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MULTIPLICATIVE REPRESENTATIONS OF SURFACE GROUPS

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List of symbols

Groups and presentations

G	generic group
∂G	boundary of G
Γ	generic surface group
$\partial\Gamma$	boundary of Γ
Γ_k	surface group of genus k
k	genus of a surface group
S	symmetric set of generators
S_k	standard symmetric set of generators of Γ_k
R	set of relators
R_*	set of fundamental relators
S^*	set of all words on S
a_i, b_i ($i = 1, \dots, k$)	standard generators of a surface group
s, t	generic generators
\bar{u}	group element represented by $u \in S^*$

Cones, cone types, matrix systems

$C(x, y)$	cone with base x and vertex y
$C(w)$	the cone type of $w \in S^*$
$\mathbf{c}, \mathbf{c}', \mathbf{d}, \mathbf{d}', \dots$	cone types (or cones with vertex e)
\mathcal{C}	set of cone types
$\mathbf{c} \xrightarrow{s} \mathbf{c}'$	transition in the CTA
$\mathbf{c} \sim C(x, y)$	the cone $C(x, y)$ has cone type \mathbf{c}
$x \cdot \mathbf{c}$	action of $x \in \Gamma$ on the cone \mathbf{c}
$V_{\mathbf{c}}$	vector space associated to $\mathbf{c} \in \mathcal{C}$
$H_{\mathbf{c}, \mathbf{c}', s}$	linear map associated to $\mathbf{c} \xrightarrow{s} \mathbf{c}'$
V	vector space $V = \bigoplus_{\mathbf{c}} V_{\mathbf{c}}$

Various

$\mathbb{N}, \mathbb{Z}, \mathbb{R}, \mathbb{C}$	sets of naturals, integers, reals and complex
X	the space $\Gamma \cup \partial\Gamma$
$C(X)$	continuous functions on X
$B(X)$	bounded Borel-measurable functions on X
\mathcal{H}, \mathcal{K}	Hilbert spaces
π, σ	unitary representations
ρ	\mathbb{C}^* -representation
$\tilde{\rho}$	extension of the rep.n ρ of $C(X)$ to $B(X)$
λ	left-regular representation
\mathcal{H}^∞	equivalence classes of multiplicative functions
\mathcal{H}_m	Hilbert space of multiplicative functions
π_m	multiplicative representation
\mathbf{m}	multiplicative function
$\mathbf{m}[x, y, v]$	elementary multiplicative function
$\mathfrak{T}, \mathfrak{D}$	two particular matrices

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INTRODUCTION

Harmonic analysis on some discrete groups

Let G be a locally compact second countable topological group. One branch of Harmonic Analysis is devoted to the study of the unitary dual \hat{G} , that is the space of equivalence classes of (continuous) irreducible unitary representations of G . In particular, one hopes to exhibit a sufficiently large family of irreducible representations to be able to “decompose” any other representation using the known ones. Another problem is to find a reasonable parametrization of \hat{G} . This program works very well for compact groups and also for many Lie groups.

When one passes to (infinite) discrete groups “terrible things” can happen ([Fol16, p. 230]).

Infinite discrete groups do not have irreducible representations which are strongly contained in the regular representation (cf. [CFT74]). The problem of decomposing the regular representation of these groups into irreducible ones leads to the notions of *direct integral* together with the one of *weak containment*.

A typical example is a free non-abelian group \mathbb{F} on a (finite or infinite) set of generators.

Without going into any detail, it is known that \mathbb{F} is *not* a type I group (see for example [Dix77]): among other things, this means that a given unitary representation may be decomposed as a direct integral of irreducibles in many “inequivalent” ways. Also, $\hat{\mathbb{F}}$ cannot be parametrized by any standard Borel space (cf. [Gli61]), which means in practice that one cannot hope for a parametrization that one could actually work with.

Another remarkable example of discrete groups which are not Type I is given by surface groups.

A surface group Γ_k of genus k is (isomorphic to) the fundamental group

of a closed orientable surface of genus $k \geq 2$; it is given by the standard presentation

$$\Gamma_k = \langle a_1, b_1, \dots, a_k, b_k \mid [a_1, b_1] \cdots [a_k, b_k] = e \rangle.$$

The Cayley graph of this group with respect to the standard symmetric set of generators $\{a_1^{\pm 1}, b_1^{\pm 1}, \dots, a_k^{\pm 1}, b_k^{\pm 1}\}$ is isomorphic to a tiling of the hyperbolic plane by hyperbolic octagons and it is a hyperbolic geodesic space. The group Γ_k satisfies $C'(\frac{1}{6})$ -small cancellation condition and it is Gromov-hyperbolic.

The fact that Γ_k is not type I follows, for example, from a Theorem proved by Thoma ([Fol16, Thm. 7.8 (d)]): a discrete group is type I if and only if it possesses an abelian normal subgroup of finite index.

This thesis is devoted to the study of a class of unitary representations of Γ_k which are *tempered*, i.e. weakly contained in the regular representation: we show how to construct them and we realize them as representations of a certain crossed product C*-algebra.

Boundary representations

Let G be a torsion free not almost cyclic hyperbolic group with boundary ∂G . Then G naturally acts on ∂G and we denote the action by $(g, \omega) \mapsto g\omega$ for $g \in G$ and $\omega \in \partial G$. Let $C(\partial G)$ be the space of the complex-valued continuous functions on the boundary; the group G acts on the algebra $C(\partial G)$ by left translation. One can construct the so-called crossed product C*-algebra $G \rtimes C(\partial G)$ by defining a norm on $C_c(G, C(\partial G))$, the space of linear combinations of compactly supported $C(\partial G)$ -valued functions on G . These norm depends the pairs of representations (π, ρ) , where π is a unitary representation of G and ρ is a representation of the commutative C*-algebra $C(\partial G)$ satisfying the covariant condition:

$$\pi(g)\rho(f)\pi(g^{-1}) = \rho(g.f),$$

where $g \in G$, $f \in C(\partial G)$ and $g.f$ denotes the function $\omega \mapsto f(g^{-1}\omega)$, for $\omega \in \partial G$. A *boundary representation* is precisely a representation of the C*-algebra $G \rtimes C(\partial G)$, i.e. a pair (π, ρ) as above. We usually denote $\pi \rtimes \rho$ the representation associated with the pair (π, ρ) .

Boundary representations provide some information on possible decompositions of a tempered unitary representation of a group. In fact, given a torsion free not almost cyclic hyperbolic group G , the group acts amenably in Zimmer's sense on its boundary ([Ada94]), the reduced C*-algebra is simple and there is an embedding $C_{\text{red}}^*(G) \hookrightarrow G \rtimes C(\partial G)$ ([IKS13]). Therefore,

every *tempered* unitary representation (which is precisely a representation of $C_{\text{red}}^*(G)$) extends to a representation of the crossed product; vice versa, every representation of the crossed product, when restricted to the group, turns out to be tempered (cf. for more details [IKS13, Sect. 4]). In general, the space \mathcal{H}_π of the unitary representation π may need to be enlarged. Still, by the general theory of crossed products (cf. [Tak13]), there exists a quasi-invariant measure μ on ∂G (depending on π) such that \mathcal{H}_π can be isometrically included in the direct integral $\mathcal{H}_{\partial G} = \int_{\partial G}^{\oplus} \mathcal{H}_\omega d\mu(\omega)$, where $\omega \mapsto \mathcal{H}_\omega$ is a measurable field of Hilbert spaces on the boundary.

A brief history

As before mentioned, the literature on tempered unitary representations of free groups has focused on constructing some concrete examples: [Yos51], [FTP82], [FTP83], [FTS94], as well as [Pas01] and [Pas02].

Some authors developed methods for decomposing the regular representation into some particular set of irreducibles and tried to study the properties of the different decompositions in connection with random walks on trees (e.g. [FTP82] and [FTP83] for the isotropic random walk and [FTS94] for the anisotropic random walk). The idea is roughly the following: let μ be a finitely supported symmetric measure on G , consider the random walk with law μ . The operator $R_\mu: f \mapsto f * \mu$, defined for $f \in \ell^2(G)$, commutes with the left regular representation of G , hence the spectral decomposition of R_μ gives a decomposition of $\ell^2(G)$ as a direct integral of (“irreducible”) representations.

The class of multiplicative representations was first introduced by M.G. Kuhn and T. Steger for finitely generated non-abelian free groups in the paper [KS04]. These representations act on the “space of vector-valued multiplicative functions”. Multiplicative representations are naturally realized as boundary representations; the crossed-product representation obtained from this construction is proved to be irreducible. This class of representations is intrinsically linked with the action of the free group on its Cayley graph (a homogeneous locally finite tree) and it is actually quite general: through this construction, in fact, it is possible to retrieve many concrete examples of unitary representations of free groups already present in literature (as explained in [KS04, Sect. 6]). However, the irreducibility of the representation restricted to the free group does not automatically follow from the irreducibility of the boundary representation. This last problem is treated in [KSS16], where the growth of the matrix coefficients of the multiplicative representations is studied and a criterion for the irreducibility of the

representation restricted to the free group is presented.

A further motivation that led to the study of boundary representation comes from the following problem. Let G be a torsion-free not almost cyclic hyperbolic group: as before mentioned, every tempered unitary representation of G can be extended to a representation of $G \rtimes C(\partial G)$. But how many extensions can there be?

Let us focus on the case of a free group. Given a tempered unitary representation (π, \mathcal{H}) of G , a *boundary realization* of π is given by $(\iota, \pi', \rho', \mathcal{H}')$ where $(\pi', \rho', \mathcal{H}')$ is a boundary representation of G and $\iota: \mathcal{H} \rightarrow \mathcal{H}'$ is an isometric G -inclusion. Moreover, one usually requires that the space \mathcal{H}' is generated as a $(G, C(\partial G))$ -space by the image $\iota(\mathcal{H})$. In the case where ι is unitary, the boundary realization is said to be *perfect*. In [KS01], a tempered irreducible unitary representation of a free group is said to satisfy

- *monotony* if it has a unique boundary realization (up to equivalence) which is perfect;
- *duplicity* if it has precisely two inequivalent perfect boundary realizations;
- *oddity* if it has a unique boundary realization (up to equivalence) which is not perfect.

In [KS01], Kuhn and Steger conjectured that a tempered unitary representation of a free group which is irreducible satisfies one of the three above cases.

Some interesting steps in this problem were made by A. Boyer and Ł. Garncarek in [BG16]. In the previous paper [Gar14], the second author introduces a wide class of unitary representations of hyperbolic group extending a construction presented in [BM11] for fundamental groups of compact negatively curved manifolds. In [BG16], the two authors prove that this class of representations satisfies what they call “Asymptotic Orthogonality Theorem”. The construction of [Gar14] applies to free groups (which are hyperbolic), so the authors of [BG16] used their result to prove that this class of free group representations satisfies monotony.

It is worth noticing that the authors of [Gar14] and [BG16] reserve the term “boundary representation” for the quasi-regular representation (cf. Sect. 4.1.3), while we mean a representation of the crossed product C^* -algebra $G \rtimes C(\partial G)$ (cf. Def. 4.2). Since the continuous functions act on the L^2 space as multipliers, the quasi-regular representation corresponds to a representation of the crossed product.

Another direction that one can take is to try to generalize the construction of multiplicative representations to other groups. The class of multiplicative representation of free groups of [KS04] can be made stable under the main operations that one usually performs on unitary representations, such as change of generator set, restriction to a subgroup or induction to a finite index supergroup: this is shown in [IKS12]. Exploiting these stability properties, the same authors are able to extend the class of multiplicative representations to *virtually free groups* (in [IKS13]). Virtually free groups are non-cocompact lattices in $\mathrm{PSL}_2(\mathbb{R})$: multiplicative representations are obtained by induction from a finite index subgroup.

The paper [IKS13] is a first step in the project of extending the class of multiplicative representations to hyperbolic groups.

The construction of [IKS13], though, breaks in the case of a cocompact lattice in $\mathrm{PSL}_2(\mathbb{R})$: these groups do not contain a free subgroup of finite index, thus there is no way to get multiplicative representations by induction. Moreover, their Cayley graphs contain circles, so that multiplicative functions cannot be defined exploiting the uniqueness of the geodesic joining two points as in [KS04]. Hence, the need for a new construction of the space of multiplicative functions which mimics the one presented in [KS04], but with the due consistent modifications required by the combinatoric and geometric properties of these lattices. The first steps in this direction were undertaken by Kuhn and Steger together with A. Iozzi.

The present work tries to take a further step forward by defining multiplicative representations for the class of *surface groups*, a very concrete case of a cocompact lattice in $\mathrm{PSL}_2(\mathbb{R})$. Some of our proofs are carried out for the case of a surface group of genus 2, but we strongly believe that the general case does not introduce relevant difficulties. The only really problematic point lies in the proof of the irreducibility of a certain matrix, which we obtained through computational methods for the case of genus 2.

Overview of the main results

Let Γ be a surface group with a distinguished symmetric set of generators S . We will write Γ_k, S_k whenever we want to make explicit reference to the genus $k \geq 2$. Fix on Γ the word length $|\cdot|$ and the word metric d associated to S and consider the Cayley graph $\mathcal{G} = \mathcal{G}_S(\Gamma)$ equipped with the usual edge-path metric. The group Γ satisfies the $C'(\frac{1}{6})$ -small cancellation condition: there is an explicit classification of the forms of geodesic triangles and geodesic digons in the Cayley graph of a surface group, proved by R. Strebel in [Str90].

Theorem ([Str90]). *Let $G = \langle S \mid R \rangle$ be a finite presentation of a group satisfying condition $C'(\frac{1}{6})$. A simple geodesic triangle Δ in $\mathcal{G}_S(G)$ has one of the forms I_1, \dots, V in Figure 1.*

In Chapter 1 we introduce the notion of *cone* in the Cayley graph \mathcal{G} defining it through the choice of a base point $x \in \Gamma$ and a vertex $y \in \Gamma$: the cone $C(x, y)$ is the set of all vertices $z \in \Gamma$ which lie on a geodesic starting at the base point x and passing through the vertex y ; this is equivalent to the condition that $d(x, z) = d(x, y) + d(y, z)$. The group Γ acts on the set of cones \mathfrak{C} by left translation: we define a *cone type* \mathbf{c} as an orbit of this action (and we usually identify a cone type with its unique representative having vertex $y = e$). We introduce the subdivision of a cone into subcones and we study the intersection of two different cones in the graph in Section 1.8. It turns out that this intersection equals, up to finitely many points, the union of subcones of the two.

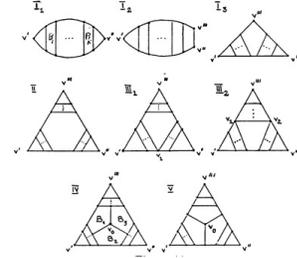


Figure 1: Strebel's classification of geodesic triangles ([Str90, Figure 11]). See Chap. 1 for a better figure.

Theorem (1.34). *Let $C(x_1, y_1)$ and $C(x_2, y_2)$ be cones in the Cayley graph of the group Γ_2 . Then, if not empty nor finite, $C(x_1, y_1) \cap C(x_2, y_2)$ is equal, up to finitely many points, to a union $\bigcup_i C(x'_i, z_i)$ of subcones of both cones.*

This is of great importance for what follows since we introduce some functions supported on cones and we consider equivalent two functions that are equal up to finitely many values.

In Chapter 2 we recall the notion of automatic group and we study the automatic structure for surface groups provided by the cone types: the set of cone types \mathcal{C} gives the set of states of the so-called Cone Type Automaton (CTA); the transitions $\mathbf{c} \xrightarrow{s} \mathbf{c}'$ are labelled by the generators $s \in S$. In the case of a surface group, the CTA is *ergodic*: this means that a natural graph associated to this automaton has a strong connectedness property: for every pair of cone types $\mathbf{c}, \mathbf{c}' \in \mathcal{C}$, we can find a sequence of transitions $\mathbf{c} \rightarrow \dots \rightarrow \mathbf{c}'$ in the CTA starting from \mathbf{c} and ending in \mathbf{c}' .

In Chapter 3 we introduce the notion of *matrix system*, which is essentially a set of linear maps $H_{\mathbf{c}, \mathbf{c}', s}: V_{\mathbf{c}} \rightarrow V_{\mathbf{c}'}$ between finite-dimensional complex vector spaces $V_{\mathbf{c}}$ indexed by the cone types. Each map $H_{\mathbf{c}, \mathbf{c}', s}$ is associated to a transition $\mathbf{c} \xrightarrow{s} \mathbf{c}'$ in the Cone Type Automaton, for $\mathbf{c}, \mathbf{c}' \in \mathcal{C}$

and $s \in S$ (thus, a matrix system depends on the set of cone types and on a set of generators and inverses).

We have a special interest for those that we call *scalar systems*: $V_{\mathbf{c}} \simeq \mathbb{C}$ for every $\mathbf{c} \in \mathcal{C}$ and the maps $H_{\mathbf{c}, \mathbf{c}', s}$ are represented by positive scalars.

The space of multiplicative functions \mathcal{H}_0^∞ on a surface group Γ depends on the choice of a matrix system as a set of parameters. These functions take values in the finite-dimensional vector space $V = \bigoplus_{\mathbf{c}} V_{\mathbf{c}}$. The multiplicative functions are defined as linear combination of some elementary functions \mathfrak{m} which satisfy a recurrence relation (more of this in Def. 3.13). Two multiplicative functions are equivalent if the set of points on which they differ is finite; the space of equivalence classes of multiplicative functions is denoted by \mathcal{H}^∞ . We introduce an *inner product* on \mathcal{H}^∞ : this depends on the choice of a sesquilinear form B on the finite-dimensional space V where the multiplicative functions take values:

$$\langle f_1, f_2 \rangle = \lim_{\epsilon \downarrow 0} \epsilon \sum_{z \in \Gamma} B(f_1(z), f_2(z)) e^{-\epsilon|z|},$$

for $f_1, f_2 \in \mathcal{H}^\infty$. Even if it is easy to show that the inner product does not depend on the representatives for the multiplicative functions, the definition involves a limit for ϵ going to zero: it is then necessary to prove that there exists a sesquilinear form B_0 such that the inner product is not infinite nor identically zero (which is not immediate). A consistent part of the Chapter is then devoted to this proof for the case of genus 2 and the choice of a scalar system (Theorem 3.30).

Theorem (3.30). *Let Γ_2 be the surface group of genus 2, fix a scalar system $\{H_{\mathbf{c}, \mathbf{c}', s}\}$. Then there exists a sesquilinear form B_0 and $\mathfrak{m}_1, \mathfrak{m}_2$ elementary multiplicative functions such that $\langle \mathfrak{m}_1, \mathfrak{m}_2 \rangle \neq 0$.*

A certain recursion relation holds for the elementary multiplicative functions: this can be formalized by means of a (quite big) transition matrix \mathfrak{T} . In the case of genus 2 and a scalar system \mathfrak{T} is positive and irreducible, hence we can apply Perron-Frobenius Theorem, which allows us to compute the limit by means of an eigenvector of \mathfrak{T} associated to the Perron-Frobenius eigenvalue.

The multiplicative representation $\pi_{\mathfrak{m}}$ is then easily defined: we complete the space of multiplicative functions \mathcal{H}^∞ to a Hilbert space $\mathcal{H}_{\mathfrak{m}}$ using the inner product introduced and we let the group Γ act by left translations: $(\pi_{\mathfrak{m}}(g)f)(z) = f(g^{-1}z)$ for $g, z \in \Gamma$ and $f \in \mathcal{H}^\infty$. Since the norm induced by the inner product is translation-invariant, the representation $\pi_{\mathfrak{m}}$ is actually unitary.

Then, we present a proof of the weak containment of the multiplicative representation in the left regular one: we show that the functions of positive type of the latter pointwise converge to the ones of the former.

Theorem (3.59). *Let $\Gamma = \Gamma_2$ be the surface group of genus 2 with a symmetric set of generators S , let π_m be the multiplicative representation associated with a scalar system. Then π_m is weakly contained in the regular representation λ of Γ .*

Chapter 4 starts with some recalls: the definition of covariant system and crossed product C*-algebra. Assume that a group G is acting continuously by automorphisms on a C*-algebra \mathcal{A} : for $g \in G$ and $f \in \mathcal{A}$ we write $g.f$ for this action. Let (π, ρ) denote respectively a unitary representation of G and a C*-representation of \mathcal{A} on the same space: if

$$\pi(g)\rho(f)\pi(g^{-1}) = \rho(g.f), \quad \text{for } g \in G, f \in \mathcal{A},$$

we say that (π, ρ) is a covariant representation of the dynamical system (G, \mathcal{A}) or a representation of the crossed product C*-algebra $G \ltimes \mathcal{A}$. Given a scalar system associated to a surface group $\Gamma = \Gamma_2$, we introduce a family of covariant representations $(\pi_\epsilon, \rho_\epsilon)$ of the system $(\Gamma, C(X))$, where $X := \Gamma \cup \partial\Gamma$ is the compactification of Γ and $C(X)$ is the algebra of continuous complex-valued functions on X . These representations act on a subspace of the tensor product Hilbert space $\ell^2(\Gamma) \otimes V$. Representations of a C*-algebra \mathcal{A} correspond to positive functionals on \mathcal{A} of norm one. Since closed balls in a dual space are weak* compact, we can pass to a convergent subsequence $(\pi_{\epsilon_j}, \rho_{\epsilon_j})$ of the family of representations that we defined and thus we get a “limit” representation (π, ρ) . In particular, (π, ρ) does not depend on the subsequence ϵ_j and it is actually a representation of $\Gamma \ltimes C(\partial\Gamma)$: these facts are proved in the two following theorems.

Theorem (4.11). *Let $\Gamma = \Gamma_2$ be the surface group of genus 2, let (π, ρ) be a limit for a subsequence of representations $(\pi_{\epsilon_j}, \rho_{\epsilon_j})$ as above. Then (π, ρ) defines a boundary representation, i.e. ρ depends only on the values of the functions on the boundary $\partial\Gamma$.*

Theorem (4.19). *Let $\Gamma = \Gamma_2$ be the surface group of genus 2, let (π, ρ) be a limit for a subsequence of representations $(\pi_{\epsilon_j}, \rho_{\epsilon_j})$ as above. Then (π, ρ) does not depend on the subsequence of representations chosen.*

In the final Section 4.5 we prove that the Γ -part of this representation, the unitary representation π , is equivalent to the multiplicative representation π_m defined in Chapter 3: the space of multiplicative functions can then be isometrically embedded in a Hilbert space on which $\Gamma \ltimes C(\partial\Gamma)$ acts.

Theorem (4.23). *Let $\Gamma = \Gamma_2$ be the surface group of genus 2 with a scalar system, let (π, ρ) be the limit representation of the family $(\pi_\epsilon, \rho_\epsilon)$. Then the unitary representation π is equivalent to the multiplicative representation π_m .*

A boundary representation is irreducible if it has no non-trivial subspace invariant for both the Γ -action and the $C(\partial\Gamma)$ -action.

Chapter 5 is devoted to the proof Theorem 5.1, which claims the irreducibility of the boundary representation defined in Chapter 4 for the case of a surface group of genus 2 and a scalar matrix system.

Theorem (5.1). *Let $\Gamma = \Gamma_2$ be the surface group of genus 2 with a scalar system, let (π, ρ) be the limit boundary representation of the family $(\pi_\epsilon, \rho_\epsilon)$. Then the the boundary representation $\pi \times \rho$ of $\Gamma \times C(\partial\Gamma)$ is irreducible.*

This proof relies on the fact that the matrix \mathfrak{T} associated with the multiplicative functions according to the structure of the group and the cone types is irreducible. We exploit the unicity (up to scaling) of the Perron-Frobenius eigenvector showing that this guarantees that every projection P on the space of the representation which intertwines both the action of the group and the action of the algebra is trivial. By a version of Schur's Lemma, this guarantees the irreducibility of the crossed-product representation.

Further developments

At this point, further work can be done on multiplicative representations of surface groups. First, the question of what happens in genus $k > 2$ still remains open: is it possible to prove by abstract methods the irreducibility of the transition matrix?

One could also study the conditions that one needs to impose on a general matrix system to be able to define multiplicative representations and the behavior of the representations obtained when one changes the matrix system (a similar work to the one carried out for free groups in [KS04]).

The irreducibility of the unitary representation π_m has not been studied in this work: it seems that it requires a considerable work dealing with the positive functions associated with it. It would be, though, a natural prosecution of the present work trying to investigate the conditions that guarantee the irreducibility of the Γ -part of the boundary representation.

Irreducibility is surprisingly related to the following question (cf. the preprint [HKS17]): in how many ways can a multiplicative representation of a surface group be extended to a boundary representation? The explicit

construction that we present in Chapter 4 is in fact just one way to get a boundary representation whose Γ -part is equivalent to π_m . Are there others? This would mean to address in the case of surface groups an analog of Kuhn and Steger's conjecture stated in [KS01] for free groups (multiplicative representations satisfy one between monotony, duplicity, oddity).

Moreover, one could ask if it is possible to extend multiplicative representations to more general hyperbolic groups. The case of *virtually fuchsian groups* is the first that comes to mind. A group G is virtually fuchsian if it has a finite index subgroup which is isomorphic to a surface group (cf. [KB02, Sect. 5]). An important Theorem (that goes under the names of Tukia-Gabai-Freden-Casson-Jungreis) tells that virtually fuchsian groups are precisely those Gromov-hyperbolic groups whose boundary is homeomorphic to a circle ([KB02, Thm. 5.4]). Induction from a finite index surface subgroup could be a way of defining multiplicative representations for virtually fuchsian groups. Though, some preliminary study on the stability properties of multiplicative representations of surface groups (analogous to the one conducted for free groups in [IKS12]) would be necessary: how does this class of representations behave under change of generation, restriction to a finite index subgroup and induction to a supergroup?

CHAPTER 1

GEOMETRIC STRUCTURE OF SURFACE GROUPS

This Chapter devoted to a duplicitous task: we first recall some basics notions from Geometric Group Theory, then we present in more detail how these notions apply to the case of a surface group. Some a quite specific result is proved in the last part of this Chapter.

The main references for this Chapter are [EPC⁺92], [Ohs02], the foreword of [Oll05], [BP03], [KB02] and [Str90], which appears as an appendix of [GdlH90].

1.1 GROUP PRESENTATIONS

This Section mainly refers to [Ohs02, Chap. 1] and [EPC⁺92, Chap. 2].

An **alphabet** A is a finite set. An element of an alphabet $a \in A$ is called a **letter**. A **word** w on A is a finite sequence of letters of A : given $n \in \mathbb{N}$, a word can be defined as a map $\{1, \dots, n\} \rightarrow A$. The number n is the **length** of w , denoted by $\ell(w)$; we define for $n = 0$ the **empty word** ε . The set A^n is composed of all word of length n on A ; the set A^0 is formed simply by ε . We define

$$A^* := \bigcup_{n=0}^{\infty} A^n$$

the set of all words on A .

We denote a word w as a sequence of letters: if w sends each $i \in \{1, \dots, n\}$ to $a_i \in A$, i.e. $w(i) = a_i$, then $w = a_1 \cdots a_n$.

Given two words w, v in A^* having lengths n, m respectively, we define the concatenation wv as the word $\{1, \dots, n + m\} \rightarrow A$ sending i to $w(i)$ if $i \in \{1, \dots, n\}$ and i to $v(n - i)$ if $i \in \{n + 1, \dots, n + m\}$.

