Constrained BSDEs driven by a non quasi-left-continuous random measure and optimal control of PDMPs on bounded domains

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Abstract

We consider an optimal control problem for piecewise deterministic Markov processes (PDMPs) on a bounded state space. A pair of controls acts continuously on the deterministic flow and on the two transition measures (in the interior and from the boundary of the domain) describing the jump dynamics of the process. For this class of control problems, the value function can be characterized as the unique viscosity solution to the corresponding fully-nonlinear Hamilton-Jacobi-Bellman equation with a non-local type boundary condition.

By means of the recent control randomization method, we are able to provide a probabilistic representation for the value function in terms of a constrained backward stochastic differential equation (BSDE), known as nonlinear Feynman-Kac formula. This result considerably extends the existing literature, where only the case with no jumps from the boundary is considered. The additional boundary jump mechanism is described in terms of a non quasi-left-continuous random measure and induces predictable jumps in the PDMP's dynamics. The existence and uniqueness results for BSDEs driven by such a random measure are non trivial, even in the unconstrained case, as emphasized in the recent work [2].

Keywords: Backward stochastic differential equations, optimal control problems, piecewise deterministic Markov processes, non quasi-left-continuous random measure, randomization of controls.

MSC 2010: 93E20, 60H10, 60J25.

1 Introduction

Piecewise deterministic Markov processes evolve by means of random jumps at random times, while the behavior between jumps is described by a deterministic flow. We consider infinite-horizon optimal control problems of PDMPs where the control acts continuously on the jump dynamics as well as on the deterministic flow. We deal with PDMPs with bounded state space: whenever the process hits the boundary, it immediately jumps into the interior of the domain. Control problems for this type of processes arise in many contexts, among which queuing and inventory systems, maintenance-replacement models, and many other areas of engineering and operations research. For instance, in [14] the authors solve a capacity expansion control problem by modeling the corresponding state process as a controlled PDMP with bounded state space, and analogous techniques are used in [16] to deal with an optimal consumption and exploration problem (see Example 2.1); we refer the interested reader to the books [12], [13], and the references therein, for a detailed overview on the possible applications of PDMPs models in optimization problems.

Our aim is to find a nonlinear Feynman-Kac representation formula for the value function, in terms of a suitable constrained BSDE. It is worth mentioning that the probability measures describing the distribution of the controlled PDMP are in general not absolutely continuous with

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respect to the law of a given, uncontrolled process (roughly speaking, the control problem is non-dominated). This is reflected in the fully nonlinear character of the associated HJB equation, and prevents the use of standard BSDE techniques. For this reason, we shall extend to the present framework the so-called *randomization method*, recently introduced by [23] in the diffusive context, to represent the solutions of fully nonlinear integro-partial differential equations by means of a new class of BSDEs with nonpositive jumps, and later developed for other types of control problems, see for instance [24], [17], [11], [6]. In the non-diffusive framework, the correct formulation of the randomization method requires some efforts and different techniques from the diffusive case, since the controlled process is naturally described only in terms of its local characteristics and not as a solution to some stochastic differential equation. A first step in the generalization of the method to the non-diffusive framework was done in [4], where optimal control for pure jump Markov processes was considered; afterwards, the randomization techniques have been implemented in [3] to solve PDMPs optimal control problems on unbounded state spaces. In the present paper we are interested to extend those results to the case of optimal control problems for general PDMPs on bounded state spaces, where additional forced jumps appear whenever the process hits the boundary. The jump mechanism from the boundary plays a fundamental role as it leads, among other things, to the study of BSDEs driven by a non quasi-left-continuous random measure. For such general backward equations, the existence and uniqueness of a solution is particularly tricky, and counterexamples can be obtained even in simple cases, see [10]. Only recently, well-posedness results have been obtained on this subject, see [9], [8], and [2], even if limited to the case of unconstrained BSDEs, under a specific condition involving the Lipschitz constants of the BSDE generator and the size of the predictable jumps.

Let us describe our setting in more detail. Let E be an open bounded subset of $\mathbb{R}^d$, with Borel σ-algebra $\mathcal{E}$. Roughly speaking, a controlled PDMP on $(E, \mathcal{E})$ is described by specifying its local characteristics, namely a vector field $h(x, a_0)$, a jump rate $\lambda(x, a_0)$, and two transition probability measures $Q(x, a_0, dy)$ and $R(x, a_\Gamma, dy)$ prescribing the positions of the process at the jump times, respectively starting from the interior and from the boundary of the domain. The local characteristics depend on some initial value $x \in E$ and on the parameters $a_0 \in A_0$, $a_\Gamma \in A_\Gamma$, where $(A_0, \mathcal{A}_0)$ and $(A_\Gamma, \mathcal{A}_\Gamma)$ are two general measurable spaces, denoting respectively the space of control actions in the interior and on the boundary of the domain. The control procedure consists in choosing a pair of strategies: a *piecewise open-loop policy* controlling the motion in the interior of the domain, i.e. a measurable function only depending on the last jump time T_n and post jump position E_n, and a boundary control belonging to the set of *feedback policies*, that only depends on the position of the process just before the jump time. The above formulation of the control problem is used in many papers as well as books, see for instance [12], [13]. The class of admissible control laws $\mathcal{A}_{ad}$ will be the set of all $\mathcal{A}_0 \otimes \mathcal{A}_\Gamma$-measurable maps $\boldsymbol{\alpha} = (\alpha^0, \alpha^\Gamma)$, with $\alpha^\Gamma : \partial E \to A_\Gamma$, and $\alpha^0 : [0, \infty) \times E \to A_0$ such that $\alpha^0_t = \alpha^0_0(t, x)\, \mathbb{1}_{[0,\, T_1)}(t) + \sum_{n=1}^\infty \alpha^0_n(t - T_n, E_n)\, \mathbb{1}_{[T_n,\, T_{n+1})}(t)$. The controlled process X is defined as

$$X_t = \begin{cases} \phi^{\alpha^0}(t, x) & \text{if } t \in [0,\, T_1), \\ \phi^{\alpha^0}(t - T_n, E_n) & \text{if } t \in [T_n,\, T_{n+1}),\ n \in \mathbb{N} \setminus \{0\}, \end{cases}$$

where $\phi^{\alpha^0}(t, x) = \phi(t, x, \alpha^0_t)$ is the unique solution to the ordinary differential equation $\dot{x}(t) = h(x(t), \alpha^0(t))$, with $x(0) = x$. For every starting point $x \in E$ and for each $\boldsymbol{\alpha} \in \mathcal{A}_{ad}$, one can introduce the unique probability measure $\mathbb{P}^x_{\boldsymbol{\alpha}}$ such that the conditional survival function of the inter-jump times and the distribution of the post jump positions of X under $\mathbb{P}^x_{\boldsymbol{\alpha}}$ are given by (2.3)-(2.4)-(2.5). We denote by $\mathbb{E}^x_{\boldsymbol{\alpha}}$ the expectation under $\mathbb{P}^x_{\boldsymbol{\alpha}}$. In the classical infinite-horizon control problem the goal is to minimize over all control laws $\boldsymbol{\alpha}$ a functional cost of the form

$$J(x,\boldsymbol{\alpha}) = \mathbb{E}^x_{\boldsymbol{\alpha}}\Big[\int_{(0,\infty)} e^{-\delta\, s}\, f(X_s, \alpha^0_s)\, ds + \int_{(0,\infty)} e^{-\delta\, s}\, c(X_{s-}, \alpha^\Gamma(X_{s-}))\, dp^*_s\Big], \tag{1.1}$$

where f is a given real function on $\bar{E}\times A_0$ representing the running cost, c is a given real function on $\partial E \times A_\Gamma$ that provides a cost every time the process hits the boundary, $\delta \in (0,\,\infty)$ is a discount factor, while the process p^*_s counts the number of times the boundary is hit (see (2.2)). The value function of the control problem is defined in the usual way:

$$V(x) = \inf_{\boldsymbol{\alpha}\in\mathcal{A}_{ad}} J(x,\boldsymbol{\alpha}), \quad x \in E. \tag{1.2}$$

Under suitable assumptions on the cost functions f, c, and on the local characteristics h, λ, Q, R, V is known to be the unique continuous viscosity solution on $[0,\,\infty)\times\bar{E}$ of the Hamilton-Jacobi-Bellman (HJB) equation with boundary non-local condition:

$$\begin{cases} \delta v(x) = \inf_{a_0\in A_0}\big(h(x,a_0)\cdot\nabla v(x) + \lambda(x,a_0)\int_E (v(y)-v(x))\,Q(x,a_0,dy) + f(x,a_0)\big)\,,\; x\in E,\\ v(x) = \min_{a_\Gamma\in A_\Gamma}\big(\int_E (v(y)-v(x))\,R(x,a_\Gamma,dy) + c(x,a_\Gamma)\big),\; x \in \partial E. \end{cases} \tag{1.3}$$

We apply the randomization approach to this framework. The fundamental idea consists in the so-called *randomization of the control*: roughly speaking, we replace the state trajectory and the associated pair of controls $(X_s, \alpha^0_s, \alpha^\Gamma_s)$ by an (uncontrolled) PDMP (X_s, I_s, J_s). The process I (resp. J) is chosen to be a pure jump process with values in the space of control actions A_0 (resp. A_Γ), with an intensity $\lambda_0(db)$ (resp. $\lambda_\Gamma(dc)$), which is arbitrary but finite and with full support. In particular, the PDMP (X, I, J) is constructed on a new probability space by means of a different triplet of local characteristics and takes values on the enlarged space $E\times A_0\times A_\Gamma$ (or, equivalently, by assigning the compensator $\tilde{p}(ds\,dy\,db\,dc)$). For any starting point (x, a_0, a_Γ) in $E\times A_0\times A_\Gamma$, let $\mathbb{P}^{x,a_0,a_\Gamma}$ the corresponding law. Then we introduce an auxiliary optimal control problem where we control the intensity of the processes I and J: using a Girsanov's type theorem for point processes, for any pair of predictable, bounded and positive processes $(\nu^0_t(b), \nu^\Gamma_t(c))$, we construct a probability measure $\mathbb{P}^{x,a_0,a_\Gamma}_{\nu^0,\nu^\Gamma}$ under which the compensator of I (resp. J) is given by $\nu^0_t(b)\,\lambda_0(db)\,dt$ (resp. $\nu^\Gamma_t(c)\,\lambda_\Gamma(dc)\,dt$). It is worth mentioning that the applicability of the Girsanov theorem to the present framework, i.e. when the compensator $\tilde{p}$ is a non quasi-left-continuous random measure, requires the validity of an additional condition involving the intensity control fields (ν^0, ν^Γ) and the predictable jumps of $\tilde{p}$, see (3.11). The aim of the new control problem (called *randomized* or *dual* control problem) is to minimize the functional

$$J(x, a_0, a_\Gamma, \nu^0, \nu^\Gamma) = \mathbb{E}^{x,a_0,a_\Gamma}_{\nu^0,\nu^\Gamma}\Big[\int_{(0,\infty)} e^{-\delta\, s}\, f(X_s, I_s)\, ds + \int_{(0,\infty)} e^{-\delta\, s}\, c(X_{s-}, J_{s-})\, dp^*_s\Big] \tag{1.4}$$

over all possible choices of ν^0, ν^Γ. Firstly, we give a probabilistic representation of the value function of the randomized control problem, denoted $V^*(x, a_0, a_\Gamma)$, in terms of a constrained BSDE, that is an equation over infinite horizon of the form (4.4) with the sign constraints (4.5)-(4.6). The random measure $q = p-\tilde{p}$ driving the BSDE is the compensated measure associated to the jumps of (X, I, J). In particular, the compensator $\tilde{p}$ has predictable jumps $\tilde{p}(\{t\}\times dy\,db\,dc) = \mathbb{1}_{X_{t-}\in\partial E}$, so that the BSDE is driven by a non quasi-left-continuous random measure. The existence and uniqueness of a maximal solution $(Y^{x,a_0,a_\Gamma}, Z^{x,a_0,a_\Gamma}, K^{x,a_0,a_\Gamma})$ to equation (4.4)-(4.5)-(4.6) are obtained by means of a penalization approach, and by suitably extending the existence and uniqueness theorem recently given in [2] for unconstrained BSDEs. Then, we prove that Y^{x,a_0,a_Γ} at the initial time represents the randomized value function $V^*(x, a_0, a_\Gamma)$. All this is collected in Theorem 4.3. At this point, we aim at proving that Y_0^{x,a_0,a_Γ} also provides a nonlinear Feynman-Kac representation to the value function (1.2) of our original optimal control problem. To this end, we introduce the deterministic real function on $E\times A_0\times A_\Gamma$ defined by $v(x,a_0,a_\Gamma) := Y_0^{x,a_0,a_\Gamma}$, and we prove that v does not depend on its two last arguments, is a bounded and continuous function on E, and that $v(X_s) = Y_s^{x,a_0,a_\Gamma}$ for all $s \geq 0$, see Theorem 5.1. Then, we show that v is a viscosity solution to (1.3), so that, by the uniqueness of the solution to the HJB equation (1.3), we can conclude that

$$Y_0^{x,a_0,a_\Gamma} = V^*(x,a_0,a_\Gamma) = V(x). \tag{1.5}$$

This constitutes the main result of the paper and is stated in Theorem 5.2. The nonlinear Feynman-Kac formula (1.5) can be used to design algorithms based on the numerical approximation of the solution to the constrained BSDE (4.4)-(4.5)-(4.6), and therefore to get probabilistic numerical approximations for the value function of the considered optimal control problem. Recently, numerical schemes for constrained BSDEs have been proposed and analyzed in the diffusive framework, see [22], and in the PDMPs context as well, see [1].

The paper is organized as follows. In Section 2 we introduce the optimal control problem (1.2), and we discuss its solvability. Section 3 is devoted to the formulation of the randomized optimal control problem (1.4). In Section 4 we introduce the constrained BSDE (4.4)-(4.5)-(4.6), we show that it admits a unique maximal solution (Y,Z,K) in a certain class of processes, and that Y_0 coincides with the value function of the randomized optimal control problem. Then, in Section 5 we prove that Y_0 also provides a viscosity solution to (1.3).

2 Optimal control of PDMPs on bounded domains

In this section we formulate the optimal control problem for piecewise deterministic Markov processes on bounded domains, and we discuss its solvability. The PDMP state space E is an open bounded subset of $\mathbb{R}^d$ of class $W^{2,\infty}$, and $\mathcal{E}$ the corresponding Borel σ-algebra. Moreover, we introduce two Borel spaces (i.e. topological spaces homeomorphic to Borel subsets of compact metric spaces) A_0, A_Γ, endowed with their Borel σ-algebras $\mathcal{A}_0$ and $\mathcal{A}_\Gamma$, that are respectively the space of control actions in the interior and on the boundary of the domain. Given a topological space F, in the sequel we will denote by $\mathbb{C}_b(F)$ (resp. $\mathbb{C}_b^1(F)$) the set of all bounded continuous functions (resp. all bounded differentiable functions whose derivative is continuous) on F.

A controlled PDMP on $(E,\mathcal{E})$ is described by means of a set of local characteristics (h,λ,Q,R), with h,λ functions on $\bar{E}\times A_0$, and Q, R probability transition measures in E respectively from $\bar{E}\times A_0$ and from $\partial E\times A_\Gamma$. We assume the following.

(HhλQR)

(i) $h : \bar{E}\times A_0 \to \mathbb{R}^d$ and $\lambda : \bar{E}\times A_0 \to \mathbb{R}_+$ are continuous and bounded functions, Lipschitz continuous on $\bar{E}$, uniformly in A_0.

(ii) Q (resp. R) maps $\bar{E}\times A_0$ (resp. $\partial E\times A_\Gamma$) into the set of probability measures on $(E,\mathcal{E})$, and is a continuous stochastic kernel. Moreover, for all $v\in\mathbb{C}_b(E)$, the maps $x\mapsto \int_E v(y)\,Q(x,a_0,dy)$ and $x\mapsto\int_E v(y)\,R(x,a_\Gamma,dy)$ are Lipschitz continuous, uniformly in $a_0\in A_0$ and in $a_\Gamma\in A_\Gamma$, respectively.

The controlled PDMP X with state space E and local characteristics (ϕ,λ,Q,R) will be constructed in a canonical way. To this end, let $\bar{\Omega}'$ be the set of sequences $\bar{\omega}' = (t_n,e_n)_{n\geq 1}$ contained in $((0,\infty)\times E)\cup\{(\infty,\Delta)\}$, where $\Delta\notin E$, is adjoined to E as an isolated point, such that $t_n\leq t_{n+1}$, and $t_n < t_{n+1}$ if $t_n<\infty$. We set $\bar{\Omega} = E\times\bar{\Omega}'$, where $\bar{\omega} = (x,\bar{\omega}') = (x,t_1,e_1,t_2,e_2,...)$. On the sample space $\bar{\Omega}$ we define the canonical functions $T_n : \bar{\Omega}\to(0,\infty]$, $E_n : \bar{\Omega}\to E\cup\{\Delta\}$ as follows:

$T_0(\bar\omega) = 0$, $E_0(\bar\omega) = x$, and for $n \geq 1$, $T_n(\bar\omega) = t_n$, $E_n(\bar\omega) = e_n$, and $T_\infty(\bar\omega) = \lim_{n\to\infty} t_n$. We also define the integer-valued random measure μ on $(0,\infty) \times E$ as

$$\mu(ds\,dy) = \sum_{n\in\mathbb{N}} \mathbb{1}_{\{T_n, E_n\}}(ds\,dy).$$

The class of admissible control maps $\mathcal{A}_{ad}$ is the set of all $\mathcal{A}_0 \otimes \mathcal{A}_\Gamma$-measurable maps $\boldsymbol{\alpha} = (\alpha^0, \alpha^\Gamma)$, where $\alpha^0 : [0,\infty) \times E \to A_0$ is a piecewise open-loop function of the form $\alpha^0_t = \alpha^0_0(t,x)\,\mathbb{1}_{[0,T_1)}(t) + \sum_{n=1}^\infty \alpha^0_n(t - T_n, E_n)\,\mathbb{1}_{[T_n, T_{n+1})}(t)$, and $\alpha^\Gamma : \partial E \to A_\Gamma$ is a feedback policy. Let $\phi^U(t,x)$, with U any $\mathcal{A}_0$-measurable function, be the unique solution to the ordinary differential equation $\dot y(t) = h(y(t), U(t))$, $y(0) = x \in E$. Then the controlled process $X : \bar\Omega \times [0,\infty) \to \bar E \cup \{\Delta\}$ is defined setting

$$X_t = \begin{cases} \phi^{\alpha^0_0}(t,x) & \text{if } t \in [0, T_1), \\ \phi^{\alpha^0_n}(t - T_n, E_n) & \text{if } t \in [T_n, T_{n+1}),\ n \in \mathbb{N} \setminus \{0\}. \end{cases} \tag{2.1}$$

Finally, we introduce the process

$$p^*_s := \sum_{n=1}^\infty \mathbb{1}_{\{s \geqslant T_n\}}\,\mathbb{1}_{\{X_{T_n-} \in \partial E\}}, \tag{2.2}$$

that counts the number of times that the process hits the boundary.

Set $\bar{\mathcal{F}}_0 = \mathcal{E} \otimes \{\emptyset, \bar\Omega'\}$ and, for all $t \geq 0$, $\bar{\mathcal{G}}_t = \sigma(\mu((0,s] \times B) : s \in (0,t], B \in \mathcal{E})$. For all t, let $\bar{\mathcal{F}}_t$ be the σ-algebra generated by $\bar{\mathcal{F}}_0$ and $\bar{\mathcal{G}}_t$. We denote by $\bar{\mathcal{F}}_\infty$ the σ-algebra generated by the all σ-algebras $\bar{\mathcal{F}}_t$. In the following all the concepts of measurability for stochastic processes will refer to the right-continuous, natural filtration $\bar{\mathbb{F}} = (\bar{\mathcal{F}}_t)_{t\geq 0}$. By the symbol $\bar{\mathcal{P}}$ we will denote the σ-algebra of $\bar{\mathbb{F}}$-predictable subsets of $[0,\infty) \times \bar\Omega$.

