

NEW EXAMPLES OF CALABI–YAU THREEFOLDS AND GENUS ZERO SURFACES

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ABSTRACT. We classify the subgroups of the automorphism group of the product of 4 projective lines admitting an invariant anticanonical smooth divisor on which the action is free. As a first application, we describe new examples of Calabi–Yau 3-folds with small Hodge numbers. In particular, the Picard number is 1 and the number of moduli is 5. Furthermore, the fundamental group is non-trivial. We also construct a new family of minimal surfaces of general type with geometric genus zero, $K^2 = 3$ and fundamental group of order 16. We show that this family dominates an irreducible component of dimension 4 of the moduli space of the surfaces of general type.

1. INTRODUCTION

A smooth ample divisor in a Calabi–Yau 3-fold is a minimal surface of general type. This simple observation yields a bridge between two important classes of algebraic varieties; a bridge that has had many applications. The most famous example is the construction of the first Calabi–Yau 3-fold with nonabelian fundamental group by Beauville in [Bea99], obtained by extending Reid’s construction in [Rei] of a Campedelli surface with fundamental group isomorphic to the group of quaternions Q_8 . Beauville shows that the surfaces constructed by Reid are all “rigid ample surfaces” (i.e., smooth ample divisors S in Y such that $h^0(\mathcal{O}_Y(S)) = 1$) in the 3-fold he constructs.

Beauville also points out that a rigid ample surface in a Calabi–Yau 3-fold is a surface with $p_g = 0$, which is one of the most interesting classes of surfaces of general type. He also mentions that whereas “for surfaces with $p_g = 0$ and $K^2 = 1$ or 2 we have a great deal of information (...) little is known about surfaces with $p_g = 0$ and $K^2 = 3, 4, 5$.” He refers to Inoue’s examples with fundamental group $Q_8 \oplus (\mathbb{Z}_2)^{K^2-2}$, asking if they are rigid ample surfaces in a Calabi–Yau 3-fold. [NP11] proves that the answer is affirmative when $K^2 = 3$.

Nowadays, we know a bit more on surfaces of general type with $p_g = 0$, but not that much. We know all possible algebraic fundamental groups of a minimal surface of general type with $p_g = 0$ and $K^2 = 1, 2$, a short list of finite groups, and the cases of bigger order are classified. But we know very little about the next cases. For sure, a similar result

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is not possible for $K^2 \geq 4$, since there are examples with infinite algebraic fundamental group: see [BCP11] for a more precise account on the state of the art in this research area.

There is an old standing conjecture by Miles Reid [Rei79, Conjecture 4] which, by a result of Mendes Lopes and Pardini [MP07, Theorem 1.2], would imply that all surfaces of general type with $p_g = 0$ and $K^2 = 3$ have finite algebraic fundamental group. If the conjecture is true, it should be possible to extend the results of the case $K^2 \leq 2$ to this case. In particular, one could hope to classify the surfaces with the biggest possible fundamental groups. There is a popular conjecture - not written anywhere, but usually attributed to M. Reid - asserting that the maximal order should be 16.

In the literature, there are 3 families of such surfaces with fundamental group of cardinality 16: one with fundamental group $Q_8 \oplus \mathbb{Z}_2$ ([BC10] and [NP11]), one with fundamental group $\mathbb{Z}_2^2 \oplus \mathbb{Z}_4$ ([MP04]), one constructed very recently ([BC12]) with fundamental group the central product of the dihedral group with 8 elements and \mathbb{Z}_4 . The first two families dominate irreducible components of the moduli space ([BC10], [Che12]). In this paper, we construct a fourth family, dominating an irreducible component. More examples of surfaces with $p_g = 0$ and $K^2 = 3$ have been recently constructed in [BP12], [CS10], [PPS10], [PPS09], [KL10] and [Rit10]. In all these examples the fundamental group is smaller or unknown.

On the Calabi–Yau side, physicists have focused recently on Calabi–Yau’s with small Hodge numbers $(h^{1,1}, h^{1,2})$: see, for instance, [Bra11], [CD10], [BCD10], [Dav11], [FS11] and [BDS12]. In [BF11], the authors describe some new examples of Calabi–Yau varieties. They are given as quotients of anticanonical sections of Fano varieties by finite groups G acting freely. The Fano varieties are products of del Pezzo surfaces of various degrees. In particular, for the product X of four complex projective lines, there exists a Calabi–Yau Y with Hodge numbers $(h^{1,1}, h^{1,2}) = (1, 5)$ and fundamental group isomorphic to $\mathbb{Z}_8 \oplus \mathbb{Z}_2$.

In [BF11] an upper bound on the order of G - depending on X - has been found. This bound is maximal (equal to 16) if and only if $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. It is then natural to ask which finite groups - among those of order 16 - yield free and smooth quotients of Calabi–Yau manifolds. These quotients are again Calabi–Yau threefolds but have smaller height and non-trivial fundamental group. In this paper we investigate all possible actions, and come up with new non-isomorphic examples.

The main results of this work are the following. We construct new families of Calabi–Yau manifolds with small Hodge numbers. More precisely, we construct 4 families of Calabi–Yau 3-folds with fundamental group of order 16 and Hodge numbers $h^{1,1} = 1$, $h^{1,2} = 5$. We show that for two of these families, no Calabi–Yau in the family contains a rigid ample divisor with $K^2 = 3$. On the contrary, in the other 2 cases such a divisor exists, giving 2 families of surfaces of general type with $p_g = 0$. One of these families is the family studied in [NP11]. The other family is a family of minimal surfaces of general type with $p_g = 0$, $K^2 = 3$ and fundamental group $\mathbb{Z}_4 \times \mathbb{Z}_4$: there is no example of a surface with the same topological type in the literature. We also show that this family dominates an irreducible component of the moduli space of the surfaces of general type.

The method is the following. Consider the 4-fold $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$. Assume that G is a finite subgroup of $\text{Aut}(X)$, and let Y be a smooth divisor in $|O_X(2, 2, 2, 2)|$, which is G -invariant and such that the action of G on Y is free. Then the quotient Y/G is also a smooth Calabi–Yau 3-fold. In this situation we say that (Y, G) is an admissible pair.

We classify all subgroups G of $\text{Aut}(X)$ which appear in an admissible pair (Y, G) . More precisely, we see that for each isomorphism class of groups, there is, up to conjugacy, at most one possible subgroup of $\text{Aut}(X)$, which may form an admissible pair. We determine the groups which appear in an admissible pair, and we give examples of admissible pairs in each case. The Hodge numbers of Y/G depend only on G , and we compute them in all cases.

This classification leads to exactly 4 groups of order 16. In the two Abelian cases, we show that G does not act on any divisor in $|O_X(1, 1, 1, 1)|$; in other words, Y/G has no rigid ample divisor with $K^2 = 3$. In the case of $G = \mathbb{Z}_4 \times \mathbb{Z}_4$, such a divisor exists, yielding a 4-dimensional family of surfaces of general type with fundamental group G . We show that this family dominates an irreducible component of the moduli space. A similar result holds in the last case $G = \mathbb{Z}_2 \oplus Q_8$; we skipped this case because that family was already completely studied in [NP11] and [BC10].

2. AUTOMORPHISMS OF $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$

Every $g \in \text{Aut}(X)$ acts on the 4 factors (see, for instance, [BF11]) giving a surjective homomorphism $\pi: \text{Aut}(X) \rightarrow S_4$ with kernel $\text{PGL}(2)^{\times 4}$. On the other hand the permutations of the factors give an inclusion $S_4 \hookrightarrow \text{Aut}(X)$ splitting π and therefore giving a structure of semidirect product

$$\text{Aut}(X) \cong S_4 \ltimes \text{PGL}(2)^{\times 4}$$

Concretely this gives, $\forall g \in \text{Aut} X$, a unique decomposition $g = (A_i) \circ \sigma$ where $\sigma = \pi(g)$, $(A_i) = (A_1, A_2, A_3, A_4) \in \text{PGL}(2)^{\times 4}$, and $\sigma(A_i)\sigma^{-1} = (A_{\sigma(i)})$.

If $g = (A_i) \circ \sigma$ and $h = (B_i) \circ \tau$,

$$h \circ g = (B_i A_{\tau(i)}) \circ (\tau \circ \sigma),$$

$$h \circ g \circ h^{-1} = (B_i A_{\tau(i)} B_{(\tau \circ \sigma \circ \tau^{-1})(i)}^{-1}) \circ (\tau \circ \sigma \circ \tau^{-1}).$$

For the purpose of what follows, we will denote by A and B the automorphisms of \mathbb{P}^1 that are represented, respectively, by

$$(x_0 : x_1) \mapsto (x_0 : -x_1) \quad \text{and} \quad (x_0 : x_1) \mapsto (x_1 : x_0),$$

where $(x_0 : x_1)$ are projective coordinates on \mathbb{P}^1 . It is easy to see that A and B have order 2. The group $\langle B \rangle$ generated by B is conjugated to the group $\langle A \rangle$ generated by A and this is true for every subgroup of $\text{PGL}(2)$ of order 2. In general, the following holds true.