Under the law given by concatenation, the set A^* becomes a monoid with identity ε ; in fact, it is isomorphic to the free monoid on the set A .

Fix a finite set S^+ , define $S^- := \{s^{-1} \mid s \in S\}$, i.e. the set of all formal inverses s^{-1} of elements $s \in S$; the requirement is that S^+ and S^- are disjoint. Define moreover $S := S^+ \cup S^-$. We say that S is **symmetric**.

We interpret S as an alphabet and we consider the monoid S^* of all words on S .

Given a word on S , say $w = s_1 \cdots s_n$, we define its **formal inverse** as $w^{-1} = s_n^{-1} \cdots s_1^{-1}$.

A word $w \in S^*$ is **reduced** if no pair of adjacent letters in w is of the form tt^{-1} or $t^{-1}t$ for some $t \in S$.

We denote by $\mathbb{F}(S)$ the free group on the set S^+ . There is a bijection between $\mathbb{F}(S)$ and the set of all reduced words on S ; thus, we think of an element of $\mathbb{F}(S)$ as a reduced word on S . Given a word $w \in S^*$, we can associate with it a unique reduced word. Thus we have a natural map $S^* \rightarrow \mathbb{F}(S)$, the reduction. The composition of reduction and the operations of concatenation and formal inversion of words as above gives the multiplication and inversion of the group $\mathbb{F}(S)$.

Given a finite set S^+ and a finite subset $R \subset \mathbb{F}(S)$, we define the group G presented by $\langle S^+ \mid R \rangle$ as the quotient $\mathbb{F}(S)/\langle R \rangle$, where $\langle R \rangle$ denotes the normal closure of R in $\mathbb{F}(S)$, i.e. the minimal normal subgroup of $\mathbb{F}(S)$ containing the set R .

Notation 1.1. Given a word $u \in S^*$ or a reduced word $v \in \mathbb{F}(S)$, we denote by \bar{u} and \bar{v} the group element represented by the words u and v .

If $G = \langle S^+ \mid R \rangle$, then there is a natural map $\mathbb{F}(S) \rightarrow G$ which takes any reduced word $w \in \mathbb{F}(S)$ to the correspondent element $\bar{w} \in G$. Since in our definition G is finitely generated, this map is finite-to-one: for every $w \in \mathbb{F}(S)$, the set of pre-images of $\bar{w} \in G$ has finite cardinality. The kernel of the map $\mathbb{F}(S) \rightarrow G$ is the normal closure of the set $\{[r] \mid r \in R\}$, where $[\cdot]$ refer to the equivalence class of a word on S in the free group $\mathbb{F}(S)$.

Moreover, we have a natural map $S^* \rightarrow G$ which takes any word on S to the element in G represented by it. We get this last map as the composition of the reduction in $S^* \rightarrow \mathbb{F}(S)$ and the map $\mathbb{F}(S) \rightarrow G$ as above.

Let S be a finite set, let $\mathbb{F}(S)$ be the free group on S . A reduced word $w = s_1 \cdots s_n \in \mathbb{F}(S)$ is cyclically reduced if $s_1 \neq s_n^{-1}$, i.e. if all cyclic permutations of w are reduced.

The symmetrization of R , denoted by R_* , is the set of all distinct cyclic permutations of the elements in R and their inverses. We call R_* the set of **fundamental relators** of the presentation $\langle S^+ \mid R \rangle$.

1.2 CAYLEY GRAPHS

The main reference for this Section is [EPC⁺92, Chap. 2], where Cayley graphs are presented as labelled directed graphs, sometimes also called “colored” directed graphs. We refer also to [Ohs02, Chap. 1], where the graphs are considered as 1-dimensional CW-complexes: in this light, the notion of geometric realization of a Cayley graph is straightforward. We mention, moreover, the lecture notes [BP03].

Definition 1.2. Let G be a finitely generated group, let S be a symmetric set of generators. The **Cayley graph** of G with respect to S is the labelled directed graph $\mathcal{G} = \mathcal{G}_S(G)$ with

- set of vertices $V(\mathcal{G})$ given by $\{g \mid g \in G\}$;
- set of directed edges $\{(g, gs) \mid g \in G, s \in S\}$;

The edge (g, gs) is directed from the vertex g to the vertex gs and it is labelled by the generator $s \in S$.

The Cayley graph $\mathcal{G}_S(G)$ has a natural root given by the vertex corresponding to the identity $e \in \Gamma$.

We can consider \mathcal{G} as an undirected graph (still denoted by the same symbol, with an abuse of notation; the context will clarify the meaning): identify the pair of edges labelled by a generator and its inverse. Thus, the edge $\{g, gs\}$ in the undirected Cayley graph corresponds to the pair of directed edges $(g, gs), (gs, g)$ in the directed graph.

Since G is finitely generated, the undirected Cayley graph $\mathcal{G}_S(G)$ is a locally finite homogeneous graph.

The geometric realization of the (undirected) Cayley graph is the topological space obtained identifying each edge of $\mathcal{G}_S(G)$ with a copy of the real unit interval $[0, 1]$. The space thus obtained is locally compact.

Let G be a group generated by a finite symmetric set S . The graph $\mathcal{G}_S(G)$ carries a natural metric $d_{\mathcal{G}}$, where each edge has length one; the distance between two vertices is given by the minimal number of edges crossed by a path from the first to the second; we can metrize the geometric realization of \mathcal{G} (where each edge is considered as isometric to the unit real interval $[0, 1]$) by a linear distance on each edge. The space $(\mathcal{G}_S(G), d_{\mathcal{G}})$ thus obtained is a locally compact geodesic metric space.

A (directed) **path** γ in \mathcal{G} is a sequence of vertices $\gamma = \{z_0, \dots, z_n\}$ in the directed Cayley graph where $(z_{i-1}, z_i) \in E(\mathcal{G})$ for every $i = 1, \dots, n$; a path can be extended infinitely both on the left and on the right. By extending γ linearly on the edges, we get a continuous path in the geometric realization of \mathcal{G} . We say that the path γ joins x and y if $z_0 = x$ and $z_n = y$; moreover, given $z \in V(\mathcal{G})$, if there exists $i \in \{0, \dots, n\}$ such that $z_i = z$, then we say that the path γ passes through z .

The **length** of the path γ , denoted by $\ell(\gamma)$, is the number of edges crossed by γ : thus, if $\gamma = \{z_0, \dots, z_n\}$, then the length of γ is n .

A path $\gamma = \{z_0, \dots, z_n\}$ is a **geodesic path** (or simply a **geodesic**) in \mathcal{G} if the length of γ is minimal between all paths in \mathcal{G} joining the endpoints z_0, z_n of γ ; in particular: $\ell(\gamma) = d(z_0, z_n)$.

Notation 1.3. We refer to the set of geodesics joining two vertices $x, y \in V(\mathcal{G})$ by $[x, y]$. In general, $[x, y]$ contains different geodesic paths; anyway, in the case where G is a finitely generated group, $[x, y]$ is a finite set.

Every reduced word $w \in \mathbb{F}(S)$ corresponds to a (non-backtracking) path in $\mathcal{G}_S(G)$ starting at the identity $e \in G$ (viewed as a vertex of $\mathcal{G}_S(G)$) and ending in the vertex corresponding to $\bar{w} \in G$.

More generally, every (possibly non-reduced) word in the generators in S corresponds to a (possibly backtracking) path in $\mathcal{G}_S(G)$. If $\bar{w} = e$, then the path is a loop based at the vertex $e \in \mathcal{G}_S(G)$.

Given a word $w \in S^*$ and $n \leq \ell(w)$, we call the word consisting of the first n -letters of w the n -**prefix** of w ; we denote it by $w[n]$.

If $w \in S^*$, we associate to it a path in the Cayley graph $\mathcal{G}_S(G)$ defined by $\hat{w}: [0, \infty) \rightarrow \mathcal{G}_S(G)$ as follows: if $0 \leq n \leq \ell(w)$ is an integer, then $\hat{w}(n) := w[n]$; if $t \in [0, \ell(w)]$ is not an integer, we connect $\hat{w}([t])$ to $\hat{w}([t]+1)$ by a geodesic segment; for all $t \geq \ell(w)$, $\hat{w}(t) := \bar{w}$.

Definition 1.4. A word $w \in S^*$ is a **geodesic word** if the path $\hat{w}|_{[0, \ell(w)]}$ in $\mathcal{G}_S(G)$ is a geodesic in $\mathcal{G}_S(G)$ joining the identity e with \bar{w} , i.e. the length of \hat{w} is minimal between all paths in $\mathcal{G}_S(G)$ having e and \bar{w} as endpoints.

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Consider two paths $\gamma_1, \gamma_2: [0, \infty) \rightarrow \mathcal{G}_S(G)$. The **uniform distance** is defined as

$$d_u(\gamma_1, \gamma_2) := \sup_{t \in [0, \infty)} d_G(\gamma_1(t), \gamma_2(t)).$$

We get a metric on the group G associated to the symmetric generating set S (see [Ohs02, Section 1.2]):

- for every $g \in G$, the **word length** of g , denoted by $|g|_S$, is defined as

$$|g|_S := \min_{w \in \mathbb{F}(S), \bar{w}=g} \ell_S(w);$$

- the **word metric** on G is the distance defined by $d_S(g, h) := |g^{-1}h|_S$, for $g, h \in G$.

Notation 1.5. We always drop the subscripts in $|\cdot|_S$, d_S , ℓ_S and d_G , once we fixed a set of generators. The context will make the meaning clear.

The word metric on G corresponds precisely to the metric on $\mathcal{G}_S(G)$ assigning length one to every edge (cf. [Ohs02, Section 1.2]).

There is a natural action by left multiplication (indicated by the dot notation) of the group G on its Cayley graph $\mathcal{G} = \mathcal{G}_S(G)$:

- G acts on the set of vertices $V(\mathcal{G}) = G$: if $g \in G$ and $x \in V(\mathcal{G})$, then $g.x = gx$, where the latter denotes the usual multiplication in G ;
- the action on the edges: given $g \in G$ and an edge $(x, xs) \in E(\mathcal{G})$, we have $g.(x, xs) = (gx, gxs)$.

This action $G \curvearrowright \mathcal{G}_S(G)$ preserves the metric d_G in the same way as the regular action $G \curvearrowright (G, d_S)$ preserves the word metric d_S . Every $g \in G$ acts on \mathcal{G} as an isometry of the graph: we say that the action is “by isometries”.

The graph $\mathcal{G}_S(G)$ is a homogeneous space, i.e. the isometry group of \mathcal{G} acts on the graph in a transitive way: if $x_1, x_2 \in V(\mathcal{G})$, then the element $x_2x_1^{-1} \in G$ sends $x_1 \in V(\mathcal{G})$ to $x_2 \in V(\mathcal{G})$.

The change of generators of G does not affect the large-scale geometry of the space its Cayley graph, cf. [BP03, Prop. 3.3]

Proposition 1.6 ([BP03]). *Consider a group G with two finite symmetric sets of generators S_1, S_2 . Then the Cayley graphs $\mathcal{G}_{S_1}(G)$ and $\mathcal{G}_{S_2}(G)$ are quasi-isometric.*

Consider now a finitely presented group $G = \langle S \mid R \rangle$, where S is a finite set of generators and R is a finite set of relators for the presentation.

We define the **Cayley complex** $\mathcal{C}_{S,R}(G)$ associated to the presentation as the 2-dimensional complex whose 1-skeleton is given by the Cayley graph $\mathcal{G}_S(G)$ (cf. [Oll05, Foreword] and [BCS02]). We attach a 2-cell labelled by the relator $r \in R$ to each loop in $\mathcal{G}_S(G)$ whose boundary carries the letters making the relator r and we orient it so as the natural reading of the labels goes.

1.3 HYPERBOLIC GROUPS

A **geodesic space** is a metric space (X, d) where the distance between two points is realized by a geodesic connecting them, i.e. for every pair of points $x, y \in X$ there exists a continuous arc $\gamma: [0, \ell] \rightarrow X$ such that $\gamma(0) = x$, $\gamma(\ell) = y$ and $d(\gamma(s), \gamma(t)) = |s - t|$ for all $s, t \in [0, \ell]$ (cf. [Ohs02, Def. 2.7, 2.8]).

In particular, any (geometric realization of a) connected graph can be made into a geodesic space by considering the natural edge-path distance between points in the graph.

A **geodesic triangle** Δ in a geodesic metric space X is defined by a triple of points $x, y, z \in X$ and a triple of geodesic segments (i.e. images of geodesics in X) joining them, $\gamma_{x,y}, \gamma_{y,z}, \gamma_{z,x}$.

Definition 1.7. Let $\delta \geq 0$. Let (X, d) be a geodesic space and Δ a geodesic triangle in X . We say that Δ is **δ -thin** if, denoting by $\gamma_{x,y}, \gamma_{y,z}, \gamma_{z,x}$ the geodesic segments in X : $d(p, \gamma_{y,z} \cup \gamma_{z,x}) \leq \delta$ for all $p \in \gamma_{x,y}$ and similar conditions for $p \in \gamma_{y,z}$ or $p \in \gamma_{z,x}$.

Definition 1.8. Let $\delta \geq 0$. A geodesic space (X, d) is **δ -hyperbolic** if every triangle in X is δ -thin.

We say that X is **hyperbolic** if there exists $\delta \geq 0$ such that X is δ -hyperbolic.

Hyperbolicity of a geodesic space is preserved by quasi-isometries, although the specific value of the constant δ may not (see [Ohs02, Th. 2.37] and [KB02, Prop. 2.2]). In particular, since it is known that Cayley graphs of a group with respect to different (finite) generator systems are quasi-isometric (Prop. 1.6), then the following definition is well-posed.

Definition 1.9. Let G be a finitely generated group. We say that G is **hyperbolic** if the Cayley graph $\mathcal{G}_S(G)$ with respect to any finite symmetric set of generators S of G is a hyperbolic geodesic space.

If (X, d) is a metric space and $p \in X$, the **Gromov product** based at p is defined as

$$(x|y)_p := \frac{1}{2} [d(x, p) + d(y, p) - d(x, y)],$$

where $x, y \in X$ (cf. [Ohs02, Def. 2.1]). Hyperbolicity can be characterized by a condition on the Gromov product (cf. [Ohs02, Def. 2.2]): there is some $\delta \geq 0$ such that for every $x, y, z \in X$

$$(x|y)_p \geq \min\{(x|z)_p, (y|z)_p\} - \delta.$$

Notice that the δ here may not be the same as in the triangle thinnes definition.

1.4 BOUNDARIES

The main references for this Section are [Ohs02, Chapter 2] and [KB02, Section 2].

Let (X, d) be a geodesic space, fix a base point $p \in X$. Let $R_p(X)$ be the set of all geodesic rays $\gamma: [0, \infty) \rightarrow X$ emanating from p . Two such rays γ_1, γ_2 are defined to be equivalent if $\sup_{t \in [0, \infty)} d(\gamma_1(t), \gamma_2(t)) \leq \infty$. We define the **visual boundary** of X with respect to p as the set of equivalence classes of geodesic rays $\partial_p X := R_p(X) / \sim$ (cf. [Ohs02, Def. 2.62], [KB02, Def. 2.7]). The visual boundary carries the topology obtained by quotienting from the topology of uniform convergence in $R_p(X)$.

If (X, d) is a proper hyperbolic geodesic space, then, for every $p, q \in X$, the space $\partial_p X$ is homeomorphic to the space $\partial_q X$ (cf. [KB02, Prop. 2.14]).

Let (X, d) be a geodesic space, let $\{x_n\}$ be a sequence of points in X , let p be an arbitrary base point in X . We say that $\{x_n\}$ **converges to infinity** and we write $x_n \rightarrow \infty$ if $\lim_{m, n \rightarrow \infty} (x_m | x_n)_p = \infty$ (cf. [Ohs02, Def. 2.50]).

Remark 1.10. The condition $\lim_{m, n \rightarrow \infty} (x_m | x_n)_p = \infty$ is independent of the choice of the base point $p \in X$, cf. [Ohs02, Sect. 2.6].

Let

$$S(X) := \{\{x_n\} \subset X \mid x_n \rightarrow \infty\}.$$

We define an relation on $S(X)$ by posing $\{x_n\} \sim \{y_n\}$ if and only if

$$\lim_{n \rightarrow \infty} (x_n | y_n) = \infty.$$

If (X, d) is a hyperbolic geodesic space, then the relation \sim is an equivalence relation on $S(X)$ (see [Ohs02, Lemma 2.51]).

Definition 1.11 ([Ohs02],[KB02]). Let (X, d) be a hyperbolic geodesic space. The **boundary** of X , denoted by ∂X , is the set of equivalence classes of sequences in X tending to infinity:

$$\partial X := S(X) / \sim.$$

If a sequence $\{x_n\}$ in X converges to $\xi \in \partial X$, i.e. if $[\{x_n\}]_{\sim} = \xi \in \partial X$ (where $[\cdot]_{\sim}$ denotes the equivalence class of a sequence with respect to \sim), then we write $x_n \rightarrow \xi$.

The **compactification** of X is the space

$$X \cup \partial X.$$

The name comes from the fact that $X \cup \partial X$ can be endowed with a topology that makes it a compact space in which X is a dense subset ([Ohs02, Def. 2.58] and [KB02, Def. 2.13]).

Proposition 1.12 ([Ohs02]). *For a proper hyperbolic geodesic space X the boundary ∂X is a compact space.*

If X and Y are quasi-isometric hyperbolic spaces, then their boundaries ∂X and ∂Y are homeomorphic (cf. [KB02, Prop. 2.20]).

Moreover, if X is a proper hyperbolic space and $p \in X$, then the visual boundary $\partial_p X$ is homeomorphic to ∂X (cf. [Ohs02, Prop. 2.64]).

We extend the Gromov product to the points in the boundary ∂X (cf. [Ohs02, Def. 2.56]) if $x, y \in X \cup \partial X$, then we define

$$(x|y) := \lim_{x_n \rightarrow x, y_n \rightarrow y} \liminf_{n \in \mathbb{N}} (x_n | y_n),$$

where $\{x_n\}$ and $\{y_n\}$ are sequences tending to x, y respectively (they always exists, for every $x, y \in X \cup \partial X$).

Notice that we drop any reference to the base point since it is irrelevant for those properties of the product we are interested in.

The extended Gromov product on $X \cup \partial X$ satisfies the following properties (cf. [Ohs02, Lemma 2.57]).

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1. The extended Gromov product on $X \cup \partial X$, when restricted to X , coincides with the usual Gromov product.
2. $(x|y) = \infty$ if and only if $x = y \in \partial X$.
3. $(x_n|x) \rightarrow \infty$ for $n \rightarrow \infty$ if and only if $x \in \partial X$ and $x_n \rightarrow x$.
4. There exists $\delta > 0$ such that $X \cup \partial X$ is a δ -hyperbolic space, i.e. it satisfies $(x|y) \geq \min\{(x|z), (y|z)\} - \delta$ for all $x, y, z \in X \cup \partial X$.

Definition 1.13. Let G be a hyperbolic group. The **boundary** ∂G of G is defined as the (equivalence class under homeomorphisms) of the boundary of its Cayley graph $\mathcal{G}_S(G)$ with respect to some finite generator set S .

Theorem 1.14 ([Ohs02]). *If G is a finitely generated hyperbolic group and S is a finite set of generators for G , denoting with $X = \mathcal{G}_S(G)$, the spaces $X \cup \partial X$ and ∂G are compact.*

1. If G is a virtually cyclic hyperbolic group, then ∂G is made by two points.
2. If G is a virtually free group with a finite index free subgroup of rank at least 2, then the boundary ∂G is homeomorphic to a Cantor set.
3. If Γ is a cocompact lattice in a semisimple Lie group G , then Γ is hyperbolic and $\partial \Gamma$ is homeomorphic to a sphere.

Theorem 1.15 ([KB02]). *The boundary of a discrete countable hyperbolic group G satisfies one of the following.*

- G is a finite group and $\partial G = \emptyset$.
- G is virtually infinite cyclic and $\#\partial G = 2$.
- *The only other possibilities are that ∂G is a singleton or it has uncountable cardinality. In these cases, G is called a non-elementary hyperbolic group.*

Consider a group G acting on a hyperbolic metric space X . Then G acts on sequences in X : if $g \in G$ and $\{x_n\}$ is a sequence in X , then $g.\{x_n\} = \{g.x_n\}$.

In particular, if $\{x_n\}$ converges to infinity in X and $g \in G$, then the image $g.\{x_n\}$ converges to infinity in X , too.

This means that we get a well-defined action of G on the boundary ∂X (cf. [Ohs02], [KB02]); this action is by homeomorphisms.

As a special case, G acts by homeomorphisms on its boundary ∂G : this action comes from the natural action $G \curvearrowright G$ by left multiplication, which preserves the word metric on G .

1.5 SMALL CANCELLATION GROUPS

The main reference for this Section is [Str90]. We mention the foreword of [Oll05], too, as concise presentation of the main geometric properties of small cancellation groups.

Definition 1.16. Let S be a symmetric set, let $R \subseteq \mathbb{F}(S)$. An element $u \in \mathbb{F}(S)$ is a **piece** with respect to R if there exist $r_1, r_2 \in R_*$ such that $r_1 \neq r_2$ and $r_1 = uv_1, r_2 = uv_2$ for some $v_1, v_2 \in \mathbb{F}(S)$, where the products are reduced.

Definition 1.17. Let $\lambda > 0$. A finite group presentation $G = \langle S \mid R \rangle$ satisfies the **small cancellation condition** $C'(\lambda)$ if for all $r \in R_*$ and for all u prefix of r which is a piece with respect to R we have $\ell(u) < \lambda \ell(r)$.

If a finite presentation satisfies condition $C'(\lambda)$, then it satisfies condition $C'(\lambda')$ for every $\lambda' \geq \lambda$.

Consider now a finitely presented group $G = \langle S \mid R \rangle$, consider the Cayley graph $\mathcal{G}_S(G)$. Let Δ be a geodesic triangle in $\mathcal{G}_S(G)$ with vertices x, y, z and edges (images of directed geodesics) $\gamma_{x,y}, \gamma_{y,z}, \gamma_{z,x}$.

Definition 1.18. A geodesic triangle Δ in $\mathcal{G}_S(G)$ is **simple** if x, y, z are pairwise distinct and the images of the geodesics $\gamma_{x,y}, \gamma_{y,z}, \gamma_{z,x}$ have only their endpoints as common points.

A particular case of interest is given by the situation where $x \neq y = z$ (or another similar combination), i.e. where there are only two distinct vertices. In this case Δ is called **geodesic digon**.

We say that a geodesic digon Δ is **simple** if the only intersections of the images of the two geodesic edges are the endpoints.

Every geodesic triangle in $\mathcal{G}_S(G)$ can be decomposed as the union of finitely many geodesic segments, finitely many simple geodesic digons and at most one simple geodesic triangle (cf. [Str90]).

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Theorem 1.19 ([Str90]). *Let $G = \langle S \mid R \rangle$ be a finite presentation of a group satisfying condition $C'(\frac{1}{6})$. A simple geodesic triangle Δ in $\mathcal{G}_S(G)$ has one of the forms I_1, \dots, V in Figure 1.1.*

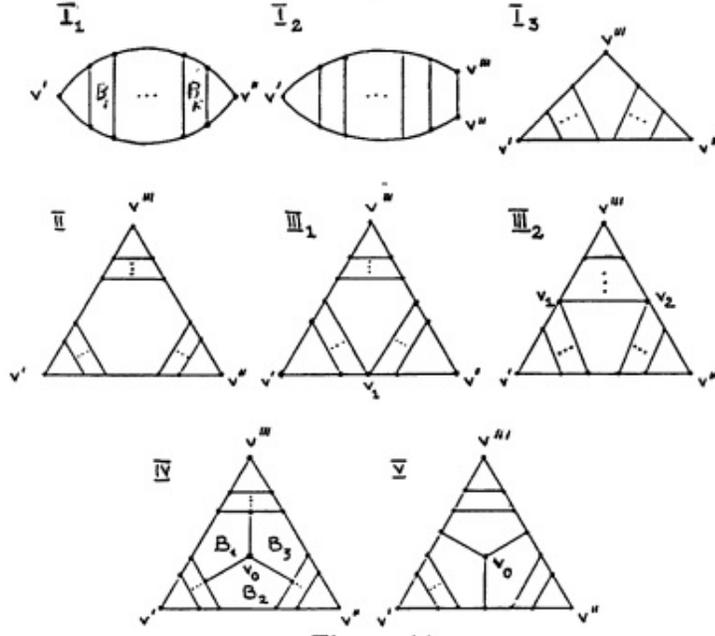


Figure 1.1: Strebel's classification of geodesic triangles in small cancellation groups, cf. [Str90, Figure 11]. More on this in the following Rem. 1.22.

As a consequence, there is $\delta \geq 0$ such that every triangle in the Cayley graph of a $C'(\frac{1}{6})$ -small cancellation group is δ -thin.

Corollary 1.20 ([Str90]). *Let G be a finitely presented group admitting a finite presentation satisfying condition $C'(\frac{1}{6})$. Then G is hyperbolic.*

Proof. Take δ as the maximum of the number of edges (we can add 1 if this is odd) in the boundary of a cell in the Cayley complex. Then, by Theorem 1.19, every geodesic triangle is δ -thin. \square

1.6 SURFACE GROUPS

Given a closed orientable surface Σ of genus $k \geq 2$, its fundamental group $\pi_1(\Sigma)$ is called a surface group and denoted with Γ_k . The **standard presentation** of Γ_k is

$$\Gamma_k := \langle a_1, b_1, \dots, a_k, b_k \mid [a_1, b_1] \cdots [a_k, b_k] = e \rangle,$$

where e here stands for the identity of the group and $[x, y] := xyx^{-1}y^{-1}$ is the commutator of two elements $x, y \in \Gamma_k$.

For the sake of brevity, we often indicate a generic surface group with the symbol Γ , dropping the genus k when unneeded.

The **standard set of generators** for Γ_k is by definition the set of generators introduced above, i.e. $S_k^+ = \{a_1, b_1, \dots, a_k, b_k\}$; it is very often convenient to make reference to the **standard symmetric set of generators**, which we denote by

$$S_k := \{a_1, b_1, \dots, a_k, b_k, a_1^{-1}, b_1^{-1}, \dots, a_k^{-1}, b_k^{-1}\}.$$

By the symbol S in the following we will mean the standard symmetric set of generators S_k , where k is omitted for brevity.

The set of **fundamental relators** of Γ_k , denoted by $(R_k)_*$, is the symmetric closure of $\{[a_1, b_1] \cdots [a_k, b_k]\}$, i.e. the smallest set of freely reduced, cyclically reduced words in $\mathbb{F}(S_k)$ (the free group on S_k^+) which is closed under cyclic permutations and inverses.

Surface groups are discrete groups, i.e. topological groups endowed with the discrete topology. They are countable groups.

The following fact has fundamental consequences in the geometry of the Cayley graphs of surface groups. It is presented as Example 6 in [Str90].