For every starting point $x \in E$ and for each $\boldsymbol{\alpha} \in \mathcal{A}_{ad}$, by Theorem 3.6 in [19], there exists a unique probability measure on $(\bar\Omega, \bar{\mathcal{F}}_\infty)$, denoted by $\mathbb{P}^x_{\boldsymbol{\alpha}}$, such that its restriction to $\bar{\mathcal{F}}_0$ is the Dirac measure concentrated at x, and the $\bar{\mathbb{F}}$-compensator under $\mathbb{P}^x_{\boldsymbol{\alpha}}$ of the measure $\mu(ds\,dy)$ is

$$\begin{aligned}\tilde p^{\boldsymbol{\alpha}}(ds\,dy) &= \sum_{n=1}^\infty \mathbb{1}_{[T_n, T_{n+1})}(s)\, R(\phi^{\alpha^0_n}(s - T_n, E_n)), \alpha^\Gamma(\phi^{\alpha^0_n}(s - T_n, E_n)), dy)\, dp^*_s \\ &+ \sum_{n=1}^\infty \mathbb{1}_{[T_n, T_{n+1})}(s)\, \lambda(\phi^{\alpha^0_n}(s - T_n, E_n)), \alpha^0_n(s - T_n, E_n))\, Q(\phi^{\alpha^0_n}(s - T_n, E_n)), \alpha^0_n(s - T_n, E_n), dy)\, ds.\end{aligned}$$

Arguing as in Proposition 2.2 in [3], one can easily see that under $\mathbb{P}^x_{\boldsymbol{\alpha}}$ the process X in (2.1) is Markovian with respect to $\bar{\mathbb{F}}$. In particular, for every $n \geq 1$, the conditional survival function of the inter-jump time $T_{n+1} - T_n$ on $\{T_n < \infty\}$ is

$$\mathbb{P}^x_{\boldsymbol{\alpha}}(T_{n+1} - T_n > s \,|\, \mathcal{F}_{T_n}) = e^{-\int_{T_n}^{T_n+s} \lambda(\phi^{\alpha^0}(r - T_n, X_{T_n}), \alpha^0_n(r - T_n, X_{T_n}))\,dr}\, \mathbb{1}_{\phi^{\alpha^0}(s, X_{T_n}) \in E}, \tag{2.3}$$

and the distribution of the post jump position $X_{T_{n+1}}$ on $\{T_n < \infty\}$ are, for any $B \in \mathcal{E}$,

$$\begin{aligned}&\mathbb{P}^x_{\boldsymbol{\alpha}}(X_{T_{n+1}} \in B \,|\, \mathcal{F}_{T_n}, T_{n+1}, \phi^{\alpha^0}(T_{n+1} - T_n, X_{T_n}) \in E) \\ &= Q(\phi^{\alpha^0}(T_{n+1} - T_n, X_{T_n}), \alpha^0_n(T_{n+1} - T_n, X_{T_n}), B),\end{aligned} \tag{2.4}$$

$$\begin{aligned}&\mathbb{P}^x_{\boldsymbol{\alpha}}(X_{T_{n+1}} \in B \,|\, \mathcal{F}_{T_n}, T_{n+1}, \phi^{\alpha^0}(T_{n+1} - T_n, X_{T_n}) \in \partial E) \\ &= R(\phi^{\alpha^0}(T_{n+1} - T_n, X_{T_n}), \alpha^\Gamma(\phi^{\alpha^0}(T_{n+1} - T_n, X_{T_n})), B).\end{aligned} \tag{2.5}$$

The infinite horizon control problem consists in minimizing over all control laws $\boldsymbol{\alpha}$ a cost functional of the form (1.1), where f is a given real function on $\bar{E} \times A_0$ representing the running cost, c is a given real function on $\partial E \times A_\Gamma$ that associates a cost to hitting the active boundary, $\delta \in (0, \infty)$ is a discounting factor. The value function of the control problem is defined in the usual way by (1.2). We ask that f and c verify the following conditions.

(Hfc) $f : \bar{E} \times A_0 \to \mathbb{R}_+$ (resp. $c : \partial E \times A_\Gamma \to \mathbb{R}_+$) is a continuous function, Lipschitz continuous on $\bar{E}$ (resp. on ∂E), uniformly in A_0 (resp. A_Γ), and bounded by the constant M_f (resp. by the constant M_c).

Moreover, set $t_*^{\alpha^0}(x) := \inf\{t \geqslant 0 : \phi^{\alpha^0}(t,x) \in \partial E,\ x \in E\}$, and $E_\varepsilon := \left\{x \in E : \inf_{\alpha^0 \in \mathcal{A}_0} t_*^{\alpha^0}(x) \geqslant \varepsilon\right\}$. We will consider the following assumption.

(H0) There exists $\varepsilon > 0$ such that $R(x, \alpha, E_\varepsilon) = 1$ for all $x \in \partial E$ and $\alpha \in \mathcal{A}_\Gamma$.

Remark 2.1. *Roughly speaking, condition* **(H0)** *says that the state process always jumps from the boundary to points of the interior of the domain whose distance from the boundary (as measured by the boundary hitting time $t_*^{\alpha^0}$) are uniformly bounded away from zero. In particular, the time interval between successive boundary jumps is at least ε, so that, for every starting point $x \in E$ and admissible control $\boldsymbol{\alpha} \in \mathcal{A}_{ad}$, the number of boundary hitting times in $[0,t]$ can be majorized as*

$$\mathbb{E}_{\boldsymbol{\alpha}}^x\left[p_t^*\right] \leqslant \frac{t}{\varepsilon} + 1 =: C^*(t), \quad \forall\, t \geq 0. \tag{2.6}$$

By the integration by parts formula for processes of finite variation (see e.g., Proposition 4.5 in [26]), this implies in particular that

$$\mathbb{E}_{\boldsymbol{\alpha}}^x\left[\int_{(0,\infty)} e^{-\delta t}\, dp_t^*\right] \leqslant \frac{1}{\delta\varepsilon} + 1 =: C^*. \tag{2.7}$$

Finally, we impose the following standard non-degeneracy assumptions, that allow to avoid difficulties arising from trajectories tangent to the boundary, see [7] for more details. We denote by $n(x)$ the normal vector to ∂E.

(HBB) For all $x \in \partial E$, if there exists $a_0 \in A_0$ such that $-h(x,a_0)\cdot n(x) \geqslant 0$, then there exists $a_0' \in A_0$ such that $-h(x,a_0')\cdot n(x) > 0$.

(HBB') For all $x \in \partial E$, if $-h(x,a_0)\cdot n(x) \geqslant 0$ for all $a_0 \in A_0$, then $-h(x,a_0)\cdot n(x) > 0$ for all $a_0 \in A_0$.

Remark 2.2. *Let us denote by d_∂ the distance to ∂E. The fact that E is of class $W^{2,\infty}$ implies, in particular, that d_∂ is of class $W^{2,\infty}$, and thus $n(x) = -\nabla d_\partial(x)$.*

Remark 2.3. *The choice of dealing with PDMPs with time-homogeneous local characteristics is not a real restriction, since time-varying coefficients can be seen as a special case of time-invariant ones. As a matter of fact, assume that the (uncontrolled) local characteristics take the form $h(t,x)$, $\lambda(t,x)\ Q(t,x,dy)$, $R(t,x,dy)$. Then, by defining the augmented state $\tilde{x} = (x^0, x)$ on the state space $\tilde{E} = R_+ \times E$, one can construct a new PDMP with local characteristics $\tilde{h}(\tilde{x}) = (1, h(x^0,x))$, $\tilde{\lambda}(\tilde{x}) = \lambda(x^0,x)$, $\tilde{Q}(\tilde{x}, \{x^0\} \times B) = Q(x^0,x,B)$, $\tilde{R}(\tilde{x}, \{x^0\} \times B) = R(x^0,x,B)$.*

Let us now consider the Hamilton-Jacobi-Bellman equation associated to the optimal control problem (1.2), that is an elliptic fully nonlinear integro-differential equation on $[0,\infty) \times \bar{E}$ with nonlocal boundary conditions

$$H^v(x, v(x), \nabla v(x)) = 0 \qquad \text{in } E, \tag{2.8}$$

$$v(x) = F^v(x) \quad \text{on } \partial E, \tag{2.9}$$

where

$$H^\psi(z,u,p) := \sup_{a_0 \in A_0} \Big\{ \delta\, u - h(z,a_0) \cdot p - f(z,a_0) - \int_E (\psi(y) - \psi(z))\, \lambda(z,a_0)\, Q(z,a_0,dy) \Big\},$$
$$F^\psi(x) := \min_{a_\Gamma \in A_\Gamma} \Big\{ c(z,a_\Gamma) + \int_E \psi(y)\, R(z,a_\Gamma,dy) \Big\}.$$

In the following the shorthand u.s.c. (resp. l.s.c.) stands for upper (resp. lower) semicontinuous.

Definition 2.1. *(i) A bounded u.s.c. function u on $\bar{E}$ is a viscosity subsolution of (2.8)-(2.9) if and only if, $\forall \phi \in \mathbb{C}^1_b(\bar{E})$, if $x_0 \in \bar{E}$ is a global maximum of $u - \phi$ one has*

$$\begin{array}{ll} H^u(x_0, u(x_0), \nabla\phi(x_0)) \leqslant 0 & \textit{if } x_0 \in E, \\ \min\{H^u(x_0, u(x_0), \nabla\phi(x_0)), u(x_0) - F^u(x_0)\} \leqslant 0 & \textit{if } x_0 \in \partial E. \end{array}$$

(ii) A bounded l.s.c. function w on $\bar{E}$ is a viscosity supersolution of (2.8)-(2.9) if and only if, $\forall \phi \in \mathbb{C}^1_b(\bar{E})$, if $x_0 \in \bar{E}$ is a global minimum of $w - \phi$ one has

$$\begin{array}{ll} H^w(x_0, w(x_0), \nabla\phi(x_0)) \geqslant 0 & \textit{if } x_0 \in E, \\ \max\{H^w(x_0, w(x_0), \nabla\phi(x_0)), w(x_0) - F^w(x_0)\} \geqslant 0 & \textit{if } x_0 \in \partial E. \end{array}$$

(iii) A viscosity solution of (2.8)-(2.9) is a continuous function which is both a viscosity subsolution and a viscosity supersolution of (2.8)-(2.9).

The following theorem collects the results of Theorems 5.8 and 7.5 in [15].

Theorem 2.1. *Let* **(HhλQR)**, **(Hfc)**, **(H0)**, **(HBB)** *and* **(HBB')** *hold, and assume that A_0, A_Γ are compact. Let $V : E \to \mathbb{R}$ be the value function of the PDMPs optimal control problem (1.2). Then V is a bounded and continuous function, and is the unique viscosity solution of (2.8)-(2.9).*

We end this section with an example of application of the PDMP optimal control theory.

Example 2.1. *Let us consider the optimal control of consumption and exploration of non-renewable resources studied in [16], Section 5. In particular, one assumes that new reserves are found at random times $T_1, T_2, \ldots$, where the inter-arrival times are exponentially distributed, and that the amount of resources found at each T_n is a random variable with given distribution. The exploration process takes place on a bounded area, and there is a maximum amount of reserves $r_m > 0$. At time t, the controller can change the exploration rate e_t and the consumption rate c_t of the current reserves. The level of current reserves decreases by the amount of consumption, and an exploration cost must be paid. Denote by A_t and r_t respectively the amount of explored land and the level of the known reserves at time t. The control process will be a pair of measurable functions $\alpha^0(t) = (c(t), e(t))$, with $e : \mathbb{R}_+ \to [\varepsilon_0, e_0]$, $c : \mathbb{R}_+ \to [0, c_0]$ where ε_0 is a small positive number. For $t \in [0, T_1)$, the state process $X = (A, r)$ evolves deterministically in $E^0_1 = [0,1) \times (0, r_m]$ according to*

$$\begin{cases} A_t = A_0 + K \int_0^t e(s) ds \\ r_t = r_0 - \int_0^t c(s) ds - \int_0^t e(s) ds, \end{cases}$$

where K is a given constant and $x_0 = (A_0, r_0)$ is a fixed value of E^0_1. The boundary state space is $\Gamma_1 = (\{1\} \times (0, r_m]) \cup ([0,1) \times \{0\})$. After each discovery, the amount of known reserves jumps to a new value according to the (uncontrolled) transition measure

$$Q(r, d\zeta) = \frac{1}{r_m - r} \mathbb{1}_{\zeta \in [r, r_m]} \mathbb{1}_{r \in (0, r_m)}\, d\zeta + \mathbb{1}_{r = r_m} \delta_{r_m + r_j}(d\zeta),$$

with controlled jump rate given by

$$\lambda(x, u^0) = \begin{cases} \lambda_0 K r(r_m - r) e & \text{if } x = (A, r) \in E_1^0 \\ 0 & \text{otherwise,} \end{cases}$$

where $u^0 = (c, e) \in [0,\, c_0] \times [\varepsilon_0,\, e_0]$ *and* λ_0 *is a fixed constant. When the process hits the boundary* $\{1\} \times (0,\, r_m]$*, there is no further exploration and the process jumps to the new state space* $E_2^0 = (0, r_m]$ *with boundary* $\Gamma_2 = \{0\}$*, where it evolves deterministically according to*

$$r_r = r_0 - \int_0^t c(s) ds, \quad r_0 \in E_2^0.$$

On the other hand, if the process hits $[0,\, 1) \times \{0\}$*, it stops and the process jumps to a cemetery state* Δ_c *and is killed. Thus, the PDMP state space is* $E = E_1^0 \cup E_2^0 \cup \{\Delta_c\}$*. The (uncontrolled) boundary transitions measures for* Γ_1 *and* Γ_2 *are respectively*

$$R_1((A, r), B) = \mathbb{1}_{r=0}\, \delta_{\Delta_c}(B) + \mathbb{1}_{A=1} \mathbb{1}_{r>0} \mathbb{1}_{B \subset E_2^0} \delta_r(B), \qquad R_2(r, B) = \mathbb{1}_{r=0}\, \delta_{\Delta_c}(B),$$

where B *is a Borel subset of* E*. The value function of the control problem is then defined as*

$$V(x_0) = \sup_{\alpha^0(\cdot)} \mathbb{E}_{\alpha^0}^{x_0} \left[\int_0^\infty e^{-\beta t} U(c(t)) dt \right],$$

where $\beta > 0$ *is a constant, and* $U(\cdot)$ *denotes the utility of consumption of the economy, which is assumed to be a continuous function. Finally, the Hamilton-Jacobi-Bellman equation associated to this problem reads*

$$\begin{aligned} &\beta\, v(A, r) = \sup_{c \in [0, c_0]} \big(U(c) - c\, \partial_r v(A, r) \big) \\ &+ \sup_{e \in [\varepsilon_0, e_0]} \left(eK\, \partial_A v(A, r) - e\, \partial_r v(A, r) - \lambda_0 K r(r_m - r) e \int_0^\infty [v(A, \zeta) - v(A, r)] Q(r, d\zeta) \right) \quad \textit{in } E_1^0, \end{aligned} \tag{2.10}$$

$$v(A, r) = F^v(A, r) \quad \textit{on } \Gamma_1, \tag{2.11}$$

where

$$F^v(A, r) := \begin{cases} \sup_{c(\cdot)} \big(\int_0^\infty e^{-\beta t} U(c(t)) dt \big) & \text{subject to } r_t = r - \int_0^t c(s) ds,\ \forall (A, r) \in \{1\} \times (0, r_m], \\ 0 & \forall (A, r) \in (0, 1] \times \{0\}. \end{cases}$$

It is easy to verify that assumptions **(Hfc)***,* **(HBB)** *and* **(HBB)'** *hold true in the present example, and that assumption* **(HhλQR)** *is verified by* λ*,* Q*. On the other hand, the PDMP state space* E *is not of class* $W^{2,\infty}$*, being the normal vector not defined at the corners of the rectangular state space* E_1^0*; nevertheless, the probability of hitting a neighborhood of these corners can be shown to be uniformly small, and one can prove the continuity of the value function for this special case. Analogously, assumptions* **(H0)** *and* **(HhλQR)** *for the boundary transition measures* R_1, R_2 *are not needed to solve this particular optimal control problem (for more details, see [16]).*

3 The randomized optimal control problem

In the present section we formulate the randomized optimal control problem. First we introduce some notations. For every $a_0 \in A_0$, we denote by $\phi(t, x, a_0)$ the unique solution to the ordinary differential equation $\dot{x}(t) = h(x(t), a_0)$, with $x(0) = x \in E$. Notice that $\phi(t, x, a_0)$ coincides with the function $\phi^U(t, x)$, introduced in Section 2, when $U(t) \equiv a_0$. We also introduce two positive measures λ_0 and λ_Γ on $(A_0, \mathcal{A}_0)$ and $(A_\Gamma, \mathcal{A}_\Gamma)$, respectively, satisfying the following assumption:

(H$\lambda_0\lambda_\Gamma$) λ_0 and λ_Γ are two finite positive measures on $(A_0, \mathcal{A}_0)$ and $(A_\Gamma, \mathcal{A}_\Gamma)$, respectively, with full topological support.

For all $t \geq 0$, $(a_0, a_\Gamma) \in A_0 \times A_\Gamma$, let us define

$$\begin{aligned} \tilde{\phi}(t, x, a_0, a_\Gamma) &:= (\phi(t, x, a_0), a_0, a_\Gamma), \quad x \in \bar{E}, \\ \tilde{\lambda}(x, a_0) &:= \lambda(x, a_0) + \lambda_0(A_0) + \lambda_\Gamma(A_\Gamma), \quad x \in \bar{E}, \\ \tilde{R}(x, a_0, a_\Gamma, dy\, db\, dc) &:= R(x, a_\Gamma, dy)\, \delta_{a_0}(db)\, \delta_{a_\Gamma}(dc), \quad x \in \partial E, \end{aligned} \tag{3.1}$$

and,

$$\tilde{Q}(x, a_0, a_\Gamma, dy\, db\, dc) := \frac{\lambda(x, a_0)\, Q(x, a_0, dy)\, \delta_{a_0}(db)\, \delta_{a_\Gamma}(dc) + \lambda_0(db)\, \delta_{a_\Gamma}(dc)\, \delta_x(dy) + \lambda_\Gamma(dc)\, \delta_{a_0}(db)\, \delta_x(dy)}{\tilde{\lambda}(x, a_0)}, \quad x \in \bar{E},$$

where, for any F topological space, δ_a denotes the Dirac measure concentrated at some point $a \in F$.

3.1 State process

Our purpose is to construct a PDMP (X, I, J) with enlarged state space $E \times A_0 \times A_\Gamma$ and local characteristics $(\tilde{\phi}, \tilde{\lambda}, \tilde{Q}, \tilde{R})$. This can be done in a canonical way, proceeding as in Section 2. So, in particular, we define Ω' as the set of sequences $\omega' = (t_n, e_n, a_n^0, a_n^\Gamma)_{n \geq 1}$ contained in $((0, \infty) \times E \times A_0 \times A_\Gamma) \cup \{(\infty, \Delta, \Delta', \Delta'')\}$, where $\Delta \notin E$, $\Delta' \notin A_0$, $\Delta'' \notin A_\Gamma$ are isolated points respectively adjoined to E, A_0 and A_Γ. In the sample space $\Omega = \Omega' \times E \times A_0 \times A_\Gamma$ we define the random variables $T_0(\omega) = 0$, $E_0(\omega) = x$, $A^0(\omega) = a_0$, $A^\Gamma(\omega) = a_\Gamma$, and the sequence of random variables $T_n : \Omega \to (0, \infty]$, $E_n : \Omega \to E \cup \{\Delta\}$, $A_n^0 : \Omega \to A_0 \cup \{\Delta'\}$, $A_n^\Gamma : \Omega \to A_\Gamma \cup \{\Delta''\}$, for $n \geq 1$, by setting $T_n(\omega) = t_n$, $E_n(\omega) = e_n$, $A_n^0(\omega) = a_n^0$, $A_n^\Gamma(\omega) = a_n^\Gamma$, with $T_\infty(\omega) = \lim_{n\to\infty} t_n$. Then, we define the process (X, I, J) on $(E \times A_0 \times A_\Gamma) \cup \{\Delta, \Delta', \Delta''\}$ as

$$(X, I, J)_t = \begin{cases} (\phi(t - T_n, E_n, A_n^0), A_n^0, A_n^\Gamma) & \text{if } T_n \leq t < T_{n+1}, \text{ for } n \in \mathbb{N}, \\ (\Delta, \Delta', \Delta'') & \text{if } t \geq T_\infty. \end{cases}$$

We also define the random measure p on $(0, \infty) \times E \times A_0 \times A_\Gamma$ as

$$p(ds\, dy\, db\, dc) = \sum_{n \in \mathbb{N}} \mathbb{1}_{\{T_n, E_n, A_n^0, A_n^\Gamma\}}(ds\, dy\, db\, dc), \tag{3.2}$$

and, for all $t \geq 0$, we introduce the σ-algebra $\mathcal{G}_t = \sigma(p((0, s] \times G) : s \in (0, t], G \in \mathcal{E} \otimes \mathcal{A}_0 \times \mathcal{A}_\Gamma)$, and the σ-algebra $\mathcal{F}_t$ generated by $\mathcal{F}_0$ and $\mathcal{G}_t$, where $\mathcal{F}_0 = \mathcal{E} \otimes \mathcal{A}_0 \otimes \mathcal{A}_\Gamma \otimes \{\emptyset, \Omega'\}$. We denote by $\mathbb{F} = (\mathcal{F}_t)_{t \geq 0}$ and $\mathcal{P}$ the corresponding filtration and predictable σ-algebra. We also denote by $\mathcal{F}_\infty$ the σ-algebra generated by the all σ-algebras $\mathcal{F}_t$.

