ON PARABOLIC EQUATIONS WITH CRITICAL ELECTROMAGNETIC POTENTIALS

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Abstract. We consider a class of parabolic equations with critical electromagnetic potentials, for which we obtain a classification of local asymptotics, unique continuation results, and an integral representation formula for solutions.

1. Introduction and statement of the main results

This paper is concerned with the following class of evolution equations with critical electromagnetic potentials

$$u_t + \left(-i\nabla + \frac{\mathbf{A}\left(\frac{x}{|x|}\right)}{|x|}\right)^2 u - \frac{a\left(\frac{x}{|x|}\right)}{|x|^2}\, u = h(x,t)u, \tag{1}$$

in $\mathbb{R}^N \times I$, for some interval $I \subset \mathbb{R}$ and for $N \geqslant 2$. Here $u = u(x,t) : \mathbb{R}^N \times I \to \mathbb{C}$, $a \in L^\infty(\mathbb{S}^{N-1}, \mathbb{R})$, $\mathbb{S}^{N-1}$ denotes the unit $(N-1)$–dimensional sphere, and $\mathbf{A} \in C^1(\mathbb{S}^{N-1}, \mathbb{R}^N)$ satisfies the following transversality condition

$$\mathbf{A}(\theta) \cdot \theta = 0 \quad \text{for all } \theta \in \mathbb{S}^{N-1}. \tag{2}$$

We always denote by $r := |x|$, $\theta = x/|x|$, so that $x = r\theta$. Under the transversality condition (2), the hamiltonian

$$\mathcal{L}_{\mathbf{A},a} := \left(-i\,\nabla + \frac{\mathbf{A}\left(\frac{x}{|x|}\right)}{|x|}\right)^2 - \frac{a\left(\frac{x}{|x|}\right)}{|x|^2} \tag{3}$$

formally acts on functions $f : \mathbb{R}^N \to \mathbb{C}$ as

$$\mathcal{L}_{\mathbf{A},a} f = -\Delta f + \frac{|\mathbf{A}\left(\frac{x}{|x|}\right)|^2 - a\left(\frac{x}{|x|}\right) - i\,\mathrm{div}_{\mathbb{S}^{N-1}} \mathbf{A}\left(\frac{x}{|x|}\right)}{|x|^2}\, f - 2i\,\frac{\mathbf{A}\left(\frac{x}{|x|}\right)}{|x|} \cdot \nabla f,$$

where $\mathrm{div}_{\mathbb{S}^{N-1}} \mathbf{A}$ denotes the Riemannian divergence of $\mathbf{A}$ on the unit sphere $\mathbb{S}^{N-1}$ endowed with the standard metric.

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The electromagnetic potential appearing in (1) is singular and homogeneous. A prototype in dimension 2 of such type of potentials is given by the Aharonov-Bohm vector potential

$$(x_1,x_2)\mapsto \boldsymbol{\mathcal{A}}(x_1,x_2)=\alpha\left(-\frac{x_2}{x_1^2+x_2^2},\frac{x_1}{x_1^2+x_2^2}\right) \tag{4}$$

which is associated to thin solenoids. If the radius of the solenoid tends to zero while the flux through it remains constantly equal to $\alpha\notin\mathbb{Z}$, then a δ-type magnetic field is produced and the so-called Aharonov-Bohm effect occurs, i.e. the magnetic potential affects charged quantum particles moving in $\mathbb{R}^2\setminus\{(0,0)\}$, even if the magnetic field $\boldsymbol{\mathcal{B}}=\operatorname{curl}\boldsymbol{\mathcal{A}}$ is zero there.

We mention that heat semigroups generated by magnetic Schrödinger operators have been studied in [4, 18]. In particular in [4] the case of a compactly supported smooth magnetic field was considered and it was shown that the large time behavior of the heat semigroup is related to a magnetic eigenvalue problem on the $(N-1)$-dimensional sphere (see (10)). In [18] it was proved that the large time behavior of magnetic heat kernels in two dimensions is determined by the flux of the magnetic field.

In the present paper, we aim at providing some unique continuation principles for problem (1) under suitable assumptions on the perturbing potential h and deriving a representation formula for solutions in the case $h\equiv 0$.

In order to establish unique continuation properties, we will describe the asymptotic behaviour of solutions near the singularity, under the assumption that, in some bounded interval I, the real-valued function h satisfies

$$\begin{cases} h,h_t\in L^r\big(I,L^{N/2}(\mathbb{R}^N)\big) \quad\text{for some } r>1, \quad h_t\in L^\infty_{\rm loc}\big(I,L^{N/2}(\mathbb{R}^N)\big), & \text{if } N\geqslant 3,\\ h,h_t\in L^r\big(I,L^{p}(\mathbb{R}^N)\big) \quad\text{for some } p,r>1, \quad h_t\in L^\infty_{\rm loc}\big(I,L^{p}(\mathbb{R}^N)\big), & \text{if } N=2,\end{cases} \tag{5}$$

and there exists $C_h>0$ such that

$$|h(x,t)|\leqslant C_h(1+|x|^{-2+\varepsilon}) \quad \text{for all } t\in I, \text{ a.e. } x\in\mathbb{R}^N, \text{ and for some } \varepsilon\in(0,2). \tag{6}$$

In particular, for $t_0\in I$ fixed, we are interested in describing the behavior of solutions along the directions $(\lambda x,t_0-\lambda^2t)$ naturally related to the heat operator. Indeed, since the unperturbed operator $u_t+\left(-i\nabla+\frac{\mathbf{A}(x/|x|)}{|x|}\right)^2u-\frac{a(x/|x|)u}{|x|^2}$ is invariant under the action $(x,t)\mapsto(\lambda x,t_0+\lambda^2t)$, we are interested in evaluating the asymptotics of

$$u(\sqrt{t_0-t}\,x,t)\quad \text{as } t\to t_0^-$$

for solutions to (1). We notice that, in the magnetic-free case $\mathbf{A}\equiv 0$, an asymptotic analysis for solutions to (1) was developed in [16].

In the description of the asymptotic behavior at the singularity of solutions to (1) a key role is played by the eigenvalues and eigenfunctions of the Ornstein-Uhlenbeck magnetic operator with singular inverse square potential

$$L_{\mathbf{A},a}:\mathcal{H}\to\mathcal{H}^\star,\quad L_{\mathbf{A},a}=\mathcal{L}_{\mathbf{A},a}+\frac{x}{2}\cdot\nabla, \tag{7}$$

defined as

$$_{\mathcal{H}^\star}\langle L_{\mathbf{A},a}v,w\rangle_{\mathcal{H}}=\int_{\mathbb{R}^N}\left(\nabla_{\mathbf{A}}v(x)\cdot\overline{\nabla_{\mathbf{A}}w(x)}-\frac{a(x/|x|)}{|x|^2}\,v(x)\overline{w(x)}\right)e^{-\frac{|x|^2}{4}}\,dx,\quad \text{for all } v,w\in\mathcal{H}.$$

Here $\mathcal{H}$ is the functional space defined as the completion of $C^\infty_{\rm c}(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ with respect to the norm

$$\|v\|_{\mathcal{H}} = \left(\int_{\mathbb{R}^N} \left(|\nabla v(x)|^2 + |v(x)|^2 + \frac{|v(x)|^2}{|x|^2} \right) e^{-\frac{|x|^2}{4}}\, dx \right)^{1/2}. \tag{8}$$

The spectrum of $L_{\mathbf{A},a}$ is related to the angular component of the operator $\mathcal{L}_{\mathbf{A},a}$ on the unit $(N-1)$–dimensional sphere $\mathbb{S}^{N-1}$, i.e. to the operator

$$T_{\mathbf{A},a} = \big(-i\,\nabla_{\mathbb{S}^{N-1}} + \mathbf{A} \big)^2 - a(\theta) = -\Delta_{\mathbb{S}^{N-1}} + \big(|\mathbf{A}|^2 - a(\theta) - i \,\mathrm{div}_{\mathbb{S}^{N-1}} \mathbf{A} \big) - 2i\, \mathbf{A} \cdot \nabla_{\mathbb{S}^{N-1}}.$$

By classical spectral theory, $T_{\mathbf{A},a}$ admits a diverging sequence of real eigenvalues with finite multiplicity $\mu_1(\mathbf{A},a) \leqslant \mu_2(\mathbf{A},a) \leqslant \cdots \leqslant \mu_k(\mathbf{A},a) \leqslant \cdots$, see [14, Lemma A.5]. The first eigenvalue $\mu_1(\mathbf{A},a)$ admits the following variational characterization:

$$\mu_1(\mathbf{A},a) = \min_{\psi \in H^1(\mathbb{S}^{N-1}) \setminus \{0\}} \frac{\int_{\mathbb{S}^{N-1}} \big[\big| \big(\nabla_{\mathbb{S}^{N-1}} + i\mathbf{A}(\theta) \big) \psi(\theta) \big|^2 - a(\theta) |\psi(\theta)|^2 \big]\, dS(\theta)}{\int_{\mathbb{S}^{N-1}} |\psi(\theta)|^2\, dS(\theta)}. \tag{9}$$

To each $k \in \mathbb{N}$, $k \geqslant 1$, we associate a $L^2\big(\mathbb{S}^{N-1}, \mathbb{C}\big)$-normalized eigenfunction ψ_k of the operator $T_{\mathbf{A},a}$ on $\mathbb{S}^{N-1}$ corresponding to the k-th eigenvalue $\mu_k(\mathbf{A},a)$, i.e. satisfying

$$\begin{cases} T_{\mathbf{A},a} \psi_k = \mu_k(\mathbf{A},a)\, \psi_k(\theta), & \text{in } \mathbb{S}^{N-1}, \\ \int_{\mathbb{S}^{N-1}} |\psi_k(\theta)|^2\, dS(\theta) = 1. \end{cases} \tag{10}$$

In the enumeration $\mu_1(\mathbf{A},a) \leqslant \mu_2(\mathbf{A},a) \leqslant \cdots \leqslant \mu_k(\mathbf{A},a) \leqslant \cdots$ we repeat each eigenvalue as many times as its multiplicity, so that exactly one eigenfunction ψ_k corresponds to each index $k \in \mathbb{N}$. Furthermore, the functions ψ_k can be chosen in such a way that they form an orthonormal basis of $L^2(\mathbb{S}^{N-1}, \mathbb{C})$. We mention that the key role played by the angular magnetic Schrödinger operator $T_{\mathbf{A},a}$ in the behaviour of the heat magnetic semigroup was already highlighted in [4].

We notice that, under the condition

$$\mu_1(\mathbf{A},a) > -\left(\frac{N-2}{2} \right)^2 \tag{11}$$

the quadratic form associated to $\mathcal{L}_{\mathbf{A},a}$ is positive definite (see [14, Lemma 2.2]), thus implying that the hamiltonian $\mathcal{L}_{\mathbf{A},a}$ is a symmetric semi-bounded operator on $L^2(\mathbb{R}^N;\mathbb{C})$ which then admits a self-adjoint extension (Friedrichs extension) with the natural form domain.

We introduce the notation

$$\alpha_k := \frac{N-2}{2} - \sqrt{\left(\frac{N-2}{2} \right)^2 + \mu_k(\mathbf{A},a)}, \quad \beta_k := \sqrt{\left(\frac{N-2}{2} \right)^2 + \mu_k(\mathbf{A},a)}, \tag{12}$$

so that $\beta_k = \frac{N-2}{2} - \alpha_k$, for all $k \in \mathbb{N}$, $k \geqslant 1$.

The first result of the present paper is the following complete description of the spectrum of the operator $L_{\mathbf{A},a}$. We mention that analogous results were proved in the case where a is a constant and $\mathbf{A} \equiv 0$ in [26, §9.3] and in the magnetic-free case $\mathbf{A} \equiv 0$ in [16, Proposition 1]; see also [5, §4.2] and [12, §2] for the non singular case.

Proposition 1.1. *The set of the eigenvalues of the operator* $L_{\mathbf{A},a}$ *is* $\big\{ \gamma_{m,k} : k, m \in \mathbb{N}, k \geqslant 1 \big\}$ *where*

$$\gamma_{m,k} = m - \frac{\alpha_k}{2}, \tag{13}$$

being α_k as in (12). Each eigenvalue $\gamma_{m,k}$ has finite multiplicity equal to

$$\#\Big\{j \in \mathbb{N}, j \geqslant 1 : \gamma_{m,k} + \frac{\alpha_j}{2} \in \mathbb{N}\Big\}$$

and a basis of the corresponding eigenspace is $\big\{V_{n,j} : j, n \in \mathbb{N}, j \geqslant 1, \gamma_{m,k} = n - \frac{\alpha_j}{2}\big\}$, where

$$V_{n,j}(x) = |x|^{-\alpha_j} P_{j,n}\Big(\frac{|x|^2}{4}\Big)\psi_j\Big(\frac{x}{|x|}\Big), \tag{14}$$

ψ_j is an eigenfunction of the operator $T_{A,a}$ associated to the j-th eigenvalue $\mu_j(\mathbf{A}, a)$ as in (10), and $P_{j,n}$ is the polynomial of degree n given by

$$P_{j,n}(t) = \sum_{i=0}^{n} \frac{(-n)_i}{\big(\frac{N}{2} - \alpha_j\big)_i} \frac{t^i}{i!},$$

denoting as $(s)_i$, for all $s \in \mathbb{R}$, the Pochhammer's symbol $(s)_i = \prod_{j=0}^{i-1}(s + j)$, $(s)_0 = 1$.

The second main result of the present paper establishes a sharp relation between the asymptotic behaviour of solutions to (1) along the directions $(\lambda x, t_0 - \lambda^2 t)$ and the spectrum of the operator $L_{\mathbf{A},a}$. Indeed we prove that

$$u(\sqrt{t_0 - t}\, x, t) \sim (t_0 - t)^{\gamma} g(x) \quad \text{as as } t \to t_0^-,$$

where γ is an eigenvalue of $L_{\mathbf{A},a}$ and g is an associated eigenfunction. In order to state precisely the result of our asymptotic analysis, we introduce the Hilbert space $\mathcal{H}_t$ defined, for every $t > 0$, as the completion of $C^\infty_{\rm c}(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ with respect to the norm

$$\|\phi\|_{\mathcal{H}_t} = \bigg(\int_{\mathbb{R}^N} \bigg(t|\nabla\phi(x)|^2 + |\phi(x)|^2 + t\frac{|\phi(x)|^2}{|x|^2}\bigg) G(x,t)\, dx\bigg)^{1/2}, \tag{15}$$

where

$$G(x,t) = t^{-N/2} \exp\Big(-\frac{|x|^2}{4t}\Big)$$

is the heat kernel of the free evolution forward equation satisfying

$$G_t - \Delta G = 0 \quad \text{and} \quad \nabla G(x,t) = -\frac{x}{2t}\, G(x,t) \quad \text{in } \mathbb{R}^N \times (0, +\infty). \tag{16}$$

For every $t > 0$, we also define the space $\mathcal{L}_t$ as the completion of $C^\infty_{\rm c}(\mathbb{R}^N)$ with respect to the norm

$$\|u\|_{\mathcal{L}_t} = \bigg(\int_{\mathbb{R}^N} |u(x)|^2 G(x,t)\, dx\bigg)^{1/2}.$$

We set $\mathcal{L} := \mathcal{L}_1$.

Theorem 1.2. *Let $N \geqslant 2$, $\mathbf{A} \in C^1(\mathbb{S}^{N-1}, \mathbb{R}^N)$ and $a \in L^\infty\big(\mathbb{S}^{N-1}, \mathbb{R}\big)$ satisfy (2) and (11). Let $u \not\equiv 0$ be a weak solution to (1) (see Definition 2.1 for the notion of weak solution) in $\mathbb{R}^N \times (t_0 - T, t_0)$ with h satisfying (5)–(6) in $I = (t_0 - T, t_0)$ for some $t_0 \in \mathbb{R}$ and $T > 0$. Then there exist $m_0, k_0 \in \mathbb{N}$, $k_0 \geqslant 1$, such that*

$$\lim_{t \to t_0^-} \widetilde{\mathcal{N}}(t) = \gamma_{m_0,k_0} = m_0 - \frac{\alpha_{k_0}}{2}, \tag{17}$$

where

$$\widetilde{\mathcal{N}}(t) = \frac{(t_0 - t)\int_{\mathbb{R}^N}\left(|\nabla_{\mathbf{A}} u(x,t)|^2 - \frac{a(x/|x|)}{|x|^2}|u(x,t)|^2 - h(x,t)|u(x,t)|^2\right)e^{-\frac{|x|^2}{4(t_0-t)}}\,dx}{\int_{\mathbb{R}^N}|u(x,t)|^2\,e^{-\frac{|x|^2}{4(t_0-t)}}\,dx}. \tag{18}$$

Furthermore, denoting as J_0 the finite set of indices $J_0 = \{(m,k) \in \mathbb{N}\times(\mathbb{N}\setminus\{0\}) : m - \frac{\alpha_k}{2} = \gamma_{m_0,k_0}\}$, for all $\tau \in (0,1)$ there holds

$$\lim_{\lambda\to 0^+}\int_\tau^1 \bigg\|\lambda^{-2\gamma_{m_0,k_0}}u(\lambda x, t_0 - \lambda^2 t) - t^{\gamma_{m_0,k_0}}\sum_{(m,k)\in J_0}\beta_{m,k}\widetilde{V}_{m,k}(x/\sqrt{t})\bigg\|_{\mathcal{H}_t}^2 dt = 0 \tag{19}$$

and

$$\lim_{\lambda\to 0^+}\sup_{t\in[\tau,1]}\bigg\|\lambda^{-2\gamma_{m_0,k_0}}u(\lambda x, t_0 - \lambda^2 t) - t^{\gamma_{m_0,k_0}}\sum_{(m,k)\in J_0}\beta_{m,k}\widetilde{V}_{m,k}(x/\sqrt{t})\bigg\|_{\mathcal{L}_t} = 0, \tag{20}$$

where $\widetilde{V}_{m,k} = V_{m,k}/\|V_{m,k}\|_{\mathcal{L}}$, $V_{m,k}$ are as in (14),

$$\begin{aligned}\beta_{m,k} = {} & \Lambda^{-2\gamma_{m_0,k_0}}\int_{\mathbb{R}^N} u(\Lambda x, t_0 - \Lambda^2)\overline{\widetilde{V}_{m,k}(x)}G(x,1)\,dx \\ & + 2\int_0^\Lambda s^{1-2\gamma_{m_0,k_0}}\left(\int_{\mathbb{R}^N} h(sx, t_0 - s^2)u(sx, t_0 - s^2)\overline{\widetilde{V}_{m,k}(x)}G(x,1)\,dx\right)ds\end{aligned} \tag{21}$$

for all $\Lambda \in (0, \Lambda_0)$ and for some $\Lambda_0 \in (0, \sqrt{T})$, and $\beta_{m,k} \neq 0$ for some $(m,k) \in J_0$.

The effect of the magnetic singular potential on the local behavior of solutions can be recognized in the values of the limit frequencies $\gamma_{m,k}$ which are directly related to the angular eigenvalues $\mu_k(\mathbf{A}, a)$ through formulas (12) and (13). We observe that the magnetic eigenvalues $\mu_k(\mathbf{A}, a)$ are indeed different from the magnetic-free eigenvalues $\mu_k(0, a)$, at least in the case of a non irrotational magnetic vector potential; e.g. in [14, Lemma A.2] it was proved that $\mu_1(\mathbf{A}, a) > \mu_1(0, a)$ if $\mathrm{curl}(\mathbf{A}/|x|) \not\equiv 0$. We also recall that, in the relevant example of a Aharonov-Bohm vector potential (4) in dimension $N = 2$ (with $a \equiv 0$), the magnetic eigenvalues are explicitly known and the limit frequencies turns out to be

$$m + \frac{|\alpha - k|}{2}, \quad m, k \in \mathbb{N},$$

showing how the asymptotic behaviour of solutions strongly relies on the circulation α.

The proof of Theorem 1.2 is based on a parabolic Almgren-Poon type monotonicity formula in the spirit of [23] combined with a blow-up analysis, see also [16]. In particular, the function (18) represents the Poon's electromagnetic parabolic counterpart of the frequency quotient introduced by Almgren in [3] and used by Garofalo and Lin [17] to prove unique continuation properties for elliptic equations with variable coefficients; indeed, both in the elliptic and in the parabolic case, monotonicity of the frequency function implies doubling properties of the solution and then the validity of unique continuation principles. As done in the proof of Theorem 1.2 and as already observed in [16] in the magnetic free case, the combination of the monotonicity argument with a blow-up analysis allows proving not only unique continuation but also the precise asymptotic description of scaled solutions near the singularity given in (19)–(20). We notice that the magnetic free results of [16] and their magnetic counterpart of the present paper generalize the classification of local asymptotics of solutions to parabolic equations with bounded coefficients obtained in [5] to the case of singular homogenous potentials (Hardy potential and homogenous electromagnetic potentials); we recall that the approach in [5] is based on recasting equations in terms of parabolic self-similar variables. We also mention [2, 7, 8, 10, 11, 15] for unique continuation results for parabolic equations with time-dependent potentials via Carleman inequalities and monotonicity methods.

