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THE CONJUGACY PROBLEM IN EXTENSIONS OF THOMPSON'S GROUP F

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ABSTRACT

We solve the twisted conjugacy problem on Thompson's group F. We also exhibit orbit undecidable subgroups of $\mathrm{Aut}(F)$, and give a proof that $\mathrm{Aut}(F)$ and $\mathrm{Aut}_+(F)$ are orbit decidable provided a certain conjecture on Thompson's group T is true. By using general criteria introduced by Bogopolski, Martino and Ventura in [5], we construct a family of free extensions of F where the conjugacy problem is unsolvable. As a byproduct of our techniques, we give a new proof of a result of Bleak-Fel'shtyn-Gonçalves in [4] showing that F has property R_∞, and which can be extended to show that Thompson's group T also has property R_∞.

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1. Introduction

Since Max Dehn formulated the three main problems in group theory in 1911, they have been a central subject of study in the theory of infinite groups. There now exists a large body of works devoted to the study of these problems. In this paper we focus on the conjugacy problem and a variant known as *the twisted conjugacy problem*. The conjugacy problem is known to be solvable for Thompson's groups F, T and V by works of Guba and Sapir [13], Belk and the second author [2] and Higman [14]. Our interest arose in the study of extensions of the group F where we find an unsolvability result. Even though Thompson himself used the groups F, T, V in the construction of finitely presented groups with unsolvable word problem, to the best of our knowledge, the result that we obtain is a first in a direct generalization of the original Thompson groups. Moreover, we also look at property R_∞ which has been under study recently and which is known to be true for the group F and one of its extensions.

We now give a more detailed description of the results. Let F be a group. We say that a subgroup $A \leqslant \mathrm{Aut}(F)$ has solvable *orbit decidability problem* (ODP) if it is decidable to determine, given $y, z \in F$, whether or not there is $\varphi \in A$

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and $g \in F$ such that

$$\varphi(z) = g^{-1}yg.$$

On the other hand, if $\varphi \in \mathrm{Aut}(F)$, we say that F has solvable *φ-twisted conjugacy problem* (TCP_φ) if it is decidable to determine, given $y, z \in F$, whether or not y is *φ-twisted conjugated* to z, i.e. whether there exists $g \in F$ such that

$$z = g^{-1}y\varphi(g). \tag{1.1}$$

More generally, we say that the group F has solvable *twisted conjugacy problem* (TCP) if (TCP_φ) is solvable for any given $\varphi \in \mathrm{Aut}(F)$.

In the recent paper [5], Bogopolski, Martino and Ventura develop a criterion to study the conjugacy problem for some extensions of groups, and found a connection of this problem with the two problems mentioned above.

Let F, G, H be finitely presented groups and consider a short exact sequence

$$1 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 1. \tag{1.2}$$

In this situation, $\alpha(F) \trianglelefteq G$ and so the conjugation map φ_g, for $g \in G$, restricts to an automorphism of F, $\varphi_g \colon F \to F$, $x \mapsto g^{-1}xg$, (which does not necessarily belong to $\mathrm{Inn}(F)$). We define the *action subgroup* of the sequence (1.2) to be the group of automorphisms

$$A_G = \{\varphi_g \mid g \in G\} \leqslant \mathrm{Aut}(F).$$

Theorem 1.1 (Bogopolski-Martino-Ventura, [5]): *Let*

$$1 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 1.$$

be an algorithmic short exact sequence of groups such that

(1) *F has solvable twisted conjugacy problem,*
(2) *H has solvable conjugacy problem, and*
(3) *for every $1 \neq h \in H$, the subgroup $\langle h \rangle$ has finite index in its centralizer $C_H(h)$, and we can compute a set of coset representatives of $\langle h \rangle$ in $C_H(h)$.*

Then, the conjugacy problem for G is solvable if and only if the action subgroup $A_G = \{\varphi_g \mid g \in G\} \leqslant \mathrm{Aut}(F)$ is orbit decidable.

Here, a short exact sequence is *algorithmic* if all the involved groups are finitely presented and given to us with an explicit finite presentation, and all the morphisms are given by the explicit images of the generators.

Condition (3) is of more technical nature. It is clearly satisfied in free groups (where the centralizer of a non-trivial element h is just the cyclic subgroup generated by its maximal root $\hat{h}$), and it is also true in torsion-free hyperbolic groups, see [5].

The goal of the present paper is to study the conjugacy problem in some extensions of Thompson's group F via Theorem 1.1 (see [5, 19] for references to similar applications of this same theorem into other families of groups).

We will assume the reader is familiar with Thompson's groups F (also denoted by $\mathrm{PL}_2(I)$, where $I = [0,1]$ is the unit interval) and T (also denoted by $\mathrm{PL}_2(S^1)$, where S^1 is the unit circle) and in any case, the comprehensive survey by Cannon, Floyd and Parry [10] is an excellent source of information for Thompson's groups.

We will employ techniques on conjugacy in the Bieri-Thompson-Stein-Strebel groups used by Kassabov and the second author in [15] and a rephrasing by Belk and the second author in [2, 18] of a conjugacy invariant of Brin and Squier [8]. The idea is to assume that the twisted conjugacy equation has a solution and use this to determine necessary conditions that a twisted conjugator should satisfy. This allows one to build some candidate conjugators which must then be tested.

With these techniques, we obtain the first result in the paper:

Theorem 1.2: *Thompson's group F has solvable twisted conjugacy problem.*

Putting together Theorems 1.1 and 1.2, this opens us to the possibility of finding extensions of F with solvable/unsolvable conjugacy problem, by detecting subgroups of $\mathrm{Aut}(F)$ which are orbit decidable/orbit undecidable:

Theorem 1.3: *Consider Thompson's group $F = \mathrm{PL}_2(I)$, a torsion-free hyperbolic group H, and let*

$$1 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 1.$$

be an algorithmic short exact sequence. The group G has solvable conjugacy problem if and only if the action subgroup $A_G \leqslant \mathrm{Aut}(F)$ is orbit decidable.

Using the previous result one can create extensions of F with unsolvable conjugacy problem.

Theorem 1.4: *There are extensions of Thompson's group F by finitely generated free groups, with unsolvable conjugacy problem.*

It is also possible to build non-trivial extensions of F with solvable conjugacy problem, provided that an open conjecture about F is true. We can do so when the action group is orbit decidable and this is where the difficulty lies. We study this in Section 4.

A group G has the *property* R_∞ if it has infinitely many distinct φ-twisted conjugacy classes, for any $\varphi \in \mathrm{Aut}(G)$. Thompson's group F was shown to have property R_∞ by Bleak, Fel'shtyn and Gonçalves in [4]. We give an alternative proof, which can be extended to Thompson's group T.

Theorem 1.5: *Thompson's group T has property R_∞.*

The paper is organized as follows. In Section 2 we introduce the groups we will be working with, we restate the twisted conjugacy problem for F and prove Theorems 1.2 and 1.3. In Section 3 we construct orbit undecidable subgroups of $\mathrm{Aut}(F)$ and exhibit free extensions of F with unsolvable conjugacy problem. In Section 4 we consider orbit decidability and construct some interesting extensions of F, which happen to have solvable conjugacy problem assuming an open conjecture on F is true. In Section 5 we show that the groups F and T have property R_∞ using ideas from Section 2. Finally, in Section 6 we analyze the extent to which the techniques of this paper generalize to other families of Thompson-like groups.

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2. The twisted conjugacy problem for F

In this section we prove Theorem 1.2. The techniques developed for this purpose will be later used in Section 5 to obtain a couple of byproducts.

2.1. Thompson's group and its automorphisms. We will look at Thompson's group F from different perspectives. The standard one is to look at F as the group $\mathrm{PL}_2(I)$ of orientation preserving piecewise-linear homeomorphisms of the unit interval $I = [0, 1]$ with a discrete (and hence finite) set of breakpoints at dyadic rational points, and such that all slopes are powers of 2 (the interval I can be replaced to an arbitrary $[p, q]$ with p, q being dyadic rationals and the resulting group is clearly isomorphic). We will also need to regard F as a subgroup of a bigger group: consider the group $\mathrm{PL}_2(\mathbb{R})$ of all orientation preserving piecewise-linear homeomorphisms of $\mathbb{R}$ with a discrete set of breakpoints at dyadic rational points and such that all slopes are powers of 2; and consider the subgroup of those elements f which are eventually integral translations, i.e. for which there exist $m_-, m_+ \in \mathbb{Z}$ and $L, R \in \mathbb{R}$ such that $f(x) = x+m_-$ for all $x \leqslant L$, and $f(x) = x + m_+$ for all $x \geqslant R$. It is straightforward to see that this subgroup of $\mathrm{PL}_2(\mathbb{R})$ is isomorphic to $\mathrm{PL}_2(I)$; see Proposition 3.1.1 in Belk and Brown [1] for an explicit isomorphism (it is interesting to note that, through this isomorphism, 2^{m_-} is the slope at the right of 0, and 2^{m_+} the slope at the left of 1). Both copies of Thompson's group will be denoted F, and it will be clear from the context which one are we talking about at any moment.

Thompson's group admits a finite presentation. The two generators are usually written x_0 and x_1, which represent the following maps on the real line:

$$x_0(t) = t + 1 \qquad x_1(t) = \begin{cases} t & \text{if } t < 0 \\ 2t & \text{if } 0 \leq t \leq 1 \\ t+1 & \text{if } t > 1. \end{cases}$$

With these generators, F admits a finite presentation with just two relations, which have lengths 10 and 14. See [10] for details. Moreover, as we will need this later, we observe that when we regard F as the group $\mathrm{PL}_2([0, 1])$, the generator x_0 has this form:

$$\theta(t) := \begin{cases} 2t & t \in \left[0, \frac{1}{4}\right] \\ t + \frac{1}{4} & t \in \left[\frac{1}{4}, \frac{1}{2}\right] \\ \frac{t}{2} + \frac{1}{2} & t \in \left[\frac{1}{2}, 1\right]. \end{cases}$$

We distinguish x_0 and θ to make it clear that the first one is seen as an element of $\mathrm{PL}_2(\mathbb{R})$ while the second is regarded as a map in $\mathrm{PL}_2([0, 1])$. The *support* of an element $f \in \mathrm{PL}_2(\mathbb{R})$ is the collection of points where f differs from the identity map, namely $\mathrm{supp}(f) = \{t \in \mathbb{R} \mid f(t) \neq t\}$.

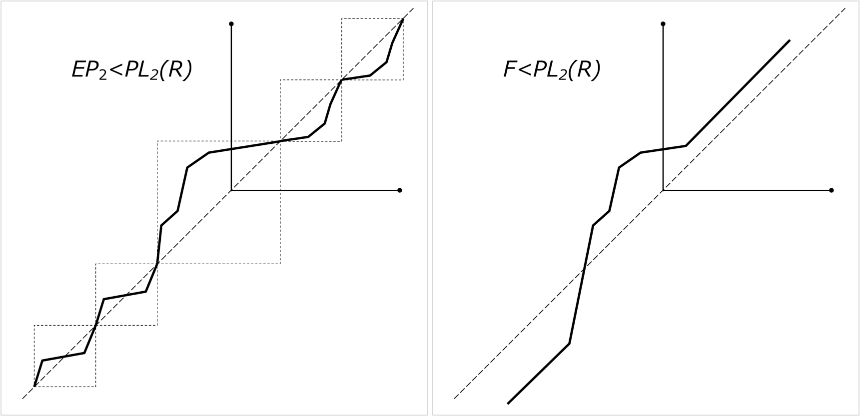
Definition 2.1: We define the following subgroups of $\mathrm{PL}_2(\mathbb{R})$*:*

(1) $\mathrm{EP}_2 = \{f \in \mathrm{PL}_2(\mathbb{R}) \mid \exists\, L, R \in \mathbb{R}$ *such that* $f(t-1) = f(t) - 1\ \forall t \leqslant L,$ *and* $f(t+1) = f(t) + 1\ \forall t \geqslant R\}$ *i.e., all functions in* $\mathrm{PL}_2(\mathbb{R})$ *which are "eventually periodic" and orientation preserving.*

(2) $F = \{f \in \mathrm{EP}_2 \mid \exists\, L, R \in \mathbb{R}, \exists\, m_-, m_+ \in \mathbb{Z}$ *such that* $f(t) = t + m_-\ \forall t \leqslant L,$ *and* $f(t) = t + m_+\ \forall t \geqslant R\}$*. As noted above,* $F \simeq \mathrm{PL}_2(I)$ *is the standard copy of Thompson's group inside* $\mathrm{PL}_2(\mathbb{R})$*.*

(3) *Let* G *be any subset of* EP_2*. For every* $-\infty \leqslant p < q \leqslant +\infty$*, define* $G(p,q)$ *to be the set of elements in* G *with support inside the interval* (p,q) *i.e.* $G(p,q) = \{g \in G \mid g(t) = t, \forall t \notin (p,q)\}$ *(so,* $G(-\infty,+\infty) = G$*). Also, define* $G^> = \{g \in G \mid g(t) > t, \forall t \in \mathbb{R}\}$ *and, similarly,* $G^<$*. When combining both notations we shall understand the inequality restricted to the support, i.e.* $G^<(p,q) = \{g \in G \mid g(t) = t,\ \forall t \notin (p,q),$ *and* $g(t) < t\ \forall t \in (p,q)\}$*. Note that, if* G *is a subgroup, then* $g \in G^>(p,q)$ *if and only if* $g^{-1} \in G^<(p,q)$*.*

At a certain point in the arguments we will also need to consider orientation reversing maps. Admitting both orientations in the definition above, one can define the group $\mathrm{PL}_2^{\pm}(\mathbb{R})$ *and the corresponding subgroup of eventually periodic functions* $\widetilde{\mathrm{EP}}_2 = \{f \in \mathrm{PL}_2^{\pm}(\mathbb{R}) \mid \exists\, L, R \in \mathbb{R}, \exists\, \epsilon \in \{1,-1\}$ *such that* $f(t-1) = f(t) - \epsilon\ \ \forall t \leqslant L,$ *and* $f(t+1) = f(t) + \epsilon\ \ \forall t \geqslant R\}$*. Note that* EP_2 *is a subgroup of* $\widetilde{\mathrm{EP}}_2$ *of index two, and* $\widetilde{\mathrm{EP}}_2 = \mathrm{EP}_2 \cup \mathcal{R} \cdot \mathrm{EP}_2$*, where* $\mathcal{R} \in \widetilde{\mathrm{EP}}_2 \setminus \mathrm{EP}_2$ *is the* reversing map, $\mathcal{R}(t) = -t$ *for all* $t \in \mathbb{R}$*.*



Convention 2.1: When talking about elements $f \in \mathrm{PL}_2(\mathbb{R})$, we say that a property $\mathcal{P}$ holds *for t positive sufficiently large* (respectively, *for t negative*

sufficiently large) to mean that there exists a number $R > 0$ such that $\mathcal{P}$ holds for every $t \geqslant R$ (respectively, there exists a number $L < 0$ such that $\mathcal{P}$ holds for every $t \leqslant L$). For example, $f \in F$ if and only if it is an integral translation for t positive sufficiently large, and for t negative sufficiently large.