Theorem 2.1 (Klein). *If G is a finite subgroup of $\text{PGL}(2)$, then G is isomorphic to \mathbb{Z}_n , D_{2n} , A_4 , S_4 or A_5 . Moreover, two isomorphic finite subgroups are conjugate.*

For every subgroup G of $\text{Aut}(X)$ we will denote by $\text{Fix}(G)$ the set of the points of X which are fixed by some nontrivial elements in G .

Remark 2.2. We are interested in automorphisms $g \in \text{Aut}(X)$ whose fixed locus is disjoint from the zero locus of a suitable section of $O_X(-K_X)$. This implies $\dim \text{Fix} \langle g \rangle = 0$. In fact $-K_X$ is ample and therefore, if C is a curve in $\text{Fix} \langle g \rangle$, then $C \cdot Y > 0$ so there is at least one fixed point of $\langle g \rangle$ lying on Y . On the other hand, $\text{Fix} \langle g \rangle$ is not empty by holomorphic Lefschetz fixed point formula.

For the holomorphic fixed point formula we refer the reader to [AB68, Theorem 4.12]. For the convenience of the reader, we recall here the following corollary which we will use.

Theorem 2.3 (Corollary of the holomorphic Lefschetz fixed point formula). *Let X be a compact complex manifold and let $f \in \text{Aut}(X)$ be an automorphism with at most a finite number of isolated nondegenerate fixed points. If we write $f^*|_{H^{0,q}(X)}$ for the endomorphism induced by f on $H^{0,q}(X)$, then*

$$\sum_{x \in \text{Fix}(f)} \frac{1}{\det(I - d_x f)} = \sum_q (-1)^q \text{Tr}(f^*|_{H^{0,q}(X)}).$$

In our case, since for $X = (\mathbb{P}^1)^4$ one has $H^{0,*}(X) = H^{0,0}(X) \cong \mathbb{C}$, the right side of the equation is equal to 1. This implies that the left side has to be different from zero and, in particular, it is necessary to have at least a fixed point to have a contribution.

It is shown in [BF11] that if $\text{Fix}(G)$ does not intersect an anticanonical divisor, then $|G|$ divides 16. We have then 4 cases to consider: groups of order 2, 4, 8 or 16.

2.1. Subgroups of order 2.

The condition in Remark 2.2 is a very restrictive condition for an automorphism of order 2. In fact, up to conjugacy, there is only one involution of $\text{Aut}(X)$ with 0-dimensional fixed locus, as the following lemma shows.

Lemma 2.4. *Assume that $g \in \text{Aut}(X)$ is an automorphism of order 2 such that $\text{Fix}(g)$ has dimension 0. Then $\pi(g) = \text{Id}$ and g is conjugate to (A, A, A, A) .*

Proof. If $g = (A_i) \circ \pi(g)$ has order 2 then $\sigma := \pi(g)$ has order at most 2 and, since $g^2 = (A_i A_{\sigma(i)}) \circ \sigma^2$, we have the relations $A_i A_{\sigma(i)} = \text{Id}$ in $\text{PGL}(2)$. Up to conjugation, we can assume $\sigma \in \{\text{Id}, (12), (12)(34)\}$. If $\sigma = (12)$, let \underline{x}_3 and \underline{x}_4 be two fixed points of A_3 and A_4 , respectively (the existence of which is guaranteed by the holomorphic Lefschetz fixed point formula). Then, every point of the form

$$P_{\underline{x}_1} := (\underline{x}_1, A_2 \underline{x}_1, \underline{x}_3, \underline{x}_4), \quad \underline{x}_1 \in \mathbb{P}^1,$$

is fixed. Indeed, since $A_1 A_2 = \text{Id}$, we get

$$P_{\underline{x}_1} \mapsto (A_1(A_2 \underline{x}_1), A_2 \underline{x}_1, A_3 \underline{x}_3, A_4 \underline{x}_4) = ((A_1 A_2) \underline{x}_1, A_2 \underline{x}_1, \underline{x}_3, \underline{x}_4) = P_{\underline{x}_1}.$$

Similarly, if $\sigma = (12)(34)$, the 2-dimensional locus consisting of the points of the form $(\underline{x}_1, A_2 \underline{x}_1, \underline{x}_2, A_4 \underline{x}_2)$ is pointwise fixed.

Accordingly, we may assume that $\sigma = \pi(g) = \text{Id}$, hence $g = (A_i)$ with (since $g^2 = \text{Id}$) $A_i^2 = \text{Id}$, for all i . If $A_i = \text{Id}$ for some i , one has at least a curve of fixed points (namely the i -th \mathbb{P}^1), so every A_i has order 2 as an element of $\text{PGL}(2)$. By Theorem 2.1, all A_i

are conjugated in $\mathrm{PGL}(2)$ to A , so there exists $B_i \in \mathrm{PGL}(2)$ such that $B_i^{-1}A_iB_i = A$. Then $(B_i)^{-1} \circ g \circ (B_i) = (A, A, A, A)$. \square

Note that if H is a nontrivial subgroup of G , then $\mathrm{Fix}(H) \subset \mathrm{Fix}(G)$. In particular, if $\dim \mathrm{Fix}(G) = 0$, every element of order 2 in G belongs to $\mathrm{Ker}(\pi)$.

2.2. Subgroups of order 4.

Up to isomorphism, there are 2 groups of order 4, namely \mathbb{Z}_4 and $\mathbb{Z}_2 \oplus \mathbb{Z}_2$.

Lemma 2.5. *Assume that $\mathbb{Z}_2 \oplus \mathbb{Z}_2 \simeq G \leq \mathrm{Aut}(X)$ satisfies $\dim \mathrm{Fix}(G) = 0$. Then G is conjugated to $\langle g, h \rangle$ with $g = (A, A, A, A)$ and $h = (B, B, B, B)$.*

Proof. Take two non-trivial elements $g, h \in G$. These have order 2 and a finite number of fixed points. By Lemma 2.4, we have $\pi(g) = \pi(h) = \mathrm{Id}$ so $g = (A_i)$ and $h = (B_i)$. Since $g^2 = h^2 = ghg^{-1}h^{-1} = \mathrm{Id}$, one obtains $A_iB_i = B_iA_i$ and $A_i^2 = B_i^2 = \mathrm{Id}$, hence $\langle A_i, B_i \rangle \leq \mathrm{PGL}(2)$ is isomorphic to a subgroup of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. If this is a proper subgroup of $\mathbb{Z}_2 \oplus \mathbb{Z}_2$, then either A_i, B_i or A_iB_i is the identity and so one of g, h or gh has at least a line of fixed points, which contradicts the assumptions. By Theorem 2.1, $\langle A_i, B_i \rangle$ is conjugated to $\langle A, B \rangle$ in $\mathrm{PGL}(2)$. Moreover the internal automorphisms of $\mathrm{PGL}(2)$ which fix the subgroup $\langle A, B \rangle$ act on $\{A, B, AB\}$ as the full group of permutations. Therefore there is a $C_i \in \mathrm{PGL}(2)$ such that $C_i^{-1}A_iC_i = A$, $C_i^{-1}B_iC_i = B$; setting $k := (C_i)$, we have $k^{-1} \langle g, h \rangle k = \langle (A), (B) \rangle$. \square

Lemma 2.6. *Assume that $g = (A_i) \circ \sigma \in \mathrm{Aut}(X)$ has order 4 and that there exists an eigensection $s \in H^0(X, \mathcal{O}_X(-K_X))$ of g such that $V(s) \cap \mathrm{Fix} \langle g \rangle = \emptyset$. Then g is conjugated to $(\mathrm{Id}, A, \mathrm{Id}, A) \circ (12)(34)$.*

Proof. Denote $G := \langle g \rangle \simeq \mathbb{Z}_4$. By Remark 2.2, $\dim \mathrm{Fix}(G) = 0$. Since $o(g) = 4$, we get $\mathrm{Fix}(G) = \mathrm{Fix}(g^2)$, so $\dim \mathrm{Fix}(g^2) = 0$. Hence by Lemma 2.4, g^2 is conjugated to (A, A, A, A) . Since

$$(A) = h^{-1}g^2h = (h^{-1}gh)^2,$$

we may assume, up to conjugation, that $g^2 = (A, A, A, A)$. Thus $g = (A_1, A_2, A_3, A_4) \circ \sigma$ with $\sigma^2 = 1$ and $A_iA_{\sigma(i)} = A$. By conjugation, we may assume $\sigma \in \{\mathrm{Id}, (12), (12)(34)\}$. The fixed points of G are

$$\mathrm{Fix}(G) = \{(P_1, P_2, P_3, P_4) : P_i \in \{(1 : 0), (0 : 1)\}\}.$$

Let us now show that if $\sigma \in \{\mathrm{Id}, (12)\}$, then at least a fixed point belongs to $V(s)$ for every invariant section s . If s is a section of $s \in H^0(X, \mathcal{O}_X(-K_X))$, then s is a polynomial of multidegree $(2, 2, 2, 2)$ in the variables $((x_{10}, x_{11}), (x_{20}, x_{21}), (x_{30}, x_{31}), (x_{40}, x_{41}))$. The condition $V(s) \cap \mathrm{Fix}(G) = \emptyset$ is equivalent to all of the coefficients of the 16 monomials $x_{1i}^2 x_{2j}^2 x_{3k}^2 x_{4l}^2$ being nonzero.

