J. Group Theory **21** (2018), 83–100
DOI 10.1515/jgth-2017-0030

Normal subgroups in limit groups of prime index

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Communicated by Dessislava H. Kochloukova

Abstract. Motivated by their study of pro-p limit groups, D. H. Kochloukova and P. A. Zalesskii formulated in [15, Remark after Theorem 3.3] a question concerning the minimum number of generators $d(N)$ of a normal subgroup N of prime index p in a non-abelian limit group G (see Question*). It is shown that the analogous question for the rational rank has an affirmative answer (see Theorem A). From this result one may conclude that the original question of Kochloukova and Zalesskii has an affirmative answer if the abelianization G^{ab} of G is torsion free and $d(G) = d(G^{\mathrm{ab}})$ (see Corollary B), or if G is a special kind of one-relator group (see Theorem D).

1 Introduction

In recent years *limit groups* (or *ω-residually free groups*) have received much attention (see [1, 3, 5, 6, 8, 9, 13, 16]). To the authors' knowledge B. Baumslag was one of the first who studied these groups – in these days under the more traditional name of *fully residually free groups* (see [2]). Indeed, this notion reflects the fact that for any limit group G and any finite subset T of G there exists a homomorphism from G to a free group F that is injective on T. Then, around twenty years ago, O. G. Kharlampovich and A. M. Myasnikov proved several structure theorems for this class of groups, which are still the principal tools for working in this area (see [10, 11]). Soon afterwards Z. Sela published his groundbreaking work on limit groups (see [17, 18] and the references therein) and also introduced the name *limit group*. More recently, O. G. Kharlampovich and A. M. Myasnikov established many properties of limit groups in [12]. Examples of limit groups include all finitely generated free groups, all finitely generated free abelian groups, and all the fundamental groups of closed oriented surfaces. Moreover, the class of limit groups is closed with respect to finitely generated subgroups and free products. This fact can be used to construct many examples. More sophisticated examples can be found in some of the articles cited above.

Recently, D. H. Kochloukova and P. A. Zalesskii introduced and studied in [15] a class of pro-p groups which might be considered as the pro-p analogue of the

The second author was supported by CNPq-Brazil.

class of limit groups and called it the class of *pro-p limit groups*. Motivated by one of their main results on pro-p limit groups they raised the following question.

Question*. Let G be a non-abelian limit group, and let U be a normal subgroup of G of prime index p. Does this imply that $d(U) > d(G)$?

Here $d(G)$ denotes the minimum number of generators of a finitely generated group G. For a pro-p group $\tilde{G}$ the minimum number of (topological) generators $d(\tilde{G})$ of $\tilde{G}$ is closely related to the cohomology group $H^1(\tilde{G}, \mathbb{F}_p)$, where $\mathbb{F}_p$ denotes the finite field with p elements, i.e., $d(\tilde{G}) = \dim_{\mathbb{F}_p}(H^1(\tilde{G}, \mathbb{F}_p))$ (see [19, Section I.4.2, Corollary of Proposition 25]). For an abstract group G such a close relation does not hold. There are two homological invariants of a finitely generated group G which can be seen as a homological approximation of $d(G)$: The *rational rank* of G given by

$$\mathrm{rk}_{\mathbb{Q}}(G) = \dim_{\mathbb{Q}}(G^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Q}) = \dim_{\mathbb{Q}}(H_1(G, \mathbb{Q})),$$

where $G^{\mathrm{ab}} = G/[G,G]$ denotes the *abelianization* of G, and

$$d(G^{\mathrm{ab}}) = \mathrm{rk}_{\mathbb{Z}}(G^{\mathrm{ab}}) = \mathrm{rk}_{\mathbb{Z}}(H_1(G, \mathbb{Z})).$$

In particular, $\mathrm{rk}_{\mathbb{Q}}(G) \leq d(G^{\mathrm{ab}}) \leq d(G)$. The main purpose of this paper is to give an affirmative answer to the analogue of Question* for the rational rank (see Theorem 3.2).

Theorem A. *Let G be a non-abelian limit group, and let U be a normal subgroup of G of prime index p. Then* $\mathrm{rk}_{\mathbb{Q}}(U) > \mathrm{rk}_{\mathbb{Q}}(G)$.

From Theorem A one concludes the following affirmative partial answer to Question* (see Corollary 3.3).

Corollary B. *Let G be a non-abelian limit group such that G^{ab} is a torsion free group and that $d(G) = d(G^{\mathrm{ab}})$. If U is a normal subgroup of G of prime index p, then $d(U) > d(G)$.*

It should be mentioned that there exist limit groups G for which $d(G^{\mathrm{ab}}) \neq d(G)$ (see Remark 4.4), and also for which G^{ab} is not torsion free (see Remark 4.5). So Corollary B gives only a partial affirmative answer to Question*.

Since every open subgroup of a pro-p group is subnormal, the affirmative answer to the analogue of Question* for pro-p limit groups has many interesting consequences. Obviously, subgroups of finite index in a discrete limit group do not have to be subnormal. Hence one may not expect that a positive solution of Question* has a similar impact as in the pro-p case. Nevertheless, Theorem A has also the following consequence (see Corollary 3.5).

Corollary C. *Let G be a non-abelian limit group, and let N be a normal subgroup of G such that G/N is infinite and nilpotent. Then* $\mathrm{rk}_{\mathbb{Q}}(N) = \infty$. *In particular, if* $\alpha\colon G \to \mathbb{Z}$ *is a non-trivial homomorphism, then* $\mathrm{rk}_{\mathbb{Q}}(\ker(\alpha)) = \infty$.

The proof of Theorem A can be modified in order to obtain an affirmative answer to Question* also for certain types of one-relator limit groups (see Theorem 4.3).

Theorem D. *Let G be a non-abelian cyclically pinched or conjugacy pinched one-relator limit group, and let U be a normal subgroup of G of prime index p. Then $d(U) > d(G)$.*

2 Limit groups

By $\mathbb{N}$ we will denote the set of positive integers, and by $\mathbb{N}_0$ the set of non-negative integers, i.e., $\mathbb{N}_0 = \{0, 1, 2, \ldots\}$. We will use the symbol "$\star$" for *free products with amalgamation* in the category of groups.

2.1 Extension of centralizers

By starting from a limit group G, there is a standard procedure to construct a limit group $G(C, m)$, where $C \subseteq G$ is a maximal cyclic subgroup of G and $m \in \mathbb{N}$. This procedure is known as *extension of centralizers*, i.e., if G is a limit group, then

$$G(C, m) = G \star_C (C \times \mathbb{Z}^m)$$

is again a limit group (see [11, Lemma 2 and Theorem 4]). For short we call a limit group G an *iterated extension of centralizers of a free group* (= i.e.c.f. group), if there exists a sequence of limit groups $(G_k)_{0 \le k \le n}$ such that

(EZ1) $G_0 = F$ is a finitely generated free group, and $G_n \simeq G$,

(EZ2) for $k \in \{0, \ldots, n-1\}$ there exists a maximal cyclic subsgroup $C_k \subseteq G_k$ and $m_k \in \mathbb{N}$ such that $G_{k+1} \simeq G_k(C_k, m_k)$.

