CALABI-YAU QUOTIENTS WITH TERMINAL SINGULARITIES

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Abstract. In this paper we are interested in quotients of Calabi-Yau threefolds with isolated singularities. In particular, we analyze the case when X/G has terminal singularities. We prove that, if G is cyclic of prime order and X/G has terminal singularities, then G has order lower than or equal to 5.

Introduction

A Calabi-Yau variety (over $\mathbb{C}$) is a projective variety X that has trivial canonical bundle and no non-zero holomorphic $p-$forms for $1 \leq p \leq \mathrm{Dim}(X) - 1$. Even for complex Calabi-Yau threefolds, the ones concerning us in this paper, very little is known about many geometric aspects. For example, a topological classification is very far to be understood and all possible sets of Betti numbers of a Calabi-Yau threefold are not known. Here we are interested in automorphisms of a Calabi-Yau variety. Most of the interesting results in this area are for particular families of Calabi-Yau varieties. For example, Wilson has proved that a Calabi-Yau manifold whose second Chern class is positive on the Kähler cone has a finite automorphism group (see [Wil11]). In [Ogu13], it is shown that the same is true if X is a Calabi-Yau threefold with Picard number 2. In this paper, we would like to say something about groups G of automorphisms of Calabi-Yau threefolds that give quotients with terminal singularities.

The main tool we will use is the holomorphic Lefschetz fixed point formula. If g is an automorphism of a complex threefold and if the fixed points of g are isolated, this formula, in its basic form, gives a relation between the traces of g^* restricted on $H^{0,k}(X)$ and some contributions that depend only on the type of the fixed point, i.e. on the local action of the isotropy group of the fixed point in a suitable neighborhood. First of all, we will use the Lefschetz formula to find conditions that have to be satisfied by prime-order automorphisms with isolated fixed points. Given a natural number $n > 1$ We will use the notation $\mathbb{Z}_n$ to denote the cyclic group of order n. In Theorem 1.8 we will see that, if we allow only terminal

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singularities for the quotient $X/\mathbb{Z}_p$, then p has to be equal to $2, 3$ or 5. Also, we are able to tell precisely the number of the fixed points for each case.

We will then focus on small automorphisms (see Definition 1.4) of order 2, 3 or 5. When g has order 2, a description of the quotient $X/\mathbb{Z}_2$ for all the possible values of $\mathrm{Dim}(\mathrm{Fix}(g))$ is given. It is shown that $\mathrm{Dim}(\mathrm{Fix}(g))$ has always pure dimension and that the quotients with terminal singularities are also the ones with isolated singularities (see Proposition 2.1 and 2.2). When the order of g is 3 or 5 and we allow only isolated singularities, there are several possibilities for the number of fixed points of g. More precisely, we prove in Proposition 2.4 that, if the order of g is 3, we have 9 fixed points if $X/\mathbb{Z}_3$ has terminal singularities or an even number of fixed points otherwise, i.e. when $X/\mathbb{Z}_3$ has Gorenstein singularities and it is a singular Calabi-Yau threefold. If g has order 5, the number of fixed points is studied when g is not symplectic (see Definition 1.1). In Proposition 2.4 it is shown that the minimal number of fixed points is achieved if and only if $X/\mathbb{Z}_5$ is terminal.

Finally, we will present some examples of automorphisms giving cyclic quotients with terminal singularities to show that each $p \in \{2, 3, 5\}$ can, in fact, occur. At the moment it seems very difficult to generalize the above results to other classes of groups. The results of Corollary 1.9 are a first step in this direction. We conclude exposition by analysing some other examples. The first one is a terminal quotient $X/(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$ which shows that we can have terminal singularities also when the group which acts is not cyclic. The others quotients X/G with Gorenstein non isolated singularities (one with G of order 32 and the other with respect to $\mathbb{Z}_2$) and a free quotient by a group of order 16 are also investigated in details.

1. General facts

Let X be a smooth Calabi-Yau threefold and let g be an automorphism of X such that $o(g) = p$ is prime. Denote by G the group generated by g and assume that $\mathrm{Dim}(\mathrm{Fix}(g)) = 0$, i.e., g has only isolated fixed points. Under this assumption, we can apply the holomorphic Lefschetz fixed point formula to g, namely:

$$\sum_{P \in \mathrm{Fix}\, g} \frac{1}{\det(\mathrm{Id} - d_P g)} = \sum_{k=0}^{3} (-1)^k \,\mathrm{Tr}\left(g^*|_{H^{0,k}(X)}\right), \tag{1}$$

where $d_P g$ is the differential map induced by g on $T_P X$. We will investigate the restriction given by this identity under some assumptions on the types of singularities of X/G.

First of all, let's try to understand the contribution given by one point to the right hand side of (1). In a neighbourhood of a fixed point P, the action of g on X can be described in terms of a linearization of g, i.e., the action of $d_P g : T_P X \to$

$T_{g(P)}X = T_PX$. The automorphism g has finite order, so we can diagonalize d_Pg, thus obtaining locally

$$\begin{bmatrix} \omega^{a_1(P)} & 0 & 0 \\ 0 & \omega^{a_2(P)} & 0 \\ 0 & 0 & \omega^{a_3(P)} \end{bmatrix},$$

where $\omega = e^{2\pi i/p}$ and $0 \le a_j(P) \le p-1$ are the exponents determined up to permutation. The local equations for the fixed locus are $z_j(1-\omega^{a_j(P)}) = 0$; if the fixed point is isolated then $a_j(P) > 0$ for $j \in \{1,2,3\}$. Call $s(P) = a_1(P) + a_2(P) + a_3(P)$.