If $\sigma = \mathrm{Id}$, then $A_j^2 = A$ for all $j = 1, \dots, 4$. Hence $A_j = \begin{bmatrix} 1 & 0 \\ 0 & \pm i \end{bmatrix}$. Then $x_{10}^2 x_{20}^2 x_{30}^2 x_{40}^2$ and $x_{10}^2 x_{20}^2 x_{30}^2 x_{41}^2$ are eigenvectors for the natural lift of the action of g on $H^0(X, \mathcal{O}_X(-K_X))$ with different eigenvalue. So they cannot both appear with nontrivial coefficient in an eigensection s , a contradiction.

If $\sigma = (12)$, then $A_1A_2 = A_2A_1 = A_3^2 = A_4^2 = A$. Hence, we get $A_3, A_4 \in \left\{ \begin{bmatrix} 1 & 0 \\ 0 & \pm i \end{bmatrix} \right\}$. If $k = (A_1, \text{Id}, \text{Id}, \text{Id}) \circ \text{Id}$, then

$$k^{-1}gk = (A_1^{-1}A_1 \text{Id}, \text{Id} A_2A_1, A_3, A_4) \circ (12) = (\text{Id}, A, A_3, A_4) \circ (12),$$

so we may take g of the form

$$g = \left(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & \pm i \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 0 & \pm i \end{bmatrix} \right) \circ (12).$$

As in the previous case, this leads to a contradiction, since $x_{10}^2x_{20}^2x_{30}^2x_{40}^2$ and $x_{10}^2x_{20}^2x_{30}^2x_{41}^2$ are eigenvectors for the natural lift of the action of g on $H^0(X, O_X(-K_X))$ with different eigenvalue.

We conclude that we must have $\sigma = (12)(34)$. Then $g = (A_1, A_2, A_3, A_4) \circ (12)(34)$ with $g^2 = (A)$. Letting $k = (A_1, \text{Id}, A_3, \text{Id}) \circ \text{Id}$, we get $k^{-1}gk = (\text{Id}, A, \text{Id}, A) \circ (12)(34)$. \square

2.3. Subgroups of order 8.

Up to isomorphism, there are 5 groups of order 8; these are $\mathbb{Z}_2^{\oplus 3}, D_8, \mathbb{Z}_8, \mathbb{Z}_2 \oplus \mathbb{Z}_4$ and Q_8 .

Lemma 2.7. *No subgroup G of $\text{Aut}(X)$ with $\dim \text{Fix}(G) = 0$ can be isomorphic to $\mathbb{Z}_2^{\oplus 3}$.*

Proof. Assume by contradiction that such a group $G \simeq \mathbb{Z}_2^{\oplus 3}$ exists. By Lemma 2.4, $\pi(G) = \{\text{Id}\}$. Then G is generated by 3 elements of order 2 of the form $g_j = (A_{ij})$ for $j = 1, 2, 3$. By Theorem 2.1, the subgroup of $\text{PGL}(2)$ generated by A_{11}, A_{12}, A_{13} cannot be isomorphic to $\mathbb{Z}_2^{\oplus 3}$. Thus $\langle A_{11}, A_{12}, A_{13} \rangle$ must be isomorphic to one of the proper subgroups of $\mathbb{Z}_2^{\oplus 3}$. We deduce that there exists a nontrivial $g \in G$ acting trivially on the first factor. Since every automorphism of \mathbb{P}^1 has fixed points, the action of g on the other two factors has some fixed points, giving a 1-dimensional locus of fixed points of g on X , contradicting $\dim \text{Fix}(G) = 0$. \square

Lemma 2.8. *Let G be a subgroup of $\text{Aut}(X)$ isomorphic to D_8 . Then, for every eigensection $s \in H^0(X, O_X(-K_X))$, we have $V(s) \cap \text{Fix}(G) \neq \emptyset$.*

Proof. Assume by contradiction $V(s) \cap \text{Fix}(G) = \emptyset$. Then by Remark 2.2 and Lemma 2.4, for every reflection $s \in G$, $\pi(g) = \text{Id}$. By Lemma 2.6, for a rotation r of order 4, $\pi(r) \neq \text{Id}$. But in D_8 every rotation is product of two reflections, $r = s_1s_2$. This is a contradiction since then $\text{Id} \neq \pi(r) = \pi(s_1)\pi(s_2) = \text{Id} \cdot \text{Id}$. \square

Lemma 2.9. *Assume that $G \simeq \mathbb{Z}_4 \oplus \mathbb{Z}_2$ is a subgroup of $\text{Aut}(X)$ such that there exists an eigensection $s \in H^0(X, O_X(-K_X))$ with $V(s) \cap \text{Fix}(G) = \emptyset$. Then, up to conjugation, $G = \langle (\text{Id}, A, \text{Id}, A) \circ (12)(34), (B, B, B, B) \rangle$.*

Proof. Assume that $G \simeq \mathbb{Z}_4 \oplus \mathbb{Z}_2$ with generators g and h . By Lemma 2.5, we may assume $g^2 = (A, A, A, A)$ and $h = (B, B, B, B)$. By Lemma 2.6, $\pi(g)$ is conjugated to $(12)(34)$ so that we may assume $g = (A_i) \circ (12)(34)$. Since

$$A_1A_2 = A_2A_1 = A_3A_4 = A_4A_3 = A,$$

and $A_iB = BA_i$, for all $i = 1, \dots, 4$ we deduce that G is conjugated to a subgroup of $\text{Aut}(X)$ given by $\langle (\text{Id}, A, \text{Id}, A) \circ (12)(34), (B, B, B, B) \rangle$, via $k := (A_1, \text{Id}, A_3, \text{Id})$. \square

Lemma 2.10. *Assume that $G \simeq \mathbb{Z}_8$ is a subgroup of $\text{Aut}(X)$ such that there exists an eigensection $s \in H^0(X, O_X(-K_X))$ with $V(s) \cap \text{Fix}(G) = \emptyset$. Then, up to conjugation, $G = \langle (\text{Id}, \text{Id}, \text{Id}, A) \circ (1324) \rangle$.*

Proof. Choose a generator $g = (A_i) \circ \sigma$ of $G \simeq \mathbb{Z}_8$. By Lemma 2.6, we may assume $g^2 = (\text{Id}, A, \text{Id}, A) \circ (12)(34)$, so σ is equal to (1324) or its inverse. Substituting g with g^{-1} we may assume $\sigma = (1324)$. Then $A_1 = A_2 = A_3^{-1} = AA_4^{-1}$, with $A_1A = AA_1$. We can then reduce to $g = (\text{Id}, \text{Id}, \text{Id}, A) \circ (1324)$ by conjugation with $k = (A_1, A_1, \text{Id}, \text{Id})$. \square

Lemma 2.11. *Assume that $G \simeq Q_8$ is a subgroup of $\text{Aut}(X)$ such that there exists an eigensection $s \in H^0(X, O_X(-K_X))$ with $V(s) \cap \text{Fix}(G) = \emptyset$. Then, up to conjugation, $G = \langle (\text{Id}, A, \text{Id}, A) \circ (12)(34), (\text{Id}, A, A, \text{Id}) \circ (13)(24) \rangle$.*

Proof. As usual for $G \simeq Q_8$, let i, j and $k = ij$ be generators of order 4. By Lemma 2.6, $\pi(i), \pi(j), \pi(k) \in \{(12)(34), (13)(24), (14)(23)\}$. Since $\pi(i)\pi(j) = \pi(k)$, we deduce that $\pi(i), \pi(j)$ and $\pi(k)$ are all distinct. We get $\pi(Q_8) = \langle (12)(34), (13)(24) \rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$. In particular, up to conjugation, $i = (\text{Id}, A, \text{Id}, A) \circ (12)(34)$ and $j = (B_i) \circ (13)(24)$ with $j^2 = (A, A, A, A)$. This forces j to be of the form $(B_1, B_2, B_1^{-1}A, B_2^{-1}A) \circ (13)(24)$ with $AB_i = B_iA$. Similarly, for $k := ij = (B_2, AB_1, B_2^{-1}A, B_1^{-1}) \circ (14)(23)$ we know that $k^2 = (A, A, A, A)$, which implies $B_2 = AB_1$. Choosing $l = (B_1, B_1, \text{Id}, \text{Id})$, we get $l^{-1}il = i$ and $l^{-1}jl = (\text{Id}, A, A, \text{Id}) \circ (13)(24)$. \square

