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Crystallographic Coxeter groups and Kac' denominator formula

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Introduction

This thesis explores the connection between a crystallographic Coxeter system (W, S) and the roots of an arbitrary Kac-Moody Lie algebra associated with a symmetrizable Generalized Cartan Matrix (GCM) with Weyl group W through the Kac' denominator formula:

$$\sum_{w \in W} (-1)^{\ell(w)} e(c(w)) = \prod_{\alpha \in \Lambda_{\mathbb{Z}}^{+}} (1 - e(\alpha))^{m_\alpha}, \tag{1}$$

Here:

1. W is a crystallographic Coxeter group;
2. $e(\alpha)$ is a formal exponential;
3. $\Lambda_{\mathbb{Z}}^{+}$ is the positive root cone;
4. $\ell : W \to \mathbb{N}_0$ is the length on (W, S);
5. $m_{\cdot} : \Lambda_{\mathbb{Z}}^{+} \to \mathbb{N}_0$ is the multiplicity function;
6. $c(w)$ is the sum of positive roots being sent by w^{-1} to negative ones.

The formula (1) is *parametrized* when we explicitly know the function $m_{\cdot}$. In this case we know for which elements of the positive cone $\Lambda_{\mathbb{Z}}^{+}$ (called *weights*) the multiplicity is different from 0; these elements are called *roots* of a Kac-Moody Lie algebra.

The formula (1) was firstly discovered and proved by I. G. Macdonald for affine Kac-Moody Lie algebras. Subsequently it was extended to the entire class of Kac-Moody Lie algebras associated with a symmetrizable GCM by V. G. Kac.

In order to understand the formula (1) better, the reader should keep in mind that the Weyl group W of a Kac-Moody Lie algebra associated with a symmetrizable GCM is a crystallographic Coxeter group (cf. [8]). The left hand-side of (1) depends entirely on the Coxeter group. One of the most important goals is to calculate this left hand-side of (1) for crystallographic Coxeter groups with ∞-decomposition (cf. Definition 0.1 and Formula (43)). This is possible because the left hand-side of (1) can be reinterpreted with the complete growth series (cf. Section 1.12.2) applied to the trace (9) and using several facts on the complete growth series of crystallographic Coxeter groups with ∞-decomposition.

The *geometric representation* of the crystallographic Coxeter group (W, S)

$$\rho : W \to GL(\mathbb{R}, V) \tag{2}$$

allows us to consider the elements $s \in S$ as reflections, called *simple reflections*, with respect to vectors $\alpha_s \in V$, called *simple roots* (cf. [16]).
Moreover the set

$$\Phi = \{w(\alpha_s) \mid s \in S, \ w \in W\} \tag{3}$$

is called the *root system* of (W, S).
We recall that every $\alpha \in \Phi$ can be written as

$$\alpha = \sum_{s \in S} k_s \alpha_s. \tag{4}$$

Moreover, if $k_s \in \mathbb{Z}_{\geq 0}$ for all $s \in S$ then α is a *positive root*, if $k_s \in \mathbb{Z}_{\leq 0}$ for all $s \in S$ then α is a *negative root*. We denote by Φ^+ the set of positive roots and by $\Phi^- = -\Phi^+$ the set of negative roots (cf. [8]).
In the root system of a Kac-Moody Lie algebra associated with a symmetrizable GCM with Weyl group W, there are two types of roots: real and imaginary. The set of *real roots* $\dot{\Phi}_{Re}$ coincides with (3), while the set of *imaginary roots* is

$$\dot{\Phi}_{Im} = \{\alpha \in \Lambda_{\mathbb{Z}} \mid \alpha \notin \Phi_{Re}, \ m_\alpha \neq 0\} \tag{5}$$

where we define $\Lambda_{\mathbb{Z}} := \mathbb{Z}\dot{\Phi}$ the *root lattice* (cf. [8]).

The left hand side of (1) can be seen as a formal power series depending entirely on (W, S) (cf. Chapter 4). It captured our attention because it involves a map

$$c : W \to \Lambda^+_{\mathbb{Z}}, \tag{6}$$

where $\Lambda^+_{\mathbb{Z}} := \mathbb{Z}_{\geq 0}\dot{\Phi}^+$ is the *positive cone*.
Since the geometric representation of W allows us to consider W as a reflections group, where $\Lambda_{\mathbb{R}} := \mathbb{R}\dot{\Phi}$ is endowed with a symmetric bilinear form $\langle , \rangle$. Therefore for all $s \in S$, we can define the *simple coroot* λ_s as the unique element of $\Lambda_{\mathbb{Z}}$ such that

$$\langle \lambda_s, \alpha_{s'} \rangle = \frac{1}{2}\delta_{ss'}\langle \alpha_s, \alpha_s \rangle, \quad s' \in S \tag{7}$$

(cf. [8]).

In Chapter 1 (cf. Section 2.4) we show some properties of the map (6), in particular that it is a 1-cocycle that can be written as:

$$c(w) = (1 - w) \cdot \omega_0 \tag{8}$$

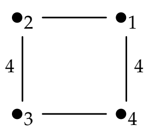
where

$$\omega_0 = \sum_{s \in S} \lambda_s \tag{9}$$

is called *trace*.

Kac′ denominator formula has been already established by I. G. McDonald (cf. [21]) for some classes of Coxeter groups. The class of Coxeter groups is split into three principal types: finite Coxeter groups, affine Coxeter groups and indefinite Coxeter groups. Kac′ denominator formula has just been parametrized for finite and affine Coxeter groups (cf. [8]) and also for some indefinite Coxeter groups (cf. [7]). Then we must work with the class of indefinite Coxeter groups. There is one class of indefinite Coxeter groups, the one of the so-called hyperbolic Coxeter groups. These groups have attracted much attention among the mathematicians in recent years as F. Lannér and J. E. Humphreys.

A Coxeter group is *cocompact* if after removing any edge in the Coxeter graph, one obtains a graph of a Coxeter group of finite or affine type. There is only one isomorphism class of cocompact crystallographic hyperbolic Coxeter systems $(\mathfrak{W}, \mathfrak{S})$, whose Coxeter graph is $\Gamma(\mathfrak{W})$:



(10)

$\Gamma(\mathfrak{W})$ sets the relations between the four generators s_1, s_2, s_3, s_4 of $\mathfrak{W}$:

1. if there are no edges linking i and j, with $i \neq j$, then $(s_i s_j)^2 = 1$;
2. if a non labelled edge links i and j, then $(s_i s_j)^3 = 1$;
3. if a labelled edge with an integer m links i and j, then $(s_i s_j)^m = 1$.

Moreover $s_i^2 = 1$ for $i = 1, \ldots, 4$.
In this thesis we spent much effort to represent $(\mathfrak{W}, \mathfrak{S})$ as a cocompact arithmetic lattice of $O^+_{\mathbb{R}}(3, 1)$, the orthogonal group of matrices with real entries that stabilizes a bilinear form of signature $(3, 1)$ and with positive entry in position $(1, 1)$, generalising a result of J. Elstrodt, F. Grunewald, J. Mennicke (cf. [11]).

In Chapter 3 we establish an explicit description of the Coxeter system $(\mathfrak{W}, \mathfrak{S})$ in the Lie group $O^+_{\mathbb{R}}(3, 1)$ using the following idea. By definition $\mathfrak{W}$ acts on the Lorentzian space of signature $(3, 1)$ (cf. Chapter 3). Then we consider a homomorphism of Lie groups (cf. [11])

$$\Psi : SL(2, \mathbb{C}) \to SO^{+}_{\mathbb{R}}(3, 1), \tag{11}$$

where $SO^{+}_{\mathbb{R}}(3, 1)$ is the special orthogonal group of matrices with real entries that stabilizes a bilinear form of signature $(3, 1)$ and with a positive entry in position $(1, 1)$.
It can be extended to an *exceptional isomorphism*

$$\tilde{\Psi} : SL(2, \mathbb{C}) \to O_{\mathbb{R}}(3, 1) \tag{12}$$

through external involutions. Indeed we considered the involution σ that acts on a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ of $SL(2, \mathbb{C})$ in the following way:

$$\sigma A \sigma = \begin{pmatrix} \overline{a} & \overline{b} \\ \overline{c} & \overline{d} \end{pmatrix}, \tag{13}$$

where $\overline{} : \mathbb{C} \to \mathbb{C}$ is the complex conjugation.

At this point we use two well-known facts. Let Q be a quadratic form and S_Q be the symmetric bilinear form relative to Q over a filed $\mathbb{K}$. Let ι be a real embedding of $\mathbb{K}$, then define the quadratic form $Q^{[\iota]}$ as $Q^{[\iota]}(x) := x^T \iota(S_Q)x$, where x^T is the transpose of x.
The following theorem describes the discrete orthogonal group constructed with the help of Q and insures that $\mathfrak{W}$ is commensurable to a subgroup of $PO_{\mathbb{R}}(3, 1)$.

Theorem 0.1 *(cf. [11]) Let $\mathbb{K}$ be a totally real number field with ring of integers $\mathcal{O}_{\mathbb{K}}$. Let Q be a quadratic form in four variables over $\mathbb{K}$ satisfying the hyperbolic signature condition. Let ι be a real embedding of $\mathbb{K}$ so that $Q^{[\iota]}$ is of signature $(1, 3)$. Define*

$$\Gamma(\mathcal{O}_{\mathbb{K}}, Q) := \iota(PO_4(\mathcal{O}_{\mathbb{K}}, Q)) < PO_4(\mathbb{R}, Q^{[\iota]}).$$

Then the following hold

1. *$\Gamma(\mathcal{O}_{\mathbb{K}}, Q)$ is a discrete subgroup of $PO_4(\mathbb{R}, Q^{[\iota]})$.*
2. *$\Gamma(\mathcal{O}_{\mathbb{K}}, Q)$ is a cocompact if and only if Q is $\mathbb{K}$-anisotropic.*

□

As quadratic form we have considered $\mathfrak{Q}(x_0, x_1, x_2, x_3) = -7x_0^2 + x_1^2 + x_2^2 + x_3^2$ in 4 variables over $\mathbb{Q}(i\sqrt{7})$ (cf. Chapter 3).

Let $\mathcal{A} := \left(\frac{a,b}{\mathbb{K}}\right)$ be the $\mathbb{K}$-quaternion algebra with basis $\mathtt{1}, \mathtt{i}, \mathtt{j}, \mathtt{k}$ such that

$$\mathtt{i}^2 = a, \ \ \mathtt{j}^2 = b, \ \ \mathtt{ij} = -\mathtt{ji} = k. \tag{14}$$

Let $\mathbb{L}$ be a field extension of $\mathbb{K}$ such that $\sqrt{a}, \ \sqrt{b} \in \mathbb{L}$, then the map

$$\phi : \mathcal{A} = \left(\frac{a,b}{\mathbb{K}}\right) \to M(2, \mathbb{L})$$

$$\phi(x_0 + x_1\mathtt{i} + x_2\mathtt{j} + x_3\mathtt{k}) = \begin{pmatrix} x_0 + x_1\sqrt{a} & x_2\sqrt{b} + x_3\sqrt{ab} \\ x_2\sqrt{b} - x_3\sqrt{ab} & x_0 - x_1\sqrt{a} \end{pmatrix} \tag{15}$$

is an injective $\mathbb{K}$-algebra homomorphism with the property that the norm of an element of $\mathcal{A}$ is the determinant of the image of that element through ϕ. (cf. [11]).
We take the map (cf. Chapter 3)

$$\phi : \mathcal{A} = \left(\frac{-1,-1}{\mathbb{Q}(i\sqrt{7})}\right) \to M(2, \mathbb{Q}(i, \ \sqrt{7})). \tag{16}$$

The following theorem describes the discrete and cocompact subgroups of $SL(2, \mathbb{C})$.

Theorem 0.2 *(cf. [11]) Let $\mathbb{K}$ an algebraic number field with exactly one pair of complex embeddings. Let $\mathcal{A}$ be a quaternion algebra over $\mathbb{K}$ which is ramified at all real embeddings of $\mathbb{K}$. For a complex embedding v_0 let $\phi : \mathcal{A} \otimes_{\mathbb{K}} \mathbb{K}_{v_0} \to M(2, \mathbb{C})$ be a $K_{v_0} = \mathbb{C}$ -algebra isomorphism. For an order $\mathcal{R} \subset \mathcal{A}$, put*

$$\Gamma = \phi(\mathcal{R}^1),$$

where R^1 is the group of the elements of R with norm 1.
The group Γ has the following properties:

1. *Γ is a discrete subgroup of $SL(2, \mathbb{C})$.*

2. *Γ is cocompact if and only if $\mathcal{A}$ is a skew field.*

□

In conclusion we exhibited the generators of $\mathfrak{W}$ in $\langle\sigma\rangle \ltimes PSL(2, \mathbb{C})$ through an explicit formula (cf. Section 3.4.6) generalizing the results of Masaaki Yoshida (cf. [35]).
Moreover, generalising some results of A. Feingold, I. Frenkel (cf. [12]), we obteined a description of the root system of a Kac-Moody Lie algebra with

Weyl group $(\mathfrak{W}, \mathfrak{S})$ defining a $\mathfrak{W}$-equivariant linear map (cf. Chapter 3)

$$\mu : \Lambda_{\mathbb{Z}} \to \{A \in M(2, \mathbb{C}) \mid det(A) \in \mathbb{R}\} \tag{17}$$

such that

$$\mu(\alpha_1) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \ \mu(\alpha_2) = \begin{pmatrix} -1-i & 0 \\ 0 & 1-i \end{pmatrix},$$
$$\mu(\alpha_3) = \begin{pmatrix} 2i & 0 \\ 0 & 2i \end{pmatrix}, \ \mu(\alpha_4) = \begin{pmatrix} 1-i & -3-\sqrt{7} \\ -3+\sqrt{7} & -1-i \end{pmatrix}, \tag{18}$$

with the property that the square of norm of a root is the determinant of the matrix that corresponds to the root through μ.
In general it is extremely difficult to determine or characterize all imaginary roots but thank to this property and to the following result of R. V. Moody (cf. [23]) about the roots of the hyperbolic Kac-Moody Lie algebras

$$\dot{\Phi}_{Im} = \{\alpha \in \Lambda_{\mathbb{Z}} \mid \langle \alpha, \alpha \rangle \leq 0\}, \tag{19}$$

it is possible in this case (cf. Chapter 3), i.e. one has

$$\mu(\dot{\Phi}_{Re}) = \{X \in \mu(\Lambda_{\mathbb{Z}}) \mid det(X) = -2, -4\}, \tag{20}$$

$$\mu(\dot{\Phi}_{Im}) = \{X \in \mu(\Lambda_{\mathbb{Z}}) \mid det(X) \geq 0\}. \tag{21}$$

In Chapter 4 we study crystallographic Coxeter systems (W, S) with a *spherically ∞-decomposition*. More precisely:

Definition 0.1 *Let $(S_\blacktriangledown, S_\blacktriangle)$ be a pair of subsets of S satisfying:*

1. *$S = S_\blacktriangledown \cup S_\blacktriangle$;*
2. *let $S_\bullet := S_\blacktriangledown \cap S_\blacktriangle$. For all $s \in S_\triangledown := S_\blacktriangledown \setminus S_\bullet$ and $t \in S_\vartriangle := S_\blacktriangle \setminus S_\bullet$ one has $m_{s,t} = \infty$, where $(m_{s,t})_{s,t \in S}$ is the Coxeter matrix of (W, S).*

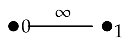
$(S_\blacktriangledown, S_\blacktriangle)$ will be called an ∞-decomposition of (W, S). □

An ∞-decomposition is said to be:

1. *non-trivial* if $S_\triangledown \neq \emptyset$ and $S_\vartriangle \neq \emptyset$;
2. a *spherical* ∞-decomposition if additionally the parabolic subgroup $W_\bullet = W_{S_\bullet}$ is finite.

Some examples of crystallographic Coxeter groups with a spherically ∞-decomposition are:

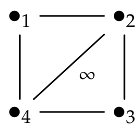
1. $\tilde{A}_1$ with Coxeter graph $\Gamma(\tilde{A}_1)$



(22)

$\Gamma(\tilde{A}_1)$ sets the relations between the two generators s_0, s_1 of $\tilde{A}_1$: the edge labelled with ∞ denotes that there are no relations between s_0 and s_1.
We can set $\mathcal{S}_\blacktriangledown = \{s_0\}$ and $\mathcal{S}_\blacktriangle = \{s_1\}$. In this case we obtain that $(\mathcal{W}_\blacktriangledown, \mathcal{S}_\blacktriangledown)$ and $(\mathcal{W}_\blacktriangle, \mathcal{S}_\blacktriangle)$ are of isomorphism class A_1 and $(\mathcal{W}_\bullet, \mathcal{S}_\bullet)$ is the trivial group.

2. $(\mathcal{W}, \mathcal{S})$ with Coxeter graph $\Gamma(\mathcal{W})$



(23)

As before, $\Gamma(\mathcal{W})$ sets the relations between the four generators s_1, s_2, s_3, s_4 of $\mathcal{W}$.
We can set $\mathcal{S}_\blacktriangledown = \{s_1, s_2, s_3\}$ and $\mathcal{S}_\blacktriangle = \{s_1, s_3, s_4\}$. In this case we obtain that $(\mathcal{W}_\blacktriangledown, \mathcal{S}_\blacktriangledown)$ and $(\mathcal{W}_\blacktriangle, \mathcal{S}_\blacktriangle)$ are of isomorphism class A_3 and $(\mathcal{W}_\bullet, \mathcal{S}_\bullet)$ is of isomorphism class $A_1 \times A_1$.

Our aim is to find an explicit formula for the left hand-side of the Kac′ denominator formula (1) for the crystallographic Coxeter groups with spherically ∞-decomposition. We use this fundamental fact:

Fact 0.1 *Let (W, S), $S = S_\blacktriangledown \cup S_\blacktriangle$, be a Coxeter system with an ∞-decomposition. Put $S_\bullet = S_\blacktriangledown \cap S_\blacktriangle$ and $W_\times = W_{S_\times}$ for $\times \in \{\blacktriangledown, \blacktriangle, \bullet\}$. Let $W_\blacktriangledown \sqcup_{W_\bullet} W_\blacktriangle$ be the free product of groups $W_\blacktriangledown$ $W_\blacktriangle$ with amalgamated group $W_\bullet$. Then the canonical map*

$$f : W_\blacktriangledown \sqcup_{W_\bullet} W_\blacktriangle \to W \tag{24}$$

is an isomorphism. □

The elements of a free product of groups with amalgamation have a unique expression in a canonical normal form (cf. [27]). Let $_\natural W^\bullet_\blacktriangledown = W^\bullet_\blacktriangledown \setminus \{1\}$ be the set of representatives of the non-trivial $W_\blacktriangledown / W^\bullet$-cosets (respectively $_\natural W^\bullet_\blacktriangle = W^\bullet_\blacktriangle \setminus \{1\}$ and $W_\blacktriangle / W^\bullet$). For every $w \in W$ there exists $t \geq 0$, $a_0, a_t \in W^\bullet_\blacktriangledown$, $a_i \in {}_\sharp W^\bullet_\blacktriangledown$ for $1 \leq i \leq t-1$, $b_j \in {}_\sharp W^\bullet_\blacktriangle$ for $1 \leq j \leq t$, and $c \in W_\bullet$ such that

$$w = a_0 b_1 \cdots b_t a_t c, \tag{25}$$

and the expression (25) is unique for the element $w \in W$.

Therefore we have

$$W = \Omega_{\blacktriangledown\blacktriangledown} \sqcup \Omega_{\blacktriangledown\blacktriangle} \sqcup \Omega_{\blacktriangle\blacktriangledown} \sqcup \Omega_{\blacktriangle\blacktriangle} \sqcup W_{\bullet}, \tag{26}$$

where

$$\Omega_{\blacktriangledown\blacktriangledown} = \{a_0 b_1 a_1 \ldots b_t a_t h \mid t \geq 0,\ a_i \in {}_{\natural}W^{\bullet}_{\blacktriangledown},\ b_i \in {}_{\natural}W^{\bullet}_{\blacktriangle},\ h \in W^{\bullet}\}, \tag{27}$$

$$\Omega_{\blacktriangledown\blacktriangle} = \{a_1 b_1 \ldots a_t b_t h \mid t \geq 1,\ a_i \in {}_{\natural}W^{\bullet}_{\blacktriangledown},\ b_i \in {}_{\natural}W^{\bullet}_{\blacktriangle},\ h \in W^{\bullet}\}, \tag{28}$$

$$\Omega_{\blacktriangle\blacktriangledown} = \{b_1 a_1 \ldots b_t a_t h \mid t \geq 1,\ a_i \in {}_{\natural}W^{\bullet}_{\blacktriangledown},\ b_i \in {}_{\natural}W^{\bullet}_{\blacktriangle},\ h \in W^{\bullet}\}, \tag{29}$$

$$\Omega_{\blacktriangle\blacktriangle} = \{b_0 a_1 b_1 \ldots a_t b_t h \mid t \geq 0,\ a_i \in {}_{\natural}W^{\bullet}_{\blacktriangledown},\ b_i \in {}_{\natural}W^{\bullet}_{\blacktriangledown},\ h \in W^{\bullet}\}. \tag{30}$$

For the Coxeter group W with an ∞-decomposition we define a *fake length function*

$$\ell_{\coprod} : W \longrightarrow \mathbb{N}_0, \tag{31}$$

such that for $w \in W$ with expression (25)

$$\ell_{\coprod}(w) := \sum_{0 \leq i \leq t} \ell_{\blacktriangledown}(a_i) + \sum_{1 \leq j \leq t} \ell_{\blacktriangle}(b_j) + \ell_{\bullet}(c), \tag{32}$$

where

$$\ell_{\times} : W_{\times} \to \mathbb{N}_0 \tag{33}$$

is the length function of $W_{\times} = W_{S_{\times}}$ for $\times \in \{\blacktriangledown, \blacktriangle, \bullet\}$.
Thus, by definition, one has $\ell(w) \leq \ell_{\coprod}(w)$ for all $w \in W$.

Theorem 0.3 *(cf. [2]) Let (W, S), $S = S_{\blacktriangledown} \cup S_{\blacktriangle}$, $S_{\bullet} = S_{\blacktriangledown} \cap S_{\blacktriangle}$, be a Coxeter system with an ∞-decomposition. Then $\ell = \ell_{\coprod}$.* □

We want to show a formula for the left hand-side of (1) when the Coxeter group W has a spherically ∞-decomposition.
We call the series

$$C_W(t) := \sum_{w \in W} t^{\ell(w)} c(w) \tag{34}$$

cocycle series for the group W. It is an element of the ring $\mathbb{Z}[\Lambda^+_{\mathbb{Z}}][\![t]\!]$ of the formal power series whose variable is t and whose coefficients come from the group ring $\mathbb{Z}[\Lambda^+_{\mathbb{Z}}]$. It is treated multiplicatively letting an element $\alpha \in \Lambda^+_{\mathbb{Z}}$ as the formal exponential $e(\alpha)$ (cf. [4]).
We define six elements.

$$\mathcal{A}_{\blacktriangledown} := \sum_{a \in {}_{\natural}W^{\bullet}_{\blacktriangledown}} t^{\ell(a)} \tag{35}$$

$$\mathcal{A}_{\blacktriangle} := \sum_{b \in_{\natural} W^{\bullet}_{\blacktriangle}} t^{\ell(b)} \tag{36}$$

$$\mathcal{A}_{\bullet} := \sum_{c \in W_{\bullet}} t^{\ell(c)} \tag{37}$$

They are elements of the ring $\mathbb{Z}[\![t]\!]$ of the formal power series whose variable is t and whose coefficients come from the ring $\mathbb{Z}$.

$$\tilde{\mathcal{A}}_{\blacktriangledown} := \sum_{a \in_{\natural} W^{\bullet}_{\blacktriangledown}} t^{\ell(a)} a \tag{38}$$

$$\tilde{\mathcal{A}}_{\blacktriangle} := \sum_{b \in_{\natural} W^{\bullet}_{\blacktriangle}} t^{\ell(b)} b \tag{39}$$

$$\tilde{\mathcal{A}}_{\bullet} := \sum_{c \in W_{\bullet}} t^{\ell(c)} c \tag{40}$$

They are elements of $\mathbb{Z}[W][\![t]\!]$ the ring of the formal power series with coefficients in the group ring $\mathbb{Z}[W]$.
Let

$$W(t) := \sum_{w \in W} t^{\ell(w)} \in \mathbb{Z}[\![t]\!] \subseteq \mathbb{Z}[W][\![t]\!] \tag{41}$$

be the *growth series* for the group W (cf. [16]) and

$$\tilde{W}(t) := \sum_{w \in W} t^{\ell(w)} w \in \mathbb{Z}[W][\![t]\!] \tag{42}$$

be the *complete growth series* for the group W (cf. [1]). These series were introduced for studying combinatorial structures in the context of infinite groups.
From (8), we write

$$C_W(t) = (W(t) - \tilde{W}(t)) \cdot \omega_0 \in \mathbb{Z}[\Lambda^+_{\mathbb{Z}}][\![t]\!] \tag{43}$$

with

$$W(t) = \mathcal{A}_{\blacktriangledown}(\sum_{i=0}(\mathcal{A}_{\blacktriangle}\mathcal{A}_{\blacktriangledown})^i)\mathcal{A}_{\bullet} + (\sum_{i=1}(\mathcal{A}_{\blacktriangle}\mathcal{A}_{\blacktriangledown})^i)\mathcal{A}_{\bullet} + (\sum_{i=1}(\mathcal{A}_{\blacktriangledown}\mathcal{A}_{\blacktriangle})^i)\mathcal{A}_{\bullet} + \mathcal{A}_{\blacktriangle}(\sum_{i=0}(\mathcal{A}_{\blacktriangledown}\mathcal{A}_{\blacktriangle})^i)\mathcal{A}_{\bullet} + \mathcal{A}_{\bullet} \tag{44}$$

and

$$\tilde{W}(t) = \tilde{\mathcal{A}}_{\blacktriangledown}(\sum_{i=0}(\tilde{\mathcal{A}}_{\blacktriangle}\tilde{\mathcal{A}}_{\blacktriangledown})^i)\tilde{\mathcal{A}}_{\bullet} + (\sum_{i=1}(\tilde{\mathcal{A}}_{\blacktriangle}\tilde{\mathcal{A}}_{\blacktriangledown})^i)\tilde{\mathcal{A}}_{\bullet} + (\sum_{i=1}(\tilde{\mathcal{A}}_{\blacktriangledown}\tilde{\mathcal{A}}_{\blacktriangle})^i)\tilde{\mathcal{A}}_{\bullet} + \tilde{\mathcal{A}}_{\blacktriangle}(\sum_{i=0}(\tilde{\mathcal{A}}_{\blacktriangledown}\tilde{\mathcal{A}}_{\blacktriangle})^i)\tilde{\mathcal{A}}_{\bullet} + \tilde{\mathcal{A}}_{\bullet} \tag{45}$$

as a formal power series of the elements (35), (36), (37), (38), (39) and (40).
In Chapter 4, we see that the formula (43) for $\tilde{A}_1$ is in analogy to the Kac'

denominator formula for $\tilde{A}_1$ shown by V. G. Kac (cf. [8]).

Chapter 1

Coxeter groups

In this chapter we introduce the general theory for the Coxeter groups. For more details we refer the reader to [16]. First of all, we recall the definition of the Coxeter groups as abstract groups. This kind of groups has a geometric representation, called also *Tits representation*, that allows us to have a concrete idea of these groups as reflection groups.
There are three principal types of Coxeter groups: finite, affine and indefinite. We will focus on the last type, since the studies about it have not been completed yet and in particular we will introduce in the third chapter a specific hyperbolic Coxeter group. This group has a certain relevance in physics, because, starting from its geometric representation, one can consider this group as a reflection group acting on the Lorentz space.

1.1 Coxeter systems

Definition 1.1 *(cf. [16]) A Coxeter system is a pair (W, S) where W is a group and $S \subset W$ is a set of generators of W satisfying exclusively relations of the form*

$$(ss')^{m(s,s')} = 1 \tag{1.1}$$

where $m(s, s) = 1$ and $m(s, s') = m(s', s) \geq 2$ for $s \neq s'$. If there are no relations between s and s' then $m(s, s') = \infty$.

The order of S, $|S|$, is called rank of the Coxeter system, the group W is a Coxeter group and the elements of S are called simple reflections. □

We shall always assume that $|S|$ is finite, even though a good part of the theory applies to arbitrary S.

Definition 1.2 *(cf. [16]) With a Coxeter system (W, S) we may associate a graph $\Gamma(W)$ called Coxeter graph as follow: the vertices of $\Gamma(W)$ are in bijection with S and an edge between the vertices s and s' is labelled by $m(s, s')$ whenever this number (∞ included) is at least 3. When $m(s, s') = 3$ the label on the edge is omitted. Then if distinct vertices s and s' are not joined, this means that $m(s, s') = 2$.* □

Example 1.1 *Let $W = S_3$ be the permutation group on the set $\{1, 2, 3\}$ and $S = \{(12), (23)\} \subset W$ be the set of transpositions, generators of W. Then (W, S) is a Coxeter system whose Coxeter graph is*

$$\bullet \!-\!\!-\!\!-\! \bullet \tag{1.2}$$

□

1.2 Length function

Since the generators $s \in S$ have order 2 in W, every $1 \neq w \in W$ can be written in the form

$$w = s_1 s_2 \dots s_r \tag{1.3}$$

for some s_i (not necessarily distinct) in S.
The expression for w is called *reduced* when r is the smallest integer such that w can be written as (1.3).

Definition 1.3 *(cf. [16]) Let (W, S) be a Coxeter system, we define a function*

$$\ell : W \to \mathbb{N}_0 \tag{1.4}$$

called length function, defined as follows:

$$\ell(w) := \begin{cases} 0 \ if \ \ w = 1 \\ r \ if \ \ w = s_1 s_2 \dots s_r \ \ is \ a \ reduced \ expression \end{cases} \tag{1.5}$$

□

The following lemma shows some elementary properties for the length function.

Lemma 1.1 *(cf. [16]) The length function satisfies the following properties:*

L1 $\ell(w) = \ell(w^{-1})$ *for all* $w \in W$;

L2 $\ell(w) = 1$ *if and only if* $w \in S$;

L3 $\ell(w) - \ell(w') \leq \ell(ww') \leq \ell(w) + \ell(w')$ *for all* $w, w' \in W$.

Proof

*L*1 If $w = s_1 \dots s_r$ then $w^{-1} = s_r \dots s_1$, so $\ell(w^{-1}) \leq \ell(w)$. Similarly for w^{-1} in place of w.

*L*2 It is trivial.

*L*3 If $w = s_1 \dots s_r$ and $w' = s'_1 \dots s'_t$ are reduced, then $ww' = s_1 \dots s_r s'_1 \dots s'_t$ and $\ell(ww') \leq \ell(w) + \ell(w')$. For the first inequality we consider $\ell(w) = \ell(ww'w'^{-1}) \leq \ell(ww') + \ell(w'^{-1})$. Then, using *L*1, $\ell(ww') \geq \ell(w) - \ell(w')$.

□

Proposition 1.1 *(cf. [16]) There is a unique group morphism* $\varepsilon : W \to \{\pm 1\}$ *such that* $\varepsilon(s) = -1$ *with* $s \in S$*. Furthermore, we have* $\varepsilon(w) = (-1)^{\ell(w)}$ *for all* $w \in W$*.*

Proof To prove the existence we only need to show that the defining equations of W are satisfies, in fact $(\varepsilon(s)\varepsilon(s'))^{m(s,s')} = 1$ for all $s, s' \in S$. Now let $w = s_1 \dots s_r$ be a reduced expression, $r = \ell(w)$, we have $\varepsilon(w) = (-1)^{\ell(w)}$. □

Corollary 1.1 *(cf. [16]) For all* $w \in W$ *and for all* $s \in S$*, we have* $\ell(ws) = \ell(w) \pm 1$ *and the same for* $\ell(sw)$*.*

Proof Using *L*2 and *L*3 of Lemma 1.1, we have that $\ell(w) - \ell(s) \leq \ell(ws) \leq \ell(w) + \ell(s)$. From Proposition 1.1, we know that $\varepsilon(ws) = -\varepsilon(w)$, therefore $\ell(ws) - \ell(w) \equiv 1 (mod\ 2)$. This proves the result. The same is for $\ell(sw)$. □

1.3 Geometric representation

In this section we recall that the *geometric representation*, called *Tits representation*, allows us to represent the Coxeter groups as groups of reflections with respect to hyperplanes over a vector space. First of all we formally introduce these notions.

Definition 1.4 *(cf. [16]) Let* V *be a vector space endowed with a symmetric bilinear* $(\cdot, \cdot)$*. A reflection is a linear operator* s *on* V *which sends some non-zero vector* α *to its negative and fixes pointwise the hyperplane* $H_\alpha = \{v \in V \mid (v, \alpha) = 0\}$ *orthogonal to* α*. We may write* $s = s_\alpha$*. The simple formula for a generic reflection is:*

$$s_\alpha(v) = v - \frac{2(v, \alpha)}{(\alpha, \alpha)}\alpha, \quad v \in V. \tag{1.6}$$

□

Let (W, S) be a Coxeter system and let $\Lambda_{\mathbb{R}}$ be an $\mathbb{R}$-vector space of dimension $|S|$. Fixed a basis $\{\alpha_s \mid s \in S\}$ in one-to-one correspondence with s, one defines a symmetric bilinear form B on $\Lambda_{\mathbb{R}}$ by

$$B(\alpha_s, \alpha_{s'}) := -\cos \frac{\pi}{m(s, s')} \tag{1.7}$$

with the convention that $B(\alpha_s, \alpha_{s'}) = -1$ when $m(s, s') = \infty$. This definition of B imposes on $\Lambda_{\mathbb{R}}$ a geometry in such a way that the *angle* between α_s and $\alpha_{s'}$ will be compatible with the given $m(s, s')$. We can observe that $B(\alpha_s, \alpha_s) = 1$ and $B(\alpha_s, \alpha_{s'}) \leq 0$ if $s \neq s'$.
Moreover let $s \in S$, the linear map

$$\rho_s : \Lambda_{\mathbb{R}} \to \Lambda_{\mathbb{R}} \tag{1.8}$$

such that $\rho_s(v) = v - 2B(\alpha_s, v)\alpha_s$, is a reflection in fact: $\rho_s(\alpha_s) = -\alpha_s$ and ρ_s fixes the hyperplane H_s orthogonal to α_s pointwise.

Remark 1.1 *(cf. [16]) Let $GL(\Lambda_{\mathbb{R}})$ be the general lienar group on $\Lambda_{\mathbb{R}}$ and $O(\Lambda_{\mathbb{R}}, B)$ be the orthogonal subgroup of $GL(\Lambda_{\mathbb{R}})$ preserving B. For all $s \in S$, ρ_s is an element of $O(\Lambda_{\mathbb{R}}, B)$.*

Proof Let consider $B(\rho_s(u), \rho_s(v)) = B(u - 2B(\alpha_s, u)\alpha_s, v - 2B(\alpha_s, v)\alpha_s) = B(u, v) - 4B(u, \alpha_s)B(\alpha_s, v) + 4B(u, \alpha_s)B(\alpha_s, v)B(\alpha_s, \alpha_s)$, $\forall u, v \in \Lambda_{\mathbb{R}}$ $\forall s \in S$. The result follows because $B(\alpha_s, \alpha_s) = 1$. Therefore, $B(\rho_s(u), \rho_s(v)) = B(u, v)$, the bilinear form B is invariant under ρ_s. □

Note that concerning the orthogonal group we use the standard notation of Dieudonné (1971), that is

$$O(\Lambda_{\mathbb{R}}, B) := \{g \in GL(\Lambda_{\mathbb{R}}) \mid B \circ g = B\}. \tag{1.9}$$

Proposition 1.2 *(cf. [16]) There is a unique group morphism $\rho : W \to O(\Lambda_{\mathbb{R}}, B)$ such that the image of any $s \in S$ is ρ_s. Moreover, for each pair $s, s' \in S$, the order of ss' in W is $m(s, s')$.* □

Corollary 1.2 *(cf. [16]) For any elements s and s' of S, the order of ss' is exactly $m(s, s')$. In particular all elements of S are distinct.*

Proof This follows from the Proposition 1.2. In fact the image of ss' is $\rho_s\rho_{s'}$, that has order $m(s, s')$. □

Proposition 1.2 gives us the geometric representation $\rho : W \to GL(\Lambda_{\mathbb{R}})$, called also *Tits representation*. The Coxeter group W can be considered as a subgroup of the orthogonal group $O(\Lambda_{\mathbb{R}}, B)$ respect to the bilinear form B defined in (1.7).

Proposition 1.3 *(cf. [16]) Let $\rho : W \to GL(\Lambda_{\mathbb{R}})$ be the geometric representation of W. $\rho(W)$ is a discrete subgroup of $GL(\Lambda_{\mathbb{R}})$.* □

1.4 Root system

The geometric representation will be useful to have a *concrete idea* of the Coxeter groups. In particular, it is considerable to analyse the action of a Coxeter group W on the vector space $\Lambda_{\mathbb{R}}$. To simplify the notation, we may write $w(\alpha_s)$ instead of $\rho(w)(\alpha_s)$.

Definition 1.5 *(cf. [16]) The root system Φ of the Coxeter system (W, S) is a set of unit vectors in $\Lambda_{\mathbb{R}}$ permuted by W:*

$$\Phi := \{w(\alpha_s) \mid w \in W, \ s \in S\}. \tag{1.10}$$

The roots α_s with $s \in S$ are called simple roots. Let α be a root, it can be uniquely written in the form

$$\alpha = \sum_{s \in S} k_s \alpha_s$$

with $k_s \in \mathbb{R}$. α is called positive (resp. negative) root, if $k_s \geq 0$ (resp. $k_s \leq 0$) for all $s \in S$ and we write $\alpha > 0$ (resp. $\alpha < 0$). The set of positive (resp. negative) roots will be denoted by Φ^+ (resp. Φ^-). □

Since W preserves the form B on $\Lambda_{\mathbb{R}}$, the roots are unit vectors.

