.

We first prove soundness, i.e. if A is $\lambda^{-1}(\ell)$-subsumed by B w.r.t. $\mathcal{T}'$, then A is also ℓ-subsumed by B w.r.t. $\mathcal{T}$. We start with an auxiliary lemma, which states that λ^{-1} is compatible with all relevant operations of L (at least in the interval $[a, 1]$).

Lemma A.1. *For all $p, q \in Ł_n$, $p' \in \lambda^{-1}(p) \cap [a, 1]$, and $q' \in \lambda^{-1}(q) \cap [a, 1]$, we have*

- $p' *_{\mathsf{L}} q' \in \lambda^{-1}(p *_{Ł_n} q) \cap [a, 1]$, *and*
- $p' \Rightarrow_{\mathsf{L}} q' \in \lambda^{-1}(p \Rightarrow_{Ł_n} q) \cap [a, 1]$.

Proof. If $p < 1$ or $q < 1$, then we have $p' = \lambda^{-1}(p)$ or $q' = \lambda^{-1}(q)$, respectively. Furthermore, we know that $p *_{Ł_n} q < 1$ and $\lambda^{-1}(p *_{Ł_n} q) \cap [a, 1]$ contains a single element. Since L contains $Ł_n$ in $[a, b]$ and all elements above b act as neutral elements for the elements in $[a, b]$ w.r.t. $*_{\mathsf{L}}$, we have $p' *_{\mathsf{L}} q' = \lambda^{-1}(p *_{Ł_n} q) \cap [a, 1]$. In the case that $p = q = 1$, we have $p' \in [b, 1]$ and $q' \in [b, 1]$, and hence $p' *_{\mathsf{L}} q' \in [b, 1] = \lambda^{-1}(1) = \lambda^{-1}(p *_{Ł_n} q)$.

For the second claim, we make a case analysis on p and q.

- If $p = q = 1$, then both p' and q' are contained in $[b, 1]$. By the properties of ordinal sums, we also have $p' \Rightarrow_{\mathsf{L}} q' \in [b, 1] = \lambda^{-1}(1) = \lambda^{-1}(p \Rightarrow_{Ł_n} q)$.
- If $p \leqslant q$, but $p = q = 1$ does not hold, then we know that $p < 1$, and hence $p' < b$. Thus, we get $p' \leqslant q'$ by the monotonicity of λ^{-1}, which implies that $p' \Rightarrow_{\mathsf{L}} q' = 1 \in \lambda^{-1}(1) = \lambda^{-1}(p \Rightarrow_{Ł_n} q)$.
- If $1 = p > q$, then $p' \Rightarrow_{\mathsf{L}} q' = q' \in \lambda^{-1}(q) \cap [a, 1] = \lambda^{-1}(p \Rightarrow_{Ł_n} q) \cap [a, 1]$.
- Finally, if $1 > p > q$, then the claim follows directly from the fact that L contains $Ł_n$ in $[a, b]$. □

For the next step, consider an $Ł_n$-interpretation $\mathcal{I}$ and define an L-interpretation $\mathcal{I}_L$ as follows:

- $\Delta^{\mathcal{I}_L} := \Delta^{\mathcal{I}}$,
- $A^{\mathcal{I}_L}(x) := \lambda^{-1}\big(A^{\mathcal{I}}(x)\big)$ for all $A \in \mathsf{N_C}$ and $x \in \Delta^{\mathcal{I}}$, and
- $r^{\mathcal{I}_L}(x, y) := \lambda^{-1}\big(r^{\mathcal{I}}(x, y)\big)$ for all $r \in \mathsf{N_R}$ and $x, y \in \Delta^{\mathcal{I}}$.

Lemma A.2. *If $\mathcal{I}$ is an $Ł_n$-model of $\mathcal{T}$, then $\mathcal{I}_L$ is an L-model of $\mathcal{T}'$.*

Proof. The axiom $\langle \top \sqsubseteq B \geqslant a \rangle$ is satisfied by the definition of $\mathcal{I}_L$. For the remaining claim, we show that $C^{\mathcal{I}_L}(x) \in \lambda^{-1}\big(C^{\mathcal{I}}(x)\big) \cap [a, 1]$ holds for all concepts C and $x \in \Delta^{\mathcal{I}}$ by induction on the structure of C. For all concept names, this holds by the definition of $\mathcal{I}_L$, and for and conjunctions, this is a consequence of Lemma A.1. We also have $\top^{\mathcal{I}_L}(x) = 1 \in \lambda^{-1}\big(\top^{\mathcal{I}}(x)\big) \cap [a, 1]$.