Remark 2.1: Observe that, for $g \in F \leqslant \mathrm{PL}_2(\mathbb{R})$, the integer m_- above (satisfying that $g(t) = t + m_-$ for t negative sufficiently large) can also be obtained as the limit $m_- = \lim_{t\to-\infty} g(t) - t$. Similarly, $g(t) = t + m_+$ for t positive sufficiently large, where $m_+ = \lim_{t\to+\infty} g(t) - t$. These two real numbers are called, respectively, the initial slope *and the* final slope *of g because, when regarded as an element of $\mathrm{PL}_2(I)$, the slopes of g on the right of the point 0 and on the left of the point 1 are, precisely, 2^{m_-} and 2^{m_+}, respectively.*

2.2. Automorphisms and transitivity on dyadics. To deal with the φ-twisted conjugacy problem for F, we first need to understand what the automorphisms of Thompson's group F look like. They have all been classified by Brin in his Theorem 1 in [6] (see also Theorem 1.2 in [9] for a more explicit version). The key idea to understand $\mathrm{Aut}(F)$ is the fact that conjugation by elements from $\widetilde{\mathrm{EP}}_2$ preserves F, and these conjugations give precisely all automorphisms of F:

Theorem 2.2 (Brin, [6]): *For Thompson's group F, the map*

$$\begin{array}{ccc} \widetilde{\mathrm{EP}}_2 & \longrightarrow & \mathrm{Aut}(F) \\ \tau & \mapsto & \begin{array}{rccc} \gamma_\tau\colon & F & \to & F \\ & g & \mapsto & \tau^{-1} g \tau, \end{array} \end{array}$$

is well defined and it is a group isomorphism, so $\mathrm{Aut}(F) \simeq \widetilde{\mathrm{EP}}_2$. Furthermore, given $\varphi \in \mathrm{Aut}(F)$ by the images of the standard generators, one can algorithmically compute the (unique) $\tau \in \widetilde{\mathrm{EP}}_2$ such that $\varphi(g) = \tau^{-1} g \tau$ for all $g \in F$.

Definition 2.2: We denote by $\mathrm{Aut}_+(F)$ the group of automorphisms of F given by conjugation by orientation preserving τ's (see Theorem 2.2); it is an index two subgroup $\mathrm{EP}_2 \simeq \mathrm{Aut}_+(F) <_2 \mathrm{Aut}(F) \simeq \widetilde{\mathrm{EP}}_2$.

Remark 2.2 (Explicit rewriting of elements of $\mathrm{Aut}(F)$): *Theorem 2.2, including its algorithmic contents, is crucial for the arguments of the present paper. Brin's original theorem establishes the isomorphism and we can algorithmically* *determine τ in the following way. Burillo and Cleary* [9] *obtain a finite presentation for* $\mathrm{Aut}(F)$ *with nine generators $\varphi_1 \dots, \varphi_9$ all expressed in terms of the standard presentation of F, and as conjugations by suitable $\tau_1, \dots, \tau_9 \in \widetilde{\mathrm{EP}}_2$, i.e. $\varphi_i = \gamma_{\tau_i}$ for $i = 1, \dots, 9$. Suppose $\varphi \in \mathrm{Aut}(F)$ is given by the images of x_0 and x_1. We can enumerate all formal words w on letters $\varphi_1, \dots, \varphi_9$ and for each one compute the images of x_0 and x_1 by $w(\varphi_1, \dots, \varphi_9)$ until they match with $\varphi(x_0)$ and $\varphi(x_1)$ (here we need to use the word problem for F); this match will happen sooner or later because $\varphi_1, \dots, \varphi_9$ do generate* $\mathrm{Aut}(F)$*. Once we have this word, it is clear that $\tau = w(\tau_1, \dots, \tau_9) \in \widetilde{\mathrm{EP}}_2$ satisfies $\gamma_\tau = \gamma_{w(\tau_1,\dots,\tau_9)} = w(\gamma_{\tau_1}, \dots, \gamma_{\tau_9}) = w(\varphi_1, \dots, \varphi_9) = \varphi$.*

The following results explains how to build PL_2-maps acting in a prescribed way on some given rational numbers. The first part gives an arithmetic condition for the existence of such a map. The second part expresses the flexibility of these groups: one can always "cut" the graphical representation of an element at a given dyadic rational, and freely "glue" the pieces to obtain new elements. This result will often be needed throughout the present paper.

Proposition 2.3 (Kassabov-Matucci, [15]): *Let η, ζ be dyadic rationals, let $\alpha, \beta \in \mathbb{Q} \cap (\eta, \zeta)$ written in the form $\alpha = \frac{2^t m}{n}$ and $\beta = \frac{2^k p}{q}$ with $t, k \in \mathbb{Z}$ and m, n, p, q odd integers such that $(m, n) = (p, q) = 1$, and let $\eta < \alpha_1 < \cdots < \alpha_r < \zeta$ and $\eta < \beta_1 < \cdots < \beta_r < \zeta$ be two finite sequences of rational numbers.*

(1) *The following are equivalent:*
   (a) *there exists $g \in \mathrm{PL}_2([\eta, \zeta])$ such that $g(\alpha) = \beta$,*
   (b) *there exists $g \in \mathrm{PL}_2(\mathbb{R})$ such that $g(\alpha) = \beta$,*
   (c) *there exists $g \in \mathrm{EP}_2$ such that $g(\alpha) = \beta$,*
   (d) *there exists $g \in F$ such that $g(\alpha) = \beta$,*
   (e) *$q = n$ and $p \equiv 2^R m \pmod{n}$ for some $R \in \mathbb{Z}$.*

   Moreover, there is an algorithm which constructs such elements g if condition (e) is satisfied.
(2) *There exists $g \in F$ with $g(\alpha_i) = \beta_i$ if and only if for every $i = 1, \dots, r$ there exists $g_i \in F$ such that $g_i(\alpha_i) = \beta_i$. Moreover, if such a g exists it can be constructed from the g_i's.*

The following is a well known standard result (see for example [15] for a proof).

Lemma 2.4: *Let $p \in \mathbb{Q}$ and $g \in \mathrm{PL}_2([p, p+1])$. Let $u, v \in (p, p+1)$ be such that $u \notin \mathrm{Fix}(g)$. Then there exists at most a unique integer m such that $g^m(u) = v$, and one can algorithmically decide it (and compute such an m if it exists).*

2.3. Restatement of the TCP . Our goal in this section is to solve the twisted conjugacy problem in F: given $\varphi \in \mathrm{Aut}(F)$ and $y, z \in F$ (all in terms of the standard presentation of F, i.e. $\varphi(x_0)$, $\varphi(x_1)$, y, z are given to us as words on x_0, x_1), we have to decide whether there exists $g \in F$ such that

$$z = g^{-1} y \varphi(g).$$

Applying Theorem 2.2, we can compute $\tau \in \widetilde{\mathrm{EP}}_2$ such that $\varphi(g) = \tau^{-1} g \tau$ for all $g \in F$, and the previous equation becomes $z = g^{-1} y(\tau^{-1} g \tau)$, that is

$$z\tau^{-1} = g^{-1}(y\tau^{-1})g.$$

Relabeling $\overline{y} := y\tau^{-1} \in \widetilde{\mathrm{EP}}_2$ and $\overline{z} := z\tau^{-1} \in \widetilde{\mathrm{EP}}_2$ to get

$$\overline{z} = g^{-1}\overline{y}g, \tag{2.1}$$

the problem reduces to the standard conjugacy problem in $\widetilde{\mathrm{EP}}_2$, but with the conjugator g forced to be chosen from $F \leqslant \widetilde{\mathrm{EP}}_2$.

Definition 2.3: Given two elements $\overline{y}, \overline{z} \in \widetilde{\mathrm{EP}}_2$, we write $\overline{y} \sim_F \overline{z}$ if they are conjugated by a conjugator in F, i.e. if there exists $g \in F$ such that $\overline{z} = g^{-1}\overline{y}g$.

Notice that if one of $\overline{y}$ and $\overline{z}$ is in EP_2 and the other is not, then equation (2.1) has no solution. Thus, we can split its study into two cases: the orientation preserving case, i.e. when $\overline{y}, \overline{z} \in \mathrm{EP}_2$ (studied in Sections 2.4, 2.5, 2.6 and 2.7) and then the orientation reversing one, i.e. when $\overline{y}, \overline{z} \in \mathcal{R} \cdot \mathrm{EP}_2$ (considered in Section 2.8). Finally, in Section 2.9 we put all pieces together.

2.4. Orientation preserving case of the TCP: periodicity boxes and building conjugators. We now deal with the equation $z = g^{-1}yg$ for $y, z \in \mathrm{EP}_2$ and $g \in F$. The argument will make use of techniques and statements in [15] and refer often to that paper.

Subsection 4.1 in [15] shows that, if $z = g^{-1}yg$ with $y, z, g \in \mathrm{PL}_2(I)$, then there exists $\varepsilon > 0$ depending only on y and z such that g is linear inside $[0, \varepsilon]^2$; the box $[0, \varepsilon]^2$ is called an *initial linearity box.* The goal of this section is to show an analog of this result inside suitable boxes $(-\infty, L]^2$ and $[R, \infty)^2$ where $y, z \in \mathrm{EP}_2$ are periodic.

The following is a first necessary condition for two maps to be conjugate to each other.

Lemma 2.5: *Let $y, z \in \mathrm{EP}_2$ be such that $y \sim_F z$. Then there exist two numbers $L, R \in \mathbb{R}$ such that $y(t) = z(t)$ for all $t \in (-\infty, L] \cup [R, \infty)$.*

Proof. Let $g \in F$ be such that $g^{-1}yg = z$. For t negative sufficiently large, we have $g(t) = t + m_-$, and so

$$z(t) = g^{-1}yg(t) = g^{-1}y(t + m_-) = g^{-1}(y(t) + m_-) = y(t) + m_- - m_- = y(t).$$

Similarly for t positive sufficiently large. ∎

We move on to prove the existence of periodicity boxes.

Lemma 2.6 (Initial and final periodicity boxes): *For every pair of elements $y, z \in \mathrm{EP}_2^{>}(-\infty, p)$ (with $-\infty < p \leqslant +\infty$), there exists a computable constant $L \in \mathbb{R}$ (depending only on y and z) such that every conjugator $g \in F$ between y and z must act as a translation inside the* initial periodicity box $(-\infty, L]^2$. *Similarly, for every pair of elements $y, z \in \mathrm{EP}_2^{>}(p, +\infty)$ (with $-\infty \leqslant p < +\infty$) and a* final periodicity box $[R, +\infty)^2$.

The exact same statement is true replacing $\mathrm{EP}_2^{>}$ *to* $\mathrm{EP}_2^{<}$.

Proof. If y and z are not equal for t positive and negative sufficiently large then, by Lemma 2.5, there is no possible conjugator $g \in F$ and there is nothing to prove. So assume they are and consider a negative sufficiently large $L \in \mathbb{R}$ such that $y(t) = z(t)$ and $y(t-1) = y(t) - 1$ (and so, $z(t-1) = z(t) - 1$), for every $t \leqslant L$ (clearly, such an L is computable). We claim that every possible $g \in F$ satisfying $g^{-1}yg = z$ must be a translation for $t \leqslant L$. By the symmetry of y and z in the definition of L and up to writing the conjugacy relation as $(g^{-1})^{-1}zg^{-1} = y$ (which changes the conjugator from g to g^{-1}), we can assume that g has non-positive translation at $-\infty$ (i.e. $g(t) = t + m_-$ for negative sufficiently large t, and with $m_- \leqslant 0$).

Assume, by contradiction, that g is not a translation map in $(-\infty, L]$. Then, there is $\lambda < L$ such that

$$g(t) = \begin{cases} t + m_- & t \leqslant \lambda \\ \alpha(t - \lambda) + \lambda + m_- & \lambda \leqslant t < \mu \end{cases}$$

for some suitable real numbers $\alpha \neq 1$, $\lambda < \mu < L$. Since z is increasing and strictly above the diagonal $\mathrm{id}(t) = t$, we can choose $r < \lambda < L$ such that $\lambda < z(r) < \mu < L$. By our choice of r, we have $y(r) = z(r)$, $y(t-1) = y(t) - 1$ and $z(t-1) = z(t) - 1$ for all $t \leqslant r$. Moreover, since $gz(t) = yg(t)$ for all $t \in \mathbb{R}$, we have

$$\alpha(z(r) - \lambda) + \lambda + m_- = gz(r) = yg(r) = y(r + m_-) = y(r) + m_- = z(r) + m_-.$$

Rearranging the terms, we have

$$\alpha(z(r) - \lambda) = z(r) - \lambda$$

and, since $z(r) - \lambda > 0$, we get $\alpha = 1$, a contradiction. Hence, $g(t) = t + m_-$ for every $t \leqslant L$ as claimed.

The symmetric argument gives a constant R establishing the final periodicity box $[R, +\infty)^2$.

If $y, z \in \mathrm{EP}_2^<$, then we apply the previous argument to y^{-1}, z^{-1} and derive the same conclusion. ∎

Remark 2.3: Note that, in the previous lemma, the constants L and R depend on y and z but not on the conjugator g. This will be crucial later.

We observe that the results of Subsection 4.2 in [15] and their proofs follow word-by-word in our generalized setting, and hence we do not reprove them. We restate Lemma 4.6 in [15] to give an example of how results appear in this context.

LEMMA 2.7: *Let $z \in \mathrm{EP}_2^<$. Let $C_F(z) = C_{\mathrm{PL}_2(\mathbb{R})}(z) \cap F$ be the set of elements in F commuting with z. Then the map $\varphi_z : C_F(z) \to \mathbb{Z}$ defined by*

$$\varphi_z(g) = \lim_{t \to -\infty} g(t) - t$$

is an injective group homomorphism. A similar statement is true for $\mathrm{EP}_2^>$.

Subsection 4.2 in [15] shows how to build a candidate conjugator g between any two elements of F after we have chosen the initial slope of g.

A *unique candidate conjugator* g between y and z with a given initial slope q, if it exists, is the unique function that one needs to test as a conjugator of y and z with initial slope q: if g fails to satisfy $g^{-1}yg = z$, then there is no conjugator of y and z with initial slope q. The proof of Corollary 4.12 in [15] can be lifted verbatim and so we only restate it in our new case.

Theorem 2.8 (Explicit conjugator): *Let $y, z \in \mathrm{EP}_2^<$. Suppose there exist $L < R$ such that y and z coincide and are periodic on $(-\infty, L] \cup [R, +\infty)$, so that $(-\infty, L]^2$ is the initial periodicity box. Let $\ell \in \mathbb{Z}_{<0}$.*

(1) *Let $g_0 \in F$ be a map which is affine inside $(-\infty, L)^2$ and such that $\lim_{t\to-\infty} g_0(t) - t = q$. Then the unique conjugator $\widehat{g} \in \mathrm{PL}_2(\mathbb{R})$ between y and z, which is affine inside $(-\infty, L)^2$ and such that $\lim_{t\to-\infty} \widehat{g}(t) - t = \ell$ is defined pointwise by*

$$\widehat{g}(t) = \lim_{r\to+\infty} y^{-r} g_0 z^r(t).$$

Moreover, the map $\widehat{g}$ is recursively constructible and y and z are always conjugate in $\mathrm{PL}_2(\mathbb{R})$ via $\widehat{g}$.