Remark 1.2 *(cf. [16]) $\Phi = -\Phi$, because $s(\alpha_s) = -\alpha_s$.* □

Theorem 1.1 *(**Tits Theorem**, cf. [16]) Let $w \in W$ and $s \in S$. If $\ell(ws) > \ell(w)$, then $w(\alpha_s) > 0$. If $\ell(ws) < \ell(w)$, then $w(\alpha_s) < 0$.* □

Corollary 1.3 *(cf. [16]) The representation $\rho : W \to O(\Lambda_{\mathbb{R}}, B)$ is faithful.*

Proof Let $w \in Ker\rho$. If $w \neq 1$, there exists $s \in S$ such that $\ell(ws) < \ell(w)$. Then Tits Theorem 1.1 says that $w(\alpha_s) < 0$. But, since $w \in Ker\rho$, then $w(\alpha_s) = \alpha_s > 0$, which is a contradiction. □

From Tits Theorem (Theorem 1.1) it directly follows that Φ is the union of Φ^+ and Φ^-.

1.5 Parabolic subgroups

Definition 1.6 *(cf. [16]) Let T be a subset of S. The pair (W_T, T) is a subgroup of (W, S) called parabolic subgroup.* □

We call ℓ_T the length function restricted to W_T. It is clear that $\ell(w) \leq \ell_T(w)$ for all $w \in W_T$.

The following theorem describes the nature of the parabolic subgroups.

Theorem 1.2 *(cf. [16])*

(a) For each subset $T \subseteq S$, the pair (W_T, T) with the given values $m(s, s')$ is a Coxeter system.

(b) Let $T \subseteq S$. If $w = s_1 \dots s_r$ with $s_i \in S$ is a reduced expression and in particular $w \in W_T$, then all $s_i \in T$. In particular $\ell(w) = \ell_T(w)$ and $W_T \cap S = T$.

(c) The assignment $T \to W_T$ defines a lattice isomorphism between the subsets of S and the parabolic subgroups of W.

(d) S is a minimal generating set for W.

Proof

(b) We proceed by induction on $\ell(w)$. If $w = 1$, then $\ell(1) = 0 = \ell_T(1)$. Let now $w = s_1 \dots s_r \neq 1$ be a reduced expression, and let $s = s_r$. By Tits Theorem (Theorem 1.1), $w(\alpha_s) < 0$. Because $w \in W_T$, then $w = t_1 \dots t_q$ is a reduced expression with $t_i \in T$. Thus

$$w(\alpha_s) = \alpha_s + \sum_{i=1}^{q} c_i \alpha_{t_i}, \quad c_i \in \mathbb{R}.$$

Since $w(\alpha_s) < 0$, then there exists i such that $s = t_i \in W_T$ and $ws = s_1 \dots s_{r-1} \in W_T$ is a reduced expression. Therefore, by induction, every $s_i \in T$. The other statements are clear.

□

Example 1.2 *Let $W = S_3$ be the permutation group on the set $\{1, 2, 3\}$ and $S = \{(12), (23)\} \subset W$ be the set of transpositions, generators of W. Then, let $T_1 = \{(12)\}$ and $T_2 = \{(23)\}$ be all non trivial subsets of S, W_{T_i} $(i = 1, 2)$ are all non-trivial parabolic subgroups of W.* □

1.6 Geometric interpretation of the length function

The geometric representation of the Coxeter groups allows to give also a geometric interpretation of the length function.

Proposition 1.4 *(cf. [16])*

(a) Let $s \in S$, then $s(\alpha_s) = -\alpha_s$ and for any $\alpha \in \Phi^+$ we have $s(\alpha) \in \Phi^+ \setminus \{\alpha_s\}$.

(b) For any $w \in W$ the length $\ell(w)$ is the number of positive roots mapped by w in negative ones.

Proof

(a) We know that $s(\alpha_s) = -\alpha_s$. Let α be a positive root distinct from α_s (therefore α cannot be a multiple of α_s). We can write $\alpha = \sum_{u \in S} k_u \alpha_u$ with $k_u > 0$ for some $u \neq s$. We observe that $s(\alpha) = \sum_{u \neq s} k_u \alpha_u + (k_s - 2B(\alpha_s, \alpha))\alpha_s$ has at least one positive coefficient $k_u > 0$ for some $u \neq s$. Since $s(\alpha)$ is a root, then it is a positive root.

(b) For $w \in W$, we define $n(w)$ to be the number of positive roots sent by w to negative ones, i.e. $n(w) := |\Pi(w)|$, where $\Pi(w) := \Phi^+ \cap w^{-1}(-\Phi^+)$.
First of all we observe that if $w(\alpha_s) < 0$ then, by part (a), $\Pi(ws) = s(\Pi(w) \setminus \{\alpha_s\})$, with $\alpha_s \in \Pi(w)$, and $n(ws) = n(w) - 1$. Otherwise if $w(\alpha_s) > 0$ then, by part (a), $\Pi(ws) = s(\Pi(w)) \sqcup \{\alpha_s\}$ and $n(ws) = n(w) + 1$.
We use induction on $\ell(w)$ to show that $\ell(w) = n(w)$. If $\ell(w) = 0$, it is trivial because $w = 1$. If $\ell(w) = 1$, then by the part (a) $n(w) = 1$. Now if $s \in S$ such that $\ell(ws) = \ell(w) - 1$. Therefore by induction $\ell(ws) = n(ws)$. We know from Tits Theorem (Theorem 1.1) that if $\ell(ws) < \ell(w)$, then $w(\alpha_s) < 0$, therefore $n(w) = n(ws) - 1$. Then $n(w) = \ell(w)$. Instead if $s \in S$ such that $\ell(ws) = \ell(w) + 1$, always by induction $\ell(w) = n(w)$. In fact we know from Tits Theorem (Theorem 1.1) that if $\ell(ws) > \ell(w)$, then $w(\alpha_s) > 0$, therefore $n(ws) = n(w) + 1$. Then $n(ws) = \ell(ws)$.

□

Proposition 1.5 *(cf. [16]) Given a reduced expression $w = s_1 \ldots s_r \in W$, $s_i \in S$, set $\alpha_i := \alpha_{s_i}$ and $\beta_i := s_r s_{r-1} \ldots s_{i+1}(\alpha_i)$, interpreting β_r to be α_r. Then $\Pi(w)$ consists of the r distinct positive roots $\beta_1, \ldots, \beta_r$.*

Proof At first we observe that β_i are positive roots for all i. In fact, if there exists i such that $\beta_i := s_r s_{r-1} \ldots s_{i+1}(\alpha_i)$ is negative, then by Tits Theorem (Theorem 1.1) $\ell(s_r \ldots s_{i+1} s_i) < \ell(s_r \ldots s_{i+1})$ and this contradicts the fact that the expression of w is reduced.

The argument is similar to show that $w(\beta_i)$ are negative roots for all i.
At the end, we prove that β_i is different from β_j for all $i \neq j$. Let suppose that $\beta_i = \beta_j$ for $i > j$. We obtain $s_{i-1} \dots s_{j+1}(\alpha_j) = -\alpha_i$, then by Tits Theorem (Theorem 1.1) $\ell(s_{i-1} \dots s_{j+1} s_j) < \ell(s_{i-1} \dots s_{j+1})$, so as before we obtain a contradiction with the hypothesis that the expression of w is reduced. □

1.7 Roots and reflections

Remind that the geometric representation $\rho : W \to GL(\Lambda_{\mathbb{R}})$ allows each $s \in S$ to act on the $\mathbb{R}$-vector space $\Lambda_{\mathbb{R}}$ as a reflection. Therefore each $s \in S$ can be associated with a root α_s. Let $\alpha \in \Phi$ such that $\alpha = w(\alpha_s)$ for some $w \in W$ and $s \in S$. The action of the element wsw^{-1} on $\Lambda_{\mathbb{R}}$ is given by:

$$wsw^{-1}(v) = w(w^{-1}(v) - 2B(w^{-1}(v), \alpha_s)\alpha_s) = v - 2B(w^{-1}(v), \alpha_s)\alpha \tag{1.11}$$

$$= v - 2B(v, w(\alpha_s))\alpha = v - 2B(v, \alpha)\alpha = s_\alpha(v).$$

wsw^{-1} depends only on α and not on w and s. It acts as a reflection on $\Lambda_{\mathbb{R}}$. We define the set $\mathcal{R}$ of all reflections s_α by

$$\mathcal{R} = \bigcup_{w \in W} wSw^{-1}. \tag{1.12}$$

Remark 1.3 *(cf. [16]) The correspondence $\alpha \to s_\alpha$ is injective for $\alpha \in \Phi^+$.*

Proof If $s_\alpha = s_\beta$, then $s_\alpha(\beta) = s_\beta(\beta)$. We obtain that $\beta = B(\alpha, \beta)\alpha$. Then $\alpha = \beta$, because they are unit vectors in Φ^+. □

Lemma 1.2 *(cf. [16]) If $\alpha, \beta \in \Phi$ and $\beta = w(\alpha)$ for some $w \in W$, then $ws_\alpha w^{-1} = s_\beta$.*

Proof This comes from the above formula (1.12) and from the fact that B is W-invariant. □

Proposition 1.6 *(cf. [16]) Let $w \in W$ and $\alpha \in \Phi^+$, then $\ell(ws_\alpha) > \ell(w)$ if and only if $w(\alpha) > 0$.*

Proof It is sufficient to prove that if $\ell(ws_\alpha) > \ell(w)$ then $w(\alpha) > 0$. We proceed by induction on $\ell(w)$. If $\ell(w) = 0$, then it is trivial. If $\ell(w) > 0$, then there exists $s \in S$ such that $\ell(sw) = \ell(w) - 1$. Then we have $\ell(ws_\alpha) > \ell(w) > \ell(sw)$. This implies that $\ell(sws_\alpha) \geq \ell(ws_\alpha) - 1 > \ell(w) - 1 = \ell(sw)$. Therefore by induction $sw(\alpha) > 0$. Assume that $w(\alpha) < 0$, this implies that $w(\alpha) = -\alpha_s$, i.e. $sw(\alpha) = \alpha_s$. From Lemma 1.2 it follows that $(sw)s_\alpha(sw)^{-1} = s$, whence we obtain a contradiction because $ws_\alpha = sw$, but $\ell(ws_\alpha) > \ell(w) > \ell(sw)$. Therefore $w(\alpha)$ must be positive. □

1.8 Strong Exchange Condition

Theorem 1.3 *(**Strong Exchange Condition**, cf. [16]) Let $w = s_1 \dots s_r$, $s_i \in S$, be a non necessary reduced expression. If there exists a reflection $t \in \mathcal{R}$ such that $\ell(wt) < \ell(w)$, then there is an index i for which $wt = s_1 \dots \widehat{s_i} \dots s_r$ (omitting s_i). If the expression for w is reduced, then the index i is unique.*

Proof Let $t = s_\alpha$ with $\alpha \in \Phi^+$. By Proposition 1.6 we have $w(\alpha) < 0$. Because $\alpha > 0$, there exists an index i such that $s_{i+1} \dots s_r(\alpha) > 0$ but $s_i \dots s_r(\alpha) < 0$, i.e. by Proposition 1.4 (a) $s_{i+1} \dots s_r(\alpha) = \alpha_i$, the simple root associated with s_i. Therefore by Lemma 1.2 $(s_{i+1} \dots s_r)t(s_r \dots s_{i+1}) = s_i$, or $wt = s_1 \dots \widehat{s_i} \dots s_r$.
If the expression is reduced $\ell(w) = r$, we suppose that there exists an other index $j > i$ such that $wt = s_1 \dots \widehat{s_i} \dots s_r = s_1 \dots \widehat{s_j} \dots s_r$, whence $s_{i+1} \dots s_j = s_i \dots s_{j-1}$ and thus $s_i \dots s_j = s_{i+1} \dots s_{j-1}$. So this contradicts the fact that the expression of w is reduced. □

Corollary 1.4 *(**Deletion Condition**, cf. [16])*

(a) Let $w = s_1 \dots s_r$, $s_i \in S$, be a non-reduced expression, then there exist indices $i < j$ such that $w = s_1 \dots \widehat{s_i} \dots \widehat{s_j} \dots s_r$.

(b) If $w = s_1 \dots s_r$, $s_i \in S$, then a reduced expression of w can be obtained by omitting an even number of s_i.

Proof

(a) Because of $\ell(w) < r$, there exists an index j such that $\ell(w's_j) < \ell(w')$, where $w' = s_1 \dots s_{j-1}$. Then by Strong Exchange Condition (Theorem 1.3) applied to w' and s_j, we get $w's_j = s_1 \dots \widehat{s_i} \dots s_{j-1}$, i.e. $w = s_1 \dots \widehat{s_i} \dots \widehat{s_j} \dots s_r$.

(b) This follows from part (a), applying as long as the expression is non-reduced.

□

Definition 1.7 *(cf. [16]) The Bruhat order is a partial ordering in W. We write that $w < w'$ if there exists a reduced expression $w' = s_1 \dots s_r$ such that $w = s_1 \dots \widehat{s_i} \dots s_r$ (or if $\ell(w) < \ell(w')$ and $w = w't$ for some $t \in \mathcal{R}$).* □

1.9 Fundamental domain for W

Let $\rho : W \to GL(\Lambda_\mathbb{R})$ be the geometric representation of the Coxeter group W and $\rho^* : W \to GL(\Lambda^*_\mathbb{R})$ be the controgradient action. Let us denote by $\langle f, \lambda \rangle$,

$f \in \Lambda_{\mathbb{R}}^*, \lambda \in \Lambda_{\mathbb{R}}$, the natural pairing in $\Lambda_{\mathbb{R}}$. Then the action of W in $\Lambda_{\mathbb{R}}^*$ is characterized by

$$\langle w(f), w(\lambda)\rangle = \langle f, \lambda\rangle, \ \ w \in W, \ \ f \in \Lambda_{\mathbb{R}}^*, \ \ \lambda \in \Lambda_{\mathbb{R}}. \tag{1.13}$$

For all $s \in S$, we can define the hyperplane

$$Z_s := \{f \in \Lambda_{\mathbb{R}}^* \mid \langle f, \alpha_s\rangle = 0\} \tag{1.14}$$

and the half-spaces

$$A_s := \{f \in \Lambda_{\mathbb{R}}^* \mid \langle f, \alpha_s\rangle > 0\} \tag{1.15}$$

$$A_s' := \{f \in \Lambda_{\mathbb{R}}^* \mid \langle f, \alpha_s\rangle < 0\} = s(A_s). \tag{1.16}$$

Let $C := \bigcap_{s\in S} A_s$, its closure is the fundamental domain for W.
Observe that s fixes Z_s pointwise. If $n = |S|$, one can identify $\Lambda_{\mathbb{R}}$ with $\mathbb{R}^n$ and so also for $\Lambda_{\mathbb{R}}^*$. Therefore, respect to the standard topology of $\mathbb{R}^n$, Z_s is closed and instead A_s and A_s' are open, from which it follows that C is also open. Call $\overline{A_s}$ the closure of A_s, i.e. $A_s \cup Z_s$, it is clear that $D := \overline{C}$ is the intersection of $\overline{A_s}$ for all $s \in S$. Moreover the action of W on $\Lambda_{\mathbb{R}}$ and $\Lambda_{\mathbb{R}}^*$ is continuous.
Let W_I be a parabolic subgroup of W ($I \subseteq S$) and define

$$C_I := \Big(\bigcap_{s\in I} Z_s\Big) \cap \Big(\bigcap_{s\notin I} A_s\Big) \tag{1.17}$$

be a subset of D. Then $C_\emptyset = C$ and $C_S = \{0\}$. W_I fixes C_I pointwise, because s fixes Z_s pointwise. Instead if $s \in S$ fixes a point $f \in C_I$, then $s \in I$, in fact $\langle f, \alpha_s\rangle = \langle s(f), s(\alpha_s)\rangle = -\langle f, \alpha_s\rangle$, from which $f \in Z_s$.
Define $U := \bigcup_{w\in W} w(D)$. U is a W-stable subset of $\Lambda_{\mathbb{R}}^*$. Next proposition claims that the family $\mathcal{C} := \{w(C_I) \mid w \in W, \ I \subset S\}$ form a partition of U. In particular U is a convex cone called *Tits cone*.

Lemma 1.3 *(cf. [16]) Let $s \in S$ and $w \in W$. Then $\ell(sw) > \ell(w)$ if and only if $w(C) \subset A_s$. Instead $\ell(sw) < \ell(w)$ if and only if $w(C) \subset A_s'$.*

Proof We use Proposition 1.6. Let $\ell(sw) < \ell(w)$, i.e. $\ell(w^{-1}s) < \ell(w^{-1})$. It is equivalent to $w^{-1}(\alpha_s) > 0$. If $f \in C$, we have that $\langle w(f), \alpha_s\rangle > 0$ if and only if $\langle f, w^{-1}(\alpha_s)\rangle > 0$, that is equivalent (by the way C is defined) to say that $w^{-1}(\alpha_s) > 0$. Then $w(C) \subset A_s$ if and only if $\ell(sw) > \ell(w)$. □

Proposition 1.7 *(cf. [16])*

(a) Let $w \in W$ and $I, J \subset S$. If $w(C_I) \cap C_J \neq \emptyset$ then $I = J$ and $w \in W_I$, so $w(C_I) = C_I$. In particular, W_I is the precise stabilizer in W of each point of C_I, and $\mathcal{C}$ is a partition of U.

(b) *D is the fundamental domain for the action of W on U: the W-orbit of each point of U meets D in exactly one point.*

(c) *The cone U is convex and every closed line segment in U meets just finitely many of the sets in the family C.*

□

1.10 Types of Coxeter groups

1.10.1 Irreducible Coxeter systems

Definition 1.8 *(cf. [16]) Let (W, S) be a Coxeter system. It is called irreducible if the Coxeter graph $\Gamma(W)$ is connected.* □

Proposition 1.8 *(cf. [16]) Let (W, S) be a Coxeter system. If $\Gamma_1, \ldots, \Gamma_n$ are connected components of the Coxeter graph $\Gamma(W)$, let $S_1, \ldots, S_n$ be the corresponding subsets of S. Then W is the direct product of the parabolic subgroups $W_{S_1}, \ldots, W_{S_n}$, and each Coxeter system (W_{S_i}, S_i) is irreducible.*

Proof The elements of S_i commute with the elements of S_j for $i \neq j$. Then each parabolic subgroup W_{S_i} centralizes each other, hence every W_{S_i} is normal in W. Moreover, the product of these subgroups contains S, i.e. it must be all W. By induction $W_{S \setminus S_i}$ is the direct product of W_{S_j} $(i \neq j)$ that intersects W_{S_i} trivially. So the product is direct. □

1.10.2 Crystallographic Coxeter groups

Definition 1.9 *(cf. [16]) Let W be a Coxeter group and $\rho : W \to GL(\Lambda_{\mathbb{R}})$ be the geometric representation. W is crystallographic relative to ρ if W stabilizes a lattice L in $\Lambda_{\mathbb{R}}$.* □

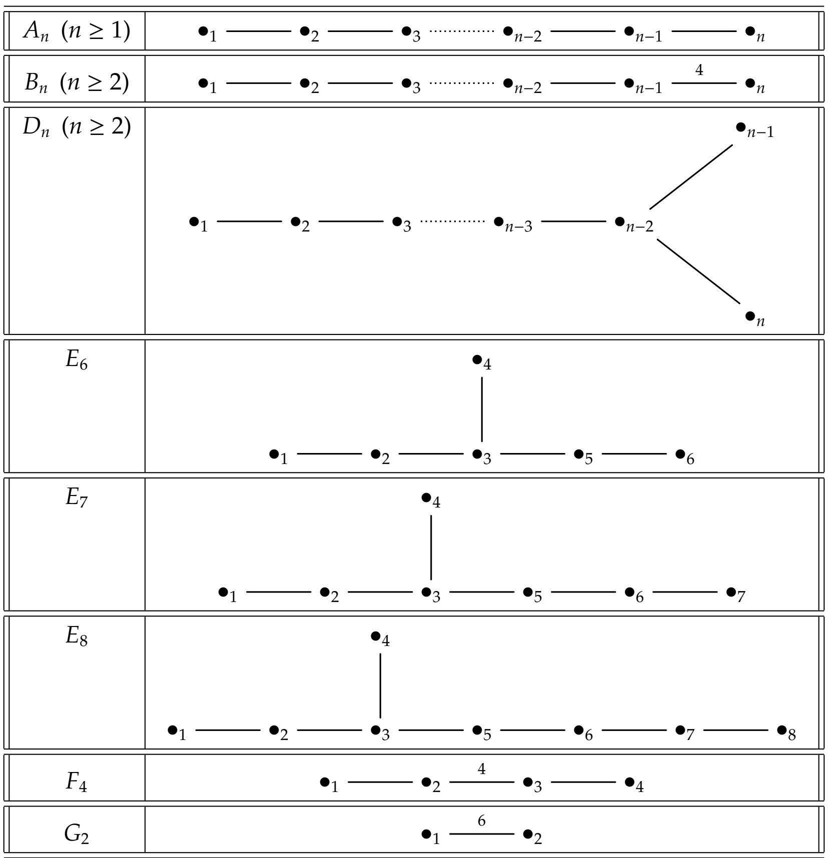
The following proposition gives a condition for W to be crystallographic.

Proposition 1.9 *(cf. [16]) Let $\Gamma(W)$ be the Coxeter graph associated with the Coxeter group W. If $\Gamma(W)$ contains no circuit, W is crystallographic (relative to ρ) if and only if $m(s, t) \in \{2, 3, 4, 6, \infty\}$ for all $s, t \in S$ with $s \neq t$. Otherwise W is crystallographic (relative to ρ) if and only if $m(s, t) \in \{2, 3, 4, 6, \infty\}$ for all $s, t \in S$ with $s \neq t$ and, moreover, for each circuit in $\Gamma(W)$, the number of edges labelled 4 (resp. 6) is even.* □

We are interested in the crystallographic Coxeter groups because they are exactly the Weyl groups of the Kac-Moody Lie algebras (cf. [8]).

1.10.3 Coxeter groups and bilinear forms

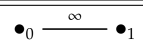
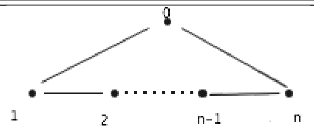
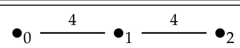
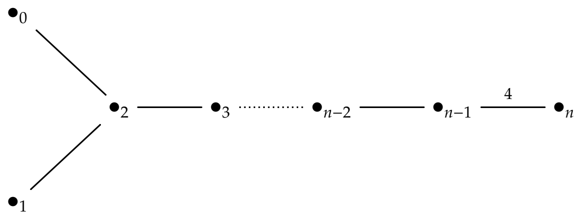
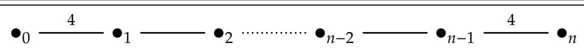
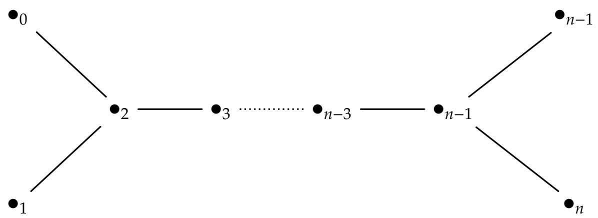
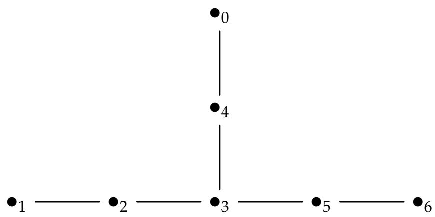
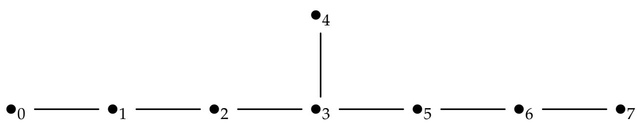
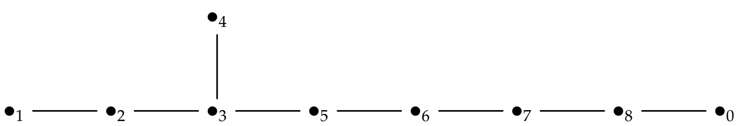
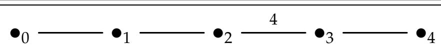
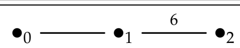
Let (W, S) be a Coxeter system and $\rho : W \to GL(\Lambda_{\mathbb{R}})$ be the geometric representation of W. Let $\{\alpha_s \mid s \in S\}$ be a fixed basis of $\Lambda_{\mathbb{R}}$, one defines the following bilinear form $B(\alpha_s, \alpha_{s'}) = -\cos\frac{\pi}{m(s,s')}$ in (cf. Section 1.3). This bilinear form can be seen as the bilinear form associated with the Coxeter graph $\Gamma(W)$ of W. Through B it is possible to distinguish certain types of Coxeter groups.
The graph $\Gamma(W)$ is said to be of *positive type* when the bilinear form B is positive definite and of *positive semidefinite type* when B is positive semidefinite but not positive definite (cf. [16]).
It is well know that the following graphs are the only connected Coxeter graphs of positive type associated with crystallographic Coxeter groups.



(1.18)

The Coxeter groups represented in Table 1.18 are the only crystallographic Coxeter groups called of *finite type* (cf. [16]).
The Coxeter groups of finite type can be represented through their geometric representation as finite reflection groups acting on the Euclidean space (cf. [16]).

Instead the following graphs are the only connected Coxeter graphs of positive semidefinite type.

$\tilde{A}_1$	
$\tilde{A}_n \ (n \geq 2)$	
$\tilde{B}_2 = \tilde{C}_2$	
$\tilde{B}_n \ (n \geq 3)$	
$\tilde{C}_n \ (n \geq 3)$	
$\tilde{D}_n \ (n \geq 4)$	
$\tilde{E}_6$	
$\tilde{E}_7$	
$\tilde{E}_8$	
$\tilde{F}_4$	
$\tilde{G}_2$	

(1.19)

The Coxeter groups represented in Table 1.19 are the only crystallographic

Coxeter groups called of *finite type* (cf. [16]).
The Coxeter groups of affine type can be represented through their geometric representation as infinite groups generated by affine reflections in the Euclidean space (cf. [16]).

1.11 Hyperbolic Coxeter groups

Let (W, S) be an irreducible Coxeter system and B the associated nondegenerate bilinear form. B allows to identify the vector space $\Lambda_{\mathbb{R}}$, on which the group W acts, and its dual $\Lambda^*_{\mathbb{R}}$.
Let $\{\lambda_s \mid s \in S\}$ be the dual basis to the basis $\{\alpha_s \mid s \in S\}$, relative to B.
Remind that the cone C defined in Section 1.9 is

$$C = \{\lambda \in \Lambda_{\mathbb{R}} \mid B(\lambda, \alpha_s) > 0 \ \ for\ all\ \ s \in S\} = \{\sum c_s\lambda_s \mid c_s > 0\}. \tag{1.20}$$

(presented as a subset of $\Lambda_{\mathbb{R}}$). Moreover observe that all λ_s lie in the closure D of C, that is the fundamental domain for the action of W on $U = \cup_{w\in W} w(C)$. D is the *convex hull* of the vectors λ_s.

Definition 1.10 *(cf. [16]) Let (W, S) be a Coxeter system acting on the vector space $\Lambda_{\mathbb{R}}$ of dimension n and let B be the associated nondegenerate bilinear form. (W, S) is called hyperbolic Coxeter system if B has signature $(n - 1, 1)$ and $B(\lambda, \lambda) < 0$ for all $\lambda \in C$. The Coxeter group W is said to be hyperbolic.* □

Definition 1.10 forces $B(\lambda, \lambda) < 0$ for all $\lambda \in D$, then, in particular, this property is true also for the vectors λ_s of the dual basis.

The following proposition allows us to recognize a hyperbolic Coxeter group from its Coxeter graph.

Proposition 1.10 *(cf. [16]) Let (W, S) be an irreducible Coxeter system, with Coxeter graph $\Gamma(W)$ and associated bilinear form B. It is hyperbolic if and only if satisfies the following conditions:*

(a) B is nondegenerate, but not positive definite.

(b) For each $s \in S$, the Coxeter graph obtained by removing s from Γ is of positive type or of positive semidefinite type.

□

The most significant facts about the classification are these: the hyperbolic Coxeter groups exist only in ranks 3 to 10 and there are only finitely many in each ranks 4 to 10 (cf. [16]).

1.11.1 Cocompact Coxeter group

Among the hyperbolic Coxeter groups we want focus our attention about the hyperbolic Coxeter groups that are also cocompact.
When B is nondegenerate W is a discrete subgroup of the corresponding orthogonal group $G := O(\Lambda_{\mathbb{R}}, B)$ (cf. [16]). G is a real Lie group endowed with a Haar measure then one can define the volume for the homogeneous space G/W. It is known that the volume is finite if and only if B is positive definite, so when W is finite, or when B has signature $(n-1, 1)$ and $B(\lambda, \lambda) < 0$ for all $\lambda \in C$, so when W is hyperbolic. To say that the volume is finite is the same to say that W is a *lattice* in G. The following theorem is a condition to distinguish those hyperbolic lattices W (called *cocompact hyperbolic*) in G such that G/W is compact.

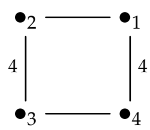
Theorem 1.4 *(cf. [16]) W is a cocompact hyperbolic Coxeter group if and only if both conditions hold:*

(a) B is nondegenerate, but not positive definite.

(b) For each $s \in S$, the Coxeter graph obtained by removing s from Γ is positive definite.

□

Cocompact hyperbolic Coxeter groups exist only in ranks $3, 4, 5$ (cf. [16]), but there is only one isomorphism class of cocompact crystallographic hyperbolic Coxeter systems: $(\mathfrak{W}, \mathfrak{S})$ with Coxeter graph $\Gamma(\mathfrak{W})$



(1.21)

In Chapter 3 we will spent much effort to represent $(\mathfrak{W}, \mathfrak{S})$ as a cocompact arithmetic lattice of $O^{+}_{\mathbb{R}}(3, 1)$, the orthogonal group of matrices with real entries that stabilizes a bilinear form of signature $(3, 1)$ and with positive entry in position $(1, 1)$, generalising a result of J. Elstrodt, F. Grunewald, J. Mennicke (cf. [11]).

1.12 Some important series for Coxeter groups

This section has the aim to remind two important series, the *growth series* (cf. [16]) and the *complete growth series* (cf. [1]), useful to study the growth of W

relative to the generating set S. These series were introduced for studying combinatorial structures in the context of infinite groups.

1.12.1 Growth series

The *growth series* is a series that, for a given set of generators, counts the number of elements of certain length (cf. [16]).
Let

$$a_n := |\{w \in W \mid \ell(w) = n\}|, \tag{1.22}$$

one defines

$$W(t) := \sum_{n \geq 0} a_n t^n = \sum_{w \in W} t^{\ell(w)} \tag{1.23}$$

that is a polynomial in the indeterminate t, said also *Poincaré polynomial*, element of the polynomial ring $\mathbb{Z}[t]$ when W is finite, otherwise is a formal power series in the indeterminate t, called also *Poincaré series*, element of the ring $\mathbb{Z}[\![t]\!]$ of the formal power series whose variable is t and whose coefficients come from the ring $\mathbb{Z}$.
There is a formula to calculate the sum of this series, as shown below.
Let $I \subseteq S$, remind that W_I is a parabolic subgroup of W, define the set

$$W^I := \{w \in W \mid \ell(ws) > \ell(w) \ \textit{for all} \ s \in I\}. \tag{1.24}$$

Proposition 1.11 *(cf. [16]) Fix $I \subseteq S$. Given $w \in W$, there exists a unique $u \in W^I$ and a unique $v \in W_I$ such that $w = uv$. Then $\ell(w) = \ell(u) + \ell(v)$. Furthermore, u is the unique element of smallest length in the coset wW_I.*

Proof Let $w \in W$ and choose a coset representative $u \in wW_I$ of smallest possible length. Then $w = uv$ for $v \in W_I$. Since $us \in wW_I$ for all $s \in I$, then $u \in W^I$, because u is of smallest possible length and $\ell(us) > \ell(u)$ for all $s \in I$. Consider the reduced expression for u and v: $u = s_1 \dots s_q$ with $s_i \in S$ and $v = s'_1 \dots s'_r$ with $s'_i \in I$. Then $\ell(w) \leq \ell(u) + \ell(v) = q + r$. If the inequality was strict, the Deletion Condition (Corollary 1.4) would allow to delete two factors s_i or s'_i in uv without changing w. If we delete two factors from u then we obtain a smaller element in wW_I. Instead if we delete two factors from v then we change v because the expression is reduced. Therefore $\ell(w) = \ell(u) + \ell(v)$.
At the end we have to prove the uniqueness of u as coset representative of smallest length.
Suppose that there exists another element $u' \in W^I$ that is also an element of wW_I different from u. We can write $u' = uv$ with $\ell(v) = r > 0$, then $v = s_1 \dots s_r$ with $s_i \in I$. But $\ell(u's_r) < \ell(u')$ contrary to $u' \in W^I$. □

For every subset $X \subseteq W$, we can define

$$X(t) := \sum_{w \in X} t^{\ell(w)}. \tag{1.25}$$

Then $W_I(t)$ is the growth series of the Coxeter group W_I, since $\ell = \ell_I$ on W_I. From Proposition 1.11 and this considerations, it follows the formula

$$W(t) = W_I(t) W^I(t). \tag{1.26}$$

Proposition 1.12 *(cf. [16])*

(a) If W is finite we have the identity

$$\sum_{I \subseteq S} (-1)^{|I|} \frac{W(t)}{W_I(t)} = \sum_{I \subseteq S} (-1)^{|I|} W^I(t) = t^N \tag{1.27}$$

where N is the number of positive roots, otherwise the right side equals 0.

(b) $W(t)$ *is an explicitly computable rational function of t.*

Proof

(a) In general, the equality of the first and second sum follows from the previous remarks.
Let W be finite and $w \in W$ be a fixed element. Set $K := \{s \in S \mid \ell(ws) > \ell(w)\}$. Then $w \in W^I$ when $I \subseteq K$, so $t^{\ell(w)}$ occurs in the sum with coefficient $\sum_{I \subseteq K} (-1)^{|I|}$. When $K \neq \emptyset$ then the coefficient $\sum_{I \subseteq K} (-1)^{|I|} = 0$, but when $K = \emptyset$, exactly when w is the element of maximal length (that exists only for finite Coxeter groups and moreover is unique (cf [16])), the coefficient is equal to t^N.
Let W be infinite. The set K is non-empty for all w, so all coefficients are 0.

(b) We prove this assertion by induction on $|S|$. If $|S| = 1$, then $W(t) = 1 + t$. Then using the equation in part (a):

$$\sum_{I \neq S} (-1)^{|I|} \frac{1}{W_I(t)} = \frac{f(t)}{W(t)}, \tag{1.28}$$

where $f(t) := t^N - (-1)^{|S|}$ when W is finite, otherwise $f(t) := -(-1)^{|S|}$.
By induction the left side is a computable rational function of t because it involves $W_I(t)$ for which $I \neq S$. Then also $W(t)$ is a computable rational function.

□

We see an example of calculation of a growth series.

Example 1.3 *Let $W = \tilde{A}_1$ the infinite dihedral group, i.e. $S = \{s_0, s_1\}$ and $m(s_0, s_1) = \infty$. From the definition of Poincaré series we have:*

$$W(t) = 1 + 2t + 2t^2 + ... \tag{1.29}$$

By Proposition 1.12, we can compute $W(t)$ as rational function. We consider all subset $I \subset S$: for $I = \emptyset$ $W_I(t) = 1$, for $I = \{s_1\}$ or $I = \{s_2\}$ $W_I(t) = 1 + t$.
Then using formula (1.28):

$$-\frac{1}{W(t)} = 1 - \frac{1}{1+t} - \frac{1}{1+t}. \tag{1.30}$$

So we obtain

$$W(t) = \frac{1+t}{1-t}. \tag{1.31}$$

□

1.12.2 Complete growth series

The *complete growth series* is another series introduced after the growth series for studying combinatorial structures in the context of infinite groups (cf. [1]). It counts the element of the group of a certain length adding them together. It is an element of the group ring.
Let (W, S) be a Coxeter system. We define the element W_n as the element of the group ring $\mathbb{Z}[W]$ formed by the sum of all elements of W of length n:

$$W_n := \sum_{\substack{w \in W \\ \ell(w)=n}} w. \tag{1.32}$$

The *complete growth series* of W is the formal power series

$$\tilde{W}(t) := \sum_{w \in W} w t^{\ell(w)} = \sum_{n=0} W_n t^n. \tag{1.33}$$

It is an element of the ring $\mathbb{Z}[W][[t]]$ of the formal power series whose variable is t and whose coefficients come from the group ring $\mathbb{Z}[W]$.

The complete growth series for Coxeter groups with respect to their standard generating set are known to be rational (cf. [30]).

Proposition 1.13 *(cf. [20]) Let (W, S) be a not finite Coxeter group. Let $\tilde{W}(t)$ be the complete growth series for W and for $I \subset S$ $\tilde{W}_I(t)$ be the complete growth series for the* *parabolic subgroup* W_I*. Then we have the identity*

$$\sum_{I\subseteq S}(-1)^{|I|}\frac{\tilde{W}(t)}{\tilde{W}_I(t)} = 0 \tag{1.34}$$

Proof The proof is similar to that one of Proposition 1.12. □

Chapter 2

Kac-Moody Lie algebras and Kac′ denominator formula

In 1967 Victor G. Kac and Robert Moody independently initiated the study of a Lie algebra associated with a matrix different from the usual Cartan Matrix: the *Generalised Cartan Matrix*. Their theory generalizes the theory of the semisimple Lie algebras. This chapter explores briefly the theory of Kac-Moody Lie algebras, highlighting the difference between the classical theory of the Lie algebras. For more details we refer the reader to [8].
Moreover in this chapter we see the connection between a crystallographic Coxeter system (W, S) and the roots of an arbitrary Kac-Moody Lie algebra associated with a symmetrizable Generalised Cartan Matrix with Weyl group W through the *Kac′ denominator formula* introduced by V. G. Kac (cf. [17]).