It remains to show the claim for an existential restriction $\exists r.C$, assuming that it already holds for C. Again by Lemma A.1 and the definition of $\mathcal{I}_L$, we know that for all $y \in \Delta^{\mathcal{I}}$ we have

$$r^{\mathcal{I}_L}(x, y) *_{\mathsf{L}} C^{\mathcal{I}_L}(y) \in \lambda^{-1}\big(r^{\mathcal{I}}(x, y) *_{Ł_n} C^{\mathcal{I}}(y)\big) \cap [a, 1].$$

Since L is finite and $(\exists r.C)^{\mathcal{I}_L}(x)$ is the supremum of all these values, it is an element of $[b, 1]$ iff one of the values $r^{\mathcal{I}}(x, y) *_{Ł_n} C^{\mathcal{I}}(y)$ is 1, and then

$$(\exists r.C)^{\mathcal{I}_L}(x) \in [b, 1] = \lambda^{-1}(1) = \lambda^{-1}\big((\exists r.C)^{\mathcal{I}}(x)\big).$$

Otherwise, none of these values is 1 and we get

$$(\exists r.C)^{\mathcal{I}_L}(x) = \lambda^{-1}\Big(\sup_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x, y) *_{Ł_n} C^{\mathcal{I}}(y)\Big) = \lambda^{-1}\big((\exists r.C)^{\mathcal{I}}(x)\big) \in [a, b)$$

by the monotonicity of λ^{-1} when restricted to $[a,b]$. This concludes the proof of the claim.

The claim immediately shows that the axioms of the form $\langle \top \sqsubseteq D \geqslant a\rangle$ in $\mathcal{T}'$ are satisfied by $\mathcal{I}_L$. Consider now an axiom of the form $\langle C \sqsubseteq D \geqslant \lambda^{-1}(p)\rangle$ in $\mathcal{T}'$. Since $\mathcal{I}$ satisfies $\mathcal{T}$, we have $C^{\mathcal{I}}(x) \Rightarrow_{\text{Ł}_n} D^{\mathcal{I}}(x) \geqslant p$ for all $x \in \Delta^{\mathcal{I}}$, and thus we get

$$C^{\mathcal{I}_L}(x) \Rightarrow_{\mathsf{L}} D^{\mathcal{I}_L}(x) \in \lambda^{-1}\big(C^{\mathcal{I}}(x) \Rightarrow_{\text{Ł}_n} D^{\mathcal{I}}(x)\big) \subseteq [\lambda^{-1}(p), 1]$$

by the above claim, Lemma A.1, and the monotonicity of λ^{-1}. For a role inclusion $\langle r_1 \circ \cdots \circ r_n \sqsubseteq r \geqslant \lambda^{-1}(p)\rangle$ in $\mathcal{T}'$, we similarly know that

$$\begin{aligned}
&\sup_{x_1,\dots,x_{n-1}\in\Delta^{\mathcal{I}_L}} r_1^{\mathcal{I}_L}(x_0,x_1) *_{\mathsf{L}} \ldots *_{\mathsf{L}} r_n^{\mathcal{I}_L}(x_{n-1},x_n) \Rightarrow_{\mathsf{L}} r^{\mathcal{I}_L}(x_0,x_n)\\
&\qquad\in \lambda^{-1}\Big(\sup_{x_1,\dots,x_{n-1}\in\Delta^{\mathcal{I}}} r_1^{\mathcal{I}}(x_0,x_1) *_{\text{Ł}_n} \ldots *_{\text{Ł}_n} r_n^{\mathcal{I}}(x_{n-1},x_n) \Rightarrow_{\text{Ł}_n} r^{\mathcal{I}}(x_0,x_n)\Big)\\
&\qquad\subseteq [\lambda^{-1}(p),1]
\end{aligned}$$

by Lemma A.1, the monotonicity of λ^{-1}, and the fact that $\mathcal{I}$ satisfies $\langle r_1 \circ \cdots \circ r_n \sqsubseteq r \geqslant p\rangle$. □

This allows us to show soundness of the reduction.

Lemma A.3. *If A is $\lambda^{-1}(\ell)$-subsumed by B w.r.t. $\mathcal{T}'$, then A is ℓ-subsumed by B w.r.t. $\mathcal{T}$.*

Proof. Let $\mathcal{I}$ be an Ł_n-model of $\mathcal{T}$ and $x \in \Delta^{\mathcal{I}}$ such that $A^{\mathcal{I}}(x) \Rightarrow_{\text{Ł}_n} B^{\mathcal{I}}(x) < \ell$. By Lemma A.2, $\mathcal{I}_L$ is an L-model of $\mathcal{T}'$. By the definition of $\mathcal{I}_L$, we know that both $A^{\mathcal{I}_L}(x)$ and $B^{\mathcal{I}_L}(x)$ satisfy the preconditions of Lemma A.1. This yields that

$$A^{\mathcal{I}_L}(x) \Rightarrow_{\mathsf{L}} B^{\mathcal{I}_L}(x) \in \lambda^{-1}\big(A^{\mathcal{I}}(x) \Rightarrow_{\text{Ł}_n} B^{\mathcal{I}}(x)\big) \cap [a,1].$$