2.4. Subgroups of order 16.

Up to isomorphism, there are 14 groups of order 16. These are:

$$\begin{aligned} & \mathbb{Z}_8 \oplus \mathbb{Z}_2, \quad \mathbb{Z}_4 \oplus \mathbb{Z}_4, \quad \mathbb{Z}_{16}, \quad \mathbb{Z}_4 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2 \oplus \mathbb{Z}_2, \quad Q_8 \oplus \mathbb{Z}_2, \quad D_8 \oplus \mathbb{Z}_2, \\ & \mathbb{Z}_4 \rtimes \mathbb{Z}_4 := \langle g, h \mid g^4 = h^4 = \text{Id}, hgh^{-1} = g^{-1} \rangle, \quad Q_{16} := \langle g, h, k \mid g^4 = h^2 = k^2 = ghk \rangle, \\ & CP(D_8, \mathbb{Z}_4) := \langle g, h, k \mid g^4 = h^2 = [g, h] = [h, k] = \text{Id}, g^2 = k^2, ghg = h \rangle, \\ & SG(16, 3) := \langle g, h, k \mid g^4 = h^2 = k^2 = [g, h] = [h, k] = \text{Id}, k g k^{-1} = gh \rangle, \\ & D_{16} := \langle g, h \mid g^8 = h^2 = \text{Id}, h^{-1}gh = g^{-1} \rangle, \\ & SD_{16} := \langle g, h \mid g^8 = h^2 = \text{Id}, h^{-1}gh = g^3 \rangle, \\ & M_{16} := \langle g, h \mid g^8 = h^2 = \text{Id}, h^{-1}gh = g^5 \rangle. \end{aligned}$$

Proposition 2.12. *Let G be a subgroup of $\text{Aut}(X)$ such that there is an eigensection $s \in H^0(X, O_X(-K_X))$, with $V(s) \cap \text{Fix}(G) = \emptyset$. Then G cannot be isomorphic to any of the following groups: $\mathbb{Z}_2^{\oplus 4}, \mathbb{Z}_4 \oplus \mathbb{Z}_2^{\oplus 2}, SG(16, 3), CP(D_8, \mathbb{Z}_4), D_8 \oplus \mathbb{Z}_2, D_{16}, SD_{16}$.*

Proof. This follows directly by Lemmas 2.7 and 2.8 since all the groups in the statement have a subgroup isomorphic either to $\mathbb{Z}_2^{\oplus 3}$ or to D_8 . \square

Proposition 2.13. *Let G be a subgroup of $\text{Aut}(X)$ such that there is an eigensection $s \in H^0(X, O_X(-K_X))$, with $V(s) \cap \text{Fix}(G) = \emptyset$. Then G is not isomorphic to \mathbb{Z}_{16} or Q_{16} .*

Proof. If $G \simeq \mathbb{Z}_{16}$, choose a generator g and set $\sigma = \pi(g)$. By Lemma 2.6, up to conjugation, $\sigma^4 = \pi(g^4) = (12)(34)$. But $(12)(34)$ has no fourth root in S_4 , a contradiction. If $G \simeq Q_{16}$ then, as in the list above, $g^4 = h^2 = ghgh$, with g of order 8 and h, gh of order 4. By the lemmas 2.6 and 2.10, we may assume $\pi(g) = (1234)$ whereas $\pi(h), \pi(gh) \in \{(12)(34), (13)(24), (14)(23)\}$. This contradicts $\pi(g)\pi(h) = \pi(gh)$. \square

Proposition 2.14. *Let G be a subgroup of $\text{Aut}(X)$ such that there is an eigensection $s \in H^0(X, \mathcal{O}_X(-K_X))$, with $V(s) \cap \text{Fix}(G) = \emptyset$. Then G is not isomorphic to M_{16} .*

Proof. If G is isomorphic to M_{16} , it has two generators g and h of order 8 and 2, respectively, such that $h^{-1}gh = g^5$. By Lemma 2.4 and Lemma 2.10, up to conjugation, we can assume that $h = (A, A, A, A)$, and $g = (A_1, A_2, A_3, A_4) \circ \sigma$ where $\sigma = (1324)$. Denote by D the element $A_1A_2A_3A_4 \in \text{PGL}(2)$. Since $o(g) = 8$, D is an involution. By direct computation, $h^{-1}gh = hgh = (AA_iA) \circ (1324)$ and $g^5 = (D, D, D, DA_4) \circ (1324)$. Since $h^{-1}gh = g^5$,

$$AA_1A = AA_2A = AA_3A = D \quad \text{and} \quad AA_4A = DA_4.$$

Therefore $A_1 = A_2 = A_3 = ADA$ and then $D = A_1A_2A_3A_4 = A_1^3A_4 = ADA_4$, so that $A_4 = (AD)^2$. Then, substituting in $AA_4A = DA_4$, we get $AADADA = DADAD$. Since $A^2 = \text{Id}$ we get $D = \text{Id}$. But then $g^4 = (D, D, D, D) = \text{Id}$, contradicting $o(g) = 8$. \square

There are 4 cases left, $\mathbb{Z}_8 \oplus \mathbb{Z}_2$, $\mathbb{Z}_4 \oplus \mathbb{Z}_4$, $Q_8 \oplus \mathbb{Z}_2$ and $\mathbb{Z}_4 \times \mathbb{Z}_4$. We will now show that they all occur and that in each case the group action is determined up to conjugacy.

Theorem 2.15. *Let G be a subgroup of $\text{Aut}(X)$ such that there is an eigensection $s \in H^0(X, \mathcal{O}_X(-K_X))$, with $V(s) \cap \text{Fix}(G) = \emptyset$. Then, up to conjugation,*

- (1) $G = \langle (\text{Id}, \text{Id}, \text{Id}, A) \circ (1324), (B, B, B, B) \rangle \simeq \mathbb{Z}_8 \oplus \mathbb{Z}_2$;
- (2) $G = \langle (\text{Id}, A, \text{Id}, A) \circ (12)(34), (\text{Id}, \text{Id}, B, B) \circ (13)(24) \rangle \simeq \mathbb{Z}_4 \oplus \mathbb{Z}_4$;
- (3) $G = \langle (\text{Id}, A, \text{Id}, A) \circ (12)(34), (\text{Id}, A, B, AB) \circ (13)(24) \rangle \simeq \mathbb{Z}_4 \times \mathbb{Z}_4$;
- (4) $G = \langle (\text{Id}, A, \text{Id}, A) \circ (12)(34), (\text{Id}, A, A, \text{Id}) \circ (13)(24), (B, B, B, B) \rangle \simeq Q_8 \oplus \mathbb{Z}_2$.

Proof. By the previous propositions, G is isomorphic to $\mathbb{Z}_8 \oplus \mathbb{Z}_2$, $\mathbb{Z}_4 \oplus \mathbb{Z}_4$, $Q_8 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_4 \times \mathbb{Z}_4$.

Case (1). Let $G \simeq \mathbb{Z}_8 \oplus \mathbb{Z}_2$ with generators g, h of order 8 and 2, respectively. By Lemma 2.10, up to conjugacy, $g = (\text{Id}, \text{Id}, \text{Id}, A) \circ (1324)$. The element h has order 2 so it is of the form $h = (B_1, B_2, B_3, B_4)$, with $B_i^2 = \text{Id}$. Since $gh = hg$,

$$B_3 = B_1, B_4 = B_2, B_2 = B_3 \text{ and } AB_1 = B_4A,$$

so that $B_i \equiv B_1$ for all i , and $B_1AB_1^{-1}A^{-1} = B_1^2 = \text{Id}$. The elements of $\text{PGL}(2)$ commuting with A leave invariant the set of fixed points of A ; they are represented by a matrix which is either diagonal (if B_1 fixes both points) or antidiagonal (if B_1 exchanges them). In the diagonal case, since $B_1^2 = \text{Id}$, we get $B_1 \in \{\text{Id}, A\}$ and then $g^4 = h$, a contradiction.

We deduce that B_1 is represented by an antidiagonal matrix

$$B_1 = \begin{bmatrix} 0 & 1 \\ f^2 & 0 \end{bmatrix}$$

for some $f \in \mathbb{C}^*$. Set $C = \begin{bmatrix} 1 & 0 \\ 0 & f \end{bmatrix}$ and $k := (C, C, C, C)$. Then

$$k^{-1}gk = g \quad \text{and} \quad k^{-1}hk = (B, B, B, B).$$

Case (2). Let $G \simeq \mathbb{Z}_4 \oplus \mathbb{Z}_4$. Let g and h be the two generators of the two factors of G . We know that the subgroup $\langle g, h^2 \rangle \simeq \mathbb{Z}_4 \oplus \mathbb{Z}_2$ is conjugated to

$$\langle (\text{Id}, A, \text{Id}, A) \circ (12)(34), (B, B, B, B) \rangle,$$

so we may assume $g = (\text{Id}, A, \text{Id}, A) \circ (12)(34)$ and $h^2 = (B, B, B, B)$. By Lemma 2.6, $\sigma := \pi(h) \in \{(12)(34), (13)(24), (14)(23)\}$. Since the same is true for $\pi(gh)$, we have $\sigma \neq (12)(34)$. By replacing h with gh , we may assume $\sigma = (13)(24)$. Since $gh = hg$ and $h^2 = (B, B, B, B)$,

$$h = (C, C, C^{-1}B, C^{-1}B) \circ (13)(24)$$

with $CAC^{-1}A^{-1} = CBC^{-1}B^{-1} = \text{Id}$. Conjugation with $k := (C, C, \text{Id}, \text{Id})$ sends g to g and h to $(\text{Id}, \text{Id}, B, B) \circ (13)(24)$.