2.1 Generalized Cartan Matrix

Definition 2.1 *(cf. [8]) A matrix $A = (A_{ij}) \in M_n(\mathbb{C})$ is a Generalized Cartan Matrix (GCM) if satisfies:*

C1 $A_{ii} = 2 \;\; i = 1, \ldots, r;$

C2 $A_{ij} \in \mathbb{Z}$ *and* $A_{ij} \leq 0$ *for* $i \neq j;$

C3 *if* $A_{ij} = 0$ *then* $A_{ji} = 0.$

□

From Definition 2.1 we can observe that the GCM is a generalization of the Cartan Matrix. In fact a Cartan Matrix satisfies the properties $C1$ and $C3$, instead $C2$ is replaced by $A_{ij} \in \{0, -1, -2, -3\}$ for $i \neq j$ and moreover one adds

that if $A_{ij} \in \{-2, -3\}$ then $A_{ji} = -1$. Conversely the entries of a GCM that do not lie in the main diagonal can assume any nonpositive integer values. Therefore a Cartan Matrix is a particular GCM.
The following definition introduce some properties that a GCM can have.

Definition 2.2 *(cf. [8]) A GCM A is indecomposable if is not equivalent to a diagonal sum*

$$\begin{pmatrix} A_1 & 0 \\ 0 & A_2 \end{pmatrix}$$

of smaller GCMs A_1 and A_2. □

Definition 2.3 *(cf. [8]) A GCM A is symmetrizable if there exists a non-singular diagonal D and a symmetric matrix B such that $A = DB$.* □

We shall always assume that the GCM that we consider are indecomposable and symmetrizable.

Proposition 2.1 *(cf. [8]) Let A be a symmetrizable indecomposable GCM. Then A can be expressed in the from $A = DB$ with $D = diag(d_1, \ldots, d_n)$ and B symmetric, where $d_i \in \mathbb{Z}_{>0}$ and $B_{ij} \in \mathbb{Q}$. D is determined by these conditions up to scalar multiple.* □

2.2 Kac-Moody Lie algebra

To define a Kac-Moody Lie algebra, at first we should remind the definition and some properties of a *minimal realisation* of a square matrix.

2.2.1 Minimal realisation of a square matrix

Definition 2.4 *(cf. [8]) Let A be an $n \times n$ matrix over $\mathbb{C}$. A realisation of A is a triple $(\mathfrak{H}, \Pi, \Pi^\vee)$ such that:*

1. *$\mathfrak{H}$ is a finite dimensional vector space over $\mathbb{C}$;*
2. *$\Pi^\vee = \{h_1, \ldots, h_n\}$ is a linearly independent subset of $\mathfrak{H}$;*
3. *$\Pi = \{\alpha_1, \ldots, \alpha_n\}$ is a linearly independent subset of $\mathfrak{H}^*$, the dual of $\mathfrak{H}$;*
4. *$\alpha_j(h_i) = A_{ij}\ \forall i, j$.*

□

Proposition 2.2 *(cf. [8]) If $(\mathfrak{H}, \Pi, \Pi^\vee)$ is a realisation of A then $dim\mathfrak{H} \geq n - rankA$.* □

Definition 2.5 *(cf. [8]) A realisation of A is called minimal if*

$$dim\mathfrak{H} = 2n - rankA.$$

□

Proposition 2.3 *(cf. [8]) Any $n\times n$ matrix over $\mathbb{C}$ has a minimal realisation. Moreover it is univocally determined up less to isomorphism.* □

2.2.2 Definition of Kac-Moody Lie algebras

The definition of a Kac-Moody Lie algebra can be understood if it is clear the definition of a Lie algebra.

Definition 2.6 *(cf. [8]) A Lie algebra is a vector space over a field $\mathbb{K}$ endowed with a bilinear function*

$$[*,*] : \mathfrak{L} \times \mathfrak{L} \longrightarrow \mathfrak{L} \tag{2.1}$$

called commutator satisfying the following axioms:

1. $[x, x] = 0 \;\; \forall x \in \mathfrak{L}$;
2. $[x, [y, z]] + [z, [x, y]] + [y, [z, x]] = 0 \;\; \forall x, y, z \in \mathfrak{L}$.

Axiom 2 is called Jacobi identity. □

Let $\mathfrak{L}$ be a Lie algebra, for every $x \in \mathfrak{L}$ one defines a $\mathbb{K}$-linear map

$$ad_x : \mathfrak{L} \to \mathfrak{L} \tag{2.2}$$

such that $ad_x(y) := [x, y]$.

Definition 2.7 *(cf. [8]) Let A be a $n \times n$ GCM of rank r and $(\mathfrak{H}, \Pi, \Pi^\vee)$ be a minimal realisation of A. A Kac-Moody algebra is a complex Lie algebra $\mathfrak{L}$ generated by $\mathfrak{H}$ and by elements $e_1, \dots, e_n, f_1, \dots, f_n$ satisfying:*

L1 $[h, h'] = 0 \; \forall h, h' \in \mathfrak{H}$;

L2 $[e_i, f_j] = \delta_{ij} h_i$;

L3 $[h, e_i] = \alpha_i(h) e_i$;

L4 $[h, f_i] = -\alpha_i(h) f_i$;

L5 $[e_k, d^-_{ij}] = 0$, *where* $d^-_{ij} = (ad_{e_i})^{1-A_{ij}} e_j$;

L6 $[f_k, d^+_{ij}] = 0$, *where* $d^+_{ij} = (ad_{f_i})^{1-A_{ij}} f_j$.

The first four relations are called Chevalley-Serre relations and the last two are the Serre relations. We can write $\mathfrak{L} = \mathfrak{L}(A)$. □

From Definition 2.7 we observe that a GCM A defines univocally (up less to isomorphism) a Kac-Moody Lie algebra $\mathfrak{L}(A)$.
As $\mathbb{C}$-vector space $\mathfrak{L}(A)$ can be written as the direct sum of $\mathfrak{H}$, $\mathfrak{N}^-$ the $\mathbb{C}$-vector space generated by $f_1, \dots, f_n$ and $\mathfrak{N}^+$ the $\mathbb{C}$-vector space by $e_1, \dots, e_n$

$$\mathfrak{L}(A) = \mathfrak{N}^- \oplus \mathfrak{H} \oplus \mathfrak{N}^+. \tag{2.3}$$

The following proposition follows trivially from Definition 2.7.

Proposition 2.4 *(cf. [8]) If A is a Cartan matrix then $\mathfrak{L}(A)$ is the finite dimensional semisimple Lie algebra with Cartan matrix A.* □

The theory of the Kac-Moody algebras is an extension of the theory of the finite dimensional semisimple Lie algebras.

Proposition 2.5 *(cf. [8]) Let A be a GCM and $\mathfrak{L}(A)$ be the associated Kac-Moody Lie algebra as in Definition 2.7. There exist a Lie algebra $\tilde{\mathfrak{L}}(A)$ and an ideal $\mathfrak{I}$ such that $\mathfrak{L}(A) = \tilde{\mathfrak{L}}(A)/\mathfrak{I}$.* □

2.2.3 Properties of Kac-Moody Lie algebras

Since the Kac-Moody Lie algebras are a generalization of the semisimple Lie algebras, the following concepts and propositions follow by analogy with the theory of the semisimple Lie algebras.
Let $\mathfrak{L}(A)$ be a Kac-Moody Lie algebra with minimal realisation $(\mathfrak{H}, \Pi, \Pi^\vee)$ where $\Pi = \{\alpha_1, \dots, \alpha_n\}$. Define the *lattice*

$$\Lambda_{\mathbb{Z}} := \{\alpha = k_1\alpha_1 + k_2\alpha_2 + \dots k_n\alpha_n \mid k_1, \dots, k_n \in \mathbb{Z}\} \tag{2.4}$$

subgroup of $\mathfrak{H}^*$ and

$$\Lambda_{\mathbb{Z}}^+ := \{0 \neq \alpha \in \Lambda_{\mathbb{Z}} \mid k_i \geq 0 \ \forall i\}, \ \Lambda_{\mathbb{Z}}^- := \{0 \neq \alpha \in \Lambda_{\mathbb{Z}} \mid k_i \leq 0 \ \forall i\} \tag{2.5}$$

subsets of $\Lambda_{\mathbb{Z}}$ that are respectively a *positive* and a *negative cone*. The elements of the positive cone are called *weights*. For each $\alpha \in \Lambda_{\mathbb{Z}}$ define

$$\mathfrak{L}_\alpha := \{x \in \mathfrak{L}(A) \mid [h, x] = \alpha(h)x \ \forall h \in \mathfrak{H}\}. \tag{2.6}$$

Proposition 2.6 *(cf. [8])*

1. $\mathfrak{L}(A) = \bigoplus_{\alpha \in \Lambda_{\mathbb{Z}}} \mathfrak{L}_\alpha$

2. *dim* $\mathfrak{L}_\alpha$ *is finite for all* $\alpha \in \Lambda_{\mathbb{Z}}$

3. $\mathfrak{L}_0 = \mathfrak{H}$

4. *If* $\alpha \neq 0$ *then* $\mathfrak{L}_\alpha = 0$ *unless* $\alpha \in \Lambda_{\mathbb{Z}}^+$ *or* $\alpha \in \Lambda_{\mathbb{Z}}^-$

5. $[\mathfrak{L}_\alpha, \mathfrak{L}_\beta] \subset \mathfrak{L}_{\alpha+\beta}$ *for all* $\alpha, \beta \in \Lambda_{\mathbb{Z}}$

Proof These properties follow from Proposition 2.5 whereby $\mathfrak{L}(A) = \tilde{\mathfrak{L}}(A)/\mathfrak{I}$ and also from the existence of the canonical map $\pi : \tilde{\mathfrak{L}}(A) \to \mathfrak{L}(A)$ that allows to shift some properties of the semisimple Lie algebras in the Kac-Moody Lie algebras. □

Definition 2.8 *(cf. [8]) The* $\mathbb{C}$*-vector space* $\mathfrak{H}$ *is a subalgebra of* $\mathfrak{L}(A)$ *called Cartan subalgebra. An element* $\alpha \in \mathfrak{H}^*$ *is called root of* $\mathfrak{L}(A)$ *if* $\alpha \neq 0$ *and* $\mathfrak{L}_\alpha \neq 0$*. The set*

$$\dot{\Phi} = \{\alpha \in \Lambda_{\mathbb{Z}} \mid \alpha \text{ is a root}\} \tag{2.7}$$

is called root system of $\mathfrak{L}(A)$*. Every root lies in* $\dot{\Phi}^+ := \dot{\Phi} \cap \Lambda_{\mathbb{Z}}^+$*, so called positive roots, or in* $\dot{\Phi}^- := \dot{\Phi} \cap \Lambda_{\mathbb{Z}}^-$*, so called negative roots. If* α *is a root, then* $\mathfrak{L}_\alpha$ *is the root space of* α*. The dimension of* $\mathfrak{L}_\alpha$ *is called the multiplicity of* α*,* m_α*.* □

Proposition 2.7 *(cf. [8])*

1. *dim* $\mathfrak{L}_{\alpha_i} = 1 =$ *dim* $\mathfrak{L}_{-\alpha_i}$

2. *If* $k > 1$ *then dim* $\mathfrak{L}_{k\alpha_i} = 0 =$ *dim* $\mathfrak{L}_{-k\alpha_i}$

Proof This proof follows from the considerations done in the proof of Proposition 2.6. □

The roots $\alpha_1, \alpha_2, \ldots, \alpha_n$ form a basis of $\Lambda_{\mathbb{Z}}$ and are called *simple roots* of $\mathfrak{L}(A)$.

2.2.4 Kac-Moody Lie algebras associated with a symmetrizable GCM

Remind that for a semisimple Lie algebra $\mathfrak{L} = \mathfrak{L}(A)$, with A Cartan matrix, we can define a nondegenerate symmetric bilinear form, called *Killing form*,

$$K : \mathfrak{L} \times \mathfrak{L} \to \mathbb{C} \tag{2.8}$$

that is invariant in the sense that $K([x, y], z) = K(x, [y, z])$ for $x, y, z \in \mathfrak{L}$.
Also in the general case of a Kac-Moody Lie algebra $\mathfrak{L}(A)$ there exists a nondegenerate symmetric invariant bilinear form on $\mathfrak{L}(A)$ when A is a symmetrizable GCM (cf. [8]).

Let A be a symmetrizable GCM. Then $A = DB$ with $D = diag(d_1, \ldots, d_n)$ diagonal matrix and B symmetric matrix.
Let $(\mathfrak{H}, \Pi, \Pi^\vee)$ be a minimal realisation of A, where $\Pi^\vee = \{h_1, \ldots, h_n\}$ is a linearly independent set of $\mathfrak{H}$, $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ is a linearly independent set of $\mathfrak{H}^*$, $\alpha_j(h_i) = A_{ij}$ and $dim\mathfrak{H} = 2n - r$ with $r = rankA$.
Let $\mathfrak{H}'$ be the subspace of $\mathfrak{H}$ generated by $h_1, \ldots, h_n$ and $\mathfrak{H}''$ be the complement of $\mathfrak{H}'$ in $\mathfrak{H}$. Then $\mathfrak{H} = \mathfrak{H}' \oplus \mathfrak{H}''$ where $dim\mathfrak{H}' = n$ and $dim\mathfrak{H}'' = n - r$.
The bilinear form

$$\langle , \rangle : \mathfrak{H} \times \mathfrak{H} \to \mathbb{C} \tag{2.9}$$

defined by the rules:

$$\langle h_i, h_j \rangle = d_i d_j B_{ij} \;\; i, j = 1, \ldots, n \tag{2.10}$$

$$\langle h_i, x \rangle = \langle x, h_i \rangle = d_i \alpha_i(x) \;\; x \in \mathfrak{H}'' \tag{2.11}$$

$$\langle x, y \rangle = 0 \;\; x, y \in \mathfrak{H}'' \tag{2.12}$$

is a nondegenerate symmetric bilinear form on $\mathfrak{H}$ (cf. [8]). Moreover this form can also be extended to a nondegenerate symmetric invariant bilinear form on $\mathfrak{L}(A)$ (cf. [8]) that is called *standard invariant bilinear form* on $\mathfrak{L}(A)$.
Since the form (2.9) is nondegenerate on $\mathfrak{H}$, it determines a bijection $\mathfrak{H}^* \to \mathfrak{H}$ by $\alpha \to h_\alpha$ where

$$\langle h'_\alpha, h \rangle = \alpha(h) \;\; \forall h \in \mathfrak{H} \tag{2.13}$$

(cf. [8]).
The bilinear form (2.9) allows to associate with a Kac-Moody algebra $\mathfrak{L}(A)$ a reflection group with interesting properties and called *Weyl group* of $\mathfrak{L}(A)$.

2.2.5 Weyl group of $\mathfrak{L}(A)$

Proposition 2.8 *(cf. [8]) The map $s_i : \mathfrak{H} \to \mathfrak{H}$ such that $s_i(x) = x - \alpha_i(x)h_i$ for $i = 1, \ldots, n$ has the following properties:*

1. $s_i^2 = 1$;
2. $s_i(h_i) = -h_i$;
3. $s_i(x) = x$ *when* $\langle h_i, x \rangle = 0$.

Proof The first and the second property come from the formula of s_i and from $\alpha_i(h_i) = 2$. To prove the third property we must add the fact that $\langle h_i, x \rangle = d_i \alpha_i(x) = 0$ implies $\alpha_i(x) = 0$. □

The maps $s_i : \mathfrak{H} \to \mathfrak{H}$ for $i = 1, \ldots, n$ are called *simple reflections*.

Definition 2.9 *(cf. [8]) The group W of non-singular linear transformations of $\mathfrak{H}$ generated by $s_1, \ldots, s_n$ is called the Weyl group of $\mathfrak{L}(A)$.* □

Proposition 2.9 *(cf. [8]) The bilinear form $\langle , \rangle$ on $\mathfrak{H}$ is invariant under W.*

Proof Let $x, y \in \mathfrak{H}$. Then

$$\langle s_i x, s_i y\rangle = \langle x-\alpha_i(x)h_i, y-\alpha_i(y)h_i\rangle = \langle x, y\rangle-\alpha_i(x)\langle h_i, y\rangle-\alpha_i(y)\langle x, h_i\rangle+\alpha_i(x)\alpha_i(y)\langle h_i, h_i\rangle$$

$$= \langle x, y\rangle - \alpha_i(x)d_i\alpha_i(y) - \alpha_i(y)d_i\alpha_i(x) + \alpha_i(x)\alpha_i(y)2d_i = \langle x, y\rangle.$$

□

The action of W can also be defined on $\mathfrak{H}^*$ by

$$(w\alpha)h = \alpha(w^{-1}h) \;\; for \;\; w \in W, \;\; \alpha \in \mathfrak{H}^*, \;\; h \in \mathfrak{H}. \tag{2.14}$$

This action is compatible with the isomorphism $\mathfrak{H}^* \to \mathfrak{H}$ given by $\alpha \to h_\alpha$ such that $\langle h'_\alpha, h\rangle = \alpha(h)$ for all $h \in \mathfrak{H}$. In fact, let $\alpha, \beta \in \mathfrak{H}^*$ such that $w(\alpha) = \beta$. Then

$$\langle w(h'_\alpha), h\rangle = \langle h'_\alpha, w^{-1}(h)\rangle = \alpha(w^{-1}(h)) = (w\alpha)h = \beta(h) = \langle h'_\beta, h\rangle \tag{2.15}$$

for all $h \in \mathfrak{H}$. Thus $w(h'_\alpha) = h'_\beta$ (cf. [8]).

Proposition 2.10 *(cf. [8]) The action of s_i on $\mathfrak{H}^*$ is given by*

$$s_i(\beta) = \beta - \beta(h_i)\alpha_i.$$

Proof Let $x \in \mathfrak{H}$. Then

$$(s_i(\beta))x = \beta(s_i^{-1}x) = \beta(s_i x) = \beta(x - \alpha_i(x)h_i)$$

$$= \beta(x) - \alpha_i(x)\beta(h_i) = (\beta - \beta(h_i)\alpha_i)x.$$

□

Remind that an element $0 \neq \alpha \in \Lambda_{\mathbb{Z}}$ such that $\mathfrak{L}_\alpha \neq 0$ is called root of $\mathfrak{L}(A)$ and that the set Φ of all roots of $\mathfrak{L}(A)$ is called root system. Then W naturally acts on Φ. In analogy with the semisimple Lie algebras there are the following properties.

Proposition 2.11 *(cf. [8]) If $\alpha \in \Phi$, $w \in W$ then $w(\alpha) \in \Phi$. Moreover $dim\mathfrak{L}_\alpha = dim\mathfrak{L}_{w(\alpha)}$.* □

Proposition 2.12 *(cf. [8]) Let $i \neq j$. The order of $s_i s_j \in W$ is:*

$$\begin{array}{ll} 2 & \textit{if } A_{ij}A_{ji} = 0 \\ 3 & \textit{if } A_{ij}A_{ji} = 1 \\ 4 & \textit{if } A_{ij}A_{ji} = 2 \\ 6 & \textit{if } A_{ij}A_{ji} = 3 \\ \infty & \textit{if } A_{ij}A_{ji} \geq 4. \end{array}$$

□

As for the Coxeter groups one can also define a *length* for the elements of a Weyl group W respect to its generators, i.e. the simple reflections $s_1, \ldots, s_n$:

$$\ell : W \to \mathbb{N}_0. \tag{2.16}$$

(cf. Chapter 1, Section 1.2).
Moreover the number $\ell(w)$ is exactly the number of positive roots being sent by w to negative ones (cf. [8]), in analogy with the geometric interpretation of the length function for Coxeter groups (cf. Chapter 1, Section 1.6).
The following proposition gives us the connection between a crystallographic Coxeter system (W, S) and a Kac-Moody Lie algebra associated with a symmetrizable Generalized Cartan Matrix (GCM) with Weyl group W.

Proposition 2.13 *(cf. [8]) The Weyl group W of the Kac-Moody Lie algebra $\mathfrak{L}(A)$ is a crystallographic Coxeter group generated by $s_1, \ldots, s_n$ with relations:*

$$\begin{array}{rl} s_i^2 = 1 & \\ (s_is_j)^2 = 1 & \textit{if } A_{ij}A_{ji} = 0 \\ (s_is_j)^3 = 1 & \textit{if } A_{ij}A_{ji} = 1 \\ (s_is_j)^4 = 1 & \textit{if } A_{ij}A_{ji} = 2 \\ (s_is_j)^6 = 1 & \textit{if } A_{ij}A_{ji} = 3. \end{array}$$

The order of s_is_j is exactly the exponent of the previous relations. □

2.2.6 Root system

Let A be a GCM and $\mathfrak{L}(A)$ be the corresponding Kac-Moody Lie algebra. Then

$$\mathfrak{L}(A) = \mathfrak{H} \oplus \sum_{\alpha \in \dot{\Phi}} \mathfrak{L}_\alpha \tag{2.17}$$

where $\dot{\Phi} := \{\alpha \neq 0 \mid \mathfrak{L}_\alpha \neq 0\}$ is the root system of $\mathfrak{L}(A)$.
In analogy with the theory of the semisimple Lie algebras, the root system $\dot{\Phi} = \dot{\Phi}^+ \sqcup \dot{\Phi}^-$ is the disjoint union of $\dot{\Phi}^+ := \dot{\Phi} \cap \Lambda_{\mathbb{Z}}^+$ that is the set of the positive roots (*positive root system*) and $\dot{\Phi}^- := \dot{\Phi} \cap \Lambda_{\mathbb{Z}}^-$ of negative ones (*negative root*

system) (cf. [8]). $\Pi := \{\alpha_1, \ldots, \alpha_n\}$ is the set of the *simple roots*.
The map

$$m_\cdot : \Lambda^+_{\mathbb{Z}} \to \mathbb{N}_0 \tag{2.18}$$

is the *multiplicity function*, that associates with every $\alpha \in \Lambda^+_{\mathbb{Z}}$ its multiplicity m_α, i.e. the dimension of $\mathfrak{L}_\alpha$. Remind that the simple roots have multiplicity equals to 1 (cf. Proposition 2.7) and the Weyl group W acts on $\dot{\Phi}$ and preserves multiplicities (cf. Proposition 2.11). In the root system of a Kac-Moody Lie algebra associated with a symmetrizable GCM with Weyl group W, there are two types of weights: real and imaginary.

Definition 2.10 *(cf. [8]) $\alpha \in \dot{\Phi}$ is called real root if there exist $\alpha_i \in \Pi$ and $w \in W$ such that $\alpha = w(\alpha_i)$, otherwise α is called imaginary root. Call $\dot{\Phi}_{Re}$ the real root system and $\dot{\Phi}_{Im}$ the imaginary root system.* □

Definition 2.11 *(cf. [15]) Let $\alpha \in \dot{\Phi}^+$, such that $\alpha = k_1\alpha_1 + \ldots + k_r\alpha_r$. The height of α is the positive integer*

$$ht(\alpha) = \sum_{i=1}^{r} k_i.$$

□

Remark 2.1 *(cf. [8]) If α is a real root then also $-\alpha$ is it and the same is true for α imaginary root.*

Proof If α is a real root, then there exist $\alpha_i \in \Pi$ and $w \in W$ such that $\alpha = w(\alpha_i)$. Therefore $-\alpha = ws_i(\alpha_i)$. Consequently if α is an imaginary root so it is $-\alpha$. □

Proposition 2.14 *(cf. [8]) Let α be a real root. Then α has multiplicity 1. Also for $k \in \mathbb{Z}$, $k\alpha$ is a root if and only if $k = \pm 1$.*

Proof If α is a real root, then there exist $\alpha_i \in \Pi$ and $w \in W$ such that $\alpha = w(\alpha_i)$. Therefore, from Proposition 2.11, it follows that α has multiplicity 1. Moreover Proposition 2.7 says us that $k\alpha_i$ is not a root. Then $k\alpha = w(k\alpha_i)$ is not a root. □

Let $\dot{\Phi}^+_{im}$ be the positive imaginary root system.

Proposition 2.15 *(cf. [8]) If $\alpha \in \dot{\Phi}^+_{Im}$ and $w \in W$. Then $w(\alpha) \in \dot{\Phi}^+_{Im}$.*

Proof We know that W acts both on $\dot{\Phi}$ and on the set $\dot{\Phi}_{Re}$ of real roots. Therefore W acts on the set $\dot{\Phi}_{Im}$ of imaginary roots. We want to show that W cannot change the sign of an imaginary root. Let

$$\alpha = \sum_{i=1}^{n} k_i\alpha_i \qquad k_i \geq 0.$$

Then at least two k_i are positive, otherwise α is a multiple of some α_i and hence $\alpha = \alpha_i$ would be a real root.
Consider that $s_i(\alpha) = \alpha - \alpha(h_i)\alpha_i$ contains at least one simple root with positive coefficient. Then $s_i(\alpha) \in \dot{\Phi}^+_{Im}$. Since w is a product of s_i then $w(\alpha) \in \dot{\Phi}^+_{Im}$. □

The following theorem and proposition characterize the set of positive imaginary roots and hence of imaginary roots.

Theorem 2.1 *(cf. [8]) Let $\alpha \in \dot{\Phi}^+_{Im}$. Then $k\alpha \in \dot{\Phi}^+_{Im}$ for all positive integers k.* □

Let A be a symmetrizable GCM and $\langle , \rangle$ be the standard invariant bilinear form on the Kac-Moody algebra $\mathfrak{L}(A)$. Since $\langle , \rangle$ is nondegenerate on $\mathfrak{H}$, then it determines an isomorphism $\mathfrak{H}^* \to \mathfrak{H}$ defined by $\lambda \to h'_\lambda$ such that $\lambda(x) = \langle h'_\lambda, x \rangle$ for all $x \in \mathfrak{H}$. The bilinear form $\langle , \rangle$ on $\mathfrak{H}$ can be shifted on $\mathfrak{H}^*$ by defining

$$\langle \lambda, \mu \rangle = \langle h'_\lambda, h'_\mu \rangle \tag{2.19}$$

(cf. [8]). In particular one can define $\langle \alpha, \alpha \rangle$ for $\alpha \in \dot{\Phi}$.

Proposition 2.16 *(cf. [8]) Let A be a symmetrizable GCM. If α is a real root then $\langle \alpha, \alpha \rangle > 0$. If α is an imaginary root then $\langle \alpha, \alpha \rangle \leq 0$.* □

2.3 The classification of Kac-Moody Lie algebras

Let A be an indecomposable GCM. The structure of a Kac-Moody Lie algebra $\mathfrak{L}(A)$ depends on the GCM A. Then to classify the Kac-Moody Lie algebras is equivalent to classify the corresponding GCMs.

The classifications of types of the Kac-Moody Lie algebras $\mathfrak{L}(A)$ is the following one (cf. [7]).

Finite type: A is positive definite. In this case $det(A) > 0$ and A is the Cartan matrix of a finite dimensional semisimple Lie algebra.

Affine type: A is positive semidefinite, but not positive definite. In this case $det(A) = 0$.

Indefinite type: A is neither of finite nor affine type.

The simplest indefinite types are called

Hyperbolic type: A is neither finite nor affine type, but every proper, indecomposable principal submatrix is either of finite or affine type. In this case $det(A) < 0$.

Lorentzian type: $det(A) < 0$ and A has exactly one negative eigenvalue.

The GCMs of Lorentzian type include and are larger then the GCMs of hyperbolic type.
Let A be a GCM of hyperbolic type. A is of *cocompact hyperbolic type* if every proper, indecomposable principal submatrix is of finite type.

If A has finite type, then $\mathfrak{L}(A)$ is a *finite Kac-Moody algebra* (simple Lie algebra); if A has affine type, then $\mathfrak{L}(A)$ is an *affine Kac-Moody algebra*; if A has indefinite type, then $\mathfrak{L}(A)$ is an *indefinite Kac-Moody algebra*; if A has hyperbolic type, then $\mathfrak{L}(A)$ is a *hyperbolic Kac-Moody algebra*.

Theorem 2.2 *(cf. [8]) Let A be an indecomposable GCM of finite or affine type. Then A is symmetrizable.* □

Since a crystallographic Coxeter group W is a Weyl group of a Kac-Moody Lie algebra $\mathfrak{L}(A)$ (cf. Proposition 2.13), then it is easy to verify that:

1. if $\mathfrak{L}(A)$ is of finite type then W is a finite crystallographic Coxeter group;
2. if $\mathfrak{L}(A)$ is of affine then W is an affine crystallographic Coxeter group;
3. if $\mathfrak{L}(A)$ is of hyperbolic type then W is a hyperbolic crystallographic Coxeter group.

The following theorems show some properties of the root systems of the different types of Kac-Moody Lie algebras.

Theorem 2.3 *(cf. [8]) Let A be an indecomposable GCM.*

1. *If A has finite type then $\mathfrak{L}(A)$ has no imaginary roots.*
2. *If A has affine type then there exists a vector u with positive entries such that $Au = 0$. u is determined up to scalar multiple. Then there is a unique u whose entries are positive integers have no common factors. Let $u = (a_1, \ldots, a_n)$. Let $\delta = a_1\alpha_1 + \ldots + a_n\alpha_n$. Then the imaginary roots of $\mathfrak{L}(A)$ are the elements $k\delta$ for $k \in \mathbb{Z}$, $k \neq 0$.*
3. *If A has indefinite type then there exists $\alpha \in \dot{\Phi}^+_{Im}$ such that $\alpha = \sum_{i=1}^n k_i\alpha_i$ with $k_i > 0$ and $\alpha(h_i) < 0$ for $i = 1, \ldots, n$.*

□

Corollary 2.1 *(cf. [8]) If A is an indecomposable GCM of affine or indefinite type then the dimension of $\mathfrak{L}(A)$ is infinite.*

Proof In both cases $\mathfrak{L}(A)$ has an imaginary root α. Then $\mathfrak{L}(A)$ has infinitely many roots $k\alpha$ for $k \in \mathbb{Z}$, $k \neq 0$, by Theorem 2.1. Since $\mathfrak{L}(A) = \mathfrak{H} \oplus \sum_{\alpha\in\Phi} \mathfrak{L}_\alpha$, then the dimension of $\mathfrak{L}(A)$ must be infinite. □

The following propositions, due to R. V. Moody, reassume the properties of the root system of a Kac-Moody Lie algebra of hyperbolic type.

Theorem 2.4 *(cf. [23]) If A is a symmetrizable and hyperbolic GCM, then*

$$\dot{\Phi}_{Im} = \{\alpha \in \Lambda_{\mathbb{Z}} \mid \langle \alpha, \alpha \rangle \leq 0\}.$$

□

Corollary 2.2 *(cf. [23]) If A is a symmetrizable and hyperbolic GCM, then $\dot{\Phi}^+_{Im}$ is a semi-group under addition.* □

In conclusion, the following theorem is one of the most important result of R. V. Moody.

Theorem 2.5 *(cf. [23]) Let A be an indecomposable symmetrizable GCM.*

$$\dot{\Phi}_{Im} = \{\alpha \in \Lambda_{\mathbb{Z}} \mid \langle \alpha, \alpha \rangle \leq 0\}$$

if and only if A has finite, affine or hyperbolic type. □

One of the open problems relative to the Kac-Moody Lie algebras is to find an effective closed formula for the dimensions of the imaginary root spaces for hyperbolic and other indefinite Kac-Moody Lie algebras.

2.4 Kac′ denominator formula

This thesis explores the connection between a crystallographic Coxeter system (W, S) and the roots of an arbitrary Kac-Moody Lie algebra associated with a symmetrizable GCM with Weyl group W through a formula, called *Kac′ denominator formula*:

$$\sum_{w\in W} (-1)^{\ell(w)} e(c(w)) = \prod_{\alpha\in\Lambda^+_{\mathbb{Z}}} (1 - e(\alpha))^{m_\alpha}, \tag{2.20}$$

Here:

1. W is a crystallographic Coxeter group;
2. $e(\alpha)$ is a formal exponential;
3. $\Lambda_{\mathbb{Z}}^{+}$ is the positive root cone;
4. $\ell : W \to \mathbb{N}_0$ is the length on (W, S);
5. $m_{\cdot} : \Lambda_{\mathbb{Z}}^{+} \to \mathbb{N}_0$ is the multiplicity function;
6. $c(w)$ is the sum of positive roots being sent by w^{-1} to negative ones.

The formula (2.20) is *parametrized* when we explicitly know the function $m_{\cdot}$. It was at first discovered and proved by I. G. Macdonald for affine Kac-Moody Lie algebras (cf. [21]) and then extended to the entire class of Kac-Moody Lie algebras associated with a symmetrizable GCM by V. G. Kac (cf. [4]).

2.4.1 The function c

The left hand-side of formula (2.20) depends crucially on the function c, then we will study its properties. Let $\Lambda_{\mathbb{R}}$ be an $\mathbb{R}$-vector space with basis the simple roots $\alpha_1, \ldots, \alpha_n$. We consider the lattice $\Lambda_{\mathbb{Z}} := \{k_1\alpha_1 + \ldots + k_n\alpha_n \mid k_1, \ldots, k_n \in \mathbb{Z}\}$. Remember that $\Lambda_{\mathbb{Z}}^{+} := \Lambda_{\mathbb{Z}_{\geq 0}}$ is the positive cone whose elements are called weights. Let Φ be the root system of W and for $w \in W$ let $\Gamma(w) := \Phi^{+} \cap w(\Phi^{-})$. The function

$$c : W \to \Lambda_{\mathbb{Z}}^{+},$$

given by

$$c(w) = \sum_{\gamma \in \Gamma(w)} \gamma,$$

maps w in the sum of positive roots being sent by w^{-1} to negative ones.

Properties of c

(c1) $c(w^{-1}) = -w^{-1}(c(w)) \;\; \forall w \in W$

Proof Let $w = s_1 \ldots s_r$ with $\ell(w) = r$, then $w^{-1} = s_r \ldots s_1$.
The positive roots that w^{-1} sends to negative ones are $\gamma_i = s_1 \cdots s_{i-1}(\alpha_i)$ for $1 \leq i \leq r$ and interpreting γ_1 to be α_1. Instead the positive roots that w sends to negative ones are $\beta_i = s_r \cdots s_{i+1}(\alpha_i)$ for $1 \leq i \leq r$ and interpreting β_r to be α_r (c.f. Proposition 1.5).
Then $c(w^{-1}) = \sum_{i=1}^{r} \beta_i$ and $c(w) = \sum_{i=1}^{r} \gamma_i$.
Observe that $w^{-1}(\gamma_i) = s_r \ldots s_1(s_1 \cdots s_{i-1}(\alpha_i)) = s_r \ldots s_i(\alpha_i) = -s_r \ldots s_{i+1}(\alpha_i) = -\beta_i$.
So $c(w^{-1}) = \sum_{i=1}^{r} \beta_i = -\sum_{i=1}^{r} w^{-1}(\gamma_i) = -w^{-1}(\sum_{i=1}^{r} \gamma_i) = -w^{-1}(c(w))$. □

(c2) Let $w = s_1 \dots s_r$ with $\ell(w) = r$, then $c(s_1 s_2 \dots s_r) = s_1 \dots s_{r-1}(\alpha_r) + c(s_1 \dots s_{r-1})$.

Proof Let γ_i as defined in the proof of (c1).
$c(s_1 s_2 \dots s_r) = \sum_{i=1}^{r} \gamma_i = \gamma_r + \sum_{i=1}^{r-1} \gamma_i = s_1 \dots s_{r-1}(\alpha_r) + c(s_1 \dots s_{r-1})$. □

(c3) Let $w \in W$ and $s \in S$ such that $\ell(ws) = \ell(w) + 1$, then $c(ws) = w(\alpha_s) + c(w)$.

Proof It follows from (c2) with $s = s_r$ and $w = s_1 \dots s_{r-1}$. □

(c4) $c(w) = 0$ if and only if $w = 1$.

Proof It is trivial, in fact $\Gamma(1) = \emptyset$. □

(c5) $c(s_\alpha) = \alpha$.

Proof It is trivial, in fact $\Gamma(s_\alpha) = \{\alpha\}$. □

(c6) $c(sw) = \alpha_s + s(c(w))$ for $s \in S$ and $w \in W$ such that $\ell(sw) = 1 + \ell(w)$.

Proof It is sufficient to observe that $\Gamma(sw) = \{\alpha_s\} \sqcup s \cdot \Gamma(w)$. □

(c7) Fixed $w = s_1 \dots s_r \in W$ with $\ell(w) = r$. One defines the map

$$c^w : \{0, 1, \dots, r\} \to \Lambda_{\mathbb{Z}}^+$$

by induction in this way:

$$c_0^w = 0, \;\; c_i^w = c_{i-1}^w + s_r \dots s_{r-(i-1)+1}(\alpha_{r-(i-1)+1}) \;\; i = 0, \dots, r$$

where $c_r^w = c(w)$.

(c8) Let $w, w' \in W$. If $w' < w$ in the Bruhat ordering then $c(w') < c(w)$, i.e. $c(w) - c(w') \in \Lambda_{\mathbb{Z}}^+$.

