An Unbounded Family of Log Calabi-Yau Pairs

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Abstract

We give an explicit example of log Calabi-Yau pairs that are log canonical and have a linearly decreasing Euler characteristic. This is constructed in terms of a degree two covering of a sequence of blow ups of three dimensional projective bundles over the Segre-Hirzebruch surfaces $\mathbb{F}_n$ for every positive integer n big enough.

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1 Introduction

A log Calabi-Yau pair (Y, D) consists of a proper variety Y and an effective $\mathbb{Q}$-divisor D such that (Y, D) is log canonical and $K_X + D$ is

$\mathbb{Q}$-linearly equivalent to zero: see, for instance, [JK16]. A Calabi-Yau variety can be viewed as $(Y, 0)$. If Y is a Fano variety such that D is $\mathbb{Q}$-linearly equivalent to the anticanonical divisor, then (Y, D) is a log Calabi-Yau pair, provided it is log canonical.

Let us take into account three dimensional log Calabi-Yau pairs. As well known, there exist finitely many deformation types of Fano threefolds. As a result, there are finitely many possible values for their Euler characteristic. Conjecturally, this should be true for the collection of all Calabi-Yau threefolds too. Here by Calabi-Yau threefold we mean a complex Kähler compact manifold with trivial canonical bundle and no p-holomorphic forms for $p = 1, 2$. Since general log Calabi-Yau pairs interpolate between these two extremes, it is natural to wonder whether they are bounded or not. In this paper, we prove the following result.

Theorem 1. *There exists an integer N_0 such that, for every $n \geq N_0$ there exists a log Calabi-Yau threefolds (Y, D) with the Euler characteristic of Y given by*

$$e(Y) = -48n - 46.$$

Moreover, Y is smooth and its Kodaira dimension is negative. Additionally, we have $K_Y + D = 0$, where D is a divisor isomorphic to a $K3$ surface.

Recently, Di Cerbo and Svaldi in [DS16] prove that log Calabi-Yau pairs are bounded. One of their assumption is that the pair are klt. Notice that there is no contradiction between their result and ours; indeed, the example in Theorem 1 is not klt but log canonical.

The proof of Theorem 1 is constructive. More specifically, we describe a collection of log Calabi-Yau threefolds with the properties mentioned above. First, take into account the Segre-Hirzebruch surface $\mathbb{F}_n$ for any positive integer n. Next, fix a suitable decomposable vector bundle on each $\mathbb{F}_n$, namely

$$\mathcal{V} := \mathcal{O}_{\mathbb{F}_n} \oplus \mathcal{O}_{\mathbb{F}_n}(2C_0 - F),$$

where C_0 is the unique effective divisor on $\mathbb{F}_n$ such that $C_0^2 = -n$ and F is the class of the fiber with respect to the $\mathbb{P}^1$-bundle structure on $\mathbb{F}_n$. For any n denote by X the scroll defined as $\mathbb{P}(\mathcal{V})$, the projective bundle of hyperplanes in $\mathcal{V}$. For futher information about these scrolls, see, for instance, [FF15].

If the linear system $|-2K_X|$ had a smooth member, then the double covering of X - branched along it - would be a smooth Calabi-Yau manifold. Unfortunately, this is not the case. The base locus of $|-2K_X|$ is given by a smooth rational curve. Luckily, the multiplicity of the generic section along the base locus is three. This requires a careful analysis of the cohomology group $H^0(X, -2K_X)$, which can be carried out more easily for n big enough.

If we blow up X along the smooth curve in the base locus of the bianticanonical system, we obtain a smooth threefold X_1. The linear system $|-2K_{X_1}|$ is not basepoint free. The base locus is given by a smooth rational curve γ_1. In order to resolve a generic section of the linear system $|-2K_X|$, we blow up X_1 along γ_1. We obtain a smooth threefold X_2. The degree two branched cover Y_2 along a smooth section of $-2K_X - 2E_1 - 4E_2$ is not normal. Taking the normalization of it is equivalent to taking the branched covering of X_2 along a smooth member of the linear system $-2K_{X_2} - 2E_2 = -2K_X - 2E_1 - 4E_2$.

Finally, in order to calculate the Euler characteristic of Y_2 for n big enough, it suffices to determine that of X_2 and that of a smooth surface in $|-2K_{X_2} - 2E_2|$. The former follows from the cohomology of blow ups along a submanifold and the latter from the Chern classes of it: see, for instance, [GH].

Our construction relies on the choice of the vector bundle $\mathcal{V} = \mathcal{O}_{\mathbb{F}_n} \oplus \mathcal{O}_{\mathbb{F}_n}(2C_0 - F)$. It is important to stress that this is only one of the possible choices in order to arrive at an unbounded family of log Calabi-Yau pairs. To be more precise, we analysed all the cases as $\mathcal{V}$ varies among the rank 2 vector bundle on $\mathbb{F}_n$ that are decomposable. Our method yields a double cover, which is a smooth Calabi-Yau threefold, only for a finite number of cases. We expect that for the great majority of the other cases the situation is similar to that presented in this paper: one can mimic the construction and obtain a log Calabi-Yau pair.

The paper is organized as follows. In Section 2 we recall some preliminary results. Section 3 is devoted to describing the bianticanonical system of the scroll X, in particular a desingularitazion of a generic section of it. At last, Section 4 concludes the exposition with the computation of the Euler characteristic, thus showing that it is in fact unbounded!

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2 Some Preliminary Results

In this section we recall some basic facts and prove some results that will be applied in what follows. For further details, the reader is referred to [H], p. 369 ff.