If G is an iterated extension of centralizers of a free group, one calls the minimum number $n \in \mathbb{N}_0$ for which there exists a sequence of limit groups $(G_k)_{0 \le k \le n}$ satisfying (EZ1) and (EZ2) the *level* of G. This number will be denoted by $\ell(G)$. For example, a finitely generated free group is an iterated extension of centralizers of a free group of level 0, and a finitely generated free abelian group is an iterated extension of centralizers of a free group of level 1.

2.2 The height of a limit group

By the second embedding theorem (see [13, Section 2.3, Theorem 2], [11, Theorem 4]), every limit group G is isomorphic to a subgroup of an i.e.c.f. group H. The *height* $\mathrm{ht}(G)$ of G is defined as

$$\mathrm{ht}(G) = \min\{\ell(H) \mid G \subseteq H,\ H \text{ an i.e.c.f. group}\}.$$

E.g., a limit group G is of height 0 if, and only if, it is a free group of finite rank, and non-cyclic finitely generated free abelian groups are of height 1.

2.3 Limit groups as fundamental groups of graph of groups

Let G be a limit group of height $\mathrm{ht}(G) = n \geq 1$, and let H be an i.e.c.f. group of level $\ell(H) = n$ such that $G \subseteq H$. Then H acts on a *tree* Γ without inversion (of edges) with two orbits on $V(\Gamma)$ and two orbits on $E(\Gamma)$, where $V(\Gamma)$ is the set of *vertices* of Γ, and $E(\Gamma)$ is the set of (oriented) *edges* of Γ (see [20, Section I.4.4]). Moreover, for $v \in V(\Gamma)$ its stabilizer H_v is either free abelian or an i.e.c.f. group of level $n-1$. For $e \in E(\Gamma)$ its stabilizer H_e is infinite cyclic.

As G is acting on Γ without inversion of edges, the fundamental theorem of Bass–Serre theory (see [20, Section I.5.4, Theorem 13]) implies that G is isomorphic to the fundamental group $\pi_1(\mathcal{G}, \Lambda, \mathcal{T})$ of a graph of groups $\mathcal{G}$ based on a connected graph Λ and $\mathcal{T}$ is a maximal subtree of Λ. For simplicity we assume that $G = \pi_1(\mathcal{G}, \Lambda, \mathcal{T})$. Since G is finitely generated, $E = E(\Lambda) \setminus E(\mathcal{T})$ must be finite. Otherwise, G would have an infinitely generated free group as a homomorphic image. Similarly, as G is finitely generated, there exists a finite set $\Omega \subseteq V(\Lambda)$ such that

$$G = \langle e, \mathcal{G}_v \mid e \in E,\ v \in \Omega \rangle. \tag{2.1}$$

Let $\mathcal{T}_0 = \mathrm{span}_{\mathcal{T}}(\Omega \cup \{o(e), t(e) \mid e \in E\})$ be the tree spanned by the set Ω and the origins and termini of the edges in E, and let Λ_0 be the subgraph of Λ satisfying $V(\Lambda_0) = V(\mathcal{T}_0)$, and $E(\Lambda_0) = E(\mathcal{T}_0) \sqcup E$. By construction, Λ_0 is a finite connected graph, and $\mathcal{T}_0$ is a maximal subtree of Λ_0. Let $\mathcal{G}' = \mathcal{G}|_{\Lambda_0}$ be the restriction of $\mathcal{G}$ to Λ_0. Then, by definition, one has a canonical homomorphism of groups $i_{\mathcal{T}} \colon \pi_1(\mathcal{G}', \Lambda_0, \mathcal{T}_0) \to \pi_1(\mathcal{G}, \Lambda, \mathcal{T})$. For $P_0 \in V(\Lambda_0)$, one has a commutative diagram

$$\begin{array}{ccc} \pi_1(\mathcal{G}', \Lambda_0, P_0) & \xrightarrow{i_{P_0}} & \pi_1(\mathcal{G}, \Lambda, P_0) \\ \downarrow & & \downarrow \\ \pi_1(\mathcal{G}', \Lambda_0, \mathcal{T}_0) & \xrightarrow{i_{\mathcal{T}}} & \pi_1(\mathcal{G}, \Lambda, \mathcal{T}), \end{array}$$

where the vertical maps are isomorphisms (see [20, Section I.5.1, Proposition 20]). As the canonical map is mapping generalized reduced words to generalized reduced words, i_{P_0} is injective (see [20, p. 50, Example (c)]), and hence $i_{\mathcal{T}}$ is injective. Thus, by (2.1), $i_{\mathcal{T}}$ is an isomorphism. As a consequence, one concludes the following property which is slightly more precise than [11, Theorem 6].

Proposition 2.1. *Let G be a limit group of height $n \geq 1$. Then G is isomorphic to the fundamental group $\pi_1(\mathcal{G}', \Lambda_0, \mathcal{T}_0)$ of a graph of groups $\mathcal{G}'$ satisfying*

(i) *Λ_0 is finite,*

(ii) *for all $v \in V(\Lambda_0)$, $\mathcal{G}'_v$ is finitely generated abelian or a limit group of height at most $n-1$,*

(iii) *for all $e \in E(\Lambda_0)$, $\mathcal{G}'_e$ is infinite cyclic or trivial.*

Proof. By the previously mentioned argument, it suffices to show that $\mathcal{G}'_v$ is finitely generated for all $v \in V(\Lambda_0)$. This follows by the argument used in [11, Proof of Theorem 6, p. 567] in connection with [4, Satz 5.8] and the equivalent statement for HNN-extensions (see [4, Satz 6.3]). □

2.4 Limit groups of small rank

As limit groups are (fully) residually free, they must be torsion free. From this property one concludes the following useful fact.

Fact 2.2. *Let G be a limit group with $\mathrm{rk}_{\mathbb{Q}}(G) = 1$. Then $G \simeq \mathbb{Z}$.*

Proof. Suppose that G is non-abelian, i.e., there exist $a, b \in G$ with $[a,b] \neq 1$. Since G is a limit group, there exists a free group F of finite rank and an epimorphism $\phi\colon G \longrightarrow F$ satisfying $\phi([a,b]) \neq 1$. As $\phi^{\mathrm{ab}}\colon G^{\mathrm{ab}} \to F^{\mathrm{ab}}$ is surjective, its induced map $\phi^{\mathrm{ab}}_{\mathbb{Q}}\colon G^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Q} \to F^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Q}$ is also surjective. Hence we have $1 = \mathrm{rk}_{\mathbb{Q}}(G) \geq \mathrm{rk}_{\mathbb{Q}}(F)$, and so $\mathrm{rk}_{\mathbb{Q}}(F) = 1$. As $\mathrm{rk}_{\mathbb{Q}}(F) = d(F^{\mathrm{ab}}) = d(F) = 1$, F must be cyclic, and, in particular, abelian. Thus $\phi([a,b]) = [\phi(a), \phi(b)] = 1$, a contradiction. Hence G is a finitely generated free abelian group. In particular, $d(G) = \mathrm{rk}_{\mathbb{Q}}(G) = 1$, and G is cyclic. □

The following property has been shown by D. H. Kochloukova in [14].

Proposition 2.3 (Kochloukova). *For a limit group G its Euler characteristic $\chi(G)$ is non-positive. Moreover, $\chi(G) = 0$ if, and only if, G is abelian.*

Limit groups with minimum number of generators less than or equal to 3 are well known. In [6] the following was shown.