By assumption, we know that the latter set cannot be $[b,1]$, and thus it must be a singleton. By the strict monotonicity of λ^{-1} when restricted to $[a,b]$, we conclude that

$$A^{\mathcal{I}_L}(x) \Rightarrow_{\mathsf{L}} B^{\mathcal{I}_L}(x) = \lambda^{-1}\big(A^{\mathcal{I}}(x) \Rightarrow_{\text{Ł}_n} B^{\mathcal{I}}(x)\big) < \lambda^{-1}(\ell).$$ □

To prove completeness, we first show an auxiliary lemma.

Lemma A.4. *For all $p, q \in \mathsf{L}$, we have*

- $\lambda(p *_{\mathsf{L}} q) = \lambda(p) *_{\text{Ł}_n} \lambda(q)$, *and*
- *if $q \geqslant a$, then $\lambda(p \Rightarrow_{\mathsf{L}} q) = \lambda(p) \Rightarrow_{\text{Ł}_n} \lambda(q)$.*

Proof. If both $p > b$ and $q > b$, then we have $\lambda(p) = \lambda(q) = 1$ and $p *_{\mathsf{L}} q \geqslant b$, and thus

$$\lambda(p *_{\mathsf{L}} q) = 1 = 1 *_{\text{Ł}_n} 1 = \lambda(p) *_{\text{Ł}_n} \lambda(q).$$

If either $p < a$ or $q < a$, then $\lambda(p) = 0$ or $\lambda(q) = 0$, respectively. Since then also $p *_{\mathsf{L}} q < a$, we obtain $\lambda(p *_{\mathsf{L}} q) = 0 = \lambda(p) *_{\text{Ł}_n} \lambda(q)$. If neither of these two cases applies, then we have $p *_{\mathsf{L}} q \in [a,b]$ and $\lambda(p *_{\mathsf{L}} q) = \lambda(p) *_{\text{Ł}_n} \lambda(q)$ since L contains Ł_n in $[a,b]$.

For the second claim, we consider the following cases.

- If $p \leqslant q$, then by the monotonicity of λ we get $\lambda(p) \leqslant \lambda(q)$, and thus

$$\lambda(p \Rightarrow_{\mathsf{L}} q) = \lambda(1) = 1 = \lambda(p) \Rightarrow_{\text{Ł}_n} \lambda(q).$$

- If $b \geqslant p > q \geqslant a$, then the claim follows directly from the fact that L contains Ł_n in $[a,b]$.

- If $p \geqslant b \geqslant q \geqslant a$ and $p > q$, then we have $\lambda(p) = 1$ and $p \Rightarrow_{\mathsf{L}} q = q \geqslant a$, and thus

$$\lambda(p \Rightarrow_{\mathsf{L}} q) = \lambda(q) = \lambda(p) \Rightarrow_{\text{Ł}_n} \lambda(q).$$

- Finally, if $p > q \geqslant b$, then $p \Rightarrow_{\mathsf{L}} q \geqslant q \geqslant b$, $\lambda(p) = \lambda(q) = 1$, and $\lambda(p \Rightarrow_{\mathsf{L}} q) = 1 = \lambda(p) \Rightarrow_{\text{Ł}_n} \lambda(q)$. $\square$

Starting from an L-interpretation $\mathcal{I}$, we can construct an Ł_n-interpretation $\mathcal{I}_n$ as follows:

- $\Delta^{\mathcal{I}_n} := \Delta^{\mathcal{I}}$,
- $A^{\mathcal{I}_n}(x) := \lambda\big(A^{\mathcal{I}}(x)\big)$ for all $A \in \mathsf{N_C}$ and $x \in \Delta^{\mathcal{I}}$, and
- $r^{\mathcal{I}_n}(x,y) := \lambda\big(r^{\mathcal{I}}(x,y)\big)$ for all $r \in \mathsf{N_R}$ and $x, y \in \Delta^{\mathcal{I}}$.

Lemma A.5. *If $\mathcal{I}$ is an L-model of $\mathcal{T}'$, then $\mathcal{I}_n$ is an Ł_n-model of $\mathcal{T}$.*

Proof. We first show the auxiliary claim that $C^{\mathcal{I}_n}(x) = \lambda\big(C^{\mathcal{I}}(x)\big)$ holds for all concepts C and $x \in \Delta^{\mathcal{I}}$ by induction on the structure of C. For all concept names, this holds by the definition of $\mathcal{I}_n$. For conjunctions, it follows directly from Lemma A.4. We also know that $\top^{\mathcal{I}_n}(x) = 1 = \lambda(1) = \lambda\big(\top^{\mathcal{I}}(x)\big)$.