Fact 1.21. *Surface groups satisfy the $C'(\frac{1}{6})$ -small cancellation condition.*

Proof. Let $S^+ = \{a_1, b_1, \dots, a_k, b_k\}$ and let $R = \left\{ \prod_{i=1}^k [a_i, b_i] \right\}$ as above. The symmetrization R_* contains $8k$ elements; the only non-trivial pieces with respect to R have length 1. The group satisfies condition $C'(\frac{1}{4k-1})$. \square

The $C'(\frac{1}{6})$ -small cancellation condition implies that surface groups are hyperbolic (Cor. 1.20); their boundary $\partial\Gamma$ is homeomorphic to the unit circle \mathbb{S}^1 (surface groups are non-elementary hyperbolic groups, see Theorem 1.15). The space $\Gamma \cup \partial\Gamma$, endowed with the topology as in [Ohs02, Def. 2.58], becomes a compactification of Γ as a topological space.

Surface groups are cocompact lattices in $\mathrm{PSL}_2(\mathbb{R})$.

1.6.1 Cayley graphs of surface groups

Consider a surface group Γ_k with the standard symmetric set of generators S_k : we denote by $\mathcal{G}_k = \mathcal{G}_{S_k}(\Gamma_k)$ its **Cayley graph** (cf. Section 1.2; we use the symbol \mathcal{G} when the genus $k \geq 2$ is either not relevant or clear from the context).

It is well known in literature that the graph \mathcal{G} is isomorphic to the dual of the 1-skeleton of a tiling of the hyperbolic plane \mathbb{H}^2 by hyperbolic $4k$ -gons with vertex angles of $\frac{\pi}{2k}$ radians; at each vertex of the graph $4k$ edges meet (and thus $4k$ different $4k$ -gons); each edge is shared by two and only two different $4k$ -gons.

A **2-cell** (or **face** or **polygon**) of the tiling is the closure of a connected component of the complement of \mathcal{G} in \mathbb{H}^2 (\mathcal{G} embeds in \mathbb{H}^2), i.e. a 2-cell of the Cayley complex $\mathcal{C}_{S,R}(\Gamma)$ (cf. Section 1.2). The labels on the boundary of such a cell, when read clockwise or counterclockwise, give a fundamental relator of the group.

The 2-cells in the Cayley complex of a surface group Γ_k with respect to S_k, R_k are $4k$ -gons. We can think of them as hyperbolic $4k$ -gons, up to an embedding of $\mathcal{C}_{S_k, R_k}(\Gamma_k)$ in the plane \mathbb{H}^2 : thus, we get a tiling of \mathbb{H}^2 with the following basic properties (cf. Figure 1.2)¹.

- At each vertex $4k$ edges meet and $4k$ polygons share that same vertex.

- Each edge is shared by two and only two polygons.

¹Figure 1.2 was found at <http://www.yann-ollivier.org/maths/primer.php>

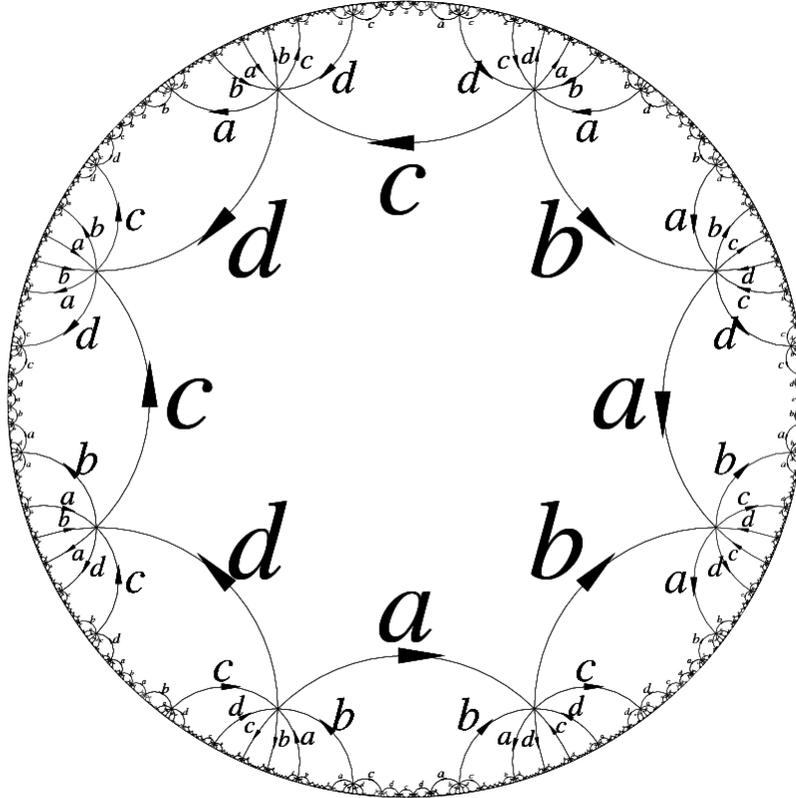


Figure 1.2: Dual of the Cayley graph of Γ_2 embedded in the hyperbolic disk (cf. <http://www.yann-ollivier.org/maths/primer.php>).

1.6.2 Digons and triangles in \mathcal{G}_k

Since the standard presentation of Γ_k satisfies the $C'(\frac{1}{6})$ -small cancellation axiom, it is known (cf. [Str90] and [Sam02]) that a geodesic triangle Δ in the Cayley graph \mathcal{G}_k can only have some of the forms in Figure 1.1 where each cell is a hyperbolic $4k$ -gon whose boundary is labelled by a fundamental relator.

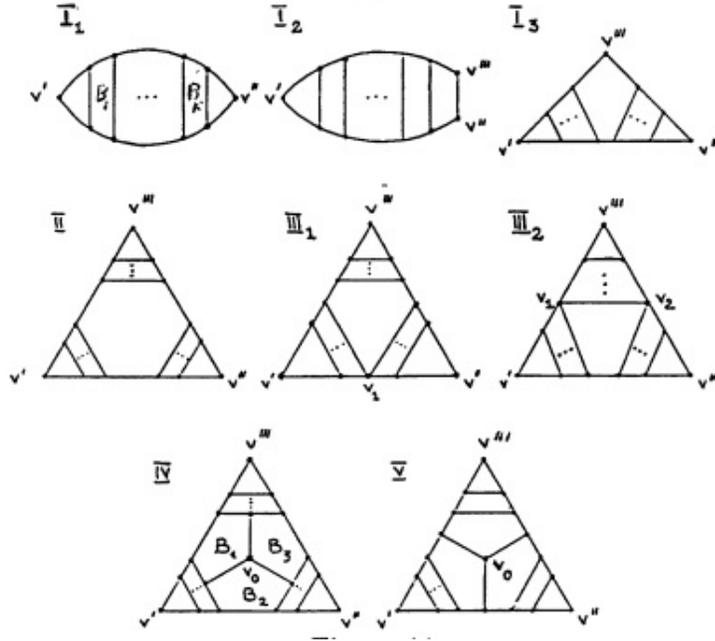


Figure 1.3: [Str90, Figure 11]. The tiles in the figure correspond to $4k$ -gons in the case of the graph \mathcal{G}_k : some edges are identified in a unique segment in the figure.

Remark 1.22. A word to interpret Figure 1.3 for the case of the Cayley complex of a surface group Γ_k is due. Each 2-cell in the figure corresponds to a tile the complex, which in our case is a $4k$ -gon. A single edge in the figures may correspond to multiple edges on the boundary of the $4k$ -gon. Anyway, since 2-cells in the complex $\mathcal{C}_{S_k, R_k}(\Gamma_k)$ are joined either at a vertex or along an edge, those edges in the figure where two tiles are joint correspond to real edges in the Cayley graph. Cases IV_1, V are excluded if we deal with the tiling associated with a surface group: in fact, there are no vertices of valency 3 in the tiling. From Figure 1.3, cases II, III_1, III_2 show that some geodesic triangles have a “central” tile, in our case is a $4k$ -gon. In particular, geodesic digons and triangles in \mathcal{G}_k are $2k$ -thin.

Lemma 1.23 (see [Str90]). *A simple geodesic triangle in \mathcal{G}_k , whose sides are three geodesics $\gamma_1, \gamma_2, \gamma_3$, has one of the forms $I_2, I_3, II, III_1, III_2$ in Figure 1.3.*

In particular, each point on one of the sides of the triangle has distance at most $2k$ from the union of the other two sides.

Lemma 1.24 ([Str90], [BCS02]). *Consider a pair of geodesics γ_1, γ_2 in \mathcal{G}_k with $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(n) = \gamma_2(n)$. Then, for every $i = 0, \dots, n$ there exists a 2-cell F_i in the tiling associated with \mathcal{G}_k such that $\gamma_1(i)$ and $\gamma_2(i)$ both belong to F_i . In particular,*

$$\sup_{t \in (0, n)} d_{\mathcal{G}_k}(\gamma_1(t), \gamma_2(t)) \leq 2k.$$

Proof. In fact, a pair of geodesics as in the hypothesis gives rise to a union of digons as in I_1 of Figure 1.3: a generic geodesic digon decomposes as a finite union of (finite) geodesic segments and simple geodesic digons. \square

We prove the following important result.

Lemma 1.25. *Consider the Cayley graph $\mathcal{G}_k = \mathcal{G}_k(\Gamma_k, S_k)$, consider a geodesic triangle Δ with vertices y_1, y_2, z and sides given by three geodesics $\gamma_1 \in [y_1, z], \gamma_2 \in [y_2, z]$ and $\eta \in [y_1, y_2]$. Assume that $|z| \gg \max\{|y_1|, |y_2|\}$. Then there exists $M = M(y_1, y_2)$ such that for all z_1 vertex on γ_1 and for all z_2 vertex on γ_2 with $|z_1|, |z_2| \geq M$ and $d(z, z_1) = d(z, z_2)$ we have $d(z_1, z_2) \leq 2k$.*

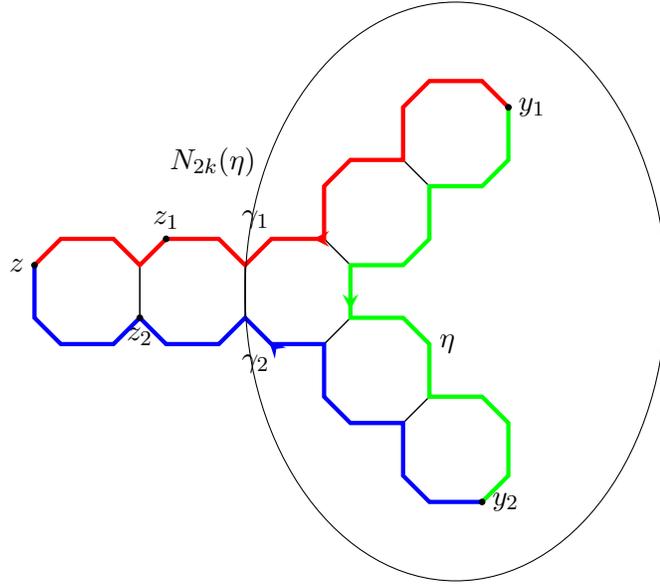


Figure 1.4: The vertices z_1, z_2 on γ_1, γ_2 respectively having $d(z, z_1) = d(z, z_2)$ and situated outside of the ellipse (which represents the set $\{|z| \leq M\}$) have distance bounded by $2k$.

Proof of Lemma 1.25. By Lemma 1.23, since $|z| \gg |y_1|, |y_2|$, the triangle with vertices y_1 , y_2 and z must have one of the forms II, III₁ or III₂ of Figure 1.3.

Let $\ell_1 = \ell(\gamma_1)$ and $\ell_2 = \ell(\gamma_2)$. By Lemma 1.23, geodesic triangles in \mathcal{G}_k are $2k$ -thin: thus, if ζ is a vertex on the geodesic γ_1 (i.e. $\zeta = \gamma_1(i)$ for some index $i \in [0, \ell_1]$), then we have that $d(\zeta, \gamma_2 \cup \eta) \leq 2k$.

Consider now a $2k$ -neighborhood $N_{2k}(\eta)$ of the image in \mathcal{G}_k of the geodesic $\eta \in [y_1, y_2]$. We define:

$$M := \max_{\tilde{\eta} \in [y_1, y_2]} \{|x| \mid x \in N_{2k}(\tilde{\eta})\}.$$

Consider two vertices z_1, z_2 on γ_1, γ_2 respectively satisfying $|z_1|, |z_2| > M$ and $d(z, z_1) = d(z, z_2)$. Then, since $d(z_1, \eta) \geq 2k$, it must be that $d(z_1, \gamma_2) \leq 2k$; similarly for z_2 . Moreover, since $d(z, z_1) = d(z, z_2)$, if we define $\tilde{\gamma}_1(i) = \gamma_1(\ell_1 - i)$ and similarly $\tilde{\gamma}_2(i) = \gamma_2(\ell_2 - i)$, then there is $i_0 \in [0, \min\{\ell_1, \ell_2\}]$ such that $z_1 = \tilde{\gamma}_1(i_0)$ and $z_2 = \tilde{\gamma}_2(i_0)$. Therefore, considering that $d(z_1, \gamma_2) \leq 2k$, that $d(z_2, \gamma_1) \leq 2k$ and that $d(z, z_1) = d(z, z_2)$, Lemma 1.23 implies that the vertices z_1 and z_2 belong to the same cell of the tiling associated with \mathcal{G}_k , so: $d(z_1, z_2) \leq 2k$ (recall that $\tilde{\gamma}_1(0) = \tilde{\gamma}_2(0) = z$). \square

From the proof of Lemma 1.25 it follows that $M = M(y_1, y_2)$ only depends on y_1, y_2 , and not on the geodesic triangle in the hypothesis.

1.7 CONES IN THE GRAPH

Let $\mathcal{G} = \mathcal{G}_S(\Gamma)$ be the Cayley graph of a surface group.

Definition 1.26. Given $x, y \in V(\mathcal{G})$, the **cone** with base point x and vertex y is

$$C(x, y) := \{z \in V(\mathcal{G}) \mid d(x, z) = d(x, y) + d(y, z)\}.$$

In other words, the cone $C(x, y)$ is the set of all vertices in \mathcal{G} which can be joined to x by a geodesic passing through the vertex y .

Remark 1.27. Since the set of vertices $V(\mathcal{G})$ is in one-to-one correspondence with the set of elements of Γ , we often consider two elements $x, y \in \Gamma$ and the cone $C(x, y)$ as a subset of Γ .

Denote by \mathfrak{C} the set of cones. We have a natural action of Γ on \mathfrak{C} : for $g \in \Gamma$ and $C(x, y) \in \mathfrak{C}$ we define

$$g.C(x, y) := C(g.x, g.y),$$

where $g.x, g.y$ denote the natural action of $g \in \Gamma$ on $x, y \in V(\mathcal{G})$ by left multiplication.

We say that the cones $C(x_1, y_1)$ and $C(x_2, y_2)$ are equivalent if there exists $g \in \Gamma$ such that $C(x_1, y_1) = g.C(x_2, y_2)$. We denote this situation by $C(x_1, y_1) \sim C(x_2, y_2)$.

Definition 1.28. A (labelled) **cone type** is an orbit of the action of Γ on the set of cones \mathfrak{C} . The set of cone types is denoted by $\mathcal{C} := \mathfrak{C}/\sim$.

If $\mathbf{c} \in \mathcal{C}$ and $C(x, y)$ is a cone in the graph, we denote by $\mathbf{c} \sim C(x, y)$ the fact that $C(x, y)$ has cone type \mathbf{c} , i.e. it belongs to this orbit.

Remark 1.29. Each cone type has a unique representative having vertex in the identity $e \in \Gamma$: in fact, for each cone $C(x, y)$, with $x, y \in V(\mathcal{G})$, we can write $C(x, y) = y.C(y^{-1}x, e)$.

With a slight abuse, we identify each cone type with this unique representative. This choice is consistent with a different definition of cone type which is widely adopted in the literature: if $x \in \Gamma$, it is customary to choose a geodesic word u representing x and to define the cone type of x as the set of all geodesic words $w \in S^*$ such that uw is a geodesic word (see Def. 2.4 in Section 2.3). It is easy to show that this definition does not depend on the choice of the geodesic word u representing x , but only on the group element itself. If one considers the set of elements in Γ represented by the geodesic words in the cone type of x , one gets precisely the cone $C(x^{-1}, e)$, as in Definition 1.26 (more on this can be found in Proposition 2.8).

1.7.1 Other definitions

In [BCS02] the cone of a vertex $x \in V(\mathcal{G})$ in a rooted graph \mathcal{G} with root $*$ is defined as the subgraph spanned by the set of vertices

$$\{y \in V(\mathcal{G}) \mid \exists \gamma \in [*, y] \text{ s.t. } \gamma(i) = x, \text{ some } i \leq \ell(\gamma)\}.$$

Two cones are considered equivalent if there exists a rooted graph isomorphism mapping the one onto the other. The cone type of a cone is then its equivalence class.

This definition of cone coincides with Definition 1.28 if we consider only cones based at the identity, i.e. cones of the kind $C(e, x)$, with $x \in \Gamma$. Such

a cone has the same cone type as $C(x^{-1}, e)$ according to Def. 1.28, while the authors of [BCS02] consider all rooted graph isomorphisms of \mathcal{G} , without any reference to the labelled graph structure: thus, they get a smaller set of cone types (“unlabelled” cone types). In our definition, on the contrary, only label preserving isomorphisms are allowed: precisely those given by the action of the group on its Cayley graph. For example, the elements $a_1 b_1 a_1^{-1}$ and $b_2 a_2 b_2^{-1}$ in Γ_2 have the same unlabelled cone type, but different labelled cone types.

In the same way, in [Gou15, Sect. 3] a countable group with a finite generating set and the induced word distance are fixed and the cone of a point x is by definition the set of all points z for which there is a geodesic from e to z going through x . The cone type of x is the set $\{x^{-1}z\}$ for z in the cone of x . This definition coincides with our Definition 1.28.

To get a smaller set of cone types in the case of a surface group, the author of [Gou15] defines the “Cannon type” of a point $x \in \Gamma_k$ as the maximal length along the last $4k$ -gon in the Cayley graph of a geodesic starting at e and ending at x . This gives precisely the same geometric/combinatorial object as the definition of cone type in [BCS02].

A combinatorial characterization of the Cannon type is presented in [Gou15]: consider the set of geodesic words $w = s_1 \cdots s_n$ representing x , let j be the length of the maximal suffix of such a geodesic word in common with a fundamental relator. The Cannon type of x is then the maximum of such j 's over all possible representations of x as a geodesic word. It is immediate to characterize the cone type as defined in 1.28.

Remark 1.30. A characterization of the cone types can be easily obtained using the same method as in [Gou15]. Fix an element $x \in \Gamma_k$, consider the set of geodesic words $w = s_1 \cdots s_n$ representing x . Let j be the length of the maximal suffix of such a geodesic word in common with a fundamental relator, let j_0 be the maximal j over all possible geodesic words representing x . Let $s_{n-j_0+1} \cdots s_n$ be the suffix of maximal length j_0 . Then the cone type of x is equal to the cone type of $s_{n-j_0+1} \cdots s_n$.

1.7.2 Subdivision into subcones

Fix $x, y \in \Gamma$ and consider the cone $C(x, y)$. Let $m \in \mathbb{N}$ be such that $\{z_1, \dots, z_m\}$ is the set of elements in $C(x, y)$ satisfying $d(e, z_j) = d(e, y) + 1$ (for $j = 1, \dots, m$). There exist $s_1, \dots, s_m \in S$ such that $z_j = y s_j$ for all $j = 1, \dots, m$ and clearly $C(x, y s_j) \subseteq C(x, y)$. Then we get the **first-level**

subcone subdivision of the cone $C(x, y)$:

$$C(x, y) \setminus \{y\} = \bigcup_{j=1}^m C(x, ys_j).$$

This union is not disjoint in general.

For each $j = 1, \dots, m$, the cone type of $C(x, ys_j)$ can be determined by the cone type of $C(x, y)$ and by the generator s_j .

In the case of a cone $C(x, e)$ with vertex the identity, the first level subcones have vertices in the generators $s_1, \dots, s_m \in S \cap C(x, e)$:

$$C(x, e) \setminus \{e\} = \bigcup_{j=1}^m C(x, s_j).$$

Recursively, this procedure gives the **n -th level subcone subdivision** of a cone: we simply reiterate the subdivision on each subcone. Thus, if $z \in C(x, y)$ and $d(y, z) = n$, the cone type of the n -th level subcone $C(x, z)$ of $C(x, y)$ depends on the cone type of $C(x, y)$ and the n generators labeling a geodesic words which joins the vertex y with the new vertex z .

1.8 INTERSECTION OF CONES

Fix a surface group Γ with S , the standard symmetric set of generators.

The word metric with respect to S is considered on Γ ; we consider moreover the Cayley graph $\mathcal{G}_S(\Gamma)$: once that a base point is fixed (the identity e will do), the terms “predecessor” and “successor” become meaningful as usual.

Notation 1.31. In the following we often equal group elements with words on the generators representing them: for example, if $x \in \Gamma$ and $w \in S^*$ we write $x = w$ to mean that the word w represents the element x , i.e. $x = \bar{w}$. This is an abuse of notation, but it should be quite clear from the context.

Notation 1.32. The symbol $\text{Pref}(R_*)$ denotes the set of all reduced words which are prefixes of some fundamental relator.

Notation 1.33. Consider two infinite subsets A, B in Γ . Then we write $A \approx B$ if the two sets differ only by a finite number of elements, i.e. there exists a radius $N \in \mathbb{N}$ and a ball $B_N(e)$ centered at e such that:

$$A \cap B_N(e)^c = B \cap B_N(e)^c.$$

Moreover, we write $A \lesssim B$ if there exists $C \subseteq B$ such that $A \approx C$. In other words, there exist a radius $N \in \mathbb{N}$ and a ball $B_N(e)$ centered at the identity such that

$$A \cap B_N(e)^c \subseteq B \cap B_N(e)^c.$$

In this Section, we prove the following important result.

Theorem 1.34. *Fix $x_1, y_1, x_2, y_2 \in \Gamma$, consider the cones $C(x_1, y_1)$ and $C(x_2, y_2)$. Then, assuming that the intersection is not finite nor empty, there exist $N \in \mathbb{N}$ and $x'_i, z_i \in \Gamma$ ($i = 1, \dots, N$) such that $C(x'_i, z_i)$ are subcones of both $C(x_1, y_1)$ and $C(x_2, y_2)$ and*

$$C(x_1, y_1) \cap C(x_2, y_2) \approx \bigcup_{i=1}^N C(x'_i, z_i).$$

1.8.1 Proof - step one

We consider first two cones with vertex in the identity. We fix $x_1, x_2 \in \Gamma$ and we deal with $C(x_1, e)$ and $C(x_2, e)$.

For each cone we consider the subdivision into first-level subcones:

$$C(x_1, e) \approx \bigcup_i C(x_1, s_i), \quad C(x_2, e) \approx \bigcup_j C(x_2, t_j),$$

where $s_i \in S \cap C(x_1, e)$ and $t_j \in S \cap C(x_2, e)$; we omit the ranges of the indexes i, j , which are finite (cf. Section 1.7.2).

It is clear that

$$\begin{aligned} C(x_1, e) \cap C(x_2, e) &\approx \left(\bigcup_i C(x_1, s_i) \right) \cap \left(\bigcup_j C(x_2, t_j) \right) \\ &= \bigcup_i \bigcup_j (C(x_1, s_i) \cap C(x_2, t_j)). \end{aligned} \tag{1.1}$$

Thus, we study the intersections

$$C(x_1, s_i) \cap C(x_2, t_j)$$

for i, j ranging over the number of first-level subcones of the two cones.

Lemma 1.35. *For each $x \in \Gamma$ and each $s \in S \cap C(x, e)$ it holds: $C(x, s) \subseteq C(e, s)$.*

Proof. Let $z \in C(x, s)$. Then

$$d(x, z) = d(x, s) + d(s, z). \quad (1.2)$$

Since $s \in C(x, e)$, the cone $C(x, s)$ is a subcone of $C(x, e)$, so that $z \in C(x, s)$ implies $z \in C(x, e)$. Therefore:

$$d(x, z) = d(x, e) + d(e, z). \quad (1.3)$$

Moreover, since $s \in C(x, e)$:

$$d(x, s) = d(x, e) + d(e, s). \quad (1.4)$$

Thus, since the right hand sides of equation 1.2 and equation 1.3 are equal:

$$d(x, e) + d(e, z) = d(x, s) + d(s, z); \quad (1.5)$$

using equation 1.4, we get:

$$d(x, e) + d(e, z) = d(x, e) + d(e, s) + d(s, z), \quad (1.6)$$

which yields

$$d(e, z) = d(e, s) + d(s, z), \quad (1.7)$$

i.e. precisely $z \in C(e, s)$. This gives the thesis. \square

If $x \in \Gamma$, then the cone $C(e, x)$ can be thought as the set of all elements $z \in \Gamma$ that admit a geodesic word whose prefix of length $|x|$ represents the element x . In other terms, these are the followers of x (in the Cayley graph $\mathcal{G}_S(\Gamma)$ with the edge-path metric).

Lemma 1.36. *Let $s, t \in S$, $s \neq t$. Then $C(e, s) \cap C(e, t) \neq \emptyset$ if and only if $s^{-1}t \in \text{Pref}(R_*)$ (cf. Not. 1.32).*

Proof. Let $z \in C(e, s) \cap C(e, t)$. Then there exists a geodesic word representing z which starts with the generator s and another geodesic word representing z which starts with the generator t :

$$z = su = tv.$$

The two geodesic words su and tv correspond to two geodesics γ_1 and γ_2 in \mathcal{G} both starting at e and ending at z . Therefore, for each $i = 0, \dots, |z|$: $\gamma_1(i)$ and $\gamma_2(i)$ are on the same $4k$ -gon (cf. Lemma 1.24).

1. GEOMETRIC STRUCTURE OF SURFACE GROUPS

In particular, $s = \gamma_1(1)$ and $t = \gamma_2(1)$ are on the same $4k$ -gon of the tiling associated to \mathcal{G}_k . Hence

$$\gamma_1(1)^{-1}\gamma_2(1) = s^{-1}t \in \text{Pref}(R_*).$$

This proves the implication holds, too: if $C(e, s) \cap C(e, t) \neq \emptyset$, then $s^{-1}t \in \text{Pref}(R_*)$.

The converse implication: if $s^{-1}t$ is a prefix of a fundamental relator, then the two edges emanating from e and labelled with s, t are on the same $4k$ -gon. Thus, there exists $z \in \Gamma$ having length $2k$ such that z admits two representations as a geodesic word: one starting with s , the other starting with t . Geometrically, we are choosing z as the vertex with two predecessors on the same $4k$ -gon individuated by s, t and with the identity on its boundary. This element is clearly contained in the intersection of the cones. \square

At this point, recalling equation 1.1, we can write:

$$C(x_1, e) \cap C(x_2, e) \lesssim \bigcup_{i,j : s_i^{-1}t_j \in \text{Pref}(R_*)} C(e, s_i) \cap C(e, t_j), \quad (1.8)$$

where $s_i \in C(x_1, e)$ and $t_j \in C(x_2, e)$ for all i, j . In fact, $C(x_1, s_i) \subseteq C(e, s_i)$ and $C(x_2, t_j) \subseteq C(e, t_j)$ by Lemma 1.35, so that:

$$C(x_1, s_i) \cap C(x_2, t_j) \subseteq C(e, s_i) \cap C(e, t_j),$$

while by Lemma 1.36 we know that the intersection $C(e, s_i) \cap C(e, t_j)$ is not empty if and only if $s_i^{-1}t_j$ is in $\text{Pref}(R_*)$.