As a consequence of Theorem 1.2 we obtain the following *strong unique continuation property* at the singularity.

Corollary 1.3. *Let u be a weak solution to (1) in $(t_0 - T, t_0)$ under the assumptions of Theorem 1.2. If*

$$u(x,t) = O\Big(\Big(|x|^2 + (t_0 - t)\Big)^k\Big) \quad \text{as } x \to 0 \text{ and } t \to t_0^-, \quad \text{for all } k \in \mathbb{N}, \tag{22}$$

then $u \equiv 0$ in $\mathbb{R}^N \times (t_0 - T, t_0)$.

The monotonicity argument developed to prove Theorem 1.2 yields as a byproduct the following *unique continuation property* with respect to time. It can be interpreted as a *backward uniqueness result* for (1) in the spirit of [21], see e.g. [19, 22].

Proposition 1.4. *Let $N \geqslant 2$, $\mathbf{A} \in C^1(\mathbb{S}^{N-1}, \mathbb{R}^N)$ and $a \in L^\infty\big(\mathbb{S}^{N-1}, \mathbb{R}\big)$ satisfy (2) and (11). Let u be a weak solution to (1) in $\mathbb{R}^N \times I$ with h satisfying (5)–(6) for some bounded interval I. If there exists $t_0 \in I$ such that*

$$u(x, t_0) = 0 \quad \text{for a.e. } x \in \mathbb{R}^N,$$

then $u \equiv 0$ in $\mathbb{R}^N \times I$.

Another main goal of this manuscript is to give an integral representation formula for magnetic caloric functions, i.e. for solutions to (1). The free heat forward equation, i.e. (1) with $\mathbf{A} \equiv 0$, $a \equiv 0$ and $h \equiv 0$, can be considered as the canonical example of diffusion equation. A well-known solution to the the Cauchy problem

$$\begin{cases} u_t = \Delta u \\ u(x,0) = f(x) \end{cases} \tag{23}$$

with datum $f \in C^0(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ is given by the integral formula:

$$u(x,t) = e^{-t\Delta} f(x) := \frac{1}{(4\pi t)^{\frac{N}{2}}} \int_{\mathbb{R}^N} e^{-\frac{|x-y|^2}{4t}} f(y)\, dy, \quad x \in \mathbb{R}^N, \quad t > 0.$$

We also refer to [6] for integral representation formulas for the heat equation in half–spaces.

With the aim of extending integral representation formulas to the electromagnetic singular case, the following theorem provides an explicit representation formula for weak solutions to (1), with $h \equiv 0$ and initial datum $u_0 \in \tilde{\mathcal{L}}$, where $\tilde{\mathcal{L}}$ is defined as the completion of $C_c^\infty(\mathbb{R}^N, \mathbb{C})$ with respect to

$$\|\varphi\|_{\tilde{\mathcal{L}}} = \bigg(\int_{\mathbb{R}^N} |\varphi(x)|^2 e^{\frac{|x|^2}{4}}\, dx\bigg)^{1/2}. \tag{24}$$

Theorem 1.5. *Let u be a weak solution (in the sense explained at the beginning of Section 7) to (1) with $h \equiv 0$ and $u(\cdot,0) = u_0 \in \tilde{\mathcal{L}}$. Then u admits the following representation: for all $t > 0$,*

$$u(x,t) = t^{-\frac{N}{2}} \int_{\mathbb{R}^N} u_0(y) K\bigg(\frac{y}{\sqrt{t}}, \frac{x}{\sqrt{t}}\bigg)\, dy,$$

where the integral at the right hand side is understood in the sense of improper multiple integrals and

$$K(x,y) = \frac{1}{2(|x||y|)^{\frac{N-2}{2}}} e^{-\frac{|x|^2+|y|^2}{4}} \sum_{k=1}^{\infty} e^{i\frac{\pi}{2}\beta_k} \psi_k\big(\tfrac{y}{|y|}\big)\overline{\psi_k\big(\tfrac{x}{|x|}\big)} J_{\beta_k}\big(\tfrac{-i|x||y|}{2}\big) \tag{25}$$

being β_k as in (12)*, ψ_k as in* (10)*, and being J_{β_k} the Bessel function of the first kind of order β_k.*

In the proof of Theorem 1.5, the critical homogeneities and the transversality condition (2) play a fundamental role. Indeed Theorem 1.5 is proved by recasting equation (1) with $h = 0$ in terms of parabolic self-similar variables (see transformation (116)) thus obtaining an equivalent parabolic equation with an Ornstein-Uhlenbeck type operator with singular homogeneous electromagnetic potentials (see (117)). Then a representation formula is obtained by expanding the transformed solution in Fourier series with respect to an orthonormal basis given by eigenfunctions of the Ornstein-Uhlenbeck type operator (see Remark 4.6 for the description of such a basis).

The present paper is organized as follows. In section 2 we give a weak formulation of problem (1). In section 3 we present some magnetic parabolic Hardy type inequalities and weighted Sobolev embeddings. Section 4 is devoted to the description of the spectrum of the operator $L_{\mathbf{A},a}$ defined in (7) and to the proof of Proposition 1.1. The parabolic monotonicity argument developed in section 5 together with the blow-up analysis of section 6 allow proving Theorem 1.2 at the end of section 6. Finally, section 7 contains the proof of the representation formula stated in Theorem 1.5.

Notation. We list below some notation used throughout the paper.

- const denotes some positive constant which may vary from formula to formula.
- dS denotes the volume element on the unit $(N-1)$-dimensional sphere $\mathbb{S}^{N-1}$.
- ω_{N-1} denotes the volume of $\mathbb{S}^{N-1}$, i.e. $\omega_{N-1} = \int_{\mathbb{S}^{N-1}} dS(\theta)$.
- For every complex number z we denote as $\mathfrak{Re}(z)$ its real part.

2. The weak formulation of the problem

The functional space $\mathcal{H}_t$ defined in (15) is related to the weighted magnetic Sobolev space $\mathcal{H}_t^{\mathbf{A}}$ defined, for every $t > 0$, as the completion of $C_c^\infty(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ with respect to the norm

$$\|u\|_{\mathcal{H}_t^{\mathbf{A}}} = \bigg(\int_{\mathbb{R}^N} \big(t|\nabla_{\mathbf{A}} u(x)|^2 + |u(x)|^2\big) G(x,t)\, dx\bigg)^{1/2},$$

where $\nabla_{\mathbf{A}} u = \nabla u + i\frac{\mathbf{A}(x/|x|)}{|x|} u$. For $\mathbf{A} \equiv 0$ we recover the Hilbert Space

$$H_t := \mathcal{H}_t^0$$

described in [16]. From [23, Proposition 3.1] (see Lemma 3.1 in section 3) it follows that

$$\text{if } N \geqslant 3 \text{ then } \mathcal{H}_t = H_t = \mathcal{H}_t^0 \text{ and the norms } \|\cdot\|_{\mathcal{H}_t}, \|\cdot\|_{H_t} \text{ are equivalent.}$$

On the other hand, if $N = 2$ we have that $\mathcal{H}_t \subset H_t$ and $\mathcal{H}_t \neq H_t$.

We also notice that $\mathcal{H}_t$ is continuously embedded into $\mathcal{H}_t^{\mathbf{A}}$; a detailed comparison between spaces $\mathcal{H}_t$ and $\mathcal{H}_t^{\mathbf{A}}$ will be performed in section 3.

We denote as $\mathcal{H}_t^\star$ the dual space of $\mathcal{H}_t$ and by ${}_{\mathcal{H}_t^\star}\langle\cdot,\cdot\rangle_{\mathcal{H}_t}$ the corresponding duality product.

In order to prove Theorem 1.2, up to a translation it is not restrictive to assume that

$$t_0 = 0.$$

Furthermore, to simplify notations and work with positive times t, it is convenient to perform the change of variable $(x,t)\mapsto(x,-t)$. Indeed, if $u(x,t)$ is solution to (1) in $\mathbb{R}^N\times(-T,0)$, then $\tilde u(x,t)=u(x,-t)$ solves

$$-\tilde u_t(x,t)+\mathcal{L}_{\mathbf{A},a}\tilde u(x,t)=h(x,-t)\tilde u(x,t) \text{ in } \mathbb{R}^N\times(0,T). \tag{26}$$

Definition 2.1. We say that $\tilde u\in L^1_{\rm loc}(\mathbb{R}^N\times(0,T))$ is a weak solution to (26) in $\mathbb{R}^N\times(0,T)$ if

$$\int_\tau^T \|\tilde u(\cdot,t)\|^2_{\mathcal{H}_t}\,dt<+\infty,\qquad \int_\tau^T \left\|\tilde u_t+\frac{\nabla_{\mathbf{A}}\tilde u\cdot x}{2t}\right\|^2_{\mathcal{H}_t^\star}dt<+\infty \text{ for all } \tau\in(0,T), \tag{27}$$

$$\begin{aligned} &{}_{\mathcal{H}_t^\star}\left\langle \tilde u_t+\frac{\nabla_{\mathbf{A}}\tilde u\cdot x}{2t},\phi\right\rangle_{\mathcal{H}_t} \\ &\quad=\int_{\mathbb{R}^N}\bigg(\nabla_{\mathbf{A}}\tilde u(x,t)\cdot\overline{\nabla_{\mathbf{A}}\phi(x)}-\frac{a(x/|x|)}{|x|^2}\,\tilde u(x,t)\overline{\phi(x)}-h(x,-t)\tilde u(x,t)\overline{\phi(x)}\bigg)G(x,t)\,dx \end{aligned} \tag{28}$$

for a.e. $t\in(0,T)$ and for each $\phi\in\mathcal{H}_t$.

Remark 2.2. In view of (2) we have that $\nabla_{\mathbf{A}}\tilde u\cdot x=\nabla\tilde u\cdot x$. Therefore, if $\tilde u\in L^1_{\rm loc}(\mathbb{R}^N\times(0,T))$ satisfies (27), then the function

$$v(x,t):=\tilde u(\sqrt{t}x,t) \tag{29}$$

satisfies

$$v\in L^2(\tau,T;\mathcal{H})\quad\text{and}\quad v_t\in L^2(\tau,T;\mathcal{H}^\star)\quad\text{for all }\tau\in(0,T), \tag{30}$$

where $\mathcal{H}:=\mathcal{H}_1$ is defined in (8). From (30) it follows that $v\in C^0([\tau,T],\mathcal{L})$ (see e.g. [24]), being $\mathcal{L}:=\mathcal{L}_1$, i.e. $\mathcal{L}$ is the completion of $C^\infty_{\rm c}(\mathbb{R}^N,\mathbb{C})$ with respect to the norm $\|v\|_{\mathcal{L}}=\big(\int_{\mathbb{R}^N}|v(x)|^2e^{-|x|^2/4}\,dx\big)^{1/2}$. Moreover the function

$$t\in[\tau,T]\mapsto\|v(t)\|^2_{\mathcal{L}}=\int_{\mathbb{R}^N}|\tilde u(x,t)|^2G(x,t)\,dx$$

is absolutely continuous and

$$\begin{aligned}\frac12\frac1{dt}\int_{\mathbb{R}^N}|\tilde u(x,t)|^2G(x,t)&=\frac12\frac1{dt}\|v(t)\|^2_{\mathcal{L}}=\mathfrak{Re}\Big[{}_{\mathcal{H}^\star}\langle v_t(\cdot,t),v(\cdot,t)\rangle_{\mathcal{H}}\Big]\\ &=\mathfrak{Re}\bigg[{}_{\mathcal{H}_t^\star}\left\langle\tilde u_t+\frac{\nabla_{\mathbf{A}}\tilde u\cdot x}{2t},\tilde u(\cdot,t)\right\rangle_{\mathcal{H}_t}\bigg]\end{aligned}$$

for a.e. $t\in(0,T)$.

Remark 2.3. If u is a weak solution to (1) in the sense of definition 2.1, then the function $v(x,t):=\tilde u(\sqrt{t}x,t)$ defined in (29) is a weak solution to

$$v_t+\frac1t\Big(-\mathcal{L}_{\mathbf{A},a}v-\frac{x}{2}\cdot\nabla v+th(\sqrt{t}x,-t)v\Big)=0,$$

in the sense that, for every $\phi \in \mathcal{H}$,

$$\begin{aligned}(31)\qquad & {}_{\mathcal{H}^\star}\langle v_t, \phi \rangle_{\mathcal{H}} \\ & = \frac{1}{t}\int_{\mathbb{R}^N}\bigg(\nabla_{\mathbf{A}} v(x,t)\cdot\overline{\nabla_{\mathbf{A}}\phi(x)} - \frac{a\big(\frac{x}{|x|}\big)}{|x|^2}\, v(x,t)\overline{\phi(x)} - t\,h(\sqrt{t}x,-t)v(x,t)\overline{\phi(x)}\bigg)G(x,1)\,dx.\end{aligned}$$

3. Magnetic parabolic Hardy type inequalities and weighted Sobolev embeddings

The following Hardy type inequality for parabolic operators was proved in [23, Proposition 3.1].

Lemma 3.1. *For every $t>0$, $N \geqslant 3$ and $u \in H_t = \mathcal{H}_t^0$ there holds*

$$\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^2}\,G(x,t)\,dx \leqslant \frac{1}{(N-2)t}\int_{\mathbb{R}^N}|u(x)|^2G(x,t)\,dx + \frac{4}{(N-2)^2}\int_{\mathbb{R}^N}|\nabla u(x)|^2\,G(x,t)\,dx.$$

In order to compare the space $\mathcal{H}_t$ with the magnetic space $\mathcal{H}_t^{\mathbf{A}}$, we recall the well-known *diamagnetic inequality*: for all $u \in \mathcal{H}_t^{\mathbf{A}}$,

$$|\nabla|u|(x)| \leqslant \big|\nabla_{\mathbf{A}}u(x)\big|, \quad \text{for a.e. } x \in \mathbb{R}^N. \tag{32}$$

Lemma 3.1 and the diamagnetic inequality (32) easily imply that

$$\text{if } N \geqslant 3 \text{ and } \mathbf{A} \in C^1(\mathbb{S}^{N-1},\mathbb{R}^N) \text{ then } \mathcal{H}_t^{\mathbf{A}} = H_t = \mathcal{H}_t \text{ for all } t>0.$$

The following lemma extends the Hardy type inequality of Lemma 3.1 to the electromagnetic case; we notice that the presence of an electromagnetic potential satisfying the positivity condition (11) allows recovering a Hardy inequality even in dimension 2.

Lemma 3.2. *Let $N \geqslant 2$, $a \in L^\infty\big(\mathbb{S}^{N-1},\mathbb{R}\big)$ and let $\mathbf{A} \in C^1(\mathbb{S}^{N-1},\mathbb{R}^N)$ satisfy the transversality condition* (2)*. For every $t>0$ and $u \in \mathcal{H}_t$, there holds*

$$\begin{aligned}\int_{\mathbb{R}^N}\bigg(|\nabla_{\mathbf{A}}u(x)|^2 - \frac{a(x/|x|)}{|x|^2}\,|u(x)|^2\bigg)G(x,t)\,dx + \frac{N-2}{4t}\int_{\mathbb{R}^N}|u(x)|^2G(x,t)\,dx \\ \geqslant \bigg(\mu_1(\mathbf{A},a) + \frac{(N-2)^2}{4}\bigg)\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^2}\,G(x,t)\,dx.\end{aligned}$$

Proof. Let $u \in C_c^\infty(\mathbb{R}^N\setminus\{0\},\mathbb{C})$. The magnetic gradient of u can be written in polar coordinates as

$$\nabla_{\mathbf{A}}u(x) = \bigg(\nabla + i\frac{\mathbf{A}(\theta)}{|x|}\bigg)u = \big(\partial_r u(r\theta)\big)\theta + \frac{1}{r}\nabla_{\mathbb{S}^{N-1}}u(r\theta) + i\frac{\mathbf{A}(\theta)}{r}u(r\theta), \quad r=|x|,\ \theta = \frac{x}{|x|},$$

hence, in view of (2),

$$|\nabla_{\mathbf{A}}u(x)|^2 = \big|\partial_r u(r\theta)\big|^2 + \frac{1}{r^2}\big|(\nabla_{\mathbb{S}^{N-1}} + i\mathbf{A})u(r\theta)\big|^2.$$

Hence

$$\int_{\mathbb{R}^N}\left(|\nabla_{\mathbf{A}}u(x)|^2-\frac{a(x/|x|)}{|x|^2}\,|u(x)|^2\right)G(x,t)\,dx \tag{33}$$
$$= t^{-\frac{N}{2}}\int_{\mathbb{S}^{N-1}}\left(\int_0^{+\infty} r^{N-1}e^{-\frac{r^2}{4t}}|\partial_r u(r\theta)|^2\,dr\right)dS(\theta)$$
$$+\,t^{-\frac{N}{2}}\int_0^{+\infty}\frac{r^{N-1}e^{-\frac{r^2}{4t}}}{r^2}\left(\int_{\mathbb{S}^{N-1}}\left[|(\nabla_{\mathbb{S}^{N-1}}+i\mathbf{A})u(r\theta)|^2-a(\theta)|u(r\theta)|^2\right]\,dS(\theta)\right)dr.$$

For all $\theta\in\mathbb{S}^{N-1}$, let $\varphi_\theta\in C^\infty_{\rm c}((0,+\infty),\mathbb{C})$ be defined as $\varphi_\theta(r)=u(r\theta)$, and $\widetilde{\varphi}_\theta\in C^\infty_{\rm c}(\mathbb{R}^N\setminus\{0\},\mathbb{C})$ be the radially symmetric function given by $\widetilde{\varphi}_\theta(x)=\varphi_\theta(|x|)$. If $N\geqslant 3$, from Lemma 3.1, it follows that

$$t^{-\frac{N}{2}}\int_{\mathbb{S}^{N-1}}\left(\int_0^{+\infty} r^{N-1}e^{-\frac{r^2}{4t}}|\partial_r u(r\theta)|^2\,dr\right)dS(\theta) \tag{34}$$
$$= t^{-\frac{N}{2}}\int_{\mathbb{S}^{N-1}}\left(\int_0^{+\infty} r^{N-1}e^{-\frac{r^2}{4t}}|\varphi_\theta'(r)|^2\,dr\right)dS(\theta)$$
$$=\frac{1}{\omega_{N-1}}\int_{\mathbb{S}^{N-1}}\left(\int_{\mathbb{R}^N}|\nabla\widetilde{\varphi}_\theta(x)|^2G(x,t)\,dx\right)dS(\theta)$$
$$\geqslant\frac{1}{\omega_{N-1}}\frac{(N-2)^2}{4}\int_{\mathbb{S}^{N-1}}\left(\int_{\mathbb{R}^N}\frac{|\widetilde{\varphi}_\theta(x)|^2}{|x|^2}G(x,t)\,dx\right)dS(\theta)$$
$$-\frac{1}{\omega_{N-1}}\frac{N-2}{4t}\int_{\mathbb{S}^{N-1}}\left(\int_{\mathbb{R}^N}|\widetilde{\varphi}_\theta(x)|^2G(x,t)\,dx\right)dS(\theta)$$
$$= t^{-\frac{N}{2}}\frac{(N-2)^2}{4}\int_{\mathbb{S}^{N-1}}\left(\int_0^{+\infty}\frac{r^{N-1}e^{-\frac{r^2}{4t}}}{r^2}|u(r\theta)|^2\,dr\right)dS(\theta)$$
$$- t^{-\frac{N}{2}}\frac{N-2}{4t}\int_{\mathbb{S}^{N-1}}\left(\int_0^{+\infty} r^{N-1}e^{-\frac{r^2}{4t}}|u(r\theta)|^2\,dr\right)dS(\theta)$$
$$=\frac{(N-2)^2}{4}\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^2}\,G(x,t)\,dx-\frac{N-2}{4t}\int_{\mathbb{R}^N}|u(x)|^2G(x,t)\,dx,$$

where ω_{N-1} denotes the volume of the unit sphere $\mathbb{S}^{N-1}$, i.e. $\omega_{N-1}=\int_{\mathbb{S}^{N-1}}dS(\theta)$. If $N=2$ we have trivially that

$$t^{-\frac{N}{2}}\int_{\mathbb{S}^{N-1}}\left(\int_0^{+\infty} r^{N-1}e^{-\frac{r^2}{4t}}|\partial_r u(r\theta)|^2\,dr\right)dS(\theta)\geqslant 0. \tag{35}$$

On the other hand, from the definition of $\mu_1(\mathbf{A},a)$ it follows that

$$\int_{\mathbb{S}^{N-1}}\left[|(\nabla_{\mathbb{S}^{N-1}}+i\mathbf{A}(\theta))u(r\theta)|^2-a(\theta)|u(r\theta)|^2\right]dS(\theta)\geqslant\mu_1(\mathbf{A},a)\int_{\mathbb{S}^{N-1}}|u(r\theta)|^2dS(\theta). \tag{36}$$

From (33), (34), (35), and (36), we deduce the stated inequality for all $u\in C^\infty_{\rm c}(\mathbb{R}^N\setminus\{0\},\mathbb{C})$. The conclusion follows by density of $C^\infty_{\rm c}(\mathbb{R}^N\setminus\{0\},\mathbb{C})$ in $\mathcal{H}_t$. □

Lemma 3.2 allows extending the Hardy type inequality of Lemma 3.1 to the case $N=2$ in the presence of a vector potential satisfying a suitable non-degeneracy condition. Indeed Lemma 3.2

implies that, if $N=2$, $t>0$ and $u\in\mathcal{H}_t$, then

$$\int_{\mathbb{R}^N} |\nabla_{\mathbf{A}} u(x)|^2\, G(x,t)\,dx \geqslant \mu_1(\mathbf{A},0)\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2}\, G(x,t)\,dx, \tag{37}$$

thus giving a Hardy type inequality if $\mu_1(\mathbf{A},0)>0$. As observe in [14, Section 2], the condition $\mu_1(\mathbf{A},0)>0$ holds if and only if

$$\Phi_{\mathbf{A}} := \frac{1}{2\pi}\int_0^{2\pi} \alpha(t)\,dt \notin \mathbb{Z}, \quad \text{where } \alpha(t) := \mathbf{A}(\cos t, \sin t)\cdot(-\sin t, \cos t). \tag{38}$$

Condition (38) is related the following Hardy inequality proved in [20]:

$$\Big(\min_{k\in\mathbb{Z}} |k-\Phi_{\mathbf{A}}|\Big)^2 \int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x|^2}\,dx \leqslant \int_{\mathbb{R}^2} \left|\nabla u(x) + i\,\frac{\mathbf{A}\big(x/|x|\big)}{|x|}\,u(x)\right|^2 dx \tag{39}$$

which holds for all functions $u\in C^\infty_{\rm c}(\mathbb{R}^N\setminus\{0\},\mathbb{C})$. Moreover $\big(\min_{k\in\mathbb{Z}}|k-\Phi_{\mathbf{A}}|\big)^2=\mu_1(\mathbf{A},0)$ is the best constant in (39).