(2) *There exists an algorithm to decide whether or not there is $g \in F$ such that $\lim_{t\to-\infty} g(t) - t = \ell$ and $g^{-1}yg = z$.*

The above result has been stated, for simplicity, for two functions $y, z \in \mathrm{EP}_2^<$. However, the same result can be stated for $y, z \in \mathrm{PL}_2^<([p_1, p_2])$ for any $p_1, p_2 \in \mathbb{Q}$, or for $y, z \in \mathrm{EP}_2^<(p, +\infty)$.

Remark 2.4: The results of this subsection do not involve dyadic rationals and slopes that are powers of 2 and are, in fact, true for other classes of groups without restrictions on the breakpoints and the slopes (for example $\mathrm{PL}_+(\mathbb{R})$, the Bieri-Thompson-Stein-Strebel groups in $\mathbb{R}$ and the corresponding subgroups with eventually periodic tails). See [15] for more details.

2.5. Orientation preserving case of the TCP: fixed points. Throughout this paper, for a subset $S \subseteq \mathbb{R}$, we denote the standard boundary of S in the Euclidean topology by ∂S. The goal of this subsection is to reduce to the case where the sets $\partial\mathrm{Fix}(y)$ and $\partial\mathrm{Fix}(z)$ do coincide. Up to suitable special cases, this will allow us to reduce to looking for potential conjugators $g \in F$ such that $\partial\mathrm{Fix}(y) = \partial\mathrm{Fix}(z) \subseteq \mathrm{Fix}(g)$, thus restricting ourselves to studying conjugacy among the corresponding intervals of y and z between any two consecutive points p and q of $\partial\mathrm{Fix}(y) = \partial\mathrm{Fix}(z)$. On each such interval y (and z) is either the identity, or has no fixed points apart from p and q and so it belongs to either $\mathrm{EP}_2^<(p, q)$ or $\mathrm{EP}_2^>(p, q)$.

Note that the sets $\partial\mathrm{Fix}(y)$ and $\partial\mathrm{Fix}(z)$ are discrete subsets of $\mathbb{Q}$, and their intersections with any finite interval $[L, R]$ are easily computable by just solving finitely many systems of linear equations. An apparent technical difficulty is that, since $y, z \in \mathrm{EP}_2$, the full sets $\partial\mathrm{Fix}(y)$ and $\partial\mathrm{Fix}(z)$ may be infinite; however, due to the periodicity, they are controlled by finite sets.

Proposition 2.9: *There is an algorithm which, given $y, z \in \mathrm{EP}_2$ being equal for t negative sufficiently large and for t positive sufficiently large, decides whether or not there exists some $g \in F$ such that $\partial\mathrm{Fix}(y) = g(\partial\mathrm{Fix}(z))$ and, in the affirmative case, it constructs such a g.*

Proof. For the given y, z we can easily compute constants $L < R$ such that, for all $t \in (-\infty, L]$, $y(t) = z(t)$ and $y(t-1) = y(t) - 1$, and such that, for all $t \in [R, +\infty)$, $y(t) = z(t)$ and $y(t+1) = y(t) + 1$. Moving L down and/or R up if necessary, we can also assume that if $\partial\mathrm{Fix}(y) \neq \emptyset$ then it has at least one point in $[L, R)$ (and similarly for z).

Now compute the finite sets of rational numbers $\partial\mathrm{Fix}(z) \cap [L, R)$, $\partial\mathrm{Fix}(y) \cap [L, R)$, $\partial\mathrm{Fix}(z) \cap [L-1, L) = \partial\mathrm{Fix}(y) \cap [L-1, L)$, and $\partial\mathrm{Fix}(z) \cap [R, R+1) = \partial\mathrm{Fix}(y) \cap [R, R+1)$; let $p, q, m, n \geqslant 0$ be their cardinals, respectively. By the periodicity of y and z outside $[L, R]$, these constitute full information about $\partial\mathrm{Fix}(y)$ and $\partial\mathrm{Fix}(z)$. Up to switching y with z, we may assume that $p \leqslant q$.

Clearly, $m = 0$ if and only if $\partial\mathrm{Fix}(y)$ and $\partial\mathrm{Fix}(z)$ have a minimum element (as opposed to having infinitely many points approaching $-\infty$). Similarly, $n = 0$ if and only if $\partial\mathrm{Fix}(y)$ and $\partial\mathrm{Fix}(z)$ have a maximum element.

If either $\partial\mathrm{Fix}(y)$ or $\partial\mathrm{Fix}(z)$ is empty then there is nothing to prove. Assume $\partial\mathrm{Fix}(y) \neq \emptyset \neq \partial\mathrm{Fix}(z)$, i.e. $1 \leqslant p \leqslant q$. We denote by a_0 (respectively, b_0) the smallest element in $\partial\mathrm{Fix}(z) \cap [L, R)$ (respectively $\partial\mathrm{Fix}(y) \cap [L, R)$) and we use it to enumerate in an order preserving way all the elements of the discrete set $\partial\mathrm{Fix}(z)$ (respectively, $\partial\mathrm{Fix}(y)$) as a_i (respectively, b_i); the index i will run over a finite, infinite or bi-infinite subset of $\mathbb{Z}$ depending on whether or not m (and/or n) is zero. With this definition, $\partial\mathrm{Fix}(z) \cap [L, R) = \{a_0 < a_1 < \cdots < a_{p-1}\}$ and $\partial\mathrm{Fix}(y) \cap [L, R) = \{b_0 < b_1 < \cdots < b_{q-1}\}$.

Note that any $g \in F$ satisfying $\partial\mathrm{Fix}(y) = g(\partial\mathrm{Fix}(z))$ must map all the a_i's bijectively to all the b_i's. In particular, if $m = 0$ then a_0 must be mapped to b_0, and if $n = 0$ then a_{p-1} must be mapped to b_{q-1} (and so a_0 to b_{q-p}). Hence, in the special case that either $m = 0$ or $n = 0$, the following claim completes the proof.

Claim 1: For every $b_i \in \partial\mathrm{Fix}(y)$, we can algorithmically decide whether or not there exists some $g \in F$ such that $\partial\mathrm{Fix}(y) = g(\partial\mathrm{Fix}(z))$ and $g(a_0) = b_i$ and, in the affirmative case, the algorithm constructs one explicitly.

The remaining case to study is when $m \neq 0 \neq n$, so that a_0 potentially could be sent to any of the b_i's by the map g. Let $\ell = \mathrm{lcm}(m, n)$ and let $[L - \ell/m, L)$ be the smallest interval to the left of L to contain ℓ points of $\partial\mathrm{Fix}(z)$. Similarly, let $[R, R + \ell/n)$ be the corresponding interval to the right of R. Consider the following two finite sets:

$$A := \partial\mathrm{Fix}(z) \cap \left[L - \tfrac{2\ell}{m}, R + \tfrac{2\ell}{n}\right),$$

$$B := \partial\mathrm{Fix}(y) \cap \left[L - \tfrac{2\ell}{m}, R + \tfrac{2\ell}{n}\right),$$

and let s_0 be the rightmost point of $\partial\mathrm{Fix}(z) \cap \left[L - \frac{2\ell}{m}, L - \frac{\ell}{m}\right)$, and let t_0 be the leftmost point of $\partial\mathrm{Fix}(z) \cap \left[R + \frac{\ell}{n}, R + \frac{2\ell}{n}\right)$. We compute A, B, s_0 and t_0 explicitly.

Claim 2: Suppose there exists a map $g \in F$ such that $\partial\mathrm{Fix}(y) = g(\partial\mathrm{Fix}(z))$ and $g(s_0) \in \left[R + \frac{k\ell}{n}, R + \frac{(k+1)\ell}{n}\right)$ for $k \geqslant 2$; then, there exists a $g' \in F$ such that $\partial\mathrm{Fix}(y) = g'(\partial\mathrm{Fix}(z))$ and $g'(s_0) \in \left[R + \frac{(k-1)\ell}{n}, R + \frac{k\ell}{n}\right)$. Similarly, if there exists $g \in F$ such that $\partial\mathrm{Fix}(y) = g(\partial\mathrm{Fix}(z))$ and $g(t_0) \in \left[L - \frac{(k+1)\ell}{n}, L - \frac{k\ell}{n}\right)$ for some $k \geqslant 2$, then there exists a $g' \in F$ such that $\partial\mathrm{Fix}(y) = g'(\partial\mathrm{Fix}(z))$ and $g'(t_0) \in \left[L - \frac{k\ell}{n}, L - \frac{(k-1)\ell}{n}\right)$.

With the help of Claim 2 we can complete the proof in the following way. Suppose there exists $g \in F$ such that $\partial\mathrm{Fix}(y) = g(\partial\mathrm{Fix}(z))$. Since $p \leqslant q$ it cannot simultaneously happen that $g(s_0) < s_0$ and $t_0 < g(t_0)$. Hence either $s_0 \leqslant g(s_0)$ or $g(t_0) \leqslant t_0$ and, in either case, a repeated application of Claim 2 implies the existence of $g' \in F$ such that $\partial\mathrm{Fix}(y) = g'(\partial\mathrm{Fix}(z))$ and $g'(A) \cap B \neq \emptyset$. This gives finitely many possibilities for $g'(a_0)$ and so, applying Claim 1 finitely many times we can decide whether or not there exists a $g \in F$ satisfying $\partial\mathrm{Fix}(y) = g(\partial\mathrm{Fix}(z))$.

Hence, it only remains to prove the above two claims.

Proof of Claim 1. We will distinguish four cases.

Case 1: $m = 0$ **and** $n = 0$**.** In this case, $\partial\mathrm{Fix}(z) = \{a_0 < a_1 < \cdots < a_{p-1}\}$ and $\partial\mathrm{Fix}(y) = \{b_0 < b_1 < \cdots < b_{q-1}\}$ and, clearly, $p = q$ and $g(a_0) = b_0$ are necessary conditions for such a g to exist. If both conditions hold, then Proposition 2.3 makes the decision for us.

Case 2: $m \geqslant 1$ **and** $n = 0$**.** This case is entirely similar to the next one.

Case 3: $m = 0$ **and** $n \geqslant 1$**.** In this case, $\partial\mathrm{Fix}(z)$ and $\partial\mathrm{Fix}(y)$ both have first elements a_0 and b_0 and infinitely many points approaching $+\infty$. As in case 1, $g(a_0) = b_0$ is a necessary condition for such a g to exist.

We have $\partial\mathrm{Fix}(z) \cap [R, R+1) = \{a_p < a_{p+1} < \cdots < a_{p+(n-1)}\}$ and that the elements in $\partial\mathrm{Fix}(z) \cap [R+1, +\infty)$ are integral translations of these: for every $j \geqslant 0$, write $j = \lambda n + \mu$ with $\lambda, \mu \geqslant 0$ integers and $\mu = 0, \ldots, n-1$, and we have $a_{p+j} = \lambda + a_{p+\mu}$. Similarly $\partial\mathrm{Fix}(y) \cap [R, R+1) = \{b_q < b_{q+1} < \cdots < b_{q+(n-1)}\}$ and for every $j \geqslant q$, we have $b_{q+j} = \lambda + b_{q+\mu}$. Moreover, from $a_p = b_q$ on, the two sequences coincide, i.e., for every $j \geqslant 0$,

$$\lambda + a_{p+\mu} = a_{p+j} = b_{q+j} = \lambda + b_{q+\mu}.$$

Now if some $g \in F$ satisfies $g(\partial\mathrm{Fix}(z)) = \partial\mathrm{Fix}(y)$, it must apply the points in an order preserving way, starting from the smallest ones, that is, $g(a_k) = b_k$ for any integer k. In particular, for $k \geqslant q \geqslant p$, we have

$$g(\lambda_1 + a_{p+\mu_1}) = g(a_{p+(k-p)}) = g(a_k) = b_k = b_{q+(k-q)} = \lambda_2 + b_{q+\mu_2},$$

where $k-p = \lambda_1 n + \mu_1$ and $k-q = \lambda_2 n + \mu_2$. Since g is of the form $g(t) = t + m_+$ with $m_+ \in \mathbb{Z}$ for t positive sufficiently large then, for large enough k, the above equation tells us that

$$\lambda_1 + a_{p+\mu_1} + m_+ = g(\lambda_1 + a_{p+\mu_1}) = \lambda_2 + b_{q+\mu_2}.$$

Therefore, $a_{p+\mu_1} - b_{q+\mu_2} = b_{q+\mu_1} - b_{q+\mu_2}$ must be an integer and so, $\mu_1 = \mu_2$, which means that $k-p$ and $k-q$ are congruent modulo n, i.e. $q-p$ is multiple of n.

Assume then this necessary condition, $q - p = \lambda n$ with $\lambda \in \mathbb{Z}$, and apply Proposition 2.3 (2) to the sequences $a_0 < \cdots < a_{p+\lambda n-1}$ and $b_0 < \cdots < b_{q-1}$ (both with q points). If there is no $g \in F$ sending the first list to the second then there is no g such that $\partial\mathrm{Fix}(y) = g(\partial\mathrm{Fix}(z))$ and we are done. Otherwise, we get a g matching these first q points, $g(a_0) = b_0, \ldots, g(a_{p+\lambda n-1}) = b_{q-1}$, and, after a final small modification, we will see that it automatically matches the rest.

Choose two dyadic numbers $a_{p+\lambda n-1} < \alpha < \beta < a_{p+\lambda n}$, choose $h \in F$ such that $h(\alpha) = g(\alpha)$ and $h(\beta) = \beta - \lambda$ (such an h exists and is effectively computable by Proposition 2.3 (2)), and let us consider the following map:

$$\widetilde{g}(t) = \begin{cases} g(t) & t \leqslant \alpha \\ h(t) & \alpha \leqslant t \leqslant \beta \\ t - \lambda & \beta \leqslant t. \end{cases}$$

By construction, $\widetilde{g}$ is continuous, piecewise linear with dyadic breakpoints, and all slopes are powers of 2; furthermore $g \in F$ and $\widetilde{g}$ is an integral translation for $t \geqslant \beta$ so, $\widetilde{g} \in F$. On the other hand,

$$\partial\mathrm{Fix}(y) \cap [L, b_{q-1}] = g(\partial\mathrm{Fix}(z) \cap [L, a_{p+\lambda n-1}]) = \widetilde{g}(\partial\mathrm{Fix}(z) \cap [L, a_{p+\lambda n-1}]),$$

and

$$\partial\mathrm{Fix}(y) \cap [b_q, +\infty) = \{b_q, b_{q+1}, \ldots\} = \{a_{p+\lambda n} - \lambda, a_{p+\lambda n+1} - \lambda, \ldots\} =$$
$$= \widetilde{g}(\{a_{p+\lambda n}, a_{p+\lambda n+1}, \ldots\}) = \widetilde{g}(\partial\mathrm{Fix}(z) \cap [a_{p+\lambda n}, +\infty)).$$

Hence, $\partial\mathrm{Fix}(y) = \widetilde{g}(\partial\mathrm{Fix}(z))$ and we are done.

Case 4: $m \geqslant 1$ and $n \geqslant 1$. The argument in this case is similar to that of Case 3 but repeated twice, up and down (and with no restriction for b_i because we have both infinitely many fixed points bigger and smaller than b_i).