Recall that a Calabi-Yau threefold has $\mathrm{h}^{0,1}(X) = \mathrm{h}^{0,2}(X) = 0$ and trivial canonical bundle. Hence there exists an everywhere non vanishing holomorphic $3-$form which will be denoted by η. Moreover, $H^{0,3}(X) = \eta \cdot \mathbb{C}$, so the action of an automorphism g on $H^{0,3}(X)$ is simply the multiplication by an element of $\mathbb{C}^*$ that is a root of unity of order $o(g)$. Therefore, the right-hand side of Equation (1) is simply $1 - \mathrm{Tr}\left(g^*|_{H^{0,3}(X)}\right) = 1 - \omega^r$ for some r such that $0 \le r \le p-1$.

Definition 1.1. *Let g be an automorphism of a Calabi-Yau threefold. We will say that g is a symplectic automorphism if $g^*|_{H^{0,3}(X)} = \mathrm{Id}$.*

The set $S(X)$ of symplectic automorphisms is easily proven to be a normal subgroup of $\mathrm{Aut}(X)$.

As the following lemma shows, if X is a Calabi-Yau threefold and if we know the local action around a fixed point we can obtain information about the action of g on $H^{0,3}(X)$.

Lemma 1.2. *If X is a Calabi-Yau threefold and $g \in \mathrm{Aut}(X)$, then one of the following holds:*

- $\mathrm{Fix}(g)$ *is empty and* $g \in S(X)$*;*
- $P \in \mathrm{Fix}(g)$ *and the action of g on the stalk $\Omega^3_{X,P}$ is the multiplication by* $\det d_Pg = \omega^{s(P)}$.
- *One has* $\mathrm{Tr}\left(g^*|_{H^{0,3}(X)}\right) = \omega^{s(P)}$ *for every* $P \in \mathrm{Fix}(g)$.

Proof. If $\mathrm{Fix}(g)$ is empty, the holomorphic Lefschetz fixed point formula gives the equation $0 = 1 - \omega^r$, so $r = 0$, and this implies $g \in S(X)$. Now, if there is a fixed point P, let's prove that the action of g on the stalk of the canonical sheaf is given by $\det(d_Pg)$. If $\rho_P(g^*)$ is the map given by g on the stalk over P of the sheaf Ω^3_X, we have the relation $\wedge^3(d_Pg)^* = \rho_P(g^*)$. But d_Pg is a linear automorphism of T_PX whose dimension is 3. This implies that $\wedge^3(d_Pg)$ is the multiplication by $\det(d_Pg)$, thus proving the claim. The last part is an easy consequence of the second. □

Let X be a complex threefold and consider a non trivial automorphism g of X of finite order. Let P be a fixed point of g.

Definition 1.3. *The age of g in P with respect to the primitive root λ of order $o(g)$ is*

$$\mathrm{age}_P(g,\lambda) := (a_1+a_2+a_3)/o(g),$$

where λ^{a_j} are the eigenvalues of $d_P g$ and $0 \le a_j \le o(g)-1$ for $j \in \{1,2,3\}$.

Recall that if V is a vector space and $f : V \to V$ is linear, f is a *quasi-reflection* if $\mathrm{Rk}(f - \mathrm{Id}) = 1$.

Definition 1.4. *A group G acting on a complex manifold is said to be small if for every $g \in G \setminus \{\mathrm{Id}\}$ and every $P \in \mathrm{Fix}(g)$ one has that $d_P g : T_P X \to T_P X$ is not a quasi-reflection.*

This condition is equivalent to asking that $\mathrm{Fix}(G)$ has codimension at least 2. The following theorem recalls some well known facts about some types of singularities (see, for example, [MG84]).

Theorem 1.5. *Let X be a complex threefold and consider a small group G that acts on X. Call $\pi : X \to X/G$ the projection on the quotient and $G_P = \mathrm{Stab}_G(\{P\})$ the isotropy group of P. Then, as set, $\mathrm{Sing}(X/G) = \mathrm{Fix}(G)/G$ and*

- *$\pi(P)$ is a Gorenstein singularity if and only if $d_P g \in \mathrm{SL}(T_P X)$ for each $g \in G_P$;*
- *$\pi(P)$ is a canonical singularity if and only if $\mathrm{age}_P(g,\lambda) \ge 1$ for each primitive λ and for each $g \in G_P$;*
- *$\pi(P)$ is a terminal singularity if and only if $\mathrm{age}_P(g,\lambda) > 1$ for each primitive λ and for each $g \in G_P$;*
- *$\pi(P)$ is a terminal singularity if and only if $\det(d_P g)$ is an eigenvalue of $d_P g$ for each $g \in G_P$.*

1.1. **Quotients with terminal singularities.** From now on, X is a Calabi-Yau threefold. Here we are interested in cyclic groups of prime order which are small and give a quotient with terminal singularities. The following calculation are inspired by the one done in [Sob00] for a Fano threefold.

Assume that g is an automorphism of a Calabi-Yau threefold with a finite number of fixed points and with prime order p. If $P \in \mathrm{Fix}(g)$, consider the sum

$$S_n(P) = \sum_{\substack{0 \le k_1,k_2,k_3 \le p-1 \\ a_1(P)k_1+a_2(P)k_2+a_3(P)k_3 \equiv_p n}} k_1k_2k_3, \tag{2}$$

where $\omega^{a_i(P)}$ are the eigenvalues of $d_P g$ and ω is a $p-$th primitive root of unity.