Given any starting point $(x, a_0, a_\Gamma) \in E \times A_0 \times A_\Gamma$, by Proposition 2.1 in [3], there exists a unique probability measure on $(\Omega, \mathcal{F}_\infty)$, denoted by $\mathbb{P}^{x, a_0, a_\Gamma}$, such that its restriction to $\mathcal{F}_0$ is $\delta_{(x, a_0, a_\Gamma)}$ and the $\mathbb{F}$-compensator of the measure $p(ds\, dy\, db\, dc)$ under $\mathbb{P}^{x, a_0, a_\Gamma}$ is the random measure

$$\tilde{p}(ds\, dy\, db\, dc) = \sum_{n \in \mathbb{N}} \mathbb{1}_{[T_n, T_{n+1})}(s)\, \Lambda(\phi(s - T_n, E_n, A_n^0), A_n^0, A_n^\Gamma, dy\, db\, dc)\, dA_s, \tag{3.3}$$

where, for all $(x, a_0, a_\Gamma) \in E \times A_0 \times A_\Gamma$,

$$\Lambda(x, a_0, a_\Gamma, dy\, db\, dc) = \tilde{Q}(x, a_0, a_\Gamma, dy\, db\, dc)\, \mathbb{1}_{x \in E} + \tilde{R}(x, a_0, a_\Gamma, dy\, db\, dc)\, \mathbb{1}_{x \in \partial E},$$

and A_s is the increasing, predictable process such that, for any $s \geq 0$,

$$dA_s(\omega) = \tilde{\lambda}(X_{s-}(\omega), I_{s-}(\omega))\, \mathbb{1}_{X_{s-}(\omega) \in E}\, ds + \mathbb{1}_{X_{s-}(\omega) \in \partial E}\, dp^*_s(\omega). \tag{3.4}$$

In particular,

$$\Delta A_t(\omega) = \tilde{p}(\omega, \{t\} \times E \times A_0 \times A_\Gamma) = \mathbb{1}_{X_{t-}(\omega) \in \partial E}, \tag{3.5}$$

$$\tilde{p}(\omega, \{t\} \times dy\, db\, dc) = \tilde{R}(X_{t-}(\omega), I_{t-}(\omega), J_{t-}(\omega), dy\, db\, dc)\, \Delta A_t(\omega). \tag{3.6}$$

Remark 3.1. *The $\mathbb{F}$-compensator of the measure $p(ds\, dy\, db\, dc)$ under $\mathbb{P}^{x, a_0, a_\Gamma}$ can be decomposed as $\tilde{p}(\omega, ds\, dy\, db\, dc) = \phi_{\omega,t}(dy\, db\, dc)\, dA_s(\omega)$, where*

$$\phi_{\omega,t}(dy\, db\, dc) := \Lambda(X_{t-}(\omega), I_{t-}(\omega), J_{t-}(\omega), dy\, db\, dc). \tag{3.7}$$

The process (X, I, J) is Markovian on $[0, \infty)$ with respect to $\mathbb{F}$. For every real-valued functions φ defined on $E \times A_0 \times A_\Gamma$, we define

$$\begin{aligned}
&\mathcal{L}\varphi(x, a_0, a_\Gamma) := h(x, a_0) \cdot \nabla_x \varphi(x, a_0, a_\Gamma) + \int_E (\varphi(y, a_0, a_\Gamma) - \varphi(x, a_0, a_\Gamma))\, \lambda(x, a_0)\, Q(x, a_0, dy) \\
&+ \int_{A_0} (\varphi(x, b, a_\Gamma) - \varphi(x, a_0, a_\Gamma))\, \lambda_0(db) + \int_{A_\Gamma} (\varphi(x, a_0, c) - \varphi(x, a_0, a_\Gamma))\, \lambda_\Gamma(dc), \quad x \in E, \\
&\mathcal{G}\varphi(x, a_0, a_\Gamma) := \int_E (\varphi(y, a_0, a_\Gamma) - \varphi(x, a_0, a_\Gamma))\, R(x, a_\Gamma, dy), \quad x \in \partial E.
\end{aligned}$$

From Theorem 26.14 in [13] it follows that $\mathcal{L}$ is the extended generator of the process (X, I, J) and $\mathcal{G}\varphi = 0$ if and only if φ belongs to the domain of $\mathcal{L}$.

3.2 The randomized control problem

The class of admissible control maps is the set $\mathcal{V} = \mathcal{V}_0 \otimes \mathcal{V}_\Gamma$, where $\mathcal{V}_0 = \{\nu^0 : \Omega \times [0, \infty) \times A_0 \to (0, \infty)\ \mathcal{P} \otimes \mathcal{A}_0\text{-measurable and bounded}\}$, and $\mathcal{V}_\Gamma = \{\nu^\Gamma : \Omega \times [0, \infty) \times A_\Gamma \to (0, \infty)\ \mathcal{P} \otimes \mathcal{A}_\Gamma\text{-measurable and bounded}\}$. For every $\boldsymbol{\nu} = (\nu^0, \nu^\Gamma) \in \mathcal{V}$, we define

$$\begin{aligned}
&\tilde{\lambda}^{\boldsymbol{\nu}}(t, x, a_0) := \lambda(x, a_0) + \int_{A_0} \nu^0_t(b)\, \lambda_0(db) + \int_{A_\Gamma} \nu^\Gamma_t(c)\, \lambda_\Gamma(dc), \\
&\tilde{Q}^{\boldsymbol{\nu}}(t, x, a_0, a_\Gamma, dy\, db\, dc) := \\
&\frac{\lambda(x, a_0)\, Q(x, a_0, dy)\, \delta_{a_0}(db)\, \delta_{a_\Gamma}(dc) + \nu^0_t(b)\, \lambda_0(db)\, \delta_{a_\Gamma}(dc)\, \delta_x(dy) + \nu^\Gamma_t(c)\, \lambda_\Gamma(dc)\, \delta_{a_0}(db)\, \delta_x(dy)}{\tilde{\lambda}^{\boldsymbol{\nu}}(t, x, a_0)}, \\
&\Lambda^{\boldsymbol{\nu}}(t, x, a_0, a_\Gamma, dy\, db\, dc) = \tilde{Q}^{\boldsymbol{\nu}}(t, x, a_0, a_\Gamma, dy\, db\, dc)\, \mathbb{1}_{x \in E} + \tilde{R}(x, a_0, a_\Gamma, dy\, db\, dc)\, \mathbb{1}_{x \in \partial E}.
\end{aligned}$$

Then, for every $\boldsymbol{\nu} \in \mathcal{V}$, we consider the predictable random measure

$$\tilde{p}^{\boldsymbol{\nu}}(ds\, dy\, db\, dc) = \sum_{n \in \mathbb{N}} \mathbb{1}_{[T_n, T_{n+1})}(s)\, \Lambda^{\boldsymbol{\nu}}(s, \phi(s - T_n, E_n, A^0_n), A^0_n, A^\Gamma_n, dy\, db\, dc)\, dA^{\boldsymbol{\nu}}_s, \tag{3.8}$$

where $A^{\boldsymbol{\nu}}$ is the increasing and predictable process given by

$$dA^{\boldsymbol{\nu}}_s = \tilde{\lambda}^{\boldsymbol{\nu}}(s, X_{s-}, I_{s-})\, \mathbb{1}_{X_{s-} \in E}\, ds + \mathbb{1}_{X_{s-} \in \partial E}\, dp^*_s.$$

In what follows we will denote $q = p - \tilde{p}$ and $q^{\boldsymbol{\nu}} = p - \tilde{p}^{\boldsymbol{\nu}}$. By the Radon-Nikodym theorem, there exist three nonnegative functions d_1, d_2, d_3 defined on $\Omega \times [0, \infty) \times E \times A_0 \times A_\Gamma$, $\mathcal{P} \otimes \mathcal{E} \otimes \mathcal{A}_0 \otimes \mathcal{A}_\Gamma$, such that $d\tilde{p}^{\boldsymbol{\nu}} = (\nu^0\, d_1 + \nu^\Gamma\, d_2 + d_3)\, d\tilde{p}$, with $d_1 + d_2 + d_3 = 1$, $\tilde{p}$-a.e., and

$$
\begin{aligned}
d_1(t, y, b, c)\, \tilde{p}(dt\, dy\, db\, dc) &= \lambda_0(db)\, \delta_{\{X_{t-}\}}(dy)\, \delta_{\{J_{t-}\}}(dc)\, \mathbb{1}_{X_{t-} \in E}\, dt, && (3.9)\\
d_2(t, y, b, c)\, \tilde{p}(dt\, dy\, db\, dc) &= \lambda_\Gamma(dc)\, \delta_{\{X_{t-}\}}(dy)\, \delta_{\{I_{t-}\}}(db)\, \mathbb{1}_{X_{t-} \in E}\, dt, && (3.10)\\
d_3(t, y, b, c)\, \tilde{p}(dt\, dy\, db\, dc) &= \lambda(X_{t-},\, I_{t-})\, Q(X_{t-},\, I_{t-},\, dy)\, \delta_{\{I_{t-}\}}(db)\, \delta_{\{J_{t-}\}}(dc)\, \mathbb{1}_{X_{t-} \in E}\, dt +\\
&\quad + R(X_{t-},\, J_{t-},\, dy)\, \delta_{\{I_{t-}\}}(db)\, \delta_{\{J_{t-}\}}(dc)\, \mathbb{1}_{X_{t-} \in \partial E}\, dp^*_t.
\end{aligned}
$$

Remark 3.2. *Without loss of generality, one can select a good version of d_1, d_2, d_3 such that*

$$
\begin{aligned}
d_1(t, y, b, c)\, \mathbb{1}_{X_{t-} \in \partial E} = d_2(t, y, b, c)\, \mathbb{1}_{X_{t-} \in \partial E} &= 0,\\
d_3(t, y, b, c)\, \mathbb{1}_{X_{t-} \in \partial E} &= \mathbb{1}_{X_{s-} \in \partial E}.
\end{aligned}
$$

For any $\boldsymbol{\nu} \in \mathcal{V}$, we introduce the Doléans-Dade exponential local martingale $L^{\boldsymbol{\nu}}$ as

$$
\begin{aligned}
L^{\boldsymbol{\nu}}_t &= e^{\int_0^t \int_{A_0} (1 - \nu^0_r(b)) \lambda_0(db)\, dr}\, e^{\int_0^t \int_{A_\Gamma} (1 - \nu^\Gamma_r(c)) \lambda_\Gamma(dc)\, dr}\cdot\\
&\cdot \prod_{n \geqslant 1 : T_n \leqslant t} (\nu^0_{T_n}(A^0_n)\, d_1(T_n, E_n, A^0_n, A^\Gamma_n) + \nu^\Gamma_{T_n}(A^\Gamma_n)\, d_2(T_n, E_n, A^0_n, A^\Gamma_n) + d_3(T_n, E_n, A^0_n, A^\Gamma_n)),
\end{aligned}
$$

for all $t \geq 0$. Notice that, when $(L^{\boldsymbol{\nu}}_t)_{t \geq 0}$ is a true martingale, for every time $T > 0$ we can define a probability measure $\mathbb{P}^{x, a_0, a_\Gamma}_{\boldsymbol{\nu}, T}$ equivalent to $\mathbb{P}^{x, a_0, a_\Gamma}$ on $(\Omega,\, \mathcal{F}_T)$ by $\mathbb{P}^{x, a_0, a_\Gamma}_{\boldsymbol{\nu}, T}(d\omega) = L^{\boldsymbol{\nu}}_T(\omega)\, \mathbb{P}^{x, a_0, a_\Gamma}(d\omega)$.

Lemma 3.1. *When $(L^{\boldsymbol{\nu}}_t)_{t \geq 0}$ is a true martingale, for every $T > 0$, the restriction of the random measure p to $(0, T] \times E \times A_0 \times A_\Gamma$ admits $\tilde{p}^{\boldsymbol{\nu}} = (\nu^0\, d_1 + \nu^\Gamma\, d_2 + d_3)\, \tilde{p}$ as compensator under $\mathbb{P}^{x,a}_{\boldsymbol{\nu}, T}$.*

Proof. We shall prove that

$$\hat{\bar{\nu}}_t = 1 \ \text{ whenever } \ \alpha_t = 1, \tag{3.11}$$

with $\alpha_t := \tilde{p}(\{t\} \times E \times A_0 \times A_\Gamma)$, $\bar{\nu}_t(y, b, c) := \nu^0_t(b)\, d_1(t, y, b, c) + \nu^\Gamma_t(c)\, d_2(t, y, b, c) + d_3(t, y, b, c)$, and $\hat{\bar{\nu}}_t := \int_{E \times A_0 \times A_\Gamma} \bar{\nu}_t(y, b, c)\, \tilde{p}(\{t\} \times dy\, db\, dc)$. Indeed, if condition (3.11) holds, then the result would be a direct application of Theorem 4.5 in [19]. Let us thus show the validity of (3.11). To this end, we start by noticing that, by Remark 3.2,

$$\bar{\nu}_s(y, b, c)\, \mathbb{1}_{X_{s-} \in \partial E} = \mathbb{1}_{X_{s-} \in \partial E}. \tag{3.12}$$

Moreover, (3.6) implies $\int_E \bar{\nu}(t, y, b, c)\, \tilde{p}(\{t\} \times dy\, db\, dc) = \int_E \bar{\nu}(t, y, I_{s-}, J_{s-})\, R(X_{s-}, J_{s-}, dy)\, \mathbb{1}_{X_{s-} \in \partial E} = \mathbb{1}_{X_{s-} \in \partial E}$, where the latter equality follows from (3.12). On the other hand, by (3.5) we have $\alpha_t = \mathbb{1}_{X_{s-} \in \partial E}$, and condition (3.11) follows. □

At this point, for every $T > 0$, let us set

$$
\begin{aligned}
\hat{Z}_t \ &:= \ \int_{E \times A_0 \times A_\Gamma} Z_t(y, b, c)\, \tilde{p}(\{t\} \times dy\, db\, dc), \qquad 0 \leq t \leq T, && (3.13)\\
||Z||^2_{\mathcal{G}^2_{\mathbf{x}, \mathbf{a_0}, \mathbf{a_\Gamma}}(\mathrm{q}; \mathbf{0}, \mathbf{T})} \ &:= \ \mathbb{E}^{x, a_0, a_\Gamma}\left[\int_{(0, T]} \int_{E \times A_0 \times A_\Gamma} \left| Z_t(y, b, c) - \hat{Z}_t\, \mathbb{1}_K(t) \right|^2 \tilde{p}(dt\, dy\, db\, dc) \right], && (3.14)
\end{aligned}
$$

and

$$
\begin{aligned}
\mathcal{G}^2_{\mathbf{x}, \mathbf{a_0}, \mathbf{a_\Gamma}}(\mathrm{q}; \mathbf{0}, \mathbf{T}) := \{ &\mathcal{P}_T \otimes \mathcal{E} \otimes \mathcal{A}_0 \otimes \mathcal{A}_\Gamma\text{-measurable maps } Z : \Omega \times [0,\, T] \times E \times A_0 \times A_\Gamma \to \mathbb{R}\\
&\text{such that } ||Z||^2_{\mathcal{G}^2_{\mathbf{x}, \mathbf{a_0}, \mathbf{a_\Gamma}}(\mathrm{q}; \mathbf{0}, \mathbf{T})} < \infty \}.
\end{aligned}
$$

We also set $\mathcal{G}^2_{\mathbf{x}, \mathbf{a}_0, \mathbf{a}_\Gamma, \mathbf{loc}}(\mathrm{q}) := \cap_{T > 0}\, \mathcal{G}^2_{\mathbf{x}, \mathbf{a_0}, \mathbf{a_\Gamma}}(\mathrm{q}; \mathbf{0}, \mathbf{T})$. We can now state the following result.

Proposition 3.2. *Let assumptions* **(HhλQR)** *and* **(H$\lambda_0\lambda_\Gamma$)** *hold. Then, for every* $(x, a_0, a_\Gamma) \in E \times A_0 \times A_\Gamma$ *and* $\boldsymbol{\nu} \in \mathcal{V}$*, under* $\mathbb{P}^{x,a_0,a_\Gamma}$ *the process* $(L_t^{\boldsymbol{\nu}})_{t\geq 0}$ *is a martingale. Moreover, for any* $T > 0$*,* $L_T^{\boldsymbol{\nu}}$ *is square integrable, and, for every* $H \in \mathcal{G}^{\boldsymbol{2}}_{\boldsymbol{x},\boldsymbol{a_0},\boldsymbol{a_\Gamma}}(\mathrm{q}; \mathbf{0}, \mathbf{T})$*, the process*

$$M_t^{\boldsymbol{\nu}} := \int_{(0,t]} \int_{E\times A_0\times A_\Gamma} H_s(y, b, c)\, q^{\boldsymbol{\nu}}(ds\, dy\, db\, dc), \quad t \in [0, T], \tag{3.15}$$

is a square integrable $\mathbb{P}^{x,a_0,a_\Gamma}_{\boldsymbol{\nu},T}$*-martingale on* $[0, T]$*. Finally, there exists a unique probability* $\mathbb{P}^{x,a_0,a_\Gamma}_{\boldsymbol{\nu}}$ *on* $(\Omega, \mathcal{F}_\infty)$*, under which* $\tilde{p}^{\boldsymbol{\nu}}$ *in* (3.8) *is the compensator of* p *in* (3.2) *on* $(0, \infty) \times E \times A_0 \times A_\Gamma$*, and such that, for any* $T > 0$*, the restriction of* $\mathbb{P}^{x,a_0,a_\Gamma}_{\boldsymbol{\nu}}$ *on* $(\Omega, \mathcal{F}_T)$ *is* $\mathbb{P}^{x,a_0,a_\Gamma}_{\boldsymbol{\nu},T}$*.*

Proof. The first part of the result is a consequence of Theorem 5.2 in [19]. The square integrability property of $L^{\boldsymbol{\nu}}$ can be proved arguing as in the proof of Lemma 3.2 in [4]. Moreover, Proposition 3.71-(a) in [20] implies that the stochastic integral $\int_{(t,T]} \int_{E\times A_0\times A_\Gamma} H_s(y, b, c)\, q(ds\, dy\, db\, dc)$ is well-defined, and, by Proposition 3.66 in [20], the process $M_t := \int_{(0,t]} \int_{E\times A_0\times A_\Gamma} H_s(y, b, c)\, q(ds\, dy\, db\, dc)$, $t \in [0, T]$, is a square integrable $\mathbb{P}^{x,a_0,a_\Gamma}_T$-martingale. Using the Burkholder-Davis-Gundy and Cauchy-Schwarz inequalities, together with the square integrability of L^{ν}_T, we see that (3.15) is a square integrable $\mathbb{P}^{x,a_0,a_\Gamma}_{T,\boldsymbol{\nu}}$-martingale. Finally, the probability measure $\mathbb{P}^{x,a_0,a_\Gamma}_{\boldsymbol{\nu}}$ on $(\Omega, \mathcal{F}_\infty)$ can be constructed as usual by means of the Kolmogorov extension theorem, see e.g. Theorem 1.1.10 in [28]. $\square$

For every $(x, a_0, a_\Gamma) \in E \times A_0 \times A_\Gamma$, the randomized optimal control problem consists in minimizing over all $\boldsymbol{\nu} \in \mathcal{V}$ the cost functional $J(x, a_0, a_\Gamma, \boldsymbol{\nu})$ defined in (1.4) (we denote by $\mathbb{E}^{x,a_0,a_\Gamma}_{\boldsymbol{\nu}}$ the expectation operator under $\mathbb{P}^{x,a_0,a_\Gamma}_{\boldsymbol{\nu}}$). The value function is given by

$$V^*(x, a_0, a_\Gamma) := \inf_{\boldsymbol{\nu}\in\mathcal{V}} J(x, a_0, a_\Gamma, \boldsymbol{\nu}). \tag{3.16}$$

4 Constrained BSDEs and probabilistic representation of V^*

In the present section we introduce a BSDE with two sign constraints on its martingale part, that will provide a probabilistic representation formula for the value function V^* in (3.16). The main novelty with respect to previous literature is that our BSDE is driven by a non quasi-left-continuous random measure. For such an equation, the proof of existence and uniqueness is a difficult task, and counterexamples can be obtained even in simple cases, see [10]. Only recently, some results in the unconstrained case have been obtained in this context, see [9], [8], [2]. In order to have an existence and uniqueness result for our BSDE, we have to impose the following additional assumption on the counting process p^* defined in (2.2).