Following the notation above, the m fixed points from $\partial\mathrm{Fix}(z) \cap [L-1, L) = \partial\mathrm{Fix}(y) \cap [L-1, L)$ are labeled and ordered as $a_{-m} < \cdots < a_{-1}$ and $b_{-m} < \cdots < b_{-1}$ (hence, $a_{-j} = b_{-j}$ for $j = 1, \ldots, m$). The elements from $\partial\mathrm{Fix}(z) \cap (-\infty, L-1)$ and $\partial\mathrm{Fix}(y) \cap (-\infty, L-1)$ are their integral translations to the left.

Now if some $g \in F$ satisfies $g(\partial\mathrm{Fix}(z)) = \partial\mathrm{Fix}(y)$ and $g(a_0) = b_i$, it must send the points a_j to the b_j in an order preserving way starting from $g(a_0) = b_i$, both up and down. Hence, two arguments exactly like in the previous case give us two necessary congruences among p, q and i, modulo n (close to $+\infty$) and modulo m (close to $-\infty$). If one of them fails, then there is no such g and we are done. If both are satisfied, then apply Proposition 2.3 (2) to a long enough tuple of a_j's and b_j's: a negative answer tells us there is no such $g \in F$, and a positive answer provides a $g \in F$ which, after two local modifications like in the previous case (one close to $+\infty$ and the other close to $-\infty$), will finally give us a $g' \in F$ such that $g'(\partial\mathrm{Fix}(z)) = \partial\mathrm{Fix}(y)$, and $g'(a_0) = b_i$.

This completes the proof of Claim 1. ∎

Proof of Claim 2. We will prove the first part of the claim; the symmetric argument for the second part is left to the reader.

Assume the existence of $g \in F$ such that $g(\partial\mathrm{Fix}(z)) = \partial\mathrm{Fix}(y)$ and $g(s_0) \in \left[R + \frac{k\ell}{n}, R + \frac{(k+1)\ell}{n}\right)$ for $k \geqslant 2$. To push $g(s_0)$ down, let us define the *reduction* map g_- by

$$g_-(t) = \begin{cases} g(t - \frac{\ell}{m}) & t < s_0 \\ g(t) - \frac{\ell}{n} & t \geq s_0. \end{cases}$$

To understand the map g_-, note that its graphical representation can be obtained from that of g by performing the following operation: remove the graph within $[s_0 - \ell/m, s_0]$, translate the graph of g defined on $[s_0, +\infty)$ by the vector $(0, -\ell/m)$ and translate the graph of g defined on $(-\infty, s_0 - \ell/m]$ by the vector $(\ell/m, 0)$. Hence, g_- is the same as g avoiding the piece over the interval $[s_0 - \ell/m, s_0]$.

The two parts of g_- to the left and to the right of s_0 are both continuous, increasing, piecewise linear, with dyadic breakpoints, with slopes being powers of two, and are eventually translations (near $-\infty$ and $+\infty$, respectively). To check whether g_- is in F it only remains to analyze what happens around the point s_0.

First of all, g_- is continuous at s_0: observe that $s_0 - \frac{\ell}{m} \in \partial\mathrm{Fix}(z)$ is exactly ℓ points to the left of s_0 in the discrete set $\partial\mathrm{Fix}(z)$; since $g(\partial\mathrm{Fix}(z)) = \partial\mathrm{Fix}(y)$ and g is an increasing function, $g(s_0 - \frac{\ell}{m})$ must be exactly ℓ points to the left of $g(s_0)$ in the discrete set $\partial\mathrm{Fix}(y)$ that is, $g(s_0 - \frac{\ell}{m}) = g(s_0) - \frac{\ell}{n}$.

Unfortunately, the slopes of g_- to the left and to the right of s_0 (i.e. the slopes of g to the left of $s_0 - \ell/m$ and to the right of s_0) may be different; and s_0 may not be a dyadic rational number. If these two facts happen simultaneously then g_- will not an element of F because of having a breakpoint at a non-dyadic point, namely s_0. This technical difficulty will be fixed later by modifying the map g_- in a suitably small neighborhood of s_0.

Before doing this, let us check that g_- fulfils our requirement. Since $g(s_0 - \frac{\ell}{m}) = g(s_0) - \frac{\ell}{n}$, the hypothesis $g(\partial\mathrm{Fix}(z)) = \partial\mathrm{Fix}(y)$ implies that

$$g_-(\partial\mathrm{Fix}(z) \cap (-\infty, s_0]) = g\left(\partial\mathrm{Fix}(z) \cap \left(-\infty, s_0 - \frac{\ell}{m}\right]\right) =$$

$$= \partial\mathrm{Fix}(y) \cap \left(-\infty, g(s_0) - \frac{\ell}{n}\right],$$

and $g_-(s_0) = g(s_0 - \frac{\ell}{m}) = g(s_0) - \frac{\ell}{n}$, and

$$g_-(\partial\mathrm{Fix}(z) \cap [s_0, +\infty)) = g(\partial\mathrm{Fix}(z) \cap [s_0, +\infty)) - \frac{\ell}{n} = \partial\mathrm{Fix}(y) \cap \left[g(s_0) - \frac{\ell}{n}, +\infty\right).$$

Hence, $g_-(\partial\mathrm{Fix}(z)) = \partial\mathrm{Fix}(y)$ and $g_-(s_0) = g(s_0) - \frac{\ell}{n} \in \left[R + \frac{(k-1)\ell}{n}, R + \frac{k\ell}{n}\right)$, as we wanted.

To complete the proof of Claim 2 we must be able to fix the above technical problem, by modifying g_- in such a way that the resulting map belongs to F, but not changing the image of any point in $\partial\mathrm{Fix}(z)$; this will be achieved by changing g_- only in a small enough neighborhood of s_0 not containing any other point of $\partial\mathrm{Fix}(z)$ (and, of course, not changing the image of s_0 itself).

Let α_1 be a dyadic point found strictly between $\alpha_2 := s_0$ and the point of A immediately to the left of s_0; and let α_3 be a dyadic point found strictly between $\alpha_2 := s_0$ and the point of A immediately to the right of s_0. Now consider the points

$$\beta_1 := g_-(\alpha_1) = g\left(\alpha_1 - \frac{\ell}{m}\right),$$

$$\beta_2 := g_-(\alpha_2) = g\left(\alpha_2 - \frac{\ell}{m}\right) = g(\alpha_2) - \frac{\ell}{n},$$

$$\beta_3 := g_-(\alpha_3) = g(\alpha_3) - \frac{\ell}{n}.$$

Since $\alpha_1 < \alpha_2 < \alpha_3$ and $\beta_1 < \beta_2 < \beta_3$ are rational points such that, for every $i = 1, 2, 3$, β_i is the image of α_i by some element in F, then we can apply Proposition 2.3 (2) and construct a function $h \in F$ such that $\beta_i = h(\alpha_i)$. Finally, define

$$g'(t) = \begin{cases} h(t) & t \in [\alpha_1, \alpha_3] \\ g_-(t) & t \notin [\alpha_1, \alpha_3]. \end{cases}$$

Clearly, $g' \in F$, $g'(s_0) = g_-(s_0)$ and $g'(\partial\mathrm{Fix}(z)) = g_-(\partial\mathrm{Fix}(z)) = \partial\mathrm{Fix}(y)$. This completes the proof of Claim 2. ∎

This finishes the proof of Proposition 2.9. ∎

Lemma 2.10: *The decidability of the following two problems is equivalent:*

(TCP) *For any two $y, z \in \mathrm{EP}_2$ we can determine whether or not there is $g \in F$ such that $g^{-1}yg = z$.*

(RTCP) *For any two $y, z \in \mathrm{EP}_2$ such that $\partial\mathrm{Fix}(y) = \partial\mathrm{Fix}(z)$ we can determine, whether or not there is $g \in F$ such that $g^{-1}yg = z$.*

Proof. Obviously, if (TCP) is decidable, then (RTCP) is decidable. Assume now that (RTCP) is decidable. By the discussion at the beginning of this subsection, if y and z are conjugate via $g \in F$, then $\partial\mathrm{Fix}(y) = g(\partial\mathrm{Fix}(z))$. By Theorem 2.9 we can decide whether or not there is a map $g \in F$ such that $\partial\mathrm{Fix}(y) = g(\partial\mathrm{Fix}(z))$. If there is no such map, then y and z are not conjugate. If there is such a $g \in F$ (and in this case Theorem 2.9 constructs it) then $\partial(\mathrm{Fix}(gzg^{-1})) = g(\partial\mathrm{Fix}(z)) = \partial\mathrm{Fix}(y)$ and we can apply (RTCP) to the two maps y and gzg^{-1} to detect whether or not they are conjugate. We note that this is the same decision problem as the one we are interested in. ∎

By Lemma 2.10 we can restrict our focus to studying (RTCP).

2.6. Orientation preserving case of the TCP: Reducing the problem to squares. We can make $\partial\mathrm{Fix}(y) = \partial\mathrm{Fix}(z)$ as done in Proposition 2.9. If $\partial\mathrm{Fix}(y) = \partial\mathrm{Fix}(z) = \emptyset$ we defer the discussion to Subsection 2.7. On the other hand, if $\partial\mathrm{Fix}(y) = \partial\mathrm{Fix}(z) \neq \emptyset$ and $g \in F$ is a conjugator between y and z, the only thing we can say is that g acts on $\partial\mathrm{Fix}(y)$ in an order preserving way. There are two possibilities:

(1) $\mathrm{Fix}(g) \neq \emptyset$.
(2) $\mathrm{Fix}(g) = \emptyset$. We can assume that $g \in F^{>}$.

Case (2) can indeed happen as is shown by the following example: take $y = z$ to be a non-trivial periodic function of period 1 with fixed points. Then the map $g(t) = t + 1 \in F^{>}$ is a conjugator for y and z having no fixed points.

We need to find if there is a conjugator g between y and z such that $g \in F^{>}$. We can assume $y \neq \mathrm{id} \neq z$, otherwise our analysis becomes trivial. We can write the supports $\mathrm{supp}(y) = \mathrm{supp}(z)$ as the union of the family $\{I_j\}$ of (possibly unbounded) intervals on which y and z have no fixed points ordered so that I_j is to the left of I_{j+1}, for every j. If this family were finite, since we are assuming $\partial\mathrm{Fix}(y) = \partial\mathrm{Fix}(z) \neq \emptyset$, then it means that $\partial\mathrm{Fix}(y)$ is finite and so g must fix the smallest element in $\partial\mathrm{Fix}(y)$ since g is order preserving, hence $\mathrm{Fix}(g) \neq \emptyset$ and this would not be the case that we are studying now. Thus we must study the case of the following proposition.

PROPOSITION 2.11: *Let $y, z \in \mathrm{EP}_2$ be such that $\mathrm{Fix}(y) = \mathrm{Fix}(z)$ and that $\partial\mathrm{Fix}(y)$ has infinitely many points. Then there are only finitely many candidate conjugators $g \in F^{>}$.*

Proof. We give out only some relevant details of how to prove this proposition. This entails generalizing of many results of this paper and of Kassabov-Matucci [15] and so we only explain how to carry out these generalizations. The main point here is noticing that we can develop a Stair Algorithm and bounding initial slopes of $g \in F$, even if at $-\infty$ the functions y, z have no initial slope.

By hypothesis, $\{I_j\}$ has infinitely many intervals and so g "shifts" them, that is $g(I_j) = I_{j+k}$, for some fixed k. Let t_j be the left endpoint of I_j. We make a series of observations:

(1) We can build candidate conjugators (Theorem 2.8) on each I_j, given a fixed initial slope at t_j,
(2) The initial slope of z on I_j coincides with the initial slope of y in the image interval $g(I_j)$,
(3) There is an "initial" box for g in I_j,
(4) We can bound the "initial" slopes of g on I_j,
(5) We can bound the initial slope of g at $-\infty$.

(1) and (2) are a straightforward calculation. (3) is a verbatim rewriting of the proof of Lemma 4.2 in [15].

(4) A standard trick from [15] is observing that

$$z = g^{-1}yg = g^{-1}y^{-r}yy^{r}g$$

and so the slope of $y^r g$ at t_j is $(y'(t_{i+k}))^r g'(t_i)$ and $y^r g$ is a conjugator for y and z on I_j. On each I_j there are only finitely many slopes for $g'(t_i^+)$ to be tested and on each one, we apply Theorem 2.8 to build candidate conjugators that we can test.

(5) Recall that a candidate conjugator g pushes all the intervals in $\mathrm{supp}(y)$ in the same direction by the "same amount of intervals in $\mathrm{supp}(y)$". In particular, the initial slope of g determines the number k such that $g(I_j) = I_{j+k}$ for every j.

We use ideas similar to Claim 2 in Proposition 2.9. Let us call J_L the left open interval on which $y = z$ and they are periodic. A similar definition can be made for J_R. Let $J_C = \mathbb{R} \setminus (J_L \cup J_R)$ the remaining central piece. Assume that there is a conjugator g between y and z which sends and interval I_j inside J_L to an interval I_{j+k+1} with the requirement that I_{j+k} is entirely contained into J_R. Using ideas similar to Claim 2 in Proposition 2.9 one can create a new conjugator $\overline{g}$ such that $\overline{g}(I_j) = I_{j+k}$.

Therefore, similarly to Claim 2 in Proposition 2.9, this allows us to reduce the study to only finitely many candidate conjugators where $g(J_C) \cap J_C \neq \emptyset$ or where the rightmost interval I_j inside J_L goes to the leftmost interval I_s of J_R (or viceversa). This argument reduces the number of initial slopes of g to be tested.

To conclude we observe that there are only finitely many slopes for g at $-\infty$ and finitely many "initial" slopes for g on the finitely many intervals I_j contained in J_C and then we can apply Theorem 2.8 on each of these intervals building finitely many candidate conjugators $g \in F^>$ which we can then test one by one. ∎

The previous result allows one to restrict to the case of looking for conjugators g with fixed points.

LEMMA 2.12: *Let $y, z \in \mathrm{EP}_2$ be such that* $\mathrm{Fix}(y) = \mathrm{Fix}(z) \neq \emptyset$ *and let g be a conjugator between y and z such that* $\mathrm{Fix}(g) \neq \emptyset$. *Then* $\mathrm{Fix}(z) \subseteq \mathrm{Fix}(g)$.

Proof. Let $a \in \mathrm{Fix}(g)$ and let b be the the smallest point of $\partial\mathrm{Fix}(z)$ such that $a < b$. Since g fixes $\mathrm{Fix}(z)$ set wise and is order-preserving, then $g(b)$ must also be the smallest point of $\partial\mathrm{Fix}(z)$ such $g(b) > a$, therefore $g(b) = b$ and so g must fix all of $\mathrm{Fix}(z)$ pointwise. ∎

We need to show that (RTCP) of Lemma 2.10 is decidable. Lemma 2.12 tells us that we can restrict ourselves to solve the problem inside the closed intervals of $\mathrm{Fix}(y) = \mathrm{Fix}(z)$.