Consider now an existential restriction $\exists r.C$ and assume that the claim holds for C. By the definition of $\mathcal{I}_n$ and Lemma A.4, we know that $r^{\mathcal{I}_n}(x,y) *_{\text{Ł}_n} C^{\mathcal{I}_n}(y) = \lambda\big(r^{\mathcal{I}}(x,y) *_{\mathsf{L}} C^{\mathcal{I}}(y)\big)$ holds for all $y \in \Delta^{\mathcal{I}}$. Since $(\exists r.C)^{\mathcal{I}_n}(x)$ is the supremum of all these values, L is finite, and λ is monotone, we have

$$(\exists r.C)^{\mathcal{I}_n}(x) = \lambda\Big(\sup_{y \in \Delta^{\mathcal{I}}} r^{\mathcal{I}}(x,y) *_{\mathsf{L}} C^{\mathcal{I}}(y)\Big) = \lambda\big((\exists r.C)^{\mathcal{I}}(x)\big),$$

which concludes the proof of the claim.

Consider now an axiom $\langle C \sqsubseteq D \geqslant p\rangle$ in $\mathcal{T}$. Since $\mathcal{I}$ is a model of $\mathcal{T}'$ and $\Rightarrow_{\mathsf{L}}$ is the residuum of $*_{\mathsf{L}}$, we have $\lambda^{-1}(p) *_{\mathsf{L}} C^{\mathcal{I}}(x) \leqslant D^{\mathcal{I}}(x)$, and thus

$$p *_{\text{Ł}_n} C^{\mathcal{I}_n}(x) = \lambda\big(\lambda^{-1}(p)\big) *_{\text{Ł}_n} \lambda\big(C^{\mathcal{I}}(x)\big) = \lambda\big(\lambda^{-1}(p) *_{\mathsf{L}} C^{\mathcal{I}}(x)\big) \leqslant \lambda\big(D^{\mathcal{I}}(x)\big) = D^{\mathcal{I}_n}(x),$$

by Lemma A.4, the above claim, and monotonicity of λ.

Similarly, for a role inclusion $\langle r_1 \circ \cdots \circ r_n \sqsubseteq r \geqslant p\rangle$ in $\mathcal{T}$, we obtain

$$\begin{aligned} p *_{\text{Ł}_n} \sup_{x_1,\dots,x_n \in \Delta^{\mathcal{I}_n}} & r_1^{\mathcal{I}_n}(x_0,x_1) *_{\text{Ł}_n} \dots *_{\text{Ł}_n} r_n^{\mathcal{I}_n}(x_{n-1},x_n) \\ &= \lambda\Big(\lambda^{-1}(p) *_{\mathsf{L}} \sup_{x_1,\dots,x_n \in \Delta^{\mathcal{I}}} r_1^{\mathcal{I}}(x_0,x_1) *_{\mathsf{L}} \dots *_{\mathsf{L}} r_n^{\mathcal{I}}(x_{n-1},x_n)\Big) \\ &\leqslant \lambda\big(r^{\mathcal{I}}(x_0,x_n)\big) \\ &= r^{\mathcal{I}_n}(x_0,x_n), \end{aligned}$$

i.e. it is satisfied by $\mathcal{I}_n$. $\square$

We can now show the completeness of the reduction of Section 3.5.

Lemma A.6. *If A is ℓ-subsumed by B w.r.t. $\mathcal{T}$, then A is $\lambda^{-1}(\ell)$-subsumed by B w.r.t. $\mathcal{T}'$.*

Proof. Consider an L-model $\mathcal{I}$ of $\mathcal{T}'$ with $A^{\mathcal{I}}(x) \Rightarrow_{\mathsf{L}} B^{\mathcal{I}}(x) < \lambda^{-1}(\ell)$ for some $x \in \Delta^{\mathcal{I}}$. By Lemma A.5, $\mathcal{I}_n$ is a model of $\mathcal{T}$. By the definition of $\mathcal{T}'$, we know that $B^{\mathcal{I}}(x) \geqslant a$. Thus, Lemma A.4 yields $A^{\mathcal{I}_n}(x) \Rightarrow_{\text{Ł}_n} B^{\mathcal{I}_n}(x) = \lambda\big(A^{\mathcal{I}}(x) \Rightarrow_{\mathsf{L}} B^{\mathcal{I}}(x)\big)$. Since $\ell > 0$ and λ is strictly monotone in $[a,b]$, this residuum is strictly smaller than $\lambda\big(\lambda^{-1}(\ell)\big) = \ell$. $\square$

Together, Lemmas A.3 and A.6 imply Lemma 3.9.