Theorem 2.4 (Fine, Gaglione, Myasnikov, Rosenberger and Spellman). *Let G be a limit group. Then:*

(a) $d(G) = 1$ *if, and only if, G is infinite cyclic.*

(b) $d(G) = 2$ *if, and only if, G is a free group of rank 2 or a free abelian group of rank 2.*

(c) $d(G) = 3$ *if, and only if, G is a free group of rank 3, a free abelian group of rank 3, or an extension of centralizers of a free group of rank 2, i.e., G has a presentation*

$$G = \langle x_1, x_2, x_3 \mid x_3^{-1} v x_3 = v \rangle,$$

where $v = v(x_1, x_2) \in F(\{x_1, x_2\})$ is non-trivial.

3 The rational rank of a limit group

The following fact will turn out to be useful for our purpose.

Lemma 3.1. *Let G_1 and G_2 be groups, and let $C = \langle c \rangle$ be an infinite cyclic group or the trivial group.*

(a) *If $G = G_1 \star_C G_2$ is a free product with amalgamation in C, then*

$$\mathrm{rk}_{\mathbb{Q}}(G) = \mathrm{rk}_{\mathbb{Q}}(G_1) + \mathrm{rk}_{\mathbb{Q}}(G_2) - \rho(G),$$

where $\rho(G) \in \{0, 1\}$. Moreover, if $C = 1$, then $\rho(G) = 0$.

(b) *If $G = \mathrm{HNN}_\phi(G_1, C, t) = \langle G_1, t \mid tct^{-1} = \phi(c) \rangle$ is an HNN-extension with equalization in $C \subseteq G_1$, then*

$$\mathrm{rk}_{\mathbb{Q}}(G) = \mathrm{rk}_{\mathbb{Q}}(G_1) + \rho(G), \tag{3.1}$$

where $\rho(G) \in \{0, 1\}$. One has an exact sequence

$$C \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\alpha} G_1^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Q} \xrightarrow{\beta} G^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Q} \to \mathbb{Q} \to 0, \tag{3.2}$$

where $\alpha(c \otimes q) = (c\phi(c)^{-1} G_1') \otimes q$, $q \in \mathbb{Q}$. Moreover,

(1) $\rho(G) = 0$ *if, and only if, α is injective,*

(2) $\rho(G) = 1$ *if, and only if, α is the 0-map.*

Proof. (a) Let $G = G_1 \star_C G_2$, and let $\iota_i : C \to G_i$, $i \in \{1, 2\}$ denote the associated embeddings. The Mayer–Vietoris sequence associated to $- \otimes_{\mathbb{Z}} \mathbb{Q}$ specializes to an exact sequence

$$
\begin{array}{ccccc}
H_1(C,\mathbb{Q}) & \xrightarrow{\alpha} & H_1(G_1,\mathbb{Q}) \oplus H_1(G_2,\mathbb{Q}) & \xrightarrow{\beta} & H_1(G,\mathbb{Q}) \\
 & & & & \downarrow \gamma \\
0 \longleftarrow H_0(G,\mathbb{Q}) & \longleftarrow & H_0(G_1,\mathbb{Q}) \oplus H_0(G_2,\mathbb{Q}) & \longleftarrow & H_0(C,\mathbb{Q}),
\end{array}
\tag{3.3}
$$

where $\alpha(c \otimes q) = ((\iota_1(c)G_1') \otimes q, -(\iota_2(c)G_2') \otimes q)$, $q \in \mathbb{Q}$. In particular, $\gamma = 0$, and this yields (a).

(b) In this case the Mayer–Vietoris sequence specializes to

$$
\begin{array}{ccccc}
H_1(C,\mathbb{Q}) & \xrightarrow{\alpha} & H_1(G_1,\mathbb{Q}) & \xrightarrow{\beta} & H_1(G,\mathbb{Q}) \\
 & & & & \downarrow \\
0 \longleftarrow H_0(G,\mathbb{Q}) & \longleftarrow & H_0(G_1,\mathbb{Q}) & \xleftarrow{\delta} & H_0(C,\mathbb{Q})
\end{array}
$$

with α as described in (b) of the statement of the lemma. In particular, $\delta = 0$ which yields (3.2), and thus also (3.1). The final remarks (1) and (2) follow from the fact that $\dim(\mathrm{im}(\alpha)) \in \{0,1\}$, and that $\dim(\mathrm{im}(\alpha)) = 1$ if, and only if, α is injective. □

From Lemma 3.1 one concludes the following.

Theorem 3.2. *Let G be a non-abelian limit group, and let U be a normal subgroup of G of prime index p. Then* $\mathrm{rk}_{\mathbb{Q}}(U) > \mathrm{rk}_{\mathbb{Q}}(G)$.

Proof. We proceed by induction on $n = \mathrm{ht}(G)$ (see Section 2.2). If $n = 0$, then G is a finitely generated free group satisfying $d(G) \geq 2$, and the Nielsen–Schreier theorem yields

$$\mathrm{rk}_{\mathbb{Q}}(U) = d(U) = p \cdot (d(G) - 1) + 1 > d(G) = \mathrm{rk}_{\mathbb{Q}}(G)$$

and hence the claim. So assume that G is a limit group of height $\mathrm{ht}(G) = n \geq 1$, and that the claim holds for all limit groups of height at most $n-1$. By Proposition 2.1, G is isomorphic to the fundamental group $\pi_1(\Upsilon, \Lambda, \mathcal{T})$ of a graph of groups Υ based on a finite connected graph Λ whose edge groups are either infinite cyclic or trivial, and whose vertex groups are either limit group of height at most $n-1$ or free abelian groups. By applying induction on $s(\Lambda) = |V(\Lambda)| + |E(\Lambda)|$ it suffices to consider the following two cases:

(I) $G = G_1 \star_C G_2$ and G_i is either a limit group of height at most $n-1$ or abelian, and C is either infinite cyclic or trivial, $i \in \{1,2\}$.

(II) $G = \mathrm{HNN}_\phi(G_1, C, t)$, where G_1 is either a limit group of height at most $n-1$ or abelian, and C is either infinite cyclic or trivial.

Case (I). Let $G = G_1 \star_C G_2$. If C is non-trivial, then either G_1 or G_2 is non-abelian. Otherwise, by a result of I. Chiswell, one would conclude that we have $\chi(G) = \chi(G_1) + \chi(G_2) - \chi(C) = 0$ and G must be abelian (see Proposition 2.3) which was excluded by hypothesis. The group G acts naturally on a tree T without inversion of edges with two orbits V_1 and V_2 on $V(T)$ and two orbits E_1 and E_2 on $E(T)$ satisfying $\bar{E_1} = E_2$. We may assume that for $e \in E_1$ one has $o(e) \in V_1$ and $t(e) \in V_2$. By hypothesis,

$$|G : UG_i| \in \{1, p\} \quad \text{and} \quad |G : UC| \in \{1, p\}, \tag{3.4}$$

and one may distinguish the following cases:

(I.1) $|G : UC| = 1$, i.e., U has one orbit on E_1 and E_2.

(I.2) $|G : UC| = p$, i.e., U has p orbits on E_1 and E_2.