Case (3). Let $G \simeq \mathbb{Z}_4 \times \mathbb{Z}_4$, $G = \langle g, h \mid g^4 = h^4 = ghgh^{-1} = \text{Id} \rangle$. Take g and h to be generators of the two \mathbb{Z}_4 . As with the case of $\mathbb{Z}_4 \oplus \mathbb{Z}_4$, we may assume $g = (\text{Id}, A, \text{Id}, A) \circ (12)(34)$, $h^2 = (B, B, B, B)$ and $\pi(h) = (13)(24)$. Since $ghg = h$,

$$h = (C, AC, BC^{-1}, ABC^{-1}) \circ (13)(24)$$

with $CAC^{-1}A^{-1} = CBC^{-1}B^{-1} = \text{Id}$. Conjugation with $k := (C, C, \text{Id}, \text{Id})$ sends g to g and h to $(\text{Id}, A, B, AB) \circ (13)(24)$.

Case (4). Let $G \simeq Q_8 \oplus \mathbb{Z}_2$. By Lemma 2.11, we can assume that $Q_8 = \langle i, j \rangle$ with $i = (\text{Id}, A, \text{Id}, A) \circ (12)(34)$ and $j = (\text{Id}, A, A, \text{Id}) \circ (13)(24)$. Let $h = (C_1, C_2, C_3, C_4)$ be the generator of the \mathbb{Z}_2 factor. Since $ih = hi$ and $jh = hj$, $C_1 = C_2 = C_3 = C_4 =: C$ with C commuting with A and $C^2 = \text{Id}$. These two conditions are satisfied if $C = \text{Id}$, $C = A$ or C is antidiagonal. If $C = \text{Id}$ or $C = A$ then $h \in Q_8$ which is impossible. Then $C = \begin{bmatrix} 0 & 1 \\ f^2 & 0 \end{bmatrix}$ for some $f \in \mathbb{C}^*$. Let $D = \begin{bmatrix} 0 & 1 \\ f & 0 \end{bmatrix}$. Conjugation by (D, D, D, D) fixes i and j and sends h to (B, B, B, B) . \square

3. HEIGHTS AND HODGE NUMBERS FOR Y/G

In what follows G is a finite subgroup of $\text{Aut}(X)$ and Y is a smooth Calabi–Yau threefold in $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ such that (Y, G) is an admissible pair, i.e., $G \leq \text{Stab}_{\text{Aut}(X)}(Y)$ and $\text{Fix}(G) \cap Y = \emptyset$. We will compute all pairs of Hodge numbers $(h^{1,1}, h^{1,2})$ and relative heights $h := h^{1,1} + h^{1,2}$ of Y/G . Every case is realized as a partial quotient of the examples of section 4. For the reader convenience we recall that, for every anticanonical divisor Y in X , $\chi(Y) = -128$ (by, e.g., [BF11, Theorem 3.1]) and that for every Calabi–Yau Z , $\chi(Z) = 2(h^{1,1}(Z) - h^{1,2}(Z))$.

Recall that, for an admissible pair (Y, G) , $\chi(Y/G) = \chi(Y)/|G|$ and

$$H^{1,1}(Y/G) = H^{1,1}(Y)^G = (\text{Pic}(X) \otimes \mathbb{C})^G = (\text{Pic}(X) \otimes \mathbb{C})^{\pi(G)}.$$

Hence it suffices to consider the image $\pi(G)$. For example, we have seen in Theorem 2.15 that, if $G \simeq \mathbb{Z}_4 \times \mathbb{Z}_4$ and (Y, G) is an admissible pair then $\pi(G) = \langle (12)(34), (13)(24) \rangle$. The action of G on $\mathbb{C}^4 = \text{Pic}(X) \otimes \mathbb{C}$ has an invariant space of dimension 1 (the diagonal) so $h^{1,1}(Y/G) = 1$.

If $|G| = 2$ one has $G \simeq \mathbb{Z}_2$, $\chi(Y/G) = -64$, $\pi(G) = \{\text{Id}\}$ and

- $(h^{1,1}(Y/G), h^{1,2}(Y/G)) = (4, 36)$ and $h(Y/G) = 40$.

If $|G| = 4$ then $\chi(Y/G) = -32$ and one of the following holds:

- $G \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$, $\pi(G) = \{\text{Id}\}$, $(h^{1,1}(Y/G), h^{1,2}(Y/G)) = (4, 20)$ and $h(Y/G) = 24$;
- $G \simeq \mathbb{Z}_4$, $\pi(G) = \langle (12)(34) \rangle$, $(h^{1,1}(Y/G), h^{1,2}(Y/G)) = (2, 18)$ and $h(Y/G) = 20$.

If G has order 8, then $\chi(Y/G) = -16$ and one of the following holds:

- $G \simeq \mathbb{Z}_8$ with $\pi(G) \simeq \langle (1234) \rangle \simeq \mathbb{Z}_4$,
 $(h^{1,1}(Y/G), h^{1,2}(Y/G)) = (1, 9)$ and $h(Y/G) = 10$;
- $G \simeq Q_8$ with $\pi(G) = \langle (12)(34), (13)(24) \rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$,
 $(h^{1,1}(Y/G), h^{1,2}(Y/G)) = (1, 9)$ and $h(Y/G) = 10$;
- $G \simeq \mathbb{Z}_4 \oplus \mathbb{Z}_2$ with $\pi(G) \simeq \langle (12)(34) \rangle \simeq \mathbb{Z}_2$,
 $(h^{1,1}(Y/G), h^{1,2}(Y/G)) = (2, 10)$ and $h(Y/G) = 12$.

Finally, if $|G| = 16$, then $\chi(Y/G) = -8$ and each quotient has the same Hodge numbers and height equal to $(1, 5)$ and 6, respectively. More precisely one of the following holds:

- $G \simeq \mathbb{Z}_8 \oplus \mathbb{Z}_2$ with $\pi(G) \simeq \langle (1234) \rangle \simeq \mathbb{Z}_4$;
- $G \simeq Q_8 \oplus \mathbb{Z}_2$ with $\pi(G) = \langle (12)(34), (13)(24) \rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$;
- $G \simeq \mathbb{Z}_4 \oplus \mathbb{Z}_4$ with $\pi(G) \simeq \langle (12)(34), (13)(24) \rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$;
- $G \simeq \mathbb{Z}_4 \times \mathbb{Z}_4$ with $\pi(G) \simeq \langle (12)(34), (13)(24) \rangle \simeq \mathbb{Z}_2 \oplus \mathbb{Z}_2$.

4. ADMISSIBLE PAIRS WITH GROUP OF ORDER 16

In this section we give examples of admissible pairs (Y, G) with G maximal, i.e., with $|G| = 16$. To be more precise, we give an example for each subgroup G of $\text{Aut}(X)$ of order 16 that acts freely on a Calabi–Yau threefold which is stabilized by G . As previously said, Y/G have Hodge numbers $(1, 5)$. By taking partial quotients, one obtains all the other examples.