From (37) it follows that

$$\text{if } N=2 \text{ and (2) and (38) hold, then that } \mathcal{H}_t=\mathcal{H}_t^{\mathbf{A}},$$

being the norms $\|\cdot\|_{\mathcal{H}_t}$, $\|\cdot\|_{\mathcal{H}_t^{\mathbf{A}}}$ equivalent. On the other hand, if $N=2$ and $\Phi_{\mathbf{A}}\in\mathbb{Z}$ (i.e. $\mu_1(\mathbf{A},0)=0$), then $\mathbf{A}$ is gauge equivalent to 0 and $\mathcal{H}_t\subset\mathcal{H}_t^{\mathbf{A}}$, $\mathcal{H}_t\neq\mathcal{H}_t^{\mathbf{A}}$; this case is actually not very interesting since it can be reduced to the magnetic free problem by a gauge transformation.

The following corollary provides a positivity condition for the quadratic form associated to the electromagnetic potential under condition (11).

Corollary 3.3. *Let $N\geqslant 2$, $a\in L^\infty\big(\mathbb{S}^{N-1},\mathbb{R}\big)$ and let $\mathbf{A}\in C^1(\mathbb{S}^{N-1},\mathbb{R}^N)$ satisfy (2) and (11). Then, for every $t>0$,*

$$\inf_{u\in\mathcal{H}_t\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\big(|\nabla_{\mathbf{A}}u(x)|^2-\frac{a(x/|x|)}{|x|^2}|u(x)|^2\big)G(x,t)\,dx+\frac{N-2}{4t}\int_{\mathbb{R}^N}|u(x)|^2G(x,t)\,dx}{\int_{\mathbb{R}^N}|\nabla u(x)|^2G(x,t)\,dx+\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^2}G(x,t)\,dx+\frac{N-2}{4t}\int_{\mathbb{R}^N}|u(x)|^2G(x,t)\,dx}$$

$$=\inf_{v\in\mathcal{H}\setminus\{0\}} \frac{\int_{\mathbb{R}^N}\big(|\nabla_{\mathbf{A}}v(x)|^2-\frac{a(x/|x|)}{|x|^2}|v(x)|^2\big)G(x,1)\,dx+\frac{N-2}{4}\int_{\mathbb{R}^N}|v(x)|^2G(x,1)\,dx}{\int_{\mathbb{R}^N}|\nabla v(x)|^2G(x,1)\,dx+\int_{\mathbb{R}^N}\frac{|v(x)|^2}{|x|^2}G(x,1)\,dx+\frac{N-2}{4}\int_{\mathbb{R}^N}|v(x)|^2G(x,1)\,dx}>0.$$

Proof. The change of variables $u(x)=v(x/\sqrt{t})$ immediately gives the equality of the two infimum levels. To prove that they are strictly positive, we argue by contradiction. Let us assume that for every $\varepsilon>0$ there exists $v_\varepsilon\in\mathcal{H}\setminus\{0\}$ such that

$$\begin{aligned}\int_{\mathbb{R}^N}\bigg(|\nabla_{\mathbf{A}}v_\varepsilon(x)|^2-\frac{a(x/|x|)}{|x|^2}|v_\varepsilon(x)|^2\bigg)G(x,1)\,dx+\frac{N-2}{4}\int_{\mathbb{R}^N}|v_\varepsilon(x)|^2G(x,1)\,dx\\ <\varepsilon\bigg(\int_{\mathbb{R}^N}|\nabla v_\varepsilon(x)|^2G(x,1)\,dx+\int_{\mathbb{R}^N}\frac{|v_\varepsilon(x)|^2}{|x|^2}G(x,1)\,dx+\frac{N-2}{4}\int_{\mathbb{R}^N}|v_\varepsilon(x)|^2G(x,1)\,dx\bigg).\end{aligned}$$

Hence

$$
\begin{aligned}
\int_{\mathbb{R}^N} \bigg(|\nabla_{\mathbf{A}} v_\varepsilon(x)|^2 - \frac{a(x/|x|)}{|x|^2}\,|v_\varepsilon(x)|^2 \bigg) G(x,1)\,dx + \frac{N-2}{4}\int_{\mathbb{R}^N} |v_\varepsilon(x)|^2 G(x,1)\,dx \\
< \varepsilon \int_{\mathbb{R}^N} \bigg(|\nabla_{\mathbf{A}} v_\varepsilon(x)|^2 + (1+\|\mathbf{A}\|_{L^\infty(\mathbb{S}^{N-1})}) \frac{|v_\varepsilon(x)|^2}{|x|^2} + \frac{N-2}{4}|v_\varepsilon(x)|^2 \bigg) G(x,1)\,dx \\
\leqslant \varepsilon (1+\|\mathbf{A}\|_{L^\infty(\mathbb{S}^{N-1})}) \int_{\mathbb{R}^N} \bigg(|\nabla_{\mathbf{A}} v_\varepsilon(x)|^2 + \frac{|v_\varepsilon(x)|^2}{|x|^2} + \frac{N-2}{4}|v_\varepsilon(x)|^2 \bigg) G(x,1)\,dx.
\end{aligned}
$$

From the above inequality and Lemma 3.2 it follows that

$$
\begin{aligned}
\bigg(\mu_1\bigg(\mathbf{A}, \frac{a}{1-(1+\|\mathbf{A}\|_{L^\infty(\mathbb{S}^{N-1})})\varepsilon}\bigg) + \frac{(N-2)^2}{4} \bigg) \int_{\mathbb{R}^N} \frac{|v_\varepsilon(x)|^2}{|x|^2}\, G(x,1)\,dx \\
\leqslant \int_{\mathbb{R}^N} \bigg(|\nabla_{\mathbf{A}} v_\varepsilon(x)|^2 - \frac{a(x/|x|)}{1-(1+\|\mathbf{A}\|_{L^\infty(\mathbb{S}^{N-1})})\varepsilon} \frac{|v_\varepsilon(x)|^2}{|x|^2} \bigg) G(x,1)\,dx \\
+ \frac{N-2}{4} \int_{\mathbb{R}^N} |v_\varepsilon(x)|^2 G(x,1)\,dx \\
< \frac{(1+\|\mathbf{A}\|_{L^\infty(\mathbb{S}^{N-1})})\varepsilon}{1-(1+\|\mathbf{A}\|_{L^\infty(\mathbb{S}^{N-1})})\varepsilon} \int_{\mathbb{R}^N} \frac{|v_\varepsilon(x)|^2}{|x|^2}\, G(x,1)\,dx,
\end{aligned}
$$

hence

$$
\mu_1\bigg(\mathbf{A}, \frac{a}{1-(1+\|\mathbf{A}\|_{L^\infty(\mathbb{S}^{N-1})})\varepsilon}\bigg) + \frac{(N-2)^2}{4} < \frac{(1+\|\mathbf{A}\|_{L^\infty(\mathbb{S}^{N-1})})\varepsilon}{1-(1+\|\mathbf{A}\|_{L^\infty(\mathbb{S}^{N-1})})\varepsilon}.
$$

From (9) it follows easily that, for fixed $\mathbf{A} \in C^1(\mathbb{S}^{N-1}, \mathbb{R}^N)$, the map $a \mapsto \mu_1(\mathbf{A}, a)$ is continuous with respect to the $L^\infty(\mathbb{S}^{N-1})$-norm. Therefore, letting $\varepsilon \to 0$ in the above inequality, we obtain $\mu_1(\mathbf{A}, a) + \frac{(N-2)^2}{4} \leqslant 0$, thus contradicting assumption (11). □

The negligibility assumption (6) allows treating h as a lower order potential, recovering for small time the positivity of the quadratic form associated to (26).

Corollary 3.4. *Let $N \geqslant 2$, $a \in L^\infty(\mathbb{S}^{N-1}, \mathbb{R})$ and let $\mathbf{A} \in C^1(\mathbb{S}^{N-1}, \mathbb{R}^N)$ satisfy* (2) *and* (11)*. Let $T > 0$, $I = (-T, 0)$ and $h \in L^\infty_{\rm loc}((\mathbb{R}^N \setminus \{0\}) \times (-T, 0))$ satisfy (6) in I. Then there exist $C_1, C_2 > 0$ and $\overline{T} > 0$ such that for every $t \in (0, \overline{T})$, $s \in (-T, 0)$, and $u \in \mathcal{H}_t$*

$$
\begin{aligned}
\int_{\mathbb{R}^N} \bigg(|\nabla_{\mathbf{A}} u(x)|^2 - \frac{a(x/|x|)}{|x|^2}\,|u(x)|^2 - |h(x,s)||u(x)|^2 \bigg) G(x,t)\,dx \\
\geqslant C_1 \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2}\, G(x,t)\,dx - \frac{C_2}{t} \int_{\mathbb{R}^N} |u(x)|^2 G(x,t)\,dx \\
\int_{\mathbb{R}^N} \bigg(|\nabla_{\mathbf{A}} u(x)|^2 - \frac{a(x/|x|)}{|x|^2}\,|u(x)|^2 - |h(x,s)||u(x)|^2 \bigg) G(x,t)\,dx + \frac{N-2}{4t} \int_{\mathbb{R}^N} |u(x)|^2 G(x,t)\,dx \\
\geqslant C_1 \bigg(\int_{\mathbb{R}^N} |\nabla u(x)|^2\, G(x,t)\,dx + \int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2}\, G(x,t)\,dx + \frac{1}{t} \int_{\mathbb{R}^N} |u(x)|^2 G(x,t)\,dx \bigg).
\end{aligned}
$$

PROOF. From (6) it follows that, for every $u \in \mathcal{H}_t$,

$$
\begin{aligned}
&\left|\int_{\mathbb{R}^N} |h(x,s)||u(x)|^2 G(x,t)\,dx\right| \\
&\leqslant C_h\bigg(\int_{\mathbb{R}^N} |u(x)|^2 G(x,t)\,dx + \int_{\mathbb{R}^N} |x|^{-2+\varepsilon}|u(x)|^2 G(x,t)\,dx\bigg) \\
&\leqslant C_h\bigg(\int_{\mathbb{R}^N} |u(x)|^2 G(x,t)\,dx + t^{\varepsilon/2}\int_{|x|\leqslant\sqrt{t}} \frac{|u(x)|^2}{|x|^2} G(x,t)\,dx \\
&\qquad + t^{-1+\varepsilon/2}\int_{|x|\geqslant\sqrt{t}} |u(x)|^2 G(x,t)\,dx\bigg) \\
&= \frac{C_h}{t}(t+t^{\varepsilon/2})\int_{\mathbb{R}^N} |u(x)|^2 G(x,t)\,dx + C_h t^{\varepsilon/2}\int_{\mathbb{R}^N} \frac{|u(x)|^2}{|x|^2} G(x,t)\,dx.
\end{aligned} \tag{40}
$$

The stated inequalities follow from (40), Lemma 3.2, Corollary 3.3, and assumption (11). □

The proof of the following inequality follows the spirit of [9, Lemma 3].

Lemma 3.5. *For every $u \in \mathcal{H}$, $|x|u \in \mathcal{L}$ and*

$$
\frac{1}{16}\int_{\mathbb{R}^N} |x|^2|u(x)|^2 G(x,1)\,dx \leqslant \int_{\mathbb{R}^N} |\nabla_{\mathbf{A}} u(x)|^2 G(x,1)\,dx + \frac{N}{4}\int_{\mathbb{R}^N} |u(x)|^2 G(x,1)\,dx.
$$

PROOF. Let $u \in C^\infty_c(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ and $w = e^{-\frac{|x|^2}{8}}u$. It follows that

$$
\nabla_{\mathbf{A}} w = e^{-\frac{|x|^2}{8}}\bigg(\Big(\nabla + i\frac{\mathbf{A}}{|x|}\Big)u - \frac{ux}{4}\bigg).
$$

Hence, by the transversality condition (2), an integration by parts yields that

$$
\begin{aligned}
\int_{\mathbb{R}^N} |\nabla_{\mathbf{A}} w|^2\,dx &= \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{4}}\left(\left|\big(\nabla + i\tfrac{\mathbf{A}}{|x|}\big)u\right|^2 + \frac{|x|^2|u|^2}{16} - \frac{1}{2}\mathfrak{Re}\left(\left(\nabla + i\tfrac{\mathbf{A}}{|x|}\right)u\cdot x\,\bar{u}\right)\right)dx \\
&= \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{4}}\left(\left|\big(\nabla + i\tfrac{\mathbf{A}}{|x|}\big)u\right|^2 + \frac{|x|^2|u|^2}{16} - \frac{1}{4}\nabla|u|^2\cdot x\right)dx \\
&= \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{4}}|\nabla_{\mathbf{A}} u|^2\,dx - \frac{1}{16}\int_{\mathbb{R}^N} e^{-\frac{|x|^2}{4}}|x|^2|u|^2\,dx + \frac{N}{4}\int_{\mathbb{R}^N} e^{-\frac{|x|^2}{4}}|u|^2\,dx \geqslant 0.
\end{aligned}
$$

The conclusion follows by density of $C^\infty_c(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ in $\mathcal{H}$. □

Using the change of variables $u(x) = v(x/\sqrt{t})$, it is easy to verify that Lemma 3.5 implies the following inequality in $\mathcal{H}_t$.

Corollary 3.6. *For every $t > 0$ and $u \in \mathcal{H}_t$, there holds*

$$
\frac{1}{16t^2}\int_{\mathbb{R}^N} |x|^2|u(x)|^2 G(x,t)\,dx \leqslant \int_{\mathbb{R}^N} |\nabla_{\mathbf{A}} u(x)|^2 G(x,t)\,dx + \frac{N}{4t}\int_{\mathbb{R}^N} |u(x)|^2 G(x,t)\,dx.
$$

The following weighted Sobolev type inequalities hold.

Lemma 3.7. *For all $v \in \mathcal{H}$ there holds $v\sqrt{G(\cdot,1)} \in L^s(\mathbb{R}^N)$ for all $s \in [2, \frac{2N}{N-2}]$ if $N \geqslant 3$ and for all $s \geqslant 2$ if $N = 2$; furthermore*

$$\left(\int_{\mathbb{R}^N} |v(x)|^s G^{\frac{s}{2}}(x,1)\,dx\right)^{\frac{2}{s}} \leqslant C_s \|v\|^2_{\mathcal{H}}$$

for some constant $C_s > 0$ independent of $v \in \mathcal{H}$. Moreover, for every $t > 0$ and $u \in \mathcal{H}_t$,

$$\left(\int_{\mathbb{R}^N} |u(x)|^s G^{\frac{s}{2}}(x,t)\,dx\right)^{\frac{2}{s}} \leqslant C_s t^{-\frac{N}{s}\left(\frac{s-2}{2}\right)} \|u\|^2_{\mathcal{H}_t},$$

for all $s \in [2, \frac{2N}{N-2}]$ if $N \geqslant 3$ and for all $s \geqslant 2$ if $N = 2$, with $C_s > 0$ as above.

Proof. Lemma 3.5 implies that, if $v \in \mathcal{H}$, then $v\sqrt{G(\cdot,1)} \in H^1(\mathbb{R}^N)$ and the first embedding follows from classical Sobolev inequalities and Lemma 3.5. The second inequality follows directly from the first one and the change of variables $u(x) = v(x/\sqrt{t})$. □

4. Spectrum of Ornstein-Uhlenbeck type operators with critical electromagnetic potentials

In this section we describe the spectral properties of the operator $L_{\mathbf{A},a} = \mathcal{L}_{\mathbf{A},a} + \frac{x}{2} \cdot \nabla$ defined in (7). In particular we extend to the general critical electromagnetic case previous analogous results obtained in [26] for $\mathbf{A} \equiv 0$ and $a \equiv \lambda$ constant and in [16] for $\mathbf{A} \equiv 0$.

In order to apply the Spectral Theorem to the operator $L_{\mathbf{A},a}$, some compactness is first needed. With this aim, following [12], we prove the following compact embedding.

Lemma 4.1. *The space $\mathcal{H}$ is compactly embedded in $\mathcal{L}$.*

Proof. Let us assume that $u_k \rightharpoonup u$ weakly in $\mathcal{H}$. From Rellich's theorem $u_k \to u$ in $L^2_{\rm loc}(\mathbb{R}^N)$, i.e. $u_k \to u$ strongly in $L^2(\Omega)$ for all $\Omega \subset\subset \mathbb{R}^N$. For every $R > 0$ and $k \in \mathbb{N}$, we have

$$\int_{\mathbb{R}^N} |u_k - u|^2 G(x,1)\,dx = A_k(R) + B_k(R) \tag{41}$$

where

$$A_k(R) = \int_{\{|x| \leqslant R\}} |u_k(x) - u(x)|^2 e^{-|x|^2/4}\,dx \to 0 \quad \text{as } k \to +\infty, \quad \text{for every } R > 0, \tag{42}$$

and

$$B_k(R) = \int_{\{|x|>R\}} |u_k(x) - u(x)|^2 G(x,1)\,dx.$$

From Lemma 3.5 and boundedness of u_k in $\mathcal{H}$, we deduce that

$$\begin{aligned} B_k(R) &\leqslant R^{-2} \int_{\{|x|>R\}} |x|^2 |u_k(x) - u(x)|^2 G(x,1)\,dx \\ &\leqslant \frac{1}{R^2}\left(16 \int_{\mathbb{R}^N} |\nabla_{\mathbf{A}}(u_k - u)(x)|^2 G(x,1)\,dx + 4N \int_{\mathbb{R}^N} |u_k(x) - u(x)|^2 G(x,1)\,dx\right) \leqslant \frac{\text{const}}{R^2}. \end{aligned} \tag{43}$$

Combining (41), (42), and (43), we obtain that $u_k \to u$ strongly in $\mathcal{L}$. □

In the proof of the representation formula for solutions stated in Theorem 1.5 and performed in section 7 also the forward Ornstein-Uhlenbeck operators $L^-_{\mathbf{A},a} = \mathcal{L}_{\mathbf{A},a} - \frac{x}{2} \cdot \nabla$ will come into play. In order to study its spectral decomposition, we introduce the forward analogue of the spaces $\mathcal{H}$ and $\mathcal{L}$. More precisely, we define the space $\tilde{\mathcal{L}}$ as in (24) and $\tilde{\mathcal{H}}$ as the completion of $C^\infty_c(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ with respect to

$$\|\varphi\|_{\tilde{\mathcal{H}}} = \left(\int_{\mathbb{R}^N} \left(|\nabla\varphi(x)|^2 + |\varphi(x)|^2 + \frac{|\varphi(x)|^2}{|x|^2} \right) e^{\frac{|x|^2}{4}} \, dx \right)^{1/2}. \tag{44}$$

Proposition 4.2. *Let $\tilde{T} : \mathcal{H} \to \tilde{\mathcal{H}}$ be defined as $\tilde{T}u(x) = e^{-\frac{|x|^2}{4}} u(x)$. Then,*

i) *$\tilde{T} : \mathcal{L} \to \tilde{\mathcal{L}}$ is an isometry;*
ii) *$\tilde{T} : \mathcal{H} \to \tilde{\mathcal{H}}$ is an isomorphism of normed vector spaces.*

Proof. We first observe that, if $\varphi = \tilde{T}u$, then

$$\int_{\mathbb{R}^N} |\varphi(x)|^2 e^{\frac{|x|^2}{4}} \, dx = \int_{\mathbb{R}^N} |u(x)|^2 e^{-\frac{|x|^2}{4}} \, dx.$$

Hence, $\|\varphi\|_{\tilde{\mathcal{L}}} = \|u\|_{\mathcal{L}} = \|\tilde{T}u\|_{\tilde{\mathcal{L}}}$ for all $u \in \mathcal{L}$. Then i) is proved.