(H0$'$) For any $(x, a_0, a_\Gamma) \in E \times A_0 \times A_\Gamma$, $t \in \mathbb{R}_+$, $\beta > 0$, there exists some $\bar{C}_\beta(t) < \infty$, only depending on t and β, such that $\mathbb{E}^{x,a_0,a_\Gamma}\left[(1 + p^*_t)\,(1 + \beta)^{p^*_t}\right] \leqslant \bar{C}_\beta(t)$.

Now, we introduce some notations. Firstly, for any $\beta \geq 0$, given the predictable increasing process A defined by (3.4), we denote by $\mathcal{E}^\beta$ the Doléans-Dade exponential of the process βA, given by

$$\mathcal{E}^\beta_t \;=\; e^{\beta\, A_t} \prod_{0<s\leq t} (1 + \beta\, \Delta A_s)\, e^{-\beta\,\Delta\, A_s}.$$

In particular, $d\mathcal{E}^\beta_s = \beta\, \mathcal{E}^\beta_{s-}\, dA_s$, $\mathcal{E}^\beta_s \geq 1$.

Remark 4.1. *Given a càdlàg process C, Itô's formula applied to $\mathcal{E}_s^{\beta}\,|C_s|^2$ yields*

$$
\begin{aligned}
d(\mathcal{E}_s^{\beta}\,|C_s|^2) &= 2\,\mathcal{E}_s^{\beta}\,C_{s-}\,dC_s + \mathcal{E}_s^{\beta}\,(\Delta C_s)^2 + \beta\,\mathcal{E}_{s-}^{\beta}\,|C_{s-}|^2\,dA_s\\
&= 2\,\mathcal{E}_s^{\beta}\,C_{s-}\,dC_s + \mathcal{E}_s^{\beta}\,(\Delta C_s)^2 + \beta\,\mathcal{E}_s^{\beta}\,(1+\beta\Delta A_s)^{-1}\,|C_{s-}|^2\,dA_s,
\end{aligned}
$$

where the last equality follows from the fact that $\mathcal{E}_{s-}^{\beta} = \mathcal{E}_s^{\beta}\,(1+\beta\,\Delta A_s)^{-1}$.

Following [20], we also define the random sets:

$$
D := \{(\omega,t) : \, p(\omega,\{t\}\times E\times A_0\times A_\Gamma) > 0\}, \tag{4.1}
$$

$$
K := \{(\omega,t) : \, \tilde p(\omega,\{t\}\times E\times A_0\times A_\Gamma) = 1\}, \tag{4.2}
$$

where p is the counting measure introduced in (3.2), and $\tilde p$ is the predictable random measure defined in (3.3). Notice that, by (3.5), $J = K = \{(\omega,t) : \Delta A_t(\omega) = 1\} = \{(\omega,t) : \, X_{t-}(\omega)\in\partial E\}$. For any stopping time τ, denote by $[[\tau]]$ the random set $\{(\omega,\tau(\omega))\}\subset\Omega\times[0,\,\infty]$. We have the following result, see e.g. Lemma 4.11 in [5].

Lemma 4.1. *Let D and K be the random sets in (4.1) and (4.2). Then there exists a sequence of totally inaccessible times $(T_n^i)_n$, with $[[T_n^i]]\cap[[T_m^i]] = \emptyset$, $n\neq m$, such that $D = K\cup(\cup_n[[T_n^i]])$ up to an evanescent set.*

Now, for any $(x,a_0,a_\Gamma)\in E\times A_0\times A_\Gamma$, and $\beta\geq 0$, we introduce the following sets.

- $\mathbf{L^2_{x,a_0,a_\Gamma}}(\mathcal{F}_\tau)$, the set of $\mathcal{F}_\tau$-measurable random variables ξ such that $\mathbb{E}\big[|\xi|^2\big] < \infty$; here $\tau\geqslant 0$ is an $\mathbb{F}$-stopping time.
- $\mathbf{S^\infty}$ the set of real-valued càdlàg adapted processes $Y = (Y_t)_{t\geqslant 0}$ which are uniformly bounded.
- $\mathbf{L^{2,\beta}_{x,a_0,a_\Gamma}}(\mathrm{p}^*;\mathbf{0},\mathbf{T})$, $T>0$, the set of real-valued progressive processes $Y = (Y_t)_{0\leqslant t\leqslant T}$ such that
$$
||Y||^2_{\mathbf{L^{2,\beta}_{x,a_0,a_\Gamma}}(\mathrm{p}^*;\mathbf{0},\mathbf{T})} := \mathbb{E}^{x,a_0,a_\Gamma}\left[\int_{(0,\,T]}\mathcal{E}_t^{\beta}\,|Y_{t-}|^2\,dA_t\right] < \infty.
$$
- $\mathcal{G}^{\mathbf{2},\beta}_{\mathbf{x},\mathbf{a_0},\mathbf{a_\Gamma}}(\mathrm{q};\mathbf{0},\mathbf{T})$, $T>0$, the set of $\mathcal{P}_T\otimes\mathcal{E}\otimes\mathcal{A}_0\otimes\mathcal{A}_\Gamma$-measurable maps $Z:\Omega\times[0,\,T]\times E\times A_0\times A_\Gamma\to\mathbb{R}$ such that
$$
||Z||^2_{\mathcal{G}^{\mathbf{2},\beta}_{\mathbf{x},\mathbf{a_0},\mathbf{a_\Gamma}}(\mathrm{q};\mathbf{0},\mathbf{T})} := \;\mathbb{E}^{x,a_0,a_\Gamma}\left[\int_{(0,\,T]}\mathcal{E}_t^{\beta}\int_{E\times A_0\times A_\Gamma}\big|Z_t(y,b,c) - \hat Z_t\,\mathbb{1}_K(t)\big|^2\,\tilde p(dt\,dy\,db\,dc)\right] \tag{4.3}
$$
is finite. We also define $\mathcal{G}^{\mathbf{2},\beta}_{\mathbf{x},\mathbf{a_0},\mathbf{a_\Gamma},\mathbf{loc}}(\mathrm{q}) := \cap_{T>0}\,\mathcal{G}^{\mathbf{2},\beta}_{\mathbf{x},\mathbf{a_0},\mathbf{a_\Gamma}}(\mathrm{q};\mathbf{0},\mathbf{T})$.
- $\mathbf{L^2}(\lambda_0)$ (resp. $\mathbf{L^2}(\lambda_\Gamma)$), the set of $\mathcal{A}_0$-measurable maps $\psi : A_0\to\mathbb{R}$ (resp. $\mathcal{A}_\Gamma$-measurable maps $\psi : A_\Gamma\to\mathbb{R}$) such that
$$
|\psi|^2_{\mathbf{L^2}(\lambda_0)} := \int_{A_0}|\psi(b)|^2\,\lambda_0(db) < \infty \quad \Big(\text{resp.}\quad |\psi|^2_{\mathbf{L^2}(\lambda_\Gamma)} := \int_{A_\Gamma}|\psi(c)|^2\,\lambda_\Gamma(dc) < \infty\Big).
$$
- $\mathbf{L^2}(\phi_{\omega,t}) = \mathbf{L}^2(E\times A_0\times A_\Gamma,\mathcal{E}\otimes\mathcal{A}_0\otimes\mathcal{A}_\Gamma,\phi_{\omega,t}(dy\,db\,dc))$, for any $(\omega,t)\in\Omega\times\mathbb{R}_+$, the set of $\mathcal{E}\otimes\mathcal{A}_0\otimes\mathcal{A}_\Gamma$-measurable maps $\zeta : E\times A_0\times A_\Gamma\to\mathbb{R}$ such that
$$
|\zeta|^2_{\mathbf{L^2}(\phi_{\omega,t})} := \int_{E\times A_0\times A_\Gamma}|\zeta(y,b,c)|^2\,\phi_{\omega,t}(dy\,db\,dc) < \infty,
$$
where $\phi_{\omega,t}(dy\,db\,dc)$ is the random measure introduced in (3.7).

- $\mathbf{K^2_{x,a_0,a_\Gamma}(0,T)}$, $T>0$, the set of nondecreasing càdlàg predictable processes $K=(K_t)_{0\leqslant t\leqslant T}$ such that $K_0=0$ and $\mathbb{E}\big[|K_T|^2\big]<\infty$. We also define $\mathbf{K^2_{x,a_0,a_\Gamma,loc}} := \cap_{T>0}\mathbf{K^2_{x,a_0,a_\Gamma}(0,T)}$.

Remark 4.2. *The norm in (4.3) is equivalent to* $\mathbb{E}^{x,a_0,a_\Gamma}\big[\sum_{s\in(0,T]}|\int_{E\times A_0\times A_\Gamma} H_s(y,b,c)\,q(\{s\}\times dy\,db\,dc)|^2\big]$.

We aim at studying the following family of BSDEs with partially nonnegative jumps over an infinite horizon, parametrized by (x,a_0,a_Γ): $\mathbb{P}^{x,a_0,a_\Gamma}$-a.s.,

$$
\begin{aligned}
Y_s^{x,a_0,a_\Gamma} &= Y_T^{x,a_0,a_\Gamma} - \delta\int_{(s,T]} Y_r^{x,a_0,a_\Gamma}\,dr + \int_{(s,T]} f(X_r,I_r)\,dr + \int_{(s,T]} c(X_{r-},J_{r-})\,dp_r^* \\
&- \int_{(s,T]}\int_{A_0} Z_r^{x,a_0,a_\Gamma}(X_r,\,b,\,J_r)\,\lambda_0(db)\,dr - \int_{(s,T]}\int_{A_\Gamma} Z_r^{x,a_0,a_\Gamma}(X_r,\,I_r,\,c)\,\lambda_\Gamma(dc)\,dr \\
&- \big(K_T^{x,a_0,a_\Gamma}-K_s^{x,a_0,a_\Gamma}\big) - \int_{(s,T]}\int_{E\times A_0\times A_\Gamma} Z_r^{x,a_0,a_\Gamma}(y,\,b,\,c)\,q(dr\,dy\,db\,dc), \quad 0\leqslant s\leqslant T<\infty, \quad (4.4)
\end{aligned}
$$

with the constraints

$$
\begin{aligned}
Z_s^{x,a_0,a_\Gamma}(X_{s-},b,J_{s-}) &\geqslant 0, \quad d\mathbb{P}^{x,a_0,a_\Gamma}\lambda_0(db)\text{ -a.e. on } [0,\infty)\times\Omega\times A_0, && (4.5)\\
Z_s^{x,a_0,a_\Gamma}(X_{s-},I_{s-},c) &\geqslant 0 \quad d\mathbb{P}^{x,a_0,a_\Gamma}\lambda_\Gamma(dc)\text{ -a.e. on } [0,\infty)\times\Omega\times A_\Gamma. && (4.6)
\end{aligned}
$$

We look for a *maximal solution* $(Y^{x,a_0,a_\Gamma},Z^{x,a_0,a_\Gamma},K^{x,a_0,a_\Gamma})\in\mathbf{S}^\infty\times\mathcal{G}^{\mathbf{2}}_{\mathbf{x,a_0,a_\Gamma,loc}}(\mathrm{q})\times\mathbf{K^2_{x,a_0,a_\Gamma,loc}}$ to (4.4)-(4.5)-(4.6), in the sense that for any other solution $(\tilde Y,\tilde Z,\tilde K)\in\mathbf{S}^\infty\times\mathcal{G}^{\mathbf{2}}_{\mathbf{x,a_0,a_\Gamma,loc}}(\mathrm{q})\times\mathbf{K^2_{x,a_0,a_\Gamma,loc}}$ to (4.4)-(4.5)-(4.6), we have $Y_t^{x,a_0,a_\Gamma}\geqslant\tilde Y_t$, $\mathbb{P}^{x,a_0,a_\Gamma}$-a.s., for all $t\geqslant 0$.

We start by considering, for every $T>0$, the family of penalized BSDEs on $[0,T]$ with zero terminal condition at time T, parametrized by the integer $n\geqslant 1$: $\mathbb{P}^{x,a_0,a_\Gamma}$-a.s.

$$
\begin{aligned}
Y_s^{T,n,x,a_0,a_\Gamma} &= -\delta\int_{(s,T]} Y_r^{T,n,x,a_0,a_\Gamma}\,dr + \int_{(s,T]} f(X_r,I_r)\,dr + \int_{(s,T]} c(X_{r-},J_{r-})\,dp_r^* \\
&- n\int_{(s,T]}\int_{A_0} [Z_r^{T,n,x,a_0,a_\Gamma}(X_r,b,J_r)]^-\,\lambda_0(db)\,dr - \int_{(s,T]}\int_{A_0} Z_r^{T,n,x,a_0,a_\Gamma}(X_r,b,J_r)\,\lambda_0(db)\,dr \\
&- n\int_{(s,T]}\int_{A_\Gamma} [Z_r^{T,n,x,a_0,a_\Gamma}(X_r,I_r,c)]^-\,\lambda_\Gamma(dc)\,dr - \int_{(s,T]}\int_{A_\Gamma} Z_r^{T,n,x,a_0,a_\Gamma}(X_r,I_r,c)\,\lambda_\Gamma(dc)\,dr \\
&- \int_{(s,T]}\int_{E\times A_0\times A_\Gamma} Z_r^{T,n,x,a_0,a_\Gamma}(y,b,c)\,q(dr\,dy\,db\,dc), \quad 0\leqslant s\leqslant T, \qquad (4.7)
\end{aligned}
$$

where $[z]^-=\max(-z,0)$ is the negative part of z. Our aim is to exploit equation (4.7) in order to construct the maximal solution $(Y^{x,a_0,a_\Gamma},Z^{x,a_0,a_\Gamma},K^{x,a_0,a_\Gamma})$, studying the limit of $(Y^{T,n},Z^{T,n})=(Y^{T,n,x,a_0,a_\Gamma},Z^{T,n,x,a_0,a_\Gamma})$ firstly as $T\to\infty$, and then as $n\to\infty$.

Remark 4.3. *The key idea of the randomization method consists in approximating the original control problem (1.2), with policies $\boldsymbol{\alpha}\in\mathcal{A}_{ad}$, by the randomized control problem (3.16), where the strategies $\boldsymbol{\alpha}$ are substituted by the pairs of piecewise constant jump processes (I,J) with controlled intensity measures $(\nu^0(b)\lambda_0(db),\nu^\Gamma(c)\lambda_\Gamma(dc))$, with (ν^0,ν^Γ) bounded functions. In particular, equation (4.7) is related to the randomized control problem whose controls (ν^0,ν^Γ) are bounded by n: roughly speaking, as soon as I (resp. J) is no more a good approximation of α^0 (resp. α^Γ), a jump of intensity measure $n\lambda_0(db)$ (resp. $n\lambda_\Gamma(dc)$) is forced to occur. The initial control problem is then recovered by sending to infinity the intensity rate n (in other words, by exploiting the density property of the class of piecewise policies in the class of the original admissible controls $\mathcal{A}_{ad}$). This gives an intuitive explanation of the form of the constrained BSDE (4.4)-(4.5)-(4.6).*

Before analyzing the asymptotic behavior of $(Y^{T,n}, Z^{T,n})$, we need to prove the existence of a unique solution to equation (4.7). This is given by the following result, and it is mainly a consequence of Theorem 4.1 in [2].

Theorem 4.2. *Under assumptions* $\mathbf{(Hh\lambda QR)}$, $\mathbf{(H}0\mathbf{)}$, $\mathbf{(H}0'\mathbf{)}$, $\mathbf{(H}\lambda_0\lambda_\Gamma\mathbf{)}$ *and* $\mathbf{(Hfc)}$, *for every* $(x,a_0,a_\Gamma) \in E\times A_0\times A_\Gamma$, $T>0$, $n\in\mathbb{N}$, *there exists of a unique solution* $(Y^{T,n,x,a_0,a_\Gamma}, Z^{T,n,x,a_0,a_\Gamma}) \in \mathbf{L}^{\mathbf{2},\beta}_{\boldsymbol{x},\boldsymbol{a}_0,\boldsymbol{a}_\Gamma}(\mathrm{p}^*;\mathbf{0},\mathbf{T}) \times \mathcal{G}^{\mathbf{2},\beta}_{\boldsymbol{x},\boldsymbol{a}_0,\boldsymbol{a}_\Gamma}(\mathrm{q};\mathbf{0},\mathbf{T})$ *to equation* (4.7), *for* $0<\beta=\beta(n)$ *large enough.*

Proof. Notice that equation (4.7) can be rewritten as: $\mathbb{P}^{x,a_0,a_\Gamma}$-a.s.,

$$\begin{aligned} Y_s^{T,n,x,a_0,a_\Gamma} &= \int_{(s,\,T]} \tilde f^n(r-s, X_{r-}, I_{r-},\, J_{r-},\, Z_r^{T,n,x,a_0,a_\Gamma})\, dA_r \\ &\quad - \int_{(s,\,T]} e^{-\delta(r-s)} \int_{E\times A_0\times A_\Gamma} Z_r^{T,n,x,a_0,a_\Gamma}(y,b,c)\, q(dr\,dy\,db\,dc), \quad s\in[0,\,T], \end{aligned} \tag{4.8}$$

where

$$\tilde f^n(t,x,a_0,a_\Gamma,\zeta) := e^{-\delta\, t}\, f^n(x,a_0,\zeta(x,\cdot,a_\Gamma),\zeta(x,a_0,\cdot))\, \mathbb{1}_{x\in E} + e^{-\delta\, t}\, c(x,a_\Gamma)\, \mathbb{1}_{x\in\partial E}, \tag{4.9}$$

with $f^n(x,a_0,\psi,\varphi) := f(x,a_0) - \int_{A_0}\{n\,[\psi(b)]^- + \psi(b)\}\,\lambda_0(db) - \int_{A_\Gamma}\{n\,[\varphi(c)]^- + \varphi(c)\}\,\lambda_\Gamma(dc)$. Under assumptions $\mathbf{(H}\lambda_0\lambda_\Gamma\mathbf{)}$ and $\mathbf{(Hfc)}$, there exists a constant L_n, depending only on n, such that

$$|f^n(x,a_0,\psi,\varphi) - f^n(x,a_0,\psi',\varphi')| \leqslant L_n\left(|\psi-\psi'|_{\mathbf{L^2}(\lambda_0)} + |\varphi-\varphi'|_{\mathbf{L^2}(\lambda_\Gamma)}\right), \tag{4.10}$$

for every $(x,a_0)\in E\times A_0$, $\psi,\,\psi'\in\mathbf{L^2}(\lambda_0)$, $\varphi,\,\varphi'\in\mathbf{L^2}(\lambda_\Gamma)$. Then, one can easily show that

$$\begin{aligned} &|\tilde f^n(t,X_{t-}(\omega),I_{t-}(\omega),J_{t-}(\omega),\zeta') - \tilde f^n(t,X_{t-}(\omega),I_{t-}(\omega),J_{t-}(\omega),\zeta)| \leq \\ &2\,L_n\bigg(\int_{E\times A_0\times A_\Gamma}\left|\tilde\zeta(y,b,c) - \Delta A_t(\omega)\int_{E\times A_0\times A_\Gamma}\tilde\zeta(\bar y,\bar b,\bar c)\,\phi_{\omega,t}(d\bar y\,d\bar b\,d\bar c)\right|^2\phi_{\omega,t}(dy\,db\,dc)\bigg)^{1/2}, \end{aligned}$$

for all $(\omega,t)\in\Omega\times[0,T]$, $\zeta,\zeta'\in\mathbf{L^2}(\phi_{\omega,t})$, with $\tilde\zeta=\zeta-\zeta'$. Finally, setting $c_1(T) = (M_f^2\vee M_c^2)||\tilde\lambda||_\infty\,T\,e^{\beta||\tilde\lambda||_\infty T}$, $c_2=(M_f^2\vee M_c^2)$, we have

$$\begin{aligned} &\mathbb{E}^{x,a_0,a_\Gamma}\left[(1+\sum_{0<t\leq T}|\Delta A_t|^2)\int_{(0,\,T]}\mathcal{E}_t^\beta\left|\tilde f^n(t,X_{t-},I_{t-},J_{t-},0)\right|^2 dA_t\right] \\ &\leq c_1(T)(1+\mathbb{E}^{x,a_0,a_\Gamma}[p_T^*]) + c_2\,\mathbb{E}^{x,a_0,a_\Gamma}[(1+p_T^*)\,(1+\beta)^{p_T^*}], \end{aligned}$$

which is finite by (2.6) and hypothesis $\mathbf{(H}0'\mathbf{)}$. We are therefore in condition to apply Theorem 4.1 in [2]. In particular, setting

$$\beta_0^n := \frac{2\,(L_n+\varepsilon)^2}{1-\varepsilon}, \quad \varepsilon\in(0,1),$$

we deduce that there exists of a unique solution $(Y^{T,n,x,a_0,a_\Gamma}, Z^{T,n,x,a_0,a_\Gamma}) \in \mathbf{L}^{\mathbf{2},\beta}_{\mathbf{x},\mathbf{a}_0,\mathbf{a}_\Gamma}(\mathrm{p}^*;\mathbf{0},\mathbf{T}) \times \mathcal{G}^{\mathbf{2},\beta}_{\mathbf{x},\mathbf{a}_0,\mathbf{a}_\Gamma}(\mathrm{q};\mathbf{0},\mathbf{T})$ to equation (4.7) for $\beta\geq\beta_0^n$. Notice that the Lipschitz constant of $\tilde f^n$ with respect to Y, that we will denote L_y, is identically zero. So, in particular, the technical assumption of Theorem 4.1 in [2], that is the existence of $\varepsilon\in(0,1)$ such that (in our framework, $\Delta A_t = \mathbb{1}_{X_{t-}\in\partial E}$)

$$2\,L_y^2\,|\Delta A_t|^2 \ \leq\ 1-\varepsilon, \qquad \mathbb{P}\text{-a.s., } \forall\, t\in[0,T],$$

here it is automatically satisfied. $\square$

Remark 4.4. *The existence result in Theorem 4.1 in [2] relies on a martingale representation theorem for marked point processes (see e.g. Theorem 5.4. in [19]), which holds for any càdlàg martingale. Notice that, being the filtration $\mathbb{F}$ right-continuous, and since any $\mathbb{F}$-martingale always admits a right-continuous adapted modification (see e.g. Corollary 2.48 in [18]), the martingale representation theorem can be applied without asking additional completeness assumptions on $\mathbb{F}$.*

We can now state the main result of this section.