Theorem 1.6. *Let X be a Calabi-Yau threefold and g an automorphism of prime order p with only isolated fixed points. Let $0 \le r \le p-1$ such that*

$$\mathrm{Tr}\left(g^*|_{H^{0,3}(X)}\right) = \omega^r.$$

Then, the equality

$$\sum_{P\in\mathrm{Fix}(g)}\left(\frac{p^3(p-1)^3}{8}-pS_0(P)\right)=\begin{cases}p^4 & \text{if } r\neq 0\\ 0 & \text{if } r=0\end{cases} \tag{3}$$

holds.

Proof. Starting from the Lefschetz fixed point formula, we have

$$\sum_{P\in\mathrm{Fix}(G)}\frac{1}{\det(I-d_Pg)}=:\Lambda(X,g)=1-\omega^r \tag{4}$$

If, as before, we call ω a root of unity and we let $\omega^{a_i(P)}$ be the eigenvalues of d_Pg one has

$$\sum_{P\in\mathrm{Fix}(G)}\frac{1}{\det(I-d_Pg)}=\sum_{P\in\mathrm{Fix}(G)}\frac{1}{(1-\omega^{a_1(P)})(1-\omega^{a_2(P)})(1-\omega^{a_3(P)})}.$$

If λ is a primitive root of unity, the following relation holds

$$\frac{1}{1-\lambda}=-\frac{1}{p}\sum_{k=1}k\lambda^k.$$

Every $\omega^{a_j(P)}$ is a primitive root. As the fixed locus has dimension 0 we have $a_j\not\equiv_p 0$. We can then write

$$\sum_{P\in\mathrm{Fix}(G)}\frac{1}{\det(I-d_Pg)}=-\frac{1}{p^3}\sum_{P\in\mathrm{Fix}(G)}\prod_{j=1}^{3}\left(\sum_{k_j}k_j\omega^{k_ja_j(P)}\right)=$$

$$=-\frac{1}{p^3}\sum_{P\in\mathrm{Fix}(G)}\sum_{0\le k_1,k_2,k_3\le p-1}k_1k_2k_3\omega^{k_1a_1(P)+k_2a_2(P)+k_3a_3(P)}.$$

The coefficient of ω^n (seeing $\mathbb{Q}(\omega)$ as a $\mathbb{Q}$ vector space with the standard basis) of the contribution to the summation given by P is precisely $-\frac{1}{p^3}S_n(P)$; so we can rewrite the left hand side of the holomorphic Lefschetz formula in the following way:

$$-\frac{1}{p^3}\sum_{P\in\mathrm{Fix}(G)}\sum_{n=0}^{p-1}\left(\sum_{\substack{0\le k_1,k_2,k_3\le p-1\\ a_1(P)k_1+a_2(P)k_2+a_3(P)k_3\equiv_p n}}k_1k_2k_3\right)\omega^n=$$

$$=-\frac{1}{p^3}\sum_{P\in\mathrm{Fix}(G)}\sum_{n=0}^{p-1}S_n(x)\omega^n.$$

We end up with the following formula:

$$\Lambda(X,g)=\sum_{x\in \mathrm{Fix}(G)}\frac{1}{\det(I-d_xg)}=-\frac{1}{p^3}\sum_{P\in \mathrm{Fix}(G)}\sum_{n=0}^{p-1}S_n(P)\omega^n. \tag{5}$$

From Equations (4) and (5) we obtain

$$\sum_{P\in \mathrm{Fix}(G)}\sum_{n=0}^{p-1}S_n(P)\omega^n=p^3\omega^r-p^3$$

$$\sum_{n=0}^{p-1}\left(\sum_{P\in \mathrm{Fix}(G)}S_n(P)\right)\omega^n+p^3-p^3\omega^r=0.$$

It is useful to separate the case $r=0$ and $r\neq 0$, i.e. the symplectic case from the non-symplectic case. Call B_n the coefficient of ω^n in the left hand side of the last equation, namely

$r\neq 0$	$r=0$
$B_n:=\sum_{P\in \mathrm{Fix}(G)}S_n(P)+\begin{cases}0 & \text{if } n\neq 0,r\\ p^3 & \text{if } n=0\\ -p^3 & \text{if } n=r\end{cases}$	$B_n:=\sum_{P\in \mathrm{Fix}(G)}S_n(P)$

The relation

$$\sum_{n=0}^{p-1}B_n\omega^n=0$$

is true if and only if the coefficients B_n are all equal. From $B_0=B_1=\cdots=B_{p-1}$ one has $B_n-B_0=0$ for all $1\leq n\leq p-1$. Hence

$$\sum_{n=1}^{p-1}(B_n-B_0)=0\iff\sum_{n=1}^{p-1}(B_n)-(p-1)B_0=0\iff\sum_{n=0}^{p-1}(B_n)-pB_0=0$$

If we solve for B_n, we have

$r\neq 0$	$r=0$
$\sum_{P\in \mathrm{Fix}(G)}\Big(\big(\sum_{n=0}^{p-1}S_n(P)\big)-pS_0(P)\Big)=p^4$	$\sum_{P\in \mathrm{Fix}(G)}\Big(\big(\sum_{n=0}^{p-1}S_n(P)\big)-pS_0(P)\Big)=0.$