To prove ii), we first observe that, if $u \in C^\infty_c(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ and $v = \tilde{T}u$, there holds

$$\nabla u(x) = e^{\frac{|x|^2}{4}} \left(\frac{x}{2} v(x) + \nabla v(x) \right),$$

and hence

$$\begin{aligned} |\nabla u(x)|^2 &= e^{\frac{|x|^2}{2}} \left(\frac{|x|^2}{4} |v(x)|^2 + |\nabla v(x)|^2 + \mathfrak{Re}(\overline{v(x)} \nabla v(x) \cdot x) \right) \\ &= e^{\frac{|x|^2}{2}} \left(\frac{|x|^2}{4} |v(x)|^2 + |\nabla v(x)|^2 + \frac{1}{2} \nabla |v|^2(x) \cdot x \right). \end{aligned}$$

Then, an integration by parts yields

$$\begin{aligned} \int_{\mathbb{R}^N} |\nabla u(x)|^2 \, e^{-\frac{|x|^2}{4}} \, dx &= \int_{\mathbb{R}^N} e^{\frac{|x|^2}{4}} \left(\frac{|x|^2}{4} |v(x)|^2 + |\nabla v(x)|^2 \right) dx + \frac{1}{2} \int_{\mathbb{R}^N} e^{\frac{|x|^2}{4}} \nabla |v|^2(x) \cdot x \, dx \\ &= \int_{\mathbb{R}^N} e^{\frac{|x|^2}{4}} \left(\frac{|x|^2}{4} |v(x)|^2 + |\nabla v(x)|^2 \right) dx - \frac{1}{2} \int_{\mathbb{R}^N} e^{\frac{|x|^2}{4}} |v(x)|^2 \left(\frac{|x|^2}{2} + N \right) dx \\ &= \int_{\mathbb{R}^N} e^{\frac{|x|^2}{4}} \left(|\nabla v(x)|^2 - \frac{N}{2} |v(x)|^2 \right) dx. \end{aligned}$$

Therefore

$$\int_{\mathbb{R}^N} e^{-\frac{|x|^2}{4}} \left(|\nabla u(x)|^2 + N|u(x)|^2 + \frac{|u(x)|^2}{|x|^2} \right) dx = \int_{\mathbb{R}^N} e^{\frac{|x|^2}{4}} \left(|\nabla v(x)|^2 + \frac{N}{2} |v(x)|^2 + \frac{|v(x)|^2}{|x|^2} \right) dx.$$

The above identity and density of $C^\infty_c(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$ in $\mathcal{H}$ and $\tilde{\mathcal{H}}$ prove statement ii). □

As a consequence, we obtain the following compact embedding for $\tilde{\mathcal{H}}$.

Corollary 4.3. *The space $\tilde{\mathcal{H}}$ is compactly embedded in $\tilde{\mathcal{L}}$.*

Proof. The conclusion follows by combining Lemma 4.1 with Proposition 4.2. □

The Hardy type inequalities established in section 3 and the embeddings discussed above allow applying the classical Spectral Theorem to the operator $L_{\mathbf{A},a}$ defined in (7).

Lemma 4.4. *For $N \geqslant 2$, let $\mathbf{A} \in C^1(\mathbb{S}^{N-1}, \mathbb{R}^N)$ satisfy (2) and let $a \in L^\infty\big(\mathbb{S}^{N-1}, \mathbb{R}\big)$ be such that (11) holds. Then the spectrum of the operator $L_{\mathbf{A},a}$ defined in (7) consists of a diverging sequence of real eigenvalues with finite multiplicity. Moreover, there exists an orthonormal basis of $\mathcal{L}$ whose elements belong to $\mathcal{H}$ and are eigenfunctions of $L_{\mathbf{A},a}$.*

Proof. By Corollary 3.3 and the Lax-Milgram Theorem, the bounded linear self-adjoint operator

$$\mathcal{T}_{\mathbf{A},a} : \mathcal{L} \to \mathcal{L}, \quad \mathcal{T}_{\mathbf{A},a} = \left(L_{\mathbf{A},a} + \frac{N-2}{4} \operatorname{Id} \right)^{-1}$$

is well defined. Moreover, by Lemma 4.1, $\mathcal{T}_{\mathbf{A},a}$ is compact. The result then follows from the Spectral Theorem. □

We are now in position to prove Proposition 1.1.

Proof of Proposition 1.1. Let γ be an eigenvalue of $L_{\mathbf{A},a}$ and $g \in \mathcal{H} \setminus \{0\}$ be a corresponding eigenfunction, so that

$$\mathcal{L}_{\mathbf{A},a} g(x) + \frac{\nabla g(x) \cdot x}{2} = \gamma\, g(x) \tag{45}$$

in a weak $\mathcal{H}$-sense. From classical regularity theory for elliptic equations, $g \in C^{1,\alpha}_{\rm loc}(\mathbb{R}^N \setminus \{0\}, \mathbb{C})$. Hence g can be expanded as

$$g(x) = g(r\theta) = \sum_{k=1}^{\infty} \phi_k(r) \psi_k(\theta) \quad \text{in } L^2(\mathbb{S}^{N-1}, \mathbb{C}),$$

where $r = |x| \in (0, +\infty)$, $\theta = x/|x| \in \mathbb{S}^{N-1}$, and

$$\phi_k(r) = \int_{\mathbb{S}^{N-1}} g(r\theta) \overline{\psi_k(\theta)}\, dS(\theta),$$

with $\{\psi_k\}_k$ as in (10). Equations (10) and (45) imply that, for every k,

$$\phi_k'' + \left(\frac{N-1}{r} - \frac{r}{2} \right) \phi_k' + \left(\gamma - \frac{\mu_k(\mathbf{A}, a)}{r^2} \right) \phi_k = 0 \quad \text{in } (0, +\infty). \tag{46}$$

The rest of the proof follows exactly as in [16, Proposition 1] . □

Denoting by $L^\alpha_m(t)$ the generalized Laguerre polynomials

$$L^\alpha_m(t) = \sum_{n=0}^{m} (-1)^n \binom{m+\alpha}{m-n} \frac{t^n}{n!},$$

we have that

$$P_{k,m}\left(\frac{|x|^2}{4} \right) = \binom{m+\beta_k}{m}^{-1} L^{\beta_k}_m \Big(\frac{|x|^2}{4} \Big), \tag{47}$$

with $\beta_k = \sqrt{\big(\frac{N-2}{2}\big)^2 + \mu_k(\mathbf{A}, a)}$. It is worth recalling the well known orthogonality relation

$$\int_0^\infty x^\alpha e^{-x} L^\alpha_n(x) L^\alpha_m(x)\, dx = \frac{\Gamma(n+\alpha+1)}{n!} \delta_{n,m}, \tag{48}$$

where $\delta_{n,m}$ denotes the Kronecker delta.

Remark 4.5. For $n, j \in \mathbb{N}$, $j \geqslant 1$, let $V_{n,j}$ be defined in (14). From the orthogonality of eigenfunctions $\{\psi_k\}_k$ in $L^2(\mathbb{S}^{N-1}, \mathbb{C})$ and the orthogonality relation for Laguerre polynomials (48), it follows that

$$\text{if } (m_1, k_1) \neq (m_2, k_2) \quad \text{then} \quad V_{m_1,k_1} \text{ and } V_{m_2,k_2} \text{ are orthogonal in } \mathcal{L}.$$

By Lemma 4.4, we conclude that an orthonormal basis of $\mathcal{L}$ is given by

$$\left\{ \widetilde{V}_{n,j} = \frac{V_{n,j}}{\|V_{n,j}\|_{\mathcal{L}}} : j, n \in \mathbb{N}, j \geqslant 1 \right\}.$$

Remark 4.6. If γ is an eigenvalue and $\varphi \in \tilde{\mathcal{H}} \setminus \{0\}$ is a corresponding eigenfunction of the operator $L^-_{\mathbf{A},a}$ defined as

$$L^-_{\mathbf{A},a} : \tilde{\mathcal{H}} \to \tilde{\mathcal{H}}^\star, \quad L^-_{\mathbf{A},a}\varphi(x) = \mathcal{L}_{\mathbf{A},a}\varphi(x) - \frac{\nabla\varphi(x) \cdot x}{2}, \tag{49}$$

i.e. $L^-_{\mathbf{A},a}\varphi = \gamma\,\varphi$ in a weak $\tilde{\mathcal{H}}$-sense, then $\gamma - \frac{N}{2}$ is an eigenvalue of the operator $L_{\mathbf{A},a}$ with $u = e^{\frac{|x|^2}{4}}\varphi$ as a corresponding eigenfunction, i.e.

$$L_{\mathbf{A},a}u(x) = \mathcal{L}_{\mathbf{A},a}u(x) + \frac{\nabla u(x) \cdot x}{2} = \left(\gamma - \frac{N}{2}\right) u(x)$$

in a weak $\mathcal{H}$-sense. It follows that the set of the eigenvalues of $L^-_{\mathbf{A},a}$ is

$$\left\{ \frac{N}{2} + m - \frac{\alpha_k}{2} : k, m \in \mathbb{N}, k \geqslant 1 \right\}.$$

Moreover, letting $U_{n,j} = e^{-\frac{|x|^2}{4}} V_{n,j}$ with $V_{n,j}$ as in (14), we have that

$$\left\{ \widetilde{U}_{n,j} = \frac{U_{n,j}}{\|U_{n,j}\|_{\tilde{\mathcal{L}}}} : j, n \in \mathbb{N}, j \geqslant 1 \right\}$$

is an orthonormal basis of $\tilde{\mathcal{L}}$.

Remark 4.7. From (47) and the orthogonality relation (48), it is easy to check that

$$\begin{aligned} \|U_{m,k}\|^2_{\tilde{\mathcal{L}}} &= \int_{\mathbb{R}^N} e^{\frac{|x|^2}{4}} |U_{m,k}|^2\,dx = \int_{\mathbb{R}^N} e^{-\frac{|x|^2}{4}} |V_{m,k}|^2\,dx \\ &= \|V_{m,k}\|^2_{\mathcal{L}} = 2^{1+2\beta_k}\Gamma(1+\beta_k) \binom{m+\beta_k}{m}^{-1}. \end{aligned}$$

5. The parabolic electromagnetic Almgren monotonicity formula

Throughout this section, we assume that $N \geqslant 2$, $\mathbf{A} \in C^1(\mathbb{S}^{N-1}, \mathbb{R}^N)$ and $a \in L^\infty(\mathbb{S}^{N-1}, \mathbb{R})$ satisfy (2) and (11), and $\tilde{u}$ is a weak solution to (26) in $\mathbb{R}^N \times (0,T)$ with h satisfying (5)–(6) in $I = (-T, 0)$. Let $C_1 > 0$ and $\overline{T} > 0$ be as in Corollary 3.4 and denote

$$\alpha = \frac{T}{2\big(\lfloor T/\overline{T} \rfloor + 1\big)},$$

where $\lfloor\cdot\rfloor$ denotes the floor function, i.e. $\lfloor x \rfloor := \max\{n \in \mathbb{Z} : \ n \leqslant x\}$. Then $(0,T) = \bigcup_{j=1}^k (a_j, b_j)$ being

$$k = 2\big(\lfloor T/\overline{T} \rfloor + 1\big) - 1, \quad a_j = (j-1)\alpha, \quad \text{and} \quad b_j = (j+1)\alpha.$$

We notice that $0 < 2\alpha < \overline{T}$ and $(a_j, b_j) \cap (a_{j+1}, b_{j+1}) = (j\alpha, (j+1)\alpha) \neq \emptyset$. For every $j = 1, \dots, k$, we define

$$\tilde{u}_j(x,t) = \tilde{u}(x, t + a_j), \quad x \in \mathbb{R}^N, \ t \in (0, 2\alpha).$$

Lemma 5.1. *For every $j = 1, \dots, k$, the function $\tilde{u}_j$ defined above is a weak solution to*

$$-(\tilde{u}_j)_t(x,t) + \mathcal{L}_{\mathbf{A},a}\tilde{u}_j(x,t) = h(x, -(t+a_j))\tilde{u}_j(x,t) \tag{50}$$

in $\mathbb{R}^N \times (0, 2\alpha)$ in the sense of Definition 2.1. Furthermore, $\tilde{v}_j(x,t) := \tilde{u}_j(\sqrt{t}x, t)$ is a weak solution to

$$(\tilde{v}_j)_t + \frac{1}{t}\bigg(-\mathcal{L}_{\mathbf{A},a}\tilde{v}_j - \frac{x}{2}\cdot\nabla\tilde{v}_j + th(\sqrt{t}x, -(t+a_j))\tilde{v}_j(x,t)\bigg) = 0 \tag{51}$$

in $\mathbb{R}^N \times (0, 2\alpha)$ in the sense of Remark 2.3.

Proof. Since the proof is very similar to the proof of [16, Lemma 4.1], we omit it here, referring to [16] for details. □

For every $j = 1, \dots, k$, we define

$$H_j(t) = \int_{\mathbb{R}^N} |\tilde{u}_j(x,t)|^2\, G(x,t)\, dx, \quad \text{for every } t \in (0, 2\alpha), \tag{52}$$

and

$$D_j(t) = \int_{\mathbb{R}^N}\bigg[|\nabla_{\mathbf{A}}\tilde{u}_j(x,t)|^2 - \frac{a(x/|x|)}{|x|^2}|\tilde{u}_j(x,t)|^2 - h(x, -(t+a_j))|\tilde{u}_j(x,t)|^2\bigg]G(x,t)dx \tag{53}$$

for a.e. $t \in (0, 2\alpha)$. From Lemma 5.1 and Remark 2.2 it follows that, for every $1 \leqslant j \leqslant k$, $H_j \in W^{1,1}_{\rm loc}(0, 2\alpha)$ and

$$H_j'(t) = 2\,\mathfrak{Re}\left[{}_{\mathcal{H}_t^\star}\Big\langle (\tilde{u}_j)_t + \frac{\nabla\tilde{u}_j\cdot x}{2t}, \tilde{u}_j(\cdot,t)\Big\rangle_{\mathcal{H}_t}\right] = 2D_j(t) \quad \text{for a.e. } t \in (0, 2\alpha). \tag{54}$$

Lemma 5.2. *Letting C_1 as in Corollary 3.4, we have that, for every $j = 1, \dots, k$, the function*

$$t \mapsto t^{-2C_1 + \frac{N-2}{2}}H_j(t)$$

is nondecreasing in $(0, 2\alpha)$.

Proof. From (54), Corollary 3.4 and the fact that $2\alpha < \overline{T}$, we have that, for all $t \in (0, 2\alpha)$,

$$H_j'(t) \geqslant \frac{1}{t}\bigg(2C_1 - \frac{N-2}{2}\bigg)H_j(t),$$

which implies that $\frac{d}{dt}\big(t^{-2C_1+\frac{N-2}{2}}H_j(t)\big) \geqslant 0$, thus concluding the proof. □

Lemma 5.3. *If $1 \leqslant j \leqslant k$ and $H_j(\bar{t}) = 0$ for some $\bar{t} \in (0, 2\alpha)$, then $H_j(t) = 0$ for all $t \in (0, \bar{t}\,]$.*

Proof. From Lemma 5.2, the function $t \mapsto t^{-2C_1+\frac{N-2}{2}}H_j(t)$ is nondecreasing in $(0, 2\alpha)$, nonnegative, and vanishing at $\bar{t}$. Hence $H_j(t) = 0$ for all $t \in (0, \bar{t}]$. □

Lemma 5.4. *If $1 \leqslant j \leqslant k$ and $T_j \in (0, 2\alpha)$ is such that $\tilde{u}_j(\cdot, T_j) \in \mathcal{H}_{T_j}$, then*

(i) $\int_\tau^{T_j}\int_{\mathbb{R}^N}\big(\big|(\tilde{u}_j)_t(x,t) + \frac{\nabla_{\mathbf{A}}\tilde{u}_j(x,t)\cdot x}{2t}\big|^2 G(x,t)\,dx\big)\,dt < +\infty$ *for all $\tau \in (0, T_j)$;*

(ii) *the function* $t \mapsto tD_j(t)$ *belongs to* $W^{1,1}_{\rm loc}(0,T_j)$ *and, for a.e.* $t \in (0,T_j)$,

$$\begin{aligned}\frac{d}{dt}\big(tD_j(t)\big) &= 2t\int_{\mathbb{R}^N}\left|(\tilde u_j)_t(x,t)+\frac{\nabla\tilde u_j(x,t)\cdot x}{2t}\right|^2 G(x,t)\,dx\\ &+\int_{\mathbb{R}^N} h(x,-(t+a_j))\bigg(\frac{N-2}{2}|\tilde u_j(x,t)|^2+\mathfrak{Re}(\tilde u_j(x,s)\overline{\nabla\tilde u_j(x,s)\cdot x})-\frac{|x|^2}{4t}|\tilde u_j(x,t)|^2\bigg)G(x,t)\,dx\\ &+t\int_{\mathbb{R}^N} h_t(x,-(t+a_j))|\tilde u_j(x,t)|^2 G(x,t)\,dx.\end{aligned}$$

Proof. Testing equation (51) with $(\tilde v_j)_t$ (this formal testing procedure can be made rigorous by a suitable approximation) and using Corollary 3.4, we obtain that, for all $t\in(0,T_j)$,

$$\begin{aligned}&\int_t^{T_j} s\bigg(\int_{\mathbb{R}^N}|(\tilde v_j)_t(x,s)|^2 G(x,1)\,dx\bigg)ds\\ &\leqslant \text{const}\bigg(\|\tilde u_j(\sqrt{T_j}\,\cdot,T_j)\|^2_{\mathcal H}+\int_{\mathbb{R}^N}|\tilde v_j(x,t)|^2G(x,1)\,dx\\ &\quad+\int_t^{T_j}\bigg(\int_{\mathbb{R}^N}h(\sqrt{s}x,-(s+a_j))\bigg(\frac{|x|^2}{8}|\tilde v_j(x,s)|^2\\ &\qquad-\frac{\mathfrak{Re}(\tilde v_j(x,s)\overline{\nabla\tilde v_j(x,s)\cdot x})}{2}-\frac{N-2}{4}|\tilde v_j(x,s)|^2\bigg)G(x,1)\,dx\bigg)ds\\ &\quad-\frac12\int_t^{T_j}s\bigg(\int_{\mathbb{R}^N}h_s(\sqrt{s}x,-(s+a_i))|\tilde v_j(x,s)|^2G(x,1)\,dx\bigg)ds\bigg).\end{aligned}$$

From (5)–(6) and Lemmas 3.5 and 3.7 we have that the integrals in the last two terms of the previous formula are finite for every $t\in(0,T_j)$. Hence we conclude that

$$(\tilde v_j)_t\in L^2(\tau,T_j;\mathcal L)\quad\text{for all }\tau\in(0,T_j).$$