Proof If $w' < w$ in the Bruhat ordering, then there exist $w_0 = w', w_1, \dots, w_r = w$ such that $w_0 < w_1 < \dots < w_r$ and $\ell(w_i) = \ell(w_{i-1}) + 1$ for $i = 1, \dots, r$, i.e. w_i is given by w_{i-1} adding a simple reflection s_i: $w_i = w_{i-1} s_i$.
By (c3), $c(w_r) = w_{r-1}(\alpha_r) + c(w_{r-1}) = w_{r-1}(\alpha_r) + w_{r-2}(\alpha_{r-1}) + c(w_{r-2}) = \dots = w_{r-1}(\alpha_r) + w_{r-2}(\alpha_{r-1}) + \dots + w_0(\alpha_1) + c(w_0)$.
$c(w_r) - c(w_0) = w_{r-1}(\alpha_r) + w_{r-2}(\alpha_{r-1}) + \dots + w_0(\alpha_1) \in \Lambda_{\mathbb{Z}}^+$, so $c(w') < c(w)$. □

Root systems

Let (W, S) be a Coxeter root system with $\rho : W \to GL(\Lambda_{\mathbb{R}})$ the geometric representation. In Definition 1.5, the root system $\Phi = \{w(\alpha_s) \mid w \in W, \;\; s \in S\}$

of (W, S) is constructed starting by the basis $\{\alpha_s \mid s \in S\}$ of unitary vectors, i.e. $B(\alpha_s, \alpha_s) = 1$. An alternative definition of the root system is the following one, because starting by unitary vectors is not always convenient.

Definition 2.12 *We call a subset $\Upsilon \subseteq \Lambda_{\mathbb{R}}$ a root system of (W, S), if*

(R1) $0 \notin \Upsilon$;

(R2) $\forall \beta \in \Upsilon$: $\mathbb{R} \cdot \beta \cap \Upsilon = \{\pm\beta\}$;

(R3) $\forall w \in W$ *one has* $w(\Upsilon) = \Upsilon$;

(R4) $\forall \gamma \in \Phi$ *there exist* $\beta_\gamma \in \Upsilon$ *and* $e \in \mathbb{R}^+$ *such that* $\gamma = e \cdot \beta_\gamma$;

(R5) $\forall \beta \in \Upsilon$ *there exist* $\gamma_\beta \in \Phi$ *and* $d \in \mathbb{R}^+$ *such that* $\beta = d \cdot \gamma_\beta$.

□

Therefore, roughly speaking, a root system Υ of (W, S) is the set of renormalized vectors $\{d_\gamma \cdot \gamma \mid \gamma \in \Phi\}$ such that $(R3)$ is satisfied.
Let $\beta \in \Upsilon$, the *reflection* s_β associated with β is

$$s_\beta(x) := x - 2 \cdot \frac{B(x, \beta)}{B(\beta, \beta)} \cdot \beta, \;\; x \in \Lambda_{\mathbb{R}}. \tag{2.21}$$

A root system Υ is said to be *crystallographic* if $\forall \alpha, \beta \in \Upsilon$

$$\langle\langle \alpha, \beta \rangle\rangle := 2 \cdot \frac{B(\alpha, \beta)}{B(\beta, \beta)} \in \mathbb{Z}. \tag{2.22}$$

and *rational crystallographic*, if it is crystallographic and for all $\beta \in \Upsilon$

$$B(\beta, \beta) \in \mathbb{Q}. \tag{2.23}$$

The canonical basis of Υ is the set

$$\Delta = \bigcup_{s \in S} (\mathbb{R}_{>0} \cdot \alpha_s \cap \Upsilon). \tag{2.24}$$

Fact 2.1 *Let Υ be a root system of the Coxeter group (W, S). Then*

(a) $\Upsilon = \{w(\beta) \mid \beta \in \Delta, w \in W\}$;

(b) (W, S) *is a crystallographic Coxeter group if and only if* $2 \cdot \frac{B(\alpha,\beta)}{B(\beta,\beta)} \in \mathbb{Z}$ *for all* $\alpha, \beta \in \Delta$.

Proof

(a) It is a direct consequence of Definition 1.5.

(b) It is sufficient to prove that if $2 \cdot \frac{B(\alpha,\beta)}{B(\beta,\beta)} \in \mathbb{Z}$ for all $\alpha, \beta \in \Delta$, then W is crystallographic.
Let consider

$$\Lambda_{\mathbb{Z}} = span_{\mathbb{Z}}\Delta \subseteq \Lambda_{\mathbb{R}}. \tag{2.25}$$

Since W is generated by S, then $\Lambda_{\mathbb{Z}}$ is a W-invariant subset of $\Lambda_{\mathbb{R}}$ and, by definition $\Delta \subset \Lambda_{\mathbb{Z}}$. Hence, by (*a*), $\Upsilon \subseteq \Lambda_{\mathbb{Z}}$. Moreover, by (*a*), for $\alpha, \beta \in \Lambda_{\mathbb{Z}}$ there exists $w \in W$ such that $w(\beta) \in \Delta$ and

$$w(\alpha - s_\beta(\alpha)) = 2 \cdot \frac{B(\alpha,\beta)}{B(\beta,\beta)} \cdot w(\beta) \in \Lambda_{\mathbb{Z}}.$$

□

We can refer to a crystallographic Coxeter group with the triple (W, S, Υ).
The lattice $\Lambda_{\mathbb{Z}} \subset \Lambda_{\mathbb{R}}$ given by (2.25) is called *root lattice* of the crystallographic Coxeter group (W, S, Υ).
From now on, we shall always consider the root system Φ of the Coxeter group (W, S) in the sense of Definition 2.12, then the vectors α_s of Φ are not necessarily unitary vectors.

Fact 2.2 *Let $\mathfrak{L}(A)$ be a Kac-Moody Lie algebra with Weyl group W. Proposition 2.13 insures that W is a crystallographic Coxeter group generated by the simple reflections. Then the root system Φ of W coincides with the real root system $\dot{\Phi}_{Re}$ of $\mathfrak{L}(A)$.* □

The rational root space

Let (W, S, Φ) be a crystallographic Coxeter group. Then we call $\Lambda_{\mathbb{Q}} := \mathbb{Q} \otimes_{\mathbb{Z}} \Lambda_{\mathbb{Z}}$ the *rational root space* of (W, S, Φ). It has the following properties.

Fact 2.3 *Let (W, S, Φ) be a crystallographic Coxeter group and $\Delta = \{\alpha_s \mid s \in S\}$ the basis of Φ, considered in the sense of Definition 2.12.*

1. *There exists $\lambda_s \in \Lambda_{\mathbb{Q}}$ for $s \in S$ such that*

$$B(\lambda_s, \alpha_{s'}) = \frac{1}{2} B(\alpha_s, \alpha_s)\delta_{s,s'}, \quad s, s' \in S, \tag{2.26}$$

where $\delta_{\cdot,\cdot}$ denotes Kronecker's function.

2. *$\alpha_s(\lambda_{s'}) = \lambda_{s'} - \delta_{s,s'}\alpha_s$ (by identifying α_s with s_α).*

Proof

1. We consider an enumeration over the elements of S: $s_1, s_2, \dots, s_n$. So using Grand-Schmidt process we can take

$$\lambda_{s_i} = \frac{1}{2}\alpha_{s_i} - \sum_{\substack{j=1 \\ j \neq i}}^{n} proj_{\lambda_{s_j}}(\alpha_{s_i}),$$

where $proj_{\lambda_{s_j}}(\alpha_{s_i})$ is the orthogonal projection of α_{s_i} over λ_{s_j}.

2. $\alpha_s(\lambda_{s'}) = \lambda_{s'} - 2\frac{B(\lambda_{s'},\alpha_s)}{B(\alpha_s,\alpha_s)}\alpha_s = \lambda_{s'} - 2\frac{\frac{1}{2}B(\alpha_s,\alpha_s)\delta_{s,s'}}{B(\alpha_s,\alpha_s)}\alpha_s = \lambda_{s'} - \delta_{s,s'}\alpha_s.$

□

The set $\Omega = \{\lambda_s \mid s \in S\}$ satisfying (2.26) will be called *dual basis* of Δ. In case that $B : \Lambda_{\mathbb{R}} \times \Lambda_{\mathbb{R}} \to \mathbb{R}$ is nondegenerate, such a basis is uniquely determined by Δ. For a dual basis Ω of Δ we put

$$\omega_0 = \sum_{s \in S} \lambda_s \tag{2.27}$$

and we call it *trace* of Ω.

The positive 1-cocycle

In this section we describe the function $c : W \to \Lambda^+_{\mathbb{Z}}$ showing an explicit formula with the following proposition.

Proposition 2.17 *Let (W, S, Φ) be a crystallographic Coxeter group and Ω be a dual basis of Δ with trace ω_0. Then for all $w \in W$ one has*

$$c(w) = (1 - w) \cdot \omega_0. \tag{2.28}$$

In particular, c is a 1-cocycle.

Proof
Let $h : W \to \Lambda_{\mathbb{Q}}$ be given by $h(w) = (1 - w) \cdot \omega_0$. h is a 1-cocycle, i.e., for all $w_1, w_2 \in W$ one has

$$h(w_1w_2) = (1-w_1w_2)\cdot\omega_0 = \omega_0 - w_1w_2\cdot\omega_0 = \omega_0 - w_1\cdot\omega_0 + w_1\cdot\omega_0 - w_1w_2\cdot\omega_0 = h(w_1) + w_1h(w_2).$$

Let $w \in W$ we prove by induction on $\ell(w)$ that $h(w) = c(w)$.
For $\ell(w) = 0$, one has $c(1) = 0 = (1 - 1) \cdot \omega_0 = h(1)$.
For $\ell(w) = 1$, one has $c(s_\alpha) = \alpha$ and $h(s_\alpha) = (1 - s_\alpha) \cdot \omega_0 = \sum_{s \in S} \omega_s - \sum_{s \in S} \alpha(\omega_s) = \sum_{s \in S} \omega_s - \sum_{s \in S} \omega_s + \alpha = \alpha$, since $\alpha = \alpha_s$. Therefore $c(s_\alpha) = h(s_\alpha)$.

Consider $w' \in W$ and assume that $c(w) = h(w)$ for all $w \in W$ with $\ell(w) \leq \ell(w')$. Let $w' = s_\alpha w$ of length $\ell(w') = \ell(w) + 1$, so

$$c(w') = c(s_\alpha w) = \alpha + s_\alpha c(w) = \alpha + s_\alpha h(w) = \alpha + s_\alpha((1-w)\cdot\omega_0) =$$

$$\alpha + \omega_0 - \alpha + (s_\alpha w)\cdot\omega_0 = (1 - s_\alpha w)\cdot\omega_0 = h(s_\alpha w).$$

Hence by induction we prove the equality $c = h$. □

We call the map $c : W \to \Lambda^+_{\mathbb{Z}}$ the *positive* 1*-cocycle* of (W, S, Φ).

Corollary 2.3 *Let $c : W \to \Lambda^+_{\mathbb{Z}}$ be the positive* 1*-cocycle of the crystallographic Coxeter group (W, S, Φ). Then c is injective.*

Proof
Let ω_0 be the trace of a dual basis Ω of Δ. Let $w_1, w_2 \in W$ such that $c(w_1) = c(w_2)$, then $(1-w_1)\cdot\omega_0 = (1-w_2)\cdot\omega_0 \Rightarrow w_1\cdot\omega_0 = w_2\cdot\omega_0 \Rightarrow w_1^{-1}w_2\cdot\omega_0 = \omega_0$. Hence $c(w_1^{-1}w_2) = (1 - w_1^{-1}w_2)\cdot\omega_0 = 0$, so $w_1^{-1}w_2 = 1 \Rightarrow w_1 = w_2$. □

To the property (c1)-(c8) we add the following one.

(c9) If $\rho := \frac{1}{2}\sum_{\alpha\in\Phi^+}\alpha$, then $w(\rho) = \rho - c(w)\ \ \forall w \in W$.

Proof We prove by induction on $\ell(w)$. Let $\ell(w) = 0$, then $w = 1$ and it is trivial. If $\ell(w) = 1$, then $w = s \in S$. Then $s(\rho) = -\frac{1}{2}\alpha_s + \frac{1}{2}\sum_{\alpha\in\Phi^+-\{\alpha_s\}}\alpha = \rho - c(s)$. We suppose that the thesis is satisfied for all $w' \in W$ with $\ell(w') < \ell(w)$. Let $s \in S$ such that $w = sw'$. Then $w(\rho) = sw'(\rho) = s(\rho - c(w')) = \rho - c(s) - s(c(w')) = \rho - c(sw')$ □

2.4.2 The characteristic power series of a crystallographic group

Let $\lambda = \sum_{s\in S} k_s\cdot\alpha_s \in \Lambda^+_{\mathbb{Z}}$, $k_s \in \mathbb{N}_0$, define

$$\underline{T}^\lambda = \prod_{s\in S} T_s^{k_s} \in \mathbb{Z}[\![T_s \mid s \in S]\!]. \tag{2.29}$$

Then

$$\chi_W(\underline{T}) = \chi_{(W,S,\Phi)}(\underline{T}) = \sum_{w\in W}(-1)^{\ell(w)}\underline{T}^{c(w)} \in \mathbb{Z}[\![T_s \mid s \in S]\!] \tag{2.30}$$

is a formal power series with coefficients in $\{-1, 0, 1\}$ and in $|S|$ independent variables. We call this series the *characteristic power series* of the crystallographic Coxeter group (W, S, Φ).
The left hand-side of Kac' denominator formula (2.20) relative to an arbitrary

Kac-Moody Lie algebra associated with a symmetrizable GCM with Weyl group W and root system Φ can be reinterpreted as the characteristic power series of a crystallographic group (W, S, Φ). Then one has the following equality:

$$\chi_W(\underline{T}) = \chi_{(W,S,\Phi)}(\underline{T}) = \sum_{w \in W} (-1)^{l(w)} \underline{T}^{c(w)} = \prod_{\alpha \in \Phi^+} (1 - \underline{T}^{\alpha})^{m_\alpha}. \tag{2.31}$$

Example 2.1 *1. Let A_1 be the Coxeter group with graph*

$$\bullet \alpha \tag{2.32}$$

Its characteristic power series is

$$\chi_{A_1}(X) = 1 - X. \tag{2.33}$$

2. *Let A_2 be the Coxeter group with graph*

$$\alpha\bullet \text{———} \bullet\beta \tag{2.34}$$

$\Phi^+ = \{\alpha, \beta, \alpha + \beta\}$ *is the positive root system of A_2. Its characteristic power series is*

$$\chi_{A_2}(X, Y) = (1 - X)(1 - Y)(1 - XY) = 1 - X - Y + X^2Y + XY^2 - X^2Y^2. \tag{2.35}$$

3. *Let A_3 be the Coxeter group with graph*

$$\alpha\bullet \text{———} \beta\bullet \text{———} \bullet\gamma \tag{2.36}$$

$\Phi^+ = \{\alpha, \beta, \gamma, \alpha + \beta, \beta + \gamma, \alpha + \beta + \gamma\}$ *is the positive root system of A_3. Its characteristic power series is*

$$\chi_{A_3}(X, Y, Z) = (1 - X)(1 - Y)(1 - Z)(1 - XY)(1 - YZ)(1 - XYZ) = \tag{2.37}$$

$$= 1 - X - Y + X^2Y + XY^2 - X^2Y^2 - Z +$$

$$+ XZ + Y^2Z - X^3Y^2Z - XY^3Z + X^3Y^3Z + YZ^2 - X^2YZ^2 +$$

$$-Y^2Z^2 + X^3Y^2Z^2 + X^2Y^4Z^2 - X^3Y^4Z^2 - XY^2Z^3 + X^2Y^2Z^3 + XY^3Z^3 - X^3Y^3Z^3 - X^2Y^4Z^3 + X^3Y^4Z^3.$$

□

2.4.3 The multiplicities

S. Berman and R. V. Moody inverted Kac′ denominator formula (2.20) and obtained an explicit formula to calculate the multiplicities of the weights (cf. [4]).
Let $\alpha \in \Lambda^+_{\mathbb{Z}}$,

$$m_\alpha = \sum_{\lambda|\alpha} \mu\Big(\frac{\alpha}{\lambda}\Big)\frac{\lambda}{\alpha} \sum_{(n)\in S(\lambda)} \Big(\prod \epsilon(c_i)^{n_i}\Big)\frac{\big((\sum n_i)-1\big)!}{\prod(n_i!)}. \tag{2.38}$$

where

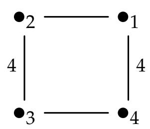
1. μ denotes the Möbius function.
2. For $\alpha, \lambda \in \Lambda^+_{\mathbb{Z}}$ we write $\lambda|\alpha$ if $\alpha = r\lambda$ for $r \in \mathbb{Z}_{\geq 0}$ and $\frac{\alpha}{\lambda} = r$.
3. Let $c_0, c_1, c_2, \dots$ be the set of elements $c(w)$, $w \in W$, written in order of increasing heights (cf. Definition 2.11)). Then $c_0 = c(1) = 0$ and $c_i = c(s_i)$ for $i = 1, \dots, r$.
4. Let $c(w)$, $\epsilon(c(w)) = (-1)^{\ell(w)+1}$.
5. $(n) = (n_1\ n_2\ n_3 \dots)$ is a sequence of nonnegative integers n_i.
6. Let $\lambda \in \Lambda^+_{\mathbb{Z}}$, then $S(\lambda) = \{(n) \mid \sum n_i c_i = \lambda\}$.

An example of calculation of multiplicities using formula (2.38) is given in Chapter 3, Example 3.5.

Chapter 3

A cocompact crystallografic hyperbolic Coxeter group

There is only one isomorphism class of cocompact crystallographic hyperbolic Coxeter systems $(\mathfrak{W}, \mathfrak{S})$, whose Coxeter graph is $\Gamma(\mathfrak{W})$:



(3.1)

In this chapter we spent much effort to represent $(\mathfrak{W}, \mathfrak{S})$ as a cocompact arithmetic lattice of $O^+_{\mathbb{R}}(3,1)$, the orthogonal group of matrices with real entries that stabilizes a bilinear form of signature $(3,1)$ and with positive entry in position $(1,1)$, generalising a result of J. Elstrodt, F. Grunewald, J. Mennicke (cf. [11]).
In fact the idea to establish an explicit description of the Coxeter system $(\mathfrak{W}, \mathfrak{S})$ in the Lie group $O^+_{\mathbb{R}}(3,1)$ starts by considering a homomorphism of Lie groups (cf. [11])

$$\Psi : SL(2,\mathbb{C}) \to SO^+_{\mathbb{R}}(3,1), \tag{3.2}$$

where $SO^+_{\mathbb{R}}(3,1)$ is the special orthogonal group of matrices with real entries that stabilizes a bilinear form of signature $(3,1)$ and with a positive entry in position $(1,1)$.
It can be extended to an *exceptional isomorphism*

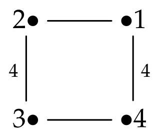
$$\tilde{\Psi} : SL(2,\mathbb{C}) \to O_{\mathbb{R}}(3,1) \tag{3.3}$$

through external involutions.
Moreover, generalising some results of A. Feingold, I. Frenkel (cf. [12]) and

using an important result of V. G. Kac (cf. [17]), we obtained also a description of the root system of a Kac-Moody Lie algebra with Weyl group $(\mathfrak{W}, \mathfrak{S})$.

3.1 The Coxeter group $(\mathfrak{W}, \mathfrak{S})$

Let $(\mathfrak{W}, \mathfrak{S})$ be the Coxeter group with Coxeter graph $\Gamma(\mathfrak{W})$



(3.4)

Then $\mathfrak{S} = \{s_1, s_2, s_3, s_4\}$ and

$$\mathfrak{W} = \langle s_1, s_2, s_3, s_4 \mid$$

$$(s_1s_2)^3 = (s_1s_3)^2 = (s_1s_4)^4 = (s_2s_3)^4 = (s_2s_4)^2 = (s_3s_4)^3 = s_i^2 = 1 \;\; i = 1, \dots, 4\rangle. \quad (3.5)$$

Let $\Lambda_{\mathbb{R}}$ be a 4-dimensional $\mathbb{R}$-vector space and $\{\alpha_i \in \Lambda_{\mathbb{R}} \mid s_i \in \mathfrak{S}\}$ be a fixed basis in one-to-one correspondence with $\mathfrak{S}$. Through the geometric representation (cf. Section 1.3) $\mathfrak{W}$ is represented as a subgroup of the orthogonal group $O(\Lambda_{\mathbb{R}}, B)$ respect to the bilinear form B defined in (1.7):

$$B := \begin{pmatrix} 1 & -\frac{1}{2} & 0 & -\frac{\sqrt{2}}{2} \\ -\frac{1}{2} & 1 & -\frac{\sqrt{2}}{2} & 0 \\ 0 & -\frac{\sqrt{2}}{2} & 1 & -\frac{1}{2} \\ -\frac{\sqrt{2}}{2} & 0 & -\frac{1}{2} & 1 \end{pmatrix}, \quad (3.6)$$

where $B_{ij} = B(\alpha_i, \alpha_j) := -\cos\left(\frac{\pi}{m_{ij}}\right)$.

Then $\mathfrak{W}$ is a reflection group generated by four simple reflections $s_i := s_{\alpha_i}$ $i = 1, \dots, 4$, where α_i are the simple roots.

In particular $\mathfrak{W}$ is a cocompact (from Theorem 1.4) crystallographic (from Proposition 1.9) hyperbolic (from Proposition 1.10) Coxeter group.

Being $\mathfrak{W}$ a crystallographic Coxeter group, then it is a Weyl group of a Kac-Moody Lie algebra (from Proposition 2.13) $\mathfrak{L}(A)$ where $A = (A_{ij})$ is the following GCM:

$$A = \begin{pmatrix} 2 & -1 & 0 & -1 \\ -1 & 2 & -1 & 0 \\ 0 & -2 & 2 & -1 \\ -2 & 0 & -1 & 2 \end{pmatrix}. \quad (3.7)$$

Moreover A is a symmetrizable GCM, in fact $A = DB'$, where

$$B' := \begin{pmatrix} 1 & -\frac{1}{2} & 0 & -\frac{1}{2} \\ -\frac{1}{2} & 1 & -\frac{1}{2} & 0 \\ 0 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{4} \\ -\frac{1}{2} & 0 & -\frac{1}{4} & \frac{1}{2} \end{pmatrix}. \tag{3.8}$$

is a symmetric matrix and $D = diag(d_i)$ is a diagonal matrix

$$D := \begin{pmatrix} 2 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 4 & 0 \\ 0 & 0 & 0 & 4 \end{pmatrix}. \tag{3.9}$$

Then (cf. Sections 2.2.4 and 2.2.6) one can define the standard bilinear form as $\langle \alpha_i, \alpha_j \rangle := \sqrt{d_i}\sqrt{d_j} B'_{ij}$ for $i, j = 1, \ldots, 4$:

1. $\langle \alpha_1, \alpha_1 \rangle = 2 = \langle \alpha_2, \alpha_2 \rangle$;
2. $\langle \alpha_3, \alpha_3 \rangle = 4 = \langle \alpha_4, \alpha_4 \rangle$;
3. $\langle \alpha_1, \alpha_2 \rangle = -1$;
4. $\langle \alpha_1, \alpha_3 \rangle = 0 = \langle \alpha_2, \alpha_4 \rangle$;
5. $\langle \alpha_1, \alpha_4 \rangle = -2 = \langle \alpha_2, \alpha_3 \rangle = \langle \alpha_3, \alpha_4 \rangle$.

The simple reflections of the Weyl group $\mathfrak{W}$, as subgroup of $O(\Lambda_{\mathbb{R}}, B)$, respect to the basis of the simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ with bilinear form $\langle \cdot, \cdot \rangle$ have the following representation:

$$s_1 = \begin{pmatrix} -1 & 1 & 0 & 2 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad s_2 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & -1 & 2 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \tag{3.10}$$

$$s_3 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 \end{pmatrix}, \quad s_4 = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 1 & -1 \end{pmatrix}. \tag{3.11}$$

3.1.1 3-dimensional Hyperbolic Space

An $\mathbb{R}$-vector space V of dimension $n+1$ endowed with a symmetric bilinear form $(\cdot, \cdot) : V \times V \to \mathbb{R}$ of signature $(n, 1)$ is called *Lorentzian space*. The quadratic hypersurface

$$\{\lambda \in V \mid (\lambda, \lambda) = -1\} \tag{3.12}$$

is a hyperboloid of two sheets, and one connected component $\mathbb{H}^n$ will be called *hyperbolic space* of dimension n. Since the tangent space of $\mathbb{H}^n$ at $\lambda \in \mathbb{H}^n$ is the orthogonal complement of λ in V, $\mathbb{H}^n$ inherits a natural Euclidean structure from $(\cdot, \cdot)$ by restriction.

Let consider in $\Lambda_{\mathbb{R}}$ (the 4-dimensional $\mathbb{R}$-vector space as defined in Section 3.1) four vectors e_0, e_1, e_2, e_3 such that

$$\langle e_i, e_j \rangle = 0 \ \forall i \neq j, \langle e_0, e_0 \rangle = -7, \ \langle e_i, e_i \rangle = 1 \ i = 1, 2, 3. \tag{3.13}$$

The simple roots $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ of the Coxeter group $\mathfrak{W}$ can be written as linear combination of e_0, e_1, e_2, e_3:

$$\alpha_1 = e_1 - e_2, \ \alpha_2 = e_2 + e_3, \ \alpha_3 = -2e_3, \ \alpha_4 = e_0 - 3e_1 - e_2 + e_3. \tag{3.14}$$

$\{e_0, e_1, e_2, e_3\}$ is also a basis for $\Lambda_{\mathbb{R}}$, then $\Lambda_{\mathbb{R}}$ is a Lorentzian space endowed with the symmetric bilinear form $\langle \cdot, \cdot \rangle$ that defines the following quadratic form

$$\mathfrak{Q}(x_0, x_1, x_2, x_3) = -7x_0^2 + x_1^2 + x_2^2 + x_3^2. \tag{3.15}$$

3.2 An *exceptional isomorphism*

In this section we shall describe the construction of an *exceptional isomorphism*

$$\tilde{\Psi} : SL(2, \mathbb{C}) \to O_{\mathbb{R}}(3, 1), \tag{3.16}$$

i.e. the construction of an isomorphism

$$\tilde{\Psi} : PSL(2, \mathbb{C}) \to PO^+_{\mathbb{R}}(3, 1), \tag{3.17}$$

We start off with certain facts about Clifford algebras. For more details we refer the reader to [11].

3.2.1 Clifford algebras

Let $\mathbb{K}$ be a field of characteristic different from 2 and V be an n-dimensional $\mathbb{K}$-vector space. Suppose $Q : V \to \mathbb{K}$ be a nondegenerate quadratic form associated with a symmetric bilinear form $\Phi_Q : V \times V \to \mathbb{K}$ such that

$$\Phi_Q = \frac{1}{2}(Q(x + y) - Q(x) - Q(y)) \tag{3.18}$$

$$Q(x) = \Phi_Q(x, x). \tag{3.19}$$

Denote by $T(V)$ the tensor algebra of V and by I the two sided ideal generated by the elements $x \otimes y + y \otimes x - 2\Phi_Q(x, y)$, where $x, y \in V$. The Clifford algebra of Q is defined as the quotient $C(Q) := T(V)/I$ (cf. [11]).
Let $e_1, \ldots, e_n$ be a basis of V, orthogonal with respect to Φ_Q. Then we have in $C(Q)$:

$$e_i^2 = Q(e_i), \;\; e_i e_j = -e_j e_i \;\; \forall i, j = 1, \ldots, n. \tag{3.20}$$

Let P_n be the set of subsets of $\{1, \ldots, n\}$. For $M \in P_n$, $M = \{i_1, \ldots, i_r\}$ with $i_1 < \ldots < i_r$, we define

$$e_M := e_{i_1} \ldots e_{i_r} \tag{3.21}$$

with the additional convention $e_\emptyset := 1$. The 2^n elements e_M form a basis for $C(Q)$. If $M = P_n$, we call e_M the *volume element* of $C(Q)$.
Denote by $\mathbb{R}^{p,q}$ the $n = p + m$-dimensional $\mathbb{R}$-vector space $\mathbb{R}^n$ with nondegenerate quadratic form Q of signature (p, q), where p is the number of positive autovalues and q of negative. In this case we write $C(p, q)$ for $C(Q)$ (cf. [13]).

Example 3.1 *(cf. [11]) For $\epsilon \in \mathbb{K}$ with $\epsilon \neq 0$ let $V_\epsilon = \mathbb{K} f_3$ be the 1-dimensional $\mathbb{K}$-vector space with basis f_3. On V_ϵ we fix the quadratic form Q_ϵ given by $Q_\epsilon(f_3) = -\epsilon$. Then the Clifford algebra $C(Q_\epsilon)$ is 2-dimensional and commutative. If $\mathbb{K} = \mathbb{R}$ and $\epsilon = 1$ then we say f_3 to be the imaginary number i. So we make the identification $C(Q_1) = \mathbb{C}$.* □

Example 3.2 *(cf. [11]) For $a, b \in \mathbb{K}$ with $a, b \neq 0$ let $H_{a,b} = \mathbb{K}\mathtt{i} \oplus \mathbb{K}\mathtt{j}$ be the 2-dimensional $\mathbb{K}$-vector space with basis $\mathtt{i}, \mathtt{j}$. On $H_{a,b}$ we fix the quadratic form $Q_{a,b}$, given by $Q_{a,b}(\alpha\mathtt{i} + \beta\mathtt{j}) = a\alpha^2 + b\beta^2$. The Clifford algebra $C(Q_{a,b})$ is 4-dimensional. $C(Q_{a,b})$ is the quaternion algebra $\mathcal{H}(a, b; \mathbb{K}) = \{\alpha + \beta\mathtt{i} + \gamma\mathtt{j} + \delta\mathtt{k} \mid \mathtt{i}^2 = a, \;\mathtt{j}^2 = b, \;\mathtt{k} = \mathtt{ij} = -\mathtt{ji}\}$.* □

3.2.2 The main involutions

Denote with $C(Q)^{op}$ the opposite algebra of $C(Q)$, i.e. an algebra with the same elements and addition operations, but with the multiplication performed in the reverse order. The Clifford algebra $C(Q)$ has a *main anti-involution*

$$^* : C(Q)^{op} \to C(Q), \tag{3.22}$$

$e_M^* := (-1)^{\frac{r(r-1)}{2}} e_M$ and a *main involution*

$$' : C(Q) \to C(Q), \tag{3.23}$$

$e'_M := (-1)^r e_M$, where r is the cardinality of M. The main anti-involution and the main involution commute, so we can also define an other anti-involution

$$^{-} : C(Q)^{op} \to C(Q), \tag{3.24}$$

that is $\overline{e_M} := e'^{*}_M = (-1)^{\frac{r(r+1)}{2}} e_M$.
We set

$$C^+(Q) := \{a \in C(Q) \mid a = a'\}, \;\; C^-(Q) := \{a \in C(Q) \mid a' = -a\}. \tag{3.25}$$

Then $C(Q) = C^+(Q) \oplus C^-(Q)$, and $C^+(Q)$ is a subalgebra of $C(Q)$, the *even Clifford algebra*, whose generators are the elements e_M where M has even cardinality.

Example 3.3 *(cf. [11]) Let $V_0 = \mathbb{K}f_0 \oplus \mathbb{K}f_1 \oplus \mathbb{K}f_2$ be a 3-dimensional $\mathbb{K}$-vector space with the quadratic form Q_0, given by $Q_0(y_0 f_0 + y_1 f_1 + y_2 f_2) = y_0^2 - y_1^2 - y_2^2$. The even Clifford algebra $C^+(Q_0)$ is the algebra of 2×2-matrices over $\mathbb{K}$, $M(2, \mathbb{K})$.*
In fact, it is easy to prove that the map $\psi : M(2, \mathbb{K}) \to C^+(Q_0)$, given by

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \to \frac{1}{2}(1 + f_0 f_1), \;\; \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \to \frac{1}{2}(f_0 f_2 - f_1 f_2),$$

$$\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \to \frac{1}{2}(f_0 f_2 + f_1 f_2), \;\; \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \to \frac{1}{2}(1 - f_0 f_1),$$

is an algebra isomorphism. Moreover, we have

$$\psi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)^* = \psi\left(\begin{pmatrix} \delta & -\beta \\ -\gamma & \alpha \end{pmatrix}\right)$$

□

Remark 3.1 *Let e_X be a fixed element in $C^-(Q)$. The conjugation action of e_X on $C^+(Q)$ allows to construct the following involution*

$$\xi_{e_X} : C^+(Q) \to C^+(Q)$$

$$\xi_{e_X}(e_M) := e_X^{-1} e_M e_X.$$

□

Definition 3.1 *(cf. [11]) For the $\mathbb{K}$-vector space V with quadratic form Q, we define the $\mathbb{K}$-vector space $\tilde{V} := V_0 \oplus V$ with quadratic form $\tilde{Q} := Q_0 \perp Q$.* □

$C(Q_0)$ can be identified with the subalgebra of $C(\tilde{Q})$ generated by f_0, f_1, f_2. The following propositions are the first step for the construction of the exceptional isomorphism (3.16).

Proposition 3.1 *(cf. [11]) The map* $\dot{}$ $: V \to C^+(\tilde{Q})$ *with* $\dot{x} := f_0 f_1 f_2 x$ *extends to an injective* $\mathbb{K}$*-algebra homomorphism* $\dot{}$ $: C(Q) \to C^+(\tilde{Q})$*. The map* $\dot{}$ *commutes with the anti-automorphism* **.*

Proof The existence of the extension of the map $\dot{}$ can be deducted from simple elementary considerations or from the universal property of Clifford algebras. The formula

$$\Big(\sum_{M \in P_n} \lambda_M e_M\Big)^{\cdot} = \sum_{M \in P_n} \lambda_M (f_0 f_1 f_2)^{\epsilon_M} e_M \tag{3.26}$$

where

$$\epsilon_M = \begin{cases} 0 & if\ \ the\ \ cardinality\ \ of\ \ M\ \ is\ \ even, \\ 1 & otherwise. \end{cases}$$

allows to prove the injectivity considering also that e_M, $M \in P_n$, form a basis of $C(Q)$ constructed from an orthogonal basis of the n dimensional vector space V. Moreover from formula (3.26) it is clear that the maps $\dot{}$ and * commute. □

Proposition 3.2 *(cf. [11]) The map* $\psi : M(2, C(Q)) \to C^+(\tilde{Q})$*,*

$$\psi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right) := \dot{\alpha}\frac{1}{2}(1+f_0f_1)+\dot{\beta}\frac{1}{2}(f_0f_2-f_1f_2)+\dot{\gamma}\frac{1}{2}(f_0f_2+f_1f_2)+\dot{\delta}\frac{1}{2}(1-f_0f_1) \tag{3.27}$$

is a $\mathbb{K}$*-algebra isomorphism and satisfies*

$$\psi\left(\begin{pmatrix} \alpha & \beta \\ \gamma & \delta \end{pmatrix}\right)^* = \psi\left(\begin{pmatrix} \delta^* & -\beta^* \\ -\gamma^* & \alpha^* \end{pmatrix}\right). \tag{3.28}$$

Proof Since the elements $\frac{1}{2}(1 + f_0 f_1)$, $\frac{1}{2}(f_0 f_2 - f_1 f_2)$, $\frac{1}{2}(f_0 f_2 + f_1 f_2)$, $\frac{1}{2}(1 - f_0 f_1)$ commute with $C(Q)$, it follows from Example 3.3 that ψ is a homomorphism. Moreover, from formula (3.26) ψ is injective. ψ is also surjective because $M(2, C(Q))$ and $C^+(\tilde{Q})$ have the same dimension. □

3.2.3 The *exceptional isomorphism*

We shall describe now the construction of the *exceptional isomorphism*

$$\Psi : SL(2, \mathbb{C}) \to O_4(\mathbb{R}, Q_1) = O_4(3, 1), \tag{3.29}$$

i.e. the construction of an isomorphism

$$\Psi : PSL(2, \mathbb{C}) \to PSO_4(\mathbb{R}, Q_1) = PSO_4(3, 1), \tag{3.30}$$

where $Q_1(x_0, x_1, x_2, x_3) := x_0^2 - x_1^2 - x_2^2 - x_3^2$ is a quadratic form.

Definition 3.2 *(cf. [11]) Let V be an n-dimensional $\mathbb{K}$-vector space with nondegenerate quadratic form Q. Then the spin-group of Q is defined as*

$$Spin_n(\mathbb{K}, Q) := \{s \in C^+(Q) \mid sVs^* \subseteq V,\ ss^* = 1\}.$$

□

Remind V_ϵ is the 1-dimensional vector space with basis f_3 endowed with the quadratic form $Q_\epsilon(f_3) = -\epsilon$ (cf. Example 3.1).

Proposition 3.3 *(cf. [11]) The $\mathbb{K}$-algebra isomorphism $\psi : M(2, C(Q_\epsilon)) \to C^+(\tilde{Q}_\epsilon)$ defined as in Proposition 3.2 restricts to a group isomorphism $\psi : SL(2, C(Q_\epsilon)) \to Spin_4(\tilde{Q}_\epsilon)$.*

Proof Let

$$A = \begin{pmatrix} a_0 + a_1 f_3 & b_0 + b_1 f_3 \\ c_0 + c_1 f_3 & d_0 + d_1 f_3 \end{pmatrix}$$

be the general element of $M(2, C(Q_\epsilon))$. Recalling the definition of ψ we find:

$$\begin{aligned} 2\psi(A) = a_0 + d_0 + (a_0 - d_0) f_0 f_1 + (b_0 + c_0) f_0 f_2 + (b_1 - c_1) f_0 f_3 \\ +(c_0 - b_0) f_1 f_2 - (b_1 + c_1) f_1 f_3 + (a_1 - d_1) f_2 f_3 + (a_1 + d_1) f_0 f_1 f_2 f_3. \end{aligned}$$

The statement follows then from Proposition 3.2 together with some elementary considerations. □

The spin-group of the quadratic form has a canonical homomorphism with the corresponding orthogonal group which is constructed as follows. Let V be an n-dimensional vector space with nondegenerate quadratic form Q. The space V is identified with a subspace of $C(Q)$ and one has that for $v \in V$: $Q(v) = vv^*$.
For $s \in Spin_n(\mathbb{K}, Q)$ then there is a linear map $\Lambda(s) \in GL(V)$ given by $\Lambda(s)(x) := sxs^*$. The computation

$$sxs^*(sxs^*)^* = sxs^*sx^*s^* = sxx^*s^* = xx^* \tag{3.31}$$

shows that $\Lambda(s) \in O_n(\mathbb{K}, Q)$. Therefore the map

$$\Lambda : Spin_n(\mathbb{K}, Q) \to O_n(\mathbb{K}, Q). \tag{3.32}$$

is a homomorphism.
We shall have a description of the image and cokernel of Λ. Let $\Omega_n(\mathbb{K}, Q)$ be the commutator subgroup of $O_n(\mathbb{K}, Q)$ and $\Gamma(Q)$ be the subgroup of $\mathbb{K}^*/\mathbb{K}^{*2}$ generated by $Q(x)Q(y)$ where $x, y \in V$ such that $Q(x) \neq 0 \neq Q(y)$. Every element

$g \in O_n(\mathbb{K}, Q)$ is the product of reflections s_{x_i} associated with anisotropic vectors x_i for $i = 1, \ldots, r$: $g = s_{x_1} \ldots s_{x_r}$. Associating with g the product $Q(x_1) \ldots Q(x_r)$ in $\mathbb{K}^*/\mathbb{K}^{*2}$, we get a well-defined homomorphism

$$\Sigma : SO_n(\mathbb{K}, Q) \to \mathbb{K}^*/\mathbb{K}^{*2} \tag{3.33}$$

which is called the *spinorial norm homomorphism*. The following proposition is contained in Dieudonné (1971), Artin (1957).