Lemma 1.37. *Given $s, t \in S$ such that $s^{-1}t \in \text{Pref}(R_*)$, there exists x (depending on s, t) such that $C(e, s) \cap C(e, t) \approx C(e, x)$.*

Proof for the case of genus 2. Since $s^{-1}t \in \text{Pref}(R_*)$, there exist $c_1, \dots, c_6 \in S$ such that $s^{-1}tc_1 \cdots c_6$ is a fundamental relator. In particular, $tc_1 \cdots c_6s^{-1}$ is again a fundamental relator. Moreover, as elements of Γ :

$$tc_1c_2c_3 = sc_6^{-1}c_5^{-1}c_4^{-1}.$$

We define $x := tc_1c_2c_3 = sc_6^{-1}c_5^{-1}c_4^{-1}$. Then $C(e, x) \subseteq C(e, s)$ and $C(e, x) \subseteq C(e, t)$.

Let now $z \in C(e, s) \cap C(e, t)$ with $s \neq t$. Then, there exist geodesic words u_1, u_2 such that

$$z = su_1 = tu_2.$$

It is possible to re-write a prefix of a geodesic word only if it contains a subword of a fundamental relator of length 4. From a geometric perspective, we have two geodesics with the same starting point, the identity, and with the same ending point z . Since at each step the two geodesics have to stay on the boundary of a common octagon (Lemma 1.24) and each geodesic can cross at most 4 edges on the same octagon, then, at least one of the two geodesics has to cross precisely 4 edges on the first octagon: in fact, the first edges s, t are different for the two geodesics, so they proceed along the boundary of the common octagon P . Since they have to remain on a common octagon even when they leave P , the possibilities are: they leave P at the same vertex or they leave it on two vertices connected by an edge. In fact, two octagons in the graph \mathcal{G}_2 can be joined at a vertex or along an edge. If the two geodesics leave the first common octagon at different vertices of its boundary, one of them crosses exactly one edge more than the other. Hence, the possibilities are: both geodesics cross 4 edges of P and they leave it at the same vertex or one of them crosses 3 edges and the other 4. Therefore, either su_1 or tu_2 have a prefix of length 4 that can be rewritten; say su_1 does: the prefix of length 4 of su_1 is precisely $sc_6^{-1}c_5^{-1}c_4^{-1}$, which is a geodesic word representing x . In other words, $z \in C(e, x)$. \square

Lemma 1.37 implies: for each i, j such that $s_i^{-1}t_j \in \text{Pref}(R_*)$ there exists $x_{i,j}$ such that

$$C(e, s_i) \cap C(e, t_j) \approx C(e, x_{i,j}).$$

This yields:

$$C(x_1, e) \cap C(x_2, e) \lesssim \bigcup_{i,j} C(e, x_{i,j}).$$

We claim now that

$$C(e, x_{i,j}) \subseteq C(x_1, e) \cap C(x_2, e).$$

This implies that

$$C(x_1, e) \cap C(x_2, e) \approx \bigcup_{i,j} C(e, x_{i,j}),$$

concluding this part of the proof.

Therefore, it is sufficient to show that $x_{i,j} \in C(x_1, e)$ and similarly $x_{i,j} \in C(x_2, e)$. Let us show the first, the second is completely analogous. Since $C(e, x_{i,j}) \subseteq C(e, s_i)$:

$$d(e, x_{i,j}) = d(e, s_i) + d(s_i, x_{i,j}); \tag{1.9}$$

since $s_i \in C(x_1, e)$:

$$d(x_1, s_i) = d(x_1, e) + d(e, s_i). \quad (1.10)$$

From equation 1.10 it follows that

$$d(x_1, s_i) + d(s_i, x_{i,j}) = d(x_1, e) + d(e, s_i) + d(s_i, x_{i,j}),$$

while using equation 1.9 we get that

$$d(x_1, s_i) + d(s_i, x_{i,j}) = d(x_1, e) + d(e, x_{i,j}).$$

If $d(x_1, s_i) + d(s_i, x_{i,j}) = d(x_1, x_{i,j})$, then we get

$$d(x_1, x_{i,j}) = d(x_1, e) + d(e, x_{i,j}),$$

which means that $x_{i,j} \in C(x_1, e)$, ending the proof. It is left to prove that $d(x_1, s_i) + d(s_i, x_{i,j}) = d(x_1, x_{i,j})$, which is equivalent to $|x_1^{-1}x_{i,j}| = |x_1| + |x_{i,j}|$.

Claim 1.38. *In the previous situation: $|x_1^{-1}x_{i,j}| = |x_1| + |x_{i,j}|$.*

Proof of Claim 1.38. Assume by contradiction that $|x_1^{-1}x_{i,j}| < |x_1| + |x_{i,j}|$: then there exist geodesic words w, u representing $x_1, x_{i,j}$ respectively such that the word $w^{-1}u$ can be shortened. Notice that u has length 4, since this is the word length of $x_{i,j}$ (see the proof of Lemma 1.37). Since a prefix of a geodesic word is again a geodesic word, the reduction in $w^{-1}u$ must happen at at the junction of the two geodesic words. Assume that $w^{-1}u = pq$, where q is such that qu is a prefix of a fundamental relator and can be reduced. Then, $w^{-1}u = pqu$ and there exists $r \in \text{Pref}(R_*)$, $\ell(r) \leq \ell(qu)$, such that qu rewrites as r . Thus, $qu = r$ as group elements, so: $u = q^{-1}r$ (again as group elements) and the latter is a geodesic word representing $x_{i,j}$ because it must have length 4. Hence, either q^{-1} starts by the letter s or it starts by the letter t . Without loss, we can suppose that it starts by s . But $w^{-1} = pq$ implies that $w = q^{-1}p^{-1}$ as a geodesic word, so there exists a geodesic word representing x_1 which starts by s . This is a contradiction, since it implies that $d(x_1, s) = |x_1| - 1$, while we assumed that $s \in C(x_1, e)$ (and so $d(x_1, s) = |x_1| + 1$). \square

We have thus concluded the proof of the fact that the intersection of two cones with the vertex in the identity is (equivalent to) a finite union of subcones.

1.8.2 Proof - step two

Let $C(x_1, y_1)$ and $C(x_2, y_2)$ be arbitrary cones. We are interested in the intersection

$$C(x_1, y_1) \cap C(x_2, y_2)$$

only up to finitely many points in it. Fix then ζ in the intersection with $|\zeta|$ large (we are interested in the intersection up to finitely many points). Then, by the classification of geodesic triangles in [Str90] (Thm. 1.19), the triangle with vertices x_1, x_2, ζ has one of the forms in Figure 1.5.

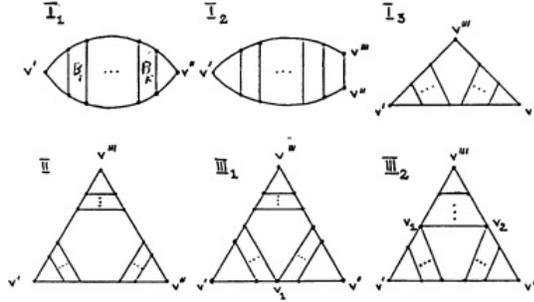


Figure 1.5: By Thm. 1.19, the triangles in \mathcal{G}_k have one of these forms.

Since we assume that $|\zeta|$ is large, then $d(\zeta, x_1)$ and $d(\zeta, x_2)$ can be taken a lot larger than $d(x_1, x_2)$, which excludes cases I_1, I_2 and I_3 and leaves us with the cases in Figure 1.6.

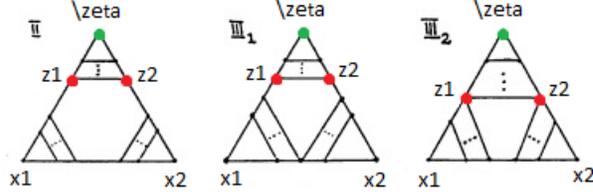


Figure 1.6: By Thm. 1.19, the triangle that we are considering can only have one of these forms. The points z_1, z_2 are at distance 1 (the segment between them corresponds to an edge in the graph).

In particular, either all geodesics $[x_1, \zeta]$ intersect a geodesic $[x_2, \zeta]$ in some point $z \in C(x_1, y_1) \cap C(x_2, y_2)$, so that $\zeta \in C(x_1, z) \cap C(x_2, z)$, or there exist z_1, z_2 in $C(x_1, y_1), C(x_2, y_2)$ respectively such that:

- $d(z_1, z_2) = 1$;
- $\zeta \in C(x_1, z_1) \cap C(x_2, z_2)$.

Figure 1.6 is quite explicatory. Each 2-cell in the figure corresponds to a tile in our complex, in our case to an octagon (for $k = 2$). A single edge in the figures may correspond to multiple edges in our case. The geodesic triangles have a “central” tile, which in our case is an octagon. Since octagons are joined at vertices or along edges, the edges in the figure where two tiles are joint correspond to actual edges in the Cayley graph. It follows that there are two vertices z_1, z_2 on the geodesics $[x_1, \zeta], [x_2, \zeta]$ respectively such that $d(z_1, z_2) = 1$. This happens on the central octagon of the geodesic triangle (cf. Figure 1.6).

Take z_1, z_2 such that $|z_1|$ and $|z_2|$ are minimal for a fixed ζ , i.e. choose them on the central octagon. By hyperbolicity, when ζ varies, the distance of these two “minimal” z_1, z_2 from x_1, x_2 is bounded from above by a constant depending only on x_1, x_2 (cf. Lemma 1.25). In other words, if P is the central polygon, hyperbolicity guarantees that $\min_{x \in P} d(e, x)$ is bounded when we fix x_1, x_2 and we vary ζ . Therefore, when we range (up to finitely

many points) with ζ over $C(x_1, y_1) \cap C(x_2, y_2)$, we find $N \in \mathbb{N}$ and finitely many pairs z_1^j, z_2^j , (the case $z_1^j = z_2^j$ is allowed), $j = 1, \dots, N$, such that $d(z_1^j, z_2^j) \in \{0, 1\}$ and:

$$\zeta \in C(x_1, y_1) \cap C(x_2, y_2) \Rightarrow \zeta \in C(x_1, z_1^j) \cap C(x_2, z_2^j), \text{ some } j.$$

Since $C(x_1, z_1^j) \cap C(x_2, z_2^j) \subseteq C(x_1, y_1) \cap C(x_2, y_2)$ for every j , this yields:

$$C(x_1, y_1) \cap C(x_2, y_2) \approx \bigcup_j C(x_1, z_1^j) \cap C(x_2, z_2^j). \quad (1.11)$$

Claim 1.39. *Let $x_1, x_2, z_1, z_2 \in \Gamma$, consider the cones $C(x_1, z_1)$ and $C(x_2, z_2)$. If $d(z_1, z_2) = 1$ and the intersection of the two cones has infinitely many elements, then*

$$C(x_1, z_1) \cap C(x_2, z_2) \approx (C(x_1, z_1) \cap C(x_2, z_1)) \cup (C(x_1, z_2) \cap C(x_2, z_2)).$$

Proof of the Claim. We are in the situation of Figure ??.

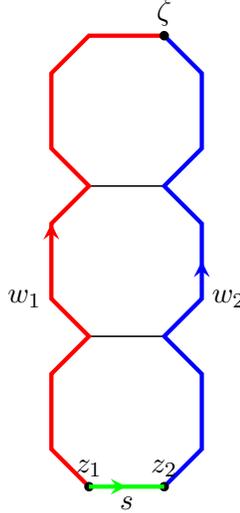


Figure 1.7: This figure corresponds to case I_2 in Figure 1.5.

Fix $\zeta \in C(x_1, z_1) \cap C(x_2, z_2)$. Thus, there are geodesic represented by the words w_1, w_2 joining z_1, z_2 with ζ . Since $d(z_1, z_2) = 1$, then there exists $s \in S$ such that $z_2 = z_1 s$. Hence, either sw_2 is a geodesic word, or $s^{-1}w_1$ is.

1. GEOMETRIC STRUCTURE OF SURFACE GROUPS

Since both w_1 and w_2 are geodesic words: $\ell(w_1) = d(z_1, \zeta)$ and $\ell(w_2) = d(z_2, \zeta)$, so: either $d(\zeta, z_1) = d(\zeta, z_2) + 1$ or $d(\zeta, z_2) = d(\zeta, z_1) + 1$. In the first case, $\ell(sw_2) = \ell(w_2) + 1 = d(z_1, \zeta)$, the other is analogous.

Thus we get: either $\zeta \in C(x_2, z_1)$ or $\zeta \in C(x_1, z_2)$. This yields that:

$$\zeta \in (C(x_1, z_1) \cap C(x_2, z_1)) \cup (C(x_1, z_2) \cap C(x_2, z_2)).$$

This proves an inclusion. The other direction of the inclusion is easy: since $d(z_1, z_2) = 1$ and the cones have non trivial intersection, either $z_1 \in C(x_2, z_2)$ or $z_2 \in C(x_1, z_1)$. \square

In the situation of equation 1.11, the Claim implies that

$$C(x_1, z_1^j) \cap C(x_2, z_2^j) \approx \left(C(x_1, z_1^j) \cap C(x_2, z_1^j) \right) \cup \left(C(x_1, z_2^j) \cap C(x_2, z_2^j) \right),$$

so, after a renumbering of the pairs $\{z_1^j, z_2^j\}$ as $\{z_j\}$ (clearly the range of j changes, but it is finite), we are left with:

$$C(x_1, y_1) \cap C(x_2, y_2) \approx \bigcup_j C(x_1, z_j) \cap C(x_2, z_j).$$

Now, we use what we know about intersection of cones with the same vertex to conclude the thesis. We recall that in Sect. 1.8.1 we proved that for $x'_1, x'_2 \in \Gamma$ there exist $N \in \mathbb{N}$ and $p_i \in \Gamma$, $i = 1, \dots, N$, (a renumbering of the $x_{i,j}$ above) such that

$$C(x'_1, e) \cap C(x'_2, e) \approx \bigcup_i C(e, p_i).$$

If we translate both cones by a fixed vertex z_j , we get:

$$C(z_j x'_1, z_j) \cap C(z_j x'_2, z_j) \approx \bigcup_i C(z_j, z_j p_i), \quad \text{for fixed } j,$$

and this holds for arbitrary x'_1, x'_2 . In particular, it must hold for $x_1 = z_j x'_1$ and $x_2 = z_j x'_2$. Hence, $C(x_1, z_j) \cap C(x_2, z_j)$ can be written as follows:

$$C(x_1, z_j) \cap C(x_2, z_j) \approx \bigcup_i C(z_j, z_j p_i), \quad \text{for fixed } j.$$

Finally:

$$C(x_1, y_1) \cap C(x_2, y_2) \approx \bigcup_j \bigcup_i C(z_j, z_j p_i).$$

Since both unions range over a finite set of indexes i, j , we can rearrange the indexes and rewrite this result as a finite union over a single set of indexes, i.e. the thesis.

CHAPTER 2

AUTOMATIC STRUCTURE OF SURFACE GROUPS

2.1 FINITE STATE AUTOMATA

2.1.1 Language theory

Let A be an alphabet, let $A^* = \bigcup_{n \geq 0} A^n$.

A **language** over A is a subset of A^* .

If p, q, u, w are words on A and $w = puq$ (juxtaposition of words), then we say that p is a **prefix** of w , u is a **subword** of w and q is a **suffix** of w . If $j \geq 0$, we denote by $w[j]$ the prefix of w having length j if $j \leq \ell(w)$ or the whole w itself otherwise.

2.1.2 Finite state automata

A **finite state automaton** (**FSA** for short) is the datum of $\mathcal{A} = (\Sigma, A, \mu, Y, \sigma_0)$ where:

- Σ is a finite set, the set of **states** of the automaton;
- A is a finite set, the **alphabet** of the automaton;
- $\mu: \Sigma \times A \rightarrow \Sigma$ is a map, called **transition function**; the result $\sigma' = \mu(\sigma, a)$ for $\sigma, \sigma' \in \Sigma$ and $a \in A$ is called a **transition** and often denoted by $\sigma \xrightarrow{a} \sigma'$, or simply by $\sigma \rightarrow \sigma'$;

-
- $Y \subseteq \Sigma$ is the set of **accepted states**;
 - $\sigma_0 \in \Sigma$ is the **starting state** (or **initial state**).

A word $w \in A^*$ is an **accepted word** for the automaton \mathcal{A} if the sequence of transitions in the automaton \mathcal{A} starting from σ_0 and with labels given by the letters of w ends in an accepted state $\sigma \in Y$: if $w = a_1 a_2 \cdots a_n$, then

$$\mu(\sigma_0, w) = \mu(\mu(\cdots \mu(\mu(\sigma, a_1), a_2) \cdots), a_n) \in Y.$$

If \mathcal{A} is a finite state automaton with alphabet A , then the **language** $L(\mathcal{A})$ **accepted** by \mathcal{A} is the set of all words on A that are accepted words for the automaton.

A language over the alphabet A is **regular** if there exists a finite state automaton \mathcal{A} over the alphabet A such that each word $w \in L$ is accepted by \mathcal{A} .

Definition 2.1. Given a finite state automaton \mathcal{A} , we associate to it a rooted labelled directed graph, denoted by $\mathfrak{G}(\mathcal{A})$, as follows.

- The set of vertices of $\mathfrak{G}(\mathcal{A})$ is given by the set of states of \mathcal{A} .
- The root of the graph is the initial state $\sigma_0 \in \Sigma$.
- The set of labelled directed edges of $\mathfrak{G}(\mathcal{A})$ is given by $\{(\sigma, \sigma') \mid \sigma, \sigma' \in \Sigma, \text{ and } \mu(\sigma, a) = \sigma' \text{ for some } a \in A\}$. The label of the edge (σ, σ') where $\sigma' = \mu(\sigma, a)$ is given by the letter $a \in A$; the edge is directed from σ to σ' .

At this point, the notation $\sigma \xrightarrow{a} \sigma'$ for a transition is quite explicative: the set of (non-trivial) transitions $\{\sigma' = \mu(\sigma, a)\}$ in \mathcal{A} is in one-to-one correspondence with the set of labelled directed edges $\{(\sigma, \mu(\sigma, a))\}$ of $\mathfrak{G}(\mathcal{A})$.

The set of paths in $\mathfrak{G}(\mathcal{A})$ starting from $\sigma_0 \in \Sigma$ and ending in an accepted state $\sigma \in Y$ is in one-to-one correspondence with the language $L(\mathcal{A})$: a word $w \in L(\mathcal{A})$ corresponds to the path starting from σ_0 and whose edges are labelled by the letters of w .

In particular, the set of geodesic paths in $\mathfrak{G}(\mathcal{A})$ is in one-to-one correspondence with the set of geodesic words in the language $L(\mathcal{A})$.

2.1.3 Two-variable languages

Consider now two alphabets A_1, A_2 .

A language over (A_1, A_2) is a set of pairs of words (w_1, w_2) where $w_1 \in A_1^*$ and $w_2 \in A_2^*$. Such a language is called a **two-variable language**.

We remark that the pair (w_1, w_2) as above can be formed by two words having a different length.

We adjoin to the alphabets A_1, A_2 a new symbol, denoted by $\$$ and called the **padding symbol**; in this way, we get new alphabets $B_1 := A_1 \cup \{\$\}$ and $B_2 := A_2 \cup \{\$\}$.

Define the **padded alphabet** associated to the pair (A_1, A_2) by $B := (B_1 \times B_2) \setminus \{(\$, \$)\}$.

A **padded word** over B is a pair $w = (w_1, w_2)$ where $w_1 \in B_1$ and $w_2 \in B_2$ are such that once the padding symbol $\$$ has occurred in w_1 or in w_2 , then the following letters of the same word are all padding symbols. Thus, for example, if $w_1 = a_1^{(1)} \cdots a_n^{(1)}$, where $a_i^{(1)} \in A_1$ for $i = 1, \dots, n$, and $a_j^{(1)} = \$$ for some $j \in \{1, \dots, n\}$, then $a_k^{(1)} = \$$ for all $k \in \{j, \dots, n\}$.

If L is a language over (A_1, A_2) , the **padded extension** $L^\$$ of L is the language over the padded alphabet B formed by the padded words $(w_1, w_2)^\$$ associated to the elements $(w_1, w_2) \in L$, where we pad the shortest between w_1, w_2 to get two words of equal lengths. More precisely, if (w_1, w_2) and $\ell(w_1) \neq \ell(w_2)$, assume that $\ell(w_1) < \ell(w_2)$: we modify the word w_1 into w'_1 adding to its end the a number of padding symbols in such a way that $\ell(w'_1) = \ell(w_2)$.

A two-variable language L over (A_1, A_2) is **regular** if its padded extension $L^\$$ is a regular language over the padded alphabet B associated to (A_1, A_2) .

A **two-variable automaton** over (A_1, A_2) is a finite state automaton over B accepting the padded extension $L^\$$ of a language L over (A_1, A_2) .

2.2 AUTOMATIC GROUPS

Let G be a finitely generated group with finite set of generators S not containing the identity. An **automatic structure** for G on S is the datum of $\{S, \mathcal{W}, \mathcal{M}_s \mid s \in S \cup \{\varepsilon\}\}$ where:

- the finite set of generators S is the **alphabet**;
- \mathcal{W} is a finite state automaton over S , called the **word acceptor**, such that the natural map $S^* \rightarrow G$, when restricted to the language $L(\mathcal{W})$ accepted by \mathcal{W} , is surjective onto G ;
- for every $s \in S \cup \{\varepsilon\}$, \mathcal{M}_s is a two-variable automaton over (S, S) such

that $(w_1, w_2) \in L(\mathcal{M}_s)$ if and only if $w_1, w_2 \in L(\mathcal{W})$ and $\overline{w_1} = \overline{w_2 s}$; we call \mathcal{W}_s the **equality recognizer** if $s = \varepsilon$ or the **multiplier** otherwise.

Definition 2.2. A finitely generated group G is **automatic** if there exists an automatic structure on some finite set of generators of the group.

The Fellow Traveller Property characterizes automatic structures for finitely generated groups.

Proposition 2.3 (Fellow Traveller Property). *A finitely generated group G has an automatic structure over a finite set of generators S if and only if there exists a regular language L over S such that*

1. *the restriction $L \rightarrow G$ of the natural map $S^* \rightarrow G$ is surjective;*
2. *there exists a constant $K > 0$ such that $d_u(\hat{w}_1, \hat{w}_2) \leq K$ for every pair $w_{1,2} \in L$ such that $p(w_1 s) = p(w_2)$ for some $s \in S \cup \{\epsilon\}$.*

An interesting class of automatic groups is formed by those admitting an automatic structure whose word acceptor recognizes the language of all geodesic words. These are called **strongly geodesically automatic** groups.

2.3 CONE TYPE AUTOMATON

2.3.1 Combinatorial definition

Definition 2.4. Given a finitely generated group G with a finite set of generators S , consider a word $u \in S^*$. The **cone type** of u is the set

$$C(u) = \{w \in S^* \mid uw \text{ is a geodesic word}\}.$$

Clearly, $C(u) \neq \emptyset$ if and only if u is a geodesic word itself.

If $x \in G$ and $u \in S^*$ is a geodesic word such that $\bar{u} = x$, then we define the cone type of x as

$$C(x) = \{\bar{w} \in G \mid w \in C(u)\}.$$

Remark 2.5. The definition is well posed: if u_1 and u_2 are geodesic words representing the same $x \in G$, i.e. $\bar{u}_1 = \bar{u}_2 = x$, then $C(u_1) = C(u_2)$: in fact, for $w \in S^*$ we have that $w \in C(u_1)$ if and only if $u_1 w$ is a geodesic word, which is equivalent to the fact that $u_2 w$ is a geodesic word, i.e. $w \in C(u_2)$.

Cannon’s Theorem ([EPC⁺92]). *If G is a hyperbolic group, then the number of distinct cone types in G is finite.*

The cone types present further symmetries. This is why the following notion of “Cannon type” is sometimes more useful than the one of cone type.

Let Γ be a surface group with symmetric set of generators S .

Given a group element $x \in \Gamma$, there are different geodesic words representing it, still, they are finite in number. Each geodesic word has a maximal suffix in common with a fundamental relator.

Definition 2.6. The **Cannon type** of $x \in \Gamma$ is the maximum of the lengths of the suffixes which are also subwords of fundamental relators among all the geodesic words representing x .

More formally: let $x \in \Gamma$, let u be a geodesic word representing x . Denote by L_u be the length of the maximal suffix w of u such that $w \in \text{Pref}(R_*)$. Then the Cannon type of x is $\max\{L_u \mid u \text{ geod. word, } \bar{u} = x\}$.

Remark 2.7. Some authors call “cone type” what we defined Cannon type of an element. It is intended that this is a notion of “unlabelled” cone type, different from the “labelled” cone type definition that we adopted. Cf. Sect. 1.7.1.

There are $2k + 1$ Cannon types in Γ_k , including the Cannon type of e , which is 0. In fact, a subword of both a geodesic word and a fundamental relator can have at most length $2k$.

In particular, there are elements of Γ having different cone types, but having the same Cannon type.

2.3.2 Equivalence of the definitions

In Definition 1.28 of Chapter 1, the cone types are equivalence classes of cones in the Cayley graph of a group; in Definition 2.4 of the present Chapter, they are sets of geodesic words (or sets of elements represented by these geodesic words). We show here that the two definitions are equivalent.

Proposition 2.8. *Let Γ be a surface group and $x \in \Gamma$. Then the cone $C(x^{-1}, e)$ as in Definition 1.26 is equal to the cone type $C(x)$ as in Definition 2.4.*

Since each cone type as in Definition 1.28 is uniquely represented by a cone with vertex in the identity (and hence identified with it), Proposition 2.8 shows that Definition 1.28 and Definition 2.4 are equivalent.

Proof of Proposition 2.8. Given an element $x \in \Gamma$ and a geodesic word u such that $\bar{u} = x$, we have that $\overline{(u^{-1})} = x^{-1}$. Thus,

$$\begin{aligned}
C(x) &= \{\bar{w} \in \Gamma \mid w \in C(u)\} \\
&= \{\bar{w} \in \Gamma \mid uw \text{ is a geodesic word}\} \\
&= \{\bar{w} \in \Gamma \mid \ell(uw) = \ell(u) + \ell(w)\} \\
&= \{\bar{w} \in \Gamma \mid d(e, \bar{u}\bar{w}) = d(e, \bar{u}) + d(\bar{u}, \bar{u}\bar{w})\} \\
&= \left\{ \bar{w} \in \Gamma \mid d(e, \bar{u}\bar{w}) = d\left(\overline{(u^{-1})}, e\right) + d(e, \bar{w}) \right\} \\
&= C\left(\overline{(u^{-1})}, e\right) = C(x^{-1}, e).
\end{aligned}$$

Now, each cone type as in 1.28 is represented by a unique cone with vertex $e \in \Gamma$. This concludes the proof. \square

We report from [Gou15] two possible descriptions of the cone type $C(x)$ (Def. 2.4) of an element $x \in \Gamma$.

Geometric description: consider the tiling of associated to \mathcal{G}_k , consider the set $[e, x]$ formed by all geodesics starting from e and ending at x . The Cannon type of x is defined as the maximal length j_0 along the last $4k$ -gon of these geodesics. The cone type of x is identified by the labels on the final j_0 edges of the geodesic realizing the maximal length. The Cannon type of an element $x \in \Gamma$ can be at most $2k$; different cone types represented as above by words of the same length correspond to the same Cannon type.