4.1. Admissible pairs with $G = \mathbb{Z}_4 \times \mathbb{Z}_4$.

By Theorem 2.15, up to conjugation $G := \langle g, h \rangle \subset \text{Aut}(X)$ with

$$(1) \quad \begin{aligned} g &:= (\text{Id}, A, \text{Id}, A,) \circ (12)(34), \\ h &:= (\text{Id}, A, B, AB) \circ (13)(24). \end{aligned}$$

G is a 2-group with 3 elements of order 2, namely $g^2 = (A, A, A, A)$, $h^2 = (B, B, B, B)$, $g^2 h^2 = (AB, AB, AB, AB)$. Therefore $\text{Fix}(G) = \text{Fix}(g^2) \cup \text{Fix}(h^2) \cup \text{Fix}(g^2 h^2)$. These are 3 disjoint sets of 16 points each:

$$(2) \quad \begin{aligned} \text{Fix}(g^2) &= \{(\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{x}_4) : \forall i \underline{x}_i \in \{(0, 1), (1, 0)\}\}, \\ \text{Fix}(h^2) &= \{(\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{x}_4) : \forall i \underline{x}_i \in \{(1, 1), (1, -1)\}\}, \\ \text{Fix}(g^2 h^2) &= \{(\underline{x}_1, \underline{x}_2, \underline{x}_3, \underline{x}_4) : \forall i \underline{x}_i \in \{(1, i), (1, -i)\}\}. \end{aligned}$$

$H^0(\mathcal{O}_X(2, 2, 2, 2))$ has basis $\{x_{10}^{a_1}x_{11}^{b_1}x_{20}^{a_2}x_{21}^{b_2}x_{30}^{a_3}x_{31}^{b_3}x_{40}^{a_4}x_{41}^{b_4} : a_1 + b_1 = \dots = a_4 + b_4 = 2\}$. Consider $Q_0, \dots, Q_5 \in H^0(\mathcal{O}_X(2, 2, 2, 2))$ given by:

$$(3) \quad \begin{aligned} Q_0 &:= \prod_{i=1}^4 x_{i0}x_{i1}, \\ Q_1 &:= (x_{11}^2x_{20}^2 + x_{10}^2x_{21}^2)(x_{31}^2x_{40}^2 + x_{30}^2x_{41}^2), \\ Q_2 &:= x_{20}x_{21}x_{30}x_{31}(x_{10}^2 + x_{11}^2)(x_{40}^2 + x_{41}^2) - x_{10}x_{11}x_{40}x_{41}(x_{20}^2 + x_{21}^2)(x_{30}^2 + x_{31}^2), \\ Q_3 &:= x_{10}x_{11}x_{30}x_{31}(x_{20}^2 + x_{21}^2)(x_{40}^2 + x_{41}^2) + x_{20}x_{21}x_{40}x_{41}(x_{10}^2 + x_{11}^2)(x_{30}^2 + x_{31}^2), \\ Q_4 &:= (x_{10}^2 + x_{11}^2)(x_{20}^2 + x_{21}^2)(x_{30}^2 + x_{31}^2)(x_{40}^2 + x_{41}^2), \\ Q_5 &:= (x_{10}^2x_{20}^2 + x_{11}^2x_{21}^2)(x_{30}^2x_{40}^2 + x_{31}^2x_{41}^2). \end{aligned}$$

Denote by $\mathcal{L} \subset |\mathcal{O}_X(2, 2, 2, 2)|$ the linear system generated by Q_0, \dots, Q_5 .

Lemma 4.1. *The linear system \mathcal{L} is 5-dimensional and has 64 base points, given by:*

$$\begin{aligned} &\{x_{10}x_{11} = x_{20}x_{21} = x_{30}^2 + x_{31}^2 = x_{40}^2 - x_{41}^2 = 0\} \cup \\ &\{x_{10}x_{11} = x_{20}x_{21} = x_{30}^2 - x_{31}^2 = x_{40}^2 + x_{41}^2 = 0\} \cup \\ &\{x_{10}^2 + x_{11}^2 = x_{20}^2 - x_{21}^2 = x_{30}x_{31} = x_{40}x_{41} = 0\} \cup \\ &\{x_{10}^2 - x_{11}^2 = x_{20}^2 + x_{21}^2 = x_{30}x_{31} = x_{40}x_{41} = 0\}. \end{aligned}$$

Proof. It is not hard to see that the Q_i are linearly independent, for example, by expanding $Q_0, Q_1, Q_2, Q_3, Q_4 - Q_1 - Q_5, Q_5$ and checking that there is no common monomial in any two of them.

A base point of \mathcal{L} must satisfy $(x_{10}^2 + x_{11}^2)(x_{20}^2 + x_{21}^2)(x_{30}^2 + x_{31}^2)(x_{40}^2 + x_{41}^2) = 0$. Assume that $x_{10}^2 + x_{11}^2 = 0$. Using Q_0 we get $x_{20}x_{21}x_{30}x_{31}x_{40}x_{41} = 0$. Notice that $x_{20}x_{21} \neq 0$ for, otherwise, $Q_1 = Q_2 = Q_3 = Q_5 = 0$ would reduce to

$$\begin{cases} x_{31}^2x_{40}^2 + x_{30}^2x_{41}^2 = 0 \\ x_{40}x_{41}(x_{30}^2 + x_{31}^2) = 0 \\ x_{30}x_{31}(x_{40}^2 + x_{41}^2) = 0 \\ x_{30}^2x_{40}^2 + x_{31}^2x_{41}^2 = 0 \end{cases}$$

which is impossible. Then $x_{30}x_{31}x_{40}x_{41} = 0$ and at least one among $x_{30}x_{31}$ and $x_{40}x_{41}$ vanish. Suppose $x_{30}x_{31} = 0$. Then Q_2 reduces to $x_{40}x_{41}(x_{20}^2 + x_{21}^2) = 0$. Arguing as above we can show that $x_{20}^2 + x_{21}^2 \neq 0$. Hence also $x_{40}x_{41} = 0$ and either Q_1 or Q_5 reduces to $x_{20}^2 - x_{21}^2 = 0$. Assuming $x_{40}x_{41} = 0$ instead of $x_{30}x_{31} = 0$ a similar argument leads to the same conclusion. The statement follows by repeating the same argument with starting assumption $x_{20}^2 + x_{21}^2 = 0$, $x_{30}^2 + x_{31}^2 = 0$ or $x_{40}^2 + x_{41}^2 = 0$. \square

Corollary 4.2. *The general $Y \in \mathcal{L}$ is a Calabi-Yau 3-fold on which G acts freely.*

Proof. Notice that G leaves invariant each of the divisors $(Q_i = 0)$, $i = 0, \dots, 5$; indeed g and h fix all of Q_i . In particular, the action of G induces an action on every $Y \in \mathcal{L}$. Moreover, the base locus of \mathcal{L} computed in Lemma 4.1 does not contain any of the fixed points of the action, given in (2); hence the action on the general $Y \in \mathcal{L}$ is free.

Consider the points $[(\pm i, 1), (\pm 1, 1), (0, 1), (0, 1)]$ of the base locus of \mathcal{L} . After localizing at the affine open set of X given by $x_{11} = x_{21} = x_{31} = x_{41} = 1$, the equation of Q_4 becomes $(x_{10}^2 + 1)(x_{20}^2 + 1)(x_{30}^2 + 1)(x_{40}^2 + 1)$. It is easy to see that this equation defines a hypersurface in \mathbb{C}^4 smooth at the points $(\pm i, \pm 1, 0, 0)$. Similarly, one checks that $(Q_4 = 0)$ is smooth at all of the 64 base points in Lemma 4.1. By Bertini's Theorem, it follows that the general $Y \in \mathcal{L}$ is smooth. By Lefschetz Hyperplane Section Theorem, Y is simply connected and since $\omega_Y = \omega_X(2, 2, 2, 2)|_Y = \mathcal{O}_Y$, Y is a Calabi–Yau. \square

4.2. Admissible pairs with $G = \mathbb{Z}_4 \oplus \mathbb{Z}_4$.

By Theorem 2.15, up to conjugation, $G := \langle g, h \rangle$ with

$$(4) \quad \begin{aligned} g &:= (\text{Id}, A, \text{Id}, A) \circ (12)(34), \\ h &:= (\text{Id}, \text{Id}, B, B) \circ (13)(24) \end{aligned}$$

The following 6 homogeneous forms

$$\begin{aligned} Q'_0 &:= \prod_{i=1}^4 x_{i0}x_{i1}, \\ Q'_1 &:= (x_{11}^2x_{20}^2 + x_{10}^2x_{21}^2)(x_{31}^2x_{40}^2 + x_{30}^2x_{41}^2), \\ Q'_2 &:= x_{20}x_{21}x_{30}x_{31}(x_{10}^2 - x_{11}^2)(x_{40}^2 - x_{41}^2) - x_{10}x_{11}x_{40}x_{41}(x_{20}^2 - x_{21}^2)(x_{30}^2 - x_{31}^2), \\ Q'_3 &:= x_{10}x_{11}x_{30}x_{31}(x_{20}^2 + x_{21}^2)(x_{40}^2 + x_{41}^2) + x_{20}x_{21}x_{40}x_{41}(x_{10}^2 + x_{11}^2)(x_{30}^2 + x_{31}^2), \\ Q'_4 &:= (x_{10}^2 + x_{11}^2)(x_{20}^2 + x_{21}^2)(x_{30}^2 + x_{31}^2)(x_{40}^2 + x_{41}^2), \\ Q'_5 &:= (x_{10}^2x_{20}^2 + x_{11}^2x_{21}^2)(x_{30}^2x_{40}^2 + x_{31}^2x_{41}^2). \end{aligned}$$

are invariant under the natural action of G on the vector space $H^0(\mathcal{O}_X(2, 2, 2, 2))$. As in the case of $\mathbb{Z}_4 \times \mathbb{Z}_4$, it can be shown that these forms span a 5-dimensional linear system whose general member is smooth, has trivial canonical bundle, is simply connected and on which G acts without fixed points.