Proposition 3.4 *(cf. [11]) Let V be an n-dimensional $\mathbb{K}$-vector space with nondegenerate quadratic form Q. Then the following statements hold.*

1. *We have $\Lambda(Spin_n(\mathbb{K}, Q)) \subset SO_n(\mathbb{K}, Q)$ and $\Sigma(SO_n(\mathbb{K}, Q)) \subset \Gamma(Q)$, the resulting sequence $1 \to \{1, -1\} \to Spin_n(\mathbb{K}, Q) \to SO_n(\mathbb{K}, Q) \to \Gamma(Q) \to 1$ is exact.*

2. *We define $SO_n^+(\mathbb{K}, Q) := Im(\Lambda)$ and get $\Omega_n(\mathbb{K}, Q) \subseteq SO_n^+(\mathbb{K}, Q)$.*

3. *If $n \geq 3$ then $\Omega_n(\mathbb{K}, Q)$ is also the commutator subgroup of $SO_n(\mathbb{K}, Q)$.*

4. *Let $n \geq 3$ and V be a vector space that contains an isotropic vector x, i.e. $x \neq 0$ such that $Q(x) = 0$, then $\Omega_n = Ker(\Sigma) = Im(\Lambda)$.*

□

Chosen as quadratic form Q_ϵ then:

Definition 3.3 *(cf. [11]) Let V_ϵ be a 1-dimensional vector space with basis f_3 and quadratic form Q_ϵ, then*

$$\Psi : SL(2, C(Q_\epsilon)) \to O_4(\mathbb{K}, \tilde{Q}_\epsilon) \tag{3.34}$$

is defined as $\Psi := \Lambda \circ \psi$. □

ψ and Ψ are usually called *exceptional isomorphims*. From Proposition 3.4 it follows:

Proposition 3.5 *(cf. [11]) The map $\Psi : SL(2, C(Q_\epsilon)) \to O_4(\mathbb{K}, \tilde{Q}_\epsilon)$ has the property*

$$\Psi(SL(2, C(Q_\epsilon))) = SO_4^+(\mathbb{K}, \tilde{Q}_\epsilon)$$

and the resulting sequence

$$1 \to \{1, -1\} \to SL(2, C(Q_\epsilon)) \to SO_4(\mathbb{K}, \tilde{Q}_\epsilon) \to \mathbb{K}/\mathbb{K}^{*2} \to 1$$

is exact. □

We give now a concrete matrix-expression for our map.

Proposition 3.6 *(cf. [11]) Let V_ϵ be the* 1*-dimensional vector space with basis f_3 and quadratic form Q_ϵ, that is $Q_\epsilon(f_3) = -\epsilon$. For*

$$A = \begin{pmatrix} a_0 + a_1 f_3 & b_0 + b_1 f_3 \\ c_0 + c_1 f_3 & d_0 + d_1 f_3 \end{pmatrix} \in SL(2, C(Q_\epsilon))$$

define

$$N1 := a_0^2 + b_0^2 + c_0^2 + d_0^2 + \epsilon(a_1^2 + b_1^2 + c_1^2 + d_1^2),$$

$$N_2 := -a_0^2 + b_0^2 - c_0^2 + d_0^2 + \epsilon(-a_1^2 + b_1^2 - c_1^2 + d_1^2),$$

$$N_3 := -a_0^2 - b_0^2 + c_0^2 + d_0^2 + \epsilon(-a_1^2 - b_1^2 + c_1^2 + d_1^2),$$

$$N_4 := a_0^2 - b_0^2 - c_0^2 + d_0^2 + \epsilon(a_1^2 - b_1^2 - c_1^2 + d_1^2),$$

$$T_1 := -a_0c_0 - b_0d_0 + \epsilon(-a_1c_1 - b_1d_1),$$

$$T_2 := a_0c_0 - b_0d_0 + \epsilon(a_1c_1 - b_1d_1),$$

$$T_3 := -a_1c_0 + a_0c_1 - b_1d_0 + b_0d_1,$$

$$T_4 := a_1c_0 - a_0c_1 - b_1d_0 + b_0d_1.$$

Then

$$\Psi(A) = \begin{pmatrix} \frac{N_1}{2} & \frac{N_2}{2} & -a_0b_0 - c_0d_0 + \epsilon(-a_1b_1 - c_1d_1) & \epsilon(a_1b_0 - a_0b_1 + c_1d_0 - c_0d_1) \\ \frac{N_3}{2} & \frac{N_4}{2} & a_0b_0 - c_0d_0 + \epsilon(a_1b_1 - c_1d_1) & \epsilon(-a_1b_0 + a_0b_1 + c_1d_0 - c_0d_1) \\ T_1 & T_2 & a_0d_0 + b_0c_0 + \epsilon(a_1d_1 + b_1c_1) & \epsilon(-a_1d_0 + a_0d_1 + b_1c_0 - b_0c_1) \\ T_3 & T_4 & a_1d_0 - a_0d_1 + b_1c_0 - b_0c_1 & a_0d_0 - b_0c_0 + \epsilon(a_1d_1 - b_1c_1) \end{pmatrix}. \tag{3.35}$$

Proof These formulas follow by straightforward computations from the previous definitions. □

3.2.4 The orthogonal group $O_4(\mathbb{R}, \mathcal{Q}_1)$

Let $\mathbb{K} = \mathbb{R}$. We take the 1-dimensional $\mathbb{R}$-vector space $V_1 = \mathbb{R}f_3$ with quadratic form $Q_1(\lambda f_3) = -\lambda^2$. Example 3.1 introduced the identification $C(Q_1) = \mathbb{C}$. Let $\mathcal{Q}_1 := \tilde{Q}_1$ be a quadratic form on the 4-dimensional vector space $\tilde{V}_1$ defined in Definition 3.1. Then we have the following statements.

Proposition 3.7 *(cf. [11]) Let $SO_4^+(\mathbb{R}, \mathcal{Q}_1)$ be the image of the homomorphism Ψ : $SL(2, \mathbb{C}) \to SO_4(\mathbb{R}, \mathcal{Q}_1)$ as defined in Proposition 3.4. Then the following holds.*

(1) $SO_4^+(\mathbb{R}, \mathcal{Q}_1)$ has index 2 *in $SO_4(\mathbb{R}, \mathcal{Q}_1)$.*

(2) $SO_4^+(\mathbb{R}, Q_1)$ *is the connected component of the identity in* $SO_4(\mathbb{R}, Q_1)$.

(3) $SO_4^+(\mathbb{R}, Q_1)$ *consists of those elements in* $SO_4(\mathbb{R}, Q_1)$ *which have a positive entry in the left upper corner.*

(4) $SO_4^+(\mathbb{R}, Q_1)$ *is equal to the commutator subgroup of* $SO_4(\mathbb{R}, Q_1)$ *and also equal to the commutator subgroup of* $O_4(\mathbb{R}, Q_1)$.

(5) The quotient map $SO_4(\mathbb{R}, Q_1) \to PSO_4(\mathbb{R}, Q_1)$ *maps* $SO_4^+(\mathbb{R}, Q_1)$ *isomorphically onto* $PSO_4(\mathbb{R}, Q_1)$.

Proof The only statements which are not clear form Proposition 3.4 are (2) and (5). Statement (2) follows from the formulas in Proposition 3.6, while (5) is obvious. □

The following proposition characterizes a particular subset of $O_4(\mathbb{R}, Q_1)$.

Proposition 3.8 *(cf. [11]) Define* $O_4^+(\mathbb{R}, Q_1)$ *to be the set of elements in* $O_4(\mathbb{R}, Q_1)$ *which have a positive entry in the left upper corner. Then the following hold:*

(1) Let $x \in \tilde{V}_1$ *be an anisotropic vector and let* $\sigma_x \in O_4(\mathbb{R}, Q_1)$ *be the corresponding reflection. Then* $-\sigma_x \in O_4^+(\mathbb{R}, Q_1)$

(2) $O_4^+(\mathbb{R}, Q_1)$ *is a subgroup of index* 2 *in* $O_4(\mathbb{R}, Q_1)$.

(3) $O_4^+(\mathbb{R}, Q_1)$ *maps isomorphically onto* $PO_4(\mathbb{R}, Q_1)$.

Proof We infer (2) from the obvious (1) and Proposition 3.7. Statement (3) is clear. □

The previous propositions described the possibility to construct the *exceptional isomorphism*

$$\Psi : SL(2, \mathbb{C}) \to O_4(\mathbb{R}, Q_1) = O_4(3, 1), \tag{3.36}$$

i.e. an isomorphism

$$\Psi : PSL(2, \mathbb{C}) \to PSO_4(\mathbb{R}, Q_1) = PSO_4(3, 1). \tag{3.37}$$

3.2.5 The orthogonal group $O_4(\mathbb{R}, \mathfrak{Q})$

We now consider the orthogonal group $O_4(\mathbb{R}, \mathfrak{Q})$ respect to the quadratic form $\mathfrak{Q}(x_0, x_1, x_2, x_3) = 7x_0^2 - x_1^2 - x_2^2 - x_3^2$, just introduced in Section 3.1.1. We will see later that it will be useful to our aim: to represent $(\mathfrak{W}, \mathfrak{S})$ as a cocompact arithmetic lattice of $O_4^+(\mathbb{R}, \mathfrak{Q})$.

Starting by the exceptional isomorphism $\Psi : SL(2,\mathbb{C}) \to O_4(\mathbb{R}, Q_1)$ (3.36), we should construct

$$\Psi : SL(2,\mathbb{C}) \to O_4(\mathbb{R}, \mathfrak{Q}). \tag{3.38}$$

At the same way of the proof of Proposition 3.6, we fix a vector space $\tilde{V}_1$ with quadratic form Q_1 and basis $\mathcal{F} = \{f_0, f_1, f_2, f_3\}$. Now, let $\Lambda_{\mathbb{R}}$ be a vector space endowed with quadratic form $\mathfrak{Q}$ and basis $\mathcal{E} = \{e_0, e_1, e_2, e_3\}$ satisfying (3.13). Changing basis, from $\mathcal{F}$ to $\mathcal{E}$, we obtain the representation of $Im\Psi$ in $O_4(\mathbb{R}, \mathfrak{Q})$:

$$N_1 = a_0^2 + b_0^2 + c_0^2 + d_0^2 + a_1^2 + b_1^2 + c_1^2 + d_1^2,$$

$$N_2 = -a_0^2 + b_0^2 - c_0^2 + d_0^2 + -a_1^2 + b_1^2 - c_1^2 + d_1^2,$$

$$N_3 = -a_0^2 - b_0^2 + c_0^2 + d_0^2 - a_1^2 - b_1^2 + c_1^2 + d_1^2,$$

$$N_4 = a_0^2 - b_0^2 - c_0^2 + d_0^2 + a_1^2 - b_1^2 - c_1^2 + d_1^2,$$

$$T_1 = -a_0c_0 - b_0d_0 + -a_1c_1 - b_1d_1$$

$$T_2 = a_0c_0 - b_0d_0 + a_1c_1 - b_1d_1,$$

$$T_3 = -a_1c_0 + a_0c_1 - b_1d_0 + b_0d_1,$$

$$T_4 = a_1c_0 - a_0c_1 - b_1d_0 + b_0d_1,$$

$$\begin{pmatrix} \frac{N_1}{2} & \frac{N_2}{2\sqrt{7}} & \frac{-a_0b_0-c_0d_0-a_1b_1-c_1d_1}{\sqrt{7}} & \frac{a_1b_0-a_0b_1+c_1d_0-c_0d_1}{\sqrt{7}} \\ \frac{\sqrt{7}N_3}{2} & \frac{N_4}{2} & a_0b_0 - c_0d_0 + a_1b_1 - c_1d_1 & -a_1b_0 + a_0b_1 + c_1d_0 - c_0d_1 \\ \sqrt{7}T_1 & T_2 & a_0d_0 + b_0c_0 + a_1d_1 + b_1c_1 & -a_1d_0 + a_0d_1 + b_1c_0 - b_0c_1 \\ \sqrt{7}T_3 & T_4 & a_1d_0 - a_0d_1 + b_1c_0 - b_0c_1 & a_0d_0 - b_0c_0 + a_1d_1 - b_1c_1 \end{pmatrix}. \tag{3.39}$$

The obtained *exceptional isomorphism* $\Psi : SL(2,\mathbb{C}) \to O_4(\mathbb{R}, \mathfrak{Q})$ has the following property:

$$\Psi(SL(2,\mathbb{C})) = SO_4^+(\mathbb{R}, \mathfrak{Q}). \tag{3.40}$$

We should extend this map to an isomorphism.
On $SL(2,\mathbb{C})$ it can be defined in a natural way an involution σ such that, let $A = \begin{pmatrix} a_0 + ia_1 & b_0 + ib_1 \\ c_0 + ic_1 & d_0 + id_1 \end{pmatrix}$ be an element of $SL(2,\mathbb{C})$,

$$\sigma A \sigma = \overline{A} := \begin{pmatrix} a_0 - ia_1 & b_0 - ib_1 \\ c_0 - ic_1 & d_0 - id_1 \end{pmatrix}. \tag{3.41}$$

We have that

$$\Psi(\overline{A}) = \begin{pmatrix} \frac{N_1}{2} & \frac{N_2}{2\sqrt{7}} & \frac{-a_0b_0-c_0d_0-a_1b_1-c_1d_1}{\sqrt{7}} & -\frac{a_1b_0-a_0b_1+c_1d_0-c_0d_1}{\sqrt{7}} \\ \frac{\sqrt{7}N_3}{2} & \frac{N_4}{2} & a_0b_0-c_0d_0+a_1b_1-c_1d_1 & -(-a_1b_0+a_0b_1+c_1d_0-c_0d_1) \\ \sqrt{7}T_1 & T_2 & a_0d_0+b_0c_0+a_1d_1+b_1c_1 & -(-a_1d_0+a_0d_1+b_1c_0-b_0c_1) \\ -\sqrt{7}T_3 & -T_4 & -(a_1d_0-a_0d_1+b_1c_0-b_0c_1) & a_0d_0-b_0c_0+a_1d_1-b_1c_1 \end{pmatrix}. \tag{3.42}$$

$$= \tau\Psi(A)\tau$$

where $\tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$.

The image of σ through Ψ is the reflection τ, i.e. an involution of trace 2. Moreover let $I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ be the identity matrix of $SL(2,\mathbb{C})$, then $\Psi(I) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} =$ $\Psi(-I)$.

In this way we extend

$$\Psi : SL(2,\mathbb{C}) \to SO_4^+(\mathbb{R},\mathfrak{Q}) \tag{3.43}$$

to an isomorphism

$$\tilde{\Psi} : \langle\sigma\rangle \ltimes PSL(2,\mathbb{C}) \to O_4^+(\mathbb{R},\mathfrak{Q}), \tag{3.44}$$

where $\ltimes$ is the semidirect product, that can also be viewed as an isomorphism

$$\tilde{\Psi} : \langle\sigma\rangle \ltimes SL(2,\mathbb{C}) \to O_4^+(\mathbb{R},\mathfrak{Q}), \ \ (mod\ \ Ker\tilde{\Psi}) \tag{3.45}$$

or

$$\tilde{\Psi} : \langle\sigma\rangle \ltimes SL(2,\mathbb{C}) \to O_4(\mathbb{R},\mathfrak{Q}), \ \ (mod\ \ Ker\tilde{\Psi}), \tag{3.46}$$

by Proposition 3.8.

3.3 The quadratic form $\mathfrak{Q}$

In Section 3.1.1 we observed that the 4-dimensional $\mathbb{R}$-vector space $\Lambda_{\mathbb{R}}$ is a Lorentzian space endowed with the symmetric bilinear form $\langle\cdot,\cdot\rangle$ that defines the following quadratic form

$$\mathfrak{Q}(x_0, x_1, x_2, x_3) = -7x_0^2 + x_1^2 + x_2^2 + x_3^2. \tag{3.47}$$

Then in this section we reassume some important facts about the theory of quadratic forms, because we will observe in the next section that their properties give us some clues to recognize groups commensurable to cocompact subgroups of the orthogonal groups. For more details we refer the reader to [24].
It is interesting to know that a good part of the theory about the quadratic forms has a strong connection with the theory about the Brauer groups.

Definition 3.4 *(cf. [25]) Let $\mathbb{K}$ be a field and let $\Omega(\mathbb{K})$ be the class of all finite dimensional, simple and central $\mathbb{K}$-algebras. Two algebras $\mathcal{A}$ and $\mathcal{B}$ are equivalent if there is a division algebra $\mathcal{D} \in \Omega(\mathbb{K})$ and a positive integers m and n such that $\mathcal{A} \simeq M(n, \mathcal{D})$ and $\mathcal{B} \simeq M(m, \mathcal{D})$.*
We write $\mathcal{A} \sim \mathcal{B}$. The equivalence class of $\mathcal{A}$ in $\Omega(\mathbb{K})$ will be denoted by $[\mathcal{A}]$.

Theorem 3.1 *(**Brauer's theorems**, cf. [25]) For a field $\mathbb{K}$, the set $\mathbf{B}(\mathbb{K}) = \{[\mathcal{A}] \mid \mathcal{A} \in \Omega(\mathbb{K})\}$ is an abelian group with the product $[\mathcal{A}][\mathcal{B}] = [\mathcal{A} \otimes \mathcal{B}]$, the unity element $[\mathbb{K}]$, and the inverse operation $[\mathcal{A}]^{-1} = [\mathcal{A}^*]$.* □

Every class in $\mathbf{B}(\mathbb{K})$ is represented by a division algebra that is unique up to isomorphism, this remark comes from Definition 3.4.

For every prime p of $\mathbb{K}$, let $\mathbb{K}_p$ denote the *p-adic completion* of $\mathbb{K}$. For an algebra $\mathcal{A}$ over $\mathbb{K}$ we put $\mathcal{A}_p = \mathcal{A} \otimes_{\mathbb{K}} \mathbb{K}_p$, the *completion* (also called *localization*) of $\mathcal{A}$ at p.
Splitting of a $\mathbb{K}$-algebra $\mathcal{A}$ means that $\mathcal{A}$ is a full matrix algebra over $\mathbb{K}$.

Theorem 3.2 *(**Local Global Principle**, cf. [28]) Let $\mathbb{K}$ be an algebraic number field and $\mathcal{A}$ be a central simple $\mathbb{K}$-algebra. If the $\mathbb{K}_p$-algebra $\mathcal{A}_p$ splits for every p then $\mathcal{A}$ spits.* □

An important application of the Local Global Principle (Theorem 3.2) is to determine the structure of the Brauer group $\mathbf{B}(\mathbb{K})$ over a number field $\mathbb{K}$. Let $\mathbb{K}$ be a number field and p be a prime of $\mathbb{K}$. If we associate with every central simple algebra $\mathcal{A}$ over $\mathbb{K}$ its completion $\mathcal{A}_p$, then we obtain the p-adic localization map of Brauer group $\mathbf{B}(\mathbb{K}) \to \mathbf{B}(\mathbb{K}_p)$. Combining the maps for all primes p of $\mathbb{K}$ we obtain the *universal localization map*

$$\mathbf{B}(\mathbb{K}) \to \bigoplus \sum_p \mathbf{B}(\mathbb{K}_p). \tag{3.48}$$

The Local-Global Principle (Theorem 3.2) can be interpreted to say that this localization is injective. So $\mathbf{B}(\mathbb{K})$ can be viewed as a subgroup of the direct sum of the local groups $\mathbf{B}(\mathbb{K}_p)$.

From the theory about Brauer groups it follows this consideration about the quadratic forms. Let Q be a quadratic form, the question Q has a non-trivial zero in a field is, of course, identical with the question whether the field is a splitting field for an algebra.

3.3.1 Quadratic forms over $\mathbb{Q}_p$

Let V be a $\mathbb{K}$-vector space and Q be a quadratic form on V, we called (V, Q) *quadratic module*. Let $x, y \in V$, the bilinear form associated with Q is

$$(x, y) = \frac{1}{2}(Q(x + y) - Q(x) - Q(y)). \tag{3.49}$$

(V, Q) is said *degenerate* if there exists $0 \neq x \in V$ such that $x.y = 0\ \forall y \in V$, otherwise (V, Q) is *nondegenerate*.
An element $0 \neq x \in V$ is said to be *isotropic* if $Q(x) = 0$.
Q *represents* an element $\alpha \in \mathbb{K}$ if there exists $0 \neq v \in V$ such that $Q(x) = \alpha$. In particular Q represents 0 if and only if the corresponding quadratic module contains a non-zero isotropic element. When Q represents 0, we say that Q is *anisotropic* over $\mathbb{K}$.

From the theory about the Brouer groups it follows this result.

Theorem 3.3 *(**Hasse-Mikowski Theorem**, cf. [10]) Let Q be a quadratic form in n variables with coefficients in $\mathbb{Q}$. Then Q represents 0 in $\mathbb{Q}$ if and only if it represents 0 in every $\mathbb{Q}_p$ for all p prime.* □

Let $n := rankQ$ be the rank of Q. If $\{e_1, \ldots, e_n\}$ is an orthogonal basis of V and put $a_i = (e_i, e_i)$, then the discriminant of Q is $d := a_1 \ldots a_n \in \mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$ and $\epsilon := \prod_{i<j} \mathbf{(}a_i, a_j\mathbf{)}$, where $\mathbf{(}a, b\mathbf{)}$ is the *Hilbert symbol* for $a, b \in \mathbb{Q}_p^*$.
The Hilbert symbol $\mathbf{(}a, b\mathbf{)}$ is given by

$$\mathbf{(}a, b\mathbf{)} := \begin{cases} 1 \ \textit{if there exists a non-trivial solution of } z^2 - ax^2 - by^2 = 0, \\ -1 \ \textit{otherwise.} \end{cases} \tag{3.50}$$

Theorem 3.4 *(cf. [10]) Let Q be a nondegenerate quadratic form in n variables with coefficients in $\mathbb{Q}$. Then Q represents 0 in $\mathbb{Q}_p$ if and only if one of the following holds:*

1. *$n = 2$ and $d = -1$ (in $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$);*

2. *$n = 3$ and $(-1, -d) = \epsilon$;*

3. *$n = 4$ and either $d \neq 1$ or $d = 1$ and $\epsilon = (-1, -d)$;*

4. *$n \geq 5$.*

□

Theorem 3.5 *(cf. [10]) Let consider the equation $aX^2 + bY^2 + cZ^2 = 0$ with $0 \neq a, b, c \in \mathbb{Z}$.*

1. *If p is an odd prime that does not divide abc, then the equation has a non-trivial solution in $\mathbb{Q}_p$.*

2. *If $p = 2$ and a, b and c are all odd, and the sum of two of them is divisible by 4, then the equation has a non-trivial solution in $\mathbb{Q}_2$.*

3. *If $p = 2$ and one of a, b and c is even, and the sum of two coefficients or the sum of all three coefficients will be divisible by 8, then the equation has a non-trivial solution in $\mathbb{Q}_2$.*

4. *If p is an odd prime and a is divisible by p and there exists an integer $r \in \mathbb{Z}$ such that $b + r^2c \equiv 0(mod\ p)$, then the equation has a non-trivial solution in $\mathbb{Q}_p$. ($-b/c$ is a quadratic residue modulo p, means that is congruent to a square mod p.)*

□

The following lemma allows us to recognize the squares in $\mathbb{Q}_p$.

Lemma 3.1 *(**Hensel's Lemma**, cf. [10]) Let $p \neq 2$ be a prime. An element $x \in \mathbb{Q}_p$ is a square if and only if it can be written $x = p^{2n}y^2$ with $n \in \mathbb{Z}$ and $y \in \mathbb{Z}_p^*$ a p-adic unit. The quotient group $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$ has order four. If $c \in Z_p^*$ is any element whose reduction modulo p is not a quadratic residue, then the set $\{1, p, c, cp\}$ is a complete set of coset representatives.*
If $p = 2$. An element $x \in \mathbb{Q}_2$ is a square if and only if it can be written $x = 2^{2n}y^2$ with $n \in \mathbb{Z}$ and y is a 2-adic unit such that $y \equiv 1(mod 8\mathbb{Z}_2)$. The quotient group $\mathbb{Q}_p^*/(\mathbb{Q}_p^*)^2$ has order eight and the set $\{\pm 1, \pm 2, \pm 5, \pm 10\}$ is a complete set of coset representatives.
□

3.3.2 $\mathfrak{Q}(x_0, x_1, x_2, x_3) = -7x_0^2 + x_1^2 + x_2^2 + x_3^2$

Let consider the quadratic form $\mathfrak{Q}(x_0, x_1, x_2, x_3) = -7x_0^2 + x_1^2 + x_2^2 + x_3^2$ introduced in Section 3.1.1. The question is if $\mathfrak{Q}$ represents 0 in $\mathbb{Q}$.
We are interesting to this property for $\mathfrak{Q}$ because in the next section, starting by

$\mathfrak{Q}$, we will construct a division algebra. By the theory about Brauer group, we know that let Q be a quadratic form, the question Q has a non-trivial zero in a field is, of course, identical with the question whether the field is a splitting field for an algebra.
Hasse-Mikowski Theorem (Theorem 3.3) allows to observe that $\mathfrak{Q}$ does not represent 0 in $\mathbb{Q}$ because for $p = 2$ it does not represent 0 in $\mathbb{Q}_2$.
In fact, referring to Theorems 3.4, we observe that the discriminant $d = -7$ is equal to 1 in $\mathbb{Q}_2^*/(\mathbb{Q}_2^*)^2$ (cf. Lemma 3.1) and $\epsilon = (-7,1)^3(1,1)^3 = 1 \neq (-1,7) = -1$ (cf. Theorem 3.5).

3.4 A representation of $\mathfrak{W}$

In this section we establish an explicit description of the Coxeter system $(\mathfrak{W}, \mathfrak{S})$ in the Lie group $O^+_{\mathbb{R}}(3,1)$ using the exceptional isomorphism (3.44) constructed in Section 3.2.5. Moreover we exhibit the generators of $\mathfrak{W}$ in $\langle\sigma\rangle \ltimes PSL(2,\mathbb{C})$ through an explicit formula and, generalising some results of A. Feingold, I. Frenkel (cf. [12]), we show a description of the root system of the Kac-Moody Lie algebra with Weyl group $(\mathfrak{W}, \mathfrak{S})$.

3.4.1 The quaternion algebra

We start off with certain facts about the quaternion algebras. Let $\mathbb{K}$ be a field of characteristic $\neq 2$.

Definition 3.5 *(cf. [25]) Let a and b be non-zero elements of $\mathbb{K}$. Let $\mathcal{A}$ be a 4-dimensional $\mathbb{K}$-vector space with basis $\mathtt{1}, \mathtt{i}, \mathtt{j}, \mathtt{k}$ and bilinear multiplication defined by the conditions that $\mathtt{1}$ is a unity element, and*

$$\mathtt{i}^2 = a, \;\; \mathtt{j}^2 = b, \;\; \mathtt{ij} = -\mathtt{ji} = \mathtt{k}. \tag{3.51}$$

$\mathcal{A} := \left(\frac{a,b}{\mathbb{K}}\right)$ is an associative $\mathbb{K}$-algebra called quaternion algebra. □

Example 3.4 $\mathbb{H}_{\mathbb{R}} := \left(\frac{-1,-1}{\mathbb{R}}\right)$ *is the classical Hamiltonian quaternion algebra.*

Lemma 3.2 *(cf. [25]) For any non-zero $a, b \in \mathbb{K}$, $\mathcal{A} = \left(\frac{a,b}{\mathbb{K}}\right)$ is a central simple $\mathbb{K}$-algebra.*

Proof $\mathcal{A}$ is a central algebra if $\mathbf{Z}(\mathcal{A}) = \mathbb{K}$. It is obvious that $\mathbf{Z}(\mathcal{A}) \supseteq \mathbb{K}$. Vice versa let $x \in \mathbf{Z}(\mathcal{A})$, $x = x_0 + x_1\mathtt{i} + x_2\mathtt{j} + x_3\mathtt{k}$. $x\mathtt{i} = \mathtt{i}x$ if and only if $x_2 = 0 = x_3$, then

$x = x_0 + x_1\mathtt{i}$. And $x\mathtt{j} = \mathtt{j}x$ if and only if $x_1 = 0$, then $x = x_0 \in \mathbb{K}$.
Let $\mathcal{I}$ a non-zero ideal of $\mathcal{A}$ and $x = x_0 + x_1\mathtt{i} + x_2\mathtt{j} + x_3\mathtt{k} \in \mathcal{I}$. Now,

$$\mathtt{j}(\mathtt{i}x - x\mathtt{i}) - (\mathtt{i}x - x\mathtt{i})\mathtt{j} = -4bx_2\mathtt{i} \in \mathcal{I},$$

$$\mathtt{k}(\mathtt{j}x - x\mathtt{j}) - (\mathtt{j}x - x\mathtt{j})\mathtt{k} = 4abx_3\mathtt{j} \in \mathcal{I},$$

$$\mathtt{i}(\mathtt{k}x - x\mathtt{k}) - (\mathtt{k}x - x\mathtt{k})\mathtt{i} = -4ax_1\mathtt{k} \in \mathcal{I}.$$

If one of x_1, x_2 or x_3 is not 0, then $\mathcal{I}$ contains a unit of $\mathcal{A}$; instead if $x_1 = x_2 = x_3 = 0$, then $0 \neq x = x_0$ is a unit belonging to $\mathcal{I}$. In all cases, $\mathcal{I} = \mathcal{A}$. □

Thus, the quaternion algebras are associative, central and simple.
In $\mathcal{A} = \left(\frac{a,b}{\mathbb{K}}\right)$ the map

$$\bar{\ }: \mathcal{A} \to \mathcal{A},\ \overline{x_0 + x_1\mathtt{i} + x_2\mathtt{j} + x_3\mathtt{k}} = x_0 - x_1\mathtt{i} - x_2\mathtt{j} - x_3\mathtt{k} \tag{3.52}$$

is called the *conjugation map*.
Moreover every element of $\mathcal{A}$ is equipped by the *norm*

$$N : \mathcal{A} \to \mathbb{K},\ N(x) = x\bar{x} \tag{3.53}$$

and by the *trace*

$$Tr : \mathcal{A} \to \mathbb{K},\ Tr(x) = x + \bar{x}. \tag{3.54}$$

These maps satisfy

$$\overline{xy} = \overline{xy},\ N(xy) = N(x)N(y) \tag{3.55}$$

for all $x, y \in \mathcal{A}$. We also have the formula

$$N(x_0 + x_1\mathtt{i} + x_2\mathtt{j} + x_3\mathtt{k}) = x_0^2 - ax_1^2 - bx_2^2 + abx_3^2. \tag{3.56}$$

Theorem 3.6 *(cf. [25]) If $\mathcal{A}$ is a quaternion algebra over $\mathbb{K}$, then $\mathcal{A}$ is either a division algebra or $\mathcal{A}$ is isomorphic to $M_2(\mathbb{K})$.*

Proof By Weddeburn's Structure Theorem for finite-dimensional simple algebras, $\mathcal{A}$ is isomorphic to a full matrix algebra $M_n(\mathcal{D})$, where $\mathcal{D}$ is a division algebra, with n and $\mathcal{D}$ uniquely determined by $\mathcal{A}$. The $\mathbb{K}$-dimension $M_n(\mathcal{D})$ is mn^2, where $m = dim_{\mathbb{K}}\mathcal{D}$. Then for the 4-dimensional quaternion algebras there are only two possibilities: $m = 4, n = 1$ or $m = 1, n = 2$. □

Definition 3.6 *(cf. [25]) If the quaternion algebra $\mathcal{A}$ over $\mathbb{K}$ is such that $\mathcal{A} \simeq M_2(\mathbb{K})$, we say that $\mathcal{A}$ splits or unramifies in $\mathbb{K}$, otherwise it does not spits or ramifies in $\mathbb{K}$.*

Let v a place of $\mathbb{K}$ and $\mathbb{K}_v$ the completion of $\mathbb{K}$ respect to v, if the quaternion $\mathbb{K}$-algebra $\mathcal{A}$ ramifies in $\mathbb{K}_v$, we say that $\mathcal{A}$ ramifies at the place v. □

The following map is the connection between a quaternion algebra and the algebra of matrix of order 2 (cf. [11]). Let $\mathbb{L}$ be a field extension of $\mathbb{K}$ with $\sqrt{a}$, $\sqrt{b} \in \mathbb{L}$ then the map

$$\phi : \mathcal{A} = \left(\frac{a,b}{\mathbb{K}}\right) \to M(2,\mathbb{L})$$

$$x_0 + x_1\mathtt{i} + x_2\mathtt{j} + x_3\mathtt{k} \to \begin{pmatrix} x_0 + x_1\sqrt{a} & x_2\sqrt{b} + x_3\sqrt{ab} \\ x_2\sqrt{b} - x_3\sqrt{ab} & x_0 - x_1\sqrt{a} \end{pmatrix} \tag{3.57}$$

is an injective $\mathbb{K}$-algebra homomorphism satisfying $N(x) = Det(\phi(x))$.

Definition 3.7 *(cf. [11]) Let $\mathcal{A}_0$ be the subspace of $\mathcal{A}$ spanned by the vectors $\mathtt{i}, \mathtt{j}, \mathtt{k}$. The elements of $\mathcal{A}_0$ are called pure quaternions in $\mathcal{A}$.* □

Let $O_{\mathbb{K}}$ be the ring of integers of $\mathbb{K}$ and O a subring of finite index in $O_{\mathbb{K}}$. (We always assume that all rings contain an identity element 1). Such subrings are usually called *orders* of $\mathbb{K}$. Let $\mathcal{A}$ be a quaternion algebra over $\mathbb{K}$. An *O-order* $\mathcal{R}$ in $\mathcal{A}$ is a subring which contains a $\mathbb{K}$-basis of $\mathcal{A}$. It is also an O-submodule of $\mathcal{A}$ and is finitely generated as O-module. An order ia an $O_{\mathbb{K}}$-order of $\mathcal{A}$. An order is called *maximal* if it is not properly contained in another order. Two orders $\mathcal{R}_1, \mathcal{R}_2$ of $\mathcal{A}$ are *equivalent* if and only if there is an invertible element $x \in \mathcal{A}$ with $\mathcal{R}_2 = x\mathcal{R}_1x^{-1}$.

Definition 3.8 *(cf. [11]) Let $\mathcal{A}$ be a quaternion algebra over $\mathbb{K}$ and $\mathcal{R}$ a subring of $\mathcal{A}$ closed under conjugation. We put*

$$\mathcal{R}^1 := \{x \in \mathcal{R} \mid N(x) = 1\} \tag{3.58}$$

and call it the norm 1 *group of $\mathcal{R}$.* □

If $\mathcal{R}$ is an order then $O_{\mathbb{K}} \subseteq \mathcal{R}$ and if $x \in \mathcal{R}$ then $x + \overline{x} \in O_{\mathbb{K}}$, hence every order is closed under conjugation and the above definition applies.

It is easy to prove that, let $\mathbb{K}$ be an algebraic number field,

$$\phi : \left(\frac{a,b}{\mathbb{K}}\right)^1 \to SL(2,\mathbb{C}) \tag{3.59}$$

$$x_0 + x_1\mathtt{i} + x_2\mathtt{j} + x_3\mathtt{k} \to \begin{pmatrix} x_0 + x_1\sqrt{a} & x_2\sqrt{b} + x_3\sqrt{ab} \\ x_2\sqrt{b} - x_3\sqrt{ab} & x_0 - x_1\sqrt{a} \end{pmatrix}$$

is an injective group homomorphism (cf. [11]).