Combinatorial description: consider a geodesic word $w = s_1 \cdots s_n$ representing x . Consider the suffix of such word $s_{n-j+1} \cdots s_n \in \text{Pref}(R_*)$ which has maximal length j . Consider now all possible ways to represent x as a geodesic word and take the suffix having maximal length j_0 among all possible geodesic words w representing x : this identifies the cone type of x , meaning that $C(x) = C(s_{n-j_0+1} \cdots s_n)$.

The precise number of cone types in Γ_k is known.

Lemma 2.9. *The number of cone types in a surface group Γ_k is $1 + 8k(2k - 1)$.*

Proof. Consider the tiling of \mathbb{H}^2 whose 1-skeleton is the dual graph of \mathcal{G}_k . We have to compute the number of vertices that lie on the boundary of a $4k$ -gon “based” at $e \in \Gamma$ (i.e. the identity is a vertex on the boundary of the $4k$ -gon). Clearly, we have the vertex e , which has a special cone type. There are precisely $4k$ points having distance 1 from e . Each of these points has one successors on the $4k$ -gon on its left and one on the $4k$ -gon on its

right. Thus, there are $8k$ vertices at distance 2 from e . Similarly, there are $8k$ vertices at distance $j = 3, \dots, 2k - 1$ from e on the polygons based at e . Finally, we have $4k$ points at distance $2k$. Thus:

$$\#\mathcal{C} = 1 + 4k + 8k(2k - 1 - 1) + 4k = 1 + 8k(2k - 1),$$

which gives the number of cone types. □

2.3.3 Cone types in genus 2

We explicitly determine the cone types in the group Γ_2 .

The relation and its inverse are

$$a_1 b_1 a_1^{-1} b_1^{-1} a_2 b_2 a_2^{-1} b_2^{-1}, \quad b_2 a_2 b_2^{-1} a_2^{-1} b_1 a_1 b_1^{-1} a_1^{-1}.$$

The set of fundamental relator in the standard presentation of Γ_2 consists of 16 elements.

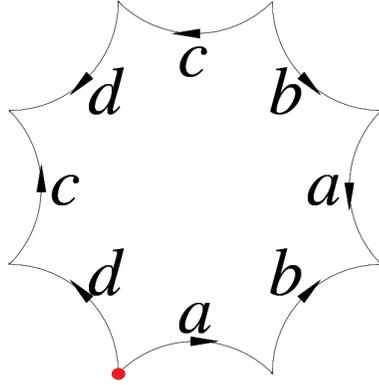


Figure 2.1: Labelled directed octagon in the Cayley graph of Γ_2 ; the red vertex represents the identity.

In the following paragraph, we use a “mod. 2” convention: $i + 1$ means 2 if $i = 1$ and 1 if $i = 2$.

The cone types in Γ_2 are:

- Cannon type 1: $C(a_i^\epsilon), C(b_i^\epsilon)$, with $i \in \{1, 2\}$ and $\epsilon \in \{\pm 1\}$;
- Cannon type 2: $C(a_i^\epsilon b_i^\epsilon), C(b_i^\epsilon a_i^\epsilon)$, for $i \in \{1, 2\}$ and $\epsilon \in \{\pm 1\}$;
 $C(a_i b_i^{-1}), C(b_i a_i^{-1}), C(a_i^{-1} b_{i+1}), C(b_i^{-1} a_{i+1})$, for $i \in \{1, 2\}$;

-
- Cannon type 3: $C(a_i b_i a_i^{-1})$, $C(a_i^{-1} b_i^{-1} a_{i+1})$, $C(b_i a_i b_i^{-1})$, $C(b_i^{-1} a_i^{-1} b_{i+1})$, for $i \in \{1, 2\}$; $C(a_i b_i^{-1} a_i^{-1})$, $C(b_i a_i^{-1} b_i^{-1})$, $C(a_i^{-1} b_{i+1} a_{i+1})$, $C(b_i^{-1} a_{i+1} b_{i+1})$, for $i \in \{1, 2\}$;
 - Cannon type 4: $C(a_i b_i a_i^{-1} b_i^{-1})$ which is the same as $C(b_{i+1} a_{i+1} b_{i+1}^{-1} a_{i+1}^{-1})$; $C(b_i a_i^{-1} b_i^{-1} a_{i+1})$ which is the same as $C(a_{i+1} b_{i+1}^{-1} a_{i+1}^{-1} b_i)$; then there are $C(a_i^{-1} b_i^{-1} a_{i+1} b_{i+1})$ which is the same as $C(b_{i+1}^{-1} a_{i+1}^{-1} b_i a_i)$, and $C(b_i^{-1} a_{i+1} b_{i+1} a_{i+1}^{-1})$ which is the same as $C(a_{i+1}^{-1} b_i a_i b_i^{-1})$; here $i \in \{1, 2\}$ with the convention as above.

The cone type of a vertex x in \mathcal{G}_k determines the number of predecessors of x (i.e. those vertices y such that $d(e, y) = d(e, x) - 1$ and (y, x) is an edge) and the number of successors of x (the vertices y such that $d(e, y) = d(e, x) + 1$ and (x, y) is an edge). Moreover, assume that $d(e, xs) = d(e, x) + 1$ for $s \in S$: then, the cone type of the successor xs of x is determined by the cone type of x and by the generator s giving the label on the edge joining x and xs .

In generic genus $k \geq 2$, each vertex in \mathcal{G}_k having Cannon type $t = 1, \dots, 2k - 1$ has a unique predecessor. The vertices having Cannon type $2k$, instead, have two possible predecessors. For a vertex of Cannon type 1 there are two possible successors having Cannon type 2, while the other successors all have Cannon type 1; for each vertex of Cannon type $t = 2, \dots, 2k - 1$ there is precisely one successor of Cannon type $t + 1$ and one successor of Cannon type 2, while the other successors have Cannon type 1. Finally, a vertex having Cannon type $2k$ has two successors whose Cannon type is 2 and the others have Cannon type 1.

2.3.4 CTA for surface groups

We define the **cone type automaton** \mathcal{A}_{ct} of Γ with respect to S , a finite state automaton on the alphabet S having the set of cone types as set of states (see [EPC⁺92], [Ohs02]).

1. The set of **states** Σ is given by all cone types \mathcal{C} and \emptyset (the failure type).
2. The **alphabet** is given by the finite symmetric set of generators S .
3. The **transition function** $\mu: \Sigma \times S \rightarrow \Sigma$ is defined as follows: let $C(u)$ be a cone type and $s \in S$: then $\mu(C(u), s) = C(us)$. We often denote a transition by $\mathbf{c} \xrightarrow{s} \mathbf{c}'$, where $\mathbf{c} = C(u)$ and $\mathbf{c}' = C(us)$.

4. The **initial/starting state** is the cone type $C(e)$, where $e \in \Gamma$ is the identity.
5. The set of **final/accepted states** is the set of by all cone types excluding the failure type.

The cone type automaton of Γ_k with respect to S_k accepts the language of all geodesic words on S_k ; in particular, the group Γ_k is *strongly geodesically automatic* and *ShortLex automatic* (cf. [EPC⁺92]).

Theorem 2.10 ([EPC⁺92]). *Let G be a hyperbolic group with a finite set of generators S . Then, there is an automatic structure on G whose word acceptor is the cone type automaton of G with respect to S .*

In particular, a finitely generated group is strongly geodesically automatic if and only if it is hyperbolic.

2.3.5 Ergodicity of the CTA

We consider the labelled directed graph $\mathfrak{G}(\mathcal{A}_{\text{ct}})$ associated to the cone type automaton as in Definition 2.1.

- the set of vertices is given by the set of states Σ of the automaton, i.e. the set of cone types \mathcal{C} ,
- there is an edge from the cone type \mathbf{c} to the cone type \mathbf{c}' if and only if there exists $s \in S$ such that $\mathbf{c}' = \mu(\mathbf{c}, s)$. The label on this edge is the generator s . Thus, the set of edges $\mathbf{c} \xrightarrow{s} \mathbf{c}'$ of the graph $\mathfrak{G}(\mathcal{A}_{\text{ct}})$ is in one-to-one correspondence with the set of transitions $\mathbf{c}' = \mu(\mathbf{c}, s)$ in the CTA.

If $\mathbf{c}' = \mu(\mathbf{c}, s)$ for some $s \in S$, then we write $\mathbf{c} \xrightarrow{s} \mathbf{c}'$ or simply $\mathbf{c} \rightarrow \mathbf{c}'$.

If \mathbf{c}, \mathbf{c}' are vertices of the graph, we write $\mathbf{c} \Rightarrow \mathbf{c}'$ if there exists a directed path $\mathbf{c} = \mathbf{d}_0, \dots, \mathbf{d}_n = \mathbf{c}'$, where \mathbf{d}_i are vertices and $\mathbf{d}_i \rightarrow \mathbf{d}_{i+1}$, $i = 0, \dots, n-1$. We denote $\mathbf{c} \xRightarrow{w} \mathbf{c}'$, where w is a word on S , if the transitions are obtained from the letters of the word w (in the right order). In this situation, we say that there is a path or a derivation from \mathbf{c} to \mathbf{c}' .

Recall that a directed graph is said to be **strongly connected** if, given any two vertices in it, there exists a directed path starting from the first and ending in the second.

Definition 2.11. The cone type automaton \mathcal{A}_{ct} is **ergodic** if the associated graph $\mathfrak{G}(\mathcal{A}_{\text{ct}})$ is strongly connected.

Proposition 2.12. *Let Γ be a surface group and S the standard set of generators. Then the graph \mathcal{A}_{ct} is strongly connected.*

Proof. Consider a generic cone type $C(w)$ in Γ represented by the geodesic word w on S .

In a surface group, we can always find a generator $s \in S$ such that $C(ws) = C(s)$. This corresponds to the fact that each vertex of any given Cannon type has at least one successor of Cannon type 1.

Moreover, we can find $C(w)$ as the last element of a derivation starting with a cone type corresponding to the Cannon type 1: there is $t \in S$ such that there exists a derivation $C(t) \Rightarrow C(w)$. In fact, if the geodesic word w is $t_1 \cdots t_n$, then we simply choose $t = t_1$: clearly, $C(t_1) \Rightarrow C(w)$, where the derivation is given by the other letters t_2, \dots, t_n of w .

Finally, we say that, for every pair of cone types corresponding to the Cannon type 1, say $C(s)$ and $C(t)$ (where $s, t \in S$), we can find a derivation $C(s) \Rightarrow C(t)$. In fact, fixing $s, t \in S$, we can find a generator $t' \in S$ such that:

- neither st' nor $t't$ are *forbidden words* (i.e. non-reduced words or subwords of a fundamental relator of length more than $2k$);
- $t't$ is not a subword of a fundamental relator (not a piece of the relation or its inverse: $t't \notin \text{Pref}(R_*)$).

The second point implies that $C(st't) = C(t)$. Thus, we choose the derivation

$$C(s) \xrightarrow{t'} C(st') \xrightarrow{t} C(st't) = C(t).$$

We conclude: given generic cone types $C(w_1), C(w_2)$, we find a derivation from $C(w_1) \Rightarrow C(s)$, where $s \in S$ and so $C(s)$ is of Cannon type 1, and we find another cone type, $C(t)$ for $t \in S$, again of Cannon type 1, such that $C(t) \Rightarrow C(w_2)$. Then, there exists a derivation $C(s) \Rightarrow C(t)$. We compose the derivations as follows:

$$C(w_1) \Rightarrow C(s) \Rightarrow C(t) \Rightarrow C(w_2).$$

This ends the proof. □

Remark 2.13. Following [Sen06], we can define an adjacency matrix associated to the graph $\mathfrak{G}(\mathcal{A}_{ct})$ whose indexes are the vertices of the graph and which tell if two vertices are connected by an edge or not. In [Sen06] it is explained that such a matrix is irreducible if and only if the graph is strongly connected.

CHAPTER 3

MULTIPLICATIVE REPRESENTATIONS

Following an analogous idea of [KS04] for a free group, we define multiplicative functions as vector-valued functions on a surface group. The behaviour of a multiplicative function is deeply connected with the structure of the cone type automaton and depends on the choice of a generating system and on a set of parameters, the so-called matrix system, which are used to form the vector space where the function takes values and a recursive procedure to calculate the value of the function on a vertex in \mathcal{G} from the value on its predecessors.

Remark 3.1. All the metric notions in the present Chapter are intended with respect to the word metric associated with a fixed symmetric set of generators.

3.1 MATRIX SYSTEMS

Fix a surface group Γ with a symmetric set of generators S . From now on, we will always consider the word metric on Γ associated to S .

Recall that the CTA can be represented as a labelled directed graph whose labelled arrows are given by the transitions.

The following definition is inspired by [KS04, Def. 3.1].

Definition 3.2. A **matrix system** $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$ is given by the following data:

-
- for each cone type $\mathbf{c} \in \mathcal{C}$ we choose a finite dimensional complex vector space $V_{\mathbf{c}}$;
 - for each transition in the cone type automaton $\mathbf{c} \xrightarrow{s} \mathbf{c}'$ we choose a linear map $H_{\mathbf{c},\mathbf{c}',s}: V_{\mathbf{c}} \rightarrow V_{\mathbf{c}'}$.

In particular, for each cone type $\mathbf{c} \in \mathcal{C}$ and each generator $s \in S \cap \mathbf{c}$ there exists $\mathbf{c}' \in \mathcal{C}$ such that $\mathbf{c} \xrightarrow{s} \mathbf{c}'$ is a transition in the cone type automaton. Thus, for each $\mathbf{c} \in \mathcal{C}$ and each $s \in S \cap \mathbf{c}$ we choose a linear map $V_{\mathbf{c}} \xrightarrow{H_{\mathbf{c},\mathbf{c}',s}} V_{\mathbf{c}'}$.

Remark 3.3. If $\mathbf{c} \in \mathcal{C}$ is a cone type and $s \in S$ is a generator not lying in \mathbf{c} , then we associate to \mathbf{c} and s the trivial linear map $H_{\mathbf{c},\mathbf{c}',s} = 0$, since the transition starting from \mathbf{c} and labelled with s goes to the failure type, thus being irrelevant for us.

Definition 3.4. Given a surface group Γ with a symmetric set of generators S and given a matrix system $\{V_{\mathbf{c}}, H_{\mathbf{c},\mathbf{c}',s}\}$ we define

$$V := \bigoplus_{\mathbf{c} \in \mathcal{C}} V_{\mathbf{c}}.$$

Remark 3.5. The space V is finite-dimensional over \mathbb{C} : in fact, the set of cone types \mathcal{C} is finite and each $V_{\mathbf{c}}$ is chosen of finite dimension.

3.1.1 Properties

In this Section we report a presentation of the properties of matrix systems as it can be found in [KS04, Sect. 3].

Let Γ be a surface group with the standard symmetric set of generators S . Fix a matrix system $\{V_{\mathbf{c}}, H_{\mathbf{c},\mathbf{c}',s}\}$.

Definition 3.6 ([KS04]). An **invariant subsystem** of $\{V_{\mathbf{c}}, H_{\mathbf{c},\mathbf{c}',s}\}$ is a family $\{W_{\mathbf{c}}\}$ of linear subspaces $W_{\mathbf{c}} \subseteq V_{\mathbf{c}}$ for all $\mathbf{c} \in \mathcal{C}$ such that $H_{\mathbf{c},\mathbf{c}',s}W_{\mathbf{c}} \subseteq W_{\mathbf{c}'}$ for $\mathbf{c}, \mathbf{c}' \in \mathcal{C}$ and $s \in S$ satisfying $\mathbf{c} \xrightarrow{s} \mathbf{c}'$. The invariant subsystem is called **trivial** if it is $\{0 \subseteq V_{\mathbf{c}} \mid \mathbf{c} \in \mathcal{C}\}$ or $\{V_{\mathbf{c}}\}$ itself.

Definition 3.7 ([KS04]). A matrix system $\{V_{\mathbf{c}}, H_{\mathbf{c},\mathbf{c}',s}\}$ is **irreducible** if it has no non-trivial invariant subsystems.

Remark 3.8 ([KS04]). A matrix system $\{V_{\mathbf{c}}, H_{\mathbf{c},\mathbf{c}',s}\}$ where $V_{\mathbf{c}} \simeq \mathbb{C}$ for all $\mathbf{c} \in \mathcal{C}$ is clearly irreducible, since the only (complex) subspaces of \mathbb{C} are the trivial ones.

Definition 3.9 ([KS04]). Consider two matrix systems $\{V_{\mathbf{c}}^{(1)}, H_{\mathbf{c}, \mathbf{c}', s}^{(1)}\}$ and $\{V_{\mathbf{c}}^{(2)}, H_{\mathbf{c}, \mathbf{c}', s}^{(2)}\}$ (associated to the same symmetric set of generators S).

A **system map** is given by a tuple of linear maps $\{J_{\mathbf{c}}\}$ where $J_{\mathbf{c}}: V_{\mathbf{c}}^{(1)} \rightarrow V_{\mathbf{c}}^{(2)}$ for $\mathbf{c} \in \mathcal{C}$ and $H_{\mathbf{c}, \mathbf{c}', s}^{(2)} J_{\mathbf{c}} = J_{\mathbf{c}'} H_{\mathbf{c}, \mathbf{c}', s}^{(1)}$ for each transition $\mathbf{c} \xrightarrow{s} \mathbf{c}'$.

A **system equivalence** (an “equivalence” for short) is a system map where each $J_{\mathbf{c}}$ is bijective.

Two matrix systems are **equivalent** if there is an equivalence between them.

Remark 3.10 ([KS04]). A system map between irreducible matrix systems is either 0 or an equivalence. In fact, the kernel of each linear map has to be a trivial subspace and so the image.

If $V_{\mathbf{c}} \simeq \mathbb{C}$, then every $H_{\mathbf{c}, \mathbf{c}', s}$ is represented by a complex scalar.

Definition 3.11. We say that a matrix system $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$ is a **scalar system** if $V_{\mathbf{c}} \simeq \mathbb{C}$ for all $\mathbf{c} \in \mathcal{C}$ and $H_{\mathbf{c}, \mathbf{c}', s}$ are non-negative scalars for all $\mathbf{c} \xrightarrow{s} \mathbf{c}'$.

Remark 3.12. A scalar system is always irreducible. Moreover, by Remark 3.10, all scalar systems (associated to a fixed symmetric set of generators for the group) are equivalent.

3.1.2 Linear maps H_{γ}

Fix a surface group Γ with the standard symmetric set of generators. Let \mathcal{G} be the Cayley graph of Γ with respect to S . Fix a matrix system $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$. Consider a cone $C(x, y)$ in \mathcal{G} and consider a geodesic starting from x , passing through y and ending in a vertex $z \in C(x, y)$. We denote by γ the part of the geodesic from y to z . There exist $s_1, \dots, s_n \in S$ (where $n = \ell(\gamma) = d(y, z)$) such that $z = ys_1 \cdots s_n$. In particular, $s_1 \cdots s_n \in S^*$ is a geodesic word associated to the geodesic γ (the labels on the edges crossed by γ are precisely the s_i). Thus, we have $\gamma(0) = y$, $\gamma(i) = ys_1 \cdots s_i$ for each $i \in \{1, \dots, n-1\}$ and $\gamma(n) = ys_1 \cdots s_n$. Denote by \mathbf{c}_0 the cone type of $C(x, y)$ and by \mathbf{c}_i the cone type of $C(x, ys_1 \cdots s_i)$. Then for each $i \in \{1, \dots, n\}$ we have the transition in the CTA

$$\mathbf{c}_{i-1} \xrightarrow{s_i} \mathbf{c}_i.$$

Hence, for each $i \in \{1, \dots, n\}$ we get the linear map

$$V_{\mathbf{c}_{i-1}} \xrightarrow{H_{\mathbf{c}_{i-1}, \mathbf{c}_i, s_i}} V_{\mathbf{c}_i}$$

and we can compose $H_{\mathbf{c}_{i-1}, \mathbf{c}_i, s_i}$ and $H_{\mathbf{c}_i, \mathbf{c}_{i+1}, s_{i+1}}$, since the range of the former lies in the domain of the latter for each $i \in \{1, \dots, n-1\}$. Thus,

$$V_{\mathbf{c}_0} \xrightarrow{H_{\mathbf{c}_0, \mathbf{c}_1, s_1}} V_{\mathbf{c}_1} \xrightarrow{H_{\mathbf{c}_1, \mathbf{c}_2, s_2}} \dots \xrightarrow{H_{\mathbf{c}_{n-1}, \mathbf{c}_n, s_n}} V_{\mathbf{c}_n}.$$

We define H_γ as the composition

$$H_\gamma := H_{\mathbf{c}_{n-1}, \mathbf{c}_n, s_n} \cdots H_{\mathbf{c}_0, \mathbf{c}_1, s_1}. \quad (3.1)$$

Geometric interpretation: fix a cone $C(x, y)$ of cone type \mathbf{c}_0 . For every geodesic of finite length starting from x , passing through y and ending in a vertex $z \in C(x, y)$, let γ denote the part connecting y with z , $n = \ell(\gamma)$ and let \mathbf{c}_n be the cone type of the subcone $C(x, z) \subseteq C(x, y)$. Then we have a linear map $H_\gamma: V_{\mathbf{c}_0} \rightarrow V_{\mathbf{c}_n}$ as in equation 3.1. In particular, the map H_γ depends only on the cone types \mathbf{c}_0 , \mathbf{c}_n and by the labels s_i on the edges crossed by the geodesic.

Combinatorial interpretation: fix a cone type $C(u)$, where u is a geodesic word, let $v \in C(u)$ be a geodesic word (i.e. uv is again a geodesic word). Then, we get a linear map associated to v , $H_v: V_{C(u)} \rightarrow V_{C(uv)}$, defined by $H_v := H_\gamma$, where γ is the geodesic joining the vertex \bar{u} with the vertex \bar{uv} associated to the geodesic word v .

3.2 MULTIPLICATIVE FUNCTIONS

Fix a surface group Γ with the standard symmetric set of generators S . Fix a matrix system $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$.

Multiplicative functions take values in $V = \bigoplus_{\mathbf{c} \in \mathcal{C}} V_{\mathbf{c}}$ (see Def. 3.4).

Recall that $[y, z]$ indicates the set of all geodesics in \mathcal{G} joining the vertices y, z .

Definition 3.13. Given a cone $C(x, y)$ in the Cayley graph \mathcal{G} , let \mathbf{c} be its cone type. Let $v \in V_{\mathbf{c}}$ be a vector. We define the **elementary multiplicative function** supported on $C(x, y)$ with starting value v as the function $\mathfrak{m}[x, y, v]: \Gamma \rightarrow V$

$$\mathfrak{m}[x, y, v](z) := \begin{cases} v & \text{if } z = y, \\ \sum_{\gamma \in [y, z]} H_\gamma v & \text{if } z \in C(x, y) \setminus \{y\}, \\ 0 & \text{otherwise.} \end{cases} \quad (3.2)$$

3. MULTIPLICATIVE REPRESENTATIONS

Remark 3.14. If \mathbf{c}' is the cone type of $C(x, z)$, then $\mathfrak{m}[x, y, v](z) \in V_{\mathbf{c}'}$. In fact, for every geodesic $\gamma \in [y, z]$ the linear map H_γ has range in $V_{\mathbf{c}'}$, so in equation 3.2 we get a sum of vectors in $V_{\mathbf{c}'}$.

Remark 3.15. We recall that we identify each cone type in Γ with the unique cone having vertex in the identity and belonging to the orbit (cf. Remark 1.29); we remind, moreover, that given $x \in \Gamma$ and \mathbf{c} a cone type, the symbol $x.\mathbf{c}$ denotes the action of x on the cone with vertex in the identity given by \mathbf{c} .

The complex vector space of finite linear combinations of elementary multiplicative functions associated with the matrix system $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$ is

$$\mathcal{H}_0^\infty(V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}) := \text{span}_{\mathbb{C}} \{ \mathfrak{m}[x, y, v] \mid x, y \in \Gamma, v \in V_{\mathbf{c}} \text{ for } \mathbf{c} = C(y^{-1}x, e) \}. \quad (3.3)$$

We will refer to this space simply by \mathcal{H}_0^∞ and the matrix system will be implicit.

Definition 3.16. A (generic) **multiplicative function** is an element of \mathcal{H}_0^∞ , i.e. a linear combination of multiplicative functions.

Remark 3.17. For $f \in \mathcal{H}_0^\infty$ and z an element in the support of f , the value $f(z)$ is not necessarily contained in a specific vector space $V_{\mathbf{c}}$ (as it holds for an elementary multiplicative function, cf. Rem. 3.14), but it is a (finite) sum in the space $V = \bigoplus_{\mathbf{c} \in \mathcal{C}} V_{\mathbf{c}}$.

Definition 3.18. Let $f_1, f_2 \in \mathcal{H}_0^\infty$ be multiplicative functions. We say that f_1 and f_2 are **equivalent**, writing $f_1 \sim f_2$, if there exists $N \in \mathbb{N}$ such that $f_1(z) = f_2(z)$ for all $z \in \Gamma$ with $|z| > N$.

Remark 3.19. Applying the previous definition to arbitrary functions $\Gamma \rightarrow V$, we get that every compactly supported V -valued function on Γ (which is not multiplicative by our definition) is equivalent to the zero multiplicative function.

Definition 3.20. The **space of multiplicative functions** is the set of equivalence classes of multiplicative functions:

$$\mathcal{H}^\infty(V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}) := \mathcal{H}_0^\infty(V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}) / \sim .$$

As before, we often denote this space simply by \mathcal{H}^∞ .

We often refer to a class of functions in cH^∞ by some explicit representative in \mathcal{H}_0^∞ .

Lemma 3.21. *Let $x, y \in \Gamma$, let $\mathbf{c} = C(y^{-1}x, e)$, let $v \in V_{\mathbf{c}}$. Consider the elementary multiplicative function $\mathbf{m}[x, y, v]$ supported on $C(x, y)$. Let $s_1, \dots, s_N \in S \cap C(x, y)$ be such that $C(x, ys_1), \dots, C(x, ys_N)$ are all the first level subcones of $C(x, y)$, let \mathbf{c}'_i be the cone type of $C(x, ys_i)$ for $i = 1, \dots, N$. Then*

$$\mathbf{m}[x, y, v] \sim \sum_{i=1}^N \mathbf{m}[x, ys_i, H_{\mathbf{c}, \mathbf{c}'_i, s_i} v],$$

i.e. the two functions are equal everywhere except for finitely many $z \in \Gamma$ (Def. 3.18).