Case (I.1). The hypothesis implies that $G = UG_1 = UG_2$, and U has one orbit on V_1 and one orbit on V_2. Thus, by the fundamental theorem of Bass–Serre theory (see [20, Section I.5.4, Theorem 13]), one has $U \simeq (U \cap G_1) \star_{(U\cap C)} (U \cap G_2)$. The hypothesis implies also that $|C : C \cap U| = p$, i.e., $C \neq 1$. Hence without loss of generality we may assume that G_1 is non-abelian, and, by induction, that $\mathrm{rk}_{\mathbb{Q}}(U \cap G_1) \geq 1 + \mathrm{rk}_{\mathbb{Q}}(G_1)$. If G_2 is also non-abelian, then we have, by induction, $\mathrm{rk}_{\mathbb{Q}}(U \cap G_2) \geq \mathrm{rk}_{\mathbb{Q}}(G_2) + 1$. Hence, by applying Lemma 3.1 (a), one concludes that

$$\begin{aligned}\mathrm{rk}_{\mathbb{Q}}(U) &\geq \mathrm{rk}_{\mathbb{Q}}(U \cap G_1) + \mathrm{rk}_{\mathbb{Q}}(U \cap G_2) - 1\\ &> \mathrm{rk}_{\mathbb{Q}}(G_1) + \mathrm{rk}_{\mathbb{Q}}(G_2) \geq \mathrm{rk}_{\mathbb{Q}}(G).\end{aligned}$$

On the other hand, if G_2 is abelian, C is a direct factor in G_2, i.e., $G_2 \simeq \mathbb{Z} \times B$, where B is a free abelian group of rank $d(G_2) - 1$. Thus the map α in (3.3) is injective, and $\mathrm{rk}_{\mathbb{Q}}(G) = \mathrm{rk}_{\mathbb{Q}}(G_1) + \mathrm{rk}_{\mathbb{Q}}(G_2) - 1$. Moreover, $\mathrm{rk}_{\mathbb{Q}}(U \cap G_2) = \mathrm{rk}_{\mathbb{Q}}(G_2)$ and Lemma 3.1 (a) yields

$$\begin{aligned}\mathrm{rk}_{\mathbb{Q}}(U) &\geq \mathrm{rk}_{\mathbb{Q}}((U \cap G_1) + \mathrm{rk}_{\mathbb{Q}}(U \cap G_2) - 1\\ &\geq \mathrm{rk}_{\mathbb{Q}}(G_1) + \mathrm{rk}_{\mathbb{Q}}(G_2) > \mathrm{rk}_{\mathbb{Q}}(G).\end{aligned}$$

Case (I.2). In this case U has p orbits on E_1 and p orbits on E_2. Moreover, $|G : UG_i| \in \{1, p\}$. Hence, by (3.4), it suffices to consider the following three cases:

(a) $|G : UG_1| = |G : UG_2| = p$.

(b) $|G : UG_1| = p$ and $G = UG_2$.

(c) $G = UG_1 = UG_2$.

Let $\Lambda = G\backslash\backslash T$ and $\widetilde{\Lambda} = U\backslash\backslash T$ be the quotient graphs of T modulo the left G- and U-action, respectively. In particular, $\Lambda = (\{v, w\}, \{\mathbf{e}, \bar{\mathbf{e}}\})$ is a line segment of length 1 and thus a tree, and $\widetilde{\Lambda}$ is connected. Moreover, one has a surjective homomorphism of graphs $\pi: \widetilde{\Lambda} \to \Lambda$.

Case (a). By hypothesis, $UG_1 = UG_2 = U$, i.e., G/U acts regularly on the vertex fibres and edge fibres of π. For $\mathbf{f} \in E(\widetilde{\Lambda})$ one has either $o(\mathbf{f}) \in \pi^{-1}(\{v\})$ or $t(\mathbf{f}) \in \pi^{-1}(\{v\})$. If $\mathbf{f}_1$ and $\mathbf{f}_2$ satisfy the first condition, and $\tilde{v} = o(\mathbf{f}_1) = o(\mathbf{f}_2)$, then $\tilde{g}.\mathbf{f}_1 = \mathbf{f}_2$ for $\tilde{g} \in \mathrm{stab}_{G/U}(\tilde{v}) = 1$. Hence $\mathbf{f}_1 = \mathbf{f}_2$. In the latter case the same argument applies, and this shows that π is a fibration, i.e., for every $z \in V(\widetilde{\Lambda})$ the map $\pi_z : \mathfrak{st}_{\widetilde{\Lambda}}(z) \to \mathfrak{st}_{\Lambda}(\pi(z))$ is a bijection. As Λ is a tree and hence simply-connected, this implies that π is a bijection, a contradiction, showing that Case (a) is impossible.

Case (b). We may assume that $G_1 = \mathrm{stab}_G(v)$ and $G_2 = \mathrm{stab}_G(w)$. Hence, by hypothesis, $|\pi^{-1}(\{v\})| = p$ and $|\pi^{-1}(\{w\})| = 1$. Let $\tilde{w} \in V(\widetilde{\Lambda})$ with $\pi(\tilde{w}) = w$. Then $E(\widetilde{\Lambda}) = \mathfrak{st}_{\widetilde{\Lambda}}(\tilde{w}) \sqcup \overline{\mathfrak{st}_{\widetilde{\Lambda}}(\tilde{w})}$ is a star with p geometric edges.

Put $U_2 = U \cap G_2$. By hypothesis, $|G_2 : U_2| = p$ and $G_1 \subseteq U$. Choosing a set of representatives $\mathcal{R} \subseteq G_2$ for G_2/U_2, the Mayer–Vietoris sequence associated to $\mathrm{Tor}^U_\bullet(-, \mathbb{Q})$ specializes to

$$\coprod_{r\in\mathcal{R}} H_1({}^rC, \mathbb{Q}) \xrightarrow{\alpha} \coprod_{r\in\mathcal{R}} H_1({}^rG_1, \mathbb{Q}) \oplus H_1(U_2, \mathbb{Q}) \to H_1(U, \mathbb{Q}) \to 0. \tag{3.5}$$

This yields

$$\mathrm{rk}_{\mathbb{Q}}(U) = p \cdot \mathrm{rk}_{\mathbb{Q}}(G_1) + \mathrm{rk}(U_2) - \delta, \tag{3.6}$$

where $\delta = \dim(\mathrm{im}(\alpha))$. We distinguish two cases.

(1) $C = 1$. So $\delta = 0$. Then, by (3.5),

$$\mathrm{rk}_{\mathbb{Q}}(U) = p \cdot \mathrm{rk}_{\mathbb{Q}}(G_1) + \mathrm{rk}_{\mathbb{Q}}(U_2).$$

As $\mathrm{rk}_{\mathbb{Q}}(U_2) \geq \mathrm{rk}_{\mathbb{Q}}(G_2)$ with equality in case that G_2 is abelian, one concludes that $\mathrm{rk}_{\mathbb{Q}}(U) = p \cdot \mathrm{rk}_{\mathbb{Q}}(G_1) + \mathrm{rk}_{\mathbb{Q}}(U_2) > \mathrm{rk}_{\mathbb{Q}}(G_1) + \mathrm{rk}_{\mathbb{Q}}(G_2) \geq \mathrm{rk}_{\mathbb{Q}}(G)$.