Testing (51) with $(\tilde v_j)_t$ also yields

$$\begin{aligned}&\int_t^{T_i} s\bigg(\int_{\mathbb{R}^N}|(\tilde v_j)_t(x,s)|^2G(x,1)\,dx\bigg)ds\\ &\quad+\frac12\int_{\mathbb{R}^N}\bigg(|\nabla_{\mathbf A}\tilde v_j(x,t)|^2-\frac{a(x/|x|)}{|x|^2}|\tilde v_j(x,t)|^2-th(\sqrt{t}x,-(t+a_j))|\tilde v_j(x,t)|^2\bigg)G(x,1)\,dx\\ &=\frac12\int_{\mathbb{R}^N}\bigg(|\nabla_{\mathbf A}v_{0,j}(x)|^2-\frac{a(x/|x|)}{|x|^2}|v_{0,j}(x)|^2-T_jh(\sqrt{T_j}x,-(T_j+a_j))|v_{0,j}(x)|^2\bigg)G(x,1)\,dx\\ &\quad+\int_t^{T_j}\bigg(\int_{\mathbb{R}^N}h(\sqrt{s}x,-(s+a_ij))\bigg(\frac{|x|^2}{8}|\tilde v_j(x,s)|^2\\ &\qquad-\frac{\mathfrak{Re}(\tilde v_j(x,s)\overline{\nabla\tilde v_j(x,s)\cdot x})}{2}-\frac{N-2}{4}|\tilde v_j(x,s)|^2\bigg)G(x,1)\,dx\bigg)ds\\ &\quad-\frac12\int_t^{T_j}s\bigg(\int_{\mathbb{R}^N}h_s(\sqrt{s}x,-(s+a_j))|\tilde v_j(x,s)|^2G(x,1)\,dx\bigg)ds,\end{aligned}$$

for all $t \in (0, T_j)$, where $v_{0,j}(x) := \tilde u_j(\sqrt{T_j}x, T_j) \in \mathcal{H}$. Therefore the function

$$t \mapsto \int_{\mathbb{R}^N}\bigg(|\nabla_{\mathbf{A}}\tilde v_j(x,t)|^2 - \frac{a(x/|x|)}{|x|^2}\,|\tilde v_j(x,t)|^2 - th(\sqrt{t}x, -(t+a_j))|\tilde v_j(x,t)|^2\bigg)G(x,1)\,dx$$

is absolutely continuous in (τ, T_j) for all $\tau \in (0, T_j)$ and

$$\begin{aligned}\frac{d}{dt}&\int_{\mathbb{R}^N}\bigg(|\nabla_{\mathbf{A}}\tilde v_j(x,t)|^2 - \frac{a(x/|x|)}{|x|^2}\,|\tilde v_j(x,t)|^2 - th(\sqrt{t}x, -(t+a_j))|\tilde v_j(x,t)|^2\bigg)G(x,1)\,dx\\ &= 2t\int_{\mathbb{R}^N}|(\tilde v_j)_t(x,t)|^2 G(x,1)\,dx\\ &\quad - \int_{\mathbb{R}^N} h(\sqrt{t}x, -(t+a_j))\bigg(\frac{|x|^2}{4}|\tilde v_j(x,t)|^2\\ &\qquad\qquad - \mathfrak{Re}(\tilde v_j(x,s)\overline{\nabla \tilde v_j(x,s)\cdot x}) - \frac{N-2}{2}|\tilde v_j(x,t)|^2\bigg)G(x,1)\,dx\\ &\quad + t\int_{\mathbb{R}^N} h_t(\sqrt{t}x, -(t+a_j))|\tilde v_j(x,t)|^2 G(x,1)\,dx.\end{aligned}$$

The change of variables $\tilde u_j(x,t) = \tilde v_j(x/\sqrt{t}, t)$ gives the conclusion. □

For all $j = 1, \dots, k$, the *Almgren type frequency function* associated to $\tilde u_j$ is defined as

$$N_j : (0, 2\alpha) \to \mathbb{R} \cup \{-\infty, +\infty\}, \quad N_j(t) := \frac{tD_j(t)}{H_j(t)}.$$

The analysis below will show that each N_j actually assumes finite values all over $(0, 2\alpha)$ and its derivative is an integrable perturbation of a nonnegative function wherever N_j assumes finite values.

Lemma 5.5. *Let $k \in \{1, \dots, k\}$. If there exist $\beta_j, T_j \in (0, 2\alpha)$ such that*

$$\beta_j < T_j, \quad H_j(t) > 0 \text{ for all } t \in (\beta_j, T_j), \quad \text{and} \quad \tilde u_j(\cdot, T_j) \in \mathcal{H}_{T_j}, \tag{55}$$

then $N_j \in W^{1,1}_{\rm loc}(\beta_j, T_j)$ and

$$N_j'(t) = \nu_{1j}(t) + \nu_{2j}(t)$$

in a distributional sense and a.e. in (β_j, T_j) where

$$\begin{aligned}\nu_{1j}(t) = \frac{2t}{H_j^2(t)}\bigg[&\bigg(\int_{\mathbb{R}^N}\bigg|(\tilde u_j)_t(x,t) + \frac{\nabla \tilde u_j(x,t)\cdot x}{2t}\bigg|^2 G(x,t)\,dx\bigg)\bigg(\int_{\mathbb{R}^N}|\tilde u_j(x,t)|^2\,G(x,t)\,dx\bigg)\\ &-\bigg(\int_{\mathbb{R}^N}\Big((\tilde u_j)_t(x,t) + \frac{\nabla \tilde u_j(x,t)\cdot x}{2t}\Big)\overline{\tilde u_j(x,t)}G(x,t)\,dx\bigg)^2\bigg]\end{aligned}$$

and

$$\begin{aligned}\nu_{2j}(t) = \frac{1}{H_j(t)}\int_{\mathbb{R}^N} h(x,-(t+a_j))\bigg(\frac{N-2}{2}|\tilde u_j(x,t)|^2 \\ + \mathfrak{Re}\big(\tilde u_j(x,t)\overline{\nabla \tilde u_j(x,t)\cdot x}\big) - \frac{|x|^2}{4t}|\tilde u_j(x,t)|^2\bigg)G(x,t)\,dx \\ + \frac{t}{H_j(t)}\bigg(\int_{\mathbb{R}^N} h_t(x,-(t+a_j))|\tilde u_j(x,t)|^2 G(x,t)\,dx\bigg).\end{aligned}$$

Proof. From (54) and Lemma 5.4, it follows that $N_j \in W^{1,1}_{\rm loc}(\beta_j, T_j)$. From (54) it follows that

$$N_j'(t) = \frac{(tD_j(t))'H_j(t) - tD_j(t)H_j'(t)}{H_j^2(t)} = \frac{(tD_j(t))'H_j(t) - 2tD_j^2(t)}{H_j^2(t)},$$

and hence, in view of (52), (54), and Lemma 5.4, we obtain the conclusion. □

Lemma 5.6. *There exists $C_3 > 0$ such that, if $j \in \{1,\dots,k\}$ and $\beta_j, T_j \in (0, 2\alpha)$ satisfy (55), then, for every $t \in (\beta_j, T_j)$,*

$$N_j(t) \leqslant -\frac{N-2}{4} + C_3\bigg(N_j(T_j) + \frac{N-2}{4}\bigg).$$

Proof. Let us first claim that, for all $j = 1,\dots,k$, the term ν_{2j} of Lemma 5.5 can be estimated as follows:

$$|\nu_{2j}(t)| \leqslant \begin{cases} C_4\big(N_j(t) + \frac{N-2}{4}\big)\Big(t^{-1+\varepsilon/2} + \|h_t(\cdot,-(t+a_j))\|_{L^{N/2}(\mathbb{R}^N)}\Big), & \text{if } N \geqslant 3,\\ C_4\big(N_j(t) + \frac{N-2}{4}\big)\Big(t^{-1+\varepsilon/2} + \|h_t(\cdot,-(t+a_j))\|_{L^{p}(\mathbb{R}^N)}\Big), & \text{if } N = 2,\end{cases} \tag{56}$$

for a.e. $t \in (\beta_j, T_j)$, with some $C_4 > 0$ independent of t and j. Indeed, from (6) it follows that, for a.e. $t \in (\beta_j, T_j)$,

$$\begin{aligned}(57)\quad &\left|\int_{\mathbb{R}^N} h(x,-(t+a_j))\mathfrak{Re}\big(\tilde u_j(x,t)\overline{\nabla \tilde u_j(x,t)\cdot x}\big)G(x,t)\,dx\right| \\ &\leqslant C_h \int_{\mathbb{R}^N} (1+|x|^{-2+\varepsilon})|\nabla \tilde u_j(x,t)||x||\tilde u_j(x,t)|G(x,t)\,dx \\ &\leqslant C_h t \int_{\mathbb{R}^N} |\nabla \tilde u_j(x,t)|\frac{|x|}{t}|u_i(x,t)|G(x,t)\,dx \\ &\quad + C_h t^{\varepsilon/2}\int_{\{|x|\leqslant\sqrt{t}\}} |\nabla \tilde u_j(x,t)|\frac{|\tilde u_j(x,t)|}{|x|}G(x,t)\,dx \\ &\quad + C_h t^{\varepsilon/2}\int_{\{|x|\geqslant\sqrt{t}\}} |\nabla \tilde u_j(x,t)|\frac{|x|}{t}|\tilde u_j(x,t)|G(x,t)\,dx \\ &\leqslant \frac12 C_h(t+t^{\varepsilon/2})\int_{\mathbb{R}^N} |\nabla \tilde u_j(x,t)|^2 G(x,t)\,dx \\ &\quad + \frac12 C_h(t+t^{\varepsilon/2})\int_{\mathbb{R}^N} \frac{|x|^2}{t^2}|\tilde u_j(x,t)|^2 G(x,t)\,dx \\ &\quad + \frac12 C_h t^{\varepsilon/2}\int_{\mathbb{R}^N} |\nabla \tilde u_j(x,t)|^2 G(x,t)\,dx + \frac12 C_h t^{\varepsilon/2}\int_{\mathbb{R}^N} \frac{|\tilde u_j(x,t)|^2}{|x|^2}G(x,t)\,dx \\ &\leqslant \frac12 C_h t^{\varepsilon/2}(2+\overline{T}^{1-\varepsilon/2})\int_{\mathbb{R}^N} |\nabla \tilde u_j(x,t)|^2 G(x,t)\,dx \\ &\quad + \frac12 C_h t^{\varepsilon/2}(1+\overline{T}^{1-\varepsilon/2})\int_{\mathbb{R}^N} \frac{|x|^2}{t^2}|\tilde u_j(x,t)|^2 G(x,t)\,dx \\ &\quad + \frac12 C_h t^{\varepsilon/2}\int_{\mathbb{R}^N} \frac{|\tilde u_j(x,t)|^2}{|x|^2}G(x,t)\,dx,\end{aligned}$$

and

$$\begin{aligned}(58)\quad &\int_{\mathbb{R}^N} |h(x,-(t+a_j))||x|^2|\tilde u_j(x,t)|^2 G(x,t)\,dx \\ &\leqslant C_h\int_{\mathbb{R}^N} |x|^2|\tilde u_j(x,t)|^2 G(x,t)\,dx + C_h\int_{\mathbb{R}^N} |x|^{-2+\varepsilon}|x|^2|\tilde u_j(x,t)|^2 G(x,t)\,dx \\ &\leqslant C_h\int_{\mathbb{R}^N} |x|^2|\tilde u_j(x,t)|^2 G(x,t)\,dx + C_h t^{\varepsilon/2}\int_{\{|x|\leqslant\sqrt{t}\}} |\tilde u_j(x,t)|^2 G(x,t)\,dx \\ &\quad + C_h t^{-1+\varepsilon/2}\int_{\{|x|\geqslant\sqrt{t}\}} |x|^2|\tilde u_j(x,t)|^2 G(x,t)\,dx \\ &\leqslant C_h t^{-1+\varepsilon/2}(1+\overline{T}^{1-\varepsilon/2})\int_{\mathbb{R}^N} |x|^2|\tilde u_j(x,t)|^2 G(x,t)\,dx + C_h t^{\varepsilon/2}\int_{\mathbb{R}^N} |\tilde u_j(x,t)|^2 G(x,t)\,dx.\end{aligned}$$

Moreover, by Hölder's inequality and Corollary 3.7, we have that, for a.e. $t \in (\beta_j, T_j)$, if $N \geqslant 3$,

$$(59)\quad \left|\int_{\mathbb{R}^N} h_t(x,-(t+a_j))|\tilde u_j(x,t)|^2 G(x,t)\,dx\right| \leqslant C_{2^*} t^{-1}\|\tilde u_j(\cdot,t)\|^2_{\mathcal{H}_t}\|h_t(\cdot,-(t+a_j))\|_{L^{N/2}(\mathbb{R}^N)}$$

and, if $N = 2$,

$$\left| \int_{\mathbb{R}^N} h_t(x, -(t+a_j)) |\tilde{u}_j(x,t)|^2 G(x,t)\, dx \right| \leqslant C_{\frac{2p}{p-1}} t^{-1/p} \|\tilde{u}_j(\cdot,t)\|^2_{\mathcal{H}_t} \|h_t(\cdot, -(t+a_j))\|_{L^p(\mathbb{R}^N)}. \tag{60}$$

Collecting (40), (57), (58) and (59)–(60), we obtain that

$$\begin{aligned} \big|\nu_{2j}(t)\big| \leqslant \frac{\text{const}\, t^{\varepsilon/2}}{H_j(t)} \bigg(\frac{1}{t} \int_{\mathbb{R}^N} |\tilde{u}_j(x,t)|^2 G(x,t)\, dx + \int_{\mathbb{R}^N} \frac{|\tilde{u}_j(x,t)|^2}{|x|^2} G(x,t)\, dx \\ + \int_{\mathbb{R}^N} |\nabla \tilde{u}_j(x,t)|^2 G(x,t)\, dx + \frac{1}{t^2} \int_{\mathbb{R}^N} |x|^2 |\tilde{u}_j(x,t)|^2 G(x,t)\, dx \bigg) + \mathcal{R}_j(t) \end{aligned} \tag{61}$$

where

$$\mathcal{R}_j(t) = \begin{cases} \frac{C_{2^*}}{H_j(t)} \|\tilde{u}_j\|^2_{\mathcal{H}_t} \|h_t(\cdot, -(t+a_j))\|_{L^{N/2}(\mathbb{R}^N)}, & \text{if } N \geqslant 3, \\ C_{\frac{2p}{p-1}} \frac{t^{1-\frac{1}{p}}}{H_j(t)} \|\tilde{u}_j\|^2_{\mathcal{H}_t} \|h_t(\cdot, -(t+a_j))\|_{L^p(\mathbb{R}^N)}, & \text{if } N = 2. \end{cases}$$

Then estimate (56) follows from inequality (61), Corollary 3.4, and Corollary 3.6.

By Schwarz's inequality,

$$\nu_{1j} \geqslant 0 \quad \text{a.e. in } (\beta_j, T_j). \tag{62}$$

From Lemma 5.5, (62), and (56) it follows that

$$\frac{d}{dt} N_j(t) \geqslant \begin{cases} -C_4 \big(N_j(t) + \frac{N-2}{4}\big) \Big(t^{-1+\varepsilon/2} + \|h_t(\cdot, -(t+a_j))\|_{L^{N/2}(\mathbb{R}^N)} \Big), & \text{if } N \geqslant 3, \\ -C_4 \big(N_j(t) + \frac{N-2}{4}\big) \Big(t^{-1+\varepsilon/2} + \|h_t(\cdot, -(t+a_j))\|_{L^p(\mathbb{R}^N)} \Big), & \text{if } N = 2, \end{cases}$$

for a.e. $t \in (\beta_j, T_j)$. After integration, it follows that

$$N_j(t) \leqslant \begin{cases} -\frac{N-2}{4} + \Big(N_j(T_j) + \frac{N-2}{4} \Big) \exp\Big(\frac{2C_4}{\varepsilon} T_j^{\varepsilon/2} + C_4 \|h_t\|_{L^1((-T,0), L^{N/2}(\mathbb{R}^N))} \Big), & \text{if } N \geqslant 3, \\ -\frac{N-2}{4} + \Big(N_j(T_j) + \frac{N-2}{4} \Big) \exp\Big(\frac{2C_4}{\varepsilon} T_j^{\varepsilon/2} + C_4 \|h_t\|_{L^1((-T,0), L^p(\mathbb{R}^N))} \Big), & \text{if } N = 2, \end{cases}$$

for any $t \in (\beta_j, T_j)$, thus yielding the conclusion. □

As a consequence of the above monotonicity argument, we can prove the following non-vanishing properties of the the functions H_j.

Proposition 5.7. (i) *Let $j \in \{1, \dots, k\}$. If $H_j \not\equiv 0$, then $H_j(t) > 0$ for all $t \in (0, 2\alpha)$.*

(ii) *Let $j \in \{1, \dots, k\}$. Then $H_j(t) \equiv 0$ in $(0, 2\alpha)$ if and only if $H_{j+1}(t) \equiv 0$ in $(0, 2\alpha)$.*

(iii) *Let $u \not\equiv 0$ is a weak solution to (1) in $\mathbb{R}^N \times (-T, 0)$ with h satisfying (5)–(6) in $\mathbb{R}^N \times (-T, 0)$. Let $\tilde{u}(x,t) = u(x,-t)$ and let H_j be defined as in (52) for $j = 1, 2, \dots, k$. Then $H_j(t) > 0$ for all $t \in (0, 2\alpha)$ and $j = 1, \dots, k$. In particular,*

$$\int_{\mathbb{R}^N} |u(x,-t)|^2 G(x,-t)\, dx > 0 \quad \textit{for all } t \in (-T, 0). \tag{63}$$

Proof. Once Lemmas 5.3 and 5.6 are established, the proof can be done arguing as in [16, Lemmas 4.9 and 4.10, Corollary 7]. Hence we omit it. □

Proof of Proposition 1.4. Up a translation in time, it is not restrictive to assume that $I = (-T, 0)$ for some $T > 0$. Then the conclusion follows from Proposition 5.7 (iii). □

Henceforward, we assume $u \not\equiv 0$ in $\mathbb{R}^N \times (-T,0)$ (so that $\tilde u \not\equiv 0$ in $\mathbb{R}^N \times (0,T)$) and we denote, for all $t \in (0,2\alpha)$,

$$H(t) = H_1(t) = \int_{\mathbb{R}^N} |\tilde u(x,t)|^2\, G(x,t)\, dx,$$

$$D(t) = D_1(t) = \int_{\mathbb{R}^N} \bigg(|\nabla_{\mathbf{A}} \tilde u(x,t)|^2 - \frac{a\big(x/|x|\big)}{|x|^2} |\tilde u(x,t)|^2 - h(x,-t)|\tilde u(x,t)|^2 \bigg)\, G(x,t)\, dx.$$

Proposition 5.7 ensures that, if $\tilde u \not\equiv 0$ in $\mathbb{R}^N \times (0,T)$, then $H(t) > 0$ for all $t \in (0,2\alpha)$. Hence the *Almgren type frequency function*

$$\mathcal{N}(t) = N_1(t) = \frac{tD(t)}{H(t)}$$

is well defined over all $(0,2\alpha)$. Moreover, by Lemma 5.5, $\mathcal{N} \in W^{1,1}_{\rm loc}(0,2\alpha)$ and $\mathcal{N}' = \nu_1 + \nu_2$ a.e. in $(0,2\alpha)$, where $\nu_1 = \nu_{11}$ and $\nu_2 = \nu_{21}$, with ν_{11}, ν_{21} as in Lemma 5.5. By (27), $\tilde u(\cdot,t) \in \mathcal{H}_t$ for a.e. $t \in (0,T)$, hence there exists a $T_0 \in (0,2\alpha)$ such that $\tilde u(\cdot,T_0) \in \mathcal{H}_{T_0}$. We now prove the existence of the limit of $\mathcal{N}(t)$ as $t \to 0^+$.