Proof. If $z \in C(x, y)$ and $z \neq y$, then every geodesic γ joining y and z passes through one of the ys_i for $i = 1, \dots, N$. In particular, $z \in C(x, ys_i)$ for this index i . Therefore:

$$\mathbf{m}[x, ys_i, H_{\mathbf{c}, \mathbf{c}'_i, s_i} v](z) = \sum_{\gamma' \in [ys_i, z]} H_{\gamma'} \left(H_{\mathbf{c}, \mathbf{c}'_i, s_i} v \right).$$

If $z \in C(x, y) \setminus \{y\}$ and $\gamma \in [y, z]$ passes through ys_i , let γ' be the part of γ following ys_i (if $\gamma = \{y, ys_i, ys_i t_1, \dots, ys_i t_1 \cdots t_m\}$ for $t_j \in S$, then $\gamma' = \{ys_i, ys_i t_1, \dots, ys_i t_1 \cdots t_m\}$). So we can write $H_\gamma = H_{\gamma'} H_{\mathbf{c}, \mathbf{c}'_i, s_i}$ (cf. Sect. 3.1.2) and

$$\begin{aligned} \mathbf{m}[x, y, v](z) &= \sum_{\gamma \in [y, z]} H_\gamma v = \sum_{i=1}^N \sum_{\gamma' \in [ys_i, z]} H_{\gamma'} H_{\mathbf{c}, \mathbf{c}'_i, s_i} v \\ &= \sum_{i=1}^N \mathbf{m}[x, ys_i, H_{\mathbf{c}, \mathbf{c}'_i, s_i} v](z), \end{aligned}$$

This gives the thesis. □

Lemma 3.22. *Let $x, y \in \Gamma$, let $z \in C(x, y)$. If $\eta \in [y, z]$ is a geodesic, then there exists a geodesic $\tilde{\eta} \in [x, z]$ whose final part coincides with η .*

Proof. If $z \in C(x, y)$, then there exists a geodesic $\gamma \in [x, z]$ passing through y . Define γ' the first part of γ connecting x and y and define θ the second part connecting y and z . More precisely: $\gamma'(i) := \gamma(i)$ for every $i = 0, \dots, d(x, y)$, while $\theta(i) := \gamma(d(x, y) + i)$ for every $i = 0, \dots, \ell(\gamma) - d(x, y)$. Then: $d(x, z) = \ell(\gamma) = \ell(\gamma') + \ell(\theta)$.

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Consider now $\eta \in [y, z]$ as in the hypothesis, notice that $\ell(\eta) = \ell(\theta)$, since both are geodesics joining the same points. Define the path $\gamma' * \eta$ as follows: $(\gamma' * \eta)(i) := \gamma'(i)$ for $i = 0, \dots, \ell(\gamma')$ and $(\gamma' * \eta)(i) := \eta(\ell(\gamma') - i)$ for $i = \ell(\gamma') + 1, \dots, \ell(\gamma') + \ell(\eta)$.

We claim that $\gamma' * \eta$ is a geodesic joining x to z . In fact:

$$\ell(\gamma' * \eta) = \ell(\gamma') + \ell(\eta) = \ell(\gamma') + \ell(\theta) = \ell(\gamma) = d(x, z).$$

The geodesic $\tilde{\eta} := \gamma' * \eta$ is the one we looked for (its final part is defined as η). \square

Lemma 3.23. *Let $x_1, y_1 \in \Gamma$ and consider the cone $C(x_2, y_2)$. Let $x_2, y_2 \in \Gamma$, let $\mathbf{c} = C(y_2^{-1}x_2, e)$, let $v \in V_{\mathbf{c}}$. Consider the elementary multiplicative function $\mathbf{m}[x_2, y_2, v]$ supported on $C(x_2, y_2)$. Then there exist $N \in \mathbb{N}$ and subcones $C(x'_i, z_i)$ for $i = 1, \dots, N$ as in Theorem 1.34 such that*

$$\mathbf{1}_{C(x_1, y_1)} \cdot \mathbf{m}[x_2, y_2, v] \sim \sum_{i=1}^N \mathbf{m}[x'_i, z_i, v_i],$$

i.e. the two functions are equal everywhere except for finitely many elements in Γ (cf. Def. 3.18).

Proof. We denote for short $\mathbf{m} = \mathbf{m}[x_2, y_2, v]$. Then, for each $z \in C(x_2, y_2)$:

$$\mathbf{m}(z) = \sum_{\gamma \in [y_2, z]} H_{\gamma} v.$$

Let $N \in \mathbb{N}$, $x'_i, z_i \in \Gamma$ for $i = 1, \dots, N$ be such that

$$C(x_1, y_1) \cap C(x_2, y_2) \approx \bigcup_{i=1}^N C(x'_i, z_i)$$

as in Theorem 1.34. The function $\mathbf{1}_{C(x_1, y_1)} \cdot \mathbf{m}[x_2, y_2, v]$ is supported on this set.

Let $z \in \bigcup_i C(x'_i, z_i)$, consider a geodesic $\gamma \in [y_2, z]$ such that γ passes through the vertex z_i for a fixed i . Denote by γ' the part of γ joining y_2 and z_i , denote by η the part of γ joining z_i to z . Then, by definition (Sect. 3.1.2): $H_{\gamma} = H_{\eta} H_{\gamma'}$. Moreover, if $\tilde{\eta}$ is another geodesic joining z_i to z , then the product $H_{\gamma'} H_{\tilde{\eta}}$ corresponds to the geodesic $\gamma' * \tilde{\eta} \in [y_2, z]$ (cf. Lemma 3.22).

For each $i = 1, \dots, N$ we define $v_i := \sum_{\gamma \in [y_2, z_i]} H_{\gamma} v$ and $\mathbf{m}_i := \mathbf{m}[x'_i, z_i, v_i]$. Thus, for $z \in C(x'_i, z_i)$ we have $\mathbf{m}_i(z) = \sum_{\eta \in [z_i, z]} H_{\eta} v_i$.

If $z \in \bigcup_i C(x'_i, z_i)$ and $z \in C(x_2, y_2)$, then:

$$\begin{aligned}
\mathbf{m}(z) &= \sum_{\gamma \in [y_2, z]} H_\gamma v \\
&= \sum_{i=1}^N \sum_{\gamma' \in [y_2, z_i]} \sum_{\eta \in [z_i, z]} H_\eta H_{\gamma'} v \\
&= \sum_{i=1}^N \sum_{\eta \in [z_i, z]} H_\eta \sum_{\gamma' \in [y_2, z_i]} H_{\gamma'} v \\
&= \sum_{i=1}^N \sum_{\eta \in [z_i, z]} H_\eta v_i \\
&= \sum_{i=1}^N \mathbf{m}_i(z).
\end{aligned}$$

We conclude by observing that the support of $\mathbf{1}_{C(x_1, y_1)} \cdot \mathbf{m}[x_2, y_2, v]$ coincides, up to finitely many elements, with the one of $\sum_{i=1}^N \mathbf{m}_i$. This implies that the two functions are equivalent in the sense of Def. 3.18. \square

3.3 INNER PRODUCT ON \mathcal{H}^∞

Fix a surface group Γ with the standard symmetric set of generators S . Fix a matrix system $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$. We consider the word metric d on Γ , we denote by $|x|$ the word length of $x \in \Gamma$.

Choose a positive-definite sesquilinear form B on $V = \bigoplus_{\mathbf{c} \in \mathcal{C}} V_{\mathbf{c}}$ (this is also called an “inner product on V ”).

Definition 3.24. We define the **inner product** on \mathcal{H}^∞ by the formula:

$$\langle f_1, f_2 \rangle := \lim_{\epsilon \downarrow 0} \epsilon \sum_{z \in \Gamma} B(f_1(z), f_2(z)) e^{-\epsilon|z|}, \quad (3.4)$$

where $f_1, f_2 \in \mathcal{H}^\infty$.

Remark 3.25. If two multiplicative functions f_1, f_2 are equivalent in \mathcal{H}^∞ , they differ only for a finite number of values, so $\langle f_1, f_1 \rangle = \langle f_2, f_2 \rangle$ in equation 3.4: in fact, $\langle f_1, f_2 \rangle = 0$ if (one of) the functions f_1, f_2 have finite support in

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Γ (cf. Rem. 3.19) because the sum over $z \in \Gamma$ becomes finite and the limit for $\epsilon \downarrow 0$ goes to zero thanks to the ϵ factor.

Even if it is clear that the above definition is independent on the representatives for the functions f_1, f_2 , at this point it is quite not clear that the above limit exists and that it is not identically zero.

Remark 3.26. Sesquilinearity and conjugate symmetry of $\langle \cdot, \cdot \rangle$ follow immediately from the same properties of B :

$$\begin{aligned} \langle f_1, f_2 \rangle &= \lim_{\epsilon \downarrow 0} \epsilon \sum_{z \in \Gamma} B(f_1(z), f_2(z)) e^{-\epsilon|z|} \\ &= \lim_{\epsilon \downarrow 0} \epsilon \sum_{z \in \Gamma} \overline{B(f_2(z), f_1(z))} e^{-\epsilon|z|} \\ &= \overline{\langle f_2, f_1 \rangle}, \end{aligned}$$

while $\langle \cdot, \cdot \rangle$ is positive-definite (on the set of equivalence classes of multiplicative functions) because B is: since $B(f(z), f(z)) \geq 0$ for every $f \in \mathcal{H}^\infty$ and $z \in \Gamma$, the limit

$$\lim_{\epsilon \downarrow 0} \epsilon \sum_{z \in \Gamma} B(f(z), f(z)) e^{-\epsilon|z|},$$

if existent, is non-negative.

Remark 3.27. Since we hope to construct irreducible representations for Γ , we cannot pretend that the series $\sum_{z \in \Gamma} B(f_1(z), f_2(z))$ converges: in fact, it is known that, since Γ is discrete, there are no square-integrable irreducible representations of Γ (according to [CFT74], a representation of a discrete infinite group cannot be both irreducible and strongly contained in ℓ^2). Moreover, by the Monotone Convergence Theorem, we know that the series

$$\sum_{z \in \Gamma} B(f_1(z), f_2(z)) e^{-\epsilon|z|}$$

diverges when we take the limit for $\epsilon \downarrow 0$. Hence we multiply it by a term ϵ to get some hope of convergence to a non-zero limit as $\epsilon \downarrow 0$.

Remark 3.28. Definition 3.24 gives a sesquilinear form which is invariant under translation of the two functions by a fixed $g \in \Gamma$: this will be shown in Lemma 3.48.

So we can ask:

Question 3.29. Is there a sesquilinear form B on V such that the limit in equation 3.4 is not identically zero? Is such a B unique (up to some marginal condition like rescaling)?

To get a positive answer to Question 3.29 it is sufficient to show that, fixed a matrix system, for some elementary multiplicative functions the inner product in Definition 3.24 is not trivially zero. We prove this for the case of a surface group of genus 2.

Let $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$ be a fixed scalar system (cf. Def. 3.11) associated with the symmetric set of generators for Γ_2 (genus $k = 2$).

Fix two cones with vertex in the identity $\mathbf{c}_0 = C(x_1, e)$ and $\mathbf{d}_0 = C(x_2, e)$ in \mathcal{G} (we identify them with their cone types). Choose vectors $v_1 \in V_{\mathbf{c}_0}$ and $v_2 \in V_{\mathbf{d}_0}$. Consider $f_1 = \mathbf{m}[x_1, e, v_1]$, $f_2 = \mathbf{m}[x_2, e, v_2]$ elementary multiplicative functions as in Definition 3.2.

Theorem 3.30. *Let Γ_2 be the surface group of genus 2 with the standard symmetric generator set. Let $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$ be a normalized scalar system (cf. Def. 3.44). If $f_1 = \mathbf{m}[x_1, e, v_1]$ and $f_2 = \mathbf{m}[x_2, e, v_2]$ are elementary multiplicative functions supported on \mathbf{c}_0 and \mathbf{d}_0 (respectively) as above, then there exists an inner product B_0 on V such that the value of $\langle f_1, f_2 \rangle$ as in Def. 3.24 is not identically zero.*

Thus, Theorem 3.30 answers positively to Question 3.29.

The proof of Theorem 3.30 is the goal of this Section.

3.3.1 Lemmata for Theorem 3.30

We start proving some Lemmata with the aim of writing the formula 3.4 in a more suitable way to address Question 3.29.

Lemma 3.31. *Assume that we are in the hypotheses of Definition 3.24. Consider $f_1, f_2 \in \mathcal{H}^\infty$, fix an arbitrary $M \in \mathbb{N}$. Then:*

$$\langle f_1, f_2 \rangle = \lim_{\epsilon \downarrow 0} \epsilon \sum_{n=M}^{\infty} \sum_{|z|=n} B(f_1(z), f_2(z)) e^{-\epsilon n}.$$

Proof. Write $T_z^\epsilon = B(f_1(z), f_2(z)) e^{-\epsilon|z|}$. Then

$$\langle f_1, f_2 \rangle = \lim_{\epsilon \downarrow 0} \epsilon \sum_{z \in \Gamma} T_z^\epsilon.$$

Since

$$\Gamma = \bigcup_{n=0}^{\infty} \{z \in \Gamma \mid |z| = n\}$$

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and each of the sets $\{z \in \Gamma \mid |z| = n\}$ is finite, we can write

$$\sum_{z \in \Gamma} T_z^\epsilon = \sum_{n=0}^{\infty} \sum_{|z|=n} T_x^\epsilon = \sum_{n=0}^{M-1} \sum_{|z|=n} T_z^\epsilon + \sum_{n=M}^{\infty} \sum_{|z|=n} T_z^\epsilon.$$

Notice that $\sum_{n=0}^{M-1} \sum_{|z|=n} T_z^\epsilon$ has finitely many terms, so:

$$\lim_{\epsilon \downarrow 0} \epsilon \sum_{n=0}^{M-1} \sum_{|z|=n} T_z^\epsilon = 0.$$

Thus,

$$\langle f_1, f_2 \rangle = \lim_{\epsilon \downarrow 0} \epsilon \sum_{n=M}^{\infty} \sum_{|z|=n} B(f_1(z), f_2(z)) e^{-\epsilon n}$$

and the proof is completed. \square

Remark 3.32. Let \bar{V} be the complex conjugate space of V . It is well known that a sesquilinear form B on V corresponds uniquely to a linear map $\hat{B}: V \otimes \bar{V} \rightarrow \mathbb{C}$ such that $\hat{B}(v \otimes \bar{w}) = B(v, w)$ for all $v, w \in V$. Moreover, since a complex vector space of finite dimension endowed with an inner product is antilinear isomorphic to its dual space, then: $V \otimes \bar{V} \simeq (V \otimes \bar{V})^*$. The isomorphism is not canonical, but it depends on the choice of a sesquilinear form. Thus, up to the choice of a sesquilinear form ϕ on $V \otimes \bar{V}$, the linear form \hat{B} on $V \otimes \bar{V}$, which is an element of $(V \otimes \bar{V})^*$, corresponds to a unique vector $R \in V \otimes \bar{V}$ and

$$\hat{B}(v \otimes \bar{u}) = \phi(R, v \otimes \bar{u}) \tag{3.5}$$

for every $v, u \in V$.

Lemma 3.33. *Assume that we are in the hypotheses of Definition 3.24, let $f_1, f_2 \in \mathcal{H}^\infty$. Choose a sesquilinear form ϕ on $V \otimes \bar{V}$. Then there exists a unique $R \in V \otimes \bar{V}$ such that*

$$\langle f_1, f_2 \rangle = \lim_{\epsilon \downarrow 0} \epsilon \sum_{n=M}^{\infty} \phi \left(R, \sum_{|z|=n} f_1(z) \otimes \bar{f}_2(z) \right) e^{-\epsilon n}. \tag{3.6}$$

Proof. We fix $M \in \mathbb{N}$ as in Lemma 3.31 and we write

$$\langle f_1, f_2 \rangle = \lim_{\epsilon \downarrow 0} \epsilon \sum_{n=M}^{\infty} \sum_{|z|=n} B(f_1(z), f_2(z)) e^{-\epsilon n}.$$

By Remark 3.32, there exists \hat{B} linear map on $V \otimes \bar{V}$ such that for all $z \in \Gamma$: $B(f_1(z), f_2(z)) = \hat{B}(f_1(z) \otimes \bar{f}_2(z))$. Thus, by linearity of \hat{B} , we can write the inner product 3.4 as

$$\begin{aligned} \langle f_1, f_2 \rangle &= \lim_{\epsilon \downarrow 0} \epsilon \sum_{n=M}^{\infty} \sum_{|z|=n} \hat{B}(f_1(z) \otimes \bar{f}_2(z)) e^{-\epsilon n} \\ &= \lim_{\epsilon \downarrow 0} \epsilon \sum_{n=M}^{\infty} \hat{B} \left(\sum_{|z|=n} f_1(z) \otimes \bar{f}_2(z) \right) e^{-\epsilon n}. \end{aligned} \tag{3.7}$$

Since the sesquilinear form ϕ gives an anti-isomorphism between $V \otimes \bar{V}$ and its dual space, the element $\hat{B} \in (V \otimes \bar{V})^*$ corresponds to a unique element $R \in V \otimes \bar{V}$ and

$$\hat{B} \left(\sum_{|z|=n} f_1(z) \otimes \bar{f}_2(z) \right) = \phi \left(R, \sum_{|z|=n} f_1(z) \otimes \bar{f}_2(z) \right),$$

which yields:

$$\langle f_1, f_2 \rangle = \lim_{\epsilon \downarrow 0} \epsilon \sum_{n=M}^{\infty} \phi \left(R, \sum_{|z|=n} f_1(z) \otimes \bar{f}_2(z) \right) e^{-\epsilon n}.$$

□

We look for a recursive computation (possibly, linear) that gives the value of

$$\sum_{|z|=n} f_1(z) \otimes \bar{f}_2(z)$$

which is computed for $|z| = n$, in terms of the one of the same value computed for $n - 1$. If we suppose that f_1, f_2 are elementary multiplicative functions supported on cones with vertices in the identity, for an element z in the intersection of the cones the value of $f_i(z)$ ($i = 1, 2$) depends on the values of f_i on the “ancestors” of z (i.e. points z' such that $|z'| < |z|$ and there exists a geodesic in $[e, z]$ passing through z').

Notation 3.34. If $u \in V = \bigoplus_{\mathbf{c}} V_{\mathbf{c}}$ is a vector, then we denote by $u_{\mathbf{c}}$ its projection on the space $V_{\mathbf{c}}$. We often refer to this projection with the term “component”.

In particular, if $f: \Gamma \rightarrow V$ is a multiplicative function and $z \in \Gamma$, then $f(z) \in V$ and we denote by $f(z)_{\mathbf{c}}$ the projection of $f(z)$ onto $V_{\mathbf{c}}$, for $\mathbf{c} \in \mathcal{C}$.

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Notation 3.35. We recall that we identify each cone type with the unique cone with vertex in the identity belonging to the orbit. Let $\mathbf{c} = C(z, e)$ be a cone type and $x \in \Gamma$. Then, by the notation $x.\mathbf{c}$ we mean the action of x on the cone $C(z, e)$: therefore, $x.\mathbf{c} = x.C(z, e) = C(xz, x)$, which is a cone having vertex in x and with the same cone type \mathbf{c} as $C(z, e)$.

Let $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$ be a fixed scalar system associated with the symmetric set of generators for Γ_2 (genus $k = 2$).

Fix two cones with vertex in the identity $\mathbf{c}_0 = C(x_1, e)$ and $\mathbf{d}_0 = C(x_2, e)$ in \mathcal{G} . Choose vectors $v_1 \in V_{\mathbf{c}_0}$ and $v_2 \in V_{\mathbf{d}_0}$. Consider $f_1 = \mathbf{m}[x_1, e, v_1]$, $f_2 = \mathbf{m}[x_2, e, v_2]$ elementary multiplicative functions. We will assume that f_1, f_2 are these fixed functions for the rest of this Section.

Definition 3.36. Fixed a matrix system, let $\mathbf{c}_0 = C(x_1, e)$ and $\mathbf{d}_0 = C(x_2, e)$, $v_1 \in V_{\mathbf{c}_0}$, $v_2 \in V_{\mathbf{d}_0}$ and let $f_1 = \mathbf{m}[x_1, e, v_1]$, $f_2 = \mathbf{m}[x_2, e, v_2]$ be elementary multiplicative functions as above. For $\mathbf{c}, \mathbf{d} \in \mathcal{C}$ and $y \in \Gamma$ define the vector in $V_{\mathbf{c}} \otimes \bar{V}_{\mathbf{d}}$

$$S_{\mathbf{c}, \mathbf{d}, y}^{(n)} := \sum_{|z|=|zy|=n} f_1(z)_{\mathbf{c}} \otimes \bar{f}_2(zy)_{\mathbf{d}}. \quad (3.8)$$

Thus we have the vector (parametrized by $y \in \Gamma$)

$$S^{(n)} = \left(S_{\mathbf{c}, \mathbf{d}, y}^{(n)} \right)_{\mathbf{c}, \mathbf{d}, y} \in \bigoplus_{\mathbf{c}, \mathbf{d}, y} (V_{\mathbf{c}} \otimes \bar{V}_{\mathbf{d}})_y,$$

where we define for each y the space $(V_{\mathbf{c}} \otimes \bar{V}_{\mathbf{d}})_y$ as a copy of $V_{\mathbf{c}} \otimes \bar{V}_{\mathbf{d}}$ parametrized by the element y appearing in the triple $(\mathbf{c}, \mathbf{d}, y)$ and the sum ranges over all possible triples $(\mathbf{c}, \mathbf{d}, y)$ where \mathbf{c} and \mathbf{d} are cone types and $y \in \Gamma$.

Since we assume that f_1, f_2 are elementary multiplicative functions, $S^{(0)}$ has a unique non-zero component, the one corresponding to $(\mathbf{c}_0, \mathbf{d}_0, e)$, and

$$S_{\mathbf{c}_0, \mathbf{d}_0, e}^{(0)} = v_1 \otimes \bar{v}_2.$$

For $z, zy \in \Gamma$ with $|z| = |zy| = n$, the component $f_1(z)_{\mathbf{c}} \otimes \bar{f}_2(zy)_{\mathbf{d}}$ is non-zero only if $z \in \mathbf{c}_0$, $zy \in \mathbf{d}_0$, and $(\mathbf{c}, \mathbf{d}, y)$ satisfy: $z.\mathbf{c}$ is an n -th level subcone of \mathbf{c}_0 and $zy.\mathbf{d}$ is an n -th level subcone of \mathbf{d}_0 (cf. Notation 3.35).

Remark 3.37. We will see that the number of triples $(\mathbf{c}, \mathbf{d}, y)$ that actually matter for our computation is finite.

In particular, we are interested in the vectors:

$$\sum_{|z|=n} f_1(z)_{\mathbf{c}} \otimes \bar{f}_2(z)_{\mathbf{d}} = S_{\mathbf{c},\mathbf{d},e}^{(n)},$$

i.e. in the triples where $y = e$ (the identity in Γ). In fact, we can write:

$$\langle f_1, f_2 \rangle = \lim_{\epsilon \downarrow 0} \epsilon \sum_{n=M}^{\infty} \phi \left(R, \bigoplus_{\mathbf{c},\mathbf{d}} S_{\mathbf{c},\mathbf{d},e}^{(n)} \right) e^{-\epsilon n},$$

where ϕ is a sesquilinear form on $V \otimes \bar{V}$ as in Lemma 3.33.

Definition 3.38. Define the linear operator

$$\mathfrak{T}: \bigoplus_{\mathbf{c},\mathbf{d},y} (V_{\mathbf{c}} \otimes \bar{V}_{\mathbf{d}})_y \rightarrow \bigoplus_{\mathbf{c}',\mathbf{d}',y'} (V_{\mathbf{c}'} \otimes \bar{V}_{\mathbf{d}'})_{y'}$$

given by $\mathfrak{T} = \left(\mathfrak{T}_{\mathbf{c}',\mathbf{d}',y'}^{\mathbf{c},\mathbf{d},y} \right)$ where the components

$$\mathfrak{T}_{\mathbf{c}',\mathbf{d}',y'}^{\mathbf{c},\mathbf{d},y}: (V_{\mathbf{c}} \otimes \bar{V}_{\mathbf{d}})_y \rightarrow (V_{\mathbf{c}'} \otimes \bar{V}_{\mathbf{d}'})_{y'}$$

are defined as follows:

$$\mathfrak{T}_{\mathbf{c}',\mathbf{d}',y'}^{\mathbf{c},\mathbf{d},y} := \begin{cases} H_{\mathbf{c},\mathbf{c}',s} \otimes \bar{H}_{\mathbf{d},\mathbf{d}',t} & \text{if } \mathbf{c} \xrightarrow{s} \mathbf{c}' \text{ } \mathbf{d} \xrightarrow{t} \mathbf{d}' \text{ and } y' = s^{-1}yt, \\ 0 & \text{otherwise.} \end{cases} \quad (3.9)$$

There is a linear relation between $S^{(n+1)}$ and $S^{(n)}$ obtained through the operator \mathfrak{T} .