(2) $C \neq 1$. Then, by (3.6),

$$\begin{aligned}\mathrm{rk}_{\mathbb{Q}}(U) &\geq p \cdot (\mathrm{rk}_{\mathbb{Q}}(G_1) - 1) + \mathrm{rk}_{\mathbb{Q}}(U_2) \\ &\geq 2 \cdot (\mathrm{rk}_{\mathbb{Q}}(G_1) - 1) + \mathrm{rk}_{\mathbb{Q}}(U_2).\end{aligned} \tag{3.7}$$

As $d(G_1) = 1$ implies that $G = \mathbb{Z} \star_{\mathbb{Z}} G_2 \simeq G_2$, which was excluded by (I), we may assume that $d(G_1) \geq 2$. Hence

$$\mathrm{rk}_{\mathbb{Q}}(U) \geq \mathrm{rk}_{\mathbb{Q}}(G_1) + \mathrm{rk}_{\mathbb{Q}}(U_2). \tag{3.8}$$

If G_2 is non-abelian, then, by induction, $\mathrm{rk}_{\mathbb{Q}}(U_2) \geq 1+\mathrm{rk}_{\mathbb{Q}}(G_2)$. Hence we obtain the inequality $\mathrm{rk}_{\mathbb{Q}}(U) \geq 1 + \mathrm{rk}_{\mathbb{Q}}(G_1) + \mathrm{rk}_{\mathbb{Q}}(G_2) > \mathrm{rk}_{\mathbb{Q}}(G)$. If G_2 is abelian, then $G = G_1 \star_{\mathbb{Z}} (\mathbb{Z} \times B)$ for some free abelian group B of rank $d(G_2) - 1$. In particular, $\mathrm{rk}_{\mathbb{Q}}(G) = \mathrm{rk}_{\mathbb{Q}}(G_1) + d(B)$. As $\mathrm{rk}_{\mathbb{Q}}(U_2) = d(B) + 1$, estimate (3.8) yields the claim also in this case.

Case (c). By hypothesis, $\widetilde{\Lambda}$ is a graph with two vertices v and w and p geometric edges. We may assume that $G_1 = \mathrm{stab}_G(v)$ and $G_2 = \mathrm{stab}_G(w)$, and we put $U_1 = U \cap G_1$ and $U_2 = U \cap G_2$. By the same argument used in Case (b), one obtains an exact sequence

$$\begin{array}{ccccc} \coprod_{r\in\mathcal{R}} H_1({}^rC,\mathbb{Q}) & \xrightarrow{\beta} & H_1(U_1,\mathbb{Q}) \oplus H_1(U_2,\mathbb{Q}) & \longrightarrow & H_1(U,\mathbb{Q}) \\ & & & & \big\downarrow \\ & & 0 & \longleftarrow & \mathbb{Q}^{p-1}, \end{array} \tag{3.9}$$

where $\mathcal{R} \subset G$ is a set of representatives of G/U. Again we distinguish two cases.

(1) $C = 1$. Then β is the 0-map, and as $\mathrm{rk}_{\mathbb{Q}}(U_i) \geq \mathrm{rk}_{\mathbb{Q}}(G_i)$ for $i \in \{1,2\}$, one has

$$\begin{aligned} \mathrm{rk}_{\mathbb{Q}}(U) &= \mathrm{rk}_{\mathbb{Q}}(U_1) + \mathrm{rk}_{\mathbb{Q}}(U_2) + (p-1) \\ &\geq \mathrm{rk}_{\mathbb{Q}}(G_1) + \mathrm{rk}_{\mathbb{Q}}(G_2) + 1 > \mathrm{rk}_{\mathbb{Q}}(G). \end{aligned}$$

(2) $C \neq 1$. Then, by (3.9),

$$\mathrm{rk}_{\mathbb{Q}}(U) \geq \mathrm{rk}_{\mathbb{Q}}(U \cap G_1) + \mathrm{rk}_{\mathbb{Q}}(U \cap G_2) - 1. \tag{3.10}$$

If G_1 and G_2 are non-abelian, then, by induction, $\mathrm{rk}_{\mathbb{Q}}(U \cap G_1) \geq 1 + \mathrm{rk}_{\mathbb{Q}}(G_1)$ and $\mathrm{rk}_{\mathbb{Q}}(U \cap G_2) > \mathrm{rk}_{\mathbb{Q}}(G_2)$. Hence

$$\begin{aligned} \mathrm{rk}_{\mathbb{Q}}(U) &> \mathrm{rk}_{\mathbb{Q}}(G_1) + 1 + rk_{\mathbb{Q}}(G_2) - 1 \\ &= \mathrm{rk}_{\mathbb{Q}}(G_1) + \mathrm{rk}_{\mathbb{Q}}(G_2) \geq \mathrm{rk}_{\mathbb{Q}}(G). \end{aligned}$$

In case that one of the groups G_1, G_2 is abelian, not both of them can be abelian. Otherwise, the Euler characteristic $\chi(G) = \chi(G_1) + \chi(G_2) - \chi(C)$ must equal 0, and G must be abelian, which was excluded by hypothesis (see Proposition 2.3). So without loss of generality we may assume that G_1 is a non-abelian, and that G_2 is abelian. Then $G_2 \simeq \mathbb{Z} \times B$, where B is a free abelian group of rank $\mathrm{rk}_{\mathbb{Q}}(G_2)-1$, and $G \simeq G_1 \star_{\mathbb{Z}} (\mathbb{Z} \times B)$. Hence $\mathrm{rk}_{\mathbb{Q}}(G) = \mathrm{rk}_{\mathbb{Q}}(G_1)+\mathrm{rk}_{\mathbb{Q}}(G_2)-1$. Furthermore, $\mathrm{rk}_{\mathbb{Q}}(U \cap G_2) = \mathrm{rk}_{\mathbb{Q}}(G_2)$, and, by induction, $\mathrm{rk}_{\mathbb{Q}}(U \cap G_1) \geq 1+\mathrm{rk}_{\mathbb{Q}}(G_1)$. Thus, by (3.10),

$$\mathrm{rk}_{\mathbb{Q}}(U) \geq \mathrm{rk}_{\mathbb{Q}}(G_1) + 1 + \mathrm{rk}_{\mathbb{Q}}(G_2) - 1 = \mathrm{rk}_{\mathbb{Q}}(G_1) + \mathrm{rk}_{\mathbb{Q}}(G_2) > \mathrm{rk}_{\mathbb{Q}}(G).$$

Case (II). Suppose that $G = \mathrm{HNN}_\phi(G_1, C, t) = \langle G_1, t \mid tct^{-1} = \phi(c)\rangle$ is an HNN-extension with $C = \langle c \rangle$. By (3.1), one has

$$\mathrm{rk}_{\mathbb{Q}}(G_1) \leq \mathrm{rk}_{\mathbb{Q}}(G) \leq \mathrm{rk}_{\mathbb{Q}}(G_1) + 1. \tag{3.11}$$

If $C = 1$, then $G = G_1 \star \langle t \rangle$ is isomorphic to a free product. Hence the claim follows already from Case (I). So we may assume that $C \neq 1$. Note that G_1 must be non-abelian. Otherwise, one has $\chi(G) = \chi(G_1) - \chi(C) = 0$, and G must be abelian (see Proposition 2.3), a contradiction.

As in Case (I), the group G has a natural vertex transitive action on a tree T with vertex stabilizer isomorphic to G_1, and edge stabilizer isomorphic to C. In particular,

$$|G : UG_1| \in \{1, p\} \quad \text{and} \quad |G : UC| \in \{1, p\}.$$

Hence one may distinguish the following two cases:

(II.1) $G = UC$.