Theorem 4.3. *Under assumptions* **(HhλQR)**, **(H0)**, **(H0$'$)**, **(H$\lambda_0\lambda_\Gamma$)** *and* **(Hfc)**, *for every $(x, a_0, a_\Gamma) \in E \times A_0 \times A_\Gamma$, there exists a unique maximal solution $(Y^{x,a_0,a_\Gamma}, Z^{x,a_0,a_\Gamma}, K^{x,a_0,a_\Gamma}) \in \mathbf{S}^\infty \times \mathcal{G}^{\mathbf{2}}_{\boldsymbol{x,a_0,a_\Gamma,\mathrm{loc}}}(\mathrm{q}) \times \mathbf{K}^{\mathbf{2}}_{\boldsymbol{x,a_0,a_\Gamma,\mathrm{loc}}}$ to the constrained BSDE (4.4)-(4.5)-(4.6). Moreover, Y^{x,a_0,a_Γ} has the explicit representation:*

$$Y_s^{x,a_0,a_\Gamma} = \operatorname*{ess\,inf}_{\boldsymbol{\nu} \in \mathcal{V}} \mathbb{E}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}\left[\int_{(s,\infty)} e^{-\delta(r-s)} f(X_r, I_r)\, dr + \int_{(s,\infty)} e^{-\delta(r-s)} c(X_{r-}, J_{r-})\, dp_r^* \Big| \mathcal{F}_s\right], \quad (4.11)$$

for all $s \geqslant 0$. In particular, setting $s = 0$, we have the following representation formula for the value function of the randomized control problem:

$$V^*(x, a_0, a_\Gamma) = Y_0^{x,a_0,a_\Gamma}, \quad (x, a_0, a_\Gamma) \in E \times A_0 \times A_\Gamma. \quad (4.12)$$

Proof. We know from Theorem 4.2 that, for every $(x, a_0, a_\Gamma) \in E \times A_0 \times A_\Gamma$, $T > 0$, $n \in \mathbb{N}$, there exists of a unique solution $(Y^{T,n,x,a_0,a_\Gamma}, Z^{T,n,x,a_0,a_\Gamma}) \in \mathbf{L}^{\mathbf{2},\beta}_{\mathbf{x,a_0,a_\Gamma}}(\mathrm{p}^*; \mathbf{0}, \mathbf{T}) \times \mathcal{G}^{\mathbf{2},\beta}_{\mathbf{x,a_0,a_\Gamma}}(\mathrm{q}; \mathbf{0}, \mathbf{T})$ to equation (4.7), for β large enough. Our aim is now to construct the maximal solution $(Y^{x,a_0,a_\Gamma}, Z^{x,a_0,a_\Gamma}, K^{x,a_0,a_\Gamma})$, as a suitable limit of $(Y^{T,n,x,a_0,a_\Gamma}, Z^{T,n,x,a_0,a_\Gamma})$ firstly as $T \to \infty$, and then as $n \to \infty$.

We split the rest of the proof into five steps.

Step I. *Convergence of $(Y^{T,n,x,a_0,a_\Gamma})_T$.* We begin by proving the following uniform estimate: $\mathbb{P}^{x,a_0,a_\Gamma}$-a.s.,

$$0 \leq Y_s^{T,n,x,a_0,a_\Gamma} \leqslant \frac{M_f}{\delta} + C^* M_c, \quad \forall\, s \in [0, T], \quad (4.13)$$

where C^* is the constant defined in (2.7). To this end, for any $\boldsymbol{\nu} \in \mathcal{V}^n$ (the set of control maps $\boldsymbol{\nu} = (\nu^0, \nu^\Gamma)$, with both ν^0 and ν^Γ bounded by n), let us introduce the compensated martingale measure $q^{\boldsymbol{\nu}}(ds\, dy\, db\, dc) = q(ds\, dy\, db\, dc) - [(\nu_s^0(b) - 1)\, d_1(s, y, b, c) + (\nu_s^\Gamma(c) - 1)\, d_2(s, y, b, c)]\, \tilde{p}(ds\, dy\, db\, dc)$ under $\mathbb{P}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}$, with d_1 and d_2 given by respectively by (3.9) and (3.10). Taking the expectation in (4.8) under $\mathbb{P}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}$, conditional to $\mathcal{F}_s$, and since $Z^{T,n}$ is in $\mathcal{G}^{\mathbf{2},\beta}_{\mathbf{x,a_0,a_\Gamma}}(\mathrm{q}; \mathbf{0}, \mathbf{T})$, from Proposition 3.2 we get that, $\mathbb{P}^{x,a_0,a_\Gamma}$-a.s.,

$$\begin{aligned}
Y_s^{T,n} = &-\mathbb{E}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}\left[\int_{(s,\,T]}\int_{A_0} e^{-\delta\,(r-s)} \{n[Z_r^{T,n}(X_r, b, J_r)]^- + \nu_r^0(b)\, Z_r^{T,n}(X_r, b, J_r)\}\, \lambda_0(db)\, dr \Big| \mathcal{F}_s\right] \\
&- \mathbb{E}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}\left[\int_{(s,\,T]}\int_{A_\Gamma} e^{-\delta\,(r-s)} \{n[Z_r^{T,n}(X_r, I_r, c)]^- + \nu_r^\Gamma(c)\, Z_r^{T,n}(X_r, I_r, c)\}\, \lambda_\Gamma(dc)\, dr \Big| \mathcal{F}_s\right] \\
&+ \mathbb{E}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}\left[\int_{(s,\,T]} e^{-\delta\,(r-s)} f(X_r, I_r)\, dr + \int_{(s,\,T]} e^{-\delta\,(r-s)} c(X_{r-}, J_r)\, dp_r^* \Big| \mathcal{F}_s\right] \; s \in [0, T]. \quad (4.14)
\end{aligned}$$

The right-hand side of estimate (4.13) directly follows from the elementary numerical inequality $n[z]^- + \nu z \geqslant 0$ for all $z \in \mathbb{R}$, $\nu \in (0, n]$, and the boundedness of f and c.

Let us now prove that $Y^{T,n}$ is nonnegative. To this end, for $\varepsilon \in (0, 1)$, let us consider the process $\boldsymbol{\nu}^\varepsilon := (\nu^{0,\varepsilon}, \nu^{\Gamma,\varepsilon}) \in \mathcal{V}^n$ defined by:

$$\nu_s^{0,\varepsilon}(b) = n\, \mathbb{1}_{\{Z_s^{T,n}(X_{s-},b,J_{s-}) \leqslant 0\}} + \varepsilon\, \mathbb{1}_{\{0 < Z_s^{T,n}(X_{s-},b,J_{s-}) < 1\}} + \varepsilon\, Z_s^{T,n}(X_{s-}, b, J_{s-})^{-1}\, \mathbb{1}_{\{Z_s^n(X_{s-},b,J_{s-}) \geqslant 1\}}, \quad (4.15)$$

$$\nu_s^{\Gamma,\varepsilon}(c) = n\,\mathbb{1}_{\{Z_s^{T,n}(X_{s-},I_{s-},c)\leqslant 0\}} + \varepsilon\,\mathbb{1}_{\{0<Z_s^{T,n}(X_{s-},I_{s-},c)<1\}} + \varepsilon\, Z_s^{T,n}(X_{s-},I_{s-},c)^{-1}\,\mathbb{1}_{\{Z_s^{T,n}(X_{s-},I_{s-},c)\geqslant 1\}}. \tag{4.16}$$

By construction, we have

$$\begin{aligned} n[Z_s^{T,n}(X_{s-},b,J_{s-})]^- + \nu_s^{0,\varepsilon}(b)\, Z_s^n(X_{s-},b,J_{s-}) \leqslant \varepsilon, &\quad s\geqslant 0,\, b\in A_0,\\ n[Z_s^{T,n}(X_{s-},I_{s-},c)]^- + \nu_s^{\Gamma,\varepsilon}(c)\, Z_s^n(X_{s-},I_{s-},c) \leqslant \varepsilon, &\quad s\geqslant 0,\, c\in A_\Gamma. \end{aligned}$$

Thus for the choice of $\boldsymbol{\nu}=\boldsymbol{\nu}^\varepsilon$ in (4.14), denoting $C_\Lambda := \lambda_0(A_0)+\lambda_\Gamma(A_\Gamma)$, we obtain

$$Y_s^{T,n} \geqslant -\varepsilon\,\frac{1-e^{-\delta(T-s)}}{\delta}\,C_\Lambda + \mathbb{E}_{\boldsymbol{\nu}^\varepsilon}^{x,a_0,a_\Gamma}\left[\int_{(s,\,T]} e^{-\delta\,(r-s)}\,f(X_r,I_r)\,dr + \int_{(s,\,T]} e^{-\delta\,(r-s)}\,c(X_{r-},J_r)\,dp_r^*\Big|\mathcal{F}_s\right].$$

Since f, c are positive, it follows that

$$Y_s^{T,n} \geqslant \operatorname*{ess\,inf}_{\boldsymbol{\nu}\in\mathcal{V}^n} \mathbb{E}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}\left[\int_{(s,\infty)} e^{-\delta\,(r-s)}\,f(X_r,I_r)\,dr + \int_{(s,\infty)} e^{-\delta\,(r-s)}\,c(X_{r-},J_{r-})\,dp_r^*\Big|\mathcal{F}_s\right] - \frac{\varepsilon}{\delta}\,C_\Lambda. \tag{4.17}$$

We conclude by the arbitrariness of ε.

Now, let us study the convergence of $(Y^{T,n})_T$. Take $T,T'>0$, with $T<T'$, and $s\in[0,\,T]$. Then

$$|Y_s^{T',n}-Y_s^{T,n}|^2 \leqslant e^{-2\,\delta\,(T-s)}\,\mathbb{E}_{\boldsymbol{\nu}^\varepsilon}^{x,a_0,a_\Gamma}[|Y_T^{T',n}-Y_T^{T,n}|^2|\mathcal{F}_s] \overset{T,\,T'\to\infty}{\longrightarrow} 0, \tag{4.18}$$

where the convergence result follows from (4.13). Let us now consider the sequence of real-valued càdlàg adapted processes $(Y^{T,n})_T$. It follows from (4.18) that, for any $t\geqslant 0$, the sequence $(Y_t^{T,n}(\omega))_T$ is Cauchy for almost every ω, so that it converges $\mathbb{P}^{x,a}$-a.s. to some $\mathcal{F}_t$-measurable random variable Y_t^n, which is bounded by the right-hand side of (4.13). Moreover, using again (4.18) and (4.13), we see that, for any $0\leqslant S<T\wedge T'$, with $T,T'>0$, we have

$$\sup_{0\leqslant t\leqslant S}|Y_t^{T',n}-Y_t^{T,n}| \leqslant e^{-\delta\,(T\wedge T'-S)}\left(\frac{M_f}{\delta}+C^*\,M_c\right) \overset{T,T'\to\infty}{\longrightarrow} 0. \tag{4.19}$$

Since each $Y^{T,n}$ is a càdlàg process, it follows that Y^n is càdlàg, as well. Finally, from estimate (4.13) we see that Y^n is uniformly bounded and therefore belongs to $\mathbf{S}^\infty$.

Step II. *Convergence of* $(Z^{T,n,x,a_0,a_\Gamma})_T$. Let $S,\,T,\,T'>0$, with $S<T<T'$. Then, applying Itó's formula to $e^{-2\,\delta\,t}|Y_t^{T',n}-Y_t^{T,n}|^2$ between 0 and S, and taking the expectation, we get

$$\begin{aligned} &\frac{1}{2}\mathbb{E}^{x,a_0,a_\Gamma}\left[\sum_{r\in(0,\,S]} e^{-2\,\delta\,r}\left|\int_{E\times A_0\times A_\Gamma}(Z_r^{T',n}(y,b,c)-Z_r^{T,n}(y,b,c))\,q(\{r\}\times dy\,db\,dc)\right|^2\right]\\ &\leqslant e^{-2\,\delta\,S}\mathbb{E}^{x,a_0,a_\Gamma}[|Y_S^{T',n}-Y_S^{T,n}|^2] + 4(n^2+1)\,C_\Lambda\,\mathbb{E}^{x,a_0,a_\Gamma}\left[\int_0^S e^{-2\,\delta\,r}\,|Y_r^{T',n}-Y_r^{T,n}|^2\,dr\right] \overset{T,T'\to\infty}{\longrightarrow} 0, \end{aligned}$$

where the convergence to zero follows from estimate (4.19). Then, for any $S>0$, we see that $(Z_{|[0,\,S]}^{T,n})_{T>S}$ is a Cauchy sequence in the Hilbert space $\mathcal{G}^{\mathbf{2}}_{\mathbf{x},\mathbf{a_0},\mathbf{a_\Gamma}}(\mathrm{q};\mathbf{0},\,\mathbf{S})$. Therefore, we deduce the existence of $Z^n\in\mathcal{G}^{\mathbf{2}}_{\mathbf{x},\mathbf{a_0},\mathbf{a_\Gamma},\mathbf{loc}}(\mathrm{q})$ such that $(Z_{|[0,\,S]}^{T,n})_{T>S}$ converges to $Z_{|[0,\,S]}^n$ in $\mathcal{G}^{\mathbf{2}}_{\mathbf{x},\mathbf{a_0},\mathbf{a_\Gamma}}(\mathrm{q};\mathbf{0},\,\mathbf{S})$. Hence, from the convergence of $(Y^{T,n})_T$ and $(Z^{T,n})_n$, we can pass to the limit in equation (4.7) as $T\to\infty$, from which we deduce that (Y^n,Z^n) (also denoted as $(Y^{n,x,a_0,a_\Gamma},Z^{n,x,a_0,a_\Gamma})$) solves the following penalized BSDE on infinite horizon: $\mathbb{P}^{x,a_0,a_\Gamma}$-a.s.,

$$Y_s^{n,x,a_0,a_\Gamma} = Y_T^{n,x,a_0,a_\Gamma} - \delta\int_{(s,T]} Y_r^{n,x,a_0,a_\Gamma}\,dr + \int_{(s,\,T]} f(X_r,I_r)\,dr + \int_{(s,\,T]} c(X_{r-},J_{r-})\,dp_r^*$$

$$
\begin{aligned}
&- \int_{(s,\,T]} \int_{A_0} Z_r^{n,x,a_0,a_\Gamma}(X_r, b, J_r)\, \lambda_0(db)\, dr - \int_{(s,\,T]} \int_{A_\Gamma} Z_r^{n,x,a_0,a_\Gamma}(X_r, I_r, c)\, \lambda_\Gamma(dc)\, dr \\
&- \big(K_T^{n,x,a_0,a_\Gamma} - K_s^{n,x,a_0,a_\Gamma}\big) - \int_{(s,\,T]} \int_{E\times A_0\times A_\Gamma} Z_r^{n,x,a_0,a_\Gamma}(y,b,c)\, q(dr\, dy\, db\, dc), \qquad (4.20)
\end{aligned}
$$

for all $0 \leqslant s \leqslant T < \infty$, where

$$
K_s^{n,x,a_0,a_\Gamma} := n \int_0^s \bigg(\int_{A_0} [Z_r^{n,x,a_0,a_\Gamma}(X_r, b, J_r)]^- \, \lambda_0(db) + \int_{A_\Gamma} [Z_r^{n,x,a_0,a_\Gamma}(X_r, I_r, c)]^- \, \lambda_\Gamma(dc) \bigg)\, dr. \qquad (4.21)
$$

Notice that equation (4.20) can also be written as follows:

$$
\begin{aligned}
Y_s^{n,x,a_0,a_\Gamma} &= Y_T^{n,x,a_0,a_\Gamma}\, e^{-\delta(T-s)} + \int_{(s,\,T]} \tilde f^n(r-s, X_{r-}, I_{r-},\, J_{r-},\, Z_r^{n,x,a_0,a_\Gamma})\, dA_r \\
&- \int_{(s,\,T]} e^{-\delta(r-s)} \int_{E\times A_0\times A_\Gamma} Z_r^{n,x,a_0,a_\Gamma}(y,b,c)\, q(dr\, dy\, db\, dc), \quad s \in [0,\, T], \qquad (4.22)
\end{aligned}
$$

where $\tilde f^n$ is the deterministic function defined in (4.9).