Lemma 3.39. *Assume we fixed a matrix system and f_1, f_2 elementary multiplicative functions supported on two cones $\mathbf{c}_0, \mathbf{d}_0$ with vertex in the identity. For $n \in \mathbb{N}$, if $S^{(n)}$ is as in Definition 3.36 and \mathfrak{T} is as in Definition 3.38, then it holds that $S^{(n+1)} = \mathfrak{T} S^{(n)}$.*

Proof. We have to prove that

$$S_{\mathbf{c}',\mathbf{d}',y'}^{(n+1)} = \sum_{(\mathbf{c},\mathbf{d},y)} \mathfrak{T}_{\mathbf{c}',\mathbf{d}',y'}^{\mathbf{c},\mathbf{d},y} S_{\mathbf{c},\mathbf{d},y}^{(n)}.$$

Write

$$S_{\mathbf{c}',\mathbf{d}',y'}^{(n+1)} = \sum_{|z'|=|z'y'|=n+1} f_1(z')_{\mathbf{c}'} \otimes \bar{f}_2(z'y')_{\mathbf{d}'}.$$

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Recall that

$$f_1(z')_{\mathbf{c}'} = \sum_{\mathbf{c}, s: \mathbf{c} \xrightarrow{s} \mathbf{c}'} H_{\mathbf{c}, \mathbf{c}', s} f(z's^{-1})_{\mathbf{c}},$$

where $z's^{-1}$ is a predecessor of z and $\mathbf{c} \in \mathcal{C}$, $s \in S$ are such that $\mathbf{c} \xrightarrow{s} \mathbf{c}'$ in the cone type automaton. Thus, if $|z'| = n+1$ and $\mathbf{c} \xrightarrow{s} \mathbf{c}'$, then $|z's^{-1}| = n$. Similarly,

$$f_2(z'y')_{\mathbf{d}'} = \sum_{\mathbf{d}, t: \mathbf{d} \xrightarrow{t} \mathbf{d}'} H_{\mathbf{d}, \mathbf{d}', t} f(z'y't^{-1})_{\mathbf{d}},$$

where $z'y't^{-1}$ is a predecessor of $z'y'$, (so that $|z'y't^{-1}| = n$) and $\mathbf{d} \in \mathcal{C}$, $t \in S$ are such that $\mathbf{d} \xrightarrow{t} \mathbf{d}'$ in the cone type automaton. Recall, moreover, that $\mathfrak{T}_{\mathbf{c}', \mathbf{d}', y'}^{\mathbf{c}, \mathbf{d}, y}$ is zero unless $\mathbf{c} \xrightarrow{s} \mathbf{c}'$, $\mathbf{d} \xrightarrow{t} \mathbf{d}'$ and $y' = s^{-1}yt$ (cf. equation 3.9): thus, when substituting \mathfrak{T} to the tensor product of matrices, we can sum over all $\mathbf{c} \in \mathcal{C}$ and $\mathbf{d} \in \mathcal{C}$. Hence:

$$\begin{aligned} S_{\mathbf{c}', \mathbf{d}', y'}^{(n+1)} &= \sum_{|z'|=|z'y'|=n+1} f_1(z')_{\mathbf{c}'} \otimes \bar{f}_2(z'y')_{\mathbf{d}'} \\ &= \sum_{|z'|=|z'y'|=n+1} \left(\sum_{\mathbf{c} \xrightarrow{s} \mathbf{c}'} H_{\mathbf{c}, \mathbf{c}', s} f_1(z's^{-1})_{\mathbf{c}} \right) \otimes \left(\sum_{\mathbf{d} \xrightarrow{t} \mathbf{d}'} \bar{H}_{\mathbf{d}, \mathbf{d}', t} \bar{f}_2(z'y't^{-1})_{\mathbf{d}} \right) \\ &= \sum_{|z'|=|z'y'|=n+1} \sum_{\mathbf{c} \xrightarrow{s} \mathbf{c}'} \sum_{\mathbf{d} \xrightarrow{t} \mathbf{d}'} (H_{\mathbf{c}, \mathbf{c}', s} \otimes \bar{H}_{\mathbf{d}, \mathbf{d}', t}) (f_1(z's^{-1})_{\mathbf{c}} \otimes \bar{f}_2(z'y't^{-1})_{\mathbf{d}}) \\ &= \sum_{|z'|=|z'y'|=n+1} \sum_{\mathbf{c}, \mathbf{d} \in \mathcal{C}} \mathfrak{T}_{\mathbf{c}', \mathbf{d}', y'}^{\mathbf{c}, \mathbf{d}, sy't^{-1}} (f_1(z's^{-1})_{\mathbf{c}} \otimes \bar{f}_2(z'y't^{-1})_{\mathbf{d}}). \end{aligned}$$

Thus, writing $z = z's^{-1}$ and $y = sy't^{-1}$, we get:

$$\begin{aligned} S_{\mathbf{c}', \mathbf{d}', y'}^{(n+1)} &= \sum_{|z'|=|z'y'|=n+1} \sum_{\mathbf{c}} \sum_{\mathbf{d}} \mathfrak{T}_{\mathbf{c}', \mathbf{d}', y'}^{\mathbf{c}, \mathbf{d}, sy't^{-1}} (f_1(z's^{-1})_{\mathbf{c}} \otimes \bar{f}_2(z'y't^{-1})_{\mathbf{d}}) \\ &= \sum_{|z|=|zy|=n} \sum_{(\mathbf{c}, \mathbf{d}, y)} \mathfrak{T}_{\mathbf{c}', \mathbf{d}', y'}^{\mathbf{c}, \mathbf{d}, y} (f_1(z)_{\mathbf{c}} \otimes \bar{f}_2(zy)_{\mathbf{d}}) \\ &= \sum_{(\mathbf{c}, \mathbf{d}, y)} \mathfrak{T}_{\mathbf{c}', \mathbf{d}', y'}^{\mathbf{c}, \mathbf{d}, y} \sum_{|z|=|zy|=n} (f_1(z)_{\mathbf{c}} \otimes \bar{f}_2(zy)_{\mathbf{d}}) \\ &= \sum_{(\mathbf{c}, \mathbf{d}, y)} \mathfrak{T}_{\mathbf{c}', \mathbf{d}', y'}^{\mathbf{c}, \mathbf{d}, y} S_{\mathbf{c}, \mathbf{d}, y}^{(n)}, \end{aligned}$$

since the sum over $(\mathbf{c}, \mathbf{d}, y)$ does not depend now on z, zy . This proves the thesis. \square

Lemma 3.40. Consider $f_1 = \mathbf{m}[x_1, e, v_1], f_2 = \mathbf{m}[x_2, e, v_2]$. Then, in the calculation $S_{\mathbf{c}, \mathbf{d}, e}^{(n)} = (\mathfrak{T}S^{(n-1)})_{\mathbf{c}, \mathbf{d}, e}$ only the indexes $(\mathbf{c}', \mathbf{d}', y')$ of \mathfrak{T} with $|y'| \leq 2k$ contribute to the sum.

Proof. We refer to Notation 3.35 for the meaning of the symbol $x.\mathbf{c}$, where $x \in \Gamma$ and \mathbf{c} is a cone type.

In the calculation of the vector

$$S_{\mathbf{c}, \mathbf{d}, e}^{(n)} = \sum_{|z|=n} f_1(z)_{\mathbf{c}} \otimes \bar{f}_2(z)_{\mathbf{d}}.$$

the value of $f_1(z)_{\mathbf{c}} \otimes \bar{f}_2(z)_{\mathbf{d}}$ (for $|z| = n$) depends on the values of f_j , $j = 1, 2$ on the points lying on two geodesics γ_1, γ_2 joining z and e , the common vertex of the supports of f_1, f_2 . Thus, a term

$$f_1(z')_{\mathbf{c}'} \otimes \bar{f}_2(z'y')_{\mathbf{d}'}$$

gives a non-zero contribution only if $z.\mathbf{c}$ appears as a subcone of $z'.\mathbf{c}'$ and similarly $z.\mathbf{d}$ appears as a subcone of $z'y'.\mathbf{d}'$. In fact, it must be that $z \in z'.\mathbf{c}' \cap z'y'.\mathbf{d}'$ and hence $z'.\mathbf{c}' \cap z'y'.\mathbf{d}' \neq \emptyset$, which yields $\mathbf{c}' \cap y'.\mathbf{d}' \neq \emptyset$.

Now, recall that both \mathbf{c}_0 and \mathbf{d}_0 (the supports of the multiplicative functions) have vertex in the identity: hence, if $z \in z'.\mathbf{c}' \cap z'y'.\mathbf{d}'$, then there exist two geodesics $\gamma_1 \in [e, z]$ and $\gamma_2 \in [e, z]$ and $i_0 \in \{1, \dots, \ell(\gamma_1)\}$ such that $z' = \gamma_1(i_0)$ and $z'y' = \gamma_2(i_0)$. Thus, by the properties of geodesic digons in \mathcal{G}_k , the element $y' \in \Gamma$ must satisfy $|y'| \leq 2k$ (cf. Lemma 1.24: this is essentially a version of hyperbolicity for Γ_k). \square

Remark 3.41. From now on we consider only the components $(\mathbf{c}', \mathbf{d}', y')$ in \mathfrak{T} which contribute to the sum $\bigoplus_{\mathbf{c}, \mathbf{d}} S_{\mathbf{c}, \mathbf{d}, e}^{(n)}$; we can imagine that we “throw away” all the unneeded components. Hence, we get a finite-dimensional matrix, where only some of the triples $(\mathbf{c}', \mathbf{d}', y')$ appear as indexes.

Fix $\mathbf{c}_0 = C(x_1, e)$, $\mathbf{d}_0 = C(x_2, e)$ cones with vertex in the identity, let $f_1 = \mathbf{m}[x_1, e, v_1]$ and $f_2 = \mathbf{m}[x_2, e, v_2]$ as above.

Definition 3.42. We call $(\mathbf{c}', \mathbf{d}', y')$ **compatible triple** if it corresponds to a nonzero contribution to $\bigoplus_{\mathbf{c}, \mathbf{d}} S_{\mathbf{c}, \mathbf{d}, e}^{(n)}$ (for some $n \in \mathbb{N}$): there is $n \in \mathbb{N}$ such that \mathbf{c}' is the cone type of an n -th level subcone of \mathbf{c}_0 , \mathbf{d}' is the cone type of an n -th level subcone of \mathbf{d}_0 , $|y'| \leq 2k$ and $\mathbf{c}' \cap y'.\mathbf{d}' \neq \emptyset$.

Remark 3.43. The compatible triples have been explicitly determined for the surface group of genus $k = 2$ and are reported in Appendix B.

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If we start from a scalar system $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$, then, by definition, each component of \mathfrak{T} is a non-negative real number. In particular, then, Perron-Frobenius Theory applies to \mathfrak{T} , if this is irreducible (cf. [Sen06]).

The following Definition is inspired by [KS04, Def. 4.4].

Definition 3.44. We say that a scalar system $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$ is **normalized** if the spectral radius of the matrix \mathfrak{T} associated with it as in Definition 3.38 is equal to 1.

The following Remark should be compared with the proof of Cor. 4.8 in [KS04].

Remark 3.45. We can always get a normalized system starting from a scalar system. In fact, since we consider the matrix \mathfrak{T} as a finite-dimensional matrix, by changing the non-zero components $H_{\mathbf{c}, \mathbf{c}', s}$ of \mathfrak{T} , which are scalars, into $\frac{1}{\sqrt{\rho}} H_{\mathbf{c}, \mathbf{c}', s}$, where ρ is the spectral radius of \mathfrak{T} , we get a new matrix \mathfrak{T}' with spectral radius equal to 1. Thus, a rescaling of the matrix system provides an irreducible system.

3.3.2 Proof of Theorem 3.30

Throughout this Section we fix the surface group Γ_2 with the standard symmetric set of generators and a normalized scalar system $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$: we prove that in this case the inner product in Def. 3.24 is well-defined. The crucial point is that for genus 2 a scalar system gives a transition matrix \mathfrak{T} which satisfies Perron-Frobenius Theorem: the spectral radius of the matrix is a simple eigenvalue.

Perron-Frobenius Theorem ([Sen06]). *Let A be a non-negative matrix. If A is irreducible, then the spectral radius of A is a simple eigenvalue for the matrix.*

The proof of Theorem 3.30 follows from this Lemma.

Lemma 3.46. *Fix the surface group Γ_2 with the standard symmetric set of generators. If $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$ is a scalar system, then the matrix \mathfrak{T} is irreducible.*

Proof. Define the matrix \mathfrak{D} (acting on the same space as \mathfrak{T}) as follows:

$$\mathfrak{D} = \left(\mathfrak{D}_{\mathbf{c}', \mathbf{d}', y'}^{\mathbf{c}, \mathbf{d}, y} \right),$$

$$\mathfrak{D}_{\mathbf{c}', \mathbf{d}', y'}^{\mathbf{c}, \mathbf{d}, y} := \begin{cases} 0 & \text{if } \mathfrak{T}_{\mathbf{c}', \mathbf{d}', y'}^{\mathbf{c}, \mathbf{d}, y} = 0 \\ 1 & \text{if } \mathfrak{T}_{\mathbf{c}', \mathbf{d}', y'}^{\mathbf{c}, \mathbf{d}, y} \neq 0 \end{cases}. \quad (3.10)$$

Then, \mathfrak{T} is irreducible if and only if \mathfrak{D} is irreducible (cf. [Sen06]).

The matrix \mathfrak{D} for the group of genus 2 has been explicitly determined using a computer: we implemented few scripts in PythonTM to find the compatible triples (which we previously studied in Appendix B); the components of \mathfrak{D} were found implementing the conditions in equation 3.9. The PythonTM scripts can be found in Appendix C.

The irreducibility condition for \mathfrak{D} has been tested using the software MATLAB (MathWorks[®]). We used the function `perron`¹ to calculate the Perron-Frobenius eigenvalue of the matrix, which turns out to be simple. \square

We recall the statement of Theorem 3.30.

Theorem (3.30). *Let Γ_2 be the surface group of genus 2 with the standard symmetric generator set. Let $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$ be a normalized scalar system. If $f_1 = \mathbf{m}[x_1, e, v_1]$ and $f_2 = \mathbf{m}[x_2, e, v_2]$ are elementary multiplicative functions supported on \mathbf{c}_0 and \mathbf{d}_0 (respectively) as above, then there exists an inner product B_0 on V such that the value of $\langle f_1, f_2 \rangle$ as in Def. 3.24 is not identically zero.*

We can now conclude the proof of the Theorem.

Proof. Fix a sesquilinear form ϕ on $V \otimes \bar{V}$. This provides an anti-linear isomorphism between $V \otimes \bar{V}$ and $(V \otimes \bar{V})^*$ and Lemma 3.33 applies.

Fix $M \in \mathbb{N}$ as in Lemma 3.31. Applying Lemma 3.39 consecutively for $n > M$,

$$\bigoplus_{\mathbf{c}, \mathbf{d}} S_{\mathbf{c}, \mathbf{d}, e}^{(n)} = \mathfrak{T} S^{(n-1)} = \mathfrak{T}^{n-M} S^{(M)}$$

where the matrix \mathfrak{T} is finite-dimensional (Lemma 3.40 and Remark 3.41); letting R be as in Lemma 3.33:

$$\begin{aligned} \langle f_1, f_2 \rangle &= \lim_{\epsilon \downarrow 0} \epsilon \sum_{n=M}^{\infty} \phi \left(R, \bigoplus_{\mathbf{c}, \mathbf{d}} S_{\mathbf{c}, \mathbf{d}, e}^{(n)} \right) e^{-\epsilon n} \\ &= \lim_{\epsilon \downarrow 0} \epsilon \sum_{n=M}^{\infty} \phi \left(R, \mathfrak{T}^{n-M} S^{(M)} \right) e^{-\epsilon n} \\ &= \lim_{\epsilon \downarrow 0} \epsilon \sum_{n=M}^{\infty} \phi \left((\mathfrak{T}^*)^{n-M} R, S^{(M)} \right) e^{-\epsilon n}, \end{aligned}$$

¹The script can be found at <https://it.mathworks.com/matlabcentral/fileexchange/22763-perron-root-computation>.

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where \mathfrak{T}^* is the adjoint matrix of \mathfrak{T} .

Since by Lemma 3.46 \mathfrak{T} satisfies Perron-Frobenius Theorem and since by assumption it has spectral radius equal to 1, there exists $R_0 \in V \otimes \bar{V}$ such that $\mathfrak{T}^* R_0 = R_0$ and R_0 is unique up to scaling with a positive constant (it is a simple eigenvector). It follows that $(\mathfrak{T}^*)^n R_0 = R_0$ for every $n \geq 0$.

Thus, for the definition of $\langle \cdot, \cdot \rangle$ we choose the positive-definite sesquilinear form B_0 corresponding to R_0 (under the anti-isomorphism given by ϕ). This yields:

$$\begin{aligned}
 \langle f_1, f_2 \rangle &= \lim_{\epsilon \downarrow 0} \epsilon \sum_{n=M}^{\infty} \phi \left((\mathfrak{T}^*)^{n-M} R_0, S^{(M)} \right) e^{-\epsilon n} \\
 &= \lim_{\epsilon \downarrow 0} \epsilon \sum_{n=M}^{\infty} \phi \left(R_0, S^{(M)} \right) e^{-\epsilon n} \\
 &= \phi \left(R_0, S^{(M)} \right) \lim_{\epsilon \downarrow 0} \epsilon \sum_{n=M}^{\infty} e^{-\epsilon n} \\
 &= \phi \left(R_0, S^{(M)} \right) \\
 &= \hat{B}_0 \left(S^{(M)} \right),
 \end{aligned}$$

where \hat{B}_0 is the linear form on $V \otimes \bar{V}$ associated to R_0 as in 3.32 and corresponding to the sesquilinear form B_0 chosen. We remark that

$$\lim_{\epsilon \downarrow 0} \epsilon \sum_{n=M}^{\infty} e^{-\epsilon n} = \lim_{\epsilon \downarrow 0} e^{-\epsilon M} \epsilon \sum_{n=0}^{\infty} e^{-\epsilon n} = \lim_{\epsilon \downarrow 0} e^{-\epsilon M} \frac{\epsilon}{1 - e^{-\epsilon}} = 1.$$

Since the value of $\hat{B}_0 \left(S^{(M)} \right)$ can be non-zero (it depends on the multiplicative functions), the definition of equation 3.4 is not identically zero. \square

Conjecture 3.47. Let Γ be a surface group of genus $k > 2$, fix a normalized scalar system associated to the standard set of generators. We believe that the matrix \mathfrak{T} is irreducible in this case, too.

- It seems that an explicit determination of the compatible triples for this case still can be done.
- We know no abstract way to show that the matrix \mathfrak{T} is irreducible, i.e. that for every couple of compatible triples $(\mathbf{c}, \mathbf{d}, y)$ and $(\mathbf{c}', \mathbf{d}', y')$ we can find w, u geodesic words of the same length giving transitions $\mathbf{c} \xrightarrow{w} \mathbf{c}'$ and $\mathbf{d} \xrightarrow{u} \mathbf{d}'$ in the CTA so that at each intermediate step $(\mathbf{c}_i, \mathbf{d}_i, s_i^{-1} y_i t_i)$ is a compatible triple. This would ensure the irreducibility of the matrix \mathfrak{T} .

3.4 MULTIPLICATIVE REPRESENTATIONS OF A SURFACE GROUP

Given a topological group G , a unitary representation π of G on a Hilbert space \mathcal{H} is a group homomorphism $\pi: G \rightarrow \mathfrak{U}(\mathcal{H})$, where $\mathfrak{U}(\mathcal{H})$ is the group of unitary operators on \mathcal{H} , such that for each $\xi \in \mathcal{H}$ the map $g \in G \mapsto \pi(g)\xi \in \mathcal{H}$ is norm continuous.

Fix now $\Gamma = \Gamma_2$ the surface group of genus 2 with the standard symmetric set of generators S . Let $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$ be a normalized scalar system associated to S . Then we get the space of multiplicative functions $\mathcal{H}^\infty = \mathcal{H}^\infty(V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s})$. We proved in Section 3.3.2 that there exists a positive-definite sesquilinear form B_0 on $V \otimes \bar{V}$ such that the inner product 3.4 is well-defined. Complete the space \mathcal{H}^∞ with respect to the inner product (3.4) to get the Hilbert space $\mathcal{H}_m = \mathcal{H}_m(V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s})$.

Denote by $\mathfrak{U}(\mathcal{H}_m)$ the unitary operators on the Hilbert space \mathcal{H}_m .

We define now the **multiplicative representation** of Γ on the space \mathcal{H}_m

$$\pi_m: \Gamma \rightarrow \mathfrak{U}(\mathcal{H}_m).$$

Let $g \in \Gamma$. The operator $\pi_m(g): \mathcal{H}^\infty \rightarrow \mathcal{H}^\infty$ acts on $f \in \mathcal{H}^\infty$ by left translation:

$$(\pi_m(g)f)(z) := f(g^{-1}z), \quad z \in \Gamma.$$

Lemma 3.48. *For $g \in \Gamma$ the operator $\pi_m(g)$ defined above satisfies*

$$\langle \pi_m(g)f, \pi_m(g)f \rangle = \langle f, f \rangle$$

for every $f \in \mathcal{H}^\infty$.

Proof. Fix $g \in \Gamma$ and $f \in \mathcal{H}^\infty$. Then

$$\begin{aligned} \langle \pi_m(g)f, \pi_m(g)f \rangle &= \lim_{\epsilon \downarrow 0} \epsilon \sum_{x \in \Gamma} B(\pi_m(g)f(x), \pi_m(g)f(x)) e^{-\epsilon|x|} \\ &= \lim_{\epsilon \downarrow 0} \epsilon \sum_{x \in \Gamma} B(f(g^{-1}x), f(g^{-1}x)) e^{-\epsilon|x|} \end{aligned}$$

Now, we change $z = g^{-1}x$, obtaining

$$\langle \pi_m(g)f, \pi_m(g)f \rangle = \lim_{\epsilon \downarrow 0} \epsilon \sum_{z \in \Gamma} B(f(z), f(z)) e^{-\epsilon|gz|},$$

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and we notice that, since $-|g| + |z| \leq |gz| \leq |g| + |z|$,

$$e^{-\epsilon|g|}e^{-\epsilon|z|} \leq e^{-\epsilon|gz|} \leq e^{\epsilon|g|}e^{-\epsilon|z|}.$$

We remark that $B(f(x), f(x))$ is a non-negative real number: thus

$$\langle \pi_m(g)f, \pi_m(g)f \rangle \geq \lim_{\epsilon \downarrow 0} e^{-\epsilon|g|} \epsilon \sum_{z \in \Gamma} B(f(z), f(z)) e^{-\epsilon|z|} = \langle f, f \rangle$$

and

$$\langle \pi_m(g)f, \pi_m(g)f \rangle \leq \lim_{\epsilon \downarrow 0} e^{\epsilon|g|} \epsilon \sum_{z \in \Gamma} B(f(z), f(z)) e^{-\epsilon|z|} = \langle f, f \rangle.$$

This yields that $\langle \pi_m(g)f, \pi_m(g)f \rangle = \langle f, f \rangle$ for $f \in \mathcal{H}^\infty$. □

Proposition 3.49. *The map $\pi_m: \Gamma \rightarrow \mathfrak{U}(\mathcal{H}_m)$ is a unitary representation of Γ on the Hilbert space \mathcal{H}_m .*

Proof. This is an immediate consequence of Lemma 3.48. By a standard continuity argument, for $g \in \Gamma$ fixed, we can extend the operator $\pi_m(g)$ to the Hilbert space \mathcal{H}_m and the extension satisfies 3.48. Thus, we get a strongly continuous group homomorphism $\pi: \Gamma \rightarrow \mathfrak{U}(\mathcal{H}_m)$. □

3.5 TEMPEREDNESS

In this Section we prove that the representations defined in Section 3.4 are tempered, i.e. weakly contained in the regular representation of Γ .

We recall the basic definitions and properties about weak containment of unitary representations; we refer to the good presentation of these concepts given by Appendix F of [BdLHV08].

We recall that a **matrix coefficient** of a unitary representation π of a discrete group G on a Hilbert space \mathcal{H} is a function

$$g \in G \mapsto \langle \pi(g)\varphi, \psi \rangle \in \mathbb{C},$$

where $\varphi, \psi \in \mathcal{H}$ are fixed vectors. If we pick $\varphi \in \mathcal{H}$, then the matrix coefficient

$$g \mapsto \langle \pi(g)\varphi, \varphi \rangle$$

is called a function of positive type associated to π .

3.5.1 Weak containment

The following Definition is taken from Appendix F in [BdLHV08].

Definition 3.50. Let G be a discrete group and let π_1, π_2 be unitary representations of G on the Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. We say that π_1 is **weakly contained** in π_2 and we write $\pi_1 \preceq \pi_2$ if for all $\varphi \in \mathcal{H}_1$, for all $E \subseteq G$ finite subset and for all $\epsilon > 0$ there exist $\psi_1, \dots, \psi_n \in \mathcal{H}_2$ such that for all $g \in E$

$$\left| \langle \pi_1(g)\varphi, \varphi \rangle - \sum_{i=1}^n \langle \pi_2(g)\psi_i, \psi_i \rangle \right| < \epsilon.$$

Remark 3.51. Definition 3.50 means that all functions of positive type associated to π_1 can be approximated uniformly on the finite subsets of G by finite sums of functions of positive type associated to π_2 . Cf. Def. F.1.1 in [BdLHV08].

It is known that the amenability of G is equivalent to the weak containment $1_G \preceq \lambda$, where $1_G: G \rightarrow \mathbb{C}$ is the trivial representation $1_G(g) = 1$ for all $g \in G$ and $\lambda: G \rightarrow \mathfrak{U}(\ell^2(G))$ is the regular representation $\lambda(g)f(h) = f(g^{-1}h)$. In fact, the weak containment $1_G \preceq \lambda$ means precisely the existence of almost invariant vectors for λ .

Remark 3.52. A unitary representation $\pi: G \rightarrow \mathfrak{U}(\mathcal{H})$ of a discrete group G can be extended to a representation of the group algebra $\mathbb{C}[G]$: if $f = \sum_{i=1}^n \alpha_i x_i \in \mathbb{C}[G]$ with $\alpha_i \in \mathbb{C}$ and $x_i \in G$ for $i = 1, \dots, n$, then $\pi(f) = \sum_{i=1}^n \alpha_i \pi(x_i)$ is a bounded operator in $\mathfrak{B}(\mathcal{H})$; there is, moreover, an extension to the algebra $\ell^1(G)$ (cf. [Dav96, Chap. VII]).

The weak containment $\pi_1 \preceq \pi_2$ is equivalent to the following inequality (in operator norm):

$$\|\pi_1(f)\| \leq \|\pi_2(f)\|,$$

for all $f \in \mathbb{C}[G]$ or $f \in \ell^1(G)$.

Lemma 3.53. *Let π_1, π_2 be unitary representations of a discrete group G on Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2$. If for all $g \in G$ and all $\varphi \in \mathcal{H}_1$ there exists a sequence $\{\psi_n\}_n \subseteq \mathcal{H}_2$ such that*

$$\lim_{n \rightarrow \infty} \langle \pi_2(g)\psi_n, \psi_n \rangle = \langle \pi_1(g)\varphi, \varphi \rangle,$$

then $\pi_1 \preceq \pi_2$.

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Proof. The statement of the Lemma implies Definition 3.50.

Fix $\varphi \in \mathcal{H}_1$. Then, by the hypothesis, for all $g \in G$ there exists a sequence $\{\psi_n\} \subseteq \mathcal{H}_2$ such that for all $\epsilon > 0$ there exists n_0 such that

$$|\langle \pi_1(g)\varphi, \varphi \rangle - \langle \pi_2(g)\psi_n, \psi_n \rangle| < \epsilon, \quad n \geq n_0.$$

In particular, this holds uniformly for every fixed finite subset of G . Therefore Definition 3.50 applies. \square

Definition 3.54. We say that a unitary representation π of a discrete group G is **tempered** if $\pi \preceq \lambda$, where λ is the (left) regular representation of G .

Remark 3.55. When G is a discrete group, it is well-known that every function in $\ell^2(G)$ is a matrix coefficient of the regular representation of G . In fact, for every $f \in \ell^2(G)$ and every $z \in G$ we have $f(z) = \langle \lambda(z)\delta_e, f \rangle_2$.

3.5.2 Left-regular representation

Let G be a discrete group and V a finite-dimensional vector space with an inner product $B: V \times V \rightarrow \mathbb{C}$. Recall that for a function $h: G \rightarrow V$ we define the norm $\|\cdot\|_2$ as

$$\|h\|_2^2 := \sum_{z \in G} B(h(z), h(z)).$$

The space $\ell^2(G, V)$ is formed by all (equivalence classes of) functions $h: G \rightarrow V$ such that $\|h\|_2 < \infty$. We have $\ell^2(G, V) \simeq \ell^2(G) \otimes V$.

A function $f \in \ell^2(G, V)$ is a limit of finite linear combinations $\sum_i \varphi_i \otimes v_i$, where $\varphi_i \in \ell^2(G)$ are square-summable functions taking values in \mathbb{C} and $v_i \in V$ are vectors.

Consider the unitary representation $\lambda_V: G \rightarrow \mathfrak{U}(\ell^2(G) \otimes V)$ acting as follows: for $g \in G$ and $\varphi \in \ell^2(G)$, $v \in V$

$$\lambda_V(g)(\varphi \otimes v) := (\lambda(g)\varphi) \otimes v;$$

where λ refers to the left-regular representation of G on $\ell^2(G)$.

Remark 3.56. The left-regular representation on $\ell^2(G) \otimes V$ is weakly equivalent to a sum of copies of the regular representation on $\ell^2(G)$ (which, in our case, where $\dim_{\mathbb{C}} V < \infty$, consists of a finite number of terms). See for more details the proof of the main result in [Kuh94]. In particular, then, if a unitary representation ρ of a discrete group G is weakly contained in the representation $\ell^2(G) \otimes V$, then it is also weakly contained in the regular representation $\ell^2(G)$ (cf. [BdLHV08, App. F]).

3.5.3 Square-summable functions

Let $\Gamma = \Gamma_2$ be the surface group of genus 2 and S the standard symmetric set of generators. Fix a normalized scalar system system $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$ and let $V = \bigoplus_{\mathbf{c}} V_{\mathbf{c}}$ as in Definition 3.4; fix $\epsilon > 0$ and a multiplicative function $\mathbf{m} \in \mathcal{H}^\infty(V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s})$.