As in [15] we observe that if $p \in \partial\mathrm{Fix}(y)$ is a non-dyadic rational point and g is a conjugator between y and z, then $g'(p^-) = g'(p^+)$ or, in other words, the slope of g at one side of p is completely determined by the slope on its other side. This implies that the important points of $\partial\mathrm{Fix}(y)$ are the dyadic rational ones (if they exist) as they are the ones where g has freedom to have different slopes on the two sides and therefore the conjugator that we are attempting to build can be constructed by by gluing two conjugators on the two sides of a dyadic rational point of $\partial\mathrm{Fix}(y)$. In the case that $\partial\mathrm{Fix}(y)$ had no dyadic rational points, then we can compute a conjugator at a point $p \in \partial\mathrm{Fix}(y)$ and this uniquely determines the conjugator on the entire real line. Otherwise, there are dyadic rational points in $\partial\mathrm{Fix}(y)$ and we argue as following.

Let $L < R$ are two integers chosen so that y and z coincide and are periodic inside $(-\infty, L] \cup [R, +\infty)$. The case when $\partial\mathrm{Fix}(y) \cap [L, R]$ contains no dyadic rational point is dealt with as above. Similarly, if there is only one dyadic point inside $\partial\mathrm{Fix}(y) \cap [L, R]$, then we have two instances of the previous case on the two sides of the dyadic point. Otherwise, we choose p_1, p_2 with the property of being dyadic and consecutive inside $\partial\mathrm{Fix}(y)$ and such that $[p_1, p_2] \subseteq [L, R]$. With these provisions, we can use the solution of the standard conjugacy problem inside $\mathrm{PL}_2([p_1, p_2])$ using the techniques from [15]. If there is no conjugator on any of those intervals, then y and z cannot be conjugate. Otherwise, we can glue the conjugators that we find on each such interval. We then need to understand what happens outside $[L, R]$.

Let p be the rightmost dyadic point of $\partial\mathrm{Fix}(y) \cap [L, R]$. If $y, z \in \mathrm{EP}_2^{>}(p, +\infty)$ (or $y, z \in \mathrm{EP}_2^{<}(p, +\infty)$), then we deal with this case in Subsection 2.7. Otherwise, let q be the leftmost point of $\partial\mathrm{Fix}(y) \cap (R, +\infty)$. If $y(t) = z(t) = t$ on $[p, q]$, then we define $g(t) = t$ on $[p, +\infty)$ and this defines a conjugator for y and z on $[p, +\infty)$ which we can glue to the previous intervals. Otherwise, we apply the standard conjugacy problem on the interval $[p, q]$ with final slope 1 at q^- since the conjugator g has to be the identity translation on $[R, +\infty)$. If the standard conjugacy problem on $[p, q]$ has no solution, then y and z cannot be conjugate. Otherwise, if h is the conjugator on $[p, q]$ we define

$$g(t) := \begin{cases} h(t) & t \in [p, q] \\ t & [q, +\infty) \end{cases}$$

which is a well-defined map of F, since $g'(q^-) = g'(q^+) = 1$, regardless of whether or not q is dyadic. The map g defines a conjugator for y and z on $[p, +\infty)$ which we can glue to the previous intervals. A similar argument can be applied to the left of L.

2.7. Orientation preserving case of the TCP: Mather invariants. The procedure outlined in [15] to solve the conjugacy problem in Bieri-Thompson-Stein-Strebel groups requires various steps which we have studied already: (i) making $\mathrm{Fix}(y)$ and $\mathrm{Fix}(z)$ coincide (seen in Subsection 2.5) and (ii) showing that, for a possible initial slope of a conjugator in F (see Remark 2.1), there exists at most one candidate and we can compute it through an algorithm (seen in Subsection 2.4). The next natural step is to bound the number of integers $\lim_{t\to-\infty} g(t) - t$ representing possible initial slopes for which we need to build a candidate conjugator.

In order to do this, we will employ ideas to characterize conjugacy from [18], by taking very large powers of y and z and building a conjugacy invariant. In [18] a conjugacy class in F has been described by a double coset $Ay^\infty B$ where y^∞ is an element of Thompson's group T obtained by taking suitable high powers of y and A and B are two finite cyclic groups (of rotations of the circle). In the case of the twisted conjugacy problem that we are studying, the Mather invariant will be essentially defined by a product $Ay^\infty B$ where $A \cong B \cong \mathbb{Z}$.

Mather invariant construction. In what follows, we will assume that $y, z \in \mathrm{EP}_2^>$, to simplify the notation. We can define Mather invariants in the two neighborhoods of infinity (that is on $\mathrm{EP}_2(-\infty, p)$ and $\mathrm{EP}_2(q, +\infty)$ for some suitable numbers p, q), while solving the conjugacy problem between any two consecutive dyadic points of $\partial\mathrm{Fix}(y) = \partial\mathrm{Fix}(z)$.

Let $y, z \in \mathrm{EP}_2^>$ and assume that, on the intervals $(-\infty, L] \cup [R, +\infty)$, the maps y and z coincide and are periodic, for some integers $L \leqslant R$. Let $N \in \mathbb{N}$ be large enough so that $y^N((y^{-1}(L), L)) \subseteq (R, +\infty)$ and $z^N((y^{-1}(L), L)) \subseteq (R, +\infty)$. We look for an orientation preserving homeomorphism $H \in \mathrm{PL}_2(\mathbb{R})$ such that

(1) $H(y^k(L)) = k$, for any integer k, and
(2) $H(y(t)) = \lambda(H(t)) = H(t) + 1$, where $\lambda(t) = t + 1$.

To construct H, choose any PL_2-homeomorphism $H_0 : [y^{-1}(L), L] \to [-1, 0]$ with finitely many breakpoints. Then we extend it to a map $H \in \mathrm{PL}_2(\mathbb{R})$ by defining

$$H(t) := H_0(y^{-k}(t)) + k \qquad \text{if } t \in [y^{k-1}(L), y^k(L)] \text{ for some integer } k.$$

We make a series of remarks.

- By construction, it is easy to see that $H(y(t)) = \lambda(H(t))$ for any real number t.
- If we define $\overline{y} := HyH^{-1}, \overline{z} := HzH^{-1}$, we observe that, by construction, they both coincide with $\lambda(t) = t + 1$ on the intervals $(-\infty, 1] \cup [N, +\infty)$. It is also clear that $\overline{y} = \lambda$.

- We notice that $\overline{\lambda} = H\lambda H^{-1} \in \mathrm{EP}_2$. To show this, let t be positive sufficiently large so that y is periodic of period 1 and that all the calculations below make sense and define $\widetilde{t} = H_0^{-1}(t-k-1)$:

$$\overline{\lambda}(t+1) = H\lambda y^{k+2} H_0^{-1}\lambda^{-k-2}(t+1) = H\lambda y^{k+2}(\widetilde{t}) = H(y^{k+2}(\widetilde{t})+1) =$$

$$\lambda^{k'} H_0 y^{-k'}(y^{k+2}(\widetilde{t}+1)) = \lambda^{k'-1} H_0 y^{-k'+1}(y^{k+1}(\widetilde{t}+1)) + 1 = \overline{\lambda}(t)+1,$$

where k' are the jumps that y must make to bring $y^{k+2}(\widetilde{t}+1)$ back to the domain of H_0. A similar argument can be shown for t negative sufficiently large.

We define

$$C_0 := (-\infty, 0)/\mathbb{Z} \qquad C_1 := (N, \infty)/\mathbb{Z}$$

and let $p_0 : (-\infty, 0) \to C_0$ and $p_1 : (N, \infty) \to C_1$ be the natural projections. Then we define the map $\overline{y}^\infty : C_0 \to C_1$ by

$$\overline{y}^\infty([t]) := [\overline{y}^N(t)].$$

Similarly we can define the map $\overline{z}^\infty$. The maps $\overline{y}^\infty$ and $\overline{z}^\infty$ are well-defined and they do not depend on the specific N chosen (the proof is analogous to the one in Section 3 of [18]). They are called the *Mather invariants* of $\overline{y}$ and $\overline{z}$.

This induces the equation $\overline{gz}^N = \overline{y}^N\overline{g}$ which, following [18], passes to quotients and becomes

$$v_1^k \overline{z}^\infty = \overline{y}^\infty v_0^\ell \tag{2.2}$$

since all the maps $\overline{y}, \overline{z}, \overline{g}$ are in EP_2 and where $v_1 := p_1\overline{\lambda}p_1^{-1}$ is an element of Thompson's group T_{C_1} defined on the circle C_1 and induced by $\overline{\lambda}$ on C_1 by passing to quotients via the map p_1, $v_0 := p_0\overline{\lambda}p_0^{-1}$ and where ℓ, k are the initial and final slopes of g.

The following result shows that the integer solutions of equation (2.2) correspond to conjugators between y and z. The proof is an extension of the proof of Theorem 4.1 in [18].

Lemma 2.13: *Let $y, z \in \mathrm{EP}_2^{>}$. Then y and z are conjugate through an element $g \in F$ if and only if there is a pair of integers k, ℓ that satisfy equation (2.2).*

Proof. Clearly, if $g \in F$ conjugates y to z, then equation (2.2) is satisfied by the calculations above. Conversely, assume that we have a pair (k, ℓ) such that (2.2) is satisfied. We use Theorem 2.8 to find a map $g \in \mathrm{PL}_2(\mathbb{R})$ which is affine around $-\infty$, such that $\lim_{x\to-\infty} g(x) - x = \ell$ and that $yg = gz$. By conjugating via H we see that $\overline{yg} = \overline{gz}$. If x is positive sufficiently large then $\overline{y}(x) = \overline{z}(x) = x + 1$ so

$$\overline{g}(x) + 1 = \overline{yg}(x) = \overline{gz}(x) = \overline{g}(x + 1).$$

Arguing similarly at ∞ we deduce that $\overline{g} \in \mathrm{EP}_2$ and so the equation $\overline{y}^N \overline{g} = \overline{gz}^N$ passes to quotients and becoming $\overline{g}_{\mathrm{ind}} \overline{z}^\infty = \overline{y}^\infty v_0^\ell$. By using our assumption we see that $\overline{g}_{\mathrm{ind}} \overline{z}^\infty = \overline{y}^\infty v_0^\ell = v_1^k \overline{z}^\infty$ and by cancellation we obtain $\overline{g}_{\mathrm{ind}} = v_1^k$. By taking the unique lift of $\overline{g}_{\mathrm{ind}}$ and v_1^k defined on $[N, N + 1)$ and passing through the point $(N, g(N))$, we see that $\overline{g}$ and $\overline{\lambda}^k$ coincide on $[N, N + 1]$ and therefore they coincide on $[N, +\infty)$ since they are both in EP_2. Thus, $g \in F$ since $\overline{g}(x) = \overline{\lambda}(x)$ around $+\infty$. ∎

We relabel $t_0 := \overline{z}^\infty v_0^{-1} (\overline{z}^\infty)^{-1}$, $t_1 := v_1$ and and $t := \overline{y}^\infty (\overline{z}^\infty)^{-1}$ and we rewrite equation (2.2) as

$$t_1^k t_0^\ell = t \tag{2.3}$$

where $t_0, t_1, t \in T_{C_1}$. To solve equation (2.3) we will need Lemma 8.4 from [15] which we restate for the reader's convenience.

Lemma 2.14 (Kassabov-Matucci, [15]): *Let $p \in \mathbb{Q}$ and let $\mathrm{PL}_2([p, p+1])$ be the group of piecewise-linear homeomorphisms of the interval $[p, p+1]$ with finitely many breakpoints which occur at dyadic rational points and such that all their slopes are powers of 2. If $t_0, t_1, t \in \mathrm{PL}_2([p, p + 1])$, there is an algorithm which outputs one of the following two mutually exclusive cases in finite time:*

1. *Equation (2.3) has at most one solution and we compute a pair (k, ℓ) such that, if equation (2.3) is solvable, then (k, ℓ) must be its unique solution.*
2. *Equation (2.3) has infinitely many solutions which are given by the sequence of pairs (k_j, ℓ_j) where $k_j = a_1 j + b_1$ and $\ell_j = a_2 j + b_2$ for any $j \in \mathbb{Z}$ and for some integers a_1, a_2, b_1, b_2 which we can compute.*

Lemma 2.14 gives a solution for equation (2.3) in the case that t_0, t_1, t live in a copy of Thompson's group $\mathrm{PL}_2([p, p + 1])$ of functions over an interval. However, equation (2.3) needs to be solved in a copy of Thompson's group T of functions over a circle, so we will need to adapt Lemma 2.14 to our needs.

Lemma 2.15: *Let T be Thompson's group $\mathrm{PL}_2(S^1)$ and let $t_0, t_1, t \in T$. Then there is an algorithm which outputs one of the following two mutually exclusive cases in finite time:*

(1) *Equation (2.3) has at most finitely many solutions and we compute a finite set S such that, if (k, ℓ) is a solution of equation (2.3), then $\ell \in S$.*
(2) *Equation (2.3) has infinitely many solutions and we compute a sequence of solutions (k_j, ℓ_j) where $k_j = a_1 j + b_1$ and $\ell_j = a_2 j + b_2$ for any $j \in \mathbb{Z}$ and for some integers a_1, a_2, b_1, b_2.*

Proof. For a map $h \in T$, we denote by $\mathrm{Per}(h)$ the set of all periodic points of h. Obviously, $\mathrm{Fix}(h) \subseteq \mathrm{Per}(h)$. By a result of Ghys and Sergiescu [11] every element of T has at least one periodic point. For $i = 0, 1$, we find a $q_i \in \mathrm{Per}(t_i)$ be a point of period d_i. If $d = \mathrm{lcm}(d_0, d_1)$, then both t_0^d, t_1^d have fixed points and therefore $\mathrm{Per}(t_i^d) = \mathrm{Fix}(t_i^d)$.

Using the division algorithm we write $k = k'd + r$ and $\ell = \ell' d + s$ with $0 \leqslant r, s < d$ so that equation (2.3) becomes

$$(t_1^d)^{k'} (t_0^d)^{\ell'} = t_1^{-r} t t_0^{-s}. \tag{2.4}$$

By considering all possibilities for $0 \leqslant r, s < d$, equation (2.4) can be regarded as a family of d^2 equations in T. Equation (2.3) is solvable if and only if at least one of the d^2 equations (2.4) is solvable.

Up to renaming t_0^d with t_0, t_1^d with t_1 and $t_1^{-r} t t_0^{-s}$ with t, we observe that each of the equations (2.4) has the same form of equation (2.3), therefore we have reduced ourselves to study equation (2.3) with the extra assumption that both t_0 and t_1 have fixed points. We compute the full fixed point sets of t_0 and t_1. We now break the proof into two cases.

Case 1: There is a point $p \in \partial \mathrm{Fix}(t_1)$ such that $p \notin \mathrm{Fix}(t_0)$. Rewriting equation (2.3) and applying it to p, we get

$$t_0^{-\ell}(p) = t^{-1}(p). \tag{2.5}$$

Since $p \notin \mathrm{Fix}(t_0)$ and t_0 is orientation preserving, then there exists at most one number ℓ satisfying equation (2.5) by Lemma 2.4.