Let S be a smooth projective surface and denote by X the projective bundle associated to a rank 2 vector bundle $\mathcal{V}$ on S. To avoid confusion we recall that X is the projective bundle $\mathbb{P}(\mathcal{V})$ over the base S, where $\mathbb{P}(\mathcal{V})$ is the projective bundle of hyperplanes in $\mathcal{V}$. In what follows we will set τ to be $c_1(\mathcal{O}_X(1))$.

Lemma 2. *Denote by $\varphi : X \to S$ the fibration given by the projective bundle structure. Then the following identities hold:*

$$c_1(X) = 2\tau + \varphi^*(c_1(S) - c_1(\mathcal{V}));$$

$$c_2(X) = \varphi^*(c_2(S) - c_1(\mathcal{V})c_1(S)) + 2\varphi^*c_1(S)\tau;$$

$$c_3(X) = 2\varphi^*(c_2(S))\tau.$$

Proof. We have the exact sequences

$$0 \to T_{X/S} \to T_X \to \varphi^* T_S \to 0, \tag{1}$$

$$0 \to \mathcal{O}_X \to (\varphi^* \mathcal{V}^\vee) \otimes \mathcal{O}_X(1) \to T_{X/S} \to 0. \tag{2}$$

Recall also that $H^*(X)$ is generated as an $H^*(S)$-algebra by τ with the single relation

$$\tau^2 - \varphi^* c_1(\mathcal{V})\tau = 0. \tag{3}$$

We have

$$c_1((\varphi^* \mathcal{V}^\vee) \otimes \mathcal{O}_X(1)) = \varphi^* c_1(\mathcal{V}^\vee) + 2\tau = -\varphi^* c_1(\mathcal{V}) + 2\tau,$$

$$c_2((\varphi^* \mathcal{V}^\vee) \otimes \mathcal{O}_X(1)) = \varphi^* c_2(\mathcal{V}^\vee) + \varphi^* c_1(\mathcal{V}^\vee)\tau + \tau^2 = \varphi^* c_2(\mathcal{V}) - \varphi^* c_1(\mathcal{V})\tau + \tau^2.$$

By (3), this yields

$$c((\pi^* \mathcal{V}^\vee) \otimes \mathcal{O}_X(1)) = 1 + (2\tau - \varphi^* c_1(\mathcal{V})). \tag{4}$$

From the exact sequences (1) and (2), we get

$$\begin{aligned} c(X) = c(T_{X/S})\varphi^* c(T_S) = c((\varphi^* \mathcal{V}^\vee) \otimes \mathcal{O}_X(1))\varphi^* c(T_S) = \\ = (1 + (2\tau - \varphi^* c_1(\mathcal{V})))\varphi^* c(S) = \\ = 1 + [2\tau - \varphi^* c_1(\mathcal{V}) + \varphi^* c_1(S)] + [\varphi^* c_2(S) + \varphi^* c_1(S)(2\tau - \varphi^* c_1(\mathcal{V}))] + [2\varphi^* c_2(S)\tau] = \\ = 1 + [2\tau + \varphi^*(c_1(S) - c_1(\mathcal{V}))] + [\varphi^*(c_2(S) - c_1(\mathcal{V})c_1(S)) + 2\varphi^* c_1(S)\tau] + [2\varphi^* c_2(S)\tau] \end{aligned} \tag{5}$$

□

In order to determine the cohomology of line bundles on X, we are going to apply the following result. We will recall it here for the sake of completeness: see, for instance, [H], pag. 253, Ex 8.4 (a).

Lemma 3. *Let $\mathcal{V}$ be a vector bundle on a smooth surface S. Let $X = \mathbb{P}(\mathcal{V})$ and define τ as before. Then*

$$\varphi_*\mathcal{O}_X(a\tau) = 0 \qquad \text{if } a < 0,$$

$$\varphi_*\mathcal{O}_X(a\tau) = S^a(\mathcal{V}) \qquad \text{if } a \geq 0,$$

$$R^i\varphi_*\mathcal{O}_X(a\tau) = 0 \qquad \forall a \in \mathbb{Z} \text{ if } 0 < i < \mathrm{Rk}(\mathcal{V}) - 1 \text{ or if } i \geq \mathrm{Rk}(\mathcal{V}),$$

$$R^{\mathrm{Rk}(\mathcal{V})-1}\varphi_*\mathcal{O}_X(a\tau) = 0 \qquad \text{if } a > -\mathrm{Rk}(\mathcal{V}).$$

3 A Generic Member of the Bianticanonical Linear System

From now onwards, S will be the Segre-Hirzebruch surface $\mathbb{F}_n = \mathbb{P}(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(-n))$, with n positive. Recall that $\mathrm{Pic}(\mathbb{F}_n)$ is generated by C_0, the only effective divisor on S such that $C_0^2 = -n$, and F, the class of a fiber of the $\mathbb{P}^1$-bundle. Hence, without loss of generality, any decomposable vector bundle of rank 2, up to tensor product with a line bundle, can be written as $\mathcal{V} = \mathcal{O}_{\mathbb{F}_n} \oplus \mathcal{O}_{\mathbb{F}_n}(-A)$, where $A = xC_0 + yF$ and x is nonnegative. For the sake of convenience, we will denote by the same symbol a divisor on S and its pullback on X. We will denote, as before, by X the projective bundle associated to $\mathcal{V}$

Proposition 4. *Consider a divisor $D = a\tau + G$ on X where $G = bC_0 + cF$ is the pullback of a divisor on S. Then the following hold:*

i) If $A = xC_0 + yF$ with $y \geq 0$ (i.e., if A is effective), D is effective if and only if $a, b, c \geq 0$.

ii) If $A = xC_0 - yF$ with $y > 0$, D is effective if and only if $a \geq 0$ and

$$(b, c) \in \bigcup_{r=0}^{a} S_r \qquad \text{with } S_r = \{(b, c) \mid b, c \geq 0\} + (rx, -ry).$$

iii) If $A = xC_0 - yF$ with $y > 0$, the only prime and rigid divisors on X are τ, C_0 and $\tau + A$.