Step III. *Representation formula for Y^{n,x,a_0,a_Γ}.* Our aim is to prove the following representation formula:

$$
Y_s^{n,x,a_0,a_\Gamma} = \operatorname*{ess\,inf}_{\boldsymbol{\nu}\in\mathcal{V}^n} \mathbb{E}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}\bigg[\int_{(s,\infty)} e^{-\delta\,(r-s)} f(X_r, I_r)\, dr + \int_{(s,\infty)} e^{-\delta\,(r-s)} c(X_{r-}, J_{r-})\, dp_r^* \bigg| \mathcal{F}_s \bigg], \qquad (4.23)
$$

for all $s \geq 0$. As at the beginning of **Step I**, for any $\boldsymbol{\nu} \in \mathcal{V}^n$, we consider the compensated martingale measure $q^{\boldsymbol{\nu}}(ds\, dy\, db\, dc)$ under $\mathbb{P}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}$. We take the expectation in (4.22) under $\mathbb{P}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}$, conditional to $\mathcal{F}_s$. For every $T > 0$, recalling that Z^n is in $\mathcal{G}^{\mathbf{2}}_{\mathbf{x},\mathbf{a_0},\mathbf{a_\Gamma}}(\mathrm{q}; \mathbf{0}, \mathbf{T})$, from Proposition 3.2 we get

$$
\begin{aligned}
Y_s^n = &- \mathbb{E}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}\bigg[\int_{(s,\,T]} \int_{A_0} e^{-\delta\,(r-s)} \{ n[Z_r^n(X_r, b, J_r)]^- + \nu_r^0(b)\, Z_r^n(X_r, b, J_r)\}\, \lambda_0(db)\, dr \bigg| \mathcal{F}_s \bigg] \qquad (4.24) \\
&- \mathbb{E}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}\bigg[\int_{(s,\,T]} \int_{A_\Gamma} e^{-\delta\,(r-s)} \{ n[Z_r^n(X_r, I_r, c)]^- + \nu_r^\Gamma(c)\, Z_r^n(X_r, I_r, c)\}\, \lambda_\Gamma(dc)\, dr \bigg| \mathcal{F}_s \bigg] \\
&+ \mathbb{E}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}\bigg[e^{-\delta\,(T-s)}\, Y_T^n + \int_{(s,\,T]} e^{-\delta\,(r-s)} f(X_r, I_r)\, dr + \int_{(s,\,T]} e^{-\delta\,(r-s)} c(X_{r-}, J_r)\, dp_r^* \bigg| \mathcal{F}_s \bigg].
\end{aligned}
$$

From the elementary inequality $n[z]^- + \nu z \geqslant 0$, $z \in \mathbb{R}$, $\nu \in (0,\, n]$, and since Y^n is in $\mathbf{S}^\infty$, sending $T \to \infty$, we obtain, by the conditional version of the Lebesgue dominated convergence theorem,

$$
Y_s^n \leq \mathbb{E}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}\bigg[\int_{(s,\,\infty)} e^{-\delta\,(r-s)} f(X_r, I_r)\, dr + \int_{(s,\,\infty)} e^{-\delta\,(r-s)} c(X_{r-}, J_r)\, dp_r^* \bigg| \mathcal{F}_s \bigg].
$$

Hence

$$
Y_s^n \leq \operatorname*{ess\,inf}_{\boldsymbol{\nu}\in\mathcal{V}^n} \mathbb{E}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}\bigg[\int_{(s,\,\infty)} e^{-\delta\,(r-s)} f(X_r, I_r)\, dr + \int_{(s,\,\infty)} e^{-\delta\,(r-s)} c(X_{r-}, J_r)\, dp_r^* \bigg| \mathcal{F}_s \bigg]. \qquad (4.25)
$$

On the other hand, for $\varepsilon \in (0,\, 1)$, let us consider the process $\boldsymbol{\nu}^\varepsilon := (\nu^{0,\varepsilon}, \nu^{\Gamma,\varepsilon}) \in \mathcal{V}^n$ defined by (4.15)-(4.16), with $Z^{T,n}$ replaced by Z^n. Thus for this choice of $\boldsymbol{\nu} = \boldsymbol{\nu}^\varepsilon$ in (4.24), we obtain

$$
Y_s^n \geqslant -\varepsilon\, \frac{1 - e^{-\delta(T-s)}}{\delta}\, C_\Lambda
$$

$$+\,\mathbb{E}^{x,a_0,a_\Gamma}_{\boldsymbol{\nu}^\varepsilon}\bigg[e^{-\delta\,(T-s)}\,Y^n_T+\int_{(s,\,T]}e^{-\delta\,(r-s)}\,f(X_r,I_r)\,dr+\int_{(s,\,T]}e^{-\delta\,(r-s)}\,c(X_{r-},J_r)\,dp^*_r\Big|\mathcal{F}_s\bigg].$$

Letting $T\to\infty$, since f,c are bounded and $Y^{n,x,a_0,a_\Gamma}\in\mathbf{S}^\infty$, it follows that

$$Y^n_s\geqslant\operatorname*{ess\,inf}_{\boldsymbol{\nu}\in\mathcal{V}^n}\mathbb{E}^{x,a_0,a_\Gamma}_{\boldsymbol{\nu}}\bigg[\int_{(s,\infty)}e^{-\delta\,(r-s)}\,f(X_r,I_r)\,dr+\int_{(s,\infty)}e^{-\delta\,(r-s)}\,c(X_{r-},J_{r-})\,dp^*_r\Big|\mathcal{F}_s\bigg]-\frac{\varepsilon}{\delta}\,C_\Lambda. \tag{4.26}$$

Taking into account the arbitrariness of ε, the required representation of Y^{n,x,a_0,a_Γ} follows from (4.25) and (4.26).

Step IV. *Uniform estimate on* $(Z^{n,x,a_0,a_\Gamma},K^{n,x,a_0,a_\Gamma})_n$. Let us prove that, for every $T>0$, there exists a constant C, depending only on M_f, M_c, δ, T, C^*, such that

$$||Z^{n,x,a_0,a_\Gamma}||^2_{\mathcal{G}^2_{\mathbf{x},\mathbf{a_0},\mathbf{a_\Gamma}}(\mathrm{q};\mathbf{0},\,\mathbf{T})}+||K^{n,x,a_0,a_\Gamma}||^2_{\mathbf{K}^2_{\mathbf{x},\mathbf{a_0},\mathbf{a_\Gamma}}(\mathbf{0},\,\mathbf{T})}\leqslant C, \tag{4.27}$$

where K^{n,x,a_0,a_Γ} is the process defined in (4.21). Fix $T>0$. In what follows we shall denote by $C>0$ a generic positive constant depending on M_f, M_c, C^*, δ and T, which may vary from line to line. Applying Itô's formula to $|Y^n_s|^2$ between 0 and T, and taking the expectation with respect to $\mathbb{P}^{x,a_0,a_\Gamma}$, recalling also Remark 4.2, we obtain

$$\begin{aligned}&\mathbb{E}^{x,a_0,a_\Gamma}\bigg[\int_{(0,T]}\int_{E\times A_0\times A_\Gamma}\big|Z^n_s(y,b,c)-\hat Z^n_s\,\mathbb{1}_K(s)\big|^2\,\tilde p(ds\,dy\,db)\bigg]\leq-2\mathbb{E}^{x,a_0,a_\Gamma}\bigg[\int_{(0,T]}Y^n_s\,dK^n_s\bigg]\\&+\mathbb{E}^{x,a_0,a_\Gamma}[|Y^n_T|^2]+2\mathbb{E}^{x,a_0,a_\Gamma}\bigg[\int_{(0,\,T]}Y^n_s\,f(X_s,I_s)\,ds\bigg]+2\,\mathbb{E}^{x,a_0,a_\Gamma}\bigg[\int_{(0,\,T]}Y^n_{s-}\,c(X_{s-},J_{s-})\,dp^*_s\bigg]\\&+2\mathbb{E}^{x,a_0,a_\Gamma}\bigg[\sum_{s\in(0,\,T]}\bigg(\int_{E\times A_0\times A_\Gamma}Z^n_s(y,b,c)\,q(\{s\}\times dy\,db\,dc)\bigg)\,c(X_{s-},J_{s-})\,\mathbb{1}_{X_{s-}\in\partial E}\bigg].\end{aligned}$$

Using the elementary inequality $2\,a\,b\leq\gamma\,a^2+\frac{1}{\gamma}\,b^2$, with $\gamma\in\mathbb{R}_+\setminus\{0\}$, $\gamma<1$, we get

$$\begin{aligned}&(1-\gamma)\,\mathbb{E}^{x,a_0,a_\Gamma}\bigg[\int_{(0,T]}\int_{E\times A_0\times A_\Gamma}\big|Z^n_s(y,b,c)-\hat Z^n_s\,\mathbb{1}_K(s)\big|^2\,\tilde p(ds\,dy\,db)\bigg]\\&\leq\mathbb{E}^{x,a_0,a_\Gamma}[|Y^n_T|^2]-2\mathbb{E}^{x,a_0,a_\Gamma}\bigg[\int_{(0,T]}Y^n_s\,dK^n_s\bigg]+\frac{1}{\gamma}\,\mathbb{E}^{x,a_0,a_\Gamma}\bigg[\sum_{s\in(0,\,T]}|c(X_{s-},J_{s-})|^2\,\mathbb{1}_{X_{s-}\in\partial E}\bigg]\\&+2\mathbb{E}^{x,a_0,a_\Gamma}\bigg[\int_{(0,\,T]}Y^n_s\,f(X_s,I_s)\,ds\bigg]+2\,\mathbb{E}^{x,a_0,a_\Gamma}\bigg[\int_{(0,\,T]}Y^n_{s-}\,c(X_{s-},J_{s-})\,dp^*_s\bigg].\end{aligned}$$

Set now $C_Y:=\frac{M_f}{\delta}+C^*\,M_c$. Recalling the uniform estimate (4.13) on Y^n, we obtain

$$\begin{aligned}&(1-\gamma)\,\mathbb{E}^{x,a_0,a_\Gamma}\bigg[\int_{(0,T]}\int_{E\times A_0\times A_\Gamma}\big|Z^n_s(y,b,c)-\hat Z^n_s\,\mathbb{1}_K(s)\big|^2\,\tilde p(ds\,dy\,db\,dc)\bigg]\\&\leqslant\frac{1}{\gamma}\,M_c^2\,C^*(T)+C_Y^2\,+2\,C_Y\,(M_f\,T+M_c\,C^*(T))+2\,C_Y\,\mathbb{E}^{x,a_0,a_\Gamma}[K^n_T],\end{aligned}\tag{4.28}$$

where $C^*(t)$ is the deterministic function defined in (2.6). On the other hand, from (4.20), we get

$$\begin{aligned}K^n_T=&\,Y^n_T-Y^n_0-\delta\int_{(0,\,T]}Y^{n,x,a}_s\,ds+\int_{(0,\,T]}f(X_s,I_s)\,ds+\int_{(0,\,T]}c(X_{s-},J_{s-})\,dp^*_s\\&-\int_{(0,\,T]}\int_{A_0}Z^n_s(X_s,b,J_s)\,\lambda_0(db)\,ds-\int_{(0,\,T]}\int_{A_\Gamma}Z^n_s(X_s,I_s,c)\,\lambda_0(dc)\,ds\end{aligned}$$

$$- \int_{(0,T]} \int_{E\times A_0\times A_\Gamma} Z^n_s(y,b,c)\, q(ds\,dy\,db\,dc). \tag{4.29}$$

Using again the inequality $2ab \leqslant \frac{1}{\eta}a^2 + \eta b^2$, for any $\eta = \alpha, k > 0$, and taking the expectation in (4.29), we find

$$2\,\mathbb{E}^{x,a_0,a_\Gamma}[K^n_T] \leqslant 4\,C_Y + 2\,\delta\,C_Y\,T + 2\,M_f\,T + 2\,M_c\,C^*(T) + \frac{T}{\alpha}\,\lambda_0(A_0) + \frac{T}{k}\,\lambda_\Gamma(A_\Gamma) \tag{4.30}$$
$$+\,\alpha\,\mathbb{E}^{x,a_0,a_\Gamma}\Big[\int_{(0,T]}\int_{A_0} |Z^n_s(X_s,b,J_s)|^2\,\lambda_0(db)\,ds\Big] + k\,\mathbb{E}^{x,a_0,a_\Gamma}\Big[\int_{(0,T]}\int_{A_\Gamma} |Z^n_s(X_s,I_s,c)|^2\,\lambda_\Gamma(dc)\,ds\Big].$$

Plugging (4.30) into (4.28), we obtain

$$(1-\gamma)\,\mathbb{E}^{x,a_0,a_\Gamma}\Big[\int_{(0,T]}\int_{E\times A_0\times A_\Gamma} \big|Z^n_s(y,b,c) - \hat{Z}^n_s\,\mathbb{1}_K(s)\big|^2\,\tilde{p}(ds\,dy\,db)\Big] \leqslant C +$$
$$+\,(\alpha\vee k)\,C_Y(1+2\,T)\bigg(\int_{(0,T]}\Big[\int_{A_0} |Z^n_s(X_s,b,J_s)|^2\,\lambda_0(db) + \int_{A_\Gamma} |Z^{n,x,a}_s(X_s,I_s,c)|^2\,\lambda_\Gamma(dc)\Big]\,ds\bigg).$$

Choosing $\alpha = k = \frac{1-\gamma}{2\,C_Y\,(1+2\,T)}$, we get the uniform estimate for $(Z^n)_n$, and also for $(K^n)_n$ by (4.29).

Step V. *Convergence of* $(Y^{n,x,a_0,a_\Gamma}, Z^{n,x,a_0,a_\Gamma}, K^{n,x,a_0,a_\Gamma})_n$. It follows from estimate (4.13) and the representation formula (4.23), that the sequence $(Y^n)_n$ converges in a nondecreasing way to some uniformly bounded process Y. By (4.23), we then deduce the representation formula (4.11) for Y. In addition, by the uniform estimate (4.27) it follows that there exist $Z^{x,a_0,a_\Gamma} \in \mathcal{G}^{\mathbf{2}}_{\mathbf{x},\mathbf{a}_0,\mathbf{a}_\Gamma,\mathbf{loc}}(\mathrm{q})$ and a nondecreasing, predictable process K^{x,a_0,a_Γ}, with $K_0 = 0$ and $\mathbb{E}^{x,a_0,a_\Gamma}[|K^{x,a_0,a_\Gamma}_T|^2] < \infty$, such that:

- Z^{x,a_0,a_Γ} is the weak limit of $(Z^{n,x,a_0,a_\Gamma})_n$ in $\mathcal{G}^{\mathbf{2}}_{\mathbf{x},\mathbf{a}_0,\mathbf{a}_\Gamma,\mathbf{loc}}(\mathrm{q})$;
- K^{x,a_0,a_Γ}_s is the weak limit of $(K^{n,x,a_0,a_\Gamma}_s)_n$ in $\mathbf{L}^{\mathbf{2}}_{\mathbf{x},\mathbf{a}_0,\mathbf{a}_\Gamma}(\mathcal{F}_s)$, for every $s \geqslant 0$.

By Lemma 2.2 in [25], we deduce that both Y^{x,a_0,a_Γ} and K^{x,a_0,a_Γ} are càdlàg processes, so that $Y^{x,a_0,a_\Gamma} \in \mathbf{S}^\infty$ and $K^{x,a_0,a_\Gamma} \in \mathbf{K}^{\mathbf{2}}_{\mathbf{x},\mathbf{a}_0,\mathbf{a}_\Gamma,\mathbf{loc}}$. Letting $n \to \infty$ in equation (4.20), we see that $(Y^{x,a_0,a_\Gamma}, Z^{x,a_0,a_\Gamma}, K^{x,a_0,a_\Gamma})$ solves equation (4.4).

Consider now another solution $(\tilde{Y}, \tilde{Z}, \tilde{K}) \in \mathbf{S}^\infty \times \mathcal{G}^{\mathbf{2}}_{\mathbf{x},\mathbf{a}_0,\mathbf{a}_\Gamma,\mathbf{loc}}(\mathrm{q}) \times \mathbf{K}^{\mathbf{2}}_{\mathbf{x},\mathbf{a}_0,\mathbf{a}_\Gamma,\mathbf{loc}}$ to (4.4)-(4.5)-(4.6). Then, it is quite easy to check that

$$\tilde{Y}^{x,a_0,a_\Gamma}_s \le \operatorname*{ess\,inf}_{\boldsymbol{\nu}\in\mathcal{V}} \mathbb{E}^{x,a_0,a_\Gamma}_{\boldsymbol{\nu}}\Big[\int_{(s,\infty)} e^{-\delta(r-s)}\,f(X_r,I_r)\,dr + \int_{(s,\infty)} e^{-\delta(r-s)}\,c(X_{r-},J_{r-})\,dp^*_r\Big|\mathcal{F}_s\Big],$$

for all $s \geq 0$. This implies the maximality of $(Y^{x,a_0,a_\Gamma}, Z^{x,a_0,a_\Gamma}, K^{x,a_0,a_\Gamma})$.

Concerning the jump constraints, we simply notice that they are a direct consequence of the uniform estimate (4.27) on the norm $||K^{n,x,a_0,a_\Gamma}||^2_{\mathbf{K}^{\mathbf{2}}_{\mathbf{x},\mathbf{a}_0,\mathbf{a}_\Gamma}(\mathbf{0},\mathbf{T})}$.

Finally, regarding the uniqueness result, let (Y, Z, K) and (Y', Z', K') be two maximal solutions of (4.4)-(4.5)-(4.6). The component Y is unique by definition. Let us now consider the difference between the two backward equations. We get: $\mathbb{P}^{x,a_0,a_\Gamma}$-a.s.

$$\int_{(0,t]}\int_{E\times A_0\times A_\Gamma} (Z_s(y,b,c) - Z'_s(y,b,c))\,q(ds\,dy\,db\,dc)$$
$$= (K_t - K'_t) - \int_{(0,t]}\int_{A_0} (Z_s(X_s,b,J_s) - Z'_s(X_s,b,J_s))\,\lambda_0(db)\,ds \tag{4.31}$$

$$- \int_{(0,t]} \int_{A_\Gamma} (Z_s(X_s, I_s, c) - Z'_s(X_s, I_s, c))\, \lambda_\Gamma(dc)\, ds, \quad 0 \leqslant t \leqslant T < \infty.$$

Since the right-hand side of (4.31) is a predictable process with locally finite variation, while the left-hand side is a local martingale, both sides vanish, see e.g. Corollary I.3.16 in [21]. This implies in particular that $Z = Z'$ in $\mathcal{G}^{\mathbf{2}}_{\mathbf{x},\mathbf{a_0},\mathbf{a_\Gamma},\mathbf{loc}}(\mathrm{q})$, and that the component K is unique as well. $\square$

Remark 4.5. *We see from the proof of Theorem 4.3 that the wellposedness result for the infinite horizon BSDE (4.4)-(4.5)-(4.6) relies on existence and uniqueness results for the corresponding approximating families of finite horizon BSDEs (4.7), parametrized by $T > 0$. In particular, finite horizon control problems can be treated via BSDEs techniques without any additional difficulty with respect to the infinite horizon case (the treatment is even simpler, since the passage to the limit as T goes to infinity is no more needed).*

5 A BSDE representation for the value function

The aim of the present section is to prove that the value function V in (1.2) can be represented in terms of the maximal solution to the BSDE with nonnegative jumps (4.4)-(4.5)-(4.6). Firstly, we introduce the deterministic function $v : E \times A_0 \times A_\Gamma \to \mathbb{R}$ given by

$$v(x, a_0, a_\Gamma) := Y_0^{x,a_0,a_\Gamma}, \quad (x, a_0, a_\Gamma) \in E \times A_0 \times A_\Gamma. \tag{5.1}$$

Theorem 5.1. *Let assumptions* **(HhλQR)**, **(H0)**, **(H0$'$)**, **(H$\lambda_0\lambda_\Gamma$)** *and* **(Hfc)** *hold. Then, the function v in (5.1) does not depend on its last arguments:*

$$v(x, a_0, a_\Gamma) = v(x, a'_0, a'_\Gamma), \quad x \in E,\ (a_0, a'_0) \in A_0,\ (a_\Gamma, a'_\Gamma) \in A_\Gamma. \tag{5.2}$$

By an abuse of notation, we define the function v on E by

$$v(\cdot) := v(\cdot, a_0, a_\Gamma), \quad \text{for any } (a_0, a_\Gamma) \in A_0 \times A_\Gamma. \tag{5.3}$$

Then v is continuous and bounded. Moreover, v admits the representation formula: $\mathbb{P}^{x,a_0,a_\Gamma}$-a.s.,

$$v(X_s) = Y_s^{x,a_0,a_\Gamma}, \quad \forall s \geq 0. \tag{5.4}$$

Proof. We split the proof into three steps.