Lemma 5.8. *The limit $\gamma := \lim_{t\to 0^+} \mathcal{N}(t)$ exists and it is finite.*

Proof. We first observe that, in view of Corollary 3.4, $tD(t) \geqslant \big(C_1 - \frac{N-2}{4}\big) H(t)$ for all $t \in (0,2\alpha)$. Hence $\mathcal{N}(t) \geqslant C_1 - \frac{N-2}{4}$, i.e. $\mathcal{N}$ is bounded from below. Let T_0 be as above. By Schwarz's inequality, $\nu_1(t) \geqslant 0$ for a.e. $t \in (0,T_0)$. Furthermore, Lemma 5.6 and estimate (56), together with assumption (5), imply that $\nu_2 \in L^1(0,T_0)$. Hence $\mathcal{N}'(t)$ is the sum of a nonnegative function and of a L^1 function over $(0,T_0)$. Therefore, $\mathcal{N}(t) = \mathcal{N}(T_0) - \int_t^{T_0} \mathcal{N}'(s)\, ds$ admits a limit as $t \to 0^+$. We conclude by observing that such a limit is finite since $\mathcal{N}$ is bounded from below (as observed at the beginning of the proof) and from above (due to Lemma 5.6) in the interval $(0,T_0)$. $\square$

Lemma 5.9. *Let $\gamma := \lim_{t\to 0^+} \mathcal{N}(t)$ be as in Lemma 5.8. Then there exists a constant $K_1 > 0$ such that*

$$H(t) \leqslant K_1 t^{2\gamma} \quad \text{for all } t \in (0,T_0). \tag{64}$$

Furthermore, for any $\sigma > 0$, there exists a constant $K_2(\sigma) > 0$ depending on σ such that

$$H(t) \geqslant K_2(\sigma)\, t^{2\gamma+\sigma} \quad \text{for all } t \in (0,T_0). \tag{65}$$

Proof. From Lemma 5.5, Schwarz's inequality, (56), and assumption (5) we deduce that

$$\begin{aligned}
\mathcal{N}(t) - \gamma &= \int_0^t (\nu_1(s) + \nu_2(s))ds \geqslant \int_0^t \nu_2(s) ds \\
&\geqslant \begin{cases} -C_3C_4\big(\mathcal{N}(T_0) + \frac{N-2}{4}\big) \int_0^t \Big(s^{-1+\varepsilon/2} + \|h_t(\cdot,-s)\|_{L^{N/2}(\mathbb{R}^N)} \Big)\, ds, & \text{if } N \geqslant 3, \\ -C_3C_4\mathcal{N}(T_0) \int_0^t \Big(s^{-1+\varepsilon/2} + \|h_t(\cdot,-s)\|_{L^p(\mathbb{R}^N)} \Big)\, ds, & \text{if } N = 2, \end{cases} \\
&\geqslant \begin{cases} -C_3C_4\big(\mathcal{N}(T_0) + \frac{N-2}{4}\big) \Big(\frac{2}{\varepsilon} t^{\varepsilon/2} + \|h_t\|_{L^r((-T,0),L^{N/2}(\mathbb{R}^N))} t^{1-1/r} \Big), & \text{if } N \geqslant 3, \\ -C_3C_4\mathcal{N}(T_0) \Big(\frac{2}{\varepsilon} t^{\varepsilon/2} + \|h_t\|_{L^r((-T,0),L^p(\mathbb{R}^N))} t^{1-1/r} \Big), & \text{if } N = 2 \end{cases} \\
&\geqslant -C_5 t^{\delta}
\end{aligned}$$

with

$$\delta = \min\left\{\frac{\varepsilon}{2}, 1-\frac{1}{r}\right\} \tag{66}$$

for some constant $C_5 > 0$ and for all $t \in (0, T_0)$. From the above estimate and (54), we deduce that $(\log H(t))' = \frac{H'(t)}{H(t)} = \frac{2}{t}\mathcal{N}(t) \geqslant \frac{2}{t}\gamma - 2C_5 t^{-1+\delta}$, which, after integration over (t, T_0), yields (64).

Since $\gamma = \lim_{t\to 0^+}\mathcal{N}(t)$, for any $\sigma > 0$ there exists $t_\sigma > 0$ such that $\mathcal{N}(t) < \gamma + \sigma/2$ for any $t \in (0, t_\sigma)$. Hence $\frac{H'(t)}{H(t)} = \frac{2\mathcal{N}(t)}{t} < \frac{2\gamma+\sigma}{t}$ which, by integration over (t, t_σ) and continuity of H outside 0, implies (65) for some constant $K_2(\sigma)$ depending on σ. □

6. Blow-up analysis

Once the monotonicity type Lemma 5.8 is established, our next step is a blow-up analysis for scaled solutions to (26), which can be performed following the procedure developed in [16]

Let $\tilde u$ be a weak solution to (26) in $\mathbb{R}^N \times (0, T)$ with h satisfying (5)–(6) in $I = (-T, 0)$. For every $\lambda > 0$, we define

$$\tilde u_\lambda(x,t) = \tilde u(\lambda x, \lambda^2 t).$$

We observe that $\tilde u_\lambda$ weakly solves

$$-(\tilde u_\lambda)_t + \mathcal{L}_{\mathbf{A},a}\tilde u_\lambda = \lambda^2 h(\lambda x, -\lambda^2 t)\tilde u_\lambda \quad \text{in } \mathbb{R}^N \times (0, T/\lambda^2). \tag{67}$$

We can associate to the scaled equation (67) the Almgren frequency function

$$\mathcal{N}_\lambda(t) = \frac{t\, D_\lambda(t)}{H_\lambda(t)}, \tag{68}$$

where

$$D_\lambda(t) = \int_{\mathbb{R}^N}\bigg(|\nabla_{\mathbf{A}}\tilde u_\lambda(x,t)|^2 - \frac{a\big(x/|x|\big)}{|x|^2}|\tilde u_\lambda(x,t)|^2 - \lambda^2 h(\lambda x, -\lambda^2 t)|\tilde u_\lambda(x,t)|^2\bigg)\, G(x,t)\, dx,$$

$$H_\lambda(t) = \int_{\mathbb{R}^N}|\tilde u_\lambda(x,t)|^2 G(x,t)\, dx.$$

By scaling and a suitable change of variables we easily see that

$$D_\lambda(t) = \lambda^2 D(\lambda^2 t) \quad \text{and} \quad H_\lambda(t) = H(\lambda^2 t), \tag{69}$$

so that

$$\mathcal{N}_\lambda(t) = \mathcal{N}(\lambda^2 t) \quad \text{for all } t \in \Big(0, \frac{2\alpha}{\lambda^2}\Big). \tag{70}$$

Lemma 6.1. *Let $N \geqslant 2$, $\mathbf{A} \in C^1(\mathbb{S}^{N-1}, \mathbb{R}^N)$ and let $a \in L^\infty\big(\mathbb{S}^{N-1}, \mathbb{R}\big)$ satisfy* (2) *and* (11). *Let $\tilde u \not\equiv 0$ be a nontrivial weak solution to (26) in $\mathbb{R}^N \times (0, T)$ with h satisfying* (5)*–*(6) *in $I = (-T, 0)$. Let $\gamma := \lim_{t\to 0^+}\mathcal{N}(t)$ as in Lemma 5.8. Then*

(i) *γ is an eigenvalue of the operator $L_{\mathbf{A},a}$ defined in (7);*

(ii) *for every sequence $\lambda_n \to 0^+$, there exists a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and an eigenfunction g of the operator $L_{\mathbf{A},a}$ associated to the eigenvalue γ such that, for all $\tau \in (0,1)$,*

$$\lim_{k\to+\infty}\int_\tau^1 \left\|\frac{\tilde u(\lambda_{n_k}x, \lambda_{n_k}^2 t)}{\sqrt{H(\lambda_{n_k}^2)}} - t^\gamma g(x/\sqrt{t})\right\|_{\mathcal{H}_t}^2 dt = 0$$

and

$$\lim_{k\to+\infty}\sup_{t\in[\tau,1]}\left\|\frac{\tilde u(\lambda_{n_k}x,\lambda_{n_k}^2t)}{\sqrt{H(\lambda_{n_k}^2)}}-t^{\gamma}g(x/\sqrt{t})\right\|_{\mathcal{L}_t}=0.$$

PROOF. Let

$$w_\lambda(x,t):=\frac{\tilde u_\lambda(x,t)}{\sqrt{H(\lambda^2)}},\tag{71}$$

with $\lambda\in(0,\sqrt{T_0})$, so that $1<T_0/\lambda^2$. From Lemma 5.2 it follows that, for all $t\in(0,1)$,

$$\int_{\mathbb{R}^N}|w_\lambda(x,t)|^2G(x,t)\,dx=\frac{H(\lambda^2t)}{H(\lambda^2)}\leqslant t^{2C_1-\frac{N-2}{2}},\tag{72}$$

with C_1 as in Corollary 3.4. Lemma 5.6, Corollary 3.4, and (69) imply that

$$\begin{aligned}\frac1t\bigg(-\frac{N-2}{4}+C_3\Big(\mathcal{N}(T_0)+\frac{N-2}{4}\Big)\bigg)H_\lambda(t)&\geqslant\lambda^2D(\lambda^2t)\\&\geqslant\frac1t\Big(C_1-\frac{N-2}{4}\Big)H_\lambda(t)+C_1\int_{\mathbb{R}^N}\bigg(|\nabla\tilde u_\lambda(x,t)|^2+\frac{|\tilde u_\lambda(x,t)|^2}{|x|^2}\bigg)G(x,t)\,dx\end{aligned}$$

and hence, in view of (72),

$$t\int_{\mathbb{R}^N}\bigg(|\nabla w_\lambda(x,t)|^2+\frac{|w_\lambda(x,t)|^2}{|x|^2}\bigg)G(x,t)\,dx\leqslant C_1^{-1}\Big(C_3\big(\mathcal{N}(T_0)+\tfrac{N-2}{4}\big)-C_1\Big)t^{2C_1-\frac{N-2}{2}}\tag{73}$$

for a.e. $t\in(0,1)$. By scaling, we have that the family of functions

$$\widetilde{w}_\lambda(x,t)=w_\lambda(\sqrt{t}x,t)=\frac{\tilde u(\lambda\sqrt{t}x,\lambda^2t)}{\sqrt{H(\lambda^2)}}$$

satisfies

$$\int_{\mathbb{R}^N}|\widetilde{w}_\lambda(x,t)|^2G(x,1)\,dx=\int_{\mathbb{R}^N}|w_\lambda(x,t)|^2G(x,t)\,dx\tag{74}$$

and

$$\int_{\mathbb{R}^N}\bigg(|\nabla\widetilde{w}_\lambda(x,t)|^2+\frac{|\widetilde{w}_\lambda(x,t)|^2}{|x|^2}\bigg)G(x,1)\,dx=t\int_{\mathbb{R}^N}\bigg(|\nabla w_\lambda(x,t)|^2+\frac{|w_\lambda(x,t)|^2}{|x|^2}\bigg)G(x,t)\,dx.\tag{75}$$

From (72), (73), (74), and (75), we deduce that, for all $\tau\in(0,1)$,

$$\big\{\widetilde{w}_\lambda\big\}_{\lambda\in(0,\sqrt{T_0})}\text{ is bounded in }L^\infty(\tau,1;\mathcal{H})\tag{76}$$

uniformly with respect to $\lambda\in(0,\sqrt{T_0})$. Moreover $\widetilde{w}_\lambda$ solves the following weak equation: for all $\phi\in\mathcal{H}$,

$$\begin{aligned}{}_{\mathcal{H}^\star}\big\langle(\widetilde{w}_\lambda)_t,\phi\big\rangle_{\mathcal{H}}=\frac1t\int_{\mathbb{R}^N}\bigg(&\nabla_{\mathbf{A}}\widetilde{w}_\lambda(x,t)\cdot\overline{\nabla_{\mathbf{A}}\phi(x)}-\frac{a\big(\frac{x}{|x|}\big)}{|x|^2}\,\widetilde{w}_\lambda(x,t)\overline{\phi(x)}\\&-\lambda^2t\,h\Big(\lambda\sqrt{t}x,-\lambda^2t\Big)\widetilde{w}_\lambda(x,t)\overline{\phi(x)}\bigg)G(x,1)\,dx.\end{aligned}\tag{77}$$

From (6) it follows that

$$
\begin{aligned}
(78)\qquad & \lambda^2 \left| \int_{\mathbb{R}^N} h(\lambda\sqrt{t}x, -\lambda^2 t)\widetilde{w}_\lambda(x,t)\overline{\phi(x)}G(x,1)\,dx \right| \\
&\leqslant C_h \lambda^2 \int_{\mathbb{R}^N} |\widetilde{w}_\lambda(x,t)||\phi(x)|G(x,1)\,dx + C_h \frac{\lambda^\varepsilon}{t} \int_{\mathbb{R}^N} |x|^{-2+\varepsilon} |\widetilde{w}_\lambda(x,t)||\phi(x)|G(x,1)\,dx \\
&\leqslant C_h \lambda^2 \|\widetilde{w}_\lambda(\cdot,t)\|_{\mathcal{H}} \|\phi\|_{\mathcal{H}} + C_h \frac{\lambda^\varepsilon}{t} \int_{|x|\leqslant 1} \frac{|\widetilde{w}_\lambda(x,t)||\phi(x)|}{|x|^2} G(x,1)\,dx \\
&\qquad\qquad + C_h \frac{\lambda^\varepsilon}{t} \int_{|x|\geqslant 1} |\widetilde{w}_\lambda(x,t)||\phi(x)|G(x,1)\,dx \\
&\leqslant C_h \frac{\lambda^\varepsilon}{t} \big(t\,\lambda^{2-\varepsilon} + 2 \big) \|\widetilde{w}_\lambda(\cdot,t)\|_{\mathcal{H}} \|\phi\|_{\mathcal{H}}
\end{aligned}
$$

for all $\lambda \in (0, \sqrt{T_0})$ and a.e. $t \in (0,1)$. From (77) and (78), and Lemma 3.1 we deduce that

$$
\|(\widetilde{w}_\lambda)_t(\cdot,t)\|_{\mathcal{H}^\star} \leqslant \frac{\text{const}}{t} \|\widetilde{w}_\lambda(\cdot,t)\|_{\mathcal{H}} \tag{79}
$$

for all $\lambda \in (0, \sqrt{T_0})$ and a.e. $t \in (0,1)$ and for some const > 0 independent of t and λ.

In view of (76), estimate (79) yields, for all $\tau \in (0,1)$,

$$
\big\{(\widetilde{w}_\lambda)_t\big\}_{\lambda \in (0,\sqrt{T_0})} \text{ is bounded in } L^\infty(\tau,1;\mathcal{H}^\star) \tag{80}
$$

uniformly with respect to $\lambda \in (0, \sqrt{T_0})$. From (76), (80), and [25, Corollary 8], we deduce that $\big\{\widetilde{w}_\lambda\big\}_{\lambda \in (0,\sqrt{T_0})}$ is relatively compact in $C^0([\tau,1],\mathcal{L})$ for all $\tau \in (0,1)$. Therefore, by a diagonal argument, from any given sequence $\lambda_n \to 0^+$ we can extract a subsequence $\lambda_{n_k} \to 0^+$ such that

$$
\widetilde{w}_{\lambda_{n_k}} \to \widetilde{w} \quad \text{in} \quad C^0([\tau,1],\mathcal{L}) \tag{81}
$$

for all $\tau \in (0,1)$ and for some $\widetilde{w} \in \bigcap_{\tau \in (0,1)} C^0([\tau,1],\mathcal{L})$. From the fact that $1 = \|\widetilde{w}_{\lambda_{n_k}}(\cdot,1)\|_{\mathcal{L}}$ and the convergence (81) it follows that

$$
\|\widetilde{w}(\cdot,1)\|_{\mathcal{L}} = 1. \tag{82}
$$

In particular $\widetilde{w}$ is nontrivial. Furthermore, by (76) and (80), the subsequence can be chosen in such a way that also

$$
\widetilde{w}_{\lambda_{n_k}} \rightharpoonup \widetilde{w} \quad \text{weakly in } L^2(\tau,1;\mathcal{H}) \quad \text{and} \quad (\widetilde{w}_{\lambda_{n_k}})_t \rightharpoonup \widetilde{w}_t \quad \text{weakly in } L^2(\tau,1;\mathcal{H}^\star) \tag{83}
$$

for all $\tau \in (0,1)$, so that $\widetilde{w} \in \bigcap_{\tau \in (0,1)} L^2(\tau,1;\mathcal{H})$ and $\widetilde{w}_t \in \bigcap_{\tau \in (0,1)} L^2(\tau,1;\mathcal{H}^\star)$. Actually, we can prove that

$$
\widetilde{w}_{\lambda_{n_k}} \to \widetilde{w} \quad \text{strongly in} \quad L^2(\tau,1;\mathcal{H}) \quad \text{for all } \tau \in (0,1). \tag{84}
$$

To prove claim (84), we notice that (83) allows passing to the limit in (77). Since (78) and (76) imply that the perturbation term vanishes in such a limit, we conclude that

$$
{}_{\mathcal{H}^\star}\langle \widetilde{w}_t, \phi \rangle_{\mathcal{H}} = \frac{1}{t} \int_{\mathbb{R}^N} \left(\nabla_{\mathbf{A}} \widetilde{w}(x,t) \cdot \overline{\nabla_{\mathbf{A}} \phi(x)} - \frac{a\big(\frac{x}{|x|}\big)}{|x|^2} \widetilde{w}(x,t)\overline{\phi(x)} \right) G(x,1)\,dx \tag{85}
$$

for all $\phi \in \mathcal{H}$ and a.e. $t \in (0,1)$, i.e. $\widetilde{w}$ is a weak solution to

$$
\widetilde{w}_t + \frac{1}{t}\bigg(-\mathcal{L}_{\mathbf{A},a}\widetilde{w} - \frac{x}{2} \cdot \nabla \widetilde{w} \bigg) = 0.
$$

Testing the difference between (77) and (85) with $(\widetilde{w}_{\lambda_{n_k}} - \widetilde{w})$ and integrating with respect to t between τ and 1, we obtain

$$\int_\tau^1 \left(\int_{\mathbb{R}^N} \left(|\nabla_{\mathbf{A}} (\widetilde{w}_{\lambda_{n_k}} - \widetilde{w})(x,t)|^2 - \frac{a(x/|x|)}{|x|^2} |(\widetilde{w}_{\lambda_{n_k}} - \widetilde{w})(x,t)|^2 \right) G(x,1)\, dx \right) dt$$
$$= \frac{1}{2} \|\widetilde{w}_{\lambda_{n_k}}(1) - \widetilde{w}(1)\|^2_{\mathcal{L}} - \frac{\tau}{2} \|\widetilde{w}_{\lambda_{n_k}}(\tau) - \widetilde{w}(\tau)\|^2_{\mathcal{L}} - \frac{1}{2} \int_\tau^1 \left(\int_{\mathbb{R}^N} |(\widetilde{w}_{\lambda_{n_k}} - \widetilde{w})(x,t)|^2 G(x,1)\, dx \right) dt$$
$$+ \lambda_{n_k}^2 \int_\tau^1 t \left(\int_{\mathbb{R}^N} h(\lambda_{n_k} \sqrt{t} x, -\lambda_{n_k}^2 t) \mathfrak{Re}\big(\widetilde{w}_{\lambda_{n_k}}(x,t) \overline{(\widetilde{w}_{\lambda_{n_k}} - \widetilde{w})(x,t)} \big) G(x,1)\, dx \right) dt.$$

Then (78) and (81) imply that, for all $\tau \in (0,1)$,

$$\lim_{k\to+\infty} \int_\tau^1 \left(\int_{\mathbb{R}^N} \left(|\nabla_{\mathbf{A}} (\widetilde{w}_{\lambda_{n_k}} - \widetilde{w})(x,t)|^2 - \frac{a(x/|x|)}{|x|^2} |(\widetilde{w}_{\lambda_{n_k}} - \widetilde{w})(x,t)|^2 \right) G(x,1)\, dx \right) dt = 0,$$

which yields the convergence claimed in (84) in view of Corollary 3.3 and (81). Thus, we have obtained that, for all $\tau \in (0,1)$,

$$\lim_{k\to+\infty} \int_\tau^1 \|w_{\lambda_{n_k}}(\cdot,t) - w(\cdot,t)\|^2_{\mathcal{H}_t} dt = 0 \quad \text{and} \quad \lim_{k\to+\infty} \sup_{t\in[\tau,1]} \|w_{\lambda_{n_k}}(\cdot,t) - w(\cdot,t)\|_{\mathcal{L}_t} = 0, \tag{86}$$

where $w(x,t) := \widetilde{w}\big(\frac{x}{\sqrt{t}}, t\big)$ is a weak solution to

$$w_t - \mathcal{L}_{\mathbf{A},a} w = 0. \tag{87}$$

In view of (68) and (71), we can write $\mathcal{N}_\lambda$ as

$$\mathcal{N}_\lambda(t) = \frac{t \int_{\mathbb{R}^N} \Big(|\nabla_{\mathbf{A}} w_\lambda(x,t)|^2 - \frac{a(x/|x|)}{|x|^2} |w_\lambda(x,t)|^2 - \lambda^2 h\big(\lambda x, -\lambda^2 t\big) |w_\lambda(x,t)|^2 \Big) G(x,t)\, dx}{\int_{\mathbb{R}^N} |w_\lambda(x,t)|^2 G(x,t)\, dx}$$

for all $t \in (0,1)$. By (86) $w_{\lambda_{n_k}}(\cdot,t) \to w(\cdot,t)$ in $\mathcal{H}_t$ for a.e. $t \in (0,1)$, and, by (78),

$$t\lambda_{n_k}^2 \int_{\mathbb{R}^N} h(\lambda_{n_k} x, -\lambda_{n_k}^2 t) |w_{\lambda_{n_k}}(x,t)|^2 G(x,t)\, dx \to 0$$

for a.e. $t \in (0,1)$. Hence

$$\int_{\mathbb{R}^N} \Big(|\nabla_{\mathbf{A}} w_{\lambda_{n_k}}(x,t)|^2 - \frac{a(x/|x|)}{|x|^2} |w_{\lambda_{n_k}}(x,t)|^2 - \lambda_{n_k}^2 h\big(\lambda_{n_k} x, -\lambda_{n_k}^2 t\big) |w_{\lambda_{n_k}}(x,t)|^2 \Big) G(x,t)\, dx \to D_w(t) \tag{88}$$

and

$$\int_{\mathbb{R}^N} |w_{\lambda_{n_k}}(x,t)|^2 G(x,t)\, dx \to H_w(t) \tag{89}$$

as $k \to +\infty$, for a.e. $t \in (0,1)$, where

$$D_w(t) = \int_{\mathbb{R}^N} \left(|\nabla_{\mathbf{A}} w(x,t)|^2 - \frac{a(x/|x|)}{|x|^2} |w(x,t)|^2 \right) G(x,t)\, dx \quad \text{and} \quad H_w(t) = \int_{\mathbb{R}^N} |w(x,t)|^2 G(x,t)\, dx.$$

From (82) it follows that $\int_{\mathbb{R}^N} |w(x,1)|^2 G(x,1)\,dx = 1$, which, arguing as in Proposition 5.7, implies that $H_w(t) > 0$ for all $t \in (0,1)$. From (88) and (89), it follows that

$$\mathcal{N}_{\lambda_{n_k}}(t) \to \mathcal{N}_w(t) \quad \text{for a.e. } t \in (0,1), \tag{90}$$

where $\mathcal{N}_w$ is the Almgren frequency function associated to equation (87), i.e.

$$\mathcal{N}_w(t) = \frac{tD_w(t)}{H_w(t)}, \tag{91}$$

which is well defined on $(0,1)$ since $H_w(t) > 0$ for all $t \in (0,1)$ as observed above.