Proof. i) $D = a\tau + bC_0 + cF$ is effective if and only if $a \geq 0$; else we have

$$\varphi_* \mathcal{O}_X(a\tau + bC_0 + cF) = \varphi_* \mathcal{O}_X(a\tau) \otimes \mathcal{O}_S(bC_0 + cF) = 0.$$

Hence, we can assume $a \geq 0$. Doing so, we have

$$H^0(\mathcal{O}_X(D)) = \bigoplus_{r=0}^{a} H^0(\mathcal{O}_S(bC_0 + cF - rA)) \supset H^0(\mathcal{O}_S(bC_0 + cF)).$$

If $b, c \geq 0$ the divisor is effective.

ii) Assume, now, $A = xC_0 - yF$ with $y > 0$. In this case

$$V_r = H^0(\mathcal{O}_S(bC_0 + cF - rA)) = H^0(\mathcal{O}_S((b - rx)C_0 + (c + ry)F))$$

and $H^0(\mathcal{O}_X(D))$ is not zero exactly when at least one of these spaces is not zero. V_r is not zero exactly when $b \geq rx$ and $c \geq -ry$, i.e., when $(b, c) \in S_r$, so the second claim is proved.

iii) Finally assume $A = xC_0 - yF$ with $y > 0$, and consider the effective divisor $D = a\tau + bC_0 + cF$ with $a, b \geq 0$ and $-ay \leq c \leq 0$. If D is rigid then $(b, c) \in S_r$ for exactly one value or r (with $0 \leq r \leq a$). If $(b, c) \in S_r$ we can write D as $a\tau + rA + b'C_0 + c'F$ with $0 \leq c' < y$. If $r < a$ we can assume $0 \leq b' < x$ whereas, if $(b, c) \in S_a$, we can assume $b' \geq 0$. In both cases, the divisor $D = a\tau + rA + b'C_0$ is effective; hence, if $c' > 0$, we have $h^0(\mathcal{O}_X(D)) \geq 2$. This shows that we have to look for rigid divisors among the ones of the form

$$D = a\tau + rA + b'C_0,$$

where $a \geq 0, 0 \leq r < a$ and $0 \leq b' < x$ or with $a \geq 0, r = a$ and $b' \geq 0$. It is not difficult to see that, in this case, $h^0(\mathcal{O}_X(D)) = 1$, so we always get a rigid divisor. It is also easy to see that every such divisor can be written as a sum

$$a_1\tau + a_2C_0 + a_3(\tau + A),$$

which proves that τ, C_0 and $\tau + A$ are the only rigid prime divisors on X when $A = xC_0 - yF$ and $y > 0$. □

We will be interested in the case $\mathcal{V} = \mathcal{O}_{\mathbb{F}_n} \oplus \mathcal{O}_{\mathbb{F}_n}(-A)$ with $A = 2C_0 - F$. Recall that, in this case,

$$K_X = -2\tau - 2C_0 - (n+2)F - 2C_0 + F = -2\tau - 4C_0 - (n+1)F,$$

so we have

$$-2K_X = 4\tau + 8C_0 + (2n+2)F.$$

As we will see, if n is big enough, the linear system $|-2K_X|$ does not have smooth members. Thus, we need to describe more closely the base locus and the type of singularities.

Proposition 5. *The base locus of the bianticanonical linear series is given by the complete intersection σ of the rigid divisors with class $\tau + A$ and C_0.*

Proof. Since $2C_0 - F$ is not effective, by Proposition 4 there are three rigid prime divisors, namely τ, C_0 and $\tau + A$. The intersections of these three divisors are

$$\tau(\tau + A) = 0, \qquad \tau C_0 := \gamma, \qquad (\tau + A)C_0 = (\tau - (2n+1)F)_{|C_0} := \sigma.$$

The unique surface T with class given by τ is a Segre-Hirzebruch surface $\mathbb{F}_n$ with standard generators for $\mathrm{Pic}(\tau)$ given by

$$C_0|_T = \gamma_T, \qquad F|_T = f_T.$$

By standard generators, we mean a basis of effective prime divisors under which the intersection product has representative matrix

$$\begin{bmatrix} -a & 1 \\ 1 & 0 \end{bmatrix}$$

where a is the (positive) index of the Segre-Hirzebruch surface. In particular, the class of the curve γ seen in T is given by γ_T.
Denote by R the only surface whose class is $\tau + A$. One can easily see that R is again a Segre-Hirzebruch surface $\mathbb{F}_n$ if one considers the vector bundle $\mathcal{V}' = \mathcal{V} \otimes \mathcal{O}_S(A)$ and uses the identification

$$X = \mathbb{P}(\mathcal{V}) = \mathbb{P}(\mathcal{O}_S \oplus \mathcal{O}_S(-A)) = \mathbb{P}(\mathcal{O}_S \oplus \mathcal{O}_S(A)) = \mathbb{P}(\mathcal{V}').$$