The number $S_n(P)$ is the sum of the product $k_1k_2k_3$ for which $a_1(P)k_1+a_2(P)k_2+a_3(P)k_3\equiv_p n$; hence the sum of the $S_n(P)$ for $0\leq n\leq p-1$ is simply the sum of $k_1k_2k_3$ for $0\leq k_i\leq p-1$. This is equivalent to the third power of the sum of the first $p-1$ integers. In the end, this can be written as

$$\sum_{P\in \mathrm{Fix}(G)}\Big(\sum_{n=0}^{p-1}S_n(P)-pS_0(P)\Big)=\sum_{P\in \mathrm{Fix}(G)}\left(\sum_{0\leq k_i\leq p-1}k_1k_2k_3-pS_0(P)\right)=$$

$$= \sum_{P\in \mathrm{Fix}(G)} \left(\frac{p^3(p-1)^3}{8} - pS_0(P) \right).$$

This is the statement of the Theorem. □

Lemma 1.7. *Let p be a prime and let $1 \leq a, b, c \leq p-1$ such that*

$$((a+b+c) \mod p) \in \{a, b, c\}.$$

Then

$$\sum_{\substack{0\leq k_1,k_2,k_3\leq p-1 \\ ak_1+bk_2+ck_3\equiv_p 0}} k_1k_2k_3 = \frac{p}{2}\left[\frac{p^2(p-1)^2}{4} - \frac{p(p-1)(2p-1)}{6}\right].$$

Proof. See [Sob00]. □

As a consequence of Lemma 1.7, for every point $P \in \mathrm{Fix}(g)$ with

$$((a_1(P) + a_2(P) + a_3(P)) \mod p) \in \{a_1(P), a_2(P), a_3(P)\}$$

we have that $S_0(P)$ depends only on p, the order of the group and not on the values of $a_j(P)$ for $j \in \{1, 2, 3\}$. Note that this condition, by Theorem 1.5, is equivalent to ask that the image of P on the quotient is a terminal singularity.

Theorem 1.8. *Assume that X is a Calabi-Yau threefold and g is an automorphism of prime order p with at most isolated fixed points. Call $G := \langle g \rangle$ and q the number of fixed points of g. If X/G has at most terminal singularities, then one of the following holds:*

- *G acts freely on X ($q = 0$) and the action is symplectic;*
- *G has fixed points and the action of G is not symplectic and, moreover, we have $p \in \{2, 3, 5\}$.*

If the second case occurs, g has $16, 9$ or 5 fixed point if $p = 2, 3$ or 5, respectively.

Proof. Since $\mathrm{Fix}(G)$ is either empty or has dimension 0, the group G is small so the quotient X/G has at most terminal singularities. If $q = 0$, i.e. if G acts freely, by Lemma 1.2, G is symplectic. Assume that there are fixed points. Using Lemma 1.7 one has

$$\sum_{x\in \mathrm{Fix}(g)} \left(\frac{p^3(p-1)^3}{8} - pS_0(x) \right) =$$

$$= \sum_{x\in \mathrm{Fix}(g)} \left(\frac{p^3(p-1)^3}{8} - p\frac{p}{2}\left[\frac{p^2(p-1)^2}{4} - \frac{p(p-1)(2p-1)}{6}\right]\right) =$$

$$= q\left(\frac{p^3(p-1)^3}{8} - p\frac{p}{2}\left[\frac{p^2(p-1)^2}{4} - \frac{p(p-1)(2p-1)}{6}\right]\right) = q\frac{p^3(p^2-1)}{24}$$

which is equal to

$$\begin{cases} p^4 & \text{if } r \neq 0 \\ 0 & \text{if } r = 0, \end{cases}$$

by Theorem 1.6. In the end

$$q\frac{(p^2-1)}{24} = \begin{cases} p & \text{if } r \neq 0 \\ 0 & \text{if } r = 0 \end{cases}$$

So, if there are fixed points, then $r \neq 0$ and the action of G is not symplectic. If we assume to have fixed points (and hence singular points on the quotient), we have that their number is given by

$$q = 24p/(p^2-1).$$

The only values of p for which q is a positive integer are $2, 3$ and 5, for which q is $16, 9$ and 5, respectively. □

Recall that terminal singularities in dimension 3 are isolated (see, for example, [Mat02, Corollary 4-6-6]). As an easy consequence of the last theorem we have the following

Corollary 1.9. *Assume that X is a Calabi-Yau threefold and that $G \leq \operatorname{Aut}(X)$ is a small group such that X/G has at most terminal singularities. Then $|G| = 2^a 3^b 5^c$ for some $a, b, c \in \mathbb{N}$.*

2. Automorphisms of order $p \in \{2, 3, 5\}$

In Section 3 we will give some examples of Calabi-Yau threefolds with quotients having terminal singularities. Now, we will analyse the cases for which $p \in \{2, 3, 5\}$.

2.1. **Small involutions.** Here we are interested in small involutions. For this case, we will not restrict to the case of zero-dimensional fixed locus.

Proposition 2.1. *Let X be a Calabi-Yau threefold and let g be a small involution with fixed points. The following are equivalent:*

(1) g is symplectic;
(2) $\operatorname{Fix}(g)$ contains a curve;
(3) $\operatorname{Fix}(g)$ is smooth of pure dimension 1.