4.3. Admissible pairs with $G = \mathbb{Z}_8 \oplus \mathbb{Z}_2$.

By Theorem 2.15, up to conjugation, $G := \langle g, h \rangle$ with

$$(5) \quad \begin{aligned} g &:= (\text{Id}, \text{Id}, \text{Id}, A) \circ (1324), \\ h &:= (B, B, B, B) \end{aligned}$$

An explicit Calabi–Yau threefold $Y \subset X$, member of $|\mathcal{O}_X(2, 2, 2, 2)|$, invariant under the action by G and on which G acts without fixed points is given in [BF11].

4.4. Admissible pairs with $G = Q_8 \oplus \mathbb{Z}_2$.

For this example we refer to [NP11], where it is given an action of G on X which is not exactly the one described in Theorem 2.15, but a conjugated of it. Indeed in [NP11, Theorem 1.1] there is a family of hypersurfaces $Z_2 \in |\mathcal{O}_X(-K_X)|$ whose general element is smooth and such that the group acts freely on it. By Theorem 2.15, that action is conjugated to the one given here, and (Z_2, G) is an admissible pair.

5. SURFACES OF GENERAL TYPE

In this section we consider the families of admissible pairs (Y, G) of a Calabi–Yau 3-fold $Y \in |\mathcal{O}_X(2, 2, 2, 2)|$ and a group of order 16, acting on $X = \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ such that Y is invariant under this action and does not meet the fixed locus of G on X . The aim is to find a Fano 3-fold $V \in |\mathcal{O}_X(1, 1, 1, 1)|$, invariant under the action of G . Then G acts freely on the surface $T := V \cap Y$, and the quotient T/G yields families of surfaces of general type with $p_g = 0$ and $K^2 = 3$.

There are only 4 such admissible pairs. For $G = Q_8 \oplus \mathbb{Z}_2$ this has been already done in [NP11], so we skip this case. Below we show that V does not exist if G is $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ or $\mathbb{Z}_4 \oplus \mathbb{Z}_4$. The remainder of this section is concerned with describing a new irreducible component of the moduli space of canonical models of surfaces of general type with $p_g = 0$, $K^2 = 3$ and fundamental group $\mathbb{Z}_4 \times \mathbb{Z}_4$. To ease notation, let H denote the class of a divisor in $|\mathcal{O}_X(1, 1, 1, 1)|$.

Proposition 5.1. *No divisor in $|\mathcal{O}_X(H)|$ is invariant by the action of $\mathbb{Z}_8 \oplus \mathbb{Z}_2$ nor by the action of $\mathbb{Z}_4 \oplus \mathbb{Z}_4$ given in Theorem 2.15.*

Proof. Note that the actions of A and B on \mathbb{P}^1 naturally lift to $H^0(\mathcal{O}_{\mathbb{P}^1}(1))$, by taking $A(x_0) = x_0$, $A(x_1) = -x_1$, $B(x_1) = x_0$, $B(x_0) = x_1$. Using (4) and (5), these lifts induce an actions of G on $H^0(\mathcal{O}_X(H))$, since the set of 16 vectors $x_{1i}x_{2j}x_{3k}x_{4l}$, $(i, j, k, l) \in \{0, 1\}^4$ is a natural basis for $H^0(\mathcal{O}_X(H))$.

In both cases, if we denote by g_1 the automorphism induced by g and by h_1 the automorphism induced by h , a direct computation shows that $g_1h_1 = -h_1g_1$.

Assume that there is a geometrically invariant hyperplane section H . Then H is the zero locus of a section $v \in H^0(\mathcal{O}_X(H))$, which must be an eigenvector for both g_1 and h_1 . However, given that $g_1h_1 = -h_1g_1$, this is impossible. \square

Consider the case $(Y, \mathbb{Z}_4 \times \mathbb{Z}_4)$. Namely, consider the family of Calabi–Yau 3-folds $Y \in \mathcal{L} \subset |\mathcal{O}_X(2H)|$, where \mathcal{L} is the linear system generated by the quadrics (3). Let $G = \mathbb{Z}_4 \times \mathbb{Z}_4$ act on X as given in (1). Consider $V = (F_1 = 0) \in |\mathcal{O}_X(H)|$ defined by

$$(6) \quad F_1 := (x_{20}x_{30} - x_{21}x_{31})(x_{11}x_{40} + x_{10}x_{41}) - i(x_{20}x_{31} - x_{21}x_{30})(x_{10}x_{40} + x_{11}x_{41}).$$

Lemma 5.2. *V is a Fano 3-fold polarized by $-K_V = H|_V$. The singular locus of V is the set of the points such that $x_{i0}^2 - x_{i1}^2 = 0$, $\forall i$ and $x_{10}x_{20}x_{30}x_{40} = -x_{11}x_{21}x_{31}x_{41}$. Therefore, $\text{Sing}(V) \subset \text{Fix}(h^2) \subset \text{Fix}(G)$, where h is given in (1).*

Proof. The only nontrivial claim is the statement about the singularities, which is checked locally on each of the 16 affine open sets. In an affine open set, for example, the affine open set given by $x_{11} = x_{21} = x_{31} = x_{41} = 1$, it is easy to see that the singular points satisfy $x_{i0}^2 = 1$, $\forall i$ and $x_{10}x_{20}x_{30}x_{40} = -1$. Checking on all remaining affine open set can be done with Macaulay2. \square

Theorem 5.3. *The general element T in the linear system $\mathcal{L}|_V$ is a simply connected smooth minimal surface of general type with $p_g(T) = 15$, $q(T) = 0$ and $K_T^2 = 48$ on which G acts freely. The quotient $S := T/G$ is a minimal surface of general type with $p_g(S) = q(S) = 0$, $K_S^2 = 3$ and fundamental group G .*

Proof. By Lemma 5.2 and Lemma 4.1, the singular locus of V does not intersect the base locus of \mathcal{L} . Some points of the base locus of \mathcal{L} are contained in V , for example, on the affine open set given by $x_{11} = x_{21} = x_{31} = x_{41} = 1$, the point $[(0, 1), (0, 1), (1, 1), (i, 1)]$ belongs to $V \cap \text{Bs } \mathcal{L}$. Nevertheless, at this point, $\frac{\partial Q_4}{\partial x_{10}} = 0$, $\frac{\partial Q_4}{\partial x_{40}} = -4i$, $\frac{\partial F_1}{\partial x_{40}} = -1$, so $[(0, 1), (0, 1), (1, 1), (i, 1)]$ is a smooth point of $(Q_4 = 0) \cap V$. A similar computation shows that $(Q_4 = 0) \cap V$ is smooth at all the base points of $\mathcal{L}|_V$. By Bertini's theorem, the general element of $\mathcal{L}|_V$ is smooth. The general T does not contain any of the 48 points in $\text{Fix}(G)$, so the G -action induced on T is free.

By Lefschetz hyperplane section theorem T is simply connected, $q(T) = 0$; then $q(S) = 0$ and $\pi_1(S) \cong G$. By adjunction $K_T = H|_T$ is ample with self-intersection $2H^4 = 48$, and therefore K_S is also ample and $K_S^2 = 48/16 = 3$. Moreover $p_g(T) = h^0(\mathcal{O}_X(H)) - 1 = 15$, and hence $\chi(\mathcal{O}_S) = \chi(\mathcal{O}_T)/16 = \frac{1-0+15}{16} = 1$. \square

Theorem 5.3 produces a family of minimal surfaces of general type with $p_g = 0$ quotients of complete intersections $T = \{F = Q = 0\}$ in X , parametrized by an open subset P° of the 6-dimensional subspace $P := \langle F_1 \rangle \times \langle Q_0, \dots, Q_5 \rangle$ of $P' := H^0(\mathcal{O}_X(H)) \oplus H^0(\mathcal{O}_X(2H))$.

Lemma 5.4. *The action of G on X lifts to an action on P' such that P is the invariant subspace $(P')^G$. In particular, $\forall v \in P$, $T_v P'$ has an induced G -action making the natural identification of P' with it G -equivariant.*

Proof. We gave the action of G on X in (1) by giving a lift to $H^0(\mathcal{O}_X(H))$. This determines a lift to $H^0(\mathcal{O}_X(2H))$ by asking the G -equivariance of the multiplication map $H^0(\mathcal{O}_X(H)) \otimes H^0(\mathcal{O}_X(H)) \rightarrow H^0(\mathcal{O}_X(2H))$. This lifts the action of G to P' .

All vectors v in P are eigenvectors, thus the action of G on P is six copies of the same 1-dimensional representation. Twisting by its dual, we find a lift fixing all vectors in P : $P \subset (P')^G$ and the natural identification of P' with $T_v P'$ is G -equivariant.

To conclude we show that $\dim(P')^G = 6$.