Step I. *The identification property of Y^{x,a_0,a_Γ}.* A first fundamental preliminary result we have to prove is the following identification property: for every $(x, a_0, a_\Gamma) \in E \times A_0 \times A_\Gamma$, $\mathbb{P}^{x,a_0,a_\Gamma}$-a.s.,

$$Y_s^{x,a_0,a_\Gamma} = v(X_s, I_s, J_s), \quad s \geq 0, \tag{5.5}$$

where v is the deterministic function defined by (5.1). Recall from the proof of Theorem 4.3 that Y^{x,a_0,a_Γ} is constructed from Y^{T,n,x,a_0,a_Γ} (see equation (4.7)), taking firstly the limit as $T \to \infty$, and then as $n \to \infty$. Therefore, it is enough to prove property (5.5) for Y^{T,n,x,a_0,a_Γ}. For simplicity of notation, denote the pair $(Y^{T,n,x,a_0,a_\Gamma}, Z^{T,n,x,a_0,a_\Gamma})$, solution to equation (4.7), simply as $(Y^{T,n}, Z^{T,n})$. Then, we know from the fixed point argument giving the well-posedness of the penalized BSDE (4.20) (see the proof of Theorem 4.1 in [2]) that there exists a sequence $(Y^{T,n,k}, Z^{T,n,k})_k$ in $\mathbf{S}^\infty \times \mathcal{G}^{\mathbf{2}}_{\mathbf{x},\mathbf{a_0},\mathbf{a_\Gamma}}(\mathrm{q}; \mathbf{0}, \mathbf{S})$ converging to $(Y^{T,n}, Z^{T,n})$ in $\mathbf{S}^\infty \times \mathcal{G}^{\mathbf{2}}_{\mathbf{x},\mathbf{a_0},\mathbf{a_\Gamma}}(\mathrm{q}; \mathbf{0}, \mathbf{S})$, such that $(Y^{T,n,0}, Z^{T,n,0}) = (0,0)$ and

$$Y_t^{T,n,k+1} = Y_S^{T,n,k} - \delta \int_t^S Y_r^{T,n,k}\, dr + \int_{(t,S]} f(X_r, I_r)\, dr + \int_{(t,S]} c(X_{r-}, J_{r-})\, dp^*_r$$

$$
\begin{aligned}
&- n \int_t^S \int_{A_0} [Z_r^{T,n,k}(X_r, b, I_r)]^- \, \lambda_0(db)\, dr - \int_{(t,\,S]} \int_{A_0} Z_r^{T,n,k}(X_r, b, I_r)\, \lambda_0(db)\, dr,\\
&- n \int_{(t,\,S]} \int_{A_\Gamma} [Z_r^{T,n,k}(X_r, I_r, c)]^- \, \lambda_\Gamma(dc)\, dr - \int_{(t,\,S]} \int_{A_\Gamma} Z_r^{T,n,k}(X_r, I_r, c)\, \lambda_\Gamma(dc)\, dr\\
&- \int_{(t,\,S]} \int_{E\times A_0\times A_\Gamma} Z_r^{T,n,k+1}(y, b, c)\, q(dr\, dy\, db\, dc), \qquad 0 \leqslant t \leqslant S. \qquad (5.6)
\end{aligned}
$$

Let us define $v^{T,n}(x, a_0, a_\Gamma) := Y_0^{T,n}$, $v^{T,n,k}(x, a_0, a_\Gamma) := Y_0^{T,n,k}$. For $k = 0$, we have, $\mathbb{P}^{x,a_0,a_\Gamma}$-a.s.,

$$
Y_t^{T,n,1} = \mathbb{E}^{x,a_0,a_\Gamma}\Big[\int_{(t,\,S]} f(X_r, I_r)\, dr + \int_{(t,\,S]} c(X_{r-}, J_{r-})\, dp_r^* \Big| \mathcal{F}_t \Big], \quad t \in [0,\, S].
$$

Then, from the Markov property of (X, I, J) we get, $\mathbb{P}^{x,a_0,a_\Gamma}$-a.s., $Y_t^{T,n,1} = v^{T,n,1}(X_t, I_t, J_t)$, and in particular

$$
\Delta Y_t^{T,n,1} = -c(X_{t-}, J_{t-})\, \Delta p_t^* + Z_t^{T,n,1}(X_t, I_t, J_t) - \hat{Z}_t^{T,n,1}\, \Delta p_t^*, \qquad 0 \leqslant t \leqslant S.
$$

which gives

$$
Z_t^{T,n,1}(y, b, c) - \hat{Z}_t^{T,n,1}\, \mathbb{1}_{X_{t-}\in\partial E} = v^{T,n,1}(y, b, c) - v^{T,n,1}(X_{t-}, I_{t-}, J_{t-}) - c(X_{t-}, J_{t-})\, \mathbb{1}_{X_{t-}\in\partial E}.
$$

We now consider the inductive step: $1 \leqslant k \in \mathbb{N}$, and assume that $\mathbb{P}^{x,a_0,a_\Gamma}$-a.s.,

$$
\begin{aligned}
Y_t^{T,n,k} &= v^{T,n,k}(X_t, I_t, J_t)\\
Z_t^{T,n,k}(y, b, c) - \hat{Z}_t^{T,n,k}\, \mathbb{1}_{X_{t-}\in\partial E} &= v^{T,n,k}(y, b, c) - v^{T,n,k}(X_{t-}, I_{t-}, J_{t-}) - c(X_{t-}, J_{t-})\, \mathbb{1}_{X_{t-}\in\partial E}.
\end{aligned}
$$

Then, plugging the expressions above in (5.6) and computing the conditional expectation as before, by the Markov property of (X, I) we achieve that, $\mathbb{P}^{x,a_0,a_\Gamma}$-a.s.,

$$
Y_t^{T,n,k+1} = v^{T,n,k+1}(X_t, I_t, J_t).
$$

At this point, applying the Itô formula to $|Y_t^{T,n,k} - Y_t^{T,n}|^2$ and taking the supremum of t between 0 and S, one can show that $\mathbb{E}^{x,a_0,a_\Gamma}\big[\sup_{0\leqslant t\leqslant S} |Y_t^{T,n,k} - Y_t^{T,n}|^2\big] \to 0$ as k goes to infinity. Therefore, $v^{T,n,k}(x, a_0, a_\Gamma) \to v^{T,n}(x, a_0, a_\Gamma)$ as k goes to infinity, for all $(x, a_0, a_\Gamma) \in E \times A_0 \times A_\Gamma$, from which it follows that, $\mathbb{P}^{x,a_0,a_\Gamma}$-a.s., $Y_t^{T,n,x,a_0,a_\Gamma} = v^{T,n}(X_t, I_t, J_t)$. Letting $T \to \infty$, and then $n \to \infty$, we deduce property (5.5) for Y^{x,a_0,a_Γ}.

Step II. *The non-dependence of the function v on its last arguments.* Notice that, by (4.12) and (5.1), v coincides with the value function V^* of the randomized control problem introduced in (3.16). Therefore, to prove (5.2) we have to show that $V^*(x, a_0, a_\Gamma)$ does not depend on (a_0, a_Γ). In other words, given $(a_0, a_0') \in A_0$, $(a_\Gamma, a_\Gamma') \in A_\Gamma$, we have to prove that

$$
V^*(x, a_0, a_\Gamma) \;=\; V^*(x, a_0', a_\Gamma'). \qquad (5.7)
$$

Notice that (5.7) holds true if we prove the following property of the cost functional: for every $\boldsymbol{\nu} = (\nu^0, \nu^\Gamma) \in \mathcal{V}$, there exist $(\nu^{0,\varepsilon}, \nu^{\Gamma,\varepsilon})_\varepsilon \in \mathcal{V}$ such that

$$
\lim_{\varepsilon\to 0^+} J(x, a_0', a_\Gamma', \nu^{0,\varepsilon}, \nu^{\Gamma,\varepsilon}) = J(x, a_0, a_\Gamma, \nu^0, \nu^\Gamma). \qquad (5.8)
$$

As a matter of fact, suppose that property (5.8) holds. Then, we deduce that $V^*(x, a_0', a_\Gamma') \leq J(x, a_0, a_\Gamma, \nu^0, \nu^\Gamma)$, and by the arbitrariness of (ν^0, ν^Γ), we conclude that $V^*(x, a_0', a_\Gamma') \leq V^*(x, a_0, a_\Gamma)$, from which we get (5.7).

It remains to prove (5.8). This can be done proceeding as in the proof of Proposition 5.6 in [3], that is in the context of PDMPs with no jumps from the boundary, since the presence of predictable jumps does not induce here any additional technical difficulty.

From now on, we suppose that the function v is defined on E, as in (5.3). So, in particular, identity (5.5) gives the representation formula (5.4).

Step III. *The function v is bounded and continuous.* By (4.12), (5.4) and recalling the definition of V^* in (3.16), we have

$$v(x) = V^*(x, a_0, a_\Gamma) = \inf_{\boldsymbol{\nu} \in \mathcal{V}} \mathbb{E}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}\bigg[\int_{(0,\,\infty)} e^{-\delta s} f(X_s, I_s)\,ds + \int_{(0,\,\infty)} e^{-\delta s} c(X_{s-}, J_{s-})\,dp_s^*\bigg].$$

The boundedness of v then directly follows from the boundedness of f and c. In particular, $|v(x)| \leqslant \frac{M_f}{\delta} + C^* M_c$, for all $x \in E$.

Let us now prove the continuity property of v. We proceed as in [15], Section 5. Let $B(E)$ be the set of all bounded functions on E. Fix $(a_0, a_\Gamma) \in A_0 \times A_\Gamma$, and define the deterministic operator $G : B(E) \to B(E)$ as $G\psi(x) := \inf_{\boldsymbol{\nu} \in \mathcal{V}} G_{\boldsymbol{\nu}}\psi(x)$, where

$$G_{\boldsymbol{\nu}}\psi(x) := \mathbb{E}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}\bigg[\int_{(0,\,T_1]} e^{-\delta s} f(X_s, I_s)\,ds + \int_{(0,\,T_1]} e^{-\delta s} c(X_{s-}, J_{s-})\,dp_s^* + e^{-\delta T_1}\,\psi(X_{T_1})\bigg],$$

with T_1 the first jump time of the PDMP (X, I, J) under $\mathbb{P}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}$. Set $t_{\boldsymbol{\nu}}^*(x) := \inf\{t \geq 0 : X_t \in \partial E, (X_0, I_0, J_0) = (x, a_0, a_\Gamma),\ \mathbb{P}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}\text{-a.s.}\}$, and consider the sequence of Borel-measurable functions $(v_n)_{n\geqslant 0}$ defined by

$$v_{n+1}(x) = G\,v_n(x) := \inf_{\boldsymbol{\nu} \in \mathcal{V}}\bigg\{\int_0^{t_{\boldsymbol{\nu}}^*(x)} \chi^{\boldsymbol{\nu}}(s) f_0^{v_n}(X_s, I_s)\,ds + \chi^{\boldsymbol{\nu}}(t_{\boldsymbol{\nu}}^*(x)) F^{v_n}(X_{t_{\boldsymbol{\nu}}^*}, J_{t_{\boldsymbol{\nu}}^*})\bigg\},$$

where $\chi^{\boldsymbol{\nu}}(s) = e^{-\delta s}\, e^{-\int_0^s \tilde\lambda^{\boldsymbol{\nu}}(t, X_t, I_t)\,dt}$ and $f_0^{\psi}(X_s, I_s) = f(X_s, I_s) + \int_E \psi(y)\,\lambda(X_s, I_s)\,Q(X_s, I_s, dy)$, $F^{\psi}(X_{s-}, J_{s-}) = c(X_{s-}, J_{s-}) + \int_E \psi(y)\,R(X_{s-}, J_{s-}, dy)$, for any $\psi \in B(E)$. If we prove that G is a two-stage contraction mapping, then by the strong Markov property of the PDMP (X, I, J) it would follow that v is the unique fixed point of G, and therefore $v(x) = \lim_{n\to\infty} v_n(x)$, see Corollary 5.6 in [15]. Then, the continuity property of v in E would follow from the existence of two monotone sequences of continuous functions converging to v, one from above and one from below, see Lemmas 5.9 and 5.10 in [15].

It remains to prove that G^2 is a contraction in E. To this end, it is enough to show that, for any $\psi_1, \psi_2 \in B(E)$, $|G_{\boldsymbol{\nu}}^2\psi_1 - G_{\boldsymbol{\nu}}^2\psi_2| \leq \rho||\psi_1 - \psi_2||$ for some constant $\rho < 1$, independent on $\boldsymbol{\nu}$, where $||\psi|| = \max_{x\in E}\psi(x)$, $\psi \in B(E)$. Denoting by T_2 the second jump time of (X, I, J), we have

$$G_{\boldsymbol{\nu}}^2\psi(x) := \mathbb{E}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}\bigg[\int_{(0,\,T_2]} e^{-\delta s} f(X_s, I_s)\,ds + \int_{(0,\,T_2]} e^{-\delta s} c(X_{s-}, J_{s-})\,dp_s^* + e^{-\delta T_2}\,\psi(X_{T_2})\bigg],$$

so that $|G_{\boldsymbol{\nu}}^2\psi_1 - G_{\boldsymbol{\nu}}^2\psi_2| \leq \mathbb{E}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}[e^{-\delta T_2}]||\psi_1 - \psi_2||$. The fact that $\mathbb{E}_{\boldsymbol{\nu}}^{x,a_0,a_\Gamma}[e^{-\delta T_2}] \leq \rho < 1$ is a consequence of assumption **(H0)**, see the proof of Proposition 46.17 in [13] for more details. □

We can finally state our main result.

Theorem 5.2. *Let assumptions* **(HhλQR)**, **(H0)**, **(H0$'$)**, **(H$\lambda_0\lambda_\Gamma$)** *and* **(Hfc)** *hold. Then, the function v in (5.1) is a viscosity solution to (2.8)-(2.9). Therefore, if assumptions* **(HBB)**, **(HBB')** *hold and A_0, A_Γ are compact, by Theorem 2.1 we conclude that $v \equiv V$ and, for any $(x, a_0, a_\Gamma) \in E \times A_0 \times A_\Gamma$, V admits the Feynman-Kac representation formula $V(x) = Y_0^{x,a_0,a_\Gamma}$.*

Before proving Theorem 5.2, we recall the following technical result, see Proposition II.1 in [27].

Lemma 5.3. *A function $u \in \mathbb{C}_b(\bar{E})$ (resp. $w \in \mathbb{C}_b(\bar{E})$) is a sub- (resp. super-) solution to (2.8)-(2.9) if and only if, for any $\phi \in \mathbb{C}_b^1(\bar{E})$, for any x_0 global maximum (resp. global minimum) point of $u - \phi$ (resp. $w - \phi$),*

$$\begin{aligned}
&H^\phi(x_0, \phi(x_0), \nabla\phi(x_0)) \leqslant 0 && \textit{if } x_0 \in E,\\
&\min\{H^\phi(x_0, \phi(x_0), \nabla\phi(x_0)), \phi(x_0) - F^\phi(x_0)\} \leqslant 0 && \textit{if } x_0 \in \partial E\\
\big(\text{resp. } &H^\phi(x_0, \phi(x_0), \nabla\phi(x_0)) \geqslant 0 && \textit{if } x_0 \in E,\\
&\max\{H^\phi(x_0, \phi(x_0), \nabla\phi(x_0)), \phi(x_0) - F^\phi(x_0)\} \geqslant 0 && \textit{if } x_0 \in \partial E.\big)
\end{aligned}$$

Proof (of Theorem 5.2). Notice that, by Theorem 5.1, it is enough to check the viscosity sub- and super-solution properties for v in the sense of Lemma 5.3. We split the proof into two steps.

Viscosity subsolution property. Let $\bar{x} \in \bar{E}$, and let $\varphi \in C^1(\bar{E})$ be a test function such that

$$0 = (v - \varphi)(\bar{x}) = \max_{y \in \bar{E}} (v - \varphi)(y). \tag{5.9}$$

Case 1: $\bar{x} \in E$. Fix $(a_0, a_\Gamma) \in A_0 \times A_\Gamma$, set $\eta = \frac{1}{2}\, d(\bar{x}, \partial E)$, and $\tau := \inf\{t \geqslant 0 : |\phi(t, \bar{x}, a_0) - \bar{x}| \geqslant \eta\}$. Let $h > 0$. Let $Y^{\bar{x}, a_0, a_\Gamma}$ be the unique maximal solution to (4.4)-(4.5)-(4.6) under $\mathbb{P}^{\bar{x}, a_0, a_\Gamma}$. We apply the Itô formula to $e^{-\delta t}\, Y_t^{\bar{x}, a_0, a_\Gamma}$ between 0 and $\theta := \tau \wedge h \wedge T_1$, where T_1 denotes the first jump time of (X, I, J). From the constraints (4.5)-(4.6) and the fact that K is a nondecreasing process, it follows that $\mathbb{P}^{\bar{x}, a_0, a_\Gamma}$-a.s.,

$$\begin{aligned}
Y_0^{\bar{x}, a_0, a_\Gamma} \leqslant e^{-\delta \theta_m}\, Y_\theta^{\bar{x}, a_0, a_\Gamma} &+ \int_{(0,\theta]} e^{-\delta r}\, f(X_r, I_r)\, dr + \int_{(0,\theta]} e^{-\delta r}\, c(X_{r-}, J_{r-})\, dp_r^*\\
&- \int_{(0,\theta]} e^{-\delta r} \int_{E \times A_0 \times A_\Gamma} Z_r^{\bar{x}, a_0, a_\Gamma}\, q(dr\, dy\, db\, dc).
\end{aligned}$$

Applying the expectation with respect to $\mathbb{P}^{\bar{x}, a_0, a_\Gamma}$, from the identification property (5.4), together with (5.9), it follows that

$$\varphi(\bar{x}) \leqslant \mathbb{E}^{\bar{x}, a_0, a_\Gamma}\bigg[e^{-\delta\theta}\, \varphi(X_\theta) + \int_{(0,\theta]} e^{-\delta r}\, f(X_r, I_r)\, dr + \int_{(0,\theta]} e^{-\delta r}\, c(X_{r-}, J_{r-})\, dp_r^*\bigg].$$

At this point, applying Itô's formula to $e^{-\delta r}\, \varphi(X_r)$ between 0 and θ, we get

$$\begin{aligned}
&\frac{1}{h}\mathbb{E}^{\bar{x}, a_0, a_\Gamma}\bigg[\int_{(0,\theta]} e^{-\delta r}\, [\delta\, \varphi(X_r) - \mathcal{L}^{I_r}\varphi(X_r) - f(X_r, I_r)]\, dr\bigg]\\
&\quad \leq \frac{1}{h}\mathbb{E}^{\bar{x}, a_0, a_\Gamma}\bigg[e^{-\delta\theta}\, [\mathcal{R}^{J_{\theta-}}\varphi(X_{\theta-}) + c(X_{\theta-}, J_{\theta-})]\, \mathbb{1}_{X_{\theta-} \in \partial E}\bigg],
\end{aligned} \tag{5.10}$$

where

$$\mathcal{L}^{I_r}\varphi(X_r) := h(X_r, I_r) \cdot \nabla\varphi(X_r) + \int_E (\varphi(y) - \varphi(X_r))\, \lambda(X_r, I_r)\, Q(X_r, I_r, dy), \tag{5.11}$$

$$\mathcal{R}^{J_{r-}}\varphi(X_{r-}) := \int_E (\varphi(y) - \varphi(X_{r-}))\, R(X_{r-}, J_{r-}, dy). \tag{5.12}$$

Now we notice that, for every $r \in [0, \theta]$, $(X_{r-}, I_{r-}, J_{r-}) = (\phi(r, \bar{x}, a_0), a_0, a_\Gamma)$, $\mathbb{P}^{\bar{x}, a_0, a_\Gamma}$-a.s., with $\phi(r, \bar{x}, a_0) \in E$. In particular the right-hand side of (5.10) is zero. Taking into account the continuity on E of the map $z \mapsto \delta\, \varphi(z) - \mathcal{L}^{a_0}\varphi(z) - f(z, a_0)$, we see that for any $\varepsilon > 0$,

$$\frac{-\varepsilon + \delta\, \varphi(\bar{x}) - \mathcal{L}^{a_0}\varphi(\bar{x}) - f(\bar{x}, a_0)}{h}\, \mathbb{E}^{\bar{x}, a_0, a_\Gamma}\left[\frac{1 - e^{-\delta\theta}}{\delta}\right] \leqslant 0. \tag{5.13}$$

Set $g(\theta) := \frac{1-e^{-\delta\theta}}{\delta}$, $\theta \in R_+$, and $l(s,x,a_0) := \tilde\lambda(\phi(s,x,a_0),a_0))$, $(s,x,a_0) \in \mathbb{R}_+ \times E \times A_0$, with $\tilde\lambda$ the function introduced in (3.1). Denoting by f_{T_1} the distribution density of T_1 under $\mathbb{P}^{\bar x, a_0, a_\Gamma}$, we have

$$\begin{aligned}\frac{\mathbb{E}^{\bar x,a_0,a_\Gamma}\,[g(\theta)]}{h} &= \frac{1}{h}\int_0^h g(s)\, f_{T_1}(s)\, ds + \frac{g(h)}{h}\,\mathbb{P}^{\bar x,a_0,a_\Gamma}[T_1 > h]\\ &= \int_0^h \frac{1-e^{-\delta s}}{\delta\, h}\, l(s,\bar x,a_0)\, e^{-\int_0^s l(r,\bar x,a_0)\,dr}\, ds + \frac{1-e^{-\delta h}}{\delta\, h}\, e^{-\int_0^h l(r,\bar x,a_0)\,dr}. \end{aligned} \tag{5.14}$$

By the boundedness of λ, λ_0 and λ_Γ, it is easy to see that the two terms in the right-hand side of (5.14) converge respectively to zero and one when h goes to zero. Thus, passing into the limit in (5.13) as h goes to zero, from the arbitrariness of $a_0 \in A_0$, we conclude that $H^\varphi(\bar x, \varphi(\bar x), \nabla\varphi(\bar x)) \leq 0$.