Indeed the class of $c_1(\mathcal{O}_{\mathbb{P}(\mathcal{V}')}(1)) = \tau'$ is $\tau + A$ under this identification. The standard generators for $\mathrm{Pic}(R)$ are

$$C_0|_R = \gamma_R, \qquad F|_R = f_R.$$

The surface U, whose class is C_0, is also a Segre- Hirzebruch surface $\mathbb{F}_m$ with $m = 2n+1$. The standard generators for the Picard lattice are

$$(\tau + A)|_U = \gamma_U, \qquad F|_U = f_U.$$

Notice that

$$-2K_X = 4(\tau + A) + (2n+6)F,$$

so, an eventual base point of $|-2K_X|$ cannot lie outside the surface R. In fact, $(2n+6)F$ is globally generated. It is easy to prove that $\tau + A$ is not a component of $|-2K_X|$, so the base locus of the bianticanonical linear series is contained in R. In fact, let us restrict $-2K_X$ to R. This yields

$$(4\tau + 8C_0 + (2n+2)F)|_R = 8\sigma + (2n+2)f_R,$$

which shows that σ is contained in the base locus of the bianticanonical linear series. Conversely, given a point in such a base locus, it must belong to σ because it is in R and nowhere else than in σ because $|f_R|$ is globally generated in R. Therefore the claim is proved. $\square$

Remark. The curves σ and γ are the intersection of C_0 with τ and $\tau + A$, respectively. We can also see them inside these surfaces and the following table describes their classes (a "-" simply means that the curve cannot be seen in that particular surface).

	T (τ)	R ($\tau + A$)	U (C_0)
$T \cap U = \gamma$	γ_T	-	$\gamma_U + (2n+1)f_U$
$R \cap U = \sigma$	-	γ_R	γ_U

From this description (as well as from adjunction) one can see that both γ and σ are smooth curves of genus 0. Moreover, σ is rigid in both R and U, whereas γ is rigid only in T.

Proposition 6. *The generic member of the bianticanonical system has multiplicity* 3 *along the base locus.*

Proof. Define t, u and r to be the sections (uniquely determined up to scalar) such that

$$H^0(\mathcal{O}_X(\tau)) = \langle t\rangle \qquad H^0(\mathcal{O}_X(C_0)) = \langle u\rangle \qquad H^0(\mathcal{O}_X(\tau + A)) = \langle r\rangle$$

so the zero loci of t, u and r describe T, U and R, respectively. Define D_i to be $-2K_X - i(\tau + A)$. Therefore, there exists a positive integer N_0 big enough such that for $n \geq N_0$ the following hold:

$h^0(D_0) = 14n + 61$	-
$h^0(D_1) = 11n + 52$	$h^0(\mathcal{O}_X(D_0)) - h^0(\mathcal{O}_X(D_1)) = 3n + 9$
$h^0(D_2) = 8n + 40$	$h^0(\mathcal{O}_X(D_1)) - h^0(\mathcal{O}_X(D_2)) = 3n + 12$
$h^0(D_3) = 5n + 25$	$h^0(\mathcal{O}_X(D_2)) - h^0(\mathcal{O}_X(D_3)) = 3n + 15$
$h^0(D_4) = 2n + 7$	$h^0(\mathcal{O}_X(D_3)) - h^0(\mathcal{O}_X(D_4)) = 3n + 18$

Let us now describe the sections of $\mathcal{O}_X(-2K_X)) = \mathcal{O}_X(4(\tau + A) + (2n + 6)F)$.

We have the exact sequence

$$0 \longrightarrow H^0(\mathcal{O}_X(D_1)) \xhookrightarrow{-\otimes r} H^0(\mathcal{O}_X(-2K_X)) \longrightarrow H^0(\mathcal{O}_R(-2K_X)) \tag{6}$$

Notice that $-2K_X|_R = (8C_0 + (2n+2)F)|_R = 8\gamma_R + (2n+2)f_R$ hence, as $R = \mathbb{F}_n$, we have

$$h^0(\mathcal{O}_R(-2K_X)) = h^0(\mathcal{O}_{\mathbb{F}_n}(8\gamma_R + (2n+2)f_R)) = 3n+9 = h^0(\mathcal{O}_X(D_0)) - h^0(\mathcal{O}_X(D_1)).$$

Thus, the restriction map $H^0(\mathcal{O}_X(-2K_X)) \to H^0(\mathcal{O}_R(-2K_X))$ in Equation 6 is indeed surjective. Denote by V_0 a subspace of $H^0(\mathcal{O}_X(-2K_X))$ such that

$$V_0 \oplus (H^0(\mathcal{O}_X(D_1)) \otimes \langle r\rangle) \simeq H^0(\mathcal{O}_R(-2K_X)).$$