Case 2: There is a point $p \in \partial \mathrm{Fix}(t_1) \cap \mathrm{Fix}(t_0)$. If $p \notin \mathrm{Fix}(t)$, by particularizing at p we see that equation (2.3) is not solvable for any pair (k, ℓ). Otherwise, $p \in \mathrm{Fix}(t)$ and we can cut the unit circle open at $p \in \mathbb{Q}/\mathbb{Z}$ and regard t_0, t_1, t

as elements of $\mathrm{PL}_2([p,p+1])$. We can now finish the proof by applying Lemma 2.14. ∎

Remark 2.5: *The proof of Lemma 2.15 shows how to locate the pairs (k,ℓ). We need to find all periodic orbits and their periods and this can be effectively achieved by computing the Brin-Salazar revealing pairs of the tree pair diagrams of T, using the Brin-Salazar technology to compute neutral leaves and thus deducing the size of periodic orbits (see, for example, Section 4 in* [3]*).*

Remark 2.6: *To sum up this subsection, Lemma 2.13 shows that y and z are conjugate via an element of F if and only if equation (2.2) is solvable for some integers k,ℓ. Lemma 2.15 shows how to narrow down the number of pairs (k,ℓ) that we need to test. There are two possible cases:*

(i) *In the first case of Lemma 2.15 we are given a finite set S of initial slopes to test. We can use Theorem 2.8(ii) for each of the finitely many initial slopes in the set S. There is a conjugator if and only if one of the applications of Theorem 2.8(ii) returns a positive answer. If there is a conjugator, it can be built using Theorem 2.8(i).*
(ii) *In the second case of Lemma 2.15 there are infinitely many possible pairs (k,ℓ) (and we can construct explicitly an infinite family) and all of them correspond to a conjugator between y and z. We can apply Theorem 2.8(i) on a specific pair (k,ℓ) of our choice to find an explicit conjugator between y and z*

Hence, in every case we can find at least one conjugator, if it exists.

Remark 2.7: *We observe that the construction of the Mather invariant can be carried out even when y and z are elements of $\mathrm{EP}_2^{\gtrless}(p,+\infty)$ or of $\mathrm{EP}_2^{\gtrless}(-\infty,p)$ for any rational number p. All the results of the current subsection can still be recovered. For this reason, in the following we will refer to the Mather invariant regardless of the ambient set where it will be built.*

2.8. Orientation reversing case of the TCP. We now study the orientation reversing case of TCP, that is, we want to solve the equation

$$z = g^{-1} y g, \tag{2.6}$$

where $y, z \in \mathcal{R} \cdot \mathrm{EP}_2 \setminus \{\mathrm{id}\}$ and $g \in F$. The general idea that we will follow is to square the equation and attempt to solve

$$z^2 = g^{-1} y^2 g$$

so that $y^2, z^2 \in \mathrm{EP}_2$ and we can appeal to the results of the previous subsections.

Since y, z are strictly decreasing and approach $\mp\infty$ when $t \to \pm\infty$ then both y and z have exactly one fixed point each. Moreover, all possible g's fulfilling equation (2.6) must also satisfy $g(\mathrm{Fix}(z)) = \mathrm{Fix}(gzg^{-1}) = \mathrm{Fix}(y)$. By Proposition 2.3(ii), one can algorithmically decide whether or not there is $g \in F$ mapping the point $\mathrm{Fix}(z)$ to the point $\mathrm{Fix}(y)$. If there is no such g, then equation (2.6) has no solution and we are done. Otherwise, compute such a $g \in F$ and, after replacing z by gzg^{-1}, we can assume that $\mathrm{Fix}(y) = \mathrm{Fix}(z) = \{p\}$, for some $p \in \mathbb{Q}$.

We start with a special case and then move on to consider all orientation reversing maps.

Proposition 2.16: *Let $y, z \in \mathcal{R} \cdot \mathrm{EP}_2$ be such that $y^2 = z^2 = \mathrm{id}$ and $y(p) = z(p) = p$, for some $p \in \mathbb{Q}$. Then y and z are conjugate by an element of F if and only if there exists $u \in \mathbb{Z}$ such that $y^{-1}z(t) = t + u$ for t positive sufficiently large.*

Proof. The forward direction follows from a straightforward check of the behavior of y and z at neighborhoods of $\pm\infty$. For the converse, define the following map

$$g(t) := \begin{cases} t & \text{if} \quad t \in (-\infty, p] \\ y^{-1}z(t) & \text{if} \quad t \in [p, +\infty). \end{cases}$$

If $t \leqslant p$, then

$$g(t) = t = y^{-2}z^2(t) = y^{-1}(y^{-1}z)z(t) = y^{-1}gz(t)$$

since $y^2 = z^2 = \mathrm{id}$ and $z(t) \geqslant p$. On the other hand, if $t \geqslant p$, then

$$g(t) = y^{-1}z(t) = y^{-1}gz(t)$$

since $z(t) \leqslant p$. So y and z are conjugate to each other by the element $g \in \mathrm{EP}_2$. The final step is to observe that g is, in fact, in F because $g(t) = t$, for t negative sufficiently large, and $g(t) = t + u$ by construction, for t positive sufficiently large. ∎

We quickly extend an argument from [15] to reduce the number of candidate conjugators to test. The trick is to reduce the number of initial slopes that we need to test.

Lemma 2.17: *Let $\overline{y}, \overline{z} \in \mathcal{R} \cdot \mathrm{EP}_2^{<}(p, +\infty)$ and $g \in F(p, +\infty)$ and consider the equation*

$$\overline{z} = x^{-1}\overline{y}x. \tag{2.7}$$

Then $x = g$ is a solution of (2.7) if and only if there exists an integer n such that $x = \overline{y}^{2n}g \in \mathrm{EP}_2(p, +\infty)$ is the unique solution of equation (2.7) such that $(\overline{y}^{2n}g)'(p^+) \in [(y^2)'(p^+), 1]$.

Proof. This follows immediately by noticing that equation (2.7) is equivalent to

$$\overline{z} = (\overline{y}^{2n}x)^{-1}\overline{y}(\overline{y}^{2n}x).$$

To show uniqueness, we observe that in Subsection 2.3 we noticed that a solution of equation (2.7) is also a solution of the squared equation

$$\overline{z}^2 = g^{-1}\overline{y}^2 g. \tag{2.8}$$

Uniqueness follows from Theorem 2.8 applied to the squared equation (2.8). ∎

Theorem 2.18: *Let $y, z \in \mathcal{R} \cdot \mathrm{EP}_2$ be such that $y(p) = z(p) = p$, for some $p \in \mathbb{Q}$. We can decide whether or not y and z are conjugate by an element of F. If there exists a conjugator, we can construct one.*

Proof. If $y^2 = z^2 = \mathrm{id}$, then we are done by Proposition 2.16. Moreover, if y and z are conjugate via an element of F, it is immediate that $y^{-1}z(t) = t + u$, for some integer u and for any t positive sufficiently large (as observed in the proof of Proposition 2.16). Thus we can assume that $y^{-1}z$ is a translation, for t positive sufficiently large.

Assume now $y^2 \neq \mathrm{id} \neq z^2$. We can appeal to Proposition 2.9 and assume that $\mathrm{Fix}(y^2) = \mathrm{Fix}(z^2)$, up to suitable conjugation. Moreover, if there is a $g \in F$ such that $z = g^{-1}yg$, then g must fix $\mathrm{Fix}(y) = \mathrm{Fix}(z) = \{p\}$ and so $\{p\} \subseteq \mathrm{Fix}(y^2) = \mathrm{Fix}(z^2) \subseteq \mathrm{Fix}(g)$.

Let $L < R$ be two suitable integers so that y^2 and z^2 coincide and are periodic on the set $(-\infty, L] \cup [R, +\infty)$. If either L or R does not exist, then y and z

cannot be conjugate. We can apply the techniques from [15] on any two consecutive dyadic rational points p_1, p_2 of $\partial\mathrm{Fix}(y^2) \cap [L, R]$ where $y^2|_{[p_1,p_2]} \neq \mathrm{id}|_{[p_1,p_2]}$ and $z^2|_{[p_1,p_2]} \neq \mathrm{id}|_{[p_1,p_2]}$ and find (if they exist) all the finitely conjugators between $y^2|_{[p_1,p_2]}$ and $z^2|_{[p_1,p_2]}$ with initial slopes within $(y^2)'(p^+)$ and $(y^{-2})'(p^+)$. Similarly we can do on $[a, +\infty)$ where a is the rightmost dyadic rational point of $\partial\mathrm{Fix}(y^2) \cap [L, R]$ by applying Lemma 2.17 in the case that y^2 and z^2 have no fixed points on $[R, +\infty)$ (to reduce the number of initial slopes on which we can apply Theorem 2.8) or using the argument at the end of Subsection 2.6 in case y^2 and z^2 have fixed points on $[R, +\infty)$.

Thus in all cases, up to using the same trick of Lemma 2.17 to reduce the slopes to test, we apply Theorem 2.8 (or its bounded version from [15]) to build finitely many functions between any two consecutive dyadic rational points p_1, p_2 of $\partial\mathrm{Fix}(y^2)$ (respectively, on an interval of the type $[p_1, +\infty)$) and such that $y^2 \neq \mathrm{id}$ on $[p_1, p_2]$ (respectively, on an interval of the type $[p_1, +\infty)$).

We now test all these functions as conjugators between y and z in the respective intervals. If there is an interval such that none of these functions conjugates y and z, then y and z cannot be conjugate via an element of F. Otherwise, on each such interval U_s we fix a conjugator g_s between y and z.

Now we will carefully glue all these conjugators with the function that we have built in Proposition 2.16 as follows. Assume that $(p, +\infty) \setminus \mathrm{Fix}(y^2)$ is a disjoint union of ordered intervals $I_i = (a_i, b_i)$ so that $a_i < a_j$, if $i < j$. Similarly, assume that $(-\infty, p) \setminus \mathrm{Fix}(y^2)$ is a disjoint union of ordered intervals $J_i = (d_i, c_i)$ such that $c_i > c_j$, if $i < j$.

$$g(t) := \begin{cases} t & \text{if} \quad t = p \text{ or } t \in \mathrm{Fix}(y^2) \cap (-\infty, p) \\ y^{-1}z(t) & \text{if} \quad t \in \mathrm{Fix}(y^2) \cap (p, +\infty) \\ g_s(t) & \text{if} \quad t \in U_s \end{cases}$$

Since y acts on $\mathbb{R}$ in an order reversing way, it is immediate to verify that $y(a_i) = c_i = z(a_i)$, $y(c_i) = a_i = y(c_i)$, $y(b_i) = d_i = z(b_i)$ and $y(d_i) = b_i = z(d_i)$ and therefore the map g is in F. It is straightforward to observe that this map is continuous and in F and that it is a conjugator, by construction. For example, since $z([c_{i+1}, d_i]) = [b_i, a_{i+1}]$ and $y^2 = z^2 = \mathrm{id}$ on $[c_{i+1}, d_i]$ then it is clear that

$$g(t) = t = y^{-2}z^2(t) = y^{-1}(y^{-1}z)z(t) = y^{-1}gz(t)$$

for any $t \in [c_{i+1}, d_i]$. ∎

2.9. Solution of the TCP. We are now ready to prove Theorem 1.2.

Theorem 1.2. *Thompson's group F has solvable twisted conjugacy problem.*

Proof. Given $y, z \in F$ and $\varphi \in \mathrm{Aut}(F)$, we need to establish whether or not there is a $g \in F$ such that

$$z = g^{-1} y \varphi(g). \tag{2.9}$$

In Subsection 2.3 we have shown that equation (2.9) is equivalent to the equation

$$\overline{z} = g^{-1} \overline{y} g, \tag{2.10}$$

for $\overline{y}, \overline{z} \in \widetilde{\mathrm{EP}}_2$ and $g \in F$. We describe a procedure to wrap up all work of the previous subsections:

(1) If one of $\overline{y}$ and $\overline{z}$ belongs to EP_2 and the other in $\mathcal{R} \cdot \mathrm{EP}_2$, then equation (2.10) has no solution for $g \in F$, since conjugation does not change the orientation of a function.
(2) If both $\overline{y}, \overline{z} \in \mathrm{EP}_2$, then we apply the results of Subsections 2.4 through 2.7 to solve equation (2.10).
(3) If $\overline{y}, \overline{z} \in \mathcal{R} \cdot \mathrm{EP}_2$, then we apply Theorem 2.18 to solve equation (2.10).

This ends the proof of Theorem 1.2. ∎

3. Extensions of F with unsolvable conjugacy problem

In this section we recall the necessary tools of [5] needed to construct extensions of Thompson's group F with unsolvable conjugacy problem (proving Theorem 1.4).

As explained in the introduction, Bogopolski, Martino and Ventura give a criterion to study the conjugacy problem in extensions of groups (see Theorem 1.1). Applying it to the case we are interested in, let F be Thompson's group, let H be any torsion-free hyperbolic group (for example, a finitely generated free group), and consider an algorithmic short exact sequence

$$1 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 1. \tag{3.1}$$

We can then consider the *action subgroup* of the sequence, $A_G = \{\varphi_g \mid g \in G\} \leqslant \mathrm{Aut}(F)$, and Theorem 1.3 tells us that G has solvable conjugacy problem if and only if $A_G \leqslant \mathrm{Aut}(F)$ is orbit decidable. In the present section we will find orbit undecidable subgroups of $\mathrm{Aut}(F)$ and so, extensions of Thompson's group F with unsolvable conjugacy problem.

A good source of orbit undecidable subgroups in $\mathrm{Aut}(F)$ comes from the presence of $F_2 \times F_2$ via Theorem 7.4 from [5]:

THEOREM 3.1 (Bogopolski-Martino-Ventura, [5]): *Let F be a finitely generated group such that $F_2 \times F_2$ embeds in $\mathrm{Aut}(F)$ in such a way that the image B intersects trivially with $\mathrm{Stab}^*(v)$ for some $v \in F$, where*

$$\mathrm{Stab}^*(v) = \{\theta \in \mathrm{Aut}(F) \mid \theta(v) \textit{ is conjugate to } v \textit{ in } F\}.$$

Then $\mathrm{Aut}(F)$ contains an orbit undecidable subgroup.

Let us first find a copy of $F_2 \times F_2$ inside $\mathrm{Aut}(F)$ and then deal with the technical condition about avoiding the stabilizer.

We can define two maps $\varphi_{-\infty}, \varphi_{\infty} \colon \mathrm{EP}_2 \to T = \mathrm{PL}_2(S^1)$ in the following way: given $f \in \mathrm{EP}_2$ we find a negative sufficiently large integer L so that f is periodic in $(-\infty, L]$; then we pass $f|_{(L-1,\, L]}$ to the quotient modulo $\mathbb{Z}$ to obtain an element from T defined to be the image of f by $\varphi_{-\infty}$. The map $\varphi_{+\infty}$ is defined similarly but looking at a neighborhood of $+\infty$.

The maps $\varphi_{-\infty}$ and $\varphi_{+\infty}$ are clearly well-defined homomorphisms from EP_2 to T. Note also that, for $f_1, f_2 \in \mathrm{EP}_2$ and $k \in \mathbb{Z}$, if f_1 and $f_2 + k$ agree for t negative (resp. positive) sufficiently large, then $\varphi_{-\infty}(f_1) = \varphi_{-\infty}(f_2)$ (resp. $\varphi_{+\infty}(f_1) = \varphi_{+\infty}(f_2)$).

We begin by showing that both $\varphi_{-\infty}$ and $\varphi_{+\infty}$ are surjective.