Remark 3.2 *(cf. [25]) If $\mathbb{F}$ is a field extending $\mathbb{K}$, then*

$$\left(\frac{a,b}{\mathbb{K}}\right)\otimes_{\mathbb{K}}\mathbb{F}\simeq\left(\frac{a,b}{\mathbb{F}}\right).$$

□

There is a connection between the quaternion algebras and the even Clifford algebras (cf. [11]). Let $\mathbb{K}_0$ be a field of characteristic $\neq 2$ and $a, b, c \in \mathbb{K}_0$. We assume that

$$d := abc \notin \mathbb{K}_0^2 \tag{3.60}$$

Let V be a 4-dimensional $\mathbb{K}_0$-vector space with basis e_0, e_1, e_2, e_3 and quadratic form

$$Q_{a,b,c}(x_0e_0 + x_1e_1 + x_2e_2 + e_3e_3) = x_0^2 + ax_1^2 + bx_2^2 + cx_3^2. \tag{3.61}$$

$C(\mathbb{K}_0, Q_{a,b,c})$ is the Clifford algebra of $Q_{a,b,c}$ over $\mathbb{K}_0$. Since we vary the field coefficients in this section we include them in the symbol for the Clifford algebra, changing the notation of Section 3.2. The algebra of even elements $C^+(\mathbb{K}_0, Q_{a,b,c})$ is an 8-dimensional $\mathbb{K}_0$-subalgebra of $C(\mathbb{K}_0, Q_{a,b,c})$. We put

$$f := e_0e_1e_2e_3,\ \ \mathbb{K} := \mathbb{K}_0(f). \tag{3.62}$$

The element f satisfies $f^2 = abc$ and $\mathbb{K}$ is equal to the center of $C^+(\mathbb{K}_0, Q_{a,b,c})$. By the assumption (3.60) the field $\mathbb{K}$ is a quadratic extension of $\mathbb{K}_0$ and the $\mathbb{K}$-algebra $C^+(\mathbb{K}_0, Q_{a,b,c})$ is 4-dimensional with basis

$$\mathtt{1},\ \ \mathtt{i} = e_0e_1,\ \ \mathtt{j} = e_0e_2,\ \ \mathtt{k} = e_2e_1. \tag{3.63}$$

These elements satisfy the usual relations (3.51) of the standard basis of a quaternonian $\mathbb{K}$-algebra, i.e.

$$\mathtt{i}^2 = -a,\ \ \mathtt{j}^2 = -b,\ \ \mathtt{ij} = -\mathtt{ji} = \mathtt{k}. \tag{3.64}$$

We get an obvious $\mathbb{K}$-algebra isomorphism (cf. [11])

$$\Theta : \left(\frac{-a,-b}{\mathbb{K}}\right) \to C^+(\mathbb{K}_0, Q_{a,b,c}). \tag{3.65}$$

Proposition 3.9 *(cf. [25]) Let $\mathcal{A}$ be a quaternion algebra and x a non-zero element of $\mathcal{A}$. x is invertible if and only if $N(x) \neq 0$.*

Proof Let x be a non-zero invertible element of $\mathcal{A}$, since $N(x)N(x^{-1}) = N(xx^{-1}) = N(1) = 1$ then $N(x) \neq 0$. Let x be a non-zero element of $\mathcal{A}$ such that $N(x) \neq 0$. The calculation show that $x\overline{x}N(x)^{-1} = 1$, then x is an invertible element. □

A first important property of our construction is given in the following lemma.

Lemma 3.3 *(cf. [11]) Let $\mathbb{K}_0$ be any field of characteristic $\neq 2$. Let $a, b, c \in \mathbb{K}_0$ be so that $abc \notin \mathbb{K}_0^2$ and put $\mathbb{K} := \mathbb{K}_0(\sqrt{abc})$. Let $Q_{a,b,c}$ be the quadratic form in 4 variables over $\mathbb{K}_0$ defined in (3.61). Then the following statements are equivalent:*

1. *The quadratic form $Q_{a,b,c}$ is $\mathbb{K}_0$-anisotropic.*
2. *The quaternion algebra $\left(\frac{-a,-b}{\mathbb{K}}\right)$ is a skew-field.*

Proof Assume first that $Q_{a,b,c}$ is $\mathbb{K}_0$-anisotropic. This is the case if and only if the equation

$$abx_0^2 + bx_1^2 + ax_2^2 + abcx_3^2 = 0 \tag{3.66}$$

has a non-trivial $\mathbb{K}_0$-solution. Then the element $\alpha := x_3\sqrt{abc} + x_2\mathtt{i} + x_1\mathtt{j} + x_0\mathtt{k}$ is a non-zero element of $\left(\frac{-a,-b}{\mathbb{K}}\right)$ having norm 0. This α cannot be invertible in $\left(\frac{-a,-b}{\mathbb{K}}\right)$ by Proposition 3.9.

To prove the reverse implication we assume that $\left(\frac{-a,-b}{\mathbb{K}}\right) \simeq M(2,\mathbb{K})$. Since $\left(\frac{-a,-b}{\mathbb{K}}\right) \simeq \left(\frac{-a,-b}{\mathbb{K}_0}\right) \otimes_{\mathbb{K}_0} \mathbb{K}$ the $\mathbb{K}_0$-quaternion algebra $\left(\frac{-a,-b}{\mathbb{K}_0}\right)$ has to contain an element $\alpha := x_0 + x_1\mathtt{i} + x_2\mathtt{j} + x_3\mathtt{k}$ with $\alpha^2 = abc$. This equation amounts to

$$x_0 = 0, \ -ax_1^2 - bx_2^2 - abx_3^2 = abc.$$

We see that (3.66) has a non-zero solution. □

The above considerations mean that starting by a quadratic form Q, we can associate a quaternion algebra $\mathcal{A} := \mathcal{A}(Q)$ and vice versa.

Theorem 3.7 *(cf. [25]) For $\mathcal{A} := \left(\frac{a,b}{\mathbb{K}}\right)$, the following are equivalent:*

1. *$\mathcal{A} \simeq \left(\frac{1,1}{\mathbb{K}}\right)$.*
2. *$\mathcal{A}$ is not a division algebra.*
3. *$\mathcal{A}$ is isotropic as a quadratic space with the norm form.*
4. *$\mathcal{A}_0$ is isotropic as a quadratic space with the norm form.*
5. *The quadratic form $ax^2 + by^2 = 1$ has solution with $(x, y) \in \mathbb{K} \times \mathbb{K}$.*

Proof The equivalence of 1 and 2 is just a restatement of Theorem 3.6.

$2 \Rightarrow 3$ If $\mathcal{A}$ is not a division algebra, it contains a non-zero non-invertible element x. Thus $N(x) = 0$ and $\mathcal{A}$ is isotropic.

$3 \Rightarrow 4$ Let $x = x_0 + x_1\mathtt{i} + x_2\mathtt{j} + x_3x_1\mathtt{k}$ be an element of $\mathcal{A}$ such that $N(x) = 0$. If $x_0 = 0$, then $x \in \mathcal{A}_0$ and $\mathcal{A}_0$ is isotropic. If $x_0 \neq 0$, then at least one of x_1, x_2 and x_3 must be non-zero. Without loss of generality, assume that $x_1 \neq 0$. Now from $N(x) = 0$, we obtain $x_0^2 - bx_2^2 = a(x_1^2 - bx_3^2)$. Let

$$y = b(x_0x_3 + x_1x_2)\mathtt{i} + a(x_1^2 - bx_3^2)\mathtt{j} + (x_0x_1 + bx_2x_3)\mathtt{k}.$$

A straightforward calculation gives that $N(y) = 0$. Now suppose that $\mathcal{A}_0$ is anisotropic. Thus $y = 0$ and, in particular, $-ax_1^2 + abx_3^2 = 0$. Thus for $z = x_1\mathtt{i} + x_3\mathtt{k}$, $N(z) = 0$. Again, if $\mathcal{A}_0$ is anisotropic, this implies that $x_1 = 0$. This is a contradiction showing that $\mathcal{A}_0$ is isotropic.

$4 \Rightarrow 5$ An equation of the form $-ax_1^2 - bx_2^2 + abx_3^2 = 0$ holds with at least two of x_1, x_2 and x_3 non-zero. If $x_3 \neq 0$, then $x = \frac{x_2}{ax_3}$, $y = \frac{x_1}{bx_3}$ satisfy $ax^2 + by^2 = 1$. If $x_3 = 0$, then $x = \frac{1+a}{2a}$ and $y = \frac{x_2(1-a)}{2ax_1}$ satisfy $ax^2 + by^2 = 1$.

$5 \Rightarrow 2$ Let $ax_0^2 + by_0^2 = 1$. $1 + x_0\mathtt{i} + y_0\mathtt{j}$ is a non-zero element of $\mathcal{A}$ of norm 0. Then A is not a division algebra. □

3.4.2 Cocompact discrete subgroups of $SL(2, \mathbb{C})$

Let G be a semisimple real Lie group with finite center, K be a maximal compact subgroup and Γ be a torsion-free discrete subgroup of G. The discrete subgroup Γ is *cocompact* if the locally symmetric space $X := G/\Gamma$ is compact. General results by A. Borel and J-P. Serre (1963) imply that such discrete cocompact subgroups can be constructed as arithmetic subgroups of suitable algebraic groups defined over some algebraic number field (cf. [5], [29]).

The construction of discrete cocompact subgroups of $SL(2, \mathbb{C})$ is incorporated in the following theorem.

Theorem 3.8 *(cf. [11]) Let $\mathbb{K}$ be an algebraic number field with exactly one pair of complex places. Let $\mathcal{A}$ be a quaternion algebra over $\mathbb{K}$ which is ramified at all real places of $\mathbb{K}$. For a complex place v_0 let $\phi : \mathcal{A} \otimes_{\mathbb{K}} \mathbb{K}_{v_0} \to M(2, \mathbb{C})$ be a $K_{v_0} = \mathbb{C}$ -algebra isomorphism. For an order $\mathcal{R} \subset \mathcal{A}$, put*

$$\Gamma = \phi(\mathcal{R}^1),$$

where $\mathcal{R}^1$ is the group of the elements of $\mathcal{R}$ with norm 1.
The group Γ has the following properties:

1. *Γ is a discrete subgroup of $SL(2, \mathbb{C})$.*

2. Γ *is cocompact if and only if* $\mathcal{A}$ *is a skew field.*

□

Then, by Theorem 3.8, the quaternion algebras are a starting point to construct discrete cocompact subgroups of $SL(2, \mathbb{C})$.

Given a quadratic form Q in 4 variables over the totally real algebraic number field $\mathbb{K}$ and let ι be a real place of $\mathbb{K}$, we may consider the quadratic form over $\mathbb{R}$: $Q^{[i]}(x) := x^T \iota(S_Q)x$ with $x \in \mathbb{R}^4$ and S_Q the matrix that represents Q.

Definition 3.9 *(cf. [11]) We say that the quadratic form Q satisfies the hyperbolic signature condition if there is a real place ι of $\mathbb{K}$ so that $Q^{[\iota]}$ is of signature* $(1, 3)$ *and if the $Q^{[\iota]}$ for all other embeddings ι are (positive or negative) definite.* □

In the situation of Definition 3.9 there is a $g \in GL(4, \mathbb{R})$ so that

$$Q^{[\iota]} \circ g(x) = x_1^2 - x_2^2 - x_3^2 - x_4^2. \tag{3.67}$$

Going back to the map Ψ as in Definition 3.3 we may use one of these g to get a group homomorphism

$$\Psi_0 : SL(2, \mathbb{C}) \to PO_4(\mathbb{R}, Q^{[\iota]}). \tag{3.68}$$

The homomorphisms Ψ_0 for the various g are equal up to conjugation in $PO_4(\mathbb{R}, Q^{[\iota]})$, we fix one of them. Ψ_0 has the center of $SL(2, \mathbb{C})$ as kernel and finite cokernel.

The following proposition is a very useful tool to construct or recognize discrete cocompact subgroups of the ortogonal group preserving a particular quadratic form.

Proposition 3.10 *(cf. [11]) Let $\mathbb{K}_0$ be a totally number field with ring of integers $O_{\mathbb{K}_0}$. Let Q be a quadratic form in* 4 *variables over $\mathbb{K}_0$ satisfying the hyperbolic signature condition. Let ι be a real place of $\mathbb{K}_0$ so that $Q^{[\iota]}$ is of signature* $(1, 3)$*. Put $d := DetS_Q$ and $\mathbb{K} := \mathbb{K}_0(\sqrt{d})$. Then $\mathbb{K}$ is a quadratic extension of $\mathbb{K}_0$ having exactly one pair of complex places v_0 and $\overline{v}_0$, they are the extensions of ι. The $\mathbb{K}_0$ -algebra $C^+(\mathbb{K}_0, Q)$ is a quaternion algebra $\mathcal{A}(Q)$ over its center $\mathbb{K}$. It is extended from a quaternion algebra over $\mathbb{K}_0$. The $\mathbb{K}$-algebra $\mathcal{A}(Q)$ is a skew field if and only if Q is $\mathbb{K}_0$ anisotropic. Choose maps*

$$\phi : \mathcal{A}(Q) \otimes_{\mathbb{K}} \mathbb{K}_{v_0} \to M(2, \mathbb{C}), \;\; \Psi_0 : SL(2, \mathbb{C}) \to PO_4(\mathbb{R}, Q^{[\iota]}) \tag{3.69}$$

where ϕ is a $\mathbb{K}_{v_0} = \mathbb{C}$-algebra isomorphism and Ψ_0 is as in (3.68). Let $\mathcal{R}$ be an order in $\mathcal{A}(Q)$. Then the group $\Psi_0(\phi(\mathcal{R}^1))$ is $O_4(\mathbb{R}, Q^{[\iota]})$-conjugate to a group commensurable with $\iota(O_4(\mathbb{R}, Q^{[\iota]}))$. □

3.4.3 $\mathfrak{W}$ as cocompact arithmetic lattice

In this section we exhibit the group $(\mathfrak{W}, \mathfrak{S})$ as a cocompact arithmetic lattice of the orthogonal group $O_4^+(\mathbb{R}, \mathfrak{Q})$ with $\mathfrak{Q}(x_0, x_1, x_2, x_3) = 7x_0^2 - x_1^2 - x_2^2 - x_3^2$, generalising a result of J. Elstrodt, F. Grunewald, J. Mennicke (cf. [11]).
The group $\mathfrak{W}$ has a presentation as algebraic group, because it is represented as subgroup of the algebraic group $O_4(\mathbb{R}, \mathfrak{Q})$. In Section 3.1 we observed that $\mathfrak{W}$ is a crystallographic, cocompact hyperbolic Coxeter group. Therefore, if we think to use the isomorphism Ψ in (3.44), then $\mathfrak{W}$ has also a representation as discrete and cocompact subgroup of $SL(2, \mathbb{C})$. Moreover we know how to construct the discrete and cocompact subgroups of $SL(2, \mathbb{C})$ thanks to Theorem 3.8.
Now we refer to Proposition 3.10. We fix $\mathbb{K}_0 = \mathbb{Q}$ be a totally real number field and $\mathfrak{Q}(x_0, x_1, x_2, x_3) = 7x_0^2 - x_1^2 - x_2^2 - x_3^2$ be an anisotropic quadratic form over $\mathbb{Q}$ (cf. Section 3.3.2). $\mathfrak{Q}$ satisfies the hyperbolic signature.
With a similar reasoning to Section 3.4.1, we consider a $\mathbb{Q}$-vector space $\Lambda_{\mathbb{Q}}$ with basis e_0, e_1, e_2, e_3 and quadratic form $\mathfrak{Q}(x_0e_0 + x_1e_1 + x_2e_2 + x_3e_3) = 7x_0^2 - x_1^2 - x_2^2 - x_3^2$. The discriminant of $\mathfrak{Q}$ (i.e. the determinant of the matrix that represent $\mathfrak{Q}$ respect to the fixed basis) is $d = -7$. Therefore, over $\mathbb{K} = \mathbb{Q}(i\sqrt{7})$, the even Clifford algebra $C^+(\mathbb{Q}, \mathfrak{Q})$ is a quaternion algebra $\mathcal{A}(\mathfrak{Q})$ over its center $\mathbb{Q}(i\sqrt{7})$. Moreover, since $\mathfrak{Q}$ is $\mathbb{Q}$-anisotropic the quaternion algebra $\mathcal{A}(\mathfrak{Q})$ over $\mathbb{Q}(i\sqrt{7})$ is also a skew-field. It is easy to construct the following $\mathbb{Q}$-algebra isomorphism:

$$\gamma : C^+(\mathbb{Q}, \mathfrak{Q}) \to \mathcal{A}(\mathfrak{Q}) \tag{3.70}$$

$$\gamma(1) = \mathtt{1},\ \ \gamma(e_0e_1e_2e_3) = i\sqrt{7},\ \ \gamma(e_1e_2) = \mathtt{i},\ \ \gamma(e_1e_3) = \mathtt{j},\ \ \gamma(e_3e_2) = \mathtt{k}$$

such that $\mathtt{i}^2 = \mathtt{j}^2 = -1$ and $\mathtt{ij} = -\mathtt{ji}$.
Therefore, $\mathcal{A}(\mathfrak{Q})$ is a quaternion algebra over the field $\mathbb{Q}(i\sqrt{7})$:

$$\mathcal{A}(\mathfrak{Q}) = \left(\frac{-1, -1}{\mathbb{Q}(i\sqrt{7})}\right). \tag{3.71}$$

An alternative proof about the fact that the quaternion algebra $\left(\frac{-1,-1}{\mathbb{Q}(i\sqrt{7})}\right)$ is a skew-field is the following one.

Remark 3.3 *(cf. [18]) $\mathcal{S} := \left(\frac{-1,-1}{\mathbb{Q}(i\sqrt{7})}\right)$ is a skew-field.*

Proof The integer ring of $\mathbb{Q}(i\sqrt{7})$ is $\mathcal{O} = \mathbb{Z}[\frac{1+i\sqrt{7}}{2}]$. In the extension $\mathbb{Q}(i\sqrt{7})$

over $\mathbb{Q}$ the prime 2 splits as $2 = \left(\frac{1+i\sqrt{7}}{2}\right)\left(\frac{1-i\sqrt{7}}{2}\right)$. Therefore $2\mathcal{O} = \mathcal{P}\mathcal{P}'$, where $\mathcal{P}$ and $\mathcal{P}'$ are distinct prime ideals. The completion $\mathbb{K}_v$ of $\mathbb{Q}(i\sqrt{7})$ at valuation v corresponding to either of these primes is thus isomorphic to the 2-adic numbers $\mathbb{Q}_2$. [1] Thus

$$\mathcal{S} \otimes_{\mathbb{Q}(i\sqrt{7})} \mathbb{K}_v \simeq \left(\frac{-1,-1}{\mathbb{Q}_2}\right).$$

The equation $-x^2 - y^2 = z^2$ has only the trivial solution in the ring of 2-adic integers by Theorem 3.5. Then from Theorem 3.7 it follows that $\mathcal{S} \otimes_{\mathbb{Q}(i\sqrt{7})} \mathbb{K}_v$ is a skew-field. Thus $\mathcal{S}$ is ramified at v ans so $\mathcal{S}$ can not be isomorphic to $M(2, \mathbb{Q}(i\sqrt{7}))$. $\square$

Let $\mathcal{S}^1$ be the group of the elements of $\mathcal{S}$ of norm 1 and consider the composition of the map Ψ (3.43) and the map ϕ (3.57).

$$\Psi = \Psi \circ \phi : \mathcal{S}^1 \to SL(2,\mathbb{C}) \to SO_4^+(\mathbb{R}, \mathfrak{Q}) \tag{3.72}$$

$$\Psi(x_0 + x_1\mathtt{i} + x_2\mathtt{j} + x_3\mathtt{k}) = \Psi(\phi(x_0 + x_1\mathtt{i} + x_2\mathtt{j} + x_3\mathtt{k})) = \Psi\left(\begin{pmatrix} x_0 + x_1 i & x_2 i + x_3 \\ x_2 i - x_3 & x_0 - x_1 i \end{pmatrix}\right) =$$

$$\begin{pmatrix} x_0^2 + x_1^2 + x_2^2 + x_3^2 & 0 & 0 & 0 \\ 0 & x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2x_1x_2 + 2x_0x_3 & 2x_0x_2 - 2x_1x_3 \\ 0 & 2x_1x_2 - 2x_0x_3 & x_0^2 - x_1^2 + x_2^2 - x_3^2 & -2x_0x_1 - 2x_2x_3 \\ 0 & -2x_0x_2 - 2x_1x_3 & 2x_0x_1 - 2x_2x_3 & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix} \tag{3.73}$$

Let the group

$$\Gamma^1 := \left\{ \begin{pmatrix} x_0 + x_1 i & x_2 i + x_3 \\ x_2 i - x_3 & x_0 - x_1 i \end{pmatrix} \in PSL(2, \mathbb{Q}(i, \sqrt{7})) \;\middle|\; x_i \in \mathbb{Q}(i\sqrt{7}) \right\}, \tag{3.74}$$

and Δ be the group $\Psi(\Gamma^1)$, i.e. the group of matrix of type (3.73).

We want to extend this homomorphism of groups through external involutions:

$$\tilde{\Psi} = \tilde{\Psi} \circ \tilde{\phi} : \langle \xi \rangle \ltimes \mathcal{S}^1 \to \langle \sigma \rangle \ltimes \Gamma^1 \to O_4(\mathbb{R}, \mathfrak{Q}) \tag{3.75}$$

where σ is the complex conjugation, instead ξ corresponding to σ through the

[1] There are two valuations v_1, v_2 above 2 corresponding to the two finite primes $p_1 = \left(\frac{1+i\sqrt{7}}{2}\right)$ and $p_2 = \left(\frac{1-i\sqrt{7}}{2}\right)$. The completion $\mathbb{K}_v$, where v is one of v_i, $i = 1, 2$, is an extension of $\mathbb{Q}_2$, since the v_1-adic topology on $\mathbb{K} = \mathbb{Q}(i\sqrt{7})$ extends the 2-adic topology on $\mathbb{Q}$. Since $\mathbb{K} = \frac{\mathbb{Q}[X]}{(X^2+7)}$, we have that $\mathbb{Q}(i\sqrt{7})$ contains a solution for the equation $X^2 + 7 = 0$. From Hensel's Lemma (Lemma 3.1), we have that -7 is a square in $\mathbb{Q}_2$. Thus $\mathbb{K}_v \simeq \mathbb{Q}_2$.

respective maps as in Section 3.2.5: let $x_0 + x_1\mathtt{i} + x_2\mathtt{j} + x_3\mathtt{k} \in \mathcal{S}^1$

$$\xi(x_0 + x_1\mathtt{i} + x_2\mathtt{j} + x_3\mathtt{k})\xi = \overline{x_0} - \overline{x_1}\mathtt{i} - \overline{x_2}\mathtt{j} + \overline{x_3}\mathtt{k} \tag{3.76}$$

where $^{-} : \mathbb{C} \to \mathbb{C}$ is the complex conjugation. Let

$$\mathcal{R} = \left\{x_0 + x_1\mathtt{i} + x_2\mathtt{j} + x_3\mathtt{k} \in \mathcal{S} \;\middle|\; x_i \in \mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]\right\} \tag{3.77}$$

be an order in $\mathcal{S}$. By Theorem 3.8 $\phi(\mathcal{R}^1)$ is a discrete cocompact subgroup of $SL(2,\mathbb{C})$, therefore $\tilde{\Psi}(\mathcal{R}^1)$ is a discrete cocompact subgroup of $O_4(\mathbb{R},\mathfrak{Q})$.

3.4.4 The tetrahedral group $\mathfrak{T}$

Let G be a group generated by reflections in the faces of a suitable polyhedra. A subgroup of G is called *polyhedral group* if it is a subgroup of index 2 in G consisting of orientation-preserving isometries in the groups generated by reflections. When the polyhedra is a tetrahedra then the group is called *tetrahedral group* and has the following presentation

$$\langle x, y, z \mid x^m = y^n = z^p = (yz^{-1})^r = (zx^{-1})^s = (xy^{-1})^t = 1\rangle. \tag{3.78}$$

The following group

$$\mathfrak{T} = \langle x, y, z \mid x^2 = y^3 = z^4 = (yz)^2 = (zx)^3 = (xy)^4 = 1\rangle. \tag{3.79}$$

is a tetrahedral subgroup of $\mathfrak{W}$ (cf. [18]).
We will find a representation of $\mathfrak{W}$ in $O_4(\mathbb{R},\mathfrak{Q})$ (in particular in $O_4^+(\mathbb{R},\mathfrak{Q})$). Let x, y, z be three elements of the group $\langle\sigma\rangle \ltimes PSL(2,\mathbb{C})$. By straightforward calculation, imposing the relations in (3.79), we obtain the representation of the tetrahedral group $\mathfrak{T}$ in $\langle\sigma\rangle \ltimes PSL(2,\mathbb{C})$:

$$x = \begin{pmatrix} a & b \\ c & -a \end{pmatrix},\; y = \begin{pmatrix} \frac{1}{2} + \frac{1}{2}i & \frac{1}{2}i + \frac{1}{2} \\ \frac{1}{2}i - \frac{1}{2} & \frac{1}{2} - \frac{1}{2}i \end{pmatrix},\; z = \begin{pmatrix} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i & 0 \\ 0 & \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \end{pmatrix}. \tag{3.80}$$

Through the map $\tilde{\Psi}$ (3.39), we obtain the representation of the elements y and z in $O_4(\mathbb{R},\mathfrak{Q})$.

$$\tilde{\Psi}(y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \tilde{\Psi}(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}. \tag{3.81}$$

By construction, we can think the elements in (3.81) as even products of simple reflections in $SO_4^+(\mathbb{R}, \mathfrak{Q})$.

3.4.5 A representation of $\mathfrak{W}$ in $O_4^+(\mathbb{R}, \mathfrak{Q})$

We remind that in Section 3.2.5 we observed that the involution, that we called σ, acts in a natural way on $SL(2, \mathbb{C})$ and we calculated the image of σ throght $\tilde{\Psi}$, that is $\tau = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$. τ is the reflection, i.e. an involution with trace 2.
To find the representation of $\mathfrak{W}$ in $O_4(\mathbb{R}, \mathfrak{Q})$ (in particular in $O_4^+(\mathbb{R}, \mathfrak{Q})$), one can fix τ as the third simple reflection:

$$S_3 := \tau. \tag{3.82}$$

Considering the element $\Psi(z)$ in (3.81) and multiplying this element by S_3, we obtain the second simple reflection

$$S_2 := S_3\Psi(z) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \end{pmatrix}. \tag{3.83}$$

Considering then the element $\Psi(y)$ in (3.81) and multiplying this element by S_2, we obtain the first simple reflection

$$S_1 := S_2\Psi(y) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.84}$$

By straightforward calculations, imposing the relations (3.5) between the simple reflections of the Coxeter group $\mathfrak{W}$, one has the last generator of the group:

$$S_4 := \begin{pmatrix} \frac{9}{2} & \frac{3}{2} & \frac{1}{2} & -\frac{1}{2} \\ -\frac{21}{2} & -\frac{7}{2} & -\frac{3}{2} & \frac{3}{2} \\ -\frac{7}{2} & -\frac{3}{2} & \frac{1}{2} & \frac{1}{2} \\ \frac{7}{2} & \frac{3}{2} & \frac{1}{2} & \frac{1}{2} \end{pmatrix}. \tag{3.85}$$

In this description of $\mathfrak{W}$ the simple roots corresponding to the simple reflections S_1, S_2, S_3, S_4 are presented as:

$$\alpha_1 = \begin{pmatrix} 0 \\ 1 \\ -1 \\ 0 \end{pmatrix}, \alpha_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}, \alpha_3 = \begin{pmatrix} 0 \\ 0 \\ 0 \\ -2 \end{pmatrix}, \alpha_4 = \begin{pmatrix} 1 \\ -3 \\ -1 \\ 1 \end{pmatrix}. \tag{3.86}$$

Then the presentation of $\mathfrak{W}$ in $O_4^+(\mathbb{R}, \mathfrak{Q})$ is

$$\mathfrak{W} = \langle S_1, S_2, S_3, S_4 \mid$$

$$(S_1S_2)^3 = (S_1S_3)^2 = (S_1S_4)^4 = (S_2S_3)^4 = (S_2S_4)^2 = (S_3S_4)^3 = S_i^2 = 1 \;\; i = 1, \ldots, 4\rangle. \tag{3.87}$$

3.4.6 A representation of $\mathfrak{W}$ in $\langle\sigma\rangle \ltimes PSL(2, \mathbb{C})$

In this description, we use the symbols s_1, s_2, s_3, s_4 as generators of $\mathfrak{W}$ in $\langle\sigma\rangle \ltimes PSL(2, \mathbb{C})$ in place of S_1, S_2, S_3, S_4, generators of $\mathfrak{W}$ in $O_4^+(\mathbb{R}, \mathfrak{Q})$. From Section 3.4.5 and using $\tilde{\Psi}$ (3.44), it is easy to observe that

$$s_3 := \sigma = \frac{1}{2}\sigma\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix}, \tag{3.88}$$

$$s_2 := \sigma z = \sigma\begin{pmatrix} \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i & 0 \\ 0 & \frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i \end{pmatrix} = \frac{1}{\sqrt{2}}\sigma\begin{pmatrix} 1+i & 0 \\ 0 & 1-i \end{pmatrix}, \tag{3.89}$$

$$s_1 := s_2 y = \sigma z y = \sigma\begin{pmatrix} \frac{1}{\sqrt{2}}i & \frac{1}{\sqrt{2}}i \\ \frac{1}{\sqrt{2}}i & -\frac{1}{\sqrt{2}}i \end{pmatrix} = \frac{1}{\sqrt{2}}\sigma\begin{pmatrix} i & i \\ i & -i \end{pmatrix}. \tag{3.90}$$

Moreover, by straightforward calculations and by supposing that $x = s_2 s_4$, we also obtain

$$x = \begin{pmatrix} \frac{1}{\sqrt{2}}i & (\frac{1}{4} + \frac{1}{4}i)(-3 - \sqrt{7})\sqrt{2} \\ (\frac{1}{4} - \frac{1}{4}i)(3 - \sqrt{7})\sqrt{2} & -\frac{1}{\sqrt{2}}i \end{pmatrix} \tag{3.91}$$

and

$$s_4 := s_2 x = \sigma z x = \sigma\begin{pmatrix} -\frac{1}{2} + \frac{1}{2}i & -\frac{1}{2}i(3 + \sqrt{7}) \\ -\frac{1}{2}i(3 - \sqrt{7}) & -\frac{1}{2} - \frac{1}{2}i \end{pmatrix} = \frac{1}{2}\sigma\begin{pmatrix} -1+i & -3i - \sqrt{7}i \\ -3i + \sqrt{7}i & -1-i \end{pmatrix}. \tag{3.92}$$

We found a representation of $\mathfrak{W}$ in $PSL(2, \mathbb{C})$ that is only commensurable with a subgroup in $PSL(2, \mathbb{Z}[\frac{1+i\sqrt{7}}{2}])$ and the representation of $\mathfrak{W}$ in $O_4(\mathbb{R}, \mathfrak{Q})$ is commensurable with a subgroup in $O_4(\mathbb{Z}, \mathfrak{Q})$.

We show a general formula for the reflections in $\langle\sigma\rangle \ltimes PSL(2,\mathbb{C})$, generalizing the results of Masaaki Yoshida (cf. [35]). Let introduce some notations.
Let A (cf. (3.7)) be the GCM associated with the group $\mathfrak{W}$. In Section 3.1 we observed that A is diagonalizable and so that there exists a diagonal matrix D (cf. (3.9)) such that $A = DB'$ with B' (cf. (3.8)) symmetric matrix. Let

$$P = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & -3 \\ -1 & 1 & 0 & -1 \\ 0 & 1 & -2 & 1 \end{pmatrix} \tag{3.93}$$

be the change of basis matrix from the basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$ of simple roots to the basis $\{e_0, e_1, e_2, e_3\}$ introduced in Section 3.1.1. We have that $AD = P^T HP$, where

$$H = \begin{pmatrix} -14 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 2 & 0 \\ 0 & 0 & 0 & 2 \end{pmatrix}. \tag{3.94}$$

Proposition 3.11 *Notations being as above, we have that*

$$S_j = \tilde{\Psi}\left(\frac{1}{\sqrt{d_j}}\sigma\begin{pmatrix} P_{4j} + P_{3j}i & -P_{2j}i + P_{1j}\sqrt{7}i \\ -P_{2j}i - P_{1j}\sqrt{7}i & P_{4j} - P_{3j}i \end{pmatrix}\right), \tag{3.95}$$

where $P = (P_{ij})$ and $D = diag(d_j)$.

Proof Let R_j be the matrix representation of R_j respect to the basis $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$. Then we have

$$R_j = I - \delta_j^T \delta_j A^T,$$

where δ_j is the column vector with null components except 1 in the j-th and I is the unit matrix. Thus we have

$$S_j = PR_jP^{-1} = I - P\delta_j^T\delta_j A^T P^{-1} = I - P\delta_j^T\delta_j D^{-1}A^T H =$$

$$= I - \begin{pmatrix} P_{1j} \\ P_{2j} \\ P_{3j} \\ P_{4j} \end{pmatrix}\begin{pmatrix} -14P_{1j} & 2P_{2j} & 2P_{3j} & 2P_{4j} \end{pmatrix} \diagup d_j =$$

$$= \begin{pmatrix} 1+\frac{14P_{1j}^2}{d_j} & -\frac{2P_{1j}P_{2j}}{d_j} & -\frac{2P_{1j}P_{3j}}{d_j} & -\frac{2P_{1j}P_{4j}}{d_j} \\ \frac{14P_{1j}P_{2j}}{d_j} & 1-\frac{2P_{2j}^2}{d_j} & -\frac{2P_{2j}P_{3j}}{d_j} & -\frac{2P_{2j}P_{4j}}{d_j} \\ \frac{14P_{1j}P_{3j}}{d_j} & -\frac{2P_{2j}P_{3j}}{d_j} & 1-\frac{2P_{3j}^2}{d_j} & -\frac{2P_{3j}P_{4j}}{d_j} \\ \frac{14P_{1j}P_{4j}}{d_j} & -\frac{2P_{2j}P_{4j}}{d_j} & -\frac{2P_{3j}P_{4j}}{d_j} & 1-\frac{2P_{4j}^2}{d_j} \end{pmatrix}.$$

On the other hand, $AD = P^T HP$ and $A_{jj} = 2$ implies

$$\frac{1}{d_j}(-7P_{1j}^2 + P_{2j}^2 + P_{3j}^2 + P_{4j}^2) = 1.$$

These considerations and formula (3.39) of $\tilde{\Psi}$ prove the proposition. □

From formula (3.95), we also obtain a formula for s_j:

$$s_j = \frac{1}{\sqrt{d_j}}\sigma A_j \in \langle\sigma\rangle \ltimes PSL(2,\mathbb{C}) \tag{3.96}$$

where

$$A_j = \begin{pmatrix} P_{4j} + P_{3j}i & -P_{2j}i + P_{1j}\sqrt{7}i \\ -P_{2j}i - P_{1j}\sqrt{7}i & P_{4j} - P_{3j}i \end{pmatrix} \in PSL(2,\mathbb{C}). \tag{3.97}$$

Because P_{ij} are integers, then $A_j \in \Gamma^1$ (cf. (3.74)) and $\sqrt{d_j}$ is exactly the norm of the simple root α_j.