If $s \in H^0(\mathcal{O}_X(-2K_X))$, we have a decomposition of s as

$$s = r\alpha_0 + \beta_0$$

with $\alpha_0 \in H^0(\mathcal{O}_X(D_1))$ and $\beta_0 \in V_0$. In particular, β_0 does not vanish identically on R (it vanishes on γ_R and some $\mathbb{P}^1$'s transversal to γ_R). We can iterate this process by restricting α_0 on R. As before, we have the following exact sequences, namely:

$$0 \longrightarrow H^0(\mathcal{O}_X(D_2)) \xhookrightarrow{-\otimes r} H^0(\mathcal{O}_X(D_1)) \longrightarrow H^0(\mathcal{O}_R(D_1)) \longrightarrow 0$$
$$0 \longrightarrow H^0(\mathcal{O}_X(D_3)) \xhookrightarrow{-\otimes r} H^0(\mathcal{O}_X(D_2)) \longrightarrow H^0(\mathcal{O}_R(D_2)) \longrightarrow 0$$
$$0 \longrightarrow H^0(\mathcal{O}_X(D_4)) \xhookrightarrow{-\otimes r} H^0(\mathcal{O}_X(D_3)) \longrightarrow H^0(\mathcal{O}_R(D_3)) \longrightarrow 0, \tag{7}$$

where the surjectivity follows as before by inspecting the dimension of

$$H^0(\mathcal{O}_R(D_i)) = H^0(\mathcal{O}_{\mathbb{F}_n}((8-2i)\gamma_R + (2n+2+i)f_R)$$

and observing that it equals $h^0(\mathcal{O}_X(D_i)) - h^0(\mathcal{O}_X(D_{i+1}))$. Then we can create the vector spaces V_i such that

$$V_i \oplus (H^0(\mathcal{O}_X(D_{i+1})) \otimes \langle r \rangle) \simeq H^0(\mathcal{O}_R(D_i))$$

and sections $\alpha_i \in H^0(\mathcal{O}_X(D_{i+1}))$, $\beta_i \in V_i$ such that

$$\alpha_i = r\alpha_{i+1} + \beta_{i+1}.$$

Finally, the section s has the following form:

$$s = r^4\alpha_3 + r^3\beta_3 + r^2\beta_2 + r\beta_1 + \beta_0. \tag{8}$$

Notice that $D_0|_R, D_1|_R$ and $D_2|_R$ are divisors with σ_R as fixed components so β_i for $i = 0, 1, 2$ will vanish on it (with multiplicity greater than or equal to 4). But the same is not true for $D_3|_R$, which is very ample. In particular, β_3 can be chosen such that $\beta_3|_R$ vanishes at exactly 5 points of σ_R (this is equal to $\sigma_R \cdot D_3|_R$) which are free on σ_R and whose associated curve cut σ_R transversely at such points.

In particular, the generic element of $|-2K_X|$ has σ as base curve and the multiplicity of σ along the generic bianticanonical divisor is 3. $\square$

4 Blowing up the Projective Bundle

In this section we will describe a resolution of a generic member of the linear system $|-2K_X|$.

Near a point P of σ we can choose local coordinates (x, y, z) such that $x = y = 0$ is the local equation of σ near P, $x = 0$ and $y = 0$ are the local equations of R and U respectively and z is a coordinate on σ. We can also use (y, z) as local coordinates on R. We write, locally

$$s = x^3 f + x^4 g + x^2 y^4 f_1 + x y^6 f_2 + y^8 f_3,$$

where f is the local expression for β_3 and g is the local expression for α_3. We can blow up σ in X and take the strict transform $\tilde{D}$ of $D := \{s = 0\}$. Near P the blow up X_1 looks like

$$\{(x, y, z) \times (l_0 : l_1) \,|\, x l_1 - y l_0 = 0\}.$$

In the local chart $U_0 = \{l_0 \neq 0\}$ we have coordinates (x, z, l_1) with $y = x l_1$ and the local equation for the exceptional divisor E which is $x = 0$. The total transform of D has equation

$$x^3 (f + xg + x^3 l_1^4 f_1 + x^4 l_1^6 f_2 + x^5 l_1^8 f_3),$$

so

$$\overline{s} = f + xg + x^3 l_1^4 f_1 + x^4 l_1^6 f_2 + x^5 l_1^8 f_3$$

is a local equazion for $\tilde{D}$. Notice that $f(0, y, z)$ is not identically zero because f is the local expression of β_3. From the proof of Proposition 6 we have also that $\tilde{D}$ is smooth along σ and hence everywhere (since it is the strict transform of something that has base locus σ). Unfortunately, $\tilde{D}$ is not a bianticanonical divisors on X_1: the bianticanonical class is indeed $\tilde{D} + E$ so we can take a bianticanonical divisor on X_1 to be the union of $\tilde{D}$ and E. This is reduced, reducible and singular exactly along the intersection $E_1 \cdot \tilde{D}$.

Lemma 7. *The divisor E is a Segre-Hirzebruch variety $\mathbb{F}_{n+1}$.*

Proof. The curve σ is a complete intersection. More precisely, it is the intersection of the two rigid divisors C_0 and $\tau + A$. Thus, the normal bundle is $\mathcal{O}_\sigma(C_0) \oplus \mathcal{O}_\sigma(\tau + A)$. By direct computation, this is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-n) \oplus \mathcal{O}_{\mathbb{P}^1}(-2n-1)$, which proves the claim. □

The Picard group of X_1 is generated by τ, C_0, F and the exceptional divisor E_1. By construction, the restriction of τ to E_1 is zero. Moreover, the restriction of C_0 to the exceptional divisor is an integer multiple of $f_1 = F|_{E_1}$, the class of a fiber of E_1 seen as Segre-Hirzebruch surface. This follows from the intersection numbers that are calculated in the next section. Therefore, the Picard group of the exceptional divisor is generated by the restriction of E_1 and F, respectively. It is not difficult to check that the unique divisor γ_1 on E_1 such that $\gamma_1^2 = -n-1$ is given by

$$\gamma_1 = -E_{1|E_1} - (2n+1)F_{|E_1}.$$

The strict transform of the divisor $-2K_X$ is equal to $-2K_X - 3E_1$. Its intersection with E_1 is given by $3\gamma_1 + 5f_1$. This is an effective divisor on E_1, which is made up of the unique curve of self-intersection $-n-1$ and 5 disjoint fibers. Since we have

$$-2K_{X_1} = -2K_X - 2E_1 = (-2K_X - 3E_1) + E_1,$$

the sections of the bianticanonical divisor $-2K_{X_1}$ pass through the curve γ_1, which is the complete intersection of $\tau + A - E_1$ (strict transform of the divisor $\tau + A$) and E_1.