Proof. If $g \in S(X)$, then there exist local coordinates around a fixed points P such that $d_P g$ acts as

$$(z_1, z_2, z_3) \mapsto (z_1, -z_2, -z_3).$$

The fixed locus has local equation $z_2 = z_3 = 0$, and so is smooth at P and the component containing P has dimension 1. To complete the proof, one can use

Lemma 1.2 and see that the same description is true near all the fixed points of g. □

Quotients of the form X/G, where $G = \langle g \rangle$ with g a small involution, are of three types and they are described by their fixed locus.

Proposition 2.2. *Let g be a small involution on a Calabi-Yau threefold X. Call G the cyclic group generated by g. Then one of the following holds:*

- $\mathrm{Fix}(g)$ *is empty, g is symplectic and X/G is a smooth Calabi-Yau threefold.*
- $\mathrm{Dim}(\mathrm{Fix}(g)) = 0$*, g is not symplectic and X/G has precisely* 16 *singular points that are all terminal.*
- $\mathrm{Dim}(\mathrm{Fix}(g)) = 1$*, g is symplectic, X/G is a singular Calabi-Yau threefold whose singular locus has pure dimension* 1*.*

Proof. If g has empty fixed locus, the action of g is free and symplectic, proving the first part. If the fixed locus has dimension 0, the eigenvalues of $d_P g$ are all equal to -1 for each $P \in \mathrm{Fix}(g)$. This implies that g is not symplectic. Moreover, because $(1+1+1) \equiv_2 1$, all the fixed points of g gives terminal singularities on the quotients. By Theorem 1.8 the fixed points are 16 and each of them gives a singular point on the quotient. Finally, if $\mathrm{Fix}(g)$ has dimension 1, X/G is a normal projective threefold with canonical and Gorenstein singularities, i.e., it is a singular Calabi-Yau threefold. □

Remark 2.3. *Calabi-Yau threefold like the one in case* 3 *of Proposition 2.2 are interesting because a crepant resolution always exists and it is a smooth Calabi-Yau.*

2.2. **Automorphism of order three with isolated fixed points.** Assume that g is an automorphism of order 3 with isolated fixed points on a Calabi-Yau threefold. Call λ a primitive root of unity of order 3. By Lemma 1.2, for each fixed point P, $d_P g$ has the same determinant. There are three possible cases, namely $\det(d_P g) = 1, \lambda$ and λ^2. Recall that if P is an isolated fixed point we have $a_i(P) \neq 0$ for all i.

If $\det(d_P g) = 1$, i.e., if g is symplectic, then

$$(\lambda^{a(P)}, \lambda^{b(P)}, \lambda^{c(P)}) \in \{(\lambda, \lambda, \lambda), (\lambda^2, \lambda^2, \lambda^2)\}.$$

Denote by n_1 the number of fixed points such that

$$(\lambda^a(P), \lambda^b(P), \lambda^c(P)) = (\lambda, \lambda, \lambda)$$

and by n_2 the number of points of the other type and define x_1, x_2 to be the numbers

$$x_1 = \frac{1}{(1-\lambda)^3} \qquad \text{and} \qquad x_2 = \frac{1}{(1-\lambda^2)^2}.$$

It is easy to see that

$$x_1 = \frac{1}{(1-\lambda)^3} = \pm\frac{i\sqrt{3}}{9}$$

and that $\bar{x}_1 = x_2$. Hence, the holomorphic Lefschetz fixed point formula in this case is simply

$$n_1 x_1 + n_2 \bar{x}_1 = 0.$$

Being x_1 pure imaginary, one has $n_1 = n_2$. In particular, $|\,\mathrm{Fix}(g)|$ is even.

If $\det(d_P g) = \lambda$ then, for each $P \in \mathrm{Fix}(g)$, $(\lambda^{a(P)}, \lambda^{b(P)}, \lambda^{c(P)}) = (\lambda, \lambda, \lambda^2)$ up to permutations. This implies that every point will give a terminal point on the quotient. So, by Theorem 1.8, the fixed points are 9. The case for which $\det(d_P g) = \lambda^2$ is similar.

We have proved the following Proposition:

Proposition 2.4. *Let g be an automorphism of order 3 on a Calabi-Yau threefold X. Call G the cyclic group generated by g and assume that it has isolated fixed points. Then one of the following holds:*

- *g is symplectic and X/G is a singular Calabi-Yau threefold with an even number of singular points.*
- *g is not symplectic, $|\,\mathrm{Fix}(g)| = 9$ and X/G has exactly 9 singular terminal points (and no other singular points).*

2.3. **Automorphism of order five with isolated fixed points.** Consider now an automorphism g of order 5 with isolated fixed points. We have seen that if $G = \langle g \rangle$ is such that X/G has terminal singularities, then $|\,\mathrm{Fix}(G)| = 5$. Now we will show that if g is not symplectic (and it has isolated fixed points) then it has at least 5 fixed points and the minimum is achieved if and only if X/G has terminal singularities.