(II.2) $|G : UC| = p$.

Case (II.1). By hypothesis, $G = UG_1$. In particular, U is acting vertex transitively on T, and has two orbits on the set of edges, i.e., $U \backslash\backslash T$ is a loop with one vertex. Thus one has $U \simeq \mathrm{HNN}_\phi(U \cap G_1, C^p, t) = \langle U \cap G_1, t \mid t(c^p)t^{-1} = \phi(c)^p\rangle$, and as in (3.11), one concludes that

$$\mathrm{rk}_{\mathbb{Q}}(U \cap G_1) \leq \mathrm{rk}_{\mathbb{Q}}(U) \leq \mathrm{rk}_{\mathbb{Q}}(U \cap G_1) + 1. \tag{3.12}$$

Since G_1 is non-abelian and $|G_1 : U \cap G_1| = p$, it follows with induction that $\mathrm{rk}_{\mathbb{Q}}(G_1 \cap U) \geq \mathrm{rk}_{\mathbb{Q}}(G_1) + 1$. Hence

$$\mathrm{rk}_{\mathbb{Q}}(U) \geq \mathrm{rk}_{\mathbb{Q}}(G_1 \cap U) \geq \mathrm{rk}_{\mathbb{Q}}(G_1) + 1 \geq \mathrm{rk}_{\mathbb{Q}}(G). \tag{3.13}$$

Suppose that $\mathrm{rk}_{\mathbb{Q}}(U) = \mathrm{rk}_{\mathbb{Q}}(G)$. Then it follows that $\mathrm{rk}_{\mathbb{Q}}(U) = \mathrm{rk}_{\mathbb{Q}}(G_1 \cap U) = \mathrm{rk}_{\mathbb{Q}}(G_1) + 1 = \mathrm{rk}_{\mathbb{Q}}(G)$. In particular, by Lemma 3.1(b), $\rho(U) = 0$ and $\rho(G) = 1$, i.e., α is the 0-map, and α_1 is injective. However, as the map tr_1 in the diagram

$$\begin{array}{ccccccccc}
\cdots \longrightarrow & H_1(C, \mathbb{Q}) & \xrightarrow{\ \alpha\ } & H_1(G_1, Q) & \longrightarrow & H_1(G, \mathbb{Q}) & \longrightarrow \cdots \\
& \downarrow{\scriptstyle \mathrm{tr}_1} & & \downarrow{\scriptstyle \mathrm{tr}_2} & & \downarrow{\scriptstyle \mathrm{tr}_3} & \\
\cdots \longrightarrow & H_1(C^p, \mathbb{Q}) & \xrightarrow{\ \alpha_1\ } & H_1(U \cap G_1, Q) & \longrightarrow & H_1(U, \mathbb{Q}) & \longrightarrow \cdots
\end{array}$$

– which is given by the transfer – is an isomorphism, this yields a contradiction. Thus $\mathrm{rk}_{\mathbb{Q}}(U) < \mathrm{rk}_{\mathbb{Q}}(G)$.

Case (II.2). In this case U has $2p$ orbits on $E(T)$, and again one may distinguish two cases:

(a) U acts vertex transitively on T.

(b) U has p orbits on $V(T)$.

Case (a). In this case $U\backslash\backslash T$ is a bouquet of p loops, $|G_1 \colon U \cap G_1| = p$ and $C \subseteq U$. The Mayer–Vietoris sequence for $H_\bullet(-,\mathbb{Q})$ yields a commutative and exact diagram

$$\begin{array}{ccccccccc}
C \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow{\alpha} & G_1^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & G^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & \mathbb{Q} & \longrightarrow & 0 \\
{\scriptstyle \mathrm{tr}_1}\downarrow & & {\scriptstyle \mathrm{tr}_2}\downarrow & & \downarrow{\scriptstyle \mathrm{tr}_3} & & \downarrow & & \\
\coprod_{i=1}^{p} {}^{g_i}C \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow{\alpha_1} & (U \cap G_1)^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & U^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & \mathbb{Q}^p & \longrightarrow & 0,
\end{array}$$

where $\{g_1, \ldots, g_p\} \subseteq G_1$ is a set of representatives for G/U, $\mathrm{tr}_{2/3}$ is induced by the transfer, and for $c \in C$, $q \in \mathbb{Q}$, one has $\mathrm{tr}_1(c \otimes q) = \sum_{i=1}^{p} {}^{g_i}c \otimes q$. Moreover,

$$\alpha_1(({}^{g_i}c) \otimes q) = \big(g_i c \phi(c)^{-1} g_i^{-1} (U \cap G_1)'\big) \otimes q \quad \text{for } c \in C.$$

Thus, one has

$$\mathrm{rk}_{\mathbb{Q}}(U) + \dim_{\mathbb{Q}}(\mathrm{im}(\alpha_1)) = \mathrm{rk}_{\mathbb{Q}}(U \cap G_1) + p, \tag{3.14}$$

and induction implies that $\mathrm{rk}_{\mathbb{Q}}(U \cap G_1) \geq \mathrm{rk}_{\mathbb{Q}}(G_1) + 1$. Hence

$$\mathrm{rk}_{\mathbb{Q}}(U) \geq \mathrm{rk}_{\mathbb{Q}}(U \cap G_1) \geq \mathrm{rk}_{\mathbb{Q}}(G_1) + 1 \geq \mathrm{rk}_{\mathbb{Q}}(G). \tag{3.15}$$

Suppose that $\mathrm{rk}_{\mathbb{Q}}(U) = \mathrm{rk}_{\mathbb{Q}}(G)$. Then equality holds throughout (3.15). In particular, (3.14) implies that $\dim_{\mathbb{Q}}(\mathrm{im}(\alpha_1)) = p$, i.e., α_1 is injective. Hence, as tr_1 is injective, $\alpha_1 \circ \mathrm{tr}_1$ is injective. From Lemma 3.1 (b) one concludes that $\rho(G) = 1$ and $\alpha = 0$, a contraction. Thus $\mathrm{rk}_{\mathbb{Q}}(U) > \mathrm{rk}_{\mathbb{Q}}(G)$ must hold.

Case (b). In this case one has $G_1 \subseteq U$ and $C \subseteq U$. As G/U is acting vertex transitively on $\Lambda = U\backslash\backslash T$, Λ must be k-regular. Hence $|E(\Lambda)| = k \cdot |V(\Lambda)|$, forcing $k = 2$. So Λ is a 2-regular connected graph, and thus a circuit with p vertices.