Therefore, we blow up X_1 along the curve γ_1 and obtain a new variety X_2 with exceptional divisor E_2. To determine its structure, we compute the normal bundle of γ_1 which is given by

$$N_{\gamma_1/X_1} = \mathcal{O}_{\gamma_1}(E_1) \oplus \mathcal{O}_{\gamma_1}(\tau + A - E_1) \simeq \mathcal{O}_{\mathbb{P}^1}(-n-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-n).$$

Therefore, the exceptional divisor E_2 is isomorphic to $\mathbb{F}_1$. Let us denote by γ_2 and f_2 the generators of E_2 such that $\gamma_2^2 = -1$, $\gamma_2 f_2 = 1$ and $f_2^2 = 0$. As in the case of E_1, we can take f_2 to be the restriction of F to E_2. As for the other divisor, it is easy to check that

$$\gamma_2 = -E_{2|E_2} - (n+1)F_{|E_2}. \tag{9}$$

The bianticanonical divisor of X_2 is thus given by

$$-2K_{X_2} = -2K_X - 2E_1 - 2E_2 = (-2K_X - 2E_1 - 4E_2) + 2E_2.$$

Let us compute the restriction of the divisor $(-2K_X - 2E_1 - 4E_2)$ to E_2. An easy calculation shows that it is equal to $4\rho_2 + 6f_2$, which corresponds to the class of a smooth irreducible curve on $E_2 \simeq \mathbb{F}_1$. Thus, there is a smooth member of the linear system

$$2(-K_{X_2} - E_2) = -2K_X - 2E_1 - 4E_2.$$

Being $2(-K_{X_2} - E_2)$ even, we can consider the cyclic covering $\beta : Y_2 \to X_2$ of degree two with branch along a smooth member of $-2K_{X_2} - 2E_2 = 2K_X - 2E_1 - 4E_2$.

Lemma 8. *Y_2 is a smooth threefold and β^*E_2 is a $K3$ surface. Moreover the pair (Y_2, β^*E_2) is a log Calabi-Yau.*

Proof. Y_2 is clearly smooth as the branch divisor has been chosen to be smooth. Moreover, by [BPHV] pag. 55, we have also

$$K_{Y_2} = \beta^*(K_{X_2} + B_2/2) = -\beta^*(E_2) \tag{10}$$

so (Y_2, β^*E_2) is a log Calabi-Yau. Notice that $\beta^*(E_2)$ is a degree two covering of the Segre-Hirzebruch surface $\mathbb{F}_1$ branched along the intersection of E_2 with the branch divisor of the covering $\beta : Y_2 \to X_2$. We have already seen that this intersection can be written as the smooth curve $B_2 = 4\rho_2 + 6f_2$ on $E_2 \simeq \mathbb{F}_1$, i.e. it is a smooth bianticanonical curve on $\mathbb{F}_1$. This is enough to conclude that the canonical divisor of β^*E_2 is trivial. The Euler characteristic is of β^*E_2 can be calculated as $2e(E_2) - e(R_2)$, where R_2 is the ramification divisor of the restriction of β to $\beta^*(E_2)$. Since β is a degree two covering, the divisor R_2 is isomorphic to the branch divisor B_2. This is a curve of genus 9, so the Euler characteristic of R_2 is -16. We have hence $e(\beta^*E_2) = 24$ so we can conclude that $\beta^*(E_2)$ is a $K3$ surface. □

In the next section, we are going to calculate the Euler characteristic of Y_2 for every $n \geq N_0$. To conclude this section, let us prove the following result.

Theorem 9. *Let Y_2 be as above. Then we have:*

$$h^{1,0}(Y_2) = 0, \qquad h^{2,0}(Y_2) = 0, \qquad h^{3,0}(Y_2) = 0.$$

Moreover, Y_2 has negative Kodaira dimension.

Proof. We need to determine $h^{q,0}(Y_2)$ for $q \geq 1$. Recall that

$$\beta_* \mathcal{O}_{Y_2} \simeq \mathcal{O}_{X_2} \oplus \mathcal{O}_{X_2}(-B_2/2) = \mathcal{O}_{X_2} \oplus \mathcal{O}_{X_2}(K_{X_2} + E_2)$$

and that $R^q \beta_* \mathcal{F} = 0$ for all $\mathcal{F}$ coherent on Y_2 and for all $q \geq 1$. Hence, by Leray spectral sequence, we have

$$H^q(\mathcal{O}_{Y_2}) \simeq H^q(\mathcal{O}_{X_2}) \oplus H^q(\mathcal{O}_{X_2}(K_{X_2} + E_2)).$$