3.4.7 A description of the root system $\dot{\Phi}$

Remind that the group $\mathfrak{W}$ is a Weyl group of a Kac-Moody Lie algebra $\mathfrak{L}(A)$ associated with the symmetrizable GCM A (cf. (3.7)). In this section we give a presentation of the root system $\dot{\Phi}$ of $\mathfrak{L}(A)$.
Let $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ be the simple roots of $\mathfrak{W}$, generators of the root lattice $\Lambda_{\mathbb{Z}}$. We choose the basis introduced in Section 3.1.1 e_0, e_1, e_2, e_3 such that $\langle e_i, e_j\rangle = 0\ \forall i \neq j$, $\langle e_i, e_i\rangle = 1\ \ i = 1,2,3$ and $\langle e_0, e_0\rangle = -7$. Then

$$\alpha_1 = e_1 - e_2,\ \ \alpha_2 = e_2 + e_3,\ \ \alpha_3 = -2e_3,\ \ \alpha_4 = e_0 - 3e_1 - e_2 + e_3. \tag{3.98}$$

We define a $\mathbb{C}$-linear map

$$\mu : \Lambda_{\mathbb{Z}} \to \{A \in M(2,\mathbb{C}) \mid det(A) \in \mathbb{R}\} \tag{3.99}$$

$$\mu(n_0e_0 + n_1e_1 + n_2e_2 + n_3e_3) = \begin{pmatrix} -n_2 - n_3 i & n_1 - \sqrt{7}n_0 \\ n_1 + \sqrt{7}n_0 & n_2 - n_3 i \end{pmatrix}$$

Then one has

$$\mu(\alpha_1) = \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},\ \mu(\alpha_2) = \begin{pmatrix} -1-i & 0 \\ 0 & 1-i \end{pmatrix}, \tag{3.100}$$

$$\mu(\alpha_3) = \begin{pmatrix} 2i & 0 \\ 0 & 2i \end{pmatrix},\ \mu(\alpha_4) = \begin{pmatrix} 1-i & -3-\sqrt{7} \\ -3+\sqrt{7} & -1-i \end{pmatrix}.$$

The group $\mathfrak{W}$ acts on $\mu(\Lambda_{\mathbb{Z}})$ by

$$(\sigma A_{i_1} \dots \sigma A_{i_k}) \cdot X = \sigma A_{i_1} \dots \sigma A_{i_k} X A_{i_k} \sigma \dots A_{i_1} \sigma. \tag{3.101}$$

Then the map μ is $\mathfrak{W}$-equivariant and we can identify $\Lambda_{\mathbb{Z}}$ with $\mu(\Lambda_{\mathbb{Z}})$.
Moreover

$$\langle \alpha, \alpha \rangle = -det(\mu(\alpha)) = 7n_0^2 - n_1^2 - n_2^2 - n_3^2. \tag{3.102}$$

We recall the *root lattice* of $\mathfrak{L}(A)$ to be the set

$$\Lambda_{\mathbb{Z}} := \mathbb{Z}\alpha_1 \oplus \mathbb{Z}\alpha_2 \oplus \mathbb{Z}\alpha_3 \oplus \mathbb{Z}\alpha_4. \tag{3.103}$$

Referring to Section 2.4.1, the set $\Omega = \{\lambda_s \mid s \in \mathfrak{S}\}$ constructed with formula (2.26) is another canonical basis of $\mathfrak{H}^*$ and λ_s are called *simple weights*:

$$\lambda_1 = -\frac{2}{7}\alpha_1 - \frac{5}{7}\alpha_2 - \frac{4}{7}\alpha_3 - \frac{3}{7}\alpha_4,\ \lambda_2 = -\frac{5}{7}\alpha_1 - \frac{2}{7}\alpha_2 - \frac{3}{7}\alpha_3 - \frac{4}{7}\alpha_4 \tag{3.104}$$

$$\lambda_3 = -\frac{8}{7}\alpha_1 - \frac{6}{7}\alpha_2 - \frac{2}{7}\alpha_3 - \frac{5}{7}\alpha_4,\ \lambda_4 = -\frac{6}{7}\alpha_1 - \frac{8}{7}\alpha_2 - \frac{5}{7}\alpha_3 - \frac{2}{7}\alpha_4.$$

Then we call the *weight lattice* of $\mathfrak{L}(A)$ the set

$$\Lambda^*_{\mathbb{Z}} := \mathbb{Z}\omega_1 \oplus \mathbb{Z}\omega_2 \oplus \mathbb{Z}\omega_3 \oplus \mathbb{Z}\omega_4. \tag{3.105}$$

We also define the sets

$$\Lambda^+_{\mathbb{Z}} := \mathbb{Z}^+\alpha_1 \oplus \mathbb{Z}^+\alpha_2 \oplus \mathbb{Z}^+\alpha_3 \oplus \mathbb{Z}^+\alpha_4, \tag{3.106}$$

$$\Lambda^-_{\mathbb{Z}} := -\Lambda^+_{\mathbb{Z}} \tag{3.107}$$

and

$$\Lambda^{*^+}_{\mathbb{Z}} := \mathbb{Z}^+\omega_1 \oplus \mathbb{Z}^+\omega_2 \oplus \mathbb{Z}^+\omega_3 \oplus \mathbb{Z}^+\omega_4, \tag{3.108}$$

$$\Lambda^{*^-}_{\mathbb{Z}} := -\Lambda^{*^+}_{\mathbb{Z}}. \tag{3.109}$$

We also see that

$$\Lambda_{\mathbb{Z}} = \{n_0 e_0 + n_1 e_1 + n_2 e_2 + n_3 e_3 \mid n_i \in \mathbb{Z}\} \tag{3.110}$$

$$\Lambda^+_{\mathbb{Z}} = \{n_0e_0+n_1e_1+n_2e_2+n_3e_3 \mid n_0 \geq 0,\ n_1+n_2 \geq -4n_0,\ n_3 \leq n_1+n_2+5n_0\}, \quad (3.111)$$

$$\Lambda^*_{\mathbb{Z}} = \{-\frac{1}{7}n_0e_0 + n_1e_1 + n_2e_2 + n_3e_3 \mid n_i \in \mathbb{Z}\}, \quad (3.112)$$

$$\Lambda^{*^+}_{\mathbb{Z}} = \{-\frac{1}{7}n_0e_0 + n_1e_1 + n_2e_2 + n_3e_3 \mid n_0 \geq n_1 \geq n_2 \geq -n_3 \geq 0\}. \quad (3.113)$$

Remind (cf. Definition 2.10) the definition of the real root system that for the Kac-Moody Lie algebra with Weyl group $\mathfrak{W}$ is

$$\dot{\Phi}_{Re} := \mathfrak{W} \cdot \{\alpha_1, \alpha_2, \alpha_3, \alpha_4\} \quad (3.114)$$

and of the imaginary root system

$$\dot{\Phi}_{Im} := \dot{\Phi} \setminus \dot{\Phi}_{Re}. \quad (3.115)$$

We also have positive (and negative) real and imaginary roots

$$\dot{\Phi}^{\pm}_{Re} = \dot{\Phi}_{Re} \cap \dot{\Phi}^{\pm},\ \dot{\Phi}^{\pm}_{Im} = \dot{\Phi}_{Im} \cap \dot{\Phi}^{\pm}, \quad (3.116)$$

and

$$\dot{\Phi}^{\pm} = \dot{\Phi} \cap \Lambda_{\mathbb{Z}},\ \dot{\Phi}^{\pm}_{Re} = \dot{\Phi}_{Re} \cap \Lambda^{\pm}_{\mathbb{Z}},\ \dot{\Phi}^{\pm}_{Im} = \dot{\Phi}_{Im} \cap \Lambda^{\pm}_{\mathbb{Z}}. \quad (3.117)$$

In general it is extremely difficult to determine or characterize all imaginary roots, but thank to the following result of R. V. Moody (cf. [23]) about the hyperbolic Kac-Moody Lie algebras

$$\dot{\Phi}_{Im} = \{\alpha \in \Lambda_{\mathbb{Z}} \mid \langle \alpha, \alpha \rangle \leq 0\}, \quad (3.118)$$

and to the property (3.102), we have that

$$\mu(\dot{\Phi}_{Re}) = \{X \in \mu(\Lambda_{\mathbb{Z}}) \mid det(X) = -2, -4\}, \quad (3.119)$$

$$\mu(\dot{\Phi}_{Im}) = \{X \in \mu(\Lambda_{\mathbb{Z}}) \mid det(X) \geq 0\}, \quad (3.120)$$

$$\mu(\dot{\Phi}^+_{Im}) = \{X \in \mu(\Lambda_{\mathbb{Z}}) \mid 7n_0^2 - n_1^2 - n_2^2 - n_3^2 \geq 0,\ n_0 \geq 0,\ n_1+n_2 \geq -4n_0,\ n_3 \leq n_1+n_2+5n_0\}. \quad (3.121)$$

Remark 3.4

$$\dot{\Phi}_{Re} = \mathfrak{W} \cdot \{\alpha_1, \alpha_3\} \quad (3.122)$$

Proof It is sufficient to observe that $s_1s_2(\alpha_1) = \alpha_2$ and $s_3s_4(\alpha_3) = \alpha_4$. □

Moreover, using formula (2.38), we calculated the multiplicities for all $\alpha \in \Lambda^+_{\mathbb{Z}}$ such that $ht(\alpha) \leq 4$ and we obtained the following results:

1. Of height 2 and 3 there no positive imaginary roots.

2. Of height 4 there is only one imaginary root

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4.$$

   of multiplicity 3.

We give an example of calculation of multiplicities using formula (2.38).

Example 3.5 *Let* $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 \in \Lambda^+_{\mathbb{Z}}$.
There are four ways to write $\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4$ *as linear combination of* $c(w)$, $w \in W$ *with integer coefficients:*

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = c(s_1) + c(s_2) + c(s_3) + c(s_4)$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = c(s_1 s_3) + c(s_2 s_4)$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = c(s_1 s_3) + c(s_2) + c(s_4).$$

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4 = c(s_1) + c(s_3) + c(s_2 s_4).$$

Then

$$m_{\alpha_1+\alpha_2+\alpha_3+\alpha_4} = (-1)^2(-1)^2(-1)^2(-1)^2\frac{(1+1+1+1-1)!}{1!1!1!1!} + (-1)^3(-1)^3\frac{(1+1-1)!}{1!1!}$$

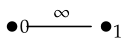
$$+(-1)^3(-1)^2(-1)^2\frac{(1+1+1-1)!}{1!1!1!} + (-1)^3(-1)^2(-1)^2\frac{(1+1+1-1)!}{1!1!1!} = 3.$$

□

Chapter 4

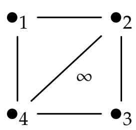
Crystallographic Coxeter groups with an ∞-decomposition and Kac′ denominator formula

In this chapter we study the crystallographic Coxeter systems (W, S) with a spherically ∞-decomposition (cf. Section 4.2).
Some examples of crystallographic Coxeter groups with a *spherically ∞-decomposition* are:
$\tilde{A}_1$



(4.1)

and $(\mathcal{W}, \mathcal{S})$ with Coxeter graph $\Gamma(\mathcal{W})$.



(4.2)

Our aim is to find an explicit formula for the left hand-side of the Kac′ denominator formula (cf. (2.20)) for the crystallographic Coxeter groups with a spherically ∞-decomposition.

4.1 Free products of groups with amalgamation

The free products of groups with amalgamation are generalization of free products of groups. In this section we remind some general notions about free products of groups with amalgamation. For more details we refer the reader to [27].

Definition 4.1 *(cf. [27]) Let $\{G_\lambda \mid \lambda \in \Lambda\}$ be a non-empty set of groups and H be a group which is isomorphic with a subgroup H_λ of G_λ by means of a monomorphism*

$$\phi_\lambda : H \to G_\lambda \qquad (\lambda \in \Lambda).$$

Considering $F := \coprod_{\lambda\in\Lambda} G_\lambda$ the free product of the G_λ and N the normal closure in F of the subset

$$\{(h^{\phi_\lambda})^{-1}h^{\phi_\mu} \mid \lambda, \mu \in \Lambda, h \in H\},$$

we call the group

$$G = F/N$$

free product group of the G_λ's with the amalgamated group H and we write

$$G = \coprod_{\substack{\lambda\in\Lambda \\ H}} G_\lambda. \tag{4.3}$$

□

The group G in (4.3) can be considered as the group generated by the G_λ, $\lambda \in \Lambda$, in which the subgroups H_λ are identified by means of ϕ_λ. Since $h^{\phi_\lambda} \equiv h^{\phi_\mu} mod N$, the subgroups NH^{ϕ_λ}/N, $\lambda \in \Lambda$, are equal in G. In general G will depend on the ϕ_λ chosen.
There are two particular cases:

1. When $H = \{1\}$ then G is a free product of groups. Therefore we may consider the free product of groups with amalgamation as a generalization of the free product of groups.

2. When we have two groups G_1 and G_2 with subgroups H_1 and H_2 that are isomorphic by means of $\phi : H_1 \to H_2$. Then we can construct the group $G_1 \coprod_H G_2$ as free product group of G_1 and G_2 with the amalgamated group H setting $H = H_1$, $\phi_1 = 1$ and $\phi_2 = \phi$.

Example 4.1 *(cf. [27]) Let $A = \langle a\rangle$ and $B = \langle b\rangle$ be cyclic groups of orders 4 and 6 respectively. We observe that in the free product of A and B*

$$A \coprod B = \langle a, b \mid a^4 = 1,\ a^2 = b^3\ \rangle$$

there are two elements a^2 and b^3 that have both order 2, therefore the subgroups $\langle a^2\rangle$ and $\langle b^3\rangle$ are isomorphic. We may form the free product G with amalgamation determined by the isomorphism $\langle a^2\rangle \to \langle b^3\rangle$. This amounts to identify a^2 and b^3. Thus

$$G = \langle a, b \mid a^4 = 1,\ a^2 = b^3\rangle. \tag{4.4}$$

The element $h = a^2 = b^3$ commutes with a and b, so it belongs to the center of G and every element of G can be written in the form

$$a^{j_1} b^{k_1} a^{j_2} b^{k_2} \cdots a^{j_r} b^{k_r} h^i,\ \ (r \geq 0), \tag{4.5}$$

where i and j_s are equal to 0 or 1 and k_s are 0, 1 or 2. We will see that every element of the free product of groups with amalgamation admits a writing as (4.5) that is called normal form. □

4.1.1 A particular case of free product with amalgamation

We will consider the case in which G is the free product of only two groups G_1 and G_2 with amalgamated group $H = G_1 \cap G_2$:

$$G = G_1 \coprod_H G_2.$$

Let $_\natural G_i \subseteq G_i$ be a set coset representatives for G_i/H containing 1 and $_\sharp G_i = {}_\natural G_i \backslash \{1\}$, $i = 1, 2$. Let $\mathcal{M}$ the free monoid generated by $_\sharp G_1 \sqcup {}_\sharp G_2 \sqcup H$, and let

$$\Omega_{G_1,G_1} = \{a_0 b_1 a_1 \ldots b_t a_t h \mid t \geq 0,\ a_i \in {}_\sharp G_1,\ b_i \in {}_\sharp G_2,\ h \in H\}, \tag{4.6}$$

$$\Omega_{G_1,G_2} = \{a_1 b_1 \ldots a_t b_t h \mid t \geq 1,\ a_i \in {}_\sharp G_1,\ b_i \in {}_\sharp G_2,\ h \in H\}, \tag{4.7}$$

$$\Omega_{G_2,G_1} = \{b_1 a_1 \ldots b_t a_t h \mid t \geq 1,\ a_i \in {}_\sharp G_1,\ b_i \in {}_\sharp G_2,\ h \in H\}, \tag{4.8}$$

$$\Omega_{G_2,G_2} = \{b_0 a_1 b_1 \ldots a_t b_t h \mid t \geq 0,\ a_i \in {}_\sharp G_1,\ b_i \in {}_\sharp G_2,\ h \in H\}. \tag{4.9}$$

Put $\Omega = \Omega_{G_1,G_1} \sqcup \Omega_{G_1,G_2} \sqcup \Omega_{G_2,G_1} \sqcup \Omega_{G_2,G_2} \sqcup H$.

Theorem 4.1 *(**Normal Form Theorem**, cf. [27) Let $G = G_1 \coprod_H G_2$, and let $[-] : \mathcal{M} \to G$ be the canonical homomorphism of monoids. Then $[-]|_\Omega : \Omega \to G$ is a bijection.* □

The Normal Form Theorem (Theorem 4.1) establishes that every element of G admits a canonical normal form and chosen the set of coset representatives, $_\sharp G_1$ and $_\sharp G_2$, the normal form is unique. In fact it is trivial to remark that the normal form of an element depends on the choice of the set of coset representatives.

Put $_\natural G = G \setminus H$, the Normal Form Theorem (Theorem 4.1) can be used to define two principal functions: a *rough length* function

$$\rho_{\coprod} : G \to \mathbb{Z}_{\geq 0} \tag{4.10}$$

and a *conductor* function

$$cond([-]) : {}_\sharp G \to {}_\sharp G_1 \coprod {}_\sharp G_2. \tag{4.11}$$

The rough length is defined by

$$\rho_{\coprod}(g) := \begin{cases} 2t+1 \;\; g = [a_0 b_1 a_1 \dots b_t a_t h] \in \Omega_{G_1,G_1} \\ 2t \;\; g = [a_1 b_1 \dots a_t b_t h] \in \Omega_{G_1,G_2} \\ 2t \;\; g = [b_1 a_1 \dots b_t a_t h] \in \Omega_{G_2,G_1} \\ 2t+1 \;\; g = [b_0 a_1 b_1 \dots a_t b_t h] \in \Omega_{G_2,G_2} \\ 0 \;\; g \in H \end{cases} \tag{4.12}$$

It does not depend on the choice of the sets of coset representatives $_\natural G_1$ and $_\natural G_2$. The conductor is defined by

$$cond([-])(g) := \begin{cases} a_0 \;\; g = [a_0 b_1 a_1 \dots b_t a_t h] \in \Omega_{G_1,G_1} \\ a_1 \;\; g = [a_1 b_1 \dots a_t b_t h] \in \Omega_{G_1,G_2} \\ b_1 \;\; g = [b_1 a_1 \dots b_t a_t h] \in \Omega_{G_2,G_1} \\ b_0 \;\; g = [b_0 a_1 b_1 \dots a_t b_t h] \in \Omega_{G_2,G_2} \end{cases} \tag{4.13}$$

On the contrary, it depends on the choice of the sets of coset representatives $_\natural G_1$ and $_\natural G_2$.
The connection between the rough length function and the conductor function of an element $g \in G \backslash H$ for $G = G_1 \coprod_H G_2$ is showed in the following proposition.

Proposition 4.1 *(cf. [2]) Let $G = G_1 \coprod_H G_2$, and $_\sharp G_1$ and $_\sharp G_2$.*

(a) Let $g \in \Omega_{G_1,G_1}$ and $g = a_0 b_1 a_1 \dots b_k a_k h$ for $k \geq 0$, $a_i \in G_1$, $b_i \in G_2$ and $h \in H$. Then $2k+1 \geq \rho_{\coprod}(g)$. Moreover, if $2k+1 = \rho_{\coprod}(g)$, then $a_0 \in cond(g)H$.

(b) Let $g \in \Omega_{G_1,G_2}$ and $g = a_1 b_1 \dots a_k b_k h$ for $k \geq 1$, $a_i \in G_1$, $b_i \in G_2$ and $h \in H$. Then $2k \geq \rho_{\coprod}(g)$. Moreover, if $2k = \rho_{\coprod}(g)$, then $a_1 \in cond(g)H$.

(c) Let $g \in \Omega_{G_2,G_1}$ and $g = b_1 a_1 \dots b_k a_k h$ for $k \geq 1$, $a_i \in G_1$, $b_i \in G_2$ and $h \in H$. Then $2k \geq \rho_{\coprod}(g)$. Moreover, if $2k+1 = \rho_{\coprod}(g)$, then $b_1 \in cond(g)H$.

(d) Let $g \in \Omega_{G_2,G_2}$ and $g = b_0 a_1 b_1 \dots a_k b_k h$ for $k \geq 0$, $a_i \in G_1$, $b_i \in G_2$ and $h \in H$. Then $2k+1 \geq \rho_{\coprod}(g)$. Moreover, if $2k+1 = \rho_{\coprod}(g)$, then $b_0 \in cond(g)H$.

Proof

(a) Suppose $2k+1 < \rho_{\coprod}(g)$. The canonical reduction process yield a normal form $g = x_0'y_1' \cdots y_k'x_k'h'$ (allowing $x_0', x_k' \in \natural G_1$). Thus, $\rho_{\coprod}(g) \leq 2k+1$, a contradiction. Therefore, $2k+1 \geq \rho_{\coprod}(g)$. If $2k+1 = \rho_{\coprod}(g)$, the canonical reduction process yield the normal form of $g \in G$. Hence, by the uniqueness of the normal form, $x_0 \in cond(g)H$. The statements (b), (c) and (d) are proved by a similar argument. □

4.2 Coxeter systems with an ∞-decomposition

Definition 4.2 *Let (W, S) be a Coxeter system and $(S_\blacktriangledown, S_\blacktriangle)$ be a pair of subsets of S satisfying:*

1. $S = S_\blacktriangledown \cup S_\blacktriangle$;
2. *for all $s \in S_\triangledown := S_\blacktriangledown \setminus S_\bullet$ and $t \in S_\vartriangle := S_\blacktriangle \setminus S_\bullet$ one has $m_{s,t} = \infty$, where $(m_{s,t})_{s,t \in S}$ is the Coxeter matrix of (W, S).*

The pair $(S_\blacktriangledown, S_\blacktriangle)$ is called an ∞-decomposition of (W, S). □

A decomposition as in Definition 4.2 is said to be:

1. *non-trivial* if $S_\triangledown \neq \emptyset$ and $S_\vartriangle \neq \emptyset$;
2. a *spherical* ∞-decomposition if additionally the parabolic subgroup $W_\bullet = W_{S_\bullet}$ is finite.

Fact 4.1 *Let (W, S), $S = S_\blacktriangledown \cup S_\blacktriangle$ be a Coxeter system with an ∞-decomposition. Put $S_\bullet = S_\blacktriangledown \cap S_\blacktriangle$ and $W_\times = W_{S_\times}$ for $\times \in \{\blacktriangledown, \blacktriangle, \bullet\}$. Then the canonical map*

$$\phi : W_\blacktriangledown \coprod_{W_\bullet} W_\blacktriangle \to W$$

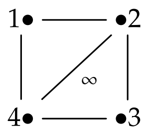
is an isomorphism. □

Example 4.2 *The Coxeter group W of isomorphism class of type $\tilde{A}_1$*

$$1\bullet \overset{\infty}{\text{———}} \bullet 2$$

has a spherically ∞-decomposition.
It is a free products of two groups $W_\blacktriangledown$ and $W_\blacktriangle$ of isomorphism class A_1, in fact $W_\bullet = \{1\}$.
□

Example 4.3 *The Coxeter group $\tilde{\mathcal{W}}$*



(4.14)

has a spherically ∞-decomposition.
This group is a free products of two groups $W_\blacktriangledown$ and $W_\blacktriangle$ of isomorphism class A_3 with amalgamated group $W_\bullet$ of isomorphism class $A_1 \times A_1$. □

Let (W, S), $S = S_\blacktriangledown \cup S_\blacktriangle$ and $S_\bullet = S_\blacktriangledown \cap S_\blacktriangle$, be a Coxeter system with an ∞-decomposition. By choosing ${}_\natural W^\bullet_\blacktriangledown = W^\bullet_\blacktriangledown \setminus \{1\}$ that is the set of representatives of the non-trivial $W_\blacktriangledown/W_\bullet$-cosets (respectively ${}_\natural W^\bullet_\blacktriangle = W^\bullet_\blacktriangle \setminus \{1\}$ and $W_\blacktriangle/W_\bullet$) one obtains, from Normal Form Theorem (Theorem 4.1), a canonical normal form for elements in W.
Let $\Omega_{\blacktriangledown,\blacktriangledown} = \Omega_{W_\blacktriangledown,W_\blacktriangledown}$, $\Omega_{\blacktriangledown,\blacktriangle} = \Omega_{W_\blacktriangledown,W_\blacktriangle}$, etc... In particular,

$$W = \Omega_{\blacktriangledown,\blacktriangledown} \sqcup \Omega_{\blacktriangledown,\blacktriangle} \sqcup \Omega_{\blacktriangle,\blacktriangledown} \sqcup \Omega_{\blacktriangle,\blacktriangle} \sqcup W_\bullet. \tag{4.15}$$

Remind that every element $w \in W$, $w \neq 1$ has a reduced expression (cf. Section 1.2):

$$w = s_1 \dots s_r, \;\; s_i \in S \tag{4.16}$$

where r is the minimal number such that this expression for w exists. Therefore the length function can be defined in the following way

$$\ell : W \to \mathbb{N}_0 \tag{4.17}$$

such that $\ell(w) = r$ and $\ell(1) = 0$ (cf. Section 1.2).
Remind also the following property of the length function, that comes from Proposition 1.11: let W' be a parabolic subgroup of W, then

$$\ell(w'w) = \ell(w') + \ell(w) \tag{4.18}$$

for all $w' \in W'$ and for all $w \in W \setminus W'$.
We can define the *conductor function* for the group W with ∞-decomposition as in (4.13):

$$cond([-]) : W \setminus W_\bullet \longrightarrow {}_\natural W^\bullet_\blacktriangledown \sqcup {}_\natural W^\bullet_\blacktriangle. \tag{4.19}$$

The function

$$c\ell = \ell \circ cond([-]) : W \setminus W_\bullet \longrightarrow \mathbb{Z}_{>1} \tag{4.20}$$

is called the *conductor length function*.

The following property is a direct consequence of Proposition 4.1 and of (4.18).

Proposition 4.2 *Let (W,S), $S = S_{\blacktriangledown} \cup S_{\blacktriangle}$, $S_{\bullet} = S_{\blacktriangledown} \cap S_{\blacktriangle}$, be a Coxeter system with ∞-decomposition.*

(a) Let $w \in \Omega_{\blacktriangledown,\blacktriangledown}$ and $w = x_0y_1 \cdots y_kx_k$ for $k \geq 0$, $x_i \in W_{\blacktriangledown}$, and $y_j \in W_{\blacktriangle}$. Then $2k+1 \geq \rho_{\coprod}(w)$. Moreover, if $2k+1 = \rho_{\coprod}(w)$, then $\ell(x_0) \geq c\ell(w)$.

(b) Let $w \in \Omega_{\blacktriangledown,\blacktriangle}$ and $w = x_1y_1 \cdots x_ky_k$ for $k \geq 1$, $x_i \in W_{\blacktriangledown}$, and $y_j \in W_{\blacktriangle}$. Then $2k \geq \rho_{\coprod}(w)$. Moreover, if $2k = \rho_{\coprod}(w)$, then $\ell(x_1) \geq c\ell(w)$.

(c) Let $w \in \Omega_{\blacktriangle,\blacktriangledown}$ and $w = y_1x_1 \cdots y_kx_k$ for $k \geq 1$, $x_i \in W_{\blacktriangledown}$, and $y_j \in W_{\blacktriangle}$. Then $2k \geq \rho_{\coprod}(w)$. Moreover, if $2k = \rho_{\coprod}(w)$, then $\ell(y_1) \geq c\ell(w)$.

(d) Let $w \in \Omega_{\blacktriangle,\blacktriangle}$ and $w = y_0x_1 \cdots x_ky_k$ for $k \geq 0$, $x_i \in W_{\blacktriangledown}$, and $y_j \in W_{\blacktriangle}$. Then $2k+1 \geq \rho_{\coprod}(w)$. Moreover, if $2k+1 = \rho_{\coprod}(w)$, then $\ell(y_0) \geq c\ell(w)$.

□

For the Coxeter group W with ∞-decomposition, a *fake length function* can be defined

$$\ell_{\coprod} : W \longrightarrow \mathbb{Z}_{\geq 0}. \tag{4.21}$$

For every element $w \in W$ there exist $t \geq 0$, $a_0, a_t \in W^{\bullet}_{\blacktriangledown}$, $a_i \in {}_{\sharp}W^{\bullet}_{\blacktriangledown}$ for $1 \leq i \leq t-1$, $b_j \in {}_{\sharp}W^{\bullet}_{\blacktriangle}$ for $1 \leq j \leq t$, and $c \in W_{\bullet}$ such that

$$w = a_0b_1 \cdots b_ta_tc, \tag{4.22}$$

and the expression (4.22) is unique for the element $w \in W$. For $w \in W$ given by (4.22) we define

$$\ell_{\coprod} = \sum_{0 \leq i \leq t} \ell(a_i) + \sum_{1 \leq j \leq t} \ell(b_j) + \ell(c). \tag{4.23}$$

Thus, by definition, one has $\ell(w) \leq \ell_{\coprod}(w)$ for all $w \in W$.

Theorem 4.2 *(cf. [2]) Let (W,S), $S = S_{\blacktriangledown} \cup S_{\blacktriangle}$, $S_{\bullet} = S_{\blacktriangledown} \cap S_{\blacktriangle}$, be a Coxeter system with an ∞-decomposition. Then $\ell = \ell_{\coprod}$.*

Proof We proceed by induction on $n = \rho_{\coprod}(w)$ in order to show that $\ell(w) = \ell_{\coprod}(w)$. For $n = 0$ one has $w \in W_{\bullet}$ and hence there is nothing to show, while for $n = 1$ the claim is a direct consequence of (4.18). Hence we may assume that one has $\ell(w') = \ell_{\coprod}(w')$ for elements $w' \in W$ satisfying $\rho_{\coprod}(w') < n$, and that $w \in W$, $\rho_{\coprod}(w) = n \geq 2$. As the cases $cond(w) \in {}_{\sharp}W^{\bullet}_{\blacktriangledown}$ and $cond(w) \in {}_{\sharp}W^{\bullet}_{\blacktriangle}$ are almost identical, we restrict our attention to the case $q = cond(w) \in {}_{\sharp}W^{\bullet}_{\blacktriangledown}$. Let $w = q\tau$. In particular, $\rho_{\coprod}(\tau) = n-1$, and thus, by hypothesis, $\ell(\tau) = \ell_{\coprod}(\tau)$.
Let $\tau = b_1a_1 \cdots b_ta_tc$ where $a_i \in {}_{\sharp}W^{\bullet}_{\blacktriangledown}$ for $1 \leq i \leq t-1$ and $a_t \in W^{\bullet}_{\blacktriangledown}$, $b_j \in {}_{\sharp}W^{\bullet}_{\blacktriangle}$ for

$1 \le j \le t$, and $c \in W_\bullet$. As $\rho_{\coprod}(\tau) \ge 1$, one has $t \ge 1$. Moreover, $n = \rho_{\coprod}(w) = 2t + 1 - \delta_{0,\ell(a_t)}$, where $\delta_{0,\ell(a_t)}$ is the Kronecker delta. Let

$$q = s_1 \cdots s_{\ell(q)},\ b_j = x_1^{(j)} \cdots x_{\ell(b_j)}^{(j)},\ a_i = u_1^{(i)} \cdots u_{\ell(a_i)}^{(i)},\ c = v_1 \cdots v_{\ell(c)}$$

be minimal expressions. Note that $\ell(a_r) = 0$ and $\ell(c) = 0$ are allowed. It suffices to show that

$$w = s_1 \cdots s_{\ell(q)} \cdot x_1^{(1)} \cdots x_{\ell(b_1)}^{(1)} \cdots\cdots u_1^{(t)} \cdots u_{\ell(a_t)}^{(t)} \cdot v_1 \cdots v_{\ell(c)}$$

is a minimal expression for w. Let $w_0 \in W$ be such that $w = s_1 w_0$. Again we proceed by induction on $k = c\ell(w) = \ell(q) \ge 1$.
If $k = 1$, then $w_0 = \tau$, and the claim holds for w_0. Hence

$$w_0 = x_1^{(1)} \cdots x_{\ell(b_1)}^{(1)} \cdots\cdots u_1^{(t)} \cdots u_{\ell(a_t)}^{(t)} \cdot v_1 \cdots v_{\ell(c)} \tag{4.24}$$

is a minimal expression for w_0. So either $\ell(w) = \ell(w_0) + 1$, and the claim holds for w, or $\ell(w) = \ell(w_0) - 1$. In the latter case the Strong Exchange Condition (Theorem 1.3) implies that one has $w = (w_0)_*^{\vee}$, where $\overset{\vee}{*}$ means omitting the involution s_* in the minimal expression (4.24). In particular, $w = b_1' a_1' \cdots b_t' a_t' c'$, $a_i \in W_\blacktriangledown$, $b_j \in W_\blacktriangle$, $c' \in W_\bullet$, and $a_t' = 1$ if $a_t = 1$, which is impossible by Proposition 4.2 (*a*) and (*b*).
Thus we may assume that $k > 1$, and that the claim holds for all elements $w' \in W$ satisfying $c\ell(w') < k$. In particular, it holds for w_0. Again either $\ell(w) = \ell(w_0) + 1$, and the claim holds for w, or $\ell(w) = \ell(w_0) - 1$. In the latter case the Strong Exchange Condition (Theorem 1.3) implies that $w = (w_0)_*^V$, where $*$ is one of the involutions in the minimal expression

$$w_0 = s_2 \cdots s_k \cdot x_1^{(1)} \cdots x_{\ell(b_1)}^{(1)} \cdots\cdots u_1^{(t)} \cdots u_{\ell(a_t)}^{(t)} \cdot v_1 \cdots v_{\ell(c)}.$$

In particular, $w = a_0' b_1' a_1' \cdots b_t' a_t'$, $a_i \in W_\blacktriangledown$, $b_j \in W_\blacktriangle$, $a_t' = 1$ if $a_t = 1$. Moreover, $a_0' = (s_2 \cdots s_k)_*^V$ and thus $\ell(a_0') < c\ell(w)$. As $\rho_{\coprod}(w) = 2t + 1 - \delta_{0,\ell(a_t)}$, part (*a*) and (*b*) of Proposition 4.2 show that this is impossible. □

Corollary 4.1 *(cf. [2]) Let (W, S), $S = S_\blacktriangledown \cup S_\blacktriangle$, $S_\bullet = S_\blacktriangledown \cap S_\blacktriangle$, be a Coxeter system with an ∞-decomposition. For $t \ge 0$, $a_0, a_t \in W_\blacktriangledown^\bullet$, $a_i \in {}_\sharp W_\blacktriangledown^\bullet$ for $1 \le i \le t-1$, $b_j \in {}_\sharp W_\blacktriangle^\bullet$ for $1 \le j \le t$ and $c \in W_\bullet$ let*

$$a_i = s_1^{(i)} \cdots s_{\ell(a_i)}^{(i)},\ b_j = x_1^{(j)} \cdots x_{\ell(b_j)}^{(j)},\ c = u_1 \cdots u_{\ell(c)},$$

$s_i, t_j, u_k \in S$, be reduced expressions. Then

$$w = s_1^{(0)} \cdots s_{\ell(a_0)}^{(0)} \cdot x_1^{(1)} \cdots x_{\ell(b_1)}^{(1)} \cdots\cdots s_1^{(t)} \cdots s_{\ell(a_t)}^{(t)} \cdot u_1 \cdots u_{\ell(c)}$$

is a reduced expression. □

4.3 Cocycle series

We recall that for a Kac-Moody Lie algebra with Weyl group W and root system Φ the Kac′ denominator formula (cf. Chapter 2) is

$$\sum_{w \in W} (-1)^{\ell(w)} e(c(w)) = \prod_{\alpha \in \Phi^+} (1 - e(\alpha))^{m_\alpha}, \tag{4.25}$$

where

$$c : W \to \Lambda_{\mathbb{Z}}^+ \tag{4.26}$$

is the cocycle (cf. Section 2.4.1) that maps an element $w \in W$ in the sum of positive roots being sent by w^{-1} to negative ones.
c can also be defined through the following formula

$$c(w) = (1 - w) \cdot \omega_0, \tag{4.27}$$

where ω_0 is the trace of dual basis of Φ (cf. Section 2.4.1).
In this section, we will observe that the left hand side of (4.25) can be interpreted as a formal power series depending entirely on (W, S).
We define six elements.

$$\mathcal{A}_\blacktriangledown := \sum_{a \in {}_\natural W^\bullet_\blacktriangledown} t^{\ell(a)} \tag{4.28}$$

$$\mathcal{A}_\blacktriangle := \sum_{b \in {}_\natural W^\bullet_\blacktriangle} t^{\ell(b)} \tag{4.29}$$

$$\mathcal{A}_\bullet := \sum_{c \in W_\bullet} t^{\ell(c)} \tag{4.30}$$

They are elements of the ring $\mathbb{Z}[\![t]\!]$ of the formal power series whose variable is t and whose coefficients come from the ring $\mathbb{Z}$.

$$\tilde{\mathcal{A}}_\blacktriangledown := \sum_{a \in {}_\natural W^\bullet_\blacktriangledown} t^{\ell(a)} a \tag{4.31}$$

$$\tilde{\mathcal{A}}_\blacktriangle := \sum_{b \in {}_\natural W^\bullet_\blacktriangle} t^{\ell(b)} b \tag{4.32}$$

$$\tilde{\mathcal{A}}_{\bullet} := \sum_{c \in W_{\bullet}} t^{\ell(c)} c \tag{4.33}$$

They are elements of $\mathbb{Z}[W][\![t]\!]$ the ring of the formal power series with coefficients in the group ring $\mathbb{Z}[W]$.
We call the series

$$C_W(t) := \sum_{w \in W} t^{\ell(w)} c(w) \tag{4.34}$$

cocycle series for the group W. It is an element of the ring $\mathbb{Z}[\Lambda^+_{\mathbb{Z}}][\![t]\!]$ of the formal power series whose variable is t and whose coefficients come from the group ring $\mathbb{Z}[\Lambda^+_{\mathbb{Z}}]$. It is treated multiplicatively letting an element $\alpha \in \Lambda^+_{\mathbb{Z}}$ as the formal exponential $e(\alpha)$ (cf. [4]).
Let

$$W(t) := \sum_{w \in W} t^{\ell(w)} \in \mathbb{Z}[\![t]\!] \subseteq \mathbb{Z}[W][\![t]\!] \tag{4.35}$$

be the *growth series* for the group W (cf. Section 1.12.1) and

$$\tilde{W}(t) := \sum_{w \in W} t^{\ell(w)} w \in \mathbb{Z}[W][\![t]\!] \tag{4.36}$$

be the *complete growth series* for the group W (cf. Section 1.12.2).
From (4.27), we write

$$C_W(t) = (W(t) - \tilde{W}(t)) \cdot \omega_0 \in \mathbb{Z}[\Lambda^+_{\mathbb{Z}}][\![t]\!] \tag{4.37}$$

with

$$W(t) = \mathcal{A}_{\blacktriangledown}(\sum_{i=0}(\mathcal{A}_{\blacktriangle}\mathcal{A}_{\blacktriangledown})^i)\mathcal{A}_{\bullet}+(\sum_{i=1}(\mathcal{A}_{\blacktriangle}\mathcal{A}_{\blacktriangledown})^i)\mathcal{A}_{\bullet}+(\sum_{i=1}(\mathcal{A}_{\blacktriangledown}\mathcal{A}_{\blacktriangle})^i)\mathcal{A}_{\bullet}+\mathcal{A}_{\blacktriangle}(\sum_{i=0}(\mathcal{A}_{\blacktriangledown}\mathcal{A}_{\blacktriangle})^i)\mathcal{A}_{\bullet}+\mathcal{A}_{\bullet} \tag{4.38}$$

and

$$\tilde{W}(t) = \tilde{\mathcal{A}}_{\blacktriangledown}(\sum_{i=0}(\tilde{\mathcal{A}}_{\blacktriangle}\tilde{\mathcal{A}}_{\blacktriangledown})^i)\tilde{\mathcal{A}}_{\bullet}+(\sum_{i=1}(\tilde{\mathcal{A}}_{\blacktriangle}\tilde{\mathcal{A}}_{\blacktriangledown})^i)\tilde{\mathcal{A}}_{\bullet}+(\sum_{i=1}(\tilde{\mathcal{A}}_{\blacktriangledown}\tilde{\mathcal{A}}_{\blacktriangle})^i)\tilde{\mathcal{A}}_{\bullet}+\tilde{\mathcal{A}}_{\blacktriangle}(\sum_{i=0}(\tilde{\mathcal{A}}_{\blacktriangledown}\tilde{\mathcal{A}}_{\blacktriangle})^i)\tilde{\mathcal{A}}_{\bullet}+\tilde{\mathcal{A}}_{\bullet} \tag{4.39}$$

as a formal power series of the elements (4.28), (4.29), (4.30), (4.31), (4.32) and (4.33).
In this section, we will see that the formula (4.37) for $\tilde{A}_1$ is in analogy to Kac' denominator formula for $\tilde{A}_1$ shown by V. G. Kac (cf. [8]).