Let $\{g_1, \ldots, g_p\} \subseteq G$ be a set of representatives for G/U. The Mayer–Vietoris sequence for $H_\bullet(-,\mathbb{Q})$ then yields a commutative and exact diagram

$$\begin{array}{ccccccccc}
C \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow{\alpha} & G_1^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & G^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & \mathbb{Q} & \longrightarrow & 0 \\
{\scriptstyle \mathrm{tr}_1}\downarrow & & {\scriptstyle \mathrm{tr}_2}\downarrow & & \downarrow{\scriptstyle \mathrm{tr}_3} & & \downarrow & & \\
\coprod_{i=1}^{p} {}^{g_i}C \otimes_{\mathbb{Z}} \mathbb{Q} & \xrightarrow{\alpha_1} & \coprod_{i=1}^{p} {}^{g_i}G_1^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & U^{\mathrm{ab}} \otimes_{\mathbb{Z}} \mathbb{Q} & \longrightarrow & \mathbb{Q} & \longrightarrow & 0,
\end{array}$$

where $\mathrm{tr}_{1/2}$ are the diagonal maps and tr_3 is the transfer. Hence

$$\mathrm{rk}_{\mathbb{Q}}(U) = 1 + p \cdot (\mathrm{rk}_{\mathbb{Q}}(G_1) - 1) + \dim_{\mathbb{Q}}(\ker(\alpha_1)). \tag{3.16}$$

Therefore, as $(p-1)\cdot(\mathrm{rk}_{\mathbb{Q}}(G_1) - 1) \geq 1$,

$$\mathrm{rk}_{\mathbb{Q}}(U) \geq \mathrm{rk}_{\mathbb{Q}}(G_1) + 1 \geq \mathrm{rk}_{\mathbb{Q}}(G). \tag{3.17}$$

Suppose that $\mathrm{rk}_{\mathbb{Q}}(U) = \mathrm{rk}_{\mathbb{Q}}(G)$. Then equality has to hold throughout (3.17), and $\dim_{\mathbb{Q}}(\ker(\alpha_1)) = 0$, $p = 2$, $\mathrm{rk}_{\mathbb{Q}}(G_1) = 2$. In particular, α_1 is injective, and by Lemma 3.1 (b), $\rho(G) = 1$, i.e., $\alpha = 0$. Hence, as tr_1 is injective, one obtains a contradiction as in Case (a) showing that $\mathrm{rk}_{\mathbb{Q}}(U) > \mathrm{rk}_{\mathbb{Q}}(G)$. □

Theorem 3.2 has the following consequence.

Corollary 3.3. *Let G be a non-abelian limit group such that G^{ab} is torsion free and that $d(G) = d(G^{\mathrm{ab}})$. Then, if U is a normal subgroup of G of prime index p, one has that $d(U) > d(G)$.*

Proof. By Theorem 3.2, $d(U) \geq \mathrm{rk}_{\mathbb{Q}}(U) > \mathrm{rk}_{\mathbb{Q}}(G) = d(G^{\mathrm{ab}}) = d(G)$. □

Obviously, if G is a free group, an abelian free group or an iterated extension of centralizers group of a free group, then $\mathrm{rk}_{\mathbb{Q}}(G) = d(G)$. In particular, in connection with Theorem 2.4 one concludes the following.

Corollary 3.4. *Let G be a non-abelian limit group with $d(G) \leq 3$ and U a normal subgroup of G of prime index p. Then $d(U) \geq \mathrm{rk}_{\mathbb{Q}}(U) > \mathrm{rk}_{\mathbb{Q}}(G) = d(G)$.*

Another consequence of Theorem 3.2 is the following.

Corollary 3.5. *Let G be a non-abelian limit group, and let N be a normal subgroup of G such that G/N is infinite and nilpotent. Then $\mathrm{rk}_{\mathbb{Q}}(N) = \infty$. In particular, $d(N) = \infty$.*

Proof. Suppose that $\mathrm{rk}_{\mathbb{Q}}(N) = n < \infty$. Let $h = h(G/N)$ be the Hirsch number of the finitely generated nilpotent group G/N. Then $h(U/N) = h(G/N)$ for every subgroup of finite index U in G satisfying $N \subset U$. In particular, one has $\mathrm{rk}_{\mathbb{Q}}(U/N) \leq h(U/N) = h$, and thus $\mathrm{rk}_{\mathbb{Q}}(U) \leq \mathrm{rk}_{\mathbb{Q}}(N) + \mathrm{rk}_{\mathbb{Q}}(U/N) \leq n + h$, contradicting Theorem 3.2. □

Remark 3.6. Obviously, the statement of Corollary 3.5 remains valid replacing "nilpotent" by "minimax".

4 One-relator limit groups

For a limit group G it is not necessarily true that $d(G) = d(G^{\mathrm{ab}})$. The following lemma shows that there exists a class of limit groups containing groups G satisfying $d(G) \neq d(G^{\mathrm{ab}})$ (see Remark 4.4) for which Question* has an affirmative answer.

Lemma 4.1. *Let $G = G_1 \star_C G_2$ be a non-abelian limit group, where G_1 and G_2 are free groups of finite rank $r(G_1)$ and $r(G_2)$, respectively, and let $C = \langle c \rangle$ be an infinite cyclic group or trivial. Then, if U is a normal subgroup of prime index p in G, one has $\mathrm{rk}_{\mathbb{Q}}(U) > d(G)$. In particular, $d(U) > d(G)$.*

Proof. If $C = 1$, G is a finitely generated free group, and there is nothing to prove. So from now on we may assume that $C \neq 1$. Then, as G is non-abelian, either G_1 or G_2 must be non-abelian. Otherwise, $\chi(G) = \chi(G_1) + \chi(G_2) - \chi(C) = 0$, and by Lemma 2.3, G must be abelian. Let T be the tree on which G acts naturally. Then, as in the proof of Theorem 3.2, Case (I), one may distinguish three cases:

(1) $G = UC$, i.e., U has one orbits on E_1 and E_2.

(2) $|G : UC| = p$, i.e., U has p orbits on E_1 and E_2,

  (b) $|G : UG_1| = p$ and $G = UG_2$,

  (c) $G = UG_1 = UG_2$.

Case (1). By hypothesis, $G = UG_1$ and $G = UG_2$ and therefore,

$$U \simeq (U \cap G_1) \star_{U \cap C} (U \cap G_2).$$

In analogy to Case (I.1) of the proof of Theorem 3.2, we may choose G_1 to be non-abelian. If G_2 is abelian, then G_2 must be cyclic, i.e., $d(G_2) = 1$. If $d(G_1) = 2$, then $d(G) \leq 3$, and the claim follows from Corollary 3.4. Thus, we may assume that $d(G_1) \geq 3$. Again, by the Nielsen–Schreier theorem (and as $d(G_1) \geq 3$), one has that $d(U \cap G_1) > 1 + d(G_1)$. As G_2 is abelian, $d(U \cap G_2) = d(G_2)$. Therefore, by Lemma 3.1 (a), one has

$$\begin{aligned} \mathrm{rk}_{\mathbb{Q}}(U) &\geq d(U \cap G_1) + d(U \cap G_2) - 1 \\ &> d(G_1) + d(G_2) \geq d(G). \end{aligned}$$

Thus, we may assume that G_2 is non-abelian. From the Nielsen–Schreier theorem one concludes that $d(U \cap G_1) \geq 1 + d(G_1)$ and $d(U \cap G_2) \geq 1 + d(G_2)$. Therefore, by Lemma 3.1 (a),

$$\begin{aligned} \mathrm{rk}_{\mathbb{Q}}(U) &\geq d(U \cap G_1) + d(U \cap G_2) - 1 \\ &\geq d(G_1) + d(G_2) + 1 > d(G). \end{aligned} \tag{4.1}$$

Case (2, b). By (3.7), one has that

$$\mathrm{rk}_{\mathbb{Q}}(U) \geq p \cdot (d(G_1) - 1) + d(G_2 \cap U). \tag{4.2}$$

If G_1 is abelian, then $d(G_1) = 1$. In this case G_2 is non-abelian. If $d(G_2) = 2$, then $d(G) \leq 3$ and the claim follows from Corollary 3.4. So we may assume that $d(G_2) \geq 3$. In particular, the Nielsen–Schreier theorem and (4.2) imply that $\mathrm{rk}_{\mathbb{Q}}(U) \geq d(G_2 \cap U) > 1 + d(G_2) \geq d(G)$.