LEMMA 3.2: *For every $a \in T$ and every dyadic rational p, there exist preimages of a by $\varphi_{-\infty}$ and $\varphi_{+\infty}$, respectively inside $\mathrm{EP}_2(-\infty, p) \leqslant \mathrm{EP}_2$ and $\mathrm{EP}_2(p, +\infty) \leqslant \mathrm{EP}_2$.*

Proof. We show the result for the case $\mathrm{EP}_2(p, +\infty)$ (the other case is completely analogous). Let $a \in T$ and choose $\widetilde{a} \in \mathrm{EP}_2$ to be any standard periodic lift of a conveniently translated up so that $p < \widetilde{a}(p+1)$. By Proposition 2.3, we can construct $g \in F$ such that $g(p) = p$ and $g(p+1) = \widetilde{a}(p+1)$. Finally, consider

$$\widehat{a}(t) = \begin{cases} t & t \leqslant p \\ g(t) & p \leqslant t \leqslant p+1 \\ \widetilde{a}(t) & p+1 \leqslant t, \end{cases}$$

which is clearly an element of $\mathrm{EP}_2(p,+\infty)$ such that $\varphi_{+\infty}(\widehat{a}) = \varphi_{+\infty}(\widetilde{a}) = a$. ∎

The following Corollary is the key observation of the current subsection.

Corollary 3.3: *The automorphism group of Thompson's group $F = \mathrm{PL}_2(I)$ contains a copy of the direct product of two free groups, $F_2 \times F_2 \leqslant \mathrm{EP}_2 \leqslant \mathrm{Aut}^+(F)$.*

Proof. It is well known that Thompson's group $T = \mathrm{PL}_2(S^1)$ contains a copy of F_2, the free group on two generators, say generated by $a, b \in T$. Apply Lemma 3.2 to obtain preimages of a and b by $\varphi_{-\infty}$, say $\widehat{a}_-, \widehat{b}_- \in \mathrm{EP}_2(-\infty,0)$, and preimages of a and b by $\varphi_{+\infty}$, say $\widehat{a}_+, \widehat{b}_+ \in \mathrm{EP}_2(0,+\infty)$. Since $\varphi_{-\infty}$ and $\varphi_{+\infty}$ are homomorphisms, we have again $\langle \widehat{a}_-, \widehat{b}_- \rangle \cong F_2 \cong \langle \widehat{a}_+, \widehat{b}_+ \rangle$. And, on the other hand, by disjointness of supports, they commute to each other and so $F_2 \times F_2 \cong \langle \widehat{a}_-, \widehat{b}_-, \widehat{a}_+, \widehat{b}_+ \rangle \leqslant \mathrm{EP}_2 \cong \mathrm{Aut}^+(F)$. ∎

We are finally ready to prove Theorem 1.4.

Theorem 1.4. *There are extensions of Thompson's group F by finitely generated free groups, with unsolvable conjugacy problem.*

Proof. We need to redo the proof of Corollary 3.3 in an algorithmic fashion while choosing our copy of $F_2 \times F_2$ inside $\mathrm{Aut}^+(F)$ carefully enough so that it satisfies the technical condition in Theorem 3.1.

Let Θ be the map obtained by repeating periodically the map θ defined in Subsection 2.1 inside each square $[k, k+1]^2$, for any integer k. Let $\alpha(t) := \Theta^2(t) \pmod 1 \in T$ and $\beta(t) := \Theta^2(t) + \frac{1}{2} \pmod 1 \in T$. By using the ping-pong lemma it is straightforward to verify that α and β generate a copy of F_2 inside T. Now take $a = \alpha^2$, $b = \beta^2$, $c = \alpha\beta\alpha^{-1}$ and $d = \beta\alpha\beta^{-1}$, which generate a copy of the free group of rank four, $F_4 \simeq \langle a, b, c, d \rangle \leqslant T$.

Using Lemma 3.2, we can find preimages of $a, b \in T$ by $\varphi_{-\infty}$, denoted by $\widehat{a}$, $\widehat{b} \in \mathrm{EP}_2(-\infty, 0) \leqslant \mathrm{EP}_2$, and preimages of $c, d \in T$ by $\varphi_{+\infty}$, denoted by $\widehat{c}$, $\widehat{d} \in \mathrm{EP}_2(0, +\infty) \leqslant \mathrm{EP}_2$. Since $\langle a, b \rangle \cong F_2 \cong \langle c, d \rangle$ and $\varphi_{-\infty}$ and $\varphi_{+\infty}$ are both group homomorphisms, we get $\langle \widehat{a}, \widehat{b} \rangle \cong F_2 \cong \langle \widehat{c}, \widehat{d} \rangle$. Moreover, the disjointness of supports gives us that $F_2 \times F_2 \cong \langle \widehat{a}, \widehat{b}, \widehat{c}, \widehat{d} \rangle \leqslant \mathrm{EP}_2$; this is the copy B of $F_2 \times F_2$ inside EP_2 (though as positive automorphisms of F via Brin's Theorem) ready to apply Theorem 3.1. Additionally, note that, by construction,

$\varphi_{-\infty}(\widehat{a}) = a$, $\varphi_{-\infty}(\widehat{b}) = b$, $\varphi_{+\infty}(\widehat{c}) = c$ and $\varphi_{+\infty}(\widehat{d}) = d$ but, at the same time, $\varphi_{+\infty}(\widehat{a}) = \varphi_{+\infty}(\widehat{b}) = \varphi_{-\infty}(\widehat{c}) = \varphi_{-\infty}(\widehat{d}) = 1_T$.

Let now $v \in F$ be the map $v(t) = t + 1$, for all $t \in \mathbb{R}$. We will show that $B \cap \mathrm{Stab}^*(v) = \{\mathrm{id}\}$. Let $\tau \in B \cap \mathrm{Stab}^*(v)$. On one hand, $\tau \in B$ and so $\tau(0) = 0$ and $\tau = w_1(\widehat{a}, \widehat{b})w_2(\widehat{c}, \widehat{d})$ for some unique reduced words $w_1(\widehat{a}, \widehat{b}) \in \langle \widehat{a}, \widehat{b} \rangle$ and $w_2(\widehat{c}, \widehat{d}) \in \langle \widehat{c}, \widehat{d} \rangle$. On the other hand, $\tau \in \mathrm{Stab}^*(v)$ and so $\tau^{-1}v\tau = g^{-1}vg$ for some $g \in F$, which implies that τg^{-1} commutes with v in EP_2. By the definition of v, the map τg^{-1} is periodic of period 1 on the entire real line, thus $\varphi_{-\infty}(\tau g^{-1}) = \varphi_{+\infty}(\tau g^{-1})$ in T. On the other hand, since $g \in F$, there exist integers m_- and m_+ such that, for negative sufficiently large t, $\tau g^{-1}(t) = \tau(t - m_-) = \tau(t) - m_-$, and for positive sufficiently large t, $\tau g^{-1}(t) = \tau(t - m_+) = \tau(t) - m_+$. Modding out these two equations by $\mathbb{Z}$ around $\pm\infty$, we get

$$\varphi_{-\infty}(\tau g^{-1}) = \varphi_{-\infty}(\tau) = \varphi_{-\infty}(w_1(\widehat{a}, \widehat{b})w_2(\widehat{c}, \widehat{d})) =$$
$$= \varphi_{-\infty}(w_1(\widehat{a}, \widehat{b}))\varphi_{-\infty}(w_2(\widehat{c}, \widehat{d})) = w_1(a, b);$$

similarly, $\varphi_{+\infty}(\tau g^{-1}) = w_2(c, d)$. Hence,

$$w_1(a, b) = \varphi_{-\infty}(\tau g^{-1}) = \varphi_{+\infty}(\tau g^{-1}) = w_2(c, d),$$

an equation holding in $\langle a, b, c, d \rangle \leqslant T$. Since this is a free group on $\{a, b, c, d\}$, we deduce that $w_1(a, b)$ and $w_2(c, d)$ are the trivial words and therefore $\tau = \mathrm{id}$.

Having shown that $B \cap \mathrm{Stab}^*(v) = \{\mathrm{id}\}$, an application of Theorem 3.1 gives us orbit undecidable subgroups of $\mathrm{Aut}^+(F)$, and Theorem 1.3 concludes the proof. ∎

Remark 3.1: The element v chosen in the previous proof is actually x_0, the first generator of the standard finite presentation defined in Subsection 2.1.

4. The orbit decidability problem for F

In this section we study the orbit decidability problem for $\mathrm{Aut}(F)$ and $\mathrm{Aut}_+(F)$. We study two different cases and use techniques which are "dual" to those of Section 2. As a consequence, provided that one knows the solvability of a certain decision problem, we can build nontrivial extensions of F with solvable conjugacy problem.

By using Theorem 2.2 and following computations similar to those in Subsection 2.3, the orbit decidability problem for $\mathrm{Aut}(F)$ can be restated as follows: given $y, z \in F$ decide whether or not there exists a $g \in \mathrm{EP}_2$ such that either

(i) $g^{-1}yg = z$, or
(ii) $g^{-1}(\mathcal{R}y\mathcal{R})g = z$.

Notice that the first equation corresponds to orbit decidability for $\mathrm{Aut}_+(F)$. Up to renaming $\mathcal{R}y\mathcal{R}$ by y, both (i) and (ii) can be regarded as an instance of (i).

4.1. Orbit decidability problem: fixed points. It is immediate to adapt Lemma 2.5 to this setting, noticing that if $y \sim_{\mathrm{EP}_2} z$ then y and z coincide around $\pm\infty$.

Remark 4.1: Since $y, z \in F$ have only finitely many intervals of fixed points, we can use the results of Subsection 2.5 and assume that $\mathrm{Fix}(y) = \mathrm{Fix}(z)$, *up to conjugating by a $g \in F$. It can be shown that if there is no $g \in F$ such that* $\mathrm{Fix}(y) = g(\mathrm{Fix}(z))$, *then there is no $h \in \mathrm{EP}_2$ such that* $\mathrm{Fix}(y) = h(\mathrm{Fix}(z))$.

Lemma 4.1: *Let $y, z \in F$ such that* $\mathrm{Fix}(y) = \mathrm{Fix}(z) \neq \emptyset$. *It is decidable to determine whether or not there is a $g \in \mathrm{EP}_2$ such that $g^{-1}yg = z$.*

Proof. If $g \in \mathrm{EP}_2$ conjugates y to z, then it must fix $\mathrm{Fix}(z)$ point wise. For any two consecutive points p_1, p_2 of $\partial\mathrm{Fix}(z)$ we can use the techniques in [15] to decide whether or not there is a $h_{p_1,p_2} \in \mathrm{PL}_2([p_1, p_2])$ conjugating $y|_{[p_1,p_2]}$ to $z|_{[p_1,p_2]}$.

Let $R = \max \mathrm{Fix}(z)$. If $R = +\infty$, then there exists a rational number p such that $y = z = \mathrm{id}$ on $[p, +\infty)$ and so we can choose $g \in \mathrm{EP}_2(R, +\infty)$ to be $g = \mathrm{id}$ to conjugate y to z. Assume now that $R < +\infty$.

By using the same idea seen in Subsection 2.9 and rewriting the equation $z = g^{-1}yg = (y^n g)^{-1} y (y^n g)$ we restrict to looking for candidate conjugators with slopes at R^+ inside $[y'(R^+), 1]$. For any power 2^α within $[y'(R^+), 1]$, we apply Theorem 2.8(ii) to build the unique conjugator $g \in \mathrm{PL}_2(R, +\infty)$ such that $g'(R^+) = 2^\alpha$. We find a finite number of conjugators $g_1, \ldots, g_s \in \mathrm{PL}_2(R, +\infty)$. Notice: by Theorem 2.8(ii) every g_i conjugates y to z, but it may not be true that $g_i \in \mathrm{EP}_2(R, +\infty)$.

There exists a positive sufficiently large number M such that, for any $t \geqslant M$, we have $y(t) = t + k = z(t)$ and that for any $i = 1, \ldots, s$ and any $t \geqslant M$, we have:

$$g_i(t) + k = yg_i(t) = g_i z(t) = g_i(t + k),$$

so that every g_i is periodic of period k on $[M, +\infty)$. To finish the proof, we only need to check if any of the g_i's is in $\mathrm{EP}_2(R, +\infty)$. To do so, we check if $g_i(t+1) = g_i(t)+1$ on the interval $[M, M+k]$. If any of them is indeed periodic of period 1, then we have found a valid conjugator, otherwise y and z are not conjugate. ∎

4.2. Orbit decidability problem: Mather invariants. We assume that $y, z \in F^{>}$ and that there exist two integers $L < R$ such that $y(t) = z(t) = t + a$ for $t \leqslant L$ and $y(t) = z(t) = t + b$ for $t \geqslant R$, for suitable integers $a, b \geqslant 1$. Up to conjugation by a suitable $g \in F$, we can assume that $L = 0$ and $R = 1$. Define the two circles

$$C_0 := (-\infty, 0)/a\mathbb{Z} \qquad C_1 := (1, \infty)/b\mathbb{Z}$$

and let $p_0 : (-\infty, 0) \to C_0$ and $p_1 : (1, \infty) \to C_1$ be the natural projections. As was done before, let N be a positive integer large enough so that $y^N(-a, 0) \subseteq (1, +\infty)$ and define the map $y^\infty : C_0 \to C_1$ by

$$y^\infty([t]) := [\overline{y}^N(t)].$$

Similarly we define z^∞ and call them the *Mather invariants* for y and z. Arguing as in Subsection 2.7 we see that, if $g^{-1}yg = z$ for $g \in \mathrm{EP}_2$, then

$$v_1 z^\infty = y^\infty v_0 \tag{4.1}$$

where v_i is an element of Thompson's group T_{C_i} induced by g on C_i, for $i = 0, 1$, and such that $v_i(t+1) = v_i(t) + 1$.

Recall that a group G has solvable *k-simultaneous conjugacy problem* (k-CP) if, for any two k-tuples $(y_1, \dots, y_k)$, $(z_1, \dots, z_k)$ of elements of G, it is decidable to say whether or not there is a $g \in G$ so that $g^{-1}y_i g = z_i$, for all $i = 1, \dots, k$. Kassabov and the second author [15] show that Thompson's group F has solvable k-CP.

Conjecture 4.2: *Thompson's group T has solvable k-CP.*

Partial results have been obtained towards this conjecture by Bleak, Kassabov and the second author in Chapter 7 of the second author's thesis [17]; it is work in progress to complete that investigation.