Case 2: $\bar x \in \partial E$. If $\varphi(\bar x) - F^\varphi(\bar x) \leq 0$ we have finished. Otherwise, suppose that $\varphi(\bar x) - F^\varphi(\bar x) > 0$. We argue similarly to the *Case 1*. Let $(x_m)_m$ in E such that $x_m \underset{m\to\infty}{\longrightarrow} \bar x$. Fix $(a_0, a_\Gamma) \in A_0 \times A_\Gamma$. Let $\eta_m := \frac{1}{2} d(x_m, \partial E)$, and $\tau_m := \inf\{t \geqslant 0 : |\phi(t,x_m,a_0) - x_m| \geqslant \eta_m\}$. Let Y^{x_m,a_0,a_Γ} be the unique maximal solution to (4.4)-(4.5)-(4.6) under $\mathbb{P}^{x_m,a_0,a_\Gamma}$. Applying the Itô formula to $e^{-\delta t}\, Y_t^{x_m,a_0,a_\Gamma}$ between 0 and $\theta_m := \tau_m \wedge T_1$, where T_1 denotes the first jump time of (X, I, J) under $\mathbb{P}^{x_m,a_0,a_\Gamma}$, and proceeding as in *Case 1*, we get

$$\begin{aligned} &\frac{1}{\tau_m}\mathbb{E}^{x_m,a_0,a_\Gamma}\left[\int_{(0,\theta_m]} e^{-\delta r}\,[\delta\,\varphi(X_r) - \mathcal{L}^{I_r}\varphi(X_r) - f(X_r, I_r)]\,dr\right]\\ &\leq \frac{1}{\tau_m}\mathbb{E}^{x_m,a_0,a_\Gamma}\left[e^{-\delta\theta_m}\,[\mathcal{R}^{J_{\theta_m -}}\varphi(X_{\theta_m -}) + c(X_{\theta_m -}, J_{\theta_m -})]\,\mathbb{1}_{X_{\theta_m -}\in\partial E}\right], \end{aligned} \tag{5.15}$$

where $\mathcal{L}^{I_r}$ and $\mathcal{R}^{J_{r-}}$ are the operators defined respectively in (5.11) and (5.12). Also in this case, for every $r \in [0,\,\theta_m]$, $(X_{r-}, I_{r-}, J_{r-}) = (\phi(r, x_m, a_0), a_0, a_\Gamma)$, $\mathbb{P}^{x_m,a_0,a_\Gamma}$-a.s., with $\phi(r, x_m, a_0) \in E$, and in particular the right-hand side of (5.15) is zero. By the continuity of the map $\Gamma(z) := \delta\,\varphi(z) - \mathcal{L}^{a_0}\varphi(z) - f(z, a_0)$, for any $\varepsilon > 0$, there exists $l = l(\varepsilon) > 0$ such that $|\Gamma(y) - \Gamma(\bar x)| \leq \varepsilon$ if $|y - \bar x| \leq l(\varepsilon)$. Thus, for ε fixed, let $m = m(\varepsilon) \in \mathbb{N}$ such that, for any $m \geq m(\varepsilon)$, $\eta_m \leq \frac{1}{2} l(\varepsilon)$ and $|x_m - \bar x| \leq \frac{1}{2} l(\varepsilon)$. By the triangle inequality, $|\phi(r, x_m, a_0) - \bar x| \leq l(\varepsilon)$. Therefore, for $m \geq m(\varepsilon)$,

$$[-\varepsilon + \delta\,\varphi(\bar x) - \mathcal{L}^{a_0}\varphi(\bar x) - f(\bar x, a_0)]\,\mathbb{E}^{x_m,a_0,a_\Gamma}\left[\frac{1-e^{-\delta\theta_m}}{\delta\tau_m}\right] \leqslant 0. \tag{5.16}$$

Now,

$$\mathbb{E}^{x_m,a_0,a_\Gamma}\left[\frac{1-e^{-\delta\theta_m}}{\delta\tau_m}\right] = \int_0^{\tau_m} \frac{1-e^{-\delta s}}{\delta\,\tau_m}\, l(s, x_m, a_0)\, e^{-\int_0^s \tilde l(r,x_m,a_0)\,dr}\, ds + \frac{1-e^{-\delta\tau_m}}{\delta\,\tau_m}\, e^{-\int_0^{\tau_m} l(r,x_m,a_0)\,dr},$$

that goes to one as m goes to infinity. Thus, passing into the limit in (5.16) as m goes to infinity, from the arbitrariness of $a_0 \in A_0$ we conclude also in this case that $H^\varphi(\bar x, \varphi(\bar x), \nabla\varphi(\bar x)) \leq 0$.

Viscosity supersolution property. Let $\bar x \in \bar E$, and let $\varphi \in C^1(\bar E)$ be a test function such that

$$0 = (v - \varphi)(\bar x) = \min_{x\in\bar E}(v - \varphi)(x). \tag{5.17}$$

Case 1: $\bar x \in E$. We can assume w.l.o.g. that $\bar x$ is a strict minimum of $v - \varphi$. For every $\eta > 0$, , we define

$$0 < \beta(\eta) := \inf_{x \in B^c(\bar x,\eta)\cap\bar E}(v - \varphi)(x), \tag{5.18}$$

where $B(\bar{x},\eta) := \{y \in E : |\bar{x}-y| < \eta\}$. We will show the result by contradiction. Assume thus that $H^\varphi(\bar{x},\varphi(\bar{x}),\nabla\varphi(\bar{x})) < 0$. Then by the continuity of H, there exists $\eta > 0$, $\beta(\eta) > 0$ and $\varepsilon \in (0,\, \beta(\eta)\delta]$ such that $H^\varphi(y,\varphi(y),\nabla\varphi(y)) \leqslant -\varepsilon$ for all $y \in B(\bar{x},\eta)$.

Let us fix $T > 0$ and define $\theta := \tau \wedge T$, where $\tau = \inf\{t \geqslant 0 : X_t \notin B(\bar{x},\eta)\}$. Moreover, let us fix $(a_0,a_\Gamma) \in A_0 \times A_\Gamma$, and consider the solution $Y^{n,\bar{x},a_0,a_\Gamma}$ to the penalized (4.20), under the probability $\mathbb{P}^{\bar{x},a_0,a_\Gamma}$. Notice that $\mathbb{P}^{\bar{x},a_0,a_\Gamma}\{\tau = 0\} = \mathbb{P}^{\bar{x},a_0,a_\Gamma}\{X_0 \notin B(\bar{x},\eta)\} = 0$. We apply Itô's formula to $e^{-\delta t}\, Y_t^{n,\bar{x},a_0,a_\Gamma}$ between 0 and θ. Then, proceeding as in the proof of the representation formula (4.23), we get the following inequality:

$$Y_0^{n,\bar{x},a_0,a_\Gamma} \geqslant \inf_{\boldsymbol{\nu}\in\mathcal{V}^n} \mathbb{E}_{\boldsymbol{\nu}}^{\bar{x},a_0,a_\Gamma}\left[e^{-\delta\theta}\, Y_\theta^{n,\bar{x},a_0,a_\Gamma} + \int_{(0,\theta]} e^{-\delta r}\, f(X_r,I_r)\,dr + \int_{(0,\theta]} e^{-\delta r}\, c(X_{r-},J_{r-})\,dp_r^*\right]. \quad (5.19)$$

Recall that $Y^{n,\bar{x},a_0,a_\Gamma}$ converges decreasingly to the maximal solution Y^{x_m,a_0,a_Γ} to the constrained BSDE (4.4)-(4.5)-(4.6). By the identification property (5.4), together with (5.17) and (5.18), from inequality (5.19) we get that there exists a strictly positive, predictable and bounded function $\boldsymbol{\nu} \in \mathcal{V}$ such that

$$\begin{aligned}\varphi(\bar{x}) &\geqslant \mathbb{E}_{\boldsymbol{\nu}}^{\bar{x},a_0,a_\Gamma}\left[e^{-\delta\theta}\,\varphi(X_\theta) + \beta\, e^{-\delta\theta}\,\mathbb{1}_{\{\tau\leqslant T\}}\right] \\ &\quad + \mathbb{E}_{\boldsymbol{\nu}}^{\bar{x},a_0,a_\Gamma}\left[\int_{(0,\theta]} e^{-\delta r}\, f(X_r,I_r)\,dr + \int_{(0,\theta]} e^{-\delta r}\, c(X_{r-},J_{r-})\,dp_r^*\right] - \frac{\varepsilon}{2\,\delta}.\end{aligned}$$

At this point, applying Itô's formula to $e^{-\delta r}\,\varphi(X_r)$ between 0 and θ, we get

$$\begin{aligned}&\mathbb{E}_{\boldsymbol{\nu}}^{\bar{x},a_0,a_\Gamma}\left[\int_{(0,\theta]} e^{-\delta r}\left[\delta\,\varphi(X_r) - \mathcal{L}^{I_r}\varphi(X_r) - f(X_r,I_r)\right]dr\right] - \beta\,\mathbb{E}_{\boldsymbol{\nu}}^{\bar{x},a_0,a_\Gamma}\left[e^{-\delta\theta}\,\mathbb{1}_{\{\tau\leqslant T\}}\right] + \frac{\varepsilon}{2} \\ &- \mathbb{E}_{\boldsymbol{\nu}}^{\bar{x},a_0,a_\Gamma}\left[\int_{(0,\theta]} e^{-\delta r}\left[\mathcal{R}^{J_{r-}}\varphi(X_{r-}) + c(X_{r-},J_{r-})\right]\mathbb{1}_{X_{r-}\in\partial E}\,dp_r^*\right] \geqslant 0, \end{aligned} \quad (5.20)$$

where $\mathcal{L}^{I_r}$ and $\mathcal{R}^{J_{r-}}$ are defined respectively in (5.11) and (5.12). Notice that, for $r \in [0,\,\theta]$, $X_{r-} \in B(\bar{x},\eta) \subset E$. In particular, $[\mathcal{R}^{J_{r-}}\varphi(X_{r-}) + c(X_{r-},J_{r-})]\mathbb{1}_{X_{r-}\in\partial E} = 0$. Moreover,

$$\delta\,\varphi(X_r) - \mathcal{L}^{I_r}\varphi(X_r) - f(X_r,I_r) \leqslant H^\varphi(X_r,\varphi(X_r),\nabla\varphi(X_r)) \leqslant -\varepsilon,$$

and therefore, from (5.20) we obtain

$$0 \leqslant \frac{\varepsilon}{2\,\delta} - \mathbb{E}_{\boldsymbol{\nu}_m}^{\bar{x},a_0,a_\Gamma}\left[\varepsilon\int_{(0,\theta]} e^{-\delta r}dr + \beta\, e^{-\delta\theta}\,\mathbb{1}_{\{\tau\leqslant T\}}\right] \leqslant -\frac{\varepsilon}{2\,\delta} + \frac{\varepsilon}{\delta}\,\mathbb{E}_{\boldsymbol{\nu}}^{\bar{x},a_0,a_\Gamma}[e^{-\delta T}\,\mathbb{1}_{\{\tau>T\}}] \leqslant -\frac{\varepsilon}{2\,\delta} + \frac{\varepsilon}{\delta}\, e^{-\delta T}.$$

Letting T go to infinity we achieve the contradiction: $0 \leqslant -\frac{\varepsilon}{2\,\delta}$.

Case 2: $\bar{x} \in \partial E$. As in the previous case, we can assume w.l.o.g. that $\bar{x}$ is a strict minimum of $v - \varphi$. Then, for every $\eta > 0$, we can define

$$0 < \beta(\eta) := \inf_{x\in\bar{B}^c(\bar{x},\eta)\cap\bar{E}} (v-\varphi)(x),$$

where $\bar{B}(\bar{x},\eta) := \{y \in \bar{E} : |\bar{x}-y| < \eta\}$. If $\varphi(\bar{x}) - F^\varphi(\bar{x}) \geq 0$ we have finished. Otherwise, assume that $\varphi(\bar{x}) - F^\varphi(\bar{x}) < 0$. We will show the result by contradiction. Assume thus that $H^\varphi(\bar{x},\varphi(\bar{x}),\nabla\varphi(\bar{x})) < 0$. Then by the continuity of H and F, there exists $\eta > 0$, $\beta(\eta) > 0$ and $\varepsilon \in (0,\, \beta(\eta)\delta]$ such that $H^\varphi(y,\varphi(y),\nabla\varphi(y)) \leqslant -\varepsilon$, and $\varphi(y) - F^\varphi(y) \leq -\varepsilon$, for all $y \in \bar{B}(\bar{x},\eta)$.

Let us fix $T > 0$ and define $\theta := \tau \wedge T$, where $\tau = \inf\{t \geqslant 0 : X_t \notin \bar{B}(\bar{x}, \eta)\}$. Arguing as in *Case 1*, we get

$$\mathbb{E}_{\boldsymbol{\nu}}^{\bar{x},a_0,a_\Gamma}\left[\int_{(0,\theta]} e^{-\delta r}\left[\delta\,\varphi(X_r) - \mathcal{L}^{I_r}\varphi(X_r) - f(X_r, I_r)\right] dr\right] + \frac{\varepsilon}{2} \tag{5.21}$$
$$- \mathbb{E}_{\boldsymbol{\nu}}^{\bar{x},a_0,a_\Gamma}\left[\int_{(0,\theta]} e^{-\delta r}\left[\mathcal{R}^{J_r}\varphi(X_{r-}) + c(X_{r-}, J_{r-})\right]\mathbb{1}_{X_{r-}\in\partial E}\, dp^*_r\right] - \beta\,\mathbb{E}_{\boldsymbol{\nu}}^{\bar{x},a_0,a_\Gamma}\left[e^{-\delta\theta}\,\mathbb{1}_{\{\tau\leqslant T\}}\right] \geqslant 0$$

for some $\boldsymbol{\nu} \in \mathcal{V}$. Noticing that, for $r \in [0,\, \theta]$,

$$\delta\,\varphi(X_r) - \mathcal{L}^{I_r}\varphi(X_r) - f(X_r, I_r) \leqslant \delta\,\varphi(X_r) - \inf_{b\in A_0}\{\mathcal{L}^{b}\varphi(X_r) + f(X_r, b)\} \leqslant -\varepsilon,$$
$$-\left(\mathcal{R}^{J_r}\varphi(X_{r-}) + c(X_{r-}, J_{r-})\right) \leq -\min_{d\in A_\Gamma}\{\mathcal{R}^{d}\varphi(X_{r-}) + c(X_{r-}, d) \leq -\varepsilon,$$

from (5.21) we obtain

$$0 \leqslant \frac{\varepsilon}{2\,\delta} - \mathbb{E}_{\boldsymbol{\nu}}^{\bar{x},a_0,a_\Gamma}\left[\varepsilon\int_{(0,\theta]} e^{-\delta r}\, dr + \varepsilon\int_{(0,\theta]} e^{-\delta r}\, dp^*_r + \beta\, e^{-\delta\theta}\,\mathbb{1}_{\{\tau\leqslant T\}}\right] \leqslant -\frac{\varepsilon}{2\,\delta} + \frac{\varepsilon}{\delta}\, e^{-\delta T}.$$

Letting T go to infinity we get the contradiction: $0 \leq -\frac{\varepsilon}{2\,\delta}$. □

References

[1] Abergel, F., Huré, C., Pham, H. Algorithmic trading in a microstructural limit order book model. Preprint HAL, https://hal.archives-ouvertes.fr/hal-01514987, 2017.

[2] Bandini, E. Existence and uniqueness for backward stochastic differential equations driven by a random measure, possibly non quasi-left-continuous. *Electr. Commun. Probab.*, **20**(71):1-13, 2015.

[3] Bandini, E. Optimal control of Piecewise Deterministic Markov Processes: a BSDE representation of the value function. *ESAIM: COCV*, **24**(1):311-354, 2018.

[4] Bandini, E., Fuhrman, M. Constrained BSDEs representation of the value function in optimal control of pure jump Markov processes. *Stochastic Proc. Appl.*, **127**(5):1441-1474, 2017.

[5] Bandini, E., Russo, F. Special weak Dirichlet processes and BSDEs driven by a random measure. *Bernoulli*, **24**(4A), 2569-2609, 2018.

[6] Bandini, E., Cosso, A., Fuhrman, M., Pham, H. Randomization method and backward SDEs for optimal control of partially observed path-dependent stochastic systems. *Ann. Appl. Probab.* **28**(3):1634-1678, 2018.

[7] Barles, G. Solutions de viscositité des équations de Hamilton-Jacobi. Mathématiques et Applications **17**. Paris, Springer-Verlag, 1994.

[8] Cohen, S., Elliott, R. Existence, uniqueness and comparisons for BSDEs in general spaces. *Ann. Probab.* **40**:2264-2297, 2012.

[9] Cohen, S., Elliott, R., Pearce, C.E.M. A general comparison theorem for backward stochastic differential equations. *Advances in Applied Probability*, **42**(3):878-898, 2010.

[10] Confortola, F., Fuhrman, M., Jacod, J. Backward stochastic differential equations driven by a marked point process: an elementary approach, with an application to optimal control. *Ann. Appl. Probab.* **26**(3):1743-1773, 2016.

[11] Cosso, A., Fuhrman, M. and Pham, H. Long time asymptotics for fully nonlinear Bellman equations: a Backward SDE approach. *Stochastic Proc. Appl.*, **126**(7):1932-1973, 2016.

[12] Costa, O.L. and Dufour, F. Continuous Average Control of Piecewise Deterministic Markov Processes. Springer Briefs in Mathematics, Springer, 2013.

[13] Davis, M.H.A. Markov models and optimization. Monographs on Statistics and Applied Probability 49, Chapman and Hall, London, 1993.

[14] Davis, M.H.A., Dempster, M.A.H., Sethi, S.P. and Vermes, D. Optimal capacity expansion under uncertainty. *Adv. Appl. Prob.*, **19**:156-176, 1987.

[15] DAVIS, M.H.A., FARID, M. Piecewise deterministic processes and viscosity solutions. *Stochastic Analysis, Control Optimization and Applications. A Volume in Honour of W. H. Fleming on Occasion of His 70th Birthday*, eds. McEneaney, W. M. et al. , Birkhäuser, 249-268, 1999.

[16] DAVIS, M.H.A., FARID, M. Optimal consumption and exploration: a case study in piecewise-deterministic Markov modelling. *Annals of Operational Research*, A Volume in Honour of W. H. Fleming on Occasion of His 70th Birthday, J.C. Baltzer AG, Science Publishers, **88**:121-137, 1999.

[17] ELIE, R., KHARROUBI, I. BSDE representations for optimal switching problems with controlled volatility. *Stoch. Dyn.* **14**(3):1450003, 15 pp., 2014.

[18] HE, S., WANG, J. and YAN, J. Semimartingale theory and stochastic calculus, Science Press Bejiing New York, 1992.

[19] JACOD, J. Multivariate point processes: predictable projection, Radon-Nikodym derivatives, representation of martingales. *Z. Wahrscheinlichkeitstheorie und Verw. Gebiete* **31**:235-253, 1974/75.

[20] JACOD, J. Calcul stocastique et problèmes de martingales, Lecture Notes in Mathematics **714**, Springer, Berlin, 1979.

[21] JACOD, J. and SHIRYAEV, A.N. Limit Theorems for Stochastic Processes, second edition, Springer, Berlin, 1987.

[22] KHARROUBI, I., LANGRENÉ, N., PHAM, H. Discrete time approximation of fully nonlinear HJB equations via BSDEs with nonpositive jumps. *Ann. Appl. Probab.* **25**(4):2301-2338, 2015.

[23] KHARROUBI, I., PHAM, H. Feynman-Kac representation for Hamilton-Jacobi-Bellman IPDE. *Ann. Probab.* **43**(4):1823-1865, 2015.

[24] KHARROUBI, I., MA, J., PHAM, H., ZHANG J. Backward SDEs with contrained jumps and quasi-variational inequalities. *Ann. Probab.* **38**(2):794-840, 2010.

[25] PENG S. Monotonic limit theorem for BSDEs and non-linear Doob-Meyer decomposition. *Probab. Theory Rel.* **16**:225-234, 2000.

[26] REVUZ, D., YOR, M. Continuous Martingales and Brownian Motion. Grundlehren der mathematischen Wissenschaften. Springer, third edition, 1999.

[27] SAYAH, A. Equations d'Hamilton-Jacobi du premier ordre avec termes intégro-différentiels. Partie I: Unicité des solutions de viscosité. Partie II: Existence de solutions de viscosité. *Comm. in Partial Differential Equations*, **16**(6-7):1075-1093, 1991.

[28] STROOCK, D.W., VARADHAN, S.S. Multidimensional diffusion processes, Springer, 2007.