On the other hand, (70) implies that $\mathcal{N}_{\lambda_{n_k}}(t) = \mathcal{N}(\lambda_{n_k}^2 t)$ for all $t \in (0,1)$ and $k \in \mathbb{N}$. Fixing $t \in (0,1)$ and passing to the limit as $k \to +\infty$, from Lemma 5.8 we obtain

$$\mathcal{N}_{\lambda_{n_k}}(t) \to \gamma \quad \text{for all } t \in (0,1). \tag{92}$$

Combining (90) and (92), we deduce that

$$\mathcal{N}_w(t) = \gamma \quad \text{for all } t \in (0,1). \tag{93}$$

Therefore $\mathcal{N}_w$ is constant in $(0,1)$ and hence $\mathcal{N}_w'(t) = 0$ for any $t \in (0,1)$. By (87) and Lemma 5.5 with $h \equiv 0$, we obtain that

$$\left(w_t(\cdot,t) + \frac{\nabla w(\cdot,t)\cdot x}{2t}, w(\cdot,t) \right)_{\mathcal{L}_t}^2 = \left\| w_t(\cdot,t) + \frac{\nabla w(\cdot,t)\cdot x}{2t} \right\|_{\mathcal{L}_t}^2 \|w(\cdot,t)\|_{\mathcal{L}_t}^2,$$

where $(\cdot,\cdot)_{\mathcal{L}_t}$ denotes the scalar product in $\mathcal{L}_t$. Then there exists a function $\beta : (0,1) \to \mathbb{R}$ such that

$$w_t(x,t) + \frac{\nabla w(x,t)\cdot x}{2t} = \beta(t) w(x,t) \quad \text{for a.e. } t \in (0,1) \text{ and a.e. } x \in \mathbb{R}^N. \tag{94}$$

Testing (87) with $\phi = w(\cdot,t)$ in the sense of (28) and taking into account (94), we obtain that

$$D_w(t) = {}_{\mathcal{H}_t^\star}\!\left\langle w_t(\cdot,t) + \frac{\nabla w(\cdot,t)\cdot x}{2t}, w(\cdot,t) \right\rangle_{\mathcal{H}_t} = \beta(t) H_w(t),$$

which, in view (91) and (93), yields $\beta(t) = \frac{\gamma}{t}$ for a.e. $t \in (0,1)$. Then (94) becomes

$$w_t(x,t) + \frac{\nabla w(x,t)\cdot x}{2t} = \frac{\gamma}{t}\, w(x,t) \quad \text{for a.e. } (x,t) \in \mathbb{R}^N \times (0,1) \tag{95}$$

in a distributional sense. From (95) and (87) we conclude that

$$\mathcal{L}_{\mathbf{A},a} w + \frac{\nabla w(x,t)\cdot x}{2t} = \frac{\gamma}{t}\, w(x,t) \tag{96}$$

for a.e. $(x,t) \in \mathbb{R}^N \times (0,1)$ and in a weak sense. From (95), it follows that, letting, for all $\eta > 0$ and a.e. $(x,t) \in \mathbb{R}^N \times (0,1)$, $w^\eta(x,t) := w(\eta x, \eta^2 t)$, there holds $\frac{dw^\eta}{d\eta} = \frac{2\gamma}{\eta} w^\eta$ a.e. and in a distributional sense. An integration yields

$$w^\eta(x,t) = w(\eta x, \eta^2 t) = \eta^{2\gamma} w(x,t) \quad \text{for all } \eta > 0 \text{ and a.e. } (x,t) \in \mathbb{R}^N \times (0,1). \tag{97}$$

The function $g(x) = w(x,1)$ satisfies $g \in \mathcal{L}$, $\|g\|_{\mathcal{L}} = 1$, and, from (97),

$$w(x,t) = w^{\sqrt{t}}\Big(\frac{x}{\sqrt{t}}, 1\Big) = t^\gamma w\Big(\frac{x}{\sqrt{t}}, 1\Big) = t^\gamma g\Big(\frac{x}{\sqrt{t}}\Big) \quad \text{for a.e. } (x,t) \in \mathbb{R}^N \times (0,1). \tag{98}$$

In particular, from (98) it follows that $g(\cdot/\sqrt{t}) \in \mathcal{H}_t$ for a.e. $t \in (0,1)$ and hence, by scaling, $g \in \mathcal{H}$. Moreover, from (96) and (98), we obtain that $g \in \mathcal{H} \setminus \{0\}$ weakly solves

$$\mathcal{L}_{\mathbf{A},a} g(x) + \frac{\nabla g(x) \cdot x}{2} = \gamma g(x),$$

i.e. γ is an eigenvalue of the operator $L_{\mathbf{A},a}$ defined in (7) and g is an eigenfunction of $L_{\mathbf{A},a}$ associated to γ. The proof is now complete. □

The next step is to determine the asymptotic behaviour of the normalization term in the blow-up family (71). To this aim, in the next two lemmas we study the limit $\lim_{t\to 0^+} t^{-2\gamma} H(t)$.

Lemma 6.2. *Under the same assumptions as in Lemma 6.1, let* $\gamma := \lim_{t\to 0^+} \mathcal{N}(t)$ *be as in Lemma 5.8. Then the limit* $\lim_{t\to 0^+} t^{-2\gamma} H(t)$ *exists and it is finite.*

Proof. We omit the proof, since it is very similar to that of [16, Lemma 5.2]. □

The limit $\lim_{t\to 0^+} t^{-2\gamma} H(t)$ is indeed strictly positive, as we prove below.

Lemma 6.3. *Under the same assumptions as in Lemma 6.1, we have that* $\lim_{t\to 0^+} t^{-2\gamma} H(t) > 0$.

Proof. Let us argue by contradiction and assume that $\lim_{t\to 0^+} t^{-2\gamma} H(t) = 0$. Let us consider the orthonormal basis of $\mathcal{L}$ introduced in Remark 4.5 and given by $\{\widetilde{V}_{n,j} : j, n \in \mathbb{N}, j \geqslant 1\}$. Since $\tilde{u}_\lambda(x,1) = \tilde{u}(\lambda x, \lambda^2) \in \mathcal{L}$ for all $\lambda \in (0, \sqrt{T_0})$, we can expand $\tilde{u}_\lambda$ as

$$\tilde{u}_\lambda(x,1) = \sum_{\substack{m,k\in\mathbb{N}\\ k\geqslant 1}} u_{m,k}(\lambda) \widetilde{V}_{m,k}(x) \quad \text{in } \mathcal{L}, \tag{99}$$

where

$$u_{m,k}(\lambda) = \int_{\mathbb{R}^N} \tilde{u}_\lambda(x,1) \overline{\widetilde{V}_{m,k}(x)} G(x,1)\, dx. \tag{100}$$

From (26) and the fact that $\widetilde{V}_{m,k}(x)$ is an eigenfuntion of the operator $L_{\mathbf{A},a}$ associated to the eigenvalue $\gamma_{m,k}$ defined in (13), we obtain that

$$u'_{m,k}(\lambda) = \frac{2}{\lambda} \gamma_{m,k} u_{m,k}(\lambda) - 2\lambda \xi_{m,k}(\lambda) \quad \text{for all } m,k \in \mathbb{N},\ k \geqslant 1, \tag{101}$$

a.e. and distributionally in $(0, \sqrt{T_0})$, where

$$\xi_{m,k}(\lambda) = \int_{\mathbb{R}^N} h(\lambda x, -\lambda^2) \tilde{u}_\lambda(x,1) \overline{\widetilde{V}_{m,k}(x)} G(x,1)\, dx. \tag{102}$$

From Lemma 6.1, γ is an eigenvalue of the operator $L_{\mathbf{A},a}$, hence, by Proposition 1.1, there exist $m_0, k_0 \in \mathbb{N}$, $k_0 \geqslant 1$, such that $\gamma = \gamma_{m_0,k_0} = m_0 - \frac{\alpha_{k_0}}{2}$. Let us denote as E_0 the associated eigenspace and by J_0 the finite set of indices $\{(m,k) \in \mathbb{N} \times (\mathbb{N} \setminus \{0\}) : \gamma = m - \frac{\alpha_k}{2}\}$, so that $\#J_0$ is equal to the multiplicity of γ and an orthonormal basis of E_0 is given by $\{\widetilde{V}_{m,k} : (m,k) \in J_0\}$.

In view of (6), for all $(m,k)\in J_0$ we can estimate $\xi_{m,k}$ as

$$
\begin{aligned}
(103)\qquad |\xi_{m,k}(\lambda)| &\leqslant C_h \int_{\mathbb{R}^N} (1+\lambda^{-2+\varepsilon}|x|^{-2+\varepsilon})|\tilde u(\lambda x,\lambda^2)||\widetilde V_{m,k}(x)|G(x,1)\,dx\\
&\leqslant C_h\bigg(\int_{\mathbb{R}^N}|\tilde u(\lambda x,\lambda^2)|^2G(x,1)\,dx\bigg)^{1/2} + C_h\lambda^{-2+\frac{\varepsilon}{2}}\int_{|x|\leqslant\lambda^{-1/2}}\frac{|\tilde u(\lambda x,\lambda^2)||\widetilde V_{m,k}(x)|}{|x|^2}G(x,1)\,dx\\
&\quad + C_h\lambda^{-1+\frac{\varepsilon}{2}}\int_{|x|\geqslant\lambda^{-1/2}}|\tilde u(\lambda x,\lambda^2)||\widetilde V_{m,k}(x)|G(x,1)\,dx\\
&\leqslant C_h(1+\lambda^{-1+\frac{\varepsilon}{2}})\sqrt{H(\lambda^2)} + C_h\lambda^{-2+\frac{\varepsilon}{2}}\bigg(\int_{\mathbb{R}^N}\tfrac{|\tilde u(\lambda x,\lambda^2)|^2}{|x|^2}G(x,1)\,dx\bigg)^{\frac12}\bigg(\int_{\mathbb{R}^N}\tfrac{|\widetilde V_{m,k}(x)|^2}{|x|^2}G(x,1)\,dx\bigg)^{\frac12}.
\end{aligned}
$$

Corollary 3.4 and Lemma 5.6 imply that

$$
\begin{aligned}
(104)\qquad \int_{\mathbb{R}^N}\frac{|\tilde u(\lambda x,\lambda^2)|^2}{|x|^2}G(x,1)\,dx &= \lambda^2\int_{\mathbb{R}^N}\frac{|\tilde u(y,\lambda^2)|^2}{|y|^2}G(y,\lambda^2)\,dy\\
&\leqslant \frac{\lambda^2}{C_1}\bigg(D(\lambda^2)+\frac{C_2}{\lambda^2}H(\lambda^2)\bigg) = \frac{H(\lambda^2)}{C_1}\big(\mathcal{N}(\lambda^2)+C_2\big) = O(H(\lambda^2))
\end{aligned}
$$

as $\lambda\to 0^+$. On the other hand, from Lemma 3.2 it follows that, for all $(m,k)\in J_0$,

$$
(105)\qquad \int_{\mathbb{R}^N}\frac{|\widetilde V_{m,k}(x)|^2}{|x|^2}G(x,1)\,dx \leqslant \bigg(\mu_1(\mathbf{A},a)+\frac{(N-2)^2}{4}\bigg)^{-1}\bigg(\gamma+\frac{N-2}{4}\bigg).
$$

Combining (103), (104), (105), and Lemma 5.9, we obtain that

$$
(106)\qquad |\xi_{m,k}(\lambda)| \leqslant O(\lambda^{-2+\frac{\varepsilon}{2}+2\gamma}) \quad \text{as } \lambda\to 0^+,
$$

for all $(m,k)\in J_0$.

In order to prove the theorem, we argue by contradiction and assume that $\lim_{t\to 0^+} t^{-2\gamma}H(t)=0$. By orthogonality of the $\widetilde V_{m,k}$'s in $\mathcal{L}$, we have that

$$
H(\lambda^2) = \sum_{\substack{n,j\in\mathbb{N}\\ j\geqslant 1}}(u_{n,j}(\lambda))^2 \geqslant (u_{m,k}(\lambda))^2 \quad \text{for all } \lambda\in(0,\sqrt{T_0}) \text{ and } m,k\in\mathbb{N},\ k\geqslant 1.
$$

Hence, $\lim_{t\to 0^+} t^{-2\gamma}H(t)=0$ implies that

$$
(107)\qquad \lim_{\lambda\to 0^+}\lambda^{-2\gamma}u_{m,k}(\lambda)=0 \quad \text{for all } m,k\in\mathbb{N},\ k\geqslant 1.
$$

Integration of (101) over (ρ,λ) yields that, for all $\lambda,\rho\in(0,\sqrt{T_0})$,

$$
(108)\qquad u_{m,k}(\rho) = \rho^{2\gamma_{m,k}}\bigg(\lambda^{-2\gamma_{m,k}}u_{m,k}(\lambda) + 2\int_\rho^\lambda s^{1-2\gamma_{m,k}}\xi_{m,k}(s)\,ds\bigg).
$$

Estimate (106) implies that, for all $(m,k)\in J_0$, the function $s\mapsto s^{1-2\gamma}\xi_{m,k}(s)$ belongs to $L^1(0,\sqrt{T_0})$. Letting $\rho\to 0^+$ in (108) and using (107), we obtain that

$$
(109)\qquad u_{m,k}(\lambda) = -2\lambda^{2\gamma}\int_0^\lambda s^{1-2\gamma}\xi_{m,k}(s)\,ds, \quad \text{for all } \lambda\in(0,\sqrt{T_0}).
$$

From (106) and (109), we deduce that, for all $(m,k)\in J_0$,

$$u_{m,k}(\lambda)=O(\lambda^{2\gamma+\frac{\varepsilon}{2}})\quad\text{as }\lambda\to 0^+. \tag{110}$$

Let us fix $0<\sigma<\frac{\varepsilon}{2}$. By Lemma 5.9, there exists $K_2(\sigma)$ such that $H(\lambda^2)\geqslant K_2(\sigma)\lambda^{2(2\gamma+\sigma)}$ for all $\lambda\in(0,\sqrt{T_0})$. Therefore, (110) implies that, for all $(m,k)\in J_0$,

$$\frac{u_{m,k}(\lambda)}{\sqrt{H(\lambda^2)}}=O(\lambda^{\frac{\varepsilon}{2}-\sigma})=o(1)\quad\text{as }\lambda\to 0^+. \tag{111}$$

On the other hand, by Lemma 6.1, for every sequence $\lambda_n\to 0^+$, there exists a subsequence $\{\lambda_{n_j}\}_{j\in\mathbb{N}}$ such that

$$\frac{\tilde u_{\lambda_{n_j}}(x,1)}{\sqrt{H(\lambda_{n_j}^2)}}\to g\quad\text{in }\mathcal{L}\quad\text{as }j\to+\infty,$$

for some eigenfunction $g\in E_0\setminus\{0\}$. Therefore, for all $(m,k)\in J_0$,

$$\frac{u_{m,k}(\lambda_{n_j})}{\sqrt{H(\lambda_{n_j}^2)}}=\left(\frac{\tilde u_{\lambda_{n_j}}(x,1)}{\sqrt{H(\lambda_{n_j}^2)}},\widetilde V_{m,k}\right)_{\mathcal{L}}\to(g,\widetilde V_{m,k})_{\mathcal{L}}\quad\text{as }j\to+\infty. \tag{112}$$

From (111) and (112) we deduce that $(g,\widetilde V_{m,k})_{\mathcal{L}}=0$ for all $(m,k)\in J_0$. Since $g\in E_0$ and $\{\widetilde V_{m,k}:(m,k)\in J_0\}$ is an orthonormal basis of E_0, this implies that $g=0$, a contradiction. □

Proof of Theorem 1.2. As already observed, up to a translation it is not restrictive to assume that $t_0=0$; then the analysis performed in this section applies to the function $\tilde u(x,t)=u(x,-t)$. In particular (17) follows from Lemma 5.8, part (i) of Lemma 6.1 and Proposition 1.1, i.e. there exists an eigenvalue $\gamma_{m_0,k_0}=m_0-\frac{\alpha_{k_0}}{2}$ of $L_{\mathbf{A},a}$, $m_0,k_0\in\mathbb{N}$, $k_0\geqslant 1$, such that $\gamma=\lim_{t\to0^+}\mathcal{N}(t)=\lim_{t\to0^-}\widetilde{\mathcal{N}}(t)=\gamma_{m_0,k_0}$. Let E_0 be the associated eigenspace and J_0 the finite set of indices $\{(m,k)\in\mathbb{N}\times(\mathbb{N}\setminus\{0\}):\gamma_{m_0,k_0}=m-\frac{\alpha_k}{2}\}$, so that $\{\widetilde V_{m,k}:(m,k)\in J_0\}$ is an orthonormal basis of E_0. Let $\{\lambda_n\}_{n\in\mathbb{N}}\subset(0,+\infty)$ such that $\lim_{n\to+\infty}\lambda_n=0$. Then, from part (ii) of Lemma 6.1 and Lemmas 6.2 and 6.3, there exist a subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$ and complex numbers $\{\beta_{n,j}:(n,j)\in J_0\}$ such that $\beta_{n,j}\neq0$ for some $(n,j)\in J_0$ and, for any $\tau\in(0,1)$,

$$\lim_{k\to+\infty}\int_\tau^1\bigg\|\lambda_{n_k}^{-2\gamma}\tilde u(\lambda_{n_k}x,\lambda_{n_k}^2t)-t^\gamma\sum_{(n,j)\in J_0}\beta_{n,j}\widetilde V_{n,j}(x/\sqrt t)\bigg\|_{\mathcal{H}_t}^2dt=0 \tag{113}$$

and

$$\lim_{k\to+\infty}\sup_{t\in[\tau,1]}\bigg\|\lambda_{n_k}^{-2\gamma}\tilde u(\lambda_{n_k}x,\lambda_{n_k}^2t)-t^\gamma\sum_{(n,j)\in J_0}\beta_{n,j}\widetilde V_{n,j}(x/\sqrt t)\bigg\|_{\mathcal{L}_t}=0. \tag{114}$$

In particular,

$$\lambda_{n_k}^{-2\gamma}\tilde u(\lambda_{n_k}x,\lambda_{n_k}^2)\underset{k\to+\infty}{\longrightarrow}\sum_{(n,j)\in J_0}\beta_{n,j}\widetilde V_{n,j}(x)\quad\text{in }\mathcal{L}. \tag{115}$$

Let us fix $\Lambda\in(0,\sqrt{T_0})$ and define $u_{m,j}$ and $\xi_{m,j}$ as in (100–102). From (115) and orthogonality of the $\widetilde V_{n,j}$'s it follows that, for any $(m,j)\in J_0$, $\lim_{k\to+\infty}\lambda_{n_k}^{-2\gamma}u_{m,j}(\lambda_{n_k})=\beta_{m,j}$. Therefore from

(108) we have that, for every $(m,j) \in J_0$,

$$\begin{aligned}\beta_{m,j} &= \Lambda^{-2\gamma} u_{m,j}(\Lambda) + 2\int_0^{\Lambda} s^{1-2\gamma}\xi_{m,j}(s)\,ds \\ &= \Lambda^{-2\gamma}\int_{\mathbb{R}^N} \tilde u(\Lambda x, \Lambda^2)\overline{\widetilde V_{m,j}(x)}G(x,1)\,dx \\ &\qquad + 2\int_0^{\Lambda} s^{1-2\gamma}\bigg(\int_{\mathbb{R}^N} h(sx,-s^2)\tilde u(sx,s^2)\overline{\widetilde V_{m,j}(x)}G(x,1)\,dx\bigg)ds.\end{aligned}$$

The above formula shows that the $\beta_{m,j}$'s depend neither on the sequence $\{\lambda_n\}_{n\in\mathbb{N}}$ nor on its subsequence $\{\lambda_{n_k}\}_{k\in\mathbb{N}}$. Then we can conclude that the convergences in (113) and (114) actually hold as $\lambda \to 0^+$. The proof is thereby complete. □