Lemma 4.3: *Let $y, z \in F^{>}$. If the 2-simultaneous conjugacy problem is solvable in Thompson's group T, then it is decidable to determine whether or not there is a $g \in \mathrm{EP}_2$ such that $g^{-1}yg = z$.*

Proof. A straightforward extension of Theorem 4.1 in [18] yields that $y \sim_{\mathrm{EP}_2} z$ if and only if there exists $v_i \in T_{C_i}$ such that $v_i(t+1) = v_i(t)+1$, for $i = 0,1$ and they satisfy equation (4.1). Since v_0 needs to be equal to $y^{-\infty} v_1 z^{\infty}$, our problem is reduced to deciding whether or not there is $v_1 \in T_{C_1}$ solving these equations:

$$\begin{aligned} v_1(t+1) &= v_1(t)+1, & \forall t \in C_1 \\ y^{-\infty} v_1 z^{\infty}(t+1) &= y^{-\infty} v_1 z^{\infty}(t)+1, & \forall t \in C_0. \end{aligned} \tag{4.2}$$

Recalling that C_0 is a circle of length a and C_1 is a circle of length b, we define $s_i : C_i \to C_i$ to be the rotation by 1 in C_i, for $i = 0,1$. The problem now becomes this: we need to decide whether or not there exists a map $v_1 \in T_{C_1}$ such that

$$\begin{gathered} v_1 s_1 = s_1 v_1 \\ y^{-\infty} v_1 z^{\infty} s_0 = s_0 y^{-\infty} v_1 z^{\infty}. \end{gathered} \tag{4.3}$$

If we relabel $y^{\infty} s_0 y^{-\infty} := y^*$ and $z^{\infty} s_0 z^{-\infty} := z^*$, equations (4.3) become

$$\begin{gathered} v_1^{-1} s_1 v_1 = s_1 \\ v_1^{-1} y^* v_1 = z^*. \end{gathered} \tag{4.4}$$

Equations (4.4) are an instance of 2-CP which is solvable by assumption. ∎

4.3. Non-trivial extensions of F with solvable conjugacy problem.

Theorem 4.4: *If Conjecture 4.2 is true for $k=2$, then $\mathrm{Aut}(F)$ and $\mathrm{Aut}_+(F)$ are orbit decidable (as subgroups of $\mathrm{Aut}(F)$). In particular, assuming that such conjecture is true, every group G in an algorithmic short exact sequence*

$$1 \longrightarrow F \xrightarrow{\alpha} G \xrightarrow{\beta} H \longrightarrow 1,$$

where $F = \mathrm{PL}_2(I)$, H is a torsion-free hyperbolic group, and the action subgroup A_G is either $\mathrm{Aut}(F)$ or $\mathrm{Aut}_+(F)$, has solvable conjugacy problem.

Proof. An application of Remark 4.1 and Lemmas 4.1 and 4.3 implies the solvability of orbit decidability for the groups $\mathrm{Aut}(F)$ and $\mathrm{Aut}_+(F)$. We verify the requirements of Theorem 1.1. By Theorem 1.2, condition (1) is satisfied. It is well known (see, for example, Proposition 4.11(b) [5]) that if H is a free group or a torsion-free hyperbolic group, conditions (2) and (3) from Theorem 1.1 are satisfied. By Theorem 4.4 we know that the action subgroup is orbit decidable, then Theorem 1.1 implies that G has solvable conjugacy problem. ∎

5. Property R_∞ in Thompson groups F and T

In this section we show that Thompson groups F and T both have property R_∞. We recall the definition of property R_∞, for the reader's convenience.

Definition 5.1: A group G has property R_∞ if for any $\varphi \in \mathrm{Aut}(G)$, there exists a sequence $\{z_i\}_{i\in\mathbb{N}}$ of pairwise distinct elements which are pairwise not φ-twisted conjugate. See also Section 1.

We know that an automorphism φ of F is obtained by conjugation in F by an element $\tau \in \widetilde{\mathrm{EP}}_2$. Moreover, we have seen in Subsection 2.3 that two elements $y, z \in F$ are φ-twisted conjugate if and only if the two elements $y\tau$ and $z\tau$ (now elements of $\widetilde{\mathrm{EP}}_2$) are conjugate by an element of F. Therefore, to prove that F has property R_∞ it is enough to show that, given $\tau \in \widetilde{\mathrm{EP}}_2$, there exists a family of elements $z_i \in F$, for all $i = 1, 2, \ldots, n, \ldots$ such that they are pairwise not φ-twisted conjugate, i.e., $z_i\tau$ and $z_j\tau$ are not conjugate by an element of F.

Assume first that $\tau \in \mathrm{EP}_2$. If two elements are conjugate by an element of F then their fixed point sets match each other. So to prove that $z_i\tau$ and $z_j\tau$ are not conjugate, it would be enough to construct the $z_i \in F$ in such a way that $z_i\tau$ has, say, a fixed point set with i connected components so that the fixed point sets for all the $z_i\tau$ would be different and the elements cannot be conjugate.

We observe that the fixed point set of $z_i\tau$ contains exactly the points $t \in \mathbb{R}$ such that $z_i(t) = \tau^{-1}(t)$. Thus, it is enough to construct a map $z_i \in F$ such that it has exactly i disjoint intervals where $z_i(t) = \tau^{-1}(t)$, thus producing i connected components for $\mathrm{Fix}(z_i\tau)$. A reader familiar with F should be able to construct easily such family z_i.

The proof above does not work if τ is orientation reversing. But it can be modified to solve this case too. Assume now that $\tau = \sigma\mathcal{R}$ with $\sigma \in \mathrm{EP}_2$. Construct the elements $z_i \in F$ similarly to the orientation preserving case using σ, but in such a way that the fixed point set for $z_i\sigma$ is symmetric with respect to the origin. More precisely, we can ensure that $\mathrm{Fix}(z_i\sigma)$ has $2i+1$ connected components given by $\{0\}$, i connected components inside $\mathbb{R}_+$ and the opposite of these components in $\mathbb{R}_-$. Moreover, we can ensure that $z_i\sigma > 0$ if and only if $t > 0$. Observe that by this symmetry, the map $\mathcal{R}z_i\sigma\mathcal{R}$ has the exact same fixed points as $z_i\sigma$ and so $\mathrm{Fix}((z_i\sigma\mathcal{R})^2) = \mathrm{Fix}((z_i\sigma)^2)$.

Using this family z_i, we see that if $z_i\tau$ and $z_j\tau$ were conjugate via an element of F, then $(z_i\sigma\mathcal{R})^2$ and $(z_j\sigma\mathcal{R})^2$ would also be, and these have a different number of connected components in their fixed-point sets, by construction, yielding a contradiction.

The argument above shows that we can recover property R_∞ for F, giving a new proof of the following result.

Theorem 5.1 (Bleak-Fel'shtyn-Gonçalves, [4]): *Thompson's group F has property R_∞.*

Remark 5.1: We observe that Gonçalves and Kochloukova [12] *generalized Theorem 5.1 to show that the Brin-Guzman generalized Thompson groups $F_{n,0}$ and their finite direct products have property R_∞. Moreover, we also note that recently Koban and Wong* [16] *have shown that the group $F \rtimes \mathbb{Z}_2$ has property R_∞.*

Since we have a characterization for $\mathrm{Aut}(T)$ also in terms of conjugation by piecewise-linear maps, the method described above to prove property R_∞ for F can be used for T as well.

Theorem 1.5. *Thompson's group T has property R_∞.*

Proof. By Theorem 1 in [6], the group $\mathrm{Aut}(T)$ can be realized by inner automorphisms and by conjugations by $\mathcal{R}$, the map which reverses the orientation.

The process will consist on constructing maps with different fixed-point sets. Consider a piecewise-linear map on $[0,1]$ whose only fixed points are 0, $\frac{1}{2}$ and 1, and also such that the graph is symmetric respect to the point $[\frac{1}{2}, \frac{1}{2}]$. Identify the endpoints to obtain a map on S^1 and hence an element of T. Call this map h_1 and consider its lift $\widetilde{h}_1 \in \mathrm{PL}_2(\mathbb{R})$. From the way we have constructed h_1, we see that $\widetilde{h}_1$ is symmetric respect $[\frac{1}{2}, \frac{1}{2}]$ inside the square $[0,1]^2$, and so $\widetilde{h}_1$ is invariant under $\mathcal{R}$, i.e., $\mathcal{R}\widetilde{h}_1\mathcal{R} = \widetilde{h}_1$ inside $\mathrm{PL}_2(\mathbb{R})$. Therefore $\mathcal{R}h_1\mathcal{R} = h_1$ in T.

Now define inductively the map h_i by subdividing the interval $[0,1]$ in its two halves and in each half define a scaled-down version of $\widetilde{h}_{i-1}$, by a factor of 2. Observe that if $i \neq j$, then h_i and h_j have different number of fixed points. For a fixed $\varepsilon \in \{0,1\}$, if $h_i\mathcal{R}^\varepsilon$ and $h_j\mathcal{R}^\varepsilon$ were conjugate in T, then $(h_i\mathcal{R}^\varepsilon)^2$ and $(h_j\mathcal{R}^\varepsilon)^2$ are also conjugate in T. We notice that $(h_i\mathcal{R})^2 = h_i^2$ and that h_i^2 and h_j^2 have different number of fixed points, so they cannot be conjugate. ∎

6. Generalizations and some questions

In this section we make a series of observations about the extent to which the material of this paper generalizes and describe some natural related questions.

6.1. Extensions of the Bieri-Thompson-Stein-Strebel groups $\mathrm{PL}_{S,G}(I)$. It seems likely that the theory developed in this paper can be generalized to a certain extent to apply to Bieri-Thompson-Stein-Strebel groups $\mathrm{PL}_{S,G}(I)$, with the computational requirements described in [15].

We recall that $\mathrm{PL}_{S,G}(I)$ is the group of piecewise-linear homeomorphisms of the unit interval I with finitely many breakpoints occurring inside $S \leqslant \mathbb{R}$, an additive subgroup of $\mathbb{R}$ containing 1, and such that the breakpoints lie in $G \leqslant U(S)$, where $U(S) = \{g \in \mathbb{R}^* \mid gS = S \text{ and } g > 0\}$.

Since our results rely on straightforward generalizations of those in [15] and [18], to generalize our algorithms to the groups $\mathrm{PL}_{S,G}(I)$ we need to observe a number of things:

(1) We define the analogues $\mathrm{PL}_{S,G}(\mathbb{R}), \widetilde{\mathrm{EP}}_{S,G}, \mathrm{EP}_{S,G}$ and observe that the existence of periodicity boxes, the construction of conjugators and moving fixed points (Subsections 2.4, 2.4 and 2.5) generalize immediately via the results in [15] (which are proved in $\mathrm{PL}_{S,G}(I)$).

(2) To reduce the number of possible "initial slopes" we need to generalize Subsection 2.7. We can do this since the material in [18] can be generalized to $\mathrm{PL}_{S,G}(I)$. The second observation that is needed to reduce slopes is the one used in the proof of Theorem 1.2, where we multiply a candidate conjugator g by a power of y^2. This shows that we need to build candidate conjugators only for slopes in $[(y^2)'(p^+), 1]$ and, by Lemma 5.4 in [15], we can show that the sets of slopes is discrete in $\mathbb{R}_+$, thereby giving us only finitely many slopes inside $[(y^2)'(p^+), 1]$. Hence, this part generalizes too.

(3) Brin's Theorem 2.2 has a non-trivial generalization in a result of Brin and Guzman [7] which describes certain classes of automorphisms of the groups $\mathrm{PL}_{\mathbb{Z}[\frac{1}{n}],\langle n\rangle}(I)$. There exist elements in the automorphism group $\mathrm{Aut}(\mathrm{PL}_{\mathbb{Z}[\frac{1}{n}],\langle n\rangle}(I))$ which are represented by conjugation via elements that are not in $\widetilde{\mathrm{EP}}_n$ (and that are called "exotic"). Therefore, we can only generalize results of the current paper by restricting the action subgroup being used. Instead of studying the full automorphism group

$\mathrm{Aut}(\mathrm{PL}_{S,G}(I))$, we can restrict to study conjugations by element of $\widetilde{\mathrm{EP}}_{S,G}$ so that we can adapt our results in a straightforward manner.

Remark 6.1: It should be noted that the tools of this paper are not generally sufficient to solve either the twisted conjugacy problem or the orbit decidability problem in any group $\mathrm{PL}_{S,G}(I)$ generalizing Thompson's group F (for example, in generalized Thompson's groups $F(n)$). This is because the full automorphism group may contain conjugations via not piecewise-linear maps.

It is however possible to give suitable reformulations of Theorems 1.2, 4.4 and 1.4 in the setting of actions whose acting group is realized by conjugations by an element of $\widetilde{\mathrm{EP}}_{S,G}$. The restatement of Theorem 4.4 will need to assume that the 2-simultaneous conjugacy problem is solvable for the groups $T_{S,G}$ and this is also work-in-progress as mentioned in Section 4.

Since the techniques used to study the twisted conjugacy problem for F arise from those used in [15] to study the simultaneous conjugacy problem for F, it is natural to ask the following question:

Question 6.1: Is the k-simultaneous twisted conjugacy problem solvable for F? More precisely, is it decidable to determine whether or not, given $\varphi \in \mathrm{Aut}(F)$ and $y_1, \dots, y_k, z_1, \dots, z_k \in F$, there exists a $g \in F$ such that $z_i = g^{-1} y_i \varphi(g)$?

6.2. Extensions of Thompson's group T. As observed at the beginning of the proof of Theorem 1.5, if $\varphi \in \mathrm{Aut}(T)$, then there exists an $\varepsilon \in \{0,1\}$ such that $\varphi(\lambda) = \mathcal{R}^\varepsilon \tau^{-1} \alpha \tau \mathcal{R}^\varepsilon$, for all $\alpha \in T$. Arguing as in Subsection 2.3, equation (1) can be rewritten as

$$g^{-1}(y\mathcal{R}^\varepsilon)g = z\mathcal{R}^\varepsilon \tag{6.1}$$

for $y, z, g \in T$ and $\varepsilon \in \{0,1\}$. To attack equation (6.1), we can start by squaring it and initially reduce ourselves to solve the equation

$$g^{-1}(y\mathcal{R}^\varepsilon)^2 g = (z\mathcal{R}^\varepsilon)^2. \tag{6.2}$$

The advantage of working with equation (6.2) is that $(y\mathcal{R}^\varepsilon)^2, (z\mathcal{R}^\varepsilon)^2 \in T$.

The conjugacy problem in T is solvable by the work of Belk and the second author in [2] and thus we can list all the conjugators in T between $(y\mathcal{R}^\varepsilon)^2$ and $(z\mathcal{R}^\varepsilon)^2$. However, there might be infinitely many of them and there is no obvious way to detect which of them will also be conjugators between $y\mathcal{R}^{\varepsilon}$ and $z\mathcal{R}^{\varepsilon}$.

We cannot use the techniques of the current paper, since there is no uniqueness given by an the "initial slope" of elements of T (although something similar may be feasible, as in Chapter 7 of [17]). We are thus led to ask:

QUESTION 6.2: Is the twisted conjugacy problem solvable in Thompson's group T?

To conclude, we mention that the orbit decidability problem for T is solvable for $\mathrm{Aut}(T)$ and $\mathrm{Aut}_+(T)$.

LEMMA 6.3: *Let T be Thompson's group $\mathrm{PL}_2(S^1)$. Then $\mathrm{Aut}(T)$ and $\mathrm{Aut}_+(T)$ are orbit decidable.*

Proof. We need to decide whether or not, given $y, z \in T$, there exists an element $g \in T$ such that at least one of the two equalities

$$z = g^{-1}yg \qquad \text{or} \qquad z = g^{-1}(\mathcal{R}y\mathcal{R})g \tag{6.3}$$

holds. This amounts to studying two distinct conjugacy problems for elements of T, each of which is solvable by the work [2]. ∎

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