First we checked that the trace of the action (1) on $H^0(\mathcal{O}_X(H))$ is zero for every non-trivial element, which implies that it is isomorphic to the regular representation $\mathbb{C}[G]$. Since the regular representation is invariant by 1-dimensional twists, also the twisted representation we are considering is isomorphic to it, and therefore $H^0(\mathcal{O}_X(H))^G$ has dimension 1 and equals $\langle F_1 \rangle$.

On the other hand, if (F, Q) is a general point in P° , $\mathcal{O}_T(2H) \cong \mathcal{O}_T(2K_T)$. Since the action of G on T has no fixed points and T is a surface of general type, by the holomorphic Lefschetz fixed point formula every linearization of the action on G on $H^0(2K_T)$ is isomorphic to 4 copies of the regular representation; this holds in particular for the linearization induced on the cokernel by the exact sequence

$$0 \rightarrow F_1 \otimes H^0(\mathcal{O}_X(H)) \oplus \langle Q \rangle \rightarrow H^0(\mathcal{O}_X(2H)) \rightarrow H^0(\mathcal{O}_T(K_T)) \rightarrow 0$$

and therefore

$$\dim(P')^G = \dim(F_1 \otimes H^0(\mathcal{O}_X(H)) \oplus \langle Q \rangle)^G + \dim H^0(\mathcal{O}_X(2H))^G = 2 + 4 = 6.$$

\square

We show now that all deformations of the complete intersection of an element of $|\mathcal{O}_X(H)|$ with an element of $|\mathcal{O}_X(2H)|$ are obtained by moving the two hypersurfaces in their linear system.

Proposition 5.5. *Let Y be a smooth element of $|\mathcal{O}_X(2H)|$, V be an element of $|\mathcal{O}_X(H)|$ such that $T = Y \cap V$ is smooth. Then all small deformations of T are embedded, i.e. the natural map*

$$\delta: H^0(\mathcal{O}_V(H)) \oplus H^0(\mathcal{O}_Y(2H)) \rightarrow H^1(\Theta_T)$$

is surjective. Moreover $h^1(\Theta_T) = 67$ and $h^2(\Theta_T) = 3$.

Proof. We compute cohomology groups of sheaves on Y and T using (in this order) the following exact sequences:

$$\begin{aligned} 0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(2H) \rightarrow \mathcal{O}_Y(2H) \rightarrow 0, \text{ and its twist by } -H \text{ and by } -2H, \\ 0 \rightarrow \mathcal{O}_Y \rightarrow \mathcal{O}_Y(H) \rightarrow \mathcal{O}_T(H) \rightarrow 0, \\ 0 \rightarrow \Theta_X(-2H) \rightarrow \Theta_X \rightarrow \Theta_{X|Y} \rightarrow 0, \text{ and its twist by } -H, \\ 0 \rightarrow \Theta_Y(-H) \rightarrow \Theta_{X|Y}(-H) \rightarrow N_{Y|X}(-H) \cong \mathcal{O}_Y(H) \rightarrow 0, \\ 0 \rightarrow \Theta_Y \rightarrow \Theta_{X|Y} \rightarrow \mathcal{O}_Y(2H) \rightarrow 0, \end{aligned}$$

recalling that $\Theta_X = \mathcal{O}_X(2, 0, 0, 0) \oplus \mathcal{O}_X(0, 2, 0, 0) \oplus \mathcal{O}_X(0, 0, 2, 0) \oplus \mathcal{O}_X(0, 0, 0, 2)$, and, since Y is a Calabi–Yau 3-fold, that $h^0(\Theta_Y) = h^3(\Omega_Y^1) = h^{1,3}(Y) = 0$. Table 1 contains the result of this computation (empty cells are zeros).

	h^0	h^1	h^2	h^3	h^4
Θ_X	12				
$\Theta_X(-H)$					
$\Theta_X(-2H)$				4	
$\Theta_X(-3H)$					
$\mathcal{O}_Y(2H)$	80				
$\mathcal{O}_Y(H)$	16				
\mathcal{O}_Y	1			1	
$\mathcal{O}_T(H) \cong \omega_T$	15		1		
$\Theta_{X Y}$	12		4		
$\Theta_{X Y}(-H)$					
$\Theta_Y(-H)$		16			
Θ_Y		68	4		
Θ_Y		68	4		

TABLE 1. Cohomology table

Now, consider the further exact sequence

$$0 \rightarrow \Theta_Y(-H) \rightarrow \Theta_Y \rightarrow \Theta_{Y|T} \rightarrow 0.$$

Then $h^1(\Theta_{Y|T}) - h^0(\Theta_{Y|T}) = 52$ and $h^2(\Theta_{Y|T}) = 4$. Finally, from

$$(7) \quad 0 \rightarrow \Theta_T \rightarrow \Theta_{Y|T} \rightarrow N_{Z|T} = \mathcal{O}_T(H) \rightarrow 0,$$

recalling that since T is of general type, $h^0(\Theta_T) = 0$, we get $h^1(\Theta_T) = 67$, $h^2(\Theta_T) = 3$.

Consider the map, δ , in the statement. It has 2 components. The first component is the composition of the restriction map $H^0(\mathcal{O}_V(H)) \rightarrow H^0(\mathcal{O}_T(H))$, which is surjective, with the coboundary map $H^0(\mathcal{O}_T(H)) \rightarrow H^1(\Theta_T)$ in the cohomology sequence of (7), whose cokernel surjects onto $H^1(\Theta_{Y|T})$. Hence the image of δ contains the kernel of the map $H^1(\Theta_T) \rightarrow H^1(\Theta_{Y|T})$. The composition of the second component with the map $H^1(\Theta_T) \rightarrow H^1(\Theta_{Y|T})$ factors as $H^0(\mathcal{O}_Y(2H)) \rightarrow H^1(\Theta_Y) \rightarrow H^1(\Theta_{Y|T})$ which is a composition of surjective maps, so surjective. It follows that δ is surjective. \square

We can finally prove our last result.

Theorem 5.6. *The family of surfaces in Theorem 5.3 dominates an irreducible component of dimension 4 of the moduli space of minimal surfaces of general type of genus 0.*

Proof. Take a smooth surface $T \in \mathcal{L}|_V$ fulfilling the conditions of Theorem 5.3 and consider the exact sequence

$$0 \rightarrow \Theta_T \rightarrow \Theta_{X|T} \rightarrow \mathcal{O}_T(H) \oplus \mathcal{O}_T(2H) \rightarrow 0.$$

The action of G on X and T induce actions on Θ_T and $\Theta_{X|T}$ for which $\Theta_T \rightarrow \Theta_{X|T}$ is G -equivariant, and therefore induces an action on the cokernel, making the coboundary map

$$d_T: H^0(\mathcal{O}_T(H) \oplus \mathcal{O}_T(2H)) \rightarrow H^1(\Theta_T),$$

which is the differential of the map sending embedded deformations of T to abstract deformations, G -equivariant. Since δ factors through d , d is surjective and so the induced map among the G -invariant subspaces

$$(8) \quad d_T^G: H^0(\mathcal{O}_T(H) \oplus \mathcal{O}_T(2H))^G \rightarrow H^1(\Theta_T)^G$$

is surjective too.

The étale map $\pi: T \rightarrow S = T/G$ induces by pull-back an isomorphism $\Theta_S \rightarrow (\pi_*\Theta_T)^G$. Since G is finite, $\pi_*\Theta_T$ splits as direct sum of invariant subsheaves, one for each character of G ; in particular inducing an isomorphism from $H^1(\Theta_S)$ to $H^1(\Theta_T)^G$.

We have then a commutative diagram

$$\begin{array}{ccc} T_v P & \xrightarrow{d_v} & H^1(\Theta_S) \\ \downarrow & & \downarrow \\ (T_v P')^G & \longrightarrow & (H^0(\mathcal{O}_T(H) \oplus \mathcal{O}_T(2H)))^G \longrightarrow H^1(\Theta_T)^G \end{array}$$

where $v \in P^\circ$ is a point corresponding to a surface $S = T/G$, d_v is the differential at v of the map from the family in Theorem 5.3 to the abstract deformations of S . Since (recall Proposition 5.5) both vertical maps are isomorphisms, and the two horizontal maps at the bottom are surjective, d_v is surjective too. This shows that the family dominates a component of the moduli space.

Note that P° has dimension 6 and multiplying the two equations by constants gives a faithful $\mathbb{C}^* \times \mathbb{C}^*$ action on P° trivial on the moduli space, so this component has dimension at most $6 - 2 = 4$. On the other hand, the expected dimension is $10\chi - 2K_S^2 = 10 - 6 = 4$; hence its dimension is 4. \square

Remark 5.7. A similar argument shows that also the family given by the action of $\mathbb{Z}_2 \oplus Q_8$ dominates an irreducible component of dimension 4 of the moduli space of minimal surfaces of general type of genus 0. As explained by the last two authors in [NP11], this was proved by Bauer and Catanese in [BC10], where an open set of that family is constructed and studied with a different method. Therefore we decided not to give here the details of our proof of that case.

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