4.3.1 Cocycle series of $\tilde{A}_1$

Let W be the Coxeter group of isomorphism class of type $\tilde{A}_1$ with Coxeter graph

$$1\bullet \overset{\infty}{\text{———}} \bullet 2 \tag{4.40}$$

As we seen in Example 4.2, it has a spherically ∞-decomposition: it is a free product of two groups

$$W_{\blacktriangledown} = \langle s_1 \mid s_1^2 = 1\rangle \tag{4.41}$$

$$W_{\blacktriangle} = \langle s_2 \mid s_2^2 = 1\rangle \tag{4.42}$$

of isomorphism class A_1, in fact $W_{\bullet} = \{1\}$.
In particular we can consider W as a free product of groups $W_{\blacktriangledown}$ and $W_{\blacktriangle}$ with amalgamation of the trivial group $W_{\bullet} = \{1\}$.

$$W \simeq W_{\blacktriangledown} \coprod_{W_{\bullet}} W_{\blacktriangle} = \langle s_1, s_2 \mid s_i^2 = 1,\ i = 1,2\rangle. \tag{4.43}$$

Normal Form Theorem (Theorem 4.1) allows a canonical normal form for the elements in W.
We write $\overline{g}$ for the representative of the coset $gW_{\bullet}$:

$$1W_{\bullet} = \{1\} = \overline{1} \tag{4.44}$$

$$s_1 W_{\bullet} = \{s_1\} = \overline{s_1} \tag{4.45}$$

$$s_2 W_{\bullet} = \{s_2\} = \overline{s_2} \tag{4.46}$$

Then:

$$\Omega_{\blacktriangledown,\blacktriangledown} = \{s_1(s_2s_1)^r \mid r \geq 0\} \tag{4.47}$$

$$\Omega_{\blacktriangledown,\blacktriangle} = \{(s_1s_2)^r \mid r \geq 1\} \tag{4.48}$$

$$\Omega_{\blacktriangle,\blacktriangledown} = \{(s_2s_1)^r \mid r \geq 1\} \tag{4.49}$$

$$\Omega_{\blacktriangle,\blacktriangle} = \{s_2(s_1s_2)^r \mid r \geq 0\} \tag{4.50}$$

and

$$W = \Omega_{\blacktriangledown,\blacktriangledown} \sqcup \Omega_{\blacktriangledown,\blacktriangle} \sqcup \Omega_{\blacktriangle,\blacktriangledown} \sqcup \Omega_{\blacktriangle,\blacktriangle} \sqcup W_{\bullet}. \tag{4.51}$$

Then the left hand-side of the Kac' denominator formula (2.20) can be divided in four parts:

$$\sum_{w\in W} (-1)^{l(w)} e^{c(w)} =$$

$$\sum_{w\in W_\bullet}(-1)^{l(w)}e^{c(w)}+\sum_{w\in\Omega_{\blacktriangledown,\blacktriangledown}}(-1)^{l(w)}e^{c(w)}+\sum_{w\in\Omega_{\blacktriangledown,\blacktriangle}}(-1)^{l(w)}e^{c(w)}+\sum_{w\in\Omega_{\blacktriangle,\blacktriangledown}}(-1)^{l(w)}e^{c(w)}+\sum_{w\in\Omega_{\blacktriangle,\blacktriangle}}(-1)^{l(w)}e^{c(w)}. \tag{4.52}$$

We calculate the following ones:

$$c(s_1(s_2s_1)^r)=(\sum_{i=0}^{r}(s_1s_2)^i)c(s_1)+s_1(\sum_{i=0}^{r-1}(s_2s_1)^i)c(s_2) \tag{4.53}$$

$$c(s_2(s_1s_2)^r)=(\sum_{i=0}^{r}(s_2s_1)^i)c(s_2)+s_2(\sum_{i=0}^{r-1}(s_1s_2)^i)c(s_1) \tag{4.54}$$

$$c((s_1s_2)^r)=(\sum_{i=0}^{r}(s_1s_2)^i)c(s_1)+s_1(\sum_{i=0}^{r}(s_2s_1)^i)c(s_2) \tag{4.55}$$

$$c((s_2s_1)^r)=(\sum_{i=0}^{r}(s_2s_1)^i)c(s_2)+s_2(\sum_{i=0}^{r}(s_1s_2)^i)c(s_1) \tag{4.56}$$

and, substituting in (4.52), obtain

$$\sum_{w\in W}(-1)^{l(w)}e^{c(w)}=$$

$$1-\sum_{r\geq 0}e^{(\sum_{i=0}^{r}(s_1s_2)^i)\alpha_1+s_1(\sum_{i=0}^{r-1}(s_2s_1)^i)\alpha_2}-\sum_{r\geq 0}e^{(s_2(\sum_{i=0}^{r-1}(s_1s_2)^i)\alpha_1+\sum_{i=0}^{r}(s_2s_1)^i)\alpha_2}+ \tag{4.57}$$
$$+\sum_{r\geq 1}e^{(\sum_{i=0}^{r}(s_1s_2)^i)\alpha_1+s_1(\sum_{i=0}^{r}(s_2s_1)^i)\alpha_2}+\sum_{r\geq 1}e^{s_2(\sum_{i=0}^{r}(s_1s_2)^i)\alpha_1+(\sum_{i=0}^{r}(s_2s_1)^i)\alpha_2}$$

The set (4.47) contains elements of odd length $2r+1$ with $r\geq 0$.
Let $w=s_1(s_2s_1)^r\in\Omega_{\blacktriangledown,\blacktriangledown}$, the positive roots being sent by w^{-1} to negative ones are:

$$\alpha_1, 2\alpha_1+\alpha_2,\ldots,(2r+1)\alpha_1+2r\alpha_2\ \ (r\geq 0). \tag{4.58}$$

Then

$$c(s_1(s_2s_1)^r)=\frac{(2r+1)(2r+2)}{2}\alpha_1+\frac{2r(2r+1)}{2}\alpha_2 \tag{4.59}$$

and

$$\sum_{w\in\Omega_{\blacktriangledown,\blacktriangledown}}(-1)^{l(w)}e^{c(w)}=-\sum_{r\geq 0}e^{\frac{(2r+1)(2r+2)}{2}\alpha_1+\frac{2r(2r+1)}{2}\alpha_2}=-\sum_{r\geq 0}X^{\frac{(2r+1)(2r+2)}{2}}Y^{\frac{2r(2r+1)}{2}}, \tag{4.60}$$

with $X=e^{\alpha_1}$ and $Y=e^{\alpha_2}$.
Similarly, if $w=s_2(s_1s_2)^r\in\Omega_{\blacktriangle,\blacktriangle}$, then

$$\sum_{w\in\Omega_{\blacktriangle,\blacktriangle}}(-1)^{l(w)}e^{c(w)}=-\sum_{r\geq 0}e^{\frac{2r(2r+1)}{2}\alpha_1+\frac{(2r+1)(2r+2)}{2}\alpha_2}=-\sum_{r\geq 0}X^{\frac{2r(2r+1)}{2}}Y^{\frac{(2r+1)(2r+2)}{2}}. \tag{4.61}$$

The set (4.48) contains elements of even length $2r$ with $r \geq 1$.
Let $w = (s_1 s_2)^r \in \Omega_{\blacktriangledown,\blacktriangle}$, the positive roots being sent by w^{-1} to negative ones are:

$$\alpha_1, 2\alpha_1 + \alpha_2, \ldots, 2r\alpha_1 + (2r-1)\alpha_2 \ \ (r \geq 1). \tag{4.62}$$

Then

$$c((s_1 s_2)^r) = \frac{2r(2r+1)}{2}\alpha_1 + \frac{(2r-1)2r}{2}\alpha_2 \tag{4.63}$$

and

$$\sum_{w \in \Omega_{\blacktriangle,\blacktriangledown}} (-1)^{l(w)} e^{c(w)} = \sum_{r \geq 1} e^{r(2r+1)\alpha_1 + (2r-1)r\alpha_2} = \sum_{r \geq 1} X^{r(2r+1)} Y^{(2r-1)r}. \tag{4.64}$$

Similarly, if $w = (s_2 s_1)^r \in \Omega_{\blacktriangle,\blacktriangledown}$:

$$\sum_{w \in \Omega_{\blacktriangledown,\blacktriangle}} (-1)^{l(w)} e^{c(w)} = \sum_{r \geq 1} e^{(2r-1)r\alpha_1 + r(2r+1)\alpha_2} = \sum_{r \geq 1} X^{(2r-1)r} Y^{r(2r+1)}. \tag{4.65}$$

Therefore the Kac' denominator formula for W of isomorphism class of type $\tilde{A}_1$, written using this reasoning, is

$$\sum_{w \in W} (-1)^{l(w)} e^{c(w)} = \tag{4.66}$$

$$1 - \sum_{r \geq 0} X^{\frac{(2r+1)(2r+2)}{2}} Y^{\frac{2r(2r+1)}{2}} - \sum_{r \geq 0} X^{\frac{2r(2r+1)}{2}} Y^{\frac{(2r+1)(2r+2)}{2}} +$$
$$+ \sum_{r \geq 1} X^{(2r-1)r} Y^{r(2r+1)} + \sum_{r \geq 1} X^{r(2r+1)} Y^{(2r-1)r}$$
$$= 1 - X\Big(\sum_{r \geq 0} X^{\frac{(2r+1)(2r+2)}{2} - 1} Y^{\frac{2r(2r+1)}{2}}\Big) - Y\Big(\sum_{r \geq 0} X^{\frac{2r(2r+1)}{2}} Y^{\frac{(2r+1)(2r+2)}{2} - 1}\Big) +$$
$$+ XY\Big(\sum_{r \geq 1} X^{(2r-1)r-1} Y^{r(2r+1)-1} + \sum_{r \geq 1} X^{r(2r+1)-1} Y^{(2r-1)r-1}\Big)$$

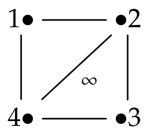
where $1 - X - Y - XY = \sum_{w \in W'} (-1)^{l(w)} e^{c(w)}$ where W' is a Coxeter group of isomorphism class $A_1 \times A_1$.
In [8], the Kac' denominator formula for W of isomorphism class $\tilde{A}_1$ is

$$\sum_{m \in \mathbb{Z}} X^{\frac{m(m-1)}{2}} Y^{\frac{m(m+1)}{2}} = \prod_{n > 0} (1 - X^n Y^n)(1 - X^{n-1} Y^n)(1 - X^n Y^{n-1}). \tag{4.67}$$

One observes that formula (4.66) is equivalent to the right part of formula (4.67).

4.3.2 Cocycle series of $\mathcal{W}$

Let $(\mathcal{W}, \mathcal{S})$ be the Coxeter group with Coxeter graph $\Gamma(\mathcal{W})$



(4.68)

As we seen in Example 4.3, it has a spherically ∞-decomposition: it is a free products of two groups

$$\mathcal{W}_{\blacktriangledown} = \langle x_1, x_2, x_3 \mid (x_1x_2)^3 = (x_2x_3)^3 = (x_1x_3)^2 = x_i^2 = 1,\ i = 1,2,3\rangle \tag{4.69}$$

$$\mathcal{W}_{\blacktriangle} = \langle y_1, y_2, y_3 \mid (y_1y_2)^3 = (y_2y_3)^3 = (y_1y_3)^2 = y_i^2 = 1,\ i = 1,2,3\rangle \tag{4.70}$$

of isomorphism class A_3 with amalgamation group

$$\mathcal{W}_{\bullet} = \langle w_1, w_2 \mid (w_1w_2)^2 = w_i^2 = 1,\ i = 1,2\rangle. \tag{4.71}$$

of isomorphism class $A_1 \times A_1$.
Now we consider the following group homomorphisms:

$$\phi_{\blacktriangledown} : \mathcal{W}_{\bullet} \to \mathcal{W}_{\blacktriangledown} \qquad \phi_{\blacktriangle} : \mathcal{W}_{\bullet} \to \mathcal{W}_{\blacktriangle} \tag{4.72}$$

$$\phi_{\blacktriangledown}(w_1) = x_1 \qquad \phi_{\blacktriangle}(w_1) = y_1$$

$$\phi_{\blacktriangledown}(w_2) = x_3 \qquad \phi_{\blacktriangle}(w_2) = y_3$$

Then

$$\mathcal{W} \simeq \mathcal{W}_{\blacktriangledown} \coprod_{\mathcal{W}_{\bullet}} \mathcal{W}_{\blacktriangle} = \tag{4.73}$$

$$\langle s_1 = x_1 = y_1, s_2 = y_2, s_3 = x_3 = y_3, s_4 = x_2 \mid$$

$$(s_1s_2)^3 = (s_1s_3)^2 = (s_1s_4)^3 = (s_2s_3)^3 = (s_3s_4)^3 = s_i^2 = 1,\ i = 1,2,3,4\rangle.$$

Normal Form Theorem (Theorem 4.1) allows a canonical normal form for the elements in $\mathcal{W}$.
Let $\mathcal{W}^{\bullet}{}_{\times} = {}^{\phi_\times}\mathcal{W}_{\bullet}$ with $\times \in \{\blacktriangledown, \blacktriangle\}$ and we write $\overline{g}$ for the representative of the coset $g\mathcal{W}^{\bullet}{}_{\times}$ (remember that a Coxeter group W of isomorphism class A_3 is the permutation group S_4 and $(i_1 \dots i_t)$ is a permutation of S_4):

$$(1)\mathcal{W}_{\bullet} = \{(1), (12), (34), (12)(34)\} = \{1, s_1, s_3, s_1s_3\} = \overline{1} \tag{4.74}$$

$$(13)\mathcal{W}_{\bullet} = \{(13), (123), (134), (1234)\} = \{s_2, s_2s_1, s_2s_3, s_2s_1s_3\} = \overline{s_2} \tag{4.75}$$

$$(14)\mathcal{W}_\bullet = \{(14),(124),(143),(1243)\} = \{s_2s_3s_2, s_2s_3s_2s_1, s_3s_2, s_3s_2s_1\} = \overline{s_3s_2} \tag{4.76}$$

$$(23)\mathcal{W}_\bullet = \{(23),(132),(234),(1342)\} = \{s_1s_2s_1, s_1s_2, ts_1s_2s_1s_3, s_1s_2s_3\} = \overline{s_1s_2} \tag{4.77}$$

$$(24)\mathcal{W}_\bullet = \{(24),(142),(243),(1432)\} = \tag{4.78}$$

$$\{s_1s_2s_3s_2s_1, s_1s_2s_3s_2, s_1s_3s_2s_1, s_1s_3s_2\} = \overline{s_1s_3s_2}$$

$$(13)(24)\mathcal{W}_\bullet = \{(13)(24),(1423),(1324),(14)(23)\} = \tag{4.79}$$

$$\{s_2s_1s_2s_3s_2s_1, s_2s_1s_2s_3s_2, s_2s_1s_3s_2s_1, s_2s_1s_3s_2\} = \overline{s_2s_1s_3s_2}$$

$$(1)\mathcal{W}_\bullet = \overline{1} \tag{4.80}$$

$$(13)\mathcal{W}_\bullet = \overline{s_4} \tag{4.81}$$

$$(14)\mathcal{W}_\bullet = \{s_4s_3s_4, s_4s_3s_4s_1, s_3s_4, s_3s_4s_1\} = \overline{s_3s_4} \tag{4.82}$$

$$(23)\mathcal{W}_\bullet = \{s_1s_4s_1, s_1s_4, s_1s_4s_1s_3, s_1s_4s_3\} = \overline{s_1s_4} \tag{4.83}$$

$$(24)\mathcal{W}_\bullet = \{s_1s_4s_3s_4s_1, s_1s_4s_3s_4, s_1s_3s_4s_1, s_1s_3s_4\} = \overline{s_1s_3s_4} \tag{4.84}$$

$$(13)(24)\mathcal{W}_\bullet = \{s_4s_1s_4s_3s_4s_1, s_4s_1s_4s_3s_4, s_4s_1s_3s_4s_1, s_4s_1s_3s_4\} = \overline{s_4s_1s_3s_4} \tag{4.85}$$

Then

$$\mathcal{A}_\blacktriangledown := \sum_{a \in {}_\natural W^\bullet_\blacktriangledown} t^{\ell(a)} = t + 2t^2 + t^3 + t^4 \tag{4.86}$$

$$\mathcal{A}_\blacktriangle := \sum_{b \in {}_\natural W^\bullet_\blacktriangle} t^{\ell(b)} = t + 2t^2 + t^3 + t^4 \tag{4.87}$$

$$\mathcal{A}_\bullet := \sum_{c \in \mathcal{W}_\bullet} t^{\ell(c)} = 1 + 2t + t^2 \tag{4.88}$$

$$\tilde{\mathcal{A}}_\blacktriangledown := \sum_{a \in {}_\natural W^\bullet_\blacktriangledown} t^{\ell(a)} a = s_2t + s_3s_2t^2 + s_1s_2t^2 + s_1s_3s_2t^3 + s_2s_1s_3s_2t^4 \tag{4.89}$$

$$\tilde{\mathcal{A}}_\blacktriangle := \sum_{b \in {}_\natural W^\bullet_\blacktriangle} t^{\ell(b)} b = s_4t + s_3s_4t^2 + s_1s_4t^2 + s_1s_3s_4t^3 + s_4s_1s_3s_4t^4 \tag{4.90}$$

$$\tilde{\mathcal{A}}_\bullet := \sum_{c \in \mathcal{W}_\bullet} t^{\ell(c)} c = 1 + s_1t + s_3t + s_1s_3t^2. \tag{4.91}$$

Then the cocycle series for $\mathcal{W}$ in terms of (4.86), (4.87), (4.88), (4.89), (4.90), (4.91) is

$$C_{\mathcal{W}}(t) = (\mathcal{W}(t) - \tilde{\mathcal{W}}(t)) \cdot \omega_0 \tag{4.92}$$

where

$$\mathcal{W}(t) = \mathcal{A}_{\blacktriangledown}(\sum_{i=0}(\mathcal{A}_{\blacktriangle}\mathcal{A}_{\blacktriangledown})^i)\mathcal{A}_{\bullet}+(\sum_{i=1}(\mathcal{A}_{\blacktriangle}\mathcal{A}_{\blacktriangledown})^i)\mathcal{A}_{\bullet}+(\sum_{i=1}(\mathcal{A}_{\blacktriangledown}\mathcal{A}_{\blacktriangle})^i)\mathcal{A}_{\bullet}+\mathcal{A}_{\blacktriangle}(\sum_{i=0}(\mathcal{A}_{\blacktriangledown}\mathcal{A}_{\blacktriangle})^i)\mathcal{A}_{\bullet}+\mathcal{A}_{\bullet} \tag{4.93}$$

is the growth series of $\mathcal{W}$ and

$$\tilde{\mathcal{W}}(t) = \tilde{\mathcal{A}}_{\blacktriangledown}(\sum_{i=0}(\tilde{\mathcal{A}}_{\blacktriangle}\tilde{\mathcal{A}}_{\blacktriangledown})^i)\tilde{\mathcal{A}}_{\bullet}+(\sum_{i=1}(\tilde{\mathcal{A}}_{\blacktriangle}\tilde{\mathcal{A}}_{\blacktriangledown})^i)\tilde{\mathcal{A}}_{\bullet}+(\sum_{i=1}(\tilde{\mathcal{A}}_{\blacktriangledown}\tilde{\mathcal{A}}_{\blacktriangle})^i)\tilde{\mathcal{A}}_{\bullet}+\tilde{\mathcal{A}}_{\blacktriangle}(\sum_{i=0}(\tilde{\mathcal{A}}_{\blacktriangledown}\tilde{\mathcal{A}}_{\blacktriangle})^i)\tilde{\mathcal{A}}_{\bullet}+\tilde{\mathcal{A}}_{\bullet} \tag{4.94}$$

is the complete growth series of $\mathcal{W}$.

Now the problem is, how this interpretation of the left hand-side of the Kac′ denominator formula can be useful to determine the roots and the multiplicities of the Kac-Moody Lie algebra that has $\mathcal{W}$ as Weyl group.
Therefore, using formula (2.38), we calculated the multiplicities for all $\alpha \in \Lambda^+_{\mathbb{Z}}$ such that $ht(\alpha) \leq 4$ and we obtained the following results:

1. Of height 2 there is only one positive imaginary root

$$\alpha_2 + \alpha_4$$

   of multiplicity 1.

2. Of height 3 there are two positive imaginary roots

$$\alpha_1 + \alpha_2 + \alpha_4$$

$$\alpha_2 + \alpha_3 + \alpha_4$$

   of multiplicity 2.

3. Of height 4 there are three positive imaginary roots of multiplicity 1, that are

$$2\alpha_1 + \alpha_2 + \alpha_4 = s_1(\alpha_2 + \alpha_4)$$

$$\alpha_2 + 2\alpha_3 + \alpha_4 = s_3(\alpha_2 + \alpha_4)$$

$$2\alpha_2 + 2\alpha_4 = 2(\alpha_2 + \alpha_4);$$

   four positive imaginary roots of multiplicity 2, that are

$$\alpha_1 + 2\alpha_2 + \alpha_4 = s_1(\alpha_1 + \alpha_2 + \alpha_4)$$

$$2\alpha_2 + \alpha_3 + \alpha_4 = s_3(\alpha_2 + \alpha_3 + \alpha_4)$$

$$\alpha_1 + \alpha_2 + 2\alpha_4 = s_4(\alpha_1 + \alpha_2 + \alpha_4)$$

$$\alpha_2 + \alpha_3 + 2\alpha_4 = s_4(\alpha_2 + \alpha_3 + \alpha_4);$$

one positive imaginary root of multiplicity 4, that is

$$\alpha_1 + \alpha_2 + \alpha_3 + \alpha_4.$$

Despite these calculations we do not find a correlations between formula (4.92) and the right hand-side of the Kac′ denominator formula.

Chapter 5

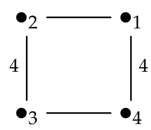
Conclusion

This thesis explored the connection between a crystallographic Coxeter system (W, S) and the roots of an arbitrary Kac-Moody Lie algebra associated with a symmetrizable Generalized Cartan Matrix (GCM) with Weyl group W through the Kac' denominator formula (cf. Chapter 2)

$$\sum_{w\in W}(-1)^{\ell(w)}e(c(w)) = \prod_{\alpha\in\Lambda^+_{\mathbb{Z}}}(1-e(\alpha))^{m_\alpha} \tag{5.1}$$

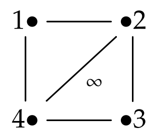
for two particular crystallographic Coxeter groups:

1. a cocompact hyperbolic Coxeter group $(\mathfrak{W}, \mathfrak{S})$ (Chapter 3)



(5.2)

2. a Coxeter group with an ∞-decomposition $(\mathcal{W}, \mathcal{S})$ (Chapter 4)



(5.3)

We know that for the Kac-Moody Lie algebras of finite and affine type the multiplicity of every roots is 1, but for the Kac-Moody Lie algebras of indefinite type the situation is vastly different, due to the exponential growth of the imaginary root spaces.

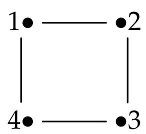
In Chapter 3 we spent much effort to represent $\mathfrak{W}$ as a cocompact arithmetic lattice of $O^+_{\mathbb{R}}(3,1)$, generalising a result of J. Elstrodt, F. Grunewald, J. Mennicke (cf. [11]). We exhibited the generators of $\mathfrak{W}$ in $\langle\sigma\rangle \ltimes PSL(2,\mathbb{C})$ through an explicit formula (cf. Section 3.4.6) generalizing the results of Masaaki Yoshida (cf. [35]). Moreover, generalising some results of A. Feingold, I. Frenkel (cf. [12]), we obteined a description of the root system of a Kac-Moody Lie algebra with Weyl group $\mathfrak{W}$.

Open problem is to find a closed formula for the multiplicities of the imaginary roots for a Kac-Moody Lie algebra with Weyl group $\mathfrak{W}$.

In Chapter 4 we presented an explicit formula for the left hand-side of the Kac' denominator formula (5.1) for the crystallographic Coxeter groups with a spherically ∞-decomposition.

Open problem is to find a connection between the explicit formula for the left hand-side of the Kac' denominator formula (5.1) for the crystallographic Coxeter groups with a spherically ∞-decomposition and the right hand-side, to determine the roots and the multiplicities of the Kac-Moody Lie algebra that has $\mathcal{W}$ as Weyl group.

In order to explain next open problem, we need to introduce some considerations.
Let $(\mathcal{W},\mathcal{S})$ be a Coxeter group with an ∞-decomposition. If we eliminate the edge labelled with ∞ in the graph (5.3) then we obtain an other crystallographic group $(\tilde{\mathcal{W}},\tilde{\mathcal{S}})$



(5.4)

Define an injective homomorphism

$$\pi : \mathcal{W} \longrightarrow \tilde{\mathcal{W}},$$

such that

$$\pi_{I_S} : \mathcal{S} \longrightarrow \tilde{\mathcal{S}}$$

is a bijection, $s_i \longrightarrow \tilde{s}_i$ $(i = 1,2,3,4)$.

Open problem is to find (if it exists) a normal form for the elements of $\tilde{\mathcal{W}}$ knowing a normal form for the elements of $\mathcal{W}$.

It is with the following observations that this problem arises and might be better formalised.

Lemma 5.1 *Let $\ell : \mathcal{W} \to \mathbb{N}_0$ be the length function of $\mathcal{W}$ and $\tilde{\ell} : \tilde{\mathcal{W}} \to \mathbb{N}_0$ be the length function of $\tilde{\mathcal{W}}$. Then $\ell(w) \geq \tilde{\ell}(\pi(w))\ \forall w \in \mathcal{W}$.*

Proof Let $w \in \mathcal{W}$ be a reduced element such that

$$w = s_{i_1} \dots s_{i_k} \underbrace{\underbrace{s_2 s_4} \dots \underbrace{s_2 s_4}}_{h} s_{i_{k+1}} \dots s_{i_r}$$

of length $\ell(w) = r + 2h$.
We distinguish two cases.
If h is even, i.e. $h = 2c$ (c can also be 0) then

$$\pi(w) = \tilde{s}_{i_1} \dots \tilde{s}_{i_k} \tilde{s}_{i_{k+1}} \dots \tilde{s}_{i_r}.$$

Therefore $\tilde{\ell}(\pi(w)) \leq r \leq \ell(w)$, the equality holds when $c = 0$.
If h is odd, i.e. $h = 2c + 1$ (c can also be 0) then

$$\pi(w) = \tilde{s}_{i_1} \dots \tilde{s}_{i_k} \tilde{s}_2 \tilde{s}_4 s_{i_{k+1}} \dots \tilde{s}_{i_r}.$$

Therefore $\tilde{\ell}(\pi(w)) \leq r + 2 \leq \ell(w)$, the equality holds when $c = 0$.
In both cases, we proved that $\ell(w) \geq \tilde{\ell}(\pi(w))$. □

Proposition 5.1 *Let $1 \neq w \in \mathcal{W}$ such that $\pi(w) = \tilde{s}_{i_1} \dots \tilde{s}_{i_r}$ is a reduced element in $\tilde{\mathcal{W}}$, then $w = s_{i_1} \dots s_{i_r}$ is reduced in $\mathcal{W}$.*

Proof We prove the proposition by induction on $\tilde{\ell}(\pi(w))$.
If $\tilde{\ell}(\pi(w)) = 1$, then $\pi(w) = \tilde{s}_i$ and so $w = s_i$ is also a reduced element.
Now we suppose that the proposition holds for all $k < \tilde{\ell}(\pi(w))$. Let $\pi(w) = \tilde{s}_{i_1} \dots \tilde{s}_{i_r}$ be a reduced element, so $\tilde{\ell}(\pi(w)) = r$. We can write $\pi(w) = \pi(w')\tilde{s}_{i_r}$, where $\pi(w') = \tilde{s}_{i_1} \dots \tilde{s}_{i_{r-1}}$ and $\tilde{\ell}(\pi(w')) = r-1$, therefore $w' = s_{i_1} \dots s_{i_{r-1}}$ is a reduced element. We have that $w = w's_{i_r}$ and $\ell(w) = \ell(w') \pm 1 = (r-1) \pm 1 \geq \tilde{\ell}(\pi(w)) = r$ by Lemma 5.1. Then $\ell(w) = r$ and $w = s_{i_1} \dots s_{i_r}$ is a reduced element of $\mathcal{W}$. □

Definition 5.1 *$\forall \tilde{w} \in \tilde{\mathcal{W}}$, we define the set*

$$\Omega_{\tilde{w}} := \{w \in \mathcal{W} \mid \ell(w) = \tilde{\ell}(\tilde{w}) \ : \ \tilde{w} = \pi(w)\} = \{w \in \pi^{-1}(\{\tilde{w}\}) \mid \ell(w) = \tilde{\ell}(\tilde{w})\}$$

□

Proposition 5.2 *Let $\tilde{w} \in \tilde{\mathcal{W}}$ be a reduced element with inside non consecutive h-sequences $\tilde{s}_2\tilde{s}_4$ or $\tilde{s}_4\tilde{s}_2$. Then $|\Omega_{\tilde{w}}| = 2^h$.*

Proof We prove by induction on h.
If $h = 0$, the reduced element $\tilde{w} \in \tilde{\mathcal{W}}$ contains no sequences $\tilde{s}_2\tilde{s}_4$ or $\tilde{s}_4\tilde{s}_2$. Then there is a unique element $w \in \mathcal{W}$ such that $\ell(w) = \tilde{\ell}(\tilde{w})$ and $\pi(w) = \tilde{w}$. Therefore $|\Omega_{\tilde{w}}| = 1 = 2^0$.
If $h = 1$, the reduced element $\tilde{w} \in \tilde{\mathcal{W}}$ contains exactly a sequence $\tilde{s}_2\tilde{s}_4$ or $\tilde{s}_4\tilde{s}_2$. Then there are exactly two elements $w_1, w_2 \in \mathcal{W}$ such that $\ell(w_i) = \tilde{\ell}(\tilde{w})$ and $\tilde{w} = \pi(w_i)$, $i = 1, 2$. In fact in $\tilde{\mathcal{W}}$ we have the equality $\tilde{s}_2\tilde{s}_4 = \tilde{s}_4\tilde{s}_2$, but in $\mathcal{W}$ $s_2s_4 \neq s_4s_2$. Therefore $|\Omega_{\tilde{w}}| = 2 = 2^1$.
Now, we suppose that the proposition holds for an element that contains non consecutive $h-1$-sequences $\tilde{s}_2\tilde{s}_4$ or $\tilde{s}_4\tilde{s}_2$ and we prove that the proposition also is true for h.
Let $\tilde{w} \in \tilde{\mathcal{W}}$ be a reduced element with h-sequences and we separate this element in two reduced elements:

$$\tilde{w} = \underbrace{\tilde{s}_{i_1} \cdots \tilde{s}_{i_k}}_{\tilde{w}'} \underbrace{\tilde{s}_2\tilde{s}_4\tilde{s}_{i_{k+1}} \cdots \tilde{s}_{i_r}}_{\tilde{w}''} .$$

$\tilde{w}'$ is a reduced element with $h-1$-sequences, therefore $|\Omega_{\tilde{w}'}| = 2^{h-1}$ and $\tilde{w}''$ is a reduced element with 1-sequence, therefore $|\Omega_{\tilde{w}''}| = 2$. If we add consecutively to $\tilde{w}'$ the element $\tilde{w}''$ to obtain $\tilde{w}$, we have that $|\Omega_{\tilde{w}}| = 2^{h-1} \cdot 2 = 2^h$. □

Proposition 5.3 *Let $\tilde{w} \in \tilde{\mathcal{W}}$, $\tilde{w} \neq 1$, be a reduced element. There exists an element $w \in \mathcal{W}$ be a reduced element in normal form (cf. Section 4.1.1) such that $\pi(w) = \tilde{w}$.*

Proof We prove this proposition by induction on $\tilde{\ell}(\tilde{w})$.
If $\tilde{\ell}(\tilde{w}) = 1$ then $\tilde{w} = \tilde{s}$, therefore we can set $w = s$ and the proposition holds trivially.
Now we suppose that the proposition holds for all $k < \tilde{\ell}(\tilde{w})$. Let $\tilde{w} = \tilde{s}_{i_1} \cdots \tilde{s}_{i_r}$ be a reduced element, so $\tilde{\ell}(\tilde{w}) = r$. We can write $\tilde{w} = \tilde{w}'\tilde{s}_{i_r}$, where $\tilde{w}' = \tilde{s}_{i_1} \cdots \tilde{s}_{i_{r-1}}$ and $\tilde{\ell}(\tilde{w}') = r-1$, therefore $w' = s_{i_1} \ldots s_{i_{r-1}}$ is reduced and in normal form such that $\pi(w') = \tilde{w}'$. Without loss of generality, we can suppose that $w' \in \Omega_{\blacktriangledown,\blacktriangledown}$, so $w' = a_0b_1a_1 \ldots b_ka_kh$ with $k \geq 0$, $a_i \in {}_{\sharp}\mathcal{W}_{\blacktriangledown}$, $b_i \in {}_{\sharp}\mathcal{W}_{\blacktriangle}$, $h \in \mathcal{W}_{\bullet}$ and $\ell(w') = r-1$.
If we consider the reduced element $\tilde{w} = \tilde{s}_{i_1} \cdots \tilde{s}_{i_r}$ with $\tilde{\ell}(\tilde{w}) = r$, then, by Proposition 5.1, $w = s_{i_1} \cdots s_{i_r} = w's_{i_r} = a_0b_1a_1 \ldots b_ka_khs_r$ is reduced. Then, remind that adding s_{i_r} the length of w is always r, we can distinguish three cases:

1. if $s_{i_r} = s_1$ or $s_{i_r} = s_3$ then $w = a_0b_1a_1 \ldots b_ka_kh'$ is in normal form;

2. if $s_{i_r} = s_4$ then $w = a_0b_1a_1 \ldots b_ka_kb_{k+1}$ is in normal form, it is necessary to observe that hs_4 is an element of ${}_{\sharp}\mathfrak{W}^{\bullet}_{\blacktriangle}$ forall $h \in \mathcal{W}\bullet$.

3. if $s_{i_r} = s_2$ then $w = a_0b_1a_1 \ldots b_ka'_kh'$ is in normal form, by straightforward

calculations and reminding that w is reduced, one observes that $a_k h s_2 = a'_k h'$ for all $a_k \in {}_{\sharp}\mathcal{W}^{\bullet}_{\blacktriangledown}$ and for all $h \in \mathcal{W}_{\bullet}$.

The proposition holds in the same way if w' is a normal element of type $\Omega_{\blacktriangledown,\blacktriangle}$, $\Omega_{\blacktriangle,\blacktriangledown}$, $\Omega_{\blacktriangle,\blacktriangle}$, $\mathcal{W}_{\bullet}$. □

Proposition 5.4 *Let $w \in \mathcal{W}$, $w \neq 1$, reduced element and in normal form (cf. Section 4.1.1) such that contains no consecutive sequences $s_2 s_4$ or $s_4 s_2$. Then $\pi(w)$ is a reduced element of the same length and has a normal form that comes from the normal form of $\mathcal{W}$.*

Proof We prove the proposition by induction on $\ell(w)$.
If $\ell(w) = 1$ then $\pi(w) = \tilde{s}$ and the proposition holds trivially.
Now we suppose that the proposition holds for all $k < \ell(w)$. Let $w = s_{i_1} \cdots s_{i_r}$ be a reduced element, so $\ell(w) = r$. Without loss of generality, we can suppose that $w \in \Omega_{\blacktriangledown,\blacktriangledown}$, so $w = a_0 b_1 a_1 \ldots b_k h$ with $k \geq 0,\ a_i \in {}_{\sharp}\mathcal{W}_{\blacktriangledown},\ b_i \in {}_{\sharp}\mathcal{W}_{\blacktriangle},\ h \in \mathcal{W}_{\bullet}$.
If we write $w = w' s_{i_r}$, we obtain by similar reasoning to those of Proposition 5.3 that w' can be of three types:

1. $w' = a_0 b_1 a_1 \ldots b_k h'$;
2. $w' = a_0 b_1 a_1 \ldots b_k$;
3. $w' = a_0 b_1 a_1 \ldots b_k a_{k-1} h$.

In all cases w' is reduced and in normal form and contains no consecutive sequences $s_2 s_4$ or $s_4 s_2$, then by induction also $\pi(w')$ is reduced, in normal form and with the same length of w'.
Then we have that $\pi(w) = \pi(w')\tilde{s}_{i_r} = \pi(a_0 b_1 a_1 \ldots b_k a_k h')\tilde{s}_{i_r}$. If for example we consider the first type for w'. First of all $\tilde{\ell}(\pi(w)) = \ell(\pi(w')) \pm 1 = \ell(w') \pm 1 = r - 1 \pm 1$. If $\tilde{\ell}(\pi(w)) = r - 2$, this means that there would be in w consecutive sequences $s_2 s_4$ or $s_4 s_2$, because $\ell(w) = r$, but this is impossible by hypothesis.
Now, we know that $\pi(w')$ is in normal form: $\pi(w') = \overline{a_0}\ \overline{b_1}\ \overline{a_1}\ \ldots\ \overline{b_k}\ \overline{h'}$ and $\pi(w) = \pi(w')\tilde{s}_{i_r} = \tilde{a_0}\tilde{b_1}\tilde{a_1} \ldots \tilde{b_k}\tilde{a_k}\tilde{h}'\tilde{s}_{i_r} = \pi(w) = \pi(a_0 b_1 a_1 \ldots b_k a_k h)$, always by similar reasoning to the proof of Proposition 5.3 also $\pi(w)$ is in normal form. □

Corollary 5.1 *Every reduced element of $\tilde{\mathcal{W}}$ has a normal form that comes from the normal form of $\mathcal{W}$.* □

We know that formula (1.34) for the growth series is valid for all crystallographic Coxeter groups, then we can use this formula for the group $\tilde{\mathcal{W}}$, but the elements of $\tilde{\mathcal{W}}$ have not a normal form, therefore it is difficult to find a formula for the cocycle series as (4.37). Then we formalise the following problem.

Open problem is to understand if it is possible to change formula (4.92) for the cocycle series of $\mathcal{W}$ to adapt it to a formula for the cocycle series of $\tilde{\mathcal{W}}$ and then how these formulas can be useful to determine the roots and the multiplicities of the roots of the Kac-Moody Lie algebras with $\mathcal{W}$ and $\tilde{\mathcal{W}}$ as Weyl groups.

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