Proof of Corollary 1.3. Let us argue by contradiction and assume that $u \not\equiv 0$ in $\mathbb{R}^N \times (t_0 - T, t_0)$. Let $k \in \mathbb{N}$ be such that $k > \gamma_{m_0,k_0}$, being γ_{m_0,k_0} as in Theorem 1.2. From (22) it follows that, for a.e. $(x,t) \in \mathbb{R}^N \times (0,1)$,

$$\lim_{\lambda\to 0^+} \lambda^{-2\gamma_{m_0,k_0}} t^{-\gamma_{m_0,k_0}} u(\lambda x, t_0 - \lambda^2 t) = 0.$$

On the other hand, Theorem 1.2 implies that, for all $t \in (0,1)$ and a.e. $x \in \mathbb{R}^N$,

$$\lim_{\lambda\to 0^+} \lambda^{-2\gamma_{m_0,k_0}} t^{-\gamma_{m_0,k_0}} u(\lambda x, t_0 - \lambda^2 t) = g(x/\sqrt{t}),$$

for some $g \in \mathcal{H} \setminus \{0\}$ eigenfunction of $L_{\mathbf{A},a}$ associated to the eigenvalue γ_{m_0,k_0}, thus giving rise to a contradiction. □

7. Proof of Theorem 1.5: representation formula of solutions

Let u be a weak solution to (1) with $h \equiv 0$. By saying that u is a weak solution to (1) we mean that the the function φ defined as

$$\varphi(x,t) = u\big(\sqrt{1+t}\,x, t\big), \quad t \geqslant 0 \tag{116}$$

is a weak solution of the following equation (which is equivalent to (1) up to the change of variable $x \mapsto \sqrt{1+t}\,x$):

$$\frac{d\varphi}{dt}(x,t) = \frac{1}{(1+t)}\Big(-\mathcal{L}_{\mathbf{A},a}\varphi(x,t) + \frac{1}{2}\nabla\varphi(x,t)\cdot x\Big), \tag{117}$$

in the sense that

$$\varphi \in L^2_{\rm loc}([0,+\infty), \tilde{\mathcal{H}}) \quad \text{and} \quad \varphi_t \in L^2_{\rm loc}([0,+\infty), \tilde{\mathcal{H}}^\star), \tag{118}$$

and

$${}_{\tilde{\mathcal{H}}^\star}\langle \varphi_t, w\rangle_{\tilde{\mathcal{H}}} = -\frac{1}{t+1}\int_{\mathbb{R}^N}\bigg(\nabla_{\mathbf{A}}\varphi(x,t)\cdot\overline{\nabla_{\mathbf{A}} w(x)} - \frac{a(x/|x|)}{|x|^2}\,\varphi(x,t)\overline{w(x)}\bigg)e^{\frac{|x|^2}{4}}\,dx$$

for all $w \in \tilde{\mathcal{H}}$, where $\tilde{\mathcal{H}}$ is defined in (44). From (118) it follows that $\varphi \in C^0([0,+\infty), \tilde{\mathcal{L}})$. Furthermore,

$$\varphi(x,0) = u(x,0) = u_0(x).$$

A representation formula for solutions u to (1) with $h \equiv 0$ can be found (in the spirit of [13]) by expanding the transformed solution φ to (117) in Fourier series with respect to an orthonormal basis of $\tilde{\mathcal{L}}$ consisting of eigenfunctions of an Ornstein-Uhlenbeck type operator perturbed with singular homogeneous electromagnetic potentials, i.e. of the operator $L^-_{\mathbf{A},a}$ defined in (49) and acting as

$$
{}_{\tilde{\mathcal{H}}^\star}\langle L^-_{\mathbf{A},a}\varphi, w\rangle_{\tilde{\mathcal{H}}} = \int_{\mathbb{R}^N} \bigg(\nabla_{\mathbf{A}}\varphi(x)\cdot\overline{\nabla_{\mathbf{A}}w(x)} - \frac{a(\frac{x}{|x|})}{|x|^2}\,\varphi(x)\overline{w(x)}\bigg)e^{\frac{|x|^2}{4}}\,dx, \tag{119}
$$

for all $\varphi, w \in \tilde{\mathcal{H}}$, where $\tilde{\mathcal{H}}^\star$ denotes the dual space of $\tilde{\mathcal{H}}$ and ${}_{\tilde{\mathcal{H}}^\star}\langle\cdot,\cdot\rangle_{\tilde{\mathcal{H}}}$ is the corresponding duality product.

Let us expand the initial datum $u_0 = u(\cdot,0) = \varphi(\cdot,0)$ in Fourier series with respect to the orthonormal basis of $\tilde{\mathcal{L}}$ introduced in Remark 4.6 as

$$
u_0 = \sum_{\substack{m,k\in\mathbb{N}\\ k\geqslant 1}} c_{m,k}\widetilde{U}_{m,k} \quad \text{in } \tilde{\mathcal{L}}, \quad \text{where } c_{m,k} = \int_{\mathbb{R}^N} u_0(x)\overline{\widetilde{U}_{m,k}(x)}e^{\frac{|x|^2}{4}}\,dx, \tag{120}
$$

and, for $t \geqslant 0$, the function $\varphi(\cdot,t)$ defined in (116) as

$$
\varphi(\cdot,t) = \sum_{\substack{m,k\in\mathbb{N}\\ k\geqslant 1}} \varphi_{m,k}(t)\widetilde{U}_{m,k} \quad \text{in } \tilde{\mathcal{L}}, \tag{121}
$$

where

$$
\varphi_{m,k}(t) = \int_{\mathbb{R}^N} \varphi(x,t)\overline{\widetilde{U}_{m,k}(x)}e^{\frac{|x|^2}{4}}\,dx.
$$

Since $\varphi(x,t)$ satisfies (117), by Remark 4.6 we obtain that $\varphi_{m,k} \in C^1([0,+\infty),\mathbb{C})$ and

$$
\varphi'_{m,k}(t) = -\frac{\tilde{\gamma}_{m,k}}{1+t}\,\varphi_{m,k}(t), \quad \varphi_{m,k}(0) = c_{m,k},
$$

with $\tilde{\gamma}_{m,k} = \frac{N}{2} + \gamma_{m,k} = \frac{N}{2} - \frac{\alpha_k}{2} + m$. Integration yields $\varphi_{m,k}(t) = c_{m,k}(1+t)^{-\tilde{\gamma}_{m,k}}$. Hence expansion (121) can be rewritten as

$$
\varphi(z,t) = \sum_{\substack{m,k\in\mathbb{N}\\ k\geqslant 1}} c_{m,k}(1+t)^{-\tilde{\gamma}_{m,k}}\widetilde{U}_{m,k}(z) \quad \text{in } \tilde{\mathcal{L}}, \quad \text{for all } t \geqslant 0.
$$

We notice that $\tilde{\gamma}_{m,k} \geqslant 0$ for all m,k, then, in view of the Parseval identity, for all $t \geqslant 0$,

$$
\|\varphi(\cdot,t)\|^2_{\tilde{\mathcal{L}}} = \sum_{\substack{m,k\in\mathbb{N}\\ k\geqslant 1}} c^2_{m,k}(1+t)^{-2\tilde{\gamma}_{m,k}} \leqslant \sum_{\substack{m,k\in\mathbb{N}\\ k\geqslant 1}} c^2_{m,k} = \|u_0\|^2_{\tilde{\mathcal{L}}}. \tag{122}
$$

In view of (120), the above series can be written as

$$
\varphi(z,t) = \sum_{\substack{m,k\in\mathbb{N}\\ k\geqslant 1}} (1+t)^{-\tilde{\gamma}_{m,k}}\bigg(\int_{\mathbb{R}^N} e^{\frac{|y|^2}{4}}u_0(y)\overline{\widetilde{U}_{m,k}(y)}\,dy\bigg)\widetilde{U}_{m,k}(z),
$$

in the sense that, for all $t \geqslant 0$, the above series converges in $\tilde{\mathcal{L}}$. Since $u_0(y)$ can be expanded as

$$
u_0(y) = u_0\big(|y|\,\tfrac{y}{|y|}\big) = \sum_{j=1}^{\infty} u_{0,j}(|y|)\psi_j\big(\tfrac{y}{|y|}\big) \quad \text{in } L^2(\mathbb{S}^{N-1}),
$$

where $u_{0,j}(|y|) = \int_{\mathbb{S}^{N-1}} u_0(|y|\theta)\overline{\psi_j(\theta)}\, dS(\theta)$, we conclude that

$$\begin{aligned}\varphi(z,t) &= \sum_{\substack{m,k\in\mathbb{N}\\ k\geqslant 1}} \frac{(1+t)^{-\tilde\gamma_{m,k}}}{\|U_{m,k}\|^2_{\tilde{\mathcal{L}}}} U_{m,k}(z)\bigg(\int_0^\infty u_{0,k}(r) r^{N-1-\alpha_k} P_{k,m}(\tfrac{r^2}{4})\, dr\bigg)\\ &= \sum_{k=1}^\infty \psi_k\big(\tfrac{z}{|z|}\big)\frac{(1+t)^{\frac{\alpha_k}{2}-\frac{N}{2}}}{2^{1+2\beta_k}\Gamma(1+\beta_k)}\Bigg[\sum_{m=0}^\infty \frac{\binom{m+\beta_k}{m}}{(1+t)^m}\times\\ &\quad\times\bigg(\int_0^\infty \frac{u_{0,k}(r)}{|rz|^{\alpha_k}} P_{k,m}\big(\tfrac{r^2}{4}\big)P_{k,m}\big(\tfrac{|z|^2}{4}\big)e^{-\frac{|z|^2}{4}} r^{N-1}dr\bigg)\Bigg].\end{aligned}$$

By [1] we know that

$$P_{k,m}\Big(\frac{r^2}{4}\Big) = \frac{\Gamma(1+\beta_k)}{\Gamma(1+\beta_k+m)} e^{\frac{r^2}{4}} r^{-\beta_k} 2^{\beta_k}\int_0^\infty e^{-t}t^{m+\frac{\beta_k}{2}} J_{\beta_k}(r\sqrt{t})\, dt,$$

where J_{β_k} is the Bessel function of the first kind of order β_k. Therefore,

$$\begin{aligned}\varphi(z,t) &= 2\sum_{k=1}^\infty \psi_k\big(\tfrac{z}{|z|}\big)(1+t)^{\frac{\alpha_k}{2}-\frac{N}{2}}\Bigg[\sum_{m=0}^\infty \frac{(1+t)^{-m}}{\Gamma(1+\beta_k+m)\Gamma(1+m)}\times\\ &\quad\times\bigg(\int_0^\infty \frac{u_{0,k}(r)}{|rz|^{\alpha_k+\beta_k}}e^{\frac{r^2}{4}}\,\bigg(\int_0^\infty\!\!\int_0^\infty e^{-s^2-s'^2}(ss')^{2m+\beta_k+1}\times\\ &\quad\times J_{\beta_k}(rs)J_{\beta_k}(|z|s')\, ds\, ds'\big) r^{N-1}dr\bigg)\Bigg]\\ &= 2\sum_{k=1}^\infty \psi_k\big(\tfrac{z}{|z|}\big)(1+t)^{\frac{\alpha_k}{2}-\frac{N}{2}}\Bigg[\int_0^\infty u_{0,k}(r)|rz|^{-\alpha_k-\beta_k}e^{\frac{r^2}{4}}r^{N-1}\times\\ &\quad\times\bigg(\int_0^\infty\!\!\int_0^\infty \frac{ss'}{e^{s^2+s'^2}}\times\\ &\quad\times\bigg(\sum_{m=0}^\infty \frac{(1+t)^{-m}}{\Gamma(1+m)\Gamma(1+\beta_k+m)}(ss')^{2m+\beta_k}\bigg)\times\ J_{\beta_k}(rs)J_{\beta_k}(|z|s')ds\, ds'\bigg)dr\Bigg]\\ &= 2\sum_{k=1}^\infty i^{-\beta_k}\psi_k\big(\tfrac{z}{|z|}\big)(1+t)^{\frac{\alpha_k}{2}-\frac{N}{2}+\frac{\beta_k}{2}}\Bigg[\int_0^\infty u_{0,k}(r)|rz|^{-\alpha_k-\beta_k}e^{\frac{r^2}{4}}r^{N-1}\times\\ &\quad\times\bigg(\int_0^\infty\!\!\int_0^\infty \frac{ss'}{e^{s^2+s'^2}}J_{\beta_k}\Big(\frac{2iss'}{\sqrt{1+t}}\Big)\ J_{\beta_k}(rs)J_{\beta_k}(|z|s')ds\, ds'\bigg)dr\Bigg],\end{aligned}$$

where in the last line we have used that

$$(1+t)^{-m}(ss')^{2m+\beta_k} = (1+t)^{\frac{\beta_k}{2}} i^{-\beta_k}(-1)^m\Big(\frac{iss'}{\sqrt{1+t}}\Big)^{2m+\beta_k},$$

and

$$\sum_{m=0}^\infty \frac{(-1)^m}{\Gamma(1+m)\Gamma(1+\beta_k+m)}\Big(\frac{iss'}{\sqrt{1+t}}\Big)^{2m+\beta_k} = J_{\beta_k}\Big(\frac{2iss'}{\sqrt{1+t}}\Big).$$

Then we have that

$$\varphi(z,t) = 2\sum_{k=1}^{\infty} \frac{e^{-i\frac{\pi}{2}\beta_k}\psi_k\big(\frac{z}{|z|}\big)}{(1+t)^{\frac{N}{4}+\frac{1}{2}}}\bigg[\int_0^\infty u_{0,k}(r)|rz|^{-\frac{N-2}{2}} e^{\frac{r^2}{4}} \mathcal{I}_{k,t}(r,|z|) r^{N-1} dr\bigg], \tag{123}$$

where

$$\mathcal{I}_{k,t}(r,|z|) = \int_0^\infty\!\!\int_0^\infty ss' e^{-s^2-s'^2} J_{\beta_k}\Big(\frac{2iss'}{\sqrt{1+t}}\Big) J_{\beta_k}(rs) J_{\beta_k}(|z|s')\, ds\, ds'.$$

From [27, formula (1), p. 395] (with $t = s'$, $p = 1$, $a = \frac{2is}{\sqrt{1+t}}$, $b = |z|$, $\nu = \beta_k$ which satisfy $\mathfrak{Re}(\nu) > -1$ and $|\arg p| < \frac{\pi}{4}$), we know that

$$\int_0^\infty s' e^{-s'^2} J_{\beta_k}\Big(\frac{2iss'}{\sqrt{1+t}}\Big) J_{\beta_k}(|z|s')\, ds' = \frac{1}{2} e^{-\frac{1}{4}(|z|^2 - \frac{4s^2}{1+t})} I_{\beta_k}\bigg(\frac{i|z|s}{\sqrt{1+t}}\bigg),$$

where I_{β_k} denotes the modified Bessel function of order β_k. Hence

$$\begin{aligned}\mathcal{I}_{k,t}(r,|z|) &= \frac{1}{2}\int_0^\infty s e^{-s^2} J_{\beta_k}(rs) e^{-\frac{|z|^2}{4}} e^{\frac{s^2}{1+t}} I_{\beta_k}\Big(\frac{i|z|s}{\sqrt{1+t}}\Big)\, ds \\ &= \frac{1}{2} e^{-i\beta_k\frac{\pi}{2}} e^{\frac{-|z|^2}{4}} \int_0^\infty s e^{-s^2} J_{\beta_k}(rs) e^{\frac{s^2}{1+t}} J_{\beta_k}\Big(\frac{-|z|s}{\sqrt{1+t}}\Big)\, ds,\end{aligned}$$

since $I_\nu(x) = e^{-\frac{1}{2}\nu\pi i} J_\nu(xe^{\frac{\pi}{2}i})$ (see e.g. [1, 9.6.3, p. 375]). We obtain

$$\mathcal{I}_{k,t}(r,|z|) = \frac{1}{2} e^{-i\beta_k\frac{\pi}{2}} e^{\frac{-|z|^2}{4}} \int_0^\infty s e^{-s^2\frac{t}{1+t}} J_{\beta_k}(rs) J_{\beta_k}\Big(\frac{-|z|s}{\sqrt{1+t}}\Big)\, ds.$$

Applying [27, formula (1), p. 395] (with $t = s$, $p^2 = \frac{t}{t+1}$, $a = r$, $b = -\frac{|z|}{\sqrt{1+t}}$, $\nu = \beta_k$ which satisfy $\mathfrak{Re}(\nu) > -1$ and $|\arg p| < \frac{\pi}{4}$) and [1, 9.6.3, p. 375], we obtain

$$\begin{aligned}\mathcal{I}_{k,t}(r,|z|) &= \frac{1}{4} e^{-\frac{\beta_k}{2}\pi i} e^{-\frac{|z|^2}{4}} e^{-\frac{r^2(1+t)+|z|^2}{4t}} \frac{(t+1)}{t} I_{\beta_k}\bigg(\frac{-r|z|\sqrt{1+t}}{2t}\bigg) \\ &= \frac{1}{4} e^{i\beta_k\pi} e^{-\frac{|z|^2}{4}} e^{-\frac{r^2(1+t)+|z|^2}{4t}} \frac{(t+1)}{t} J_{\beta_k}\bigg(\frac{-ir|z|\sqrt{1+t}}{2t}\bigg).\end{aligned}$$

From (123) and the above identity we deduce

$$\varphi(z,t) = \frac{1}{2}\sum_{k=1}^{\infty} \frac{\psi_k\big(\frac{z}{|z|}\big) e^{i\frac{\pi}{2}\beta_k}}{t(1+t)^{\frac{N}{4}-\frac{1}{2}}}\bigg[\int_0^\infty \frac{u_{0,k}(r)}{|rz|^{\frac{N-2}{2}}} e^{-\frac{|z|^2}{4t} - \frac{|z|^2}{4} - \frac{r^2}{4t}} J_{\beta_k}\bigg(\frac{-ir|z|\sqrt{1+t}}{2t}\bigg) r^{N-1}\, dr\bigg].$$

Notice that, by replacing $\int_0^\infty$ with $\int_0^R$ we obtain the series representation of the solution $\varphi_R(z,t)$ with initial datum $u_{0,R}(x) \equiv \chi_R(x) u_0(x)$, being $\chi_R(x)$ the characteristic function of the ball B_R of radius R centered at the origin. Since $u_{0,R} \to u_0$ in $\tilde{\mathcal{L}}$ as $R \to +\infty$, from (122) it follows that $\varphi(\cdot,t) = \lim_{R\to\infty} \varphi_R(\cdot,t)$ in $\tilde{\mathcal{L}}$. Recalling the definition of $u_{0,k}$ and observing that the queue of the series

$$\sum_{k=1}^{\infty} \psi_k\big(\tfrac{z}{|z|}\big)\overline{\psi_k\big(\tfrac{y}{|y|}\big)} e^{i\frac{\pi}{2}\beta_k} \frac{u_0(y)}{(|y||z|)^{\frac{N-2}{2}}} e^{-\frac{|y|^2+|z|^2(1+t)}{4t}} J_{\beta_k}\bigg(\frac{-i|y||z|\sqrt{1+t}}{2t}\bigg)$$

is uniformly convergent on compact sets (see [13, Lemma 1.2]), we can exchange integral and sum and write

$$\varphi_R(z,t) = \frac{1}{2t(1+t)^{\frac{N}{4}-\frac{1}{2}}}\int_{B_R}\frac{u_0(y)}{(|y||z|)^{\frac{N-2}{2}}}e^{-\frac{|z|^2(1+t)+|y|^2}{4t}}\left[\sum_{k=1}^{\infty} e^{i\frac{\pi}{2}\beta_k}\psi_k\big(\tfrac{z}{|z|}\big)\overline{\psi_k\big(\tfrac{y}{|y|}\big)}J_{\beta_k}\left(\frac{-i|y||z|\sqrt{1+t}}{2t}\right)\right]dy,$$

i.e.

$$\varphi_R(z,t) = t^{-\frac{N}{2}}\int_{B_R} u_0(y)K\left(\tfrac{y}{\sqrt{t}},\tfrac{z\sqrt{1+t}}{\sqrt{t}}\right)dy$$

where $K(y,z)$ is the kernel defined in (25). Letting $R\to+\infty$ we obtain that

$$\varphi(z,t) = t^{-\frac{N}{2}}\int_{\mathbb{R}^N} u_0(y)K\left(\tfrac{y}{\sqrt{t}},\tfrac{z\sqrt{1+t}}{\sqrt{t}}\right)dy,$$

where the integral at the right hand side is understood in the sense of improper multiple integrals. From (116) we conclude that, for all $t>0$,

$$u(x,t) = t^{-\frac{N}{2}}\int_{\mathbb{R}^N} u_0(y)K\left(\frac{y}{\sqrt{t}},\frac{x}{\sqrt{t}}\right)dy.$$

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