X_2 is birational to X, which is a projective bundle over $\mathbb{F}_n$ so the Hodge numbers $h^{q,0}(X_2) = h^{q,0}(X)$ are zero for $q \geq 1$. Hence we need to prove that $h^q(\mathcal{O}_{X_2}(K_{X_2} + E_2))$ is zero for $q \geq 1$ in order to conclude the proof. If $q = 3$ this is straightforward: we have

$$h^3(\mathcal{O}_{X_2}(K_{X_2} + E_2)) = h^0(\mathcal{O}_{X_2}(-E_2)) = 0 \tag{11}$$

because E_2 is effective. We have $h^p(\mathcal{O}_{X_2}(K_{X_2})) = h^{3-p}(\mathcal{O}_{X_2}) = h^{3-p}(\mathcal{O}_X)$ so,

$$h^1(\mathcal{O}_{X_2}(K_{X_2})) = h^2(\mathcal{O}_{X_2}(K_{X_2})) = 0 \qquad \text{and} \qquad h^3(\mathcal{O}_{X_2}(K_{X_2})) = 1. \tag{12}$$

To compute $H^q(K_{X_2} + E_2)$ for $q = 1, 2$, let us consider the exact sequence

$$0 \to \mathcal{O}_{X_2}(K_{X_2}) \to \mathcal{O}_{X_2}(K_{X_2} + E_2) \to \mathcal{O}_{E_2}(K_{X_2} + E_2) \to 0,$$

which yields, using also Equations 11 and 12, the exact sequences

$$0 \to H^1(\mathcal{O}_{X_2}(K_{X_2} + E_2)) \to H^1(\mathcal{O}_{E_2}(K_{X_2} + E_2)) \to 0 \tag{13}$$

$$0 \to H^2(\mathcal{O}_{X_2}(K_{X_2}+E_2)) \to H^2(\mathcal{O}_{E_2}(K_{X_2}+E_2)) \to H^3(\mathcal{O}_{X_2}(K_{X_2})) \to 0 \tag{14}$$

By adjunction, $\mathcal{O}_{E_2}(K_{X_2}+E_2)$ is the canonical divisor of K_{E_2} so $H^1(\mathcal{O}_{E_2}(K_{E_2})) = H^{1,2}(\mathbb{F}_1) = 0$ (or, alternatively, by Lemma 2.9 of [CM02]). Hence, from the exact sequence 13, also $H^1(X_2, \mathcal{O}_{X_2}(K_{X_2} + E_2))$ is zero.

Both the second and the third term of the exact sequence 14 have dimension 1 so $h^2(\mathcal{O}_{X_2}(K_{X_2}+E_2)) = 0$.

In order to see that the Kodaira dimension is $-\infty$, it is enough to observe that $-K_{Y_2}$ is effective and this follows from Equation 10. $\square$

5 The Euler Characteristic

In this section, we will calculate the Chern numbers of X_2. Recall that $X = \mathbb{P}(\mathcal{V})$ with $\mathcal{V} = \mathcal{O}_S \oplus \mathcal{O}_S(-A)$ and $A = 2C_0 - F$. If $X_1 = \mathrm{Bl}_\sigma X$, where σ is the rational curve cut out by R and U. If E_1 is the class of the exceptional divisor, we can consider the complete intersection curve cut out by the two divisors $\tau + A - E_1$ and E_1. As for the notation, denote by E_2 the exceptional divisor of the second blow up. We will apply the following lemma:

Lemma 10. *Let Z be a smooth complex threefold and let*

$$C \overset{j}{\hookrightarrow} Z,$$

where C is a smooth curve. If $Z' = \mathrm{Bl}_C(Z)$ with exceptional divisor E and blow up map $\pi : Z' \to Z$. Then the following hold:

$$c_1(Z') = \pi^* c_1(Z) - E \tag{15}$$
$$c_2(Z') = \pi^*(c_2(Z) - \eta_C) - \pi^* c_1(Z) E \tag{16}$$
$$H^*(Z') = H^*(Z) \oplus H^*(E)/H^*(C), \tag{17}$$

where η_C is the class of C in $H^4(Z)$. Moreover, if $\alpha_p \in CH^p(Z)$ and $p + q = 3$ with $q \geq 1$, then

$$E \cdot (\pi^* \alpha_2) = 0 \qquad E^2 \cdot (\pi^* \alpha_1) = -j^* \alpha_1 \qquad E^3 = -c_1(N_{C/Z}). \tag{18}$$

Proof. The first two identities can be found in [GH], p. 609. Let α_p be a class in $CH^p(Z)$ and consider the following commutative diagram

$$\begin{array}{ccc} E & \overset{\iota}{\hookrightarrow} & Bl_C(Z) \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ C & \underset{j}{\hookrightarrow} & Z \end{array}$$

If we assume that $q \geq 1$ we can write $E^q = E^{q-1} \cdot E = E^{q-1}\iota_*(1)$ so that

$$E^q{\cdot}(\pi^*\alpha_p) = (E^{q-1}\pi^*\alpha_p)\iota_*(1) = \iota^*(E^{q-1}\pi^*\alpha_p){\cdot}1 = \iota^*(E^{q-1})(\pi\circ\iota)^*\alpha_p = \\ = \iota^*(E^{q-1})(j\circ\pi)^*\alpha_p = \iota^*(E^{q-1})\pi^*(j^*\alpha_p) = \pi_*(\iota^*E)^{q-1}\cdot(j^*\alpha_p).$$

The restriction of the exceptional divisor to itself is the tautological class of E when seen as the total space of the projective bundle $\mathbb{P}(N_{C/Z}) \to C$. If we denote by $h = c_1(\mathcal{O}_{\mathbb{P}(N_{C/Z})}(1))$, we have $\iota^*(E)^{q-1} = (-1)^{q-1}h^{q-1}$. By definition we have also $\pi_*(h^{q-1}) = s_{q-2}(N_{C/Z})$, where $s_n(N_{C/Z})$ is the Segre class of level n of the vector bundle $N_{C/Z}$. To conclude, it is enough to observe that $s_1(N_{C/Z}) = -c_1(N_{C/Z})$ and $s_0(N_{C/Z}) = 1$. $\square$