So from now on we may assume that G_1 is non-abelian, i.e., $d(G_1) \geq 2$. If G_2 is abelian, then $d(U \cap G_2) = d(G_2) = 1$. So if $d(G_1) = 2$, then the claim follows by Corollary 3.4. Thus we may assume that $d(G_1) \geq 3$. Then, by (4.2) and the Nielsen–Schreier theorem, one concludes that

$$\begin{aligned} \mathrm{rk}_{\mathbb{Q}}(U) &\geq p \cdot (d(G_1) - 1) + 1 \\ &\geq d(G_1) + 2 \cdot (p - 1) \\ &> d(G_1) + 1 \geq d(G). \end{aligned}$$

So we may assume that G_2 is non-abelian. In this case the Nielsen–Schreier theorem implies that $d(U \cap G_2) \geq 1 + d(G_2)$. Then, by (4.2), one has

$$\begin{aligned} \mathrm{rk}_{\mathbb{Q}}(U) &\geq p \cdot d(G_1) - p + d(G_2) + 1 \\ &\geq d(G_1) + d(G_2) + (p - 1) \\ &\geq d(G_1) + d(G_2) + 1 > d(G). \end{aligned}$$

Case (2, c). By (3.10), one has

$$\mathrm{rk}_{\mathbb{Q}}(U) \geq d(U \cap G_1) + d(U \cap G_2) - 1.$$

Then, the proof of Case (1) can be transferred verbatim in order to show that the claim holds if one of the groups G_i, $i \in \{1, 2\}$, is abelian. So we may assume that G_1 and G_2 are non-abelian. Then the Nielsen–Schreier theorem and the same argument which was used in order to prove (4.1) implies that $\mathrm{rk}_{\mathbb{Q}}(U) > d(G)$. □

For HNN-extensions one has the following.

Lemma 4.2. *Let $G = \mathrm{HNN}_\phi(G_1, C, t) = \langle G_1, t \mid tct^{-1} = \phi(c)\rangle$ be a non-abelian limit group, where G_1 is a free group of finite rank r and C is an infinite cyclic group or trivial. Let U be a normal subgroup of G of prime index p. Then one has $\mathrm{rk}_{\mathbb{Q}}(U) > d(G)$, and, in particular, $d(U) > d(G)$.*

Proof. If C is trivial, then $G = G_1 \star \langle t\rangle$ is a free group of rank $r + 1$, and there is nothing to prove. Moreover, if $r = 1$, then $\chi(G) = 0$, and G must be abelian (see Proposition 2.3), which was excluded by hypothesis. Hence $r \geq 2$. Let T be the tree on which G acts naturally. Then one may distinguish three cases:

(1) $G = UC$.

(2) $|G : UC| = p$,

  (a) U acts vertex transitively on T,

  (b) U has p orbits on $V(T)$.

Case (1). In this case one has $U \simeq \mathrm{HNN}_\phi(U \cap G_1, C^p, t)$ (cf. the proof of Theorem 3.2, Case (II.1)). If $r = 2$, then $d(G) \leq 3$, and the claim follows from Corollary 3.4. Hence we may assume that $r \geq 3$. In this case the Nielsen–Schreier theorem implies that $d(G_1 \cap U) > r + 1$, and by (3.12), one concludes that

$$\mathrm{rk}_{\mathbb{Q}}(U) \geq d(G_1 \cap U) > r + 1 \geq d(G).$$

Case (2, a). As in Case (1), we may assume that $r \geq 3$. Since $|G_1 : G_1 \cap U| = p$ and G_1 is a non-abelian free group, the Nielsen–Schreier theorem implies that $d(U \cap G_1) > r + 1$. Hence, by (3.13),

$$\mathrm{rk}_{\mathbb{Q}}(U) \geq d(U \cap G_1) > r + 1 \geq d(G).$$

Case (2, b). As in Case (1), we may assume that $r \geq 3$. By (3.16) and the fact that $(p - 1) \cdot (d(G_1) - 1) > 1$, one concludes that

$$\mathrm{rk}_{\mathbb{Q}}(U) \geq 1 + p \cdot (d(G_1) - 1) > d(G_1) + 1 \geq d(G)$$

completing the proof of the lemma. □

As a consequence one concludes the following.

Theorem 4.3. *Let G be a non-abelian cyclically pinched or conjugacy pinched one-relator limit group* (*see* [7])*, and let U be a normal subgroup of G of prime index p. Then $d(U) > d(G)$.*

Proof. If G is a cyclically pinched one-relator group, then $G \simeq G_1 \star_C G_2$ with G_1 and G_2 free groups, and C being an infinite cyclic group being generated by a cyclically reduced word $w \in G_1$ and C is a maximal cyclic subgroup in either G_1 or G_2. Hence in this case, Lemma 4.1 yields the claim. If G is a conjugacy pinched one-relator group, then $G \simeq \mathrm{HNN}_\phi(G_1, C, t)$ with G_1 a free group, C and infinite cyclic subgroup being generated by a cyclically reduced word $w \in G_1$, and either C or $\phi(C)$ is a maximal cyclic subgroup of G_1. Then, by Lemma 4.2, one has that $d(U) > d(G)$ completing the proof. □

Remark 4.4. It should be mentioned that there exist limit groups G satisfying $d(G^{\mathrm{ab}}) \lneq d(G)$. In a post on mathoverflow the following example was given by

H. Wilton in [21]. Let $G_1 = G_2 = F_2$ be the free group of rank two, and let $C_1 = C_2 = \langle w \rangle$, where $w = a^2ba^{-1}b^{-1}$. Then w is cyclically reduced, and C_i is a maximal cyclic subgroup of G_i for $i = 1, 2$, but C_i is not a free factor. By a result of B. Baumslag (see [8, Corollary 3.6]), the corresponding double $G = G_1 \star_{C_1=C_2} G_2$ is a non-abelian limit group and has abelianization G^{ab} isomorphic to $\mathbb{Z}^3$, i.e., $d(G^{\mathrm{ab}}) = 3$. The canonical projection $\beta\colon G \to (F_2/\langle w^{F_2} \rangle) \star (F_2/\langle w^{F_2} \rangle)$ is a surjective group homomorphism. Thus, by hypothesis and Grushko's theorem, one has $d(F_2/\langle w^{F_2} \rangle \star F_2/\langle w^{F_2} \rangle) = 4$. Hence $d(G) = 4$. Note that by Lemma 4.1, $\mathrm{rk}_{\mathbb{Q}}(U) > 4$ for any normal subgroup U of G of prime index p.

Remark 4.5. The group $G = \langle a, b \rangle_{a^2b^2=c^2d^2} \langle c, d \rangle$ is the fundamental group of a closed non-orientable surface, and it is known that it is a limit group (see [8, Section 3.1]). However, $G^{\mathrm{ab}} \simeq \mathbb{Z}^3 \times C_2$, where $C_2 = \mathbb{Z}/2Z$.

Acknowledgments. The authors would like to thank H. Wilton for a very useful comment concerning an earlier version of the paper, and also the referee for his/her helpful remarks.

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Received July 25, 2016; revised May 25, 2017.

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