Recall that, given an isolated fixed point P, we have defined

$$S_n(P) = \sum_{\substack{0 \le k_1, k_2, k_3 \le p-1 \\ a_1(P)k_1 + a_2(P)k_2 + a_3(P)k_3 \equiv_p n}} k_1 k_2 k_3$$

and that, if $g \notin S(X)$, we have

$$\sum_{x \in \mathrm{Fix}(g)} \left(\frac{p^3(p-1)^3}{8} - pS_0(x) \right) = p^4$$

by Theorem 1.6. If we define

$$A := \{(4,1,1), (3,2,1), (4,2,1), (3,2,2), (4,3,1), (3,3,2), (4,4,1), (4,3,2)\} \text{ and}$$

$$B := \{(2,2,2), (4,4,3), (3,3,1), (4,4,4), (1,1,1), (4,2,2), (2,1,1), (3,3,3)\},$$

these sets correspond to all the possible values for $(a(P), b(P), c(P))$ for the case $g \notin S(X)$. A is precisely the set for $(a(P)+b(P)+c(P) \mod 5) \in \{a(P), b(P), c(P)\}$, i.e. the set corresponding to P that gives terminal singularities on X/G. By direct inspection, we see that $S_0(P)$ is equal to 175 if and only if $(a(P), b(P), c(P)) \in A$. The values that $S_0(P)$ can assume in the other case are 200 and 225. Call n, q_1, q_2 the number of points for which $S_0(P)$ is equal respectively to $175, 200$ and 225. In particular, X/G has $n + q_1 + q_2$ singular points and exactly n are terminal. The relation in Theorem 1.6 is then

$$p^4 = \sum_{x \in \mathrm{Fix}(g)} \left(\frac{p^3(p-1)^3}{8} - pS_0(x) \right) =$$

$$= n \left(\frac{p^3(p-1)^3}{8} - 175p \right) + q_1 \left(\frac{p^3(p-1)^3}{8} - 200p \right) + q_2 \left(\frac{p^3(p-1)^3}{8} - 225p \right) =$$

$$= (n+q_1+q_2) \left(\frac{p^3(p-1)^3}{8} - 175p \right) - 25pq_1 - 50pq_2 = (n+q_1+q_2)5^3 - 5^3 q_1 - 2 \cdot 5^3 q_2$$

that is

$$5 = (n + q_1 + q_2) - q_1 - 2q_2 = n - q_2 \quad \text{so that} \quad n = 5 + q_2.$$

This implies that the number of fixed points is $5 + q_1 + 2q_2$. In particular it is at least 5 and, moreover, it is equal to 5 if and only if $q_1 = q_2 = 0$.

Proposition 2.5. *Let g be a non symplectic automorphism of order 5 on a Calabi-Yau threefold X. Assume that it has a finite number of fixed points and let G be the cyclic group generated by g. Then one of the following holds:*

- $|\mathrm{Fix}(G)| = 5$ *and X/G has only terminal singularities.*
- $|\mathrm{Fix}(G)| > 5$*, X/G has 5 or more terminal singularities and at least another fixed point.*

3. Some examples

3.1. **Quotient with terminal singularities.** Here we will construct quotients of Calabi-Yau threefolds with only terminal singularities with respect to $\mathbb{Z}_3, \mathbb{Z}_5, \mathbb{Z}_2$ and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. The last example is indeed interesting because it shows that there are groups which are not cyclic but have isolated fixed points which gives terminal singularities on the quotient.

Example 1
Let X be equal to $\mathbb{P}^2 \times \mathbb{P}^2$ with projective coordinates x_i and y_i on the two factors of X. Consider the automorphism of X defined by

$$g((x_0 : x_1 : x_2) \times (y_0 : y_1 : y_2)) := (x_0 : x_1 : \lambda x_2) \times (y_0 : \lambda y_1 : \lambda^2 y_2)$$

with λ a primitive root of unity of order 3. It is easy to see that g has order 3 and that its fixed locus has 6 irreducible components, three of which are rational curves. More precisely, if $S := \{(1:0:0),(0:1:0),(0:0:1)\}$, then

$$\mathrm{Fix}(g) = (\{(x_0 : x_1 : 0) | (x_0 : x_1) \in \mathbb{P}^1\} \cup \{(0:0:1)\}) \times S.$$

If $G = \langle g \rangle$, define V to be the vector space $H^0(X, -K_X)^G$ of all $G-$invariants anticanonical sections. By direct computations, we see that the generic element of V is smooth (because V has empty base locus) and does not vanish on any of isolated points of $\mathrm{Fix}(g)$. The generic element gives an invariant Calabi-Yau Y that intersect only the components of $\mathrm{Fix}(g)$ having dimension 1. For Y generic, none of these three curves is contained in Y, so this implies that Y meets each one in exactly 3 points (by simple calculation of intersection theory). In total there are 9 fixed points of g that are on Y. By our discussion on the number of fixed points of an automorphism of order three on a Calabi-Yau threefold (see Proposition 2.4), we can conclude that $Y/G = Y/\mathbb{Z}_3$ has exactly 9 singular points and that each of them are terminal.

Example 2
Let X be equal to $\mathbb{P}^4$ with projective coordinates x_i. Take g to be the automorphism of X defined by

$$g((x_0 : x_1 : x_2 : x_3 : x_4) := (x_0 : x_1 : \lambda x_2 : \lambda^2 x_3 : \lambda^3 x_4).$$

with $\lambda^5 = 1$ and primitive. Call G the group generated by g and Y the Fermat hypersurface of degree five in $\mathbb{P}^4$. Y is a smooth Calabi-Yau threefold and is easily seen to be invariant with respect to G. The fixed locus of g on X is

$$\mathrm{Fix}(g) = \{(x_0 : x_1 : 0 : 0 : 0) | (x_0 : x_1) \in \mathbb{P}^1\} \cup$$

$$\cup \{(0:0:1:0:0),(0:0:0:1:0),(0:0:0:0:1)\}$$

and Y does not meet the isolated points of $\mathrm{Fix}(g)$. The intersection of Y with $\mathrm{Fix}(g)$ are 5 isolated points. In particular, $Y/G = Y/\mathbb{Z}_5$ has exactly 5 isolated singularities and each of them is terminal (by Proposition 2.5).