Recall that

$$c_1(X) = 2\tau + 4C_0 + (n+1)F, \tag{19}$$
$$c_2(X) = 4\tau C_0 + (2n+4)\tau F + (-2n+6)C_0F, \tag{20}$$
$$c_3(X) = 8\tau C_0 F \tag{21}$$

and that σ, the center of the first blow up, is the complete intersection of $\tau + A$ and C_0. Hence

$$N_{\sigma/X} = \mathcal{O}_\sigma(\tau + A) \oplus \mathcal{O}_\sigma(C_0)$$

and the class η_σ of σ in $H^4(X)$ is simply the class of $(\tau + A)C_0$. In order to simplify notation, we will write α to indicate both a class in X and its pullback to X_1 and X_2. The first Chern class of X_1 is simply given by $c_1(X) - E_1$ whereas

$$c_2(X_1) = c_2(X) - (\tau + A)C_0 - c_1(X)E_1.$$

We are blowing up a smooth rational curve so

$$E_1 = \mathbb{P}(N_{\sigma/X}) = \mathbb{P}(\mathcal{O}_\sigma(\tau + A) \oplus \mathcal{O}_\sigma(C_0))$$

is the Segre-Hirzebruch surface $\mathbb{F}_{n+1}$. By (15), we obtain that the Hodge structure of X_1 is pure and $h^{1,1}(X_1) = 4$. To recap, we have

$$c_1(X_1) = c_1(X) - E_1, \tag{22}$$

$$c_2(X_1) = c_2(X) - (\tau + A)C_0 - c_1(X)E_1, \tag{23}$$

$$c_3(X_1) = 10\tau C_0 F. \tag{24}$$

Moreover, the relations that characterize the intersection theory on X_1 are (here we don't report the ones coming from X) given by

$$E_1\tau = 0 \qquad E_1 C_0 F = 0 \qquad E_1^2 C_0 = n \qquad E_1^2 F = -1 \qquad E_1^3 = 3n{+}1.$$

The first relation follows simply by observing that τ and σ are disjoint so τ and E_1 do not intersect. The others follow from Lemma 10 using

$$C_0 j^*\sigma = C_0^2(\tau + A) = -n \qquad F j^*\sigma = C_0(\tau + A)F = 1.$$

and

$$c_1(N_{\sigma/X}) = C_0^2(\tau + A) + C_0(\tau + A)^2 = -(3n + 1).$$

The curve γ_1 is smooth and rational. If we blow it up, we obtain an exceptional divisor E_2 that is isomorphic to $\mathbb{F}_1$. Indeed, the normal bundle of such a curve is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(-n-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-n)$. Using the same argument as before, we have

$$c_1(X_2) = c_1(X_1) - E_2, \tag{25}$$

$$c_2(X_2) = c_2(X_1) - (\tau + A - E_1)E_1 - c_1(X_1)E_2, \tag{26}$$

$$c_3(X_2) = 12\tau C_0 F. \tag{27}$$

Continuing as before we get

$$E_2\tau = 0 \qquad E_2 C_0 F = 0 \qquad E_2 E_1 C_0 = 0 \qquad E_2 E_1 F = 0 \qquad E_2 E_1^2 = 0$$

$$E_2^2 C_0 = n \qquad E_2^2 F = -1 \qquad E_2^2 E_1 = n \qquad E_2^3 = 2n + 1$$

This is all we need to prove the following theorem

Theorem 11. *For every positive integer n big enough there exists a pair (Y, D) such that*

- *Y is a smooth threefold of negative Kodaira dimension with*
$$e(Y) - 48n - 46 \qquad \textit{and} \qquad h^{q,0}(Y) = 0 \ \textit{ for } q \geq 1;$$
- *D is a smooth K3 surface;*
- *(Y, D) is a log canonical log Calabi-Yau pair;*

Proof. Fix $n \geq N_0$ and consider the projective bundle $X = \mathbb{P}(\mathcal{V})$ over $\mathbb{F}_n$, where
$$\mathcal{V} = \mathcal{O}_{\mathbb{F}_n} \oplus \mathcal{O}_{\mathbb{F}_n}(-2C_0 + F).$$
First, blow up the projective bundle along the base locus of the bianticanonical divisor obtaining X_1. Next, blow up such a variety along the base locus of the bianticanonical divisor to obtain X_2. Take the degree two covering of X_2 with branch B_2 as described in the previous sections to finally obtain Y_2. Then one can take $Y = Y_2$ and $D = \beta^* E_2$. Everything, apart form the calculation for the Euler characteristics, have been done in the previous sections.

In order to compute the Euler characteristic, recall that if D is a smooth irreducible divisor on X_2, we have
$$c_2(D) = c_2(X_2) - c_1(X_2)D + D^2 \tag{28}$$
so
$$\mathrm{e}(D) = (c_2(X_2) - c_1(X_2)D + D^2)N_{D/X_2}. \tag{29}$$
Hence, being the branch locus B_2 a smooth element of $|-2K_{X_2} - 2E_2|$, we have that
$$e(B_2) = 48n + 70.$$
The Euler number of X_2 is given by 12 so we have, finally,
$$\mathrm{e}(Y_2) = 2\,\mathrm{e}(X_2) - \mathrm{e}(B_2) = 2 \cdot 12 - (48n + 70) = -48n - 46. \tag{30}$$
Although feasible by hands, we have done the last computation using `Magma`. □

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