Example 3
Set $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ with $(x_i : y_i)$ that are projective coordinates on the $i-$th $\mathbb{P}^1$. In [BF11] and [BFNP13] the authors study the automorphisms of Calabi-Yau manifolds embedded in X that have empty fixed locus. The authors produce a classification of all the admissible pairs in X, i.e. the pairs (Y, G) where Y is a Calabi-Yau threefold and G is a finite group of authomorphisms of X that stabilizes Y and acts freely on Y. Here we will show that one can easily constuct examples with a different kind of fixed points.

Every $g \in \mathrm{Aut}(X)$ acts on the 4 factors (see, for instance, [BF11]) giving a surjective homomorphism $\pi\colon \mathrm{Aut}(X) \to S_4$ with kernel $\mathrm{PGL}(2)^{\times 4}$. On the other hand the permutations of the factors give an inclusion $S_4 \hookrightarrow \mathrm{Aut}(X)$ splitting π and therefore giving a structure of semidirect product

$$\mathrm{Aut}(X) \cong S_4 \ltimes \mathrm{PGL}(2)^{\times 4}$$

Concretely this gives, $\forall g \in \mathrm{Aut}\, X$, a unique decomposition $g = (A_i) \circ \sigma$ where $\sigma = \pi(g)$ and $(A_i) = (A_1, A_2, A_3, A_4) \in \mathrm{PGL}(2)^{\times 4}$. Denote with A and B the automorphisms of $\mathbb{P}^1$ that send $(x_1 : y_1)$ respectively in $(x_1 : -y_1)$ and $(y_1 : x_1)$.

Call $g := (\mathrm{Id}, \mathrm{Id}, A, A) \circ (12)$ and $h := (A, A, \mathrm{Id}, \mathrm{Id}) \circ (34)$. It is easy to see that $G := \langle g, h\rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$. As an automorphism of X, g have fixed locus composed of 4 rational curves. These are

$$\{(P, P, Q_1, Q_2) \,|\, P \in \mathbb{P}^1,\ Q_1, Q_2 \in \{(1 : 0), (0 : 1)\}\}.$$

Call C a component of $\mathrm{Fix}(g)$ and consider a generic element $Y \in | - K_X|$. It is easy to see that $Y \cdot C = 4$ so generically one expect that $Y \cap C$ has 4 fixed points. A similar result holds for h. For $gh = (A, A, A, A) \circ (12)(34)$ one can see that $\mathrm{Fix}(gh)$ is composed of isolated points so generically, one expect that the generic member of $| - K_X|$ does not meet $\mathrm{Fix}(gh)$. Call V the vector space $H^0(X, -K_X)^G$ of all $G-$invariants anticanonical sections. By direct computations one see that V has empty base locus so the generic element is a smooth Calabi-Yau that admits an action of G.

We point out that the Calabi-Yau threefold Y constructed as the zero locus of the generic section of V does not contain $\mathrm{Fix}(g)$ so that it meets Y at isolated points. By Theorem 1.8, g is an involution with exactly 16 fixed points and acts as -1 on $H^{0,3}(X)$. The same is true for h and gh which have to act as Id on $H^{0,3}(X)$. Because gh is an involution and $(gh)^*|_{H^{0,3}(X)} = \mathrm{Id}$ one has that $\mathrm{Fix}(gh)$ is either empty or it has pure dimension 1. The latter cannot occour because gh has a finite number of fixed points on X so gh acts freely on Y.

By direct computation the common fixed points of g and h on X are 4, namely:

$$\{(P, P, Q, Q) \,|\, P, Q \in \{(1 : 0), (0 : 1)\}\}.$$

The generic invariant section is not zero on any of these point so, for the generic invariant Calabi-Yau Y, the fixed locus of g and the one of h are disjoint.

To conclude, for Y generic, we have given four quotients, namely: $Z_1 = Y/\langle g\rangle$, $Z_2 = Y/\langle h\rangle$, $Z_3 = Y/\langle gh\rangle$ and $Z_4 = Y/G = Y/(\mathbb{Z}_2 \oplus \mathbb{Z}_2)$. The manifold Z_3 is smooth because gh acts freely on Y; thus the map $Y \to Y/\langle gh\rangle$ is an étale cover of degree 2. The varieties Z_1, Z_2 and Z_4 have only terminal singularities by construction. Z_1 and Z_2 have exactly 16 singular points. To compute the number of singular points of Z_4 we can use Burnside's Lemma on the action of G restricted on $\mathrm{Fix}(G)$. In fact $\mathrm{Sing}(Y/G) = \mathrm{Fix}(G)/G$ because G is small. The formula for this case is

$$|\operatorname{Sing}(Y/G)| = |\operatorname{Fix}(G)/G| = \frac{1}{|G|}\sum_{g\in G} |\operatorname{Fix}(g)| =$$

$$= \frac{1}{|G|}(\operatorname{Fix}(\mathrm{Id}) + \operatorname{Fix}(g) + \operatorname{Fix}(h) + \operatorname{Fix}(gh)) = \frac{1}{4}(32 + 16 + 16 + 0) = 16.$$

In conclusion Z_4, as Z_1 and Z_2, has exactly 16 isolated and terminal singularities.

Another way to see this fact is to note that $\langle gh\rangle >$ is normal in G with quotient generated by an involution which we will denote $\hat{g}$. We can write the quotient $Z_4 = Y/G$ as $(Y/\langle gh\rangle)/(G/\langle gh\rangle) = Z_3/(G/\langle gh\rangle)$. Hence, Y/G can be seen as the quotient of a smooth Calabi-Yau threefold by the single involution $\hat{g}$, which has isolated fixed points. By our classification of fixed locus of involutions on Calabi-Yau manifolds, $\hat{g}$ has exactly 16 fixed points. Since we have

$$\operatorname{Sing}(Y/G) = \operatorname{Fix}(\hat{g})/\langle\hat{g}\rangle,$$

we obtain again that Z_4 has 16 fixed points.

3.2. **A quotient with non-isolated Gorenstein singularities.** Here we will construct a quotient of a Calabi-Yau by a group of order 32 that is contained in $S(X)$ whose singular locus has pure dimension 1. Moreover, one of its subgroups of order 16 acts freely on the same Calabi-Yau threefold. This is interesting because it shows that all the cases of Proposition 2.2 are realized.

Example 4
Use the same notation introduced in the last example. Call

$$g := (\mathrm{Id}, \mathrm{Id}, \mathrm{Id}, A) \circ (1324), h := (B, B, B, B), k := (14)(23)$$

and consider the group $G := \langle g, h, k\rangle$. The elements g, h and k satisfy the following relations

$$g^8 = h^2 = k^2 = 1 \quad gh = hg \quad kh = hk \quad gk = kg^{-1}.$$

Using these relations it is easy to see that G has 32 elements and that G is isomorphic to $D_{16} \times \mathbb{Z}_2$ where $D_{16} = \langle g, k\rangle$ is the dihedral group with 16 elements and $\mathbb{Z}_2 = \langle h\rangle$. The elements of G can be written uniquely as $g^a h^b k^c$ with $0 \le a \le 7$ and $0 \le b, c \le 1$. All the elements with $c = 1$ are involutions. If we consider the action of G on X then $\operatorname{Fix}(g^a h^b)$ has a finite number of points whereas $\operatorname{Fix}(k)$ has pure dimension 2 and the same is true for $\operatorname{Fix}(g^a h^b k)$. If we denote by $V := H^0(X, K_X)^G$ it can be seen that the Calabi-Yau Y given by the generic $s \in V$ is smooth and satisfies

$$\operatorname{Fix}(g^a h^b) \cap Y = \emptyset \qquad \operatorname{Dim}(\operatorname{Fix}(k) \cap Y) = 1$$

for all a, b. The involution k acts on Y with a fixed locus of dimension 1. By Proposition 2.2 this implies that $k \in S(Y)$. The two elements g and h do not have fixed points on Y so they are elements of $S(Y)$. This is enough to conclude that $G \leq S(Y)$. It can be shown that $\mathrm{Aut}(Y) = S(Y)$ for the generic $G-$invariant Calabi-Yau threefold.

By the Moishezon-Nakai criterion, being $Y \in |-K_X|$ ample, Y cannot be disjoint from $\mathrm{Fix}(g^a h^b k)$ that has dimension 2 on X. We know that $g^a h^b k$ is a symplectic involution so we conclude that $\mathrm{Fix}(g^a h^b k) \cap Y$ is smooth of pure dimension 1 for each a, b.
All the irreducible components of $\mathrm{Fix}(G)$ are obtained from the irreducible components of $\mathrm{Fix}(k), \mathrm{Fix}(gk), \mathrm{Fix}(hk)$ and $\mathrm{Fix}(ghk)$ using the fact that $\mathrm{Fix}(b^{-1}ab) = b(\mathrm{Fix}(a))$ and $x^{-1}(g^a h^b k)x = g^{a+2d}h^b k$ for $x \in G$. Using MAGMA, it is possible to check that there are at least 32 components and some of them are rational curves. Starting from these remarks one can use MAGMA in order to prove that there are several different orbits for the action of G on $\mathrm{Fix}(G)$ (more precisely, there are at least 4 orbits).

It is interesting to note that $H := \langle g, h\rangle \simeq \mathbb{Z}_8 \times \mathbb{Z}_2$ is a subgroup of G of index 2 and thus is a normal subgroup of G. The quotient Y/G can be viewed as $(Y/H)/(G/H)$. It is interesting to write the quotient like this because $G/H \simeq \mathbb{Z}_2$ and it is generated by an involution which we will denote $\hat{k}$. Moreover $\mathrm{Fix}(H)$ is empty (the pair (Y, H) is one of the admissible pairs studied in [BFNP13]) so $Y \to Y/H$ is an ètale cover of degree 16 and Y/H is again a smooth Calabi-Yau threefold. Hence, as we were saying before, Y/G may be viewed as the quotient of the smooth Calabi-Yau threefold Y/H by the group G/H. As the class of k in G/H, i.e. $\hat{k}$, generate the group and acts trivially on $\mathrm{Fix}(\hat{k})$, we have that

$$\mathrm{Fix}(\hat{k}) = \mathrm{Fix}(\hat{k})/\langle\hat{k}\rangle = \mathrm{Sing}(Y/G) = \mathrm{Fix}(G)/G,$$

so the singular locus of Y/G has at least 4 irreducible components and at least one of them is a rational curve.

The quotient Y/H is also interesting because it shows, as we have already seen, that there are groups which acts on Calabi-Yau threefolds without fixed points. In particular, H is a symplectic group. As already pointed out, in [BF11] and [BFNP13] one can find several examples of symplectic groups which acts freely on Calabi-Yau threefolds.

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