Definition 3.57. Define the function $\mathbf{m}_\epsilon: \Gamma \rightarrow V$ by:

$$\mathbf{m}_\epsilon(z) := \sqrt{\epsilon} e^{-\frac{\epsilon}{2}|z|} \mathbf{m}(z), \quad \text{for } z \in \Gamma. \quad (3.11)$$

Lemma 3.58. *For a given matrix system, a given $\epsilon > 0$ and a fixed multiplicative function $\mathbf{m} \in \mathcal{H}^\infty$, the function \mathbf{m}_ϵ as in Definition 3.57 belongs to $\ell^2(\Gamma, V)$.*

Proof. Using the sesquilinearity of B :

$$\begin{aligned} \|\mathbf{m}_\epsilon\|_2^2 &= \sum_{z \in \Gamma} B(\mathbf{m}_\epsilon(z), \mathbf{m}_\epsilon(z)) \\ &= \sum_{z \in \Gamma} B\left(\sqrt{\epsilon} e^{-\frac{\epsilon}{2}|z|} \mathbf{m}(z), \sqrt{\epsilon} e^{-\frac{\epsilon}{2}|z|} \mathbf{m}(z)\right) \\ &= \epsilon \sum_{z \in \Gamma} B(\mathbf{m}(z), \mathbf{m}(z)) e^{-\epsilon|z|}. \end{aligned}$$

If $\mathbf{m} \in \mathcal{H}^\infty$ (and $\mathbf{m} \neq 0$), then (by Thm. 3.30) $0 < \langle \mathbf{m}, \mathbf{m} \rangle_{\mathcal{H}^\infty} < \infty$, i.e.,

$$0 < \lim_{\epsilon \rightarrow 0^+} \epsilon \sum_{z \in \Gamma} B(\mathbf{m}(z), \mathbf{m}(z)) e^{-\epsilon|z|} < \infty.$$

In particular, the infinite sum over Γ converges. For a fixed $\epsilon > 0$ we have

$$0 < \epsilon \sum_{z \in \Gamma} B(\mathbf{m}(z), \mathbf{m}(z)) e^{-\epsilon|z|} < \infty$$

and this quantity is bounded if we vary ϵ . Thus, for a specific ϵ it holds $\|\mathbf{m}_\epsilon\|_2 < \infty$. \square

3.5.4 Temperedness of multiplicative representations

Let $\Gamma = \Gamma_2$ be the surface group of genus 2 and S the standard symmetric set of generators. Fix a normalized scalar system system $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$.

Fix $\mathbf{m} \in \mathcal{H}^\infty$. We now approximate the functions of positive type $\langle \pi_{\mathbf{m}}(g)\mathbf{m}, \mathbf{m} \rangle_{\mathcal{H}^\infty}$ associated to the multiplicative representation using the

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functions of positive type of the left-regular representation on $\ell^2\Gamma \otimes V$. This suffices to prove the weak containment of Γ in the regular representation (cf. Lemma 3.53 and Remark 3.56).

We get a sequence of functions of positive type associated to λ_V converging to the function of positive type associated to π_m . The thesis will follow from Lemma 3.53.

Theorem 3.59. *Let $\Gamma = \Gamma_2$ be the surface group of genus 2 and S the standard symmetric set of generators. Fix a normalized scalar system $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$, define the multiplicative representation π_m on \mathcal{H}_m . Then π_m is weakly contained in the regular representation λ of Γ .*

Proof. Consider the functions $\mathbf{m}_\epsilon(z) := \sqrt{\epsilon} e^{-\frac{\epsilon}{2}|z|} \mathbf{m}(z)$ as in eq. 3.11, with fixed $\epsilon > 0$ and $\mathbf{m} \in \mathcal{H}^\infty$. Compute for $g \in \Gamma$

$$\begin{aligned} \langle \lambda_V(g) \mathbf{m}_\epsilon, \mathbf{m}_\epsilon \rangle_{\ell^2\Gamma \otimes V} &= \sum_{z \in \Gamma} B(\lambda_V(g) \mathbf{m}_\epsilon(z), \mathbf{m}_\epsilon(z)) \\ &= \sum_{z \in \Gamma} B(\mathbf{m}_\epsilon(g^{-1}z), \mathbf{m}_\epsilon(z)) \\ &= \sum_{z \in \Gamma} \epsilon e^{-\frac{\epsilon}{2}(|g^{-1}z| + |z|)} B(\mathbf{m}(g^{-1}z), \mathbf{m}(z)). \end{aligned}$$

Notice that

$$e^{-\epsilon(|g| + |z|)} \leq e^{-\epsilon|gz|} \leq e^{\epsilon(|g| - |z|)}. \quad (3.12)$$

We can write the series

$$\sum_{z \in \Gamma} \epsilon e^{-\frac{\epsilon}{2}(|g^{-1}z| + |z|)} B(\mathbf{m}(g^{-1}z), \mathbf{m}(z))$$

as the sum of its real and imaginary parts: in fact, $B(\mathbf{m}(g^{-1}z), \mathbf{m}(z)) \in \mathbb{C}$. Then, the convergence of the complex series is equivalent to the convergence of both its real and its imaginary part. Thus, if

$$\begin{aligned} \Re(B(\mathbf{m}(g^{-1}z), \mathbf{m}(z))) &= r_{g,z} = r_{g,z}^+ - r_{g,z}^-, \\ \Im(B(\mathbf{m}(g^{-1}z), \mathbf{m}(z))) &= m_{g,z} = m_{g,z}^+ - m_{g,z}^-, \end{aligned}$$

then

$$B(\mathbf{m}(g^{-1}z), \mathbf{m}(z)) = r_{g,z}^+ - r_{g,z}^- + i(m_{g,z}^+ - m_{g,z}^-).$$

By the sake of a simpler notation, in what follows we drop the subscript g and simply write r_z, m_z instead of $r_{g,z}, m_{g,z}$ (similarly for the other symbols). Now the series

$$\sum_{z \in \Gamma} \epsilon e^{-\frac{\epsilon}{2}(|g^{-1}z| + |z|)} r_z^+$$

has real and positive terms; the same holds for the series with r_z^-, m_z^+, m_z^- instead of r_z^+ .

By equation 3.12:

$$\sum_{z \in \Gamma} \epsilon e^{-\frac{\epsilon}{2}(|g^{-1}z|+|z|)} r_z^+ \leq \epsilon e^{\frac{\epsilon}{2}|g|} \sum_{z \in \Gamma} r_z^+ e^{-\epsilon|z|},$$

and similarly

$$\sum_{z \in \Gamma} \epsilon e^{-\frac{\epsilon}{2}(|g^{-1}z|+|z|)} r_z^+ \geq \epsilon e^{-\frac{\epsilon}{2}|g|} \sum_{z \in \Gamma} r_z^+ e^{-\epsilon|z|}.$$

Thus:

$$\lim_{\epsilon \rightarrow 0^+} \epsilon \sum_{z \in \Gamma} e^{-\frac{\epsilon}{2}(|g^{-1}z|+|z|)} r_z^+ = \lim_{\epsilon \rightarrow 0^+} \epsilon \sum_{z \in \Gamma} e^{-\epsilon|z|} r_z^+.$$

Analogous equalities hold for r_z^-, m_z^+, m_z^- .

Recall that:

$$\begin{aligned} \langle \pi_{\mathbf{m}}(g)\mathbf{m}, \mathbf{m} \rangle_{\mathcal{H}^\infty} &= \lim_{\epsilon \rightarrow 0^+} \epsilon \sum_{z \in \Gamma} B(\mathbf{m}(g^{-1}z), \mathbf{m}(z)) e^{-\epsilon|z|} \\ &= \lim_{\epsilon \rightarrow 0^+} \epsilon \sum_{z \in \Gamma} (r_z^+ - r_z^- + i(m_z^+ - m_z^-)) e^{-\epsilon|z|} \end{aligned}$$

converges (Theorem 3.30).

Observing that $\lim_{\epsilon \rightarrow 0^+} e^{-\frac{\epsilon}{2}|g|} = \lim_{\epsilon \rightarrow 0^+} e^{\frac{\epsilon}{2}|g|} = 1$ for $g \in \Gamma$ fixed, we get:

$$\lim_{\epsilon \rightarrow 0^+} \epsilon e^{\epsilon|g|} \sum_{z \in \Gamma} B(\mathbf{m}(g^{-1}z), \mathbf{m}(z)) e^{-\epsilon|z|} = \langle \pi_{\mathbf{m}}(g)\mathbf{m}, \mathbf{m} \rangle_{\mathcal{H}^\infty};$$

similarly

$$\lim_{\epsilon \rightarrow 0^+} \epsilon e^{-\epsilon|g|} \sum_{z \in \Gamma} B(\mathbf{m}(g^{-1}z), \mathbf{m}(z)) e^{-\epsilon|z|} = \langle \pi_{\mathbf{m}}(g)\mathbf{m}, \mathbf{m} \rangle_{\mathcal{H}^\infty},$$

which yields

$$\lim_{\epsilon \rightarrow 0^+} \langle \lambda_V(g)\mathbf{m}_\epsilon, \mathbf{m}_\epsilon \rangle_{\ell^2 \Gamma \otimes V} = \langle \pi_{\mathbf{m}}(g)\mathbf{m}, \mathbf{m} \rangle_{\mathcal{H}^\infty}.$$

By Lemma 3.53 and Remark 3.56 follows the thesis. \square

CHAPTER 4

BOUNDARY REPRESENTATIONS

In the present Chapter, after recalling the notion of covariant system, crossed product C^* -algebra and some of their properties (cf. [Dav96] and [Dix77]), we introduce a class of boundary representations of the surface group Γ_2 whose unitary part is equivalent to a multiplicative representation defined as in Chapter 3.

We briefly recall in Section 4.1.3 a general construction for boundary representations and a result, Theorem 4.6 (cf. [IKS13]), which guarantees that every tempered unitary representation of a hyperbolic group can be extended to a boundary representation.

The methods used to prove Theorem 4.6 are quite abstract and they do not provide clues on how to show the irreducibility of the boundary representation if the irreducibility of the tempered representation is not previously known (as it happens in our case). Therefore, in Section 4.2 and following we explicitly determine a boundary representation whose unitary representation is equivalent to π_m as defined in Chapter 3 (the equivalence is proved in Section 4.5). The method is based on taking a limit of a family of representations obtained from the regular one, as introduced in Section 4.2.2; the limit representation, whose existence is guaranteed by a compactness argument, gives a representation of $\Gamma \rtimes C(\partial\Gamma)$ (Section 4.3); finally, the uniqueness of the limit is proved in Section 4.4.

4.1 CROSSED PRODUCTS

Let \mathcal{H} be a Hilbert space. We denote with $\mathfrak{B}(\mathcal{H})$ the algebra of all bounded linear operators on \mathcal{H} , while $\mathfrak{U}(\mathcal{H})$ indicates the group of unitary operators on \mathcal{H} .

Let \mathcal{A} be a C^* -algebra. A C^* -representation (or $*$ -representation) of \mathcal{A} on a Hilbert space \mathcal{H} is an algebra homomorphism $\rho: \mathcal{A} \rightarrow \mathfrak{B}(\mathcal{H})$ which preserves the adjoint operation (cf. [Con13, Def. 5.1, Chap. VIII]). If \mathcal{A} has identity $\mathbf{1}$, then we assume that $\rho(\mathbf{1}) = I$, the identity operator on \mathcal{H} . A cyclic vector for the representation is an element $\xi \in \mathcal{H}$ such that $\rho(\mathcal{A})\xi$ is dense in \mathcal{H} .

By Gelfand-Naimark-Segal construction ([Con13, Thm. 5.14, Chap. VIII]), every representation of \mathcal{A} with a distinguished cyclic vector corresponds to a positive linear functional $\mathcal{A} \rightarrow \mathbb{C}$, and vice versa.

4.1.1 Crossed product C^* -algebras

The main reference for this Section is [Dav96, Chap. VIII]. We restrict to the class of discrete groups, i.e. groups endowed with the discrete topology. Surface groups, in particular, are discrete groups and countable.

Given a discrete group G , a C^* -algebra \mathcal{A} and a continuous action of G on \mathcal{A} by automorphisms (the action is implicit and indicated by the dot notation, $g.a$ for $g \in G$ and $a \in \mathcal{A}$), we call (G, \mathcal{A}) a **covariant system** or a **C^* -dynamical system** (cf. [Dav96, Chap. VIII]).

Definition 4.1 ([Dav96]). A **covariant representation** of a covariant system (G, \mathcal{A}) is the datum of (π, ρ, \mathcal{H}) where

- \mathcal{H} is a Hilbert space;
- $\pi: G \rightarrow \mathfrak{U}(\mathcal{H})$ is a unitary representation of G on \mathcal{H} ;
- $\rho: \mathcal{A} \rightarrow \mathfrak{B}(\mathcal{H})$ is a C^* -representation of \mathcal{A} on \mathcal{H} ;
- $\pi(g)\rho(a)\pi(g^{-1}) = \rho(g.a)$ for all $g \in G$ and all $a \in \mathcal{A}$.

Let (G, \mathcal{A}) be a covariant system. Consider the space of compactly supported functions on G with values in \mathcal{A} , denoted by $C_c(G, \mathcal{A})$. Since G is discrete, if $F \in C_c(G, \mathcal{A})$, then there exists $n \in \mathbb{N}$, $g_1, \dots, g_n \in G$ and $a_1, \dots, a_n \in \mathcal{A}$ such that $F = \sum_{i=1}^n a_i \delta_{g_i}$, where δ_g is the Dirac function at $g \in G$.

4. BOUNDARY REPRESENTATIONS

The space $C_c(G, \mathcal{A})$ has a structure of involutive algebra defined as follows: the sum of two elements in $C_c(G, \mathcal{A})$ is given by the sum of two \mathcal{A} -valued functions, the product is obtained by extending the law

$$a_1 \delta_{g_1} \cdot a_2 \delta_{g_2} = (a_1 (g_1 \cdot a_2)) \delta_{g_1 g_2}.$$

Finally, we get an involution by the linear extension of the formula

$$(a \delta_g)^* = (g^{-1} \cdot a^*) \delta_{g^{-1}}.$$

Let (π, ρ, \mathcal{H}) be a covariant representation of (G, \mathcal{A}, α) and $F = \sum_{i=1}^n a_i \delta_{g_i}$. Define the operator in $\mathfrak{B}(\mathcal{H})$

$$(\pi \times \rho)(F) := \sum_{i=1}^n \rho(a_i) \pi(g_i).$$

For $F \in C_c(G, \mathcal{A})$ the universal norm is defined as

$$\|F\|_u := \sup_{(\pi, \rho, \mathcal{H})} \|(\pi \times \rho)(F)\|_{\mathcal{H}},$$

where the supremum is taken over all covariant representations (π, ρ, \mathcal{H}) of (G, \mathcal{A}) . The **full crossed product C*-algebra** $G \rtimes \mathcal{A}$ is the completion of $C_c(G, \mathcal{A})$ with respect to the universal norm.

Consider now a C*-representation ρ of \mathcal{A} on a Hilbert space \mathcal{H} and define a covariant representation $(\tilde{\lambda}, \tilde{\rho})$ of (G, \mathcal{A}) on $\ell^2(G) \otimes \mathcal{H}$ as follows:

$$\begin{aligned} (\tilde{\rho}(a)\varphi)(x) &= \rho(x^{-1} \cdot a)\varphi(x), \\ (\tilde{\lambda}(g)\varphi)(x) &= \varphi(g^{-1}x), \end{aligned}$$

for $a \in \mathcal{A}$, $g, x \in G$ and $\varphi \in \mathcal{H} \otimes \ell^2(G)$. This is indeed a covariant representation of the covariant system (G, \mathcal{A}) . For every $F \in C_c(G, \mathcal{A})$ the reduced norm is defined as

$$\|F\|_r := \sup_{(\rho, \mathcal{H})} \left\| (\tilde{\lambda} \times \tilde{\rho})(F) \right\|_{\mathcal{H} \otimes \ell^2(G)},$$

where the supremum is taken over all C*-representation ρ of \mathcal{A} . The **reduced crossed product C*-algebra** $G \rtimes_{\text{red}} \mathcal{A}$ is the completion of $C_c(G, \mathcal{A})$ with respect to the reduced norm.

As a special case, when $\mathcal{A} = \mathbb{C}$ we get the **group C*-algebra** $C^*(G) = G \rtimes \mathbb{C}$ and the **reduced group C*-algebra** $C^*_{\text{red}}(G) = G \rtimes_{\text{red}} \mathbb{C}$.

It is well known that the representations of the crossed product $G \rtimes \mathcal{A}$ are in one-to-one correspondence with the representations of the covariant system (G, \mathcal{A}) (cf. [Dav96, Chap. VIII]). Denoting (π, ρ, \mathcal{H}) such a covariant representation, we get a unique C^* -representation $\pi \rtimes \rho$ of $G \rtimes \mathcal{A}$ on \mathcal{H} .

Given a compact Hausdorff space X and a group action $G \curvearrowright X$ denoted by $G \times X \ni (g, \omega) \mapsto g\omega \in X$, there is a natural induced action of G on $C(X)$ by left translations: if $f \in C(X)$, then $g.f(\omega) = f(g^{-1}\omega)$ (for $g \in G$ and $\omega \in X$).

Let G be a hyperbolic group, let ∂G be its Gromov boundary: it is a compact Hausdorff space. Hence G acts on $C(\partial G)$ by left translations and $(G, C(\partial G))$ is a covariant system.

Definition 4.2 ([IKS13]). A **boundary representation** is a covariant representation of the covariant system $(G, C(\partial G))$, i.e. a pair

$$\begin{aligned} \pi: G &\rightarrow \mathfrak{U}(\mathcal{H}) \quad (\text{unitary representation of } G), \\ \rho: C(\partial G) &\rightarrow \mathfrak{B}(\mathcal{H}) \quad (C^*\text{-representation of } C(\partial G)), \end{aligned}$$

both acting on the same Hilbert space \mathcal{H} and satisfying:

$$\pi(g)\rho(f)\pi(g^{-1}) = \rho(g.f), \quad \text{for all } g \in G, f \in C(\partial G).$$

4.1.2 Amenable actions

This Section refers mainly to [AD02], [AD03] and [Zim13]. An important result is also contained in [Ada94].

Let G be a hyperbolic group, let ∂G be its boundary. There is a notion of *topological amenability* for an action of a group on a locally compact space and a notion of *measurable amenability* of an action on a measured space *in Zimmer's sense*.

Adams showed that, given a hyperbolic group G , for every quasi-invariant measure μ on ∂G the group acts amenably in Zimmer's sense on the measured space $(\partial G, \mu)$ ([Ada94]).

Theorem 4.3 ([Ada94]). *Let G be a hyperbolic group, let S be a finite set of generators. Let ∂G be the boundary of G . If μ is any finite Borel measure on ∂G which is quasi-invariant with respect to the action $G \curvearrowright \partial G$, then $G \curvearrowright (\partial G, \mu)$ is an amenable action.*

In [ADR00, Theorem 3.3.7] it is shown that this is equivalent to the topological amenability of the action $G \curvearrowright \partial G$.

The main point of interest for us lies in the theory of crossed product C^* -algebras: in fact, when a group G acts amenably on a locally compact space X , the reduced crossed product $G \rtimes_{\text{red}} C_0(X)$ is equal to the correspondent full crossed product.

Theorem 4.4 ([AD02]). *Let (G, X, μ) be a measured G -space which is amenable in Zimmer's sense. Then*

$$G \rtimes_{\text{red}} C_0(X) = G \rtimes C_0(X).$$

In particular, if Γ is a surface group, then the action $\Gamma \curvearrowright \partial\Gamma$ is topologically amenable, so, by Theorem 4.4, the two crossed products coincide: $\Gamma \rtimes_{\text{red}} C(\partial\Gamma) = \Gamma \rtimes C(\partial\Gamma)$.

4.1.3 Cocycle representations

We recall in this Section some material that can be found in [Tak13] or also in [AD03]. We refer moreover to [IKS13, Sect. 4].

Let G be a hyperbolic group, let ∂G be its boundary. There is a universal construction for representations of the crossed product C^* -algebra $G \rtimes C(\partial G)$ given by the so-called cocycle representations (cf. [Tak13], [AD03] or also [IKS13, Sect. 4]). Fix the following:

- a standard Borel space X with a positive Borel measure μ ;
- an action $G \curvearrowright X$ which preserves the measure class of μ (we say that μ is G -quasi-invariant);
- a Borel field of Hilbert spaces on X : $\omega \mapsto \mathcal{H}_\omega$, $\omega \in X$.

Denote by

$$P(\omega, g) := \frac{d\mu(g^{-1}\omega)}{d\mu(\omega)}$$

the Radon-Nikodym cocycle of the action $G \curvearrowright X$ and denote by $\text{Iso}(\mathcal{H}_{\omega_1}, \mathcal{H}_{\omega_2})$ the space of all isometries $\mathcal{H}_{\omega_1} \rightarrow \mathcal{H}_{\omega_2}$ for $\omega_1, \omega_2 \in X$. Let

$$\mathcal{H} := \int_X^\oplus \mathcal{H}_\omega d\mu(\omega)$$

be the direct integral of the field of Hilbert spaces.

A **unitary Borel cocycle** is a map

$$A: X \times G \ni (\omega, g) \mapsto A(\omega, g) \in \text{Iso}(\mathcal{H}_{g^{-1}\omega}, \mathcal{H}_\omega)$$

such that

-
- $A(\omega, g_1 g_2) = A(\omega, g_1) A(g_1^{-1} \omega, g_2)$ μ -almost everywhere;
 - the function $\omega \mapsto \langle \xi_1(\omega), A(\omega, g) \xi_2(g^{-1} \omega) \rangle$ is measurable for every $\xi_1, \xi_2 \in \mathcal{H}$ and every $g \in G$.

Define a unitary representation of G on the direct integral \mathcal{H} as follows:

$$(\pi(g)\xi)(\omega) := P^{\frac{1}{2}}(\omega, g) A(\omega, g) \xi(g^{-1} \omega), \quad (4.1)$$

for $g \in G$, $\xi \in \mathcal{H}$ and $\omega \in X$. The C^* -algebra $C(X)$ simply acts by left multiplications:

$$(\rho(f)\xi)(\omega) := f(\omega) \xi(\omega),$$

where $f \in C(X)$, $\xi \in \mathcal{H}$ and $\omega \in X$. It is easy to verify that the two representations satisfy

$$\pi(g) \rho(f) \pi(g^{-1}) = \rho(g.f),$$

where $g.f(\omega) = f(g^{-1} \omega)$ for $g \in G$, $f \in C(X)$ and $\omega \in X$.

When $\mathcal{H}_\omega \simeq \mathbb{C}$ for all $\omega \in X$ is the trivial field of Hilbert spaces and $A \equiv 1$ is the trivial cocycle, then π is the **quasi-regular representation** of G on $L^2(X, d\mu)$.

In [Tak13, Chap. X, Th. 3.8 & Th. 3.15] it is shown that the representations of the full crossed product $G \rtimes C(\partial G)$ are given exactly by the cocycle representations for some quasi-invariant measure on the boundary and some field of Hilbert spaces.

Remark 4.5. By [Dix77, Lem. 2.10.1], if $\mathcal{B} \subseteq \mathcal{A}$ are C^* -algebras and Π is a C^* -representation of \mathcal{A} on a Hilbert space \mathcal{K} , then there exists a C^* -representation $\tilde{\Pi}$ of \mathcal{B} on a Hilbert space $\tilde{\mathcal{K}}$ and an inclusion $\mathcal{K} \hookrightarrow \tilde{\mathcal{K}}$ such that $\tilde{\Pi}$ extends Π . Moreover, if Π is irreducible, then one can require $\tilde{\Pi}$ to be irreducible, too. It is worth noticing that the space \mathcal{K} may be properly included into $\tilde{\mathcal{K}}$.

If G is a discrete hyperbolic group with boundary ∂G (satisfying some additional conditions, see Thm. 4.6), this fact can be used to extend a representation of the reduced group C^* -algebra $C_{\text{red}}^*(G)$ to the whole crossed product $G \rtimes_{\text{red}} C(\partial G)$, once that an inclusion $C_{\text{red}}^*(G) \hookrightarrow G \rtimes_{\text{red}} C(\partial G)$ is provided.

Recall that the representations of $C_{\text{red}}^*(G)$ are precisely the tempered representations of G .

The following result guarantees, in particular, that every tempered representation of a surface group can be extended to a boundary representation.

Theorem 4.6 ([IKS13]). *Let G be a torsion-free not almost cyclic hyperbolic group. Every tempered representation of G is a subrepresentation of a cocycle representation. Moreover, if the representation of G is irreducible, then the cocycle representation can be taken irreducible.*

Idea of the proof, from [IKS13]. The proof relies on the existence of an inclusion

$$C_{\text{red}}^*(G) \hookrightarrow G \rtimes_{\text{red}} C(\partial G),$$

as shown in [IKS13], for the case where G is a torsion-free not almost cyclic hyperbolic group and ∂G is its boundary: in this case, the reduced C^* -algebra of G is simple. Tempered representations of G correspond one-to-one to representations of the reduced group C^* -algebra, while boundary representations are precisely the representations of $G \rtimes_{\text{red}} C(\partial G)$. \square

4.2 COVARIANT REPRESENTATIONS OF ($\Gamma, C(X)$)

Remark 4.7. All the metric notions in this Section and those following are intended with respect to the word metric associated with a symmetric set of generators.

Let $\Gamma = \Gamma_2$ be the surface group of genus 2 with a symmetric set of generators S .

In this Section we introduce a net of covariant representations of the covariant system $(\Gamma, C(\Gamma \cup \partial\Gamma))$. The representations are defined as subspaces of $\ell^2(\Gamma, V)$, where V is the finite-dimensional vector space associated to a fixed matrix system as in 3.4 and the action of the group is a restriction of the regular action, while the functions in $C(\Gamma \cup \partial\Gamma)$ act by multiplication. A compactness argument guarantees that the net we defined has a limit.

We show that the limit is actually a covariant representation of $(\Gamma, C(\partial\Gamma))$, i.e. a boundary representation (cf. Definition 4.2): only the boundary values of the functions really matter. We then show that this “limit boundary representation” is unique (i.e. it does not depend on the sequence used to calculate the limit).

We show in the later Section 4.5 that the Γ -part of the limit boundary representation is equivalent to π_m , the multiplicative representation defined

in Chapter 3. In a sense, the limit representation realizes π_m as the Γ -part of a boundary representation.

Recall that, if $\Gamma = \Gamma_2$ and $\partial\Gamma$ denotes its boundary, the topological space

$$X := \Gamma \cup \partial\Gamma \tag{4.2}$$

is compact and gives a natural compactification of Γ . In particular, Γ acts naturally on X and we have the covariant system $(\Gamma, C(X))$.

4.2.1 Action of $\ell^\infty(\Gamma)$ on $\ell^2(\Gamma, V)$

It is known that $\ell^\infty(\Gamma)$ acts by multiplication on the Hilbert space $\ell^2(\Gamma)$: for $f \in \ell^\infty(\Gamma)$, $\varphi \in \ell^2(\Gamma)$ the action is given by

$$(f \cdot \varphi)(z) = f(z)\varphi(z), \quad z \in \Gamma.$$

The action of $\ell^\infty(\Gamma)$ extends to the Hilbert space tensor product $\ell^2(\Gamma, V) \simeq \ell^2(\Gamma) \otimes V$ as follows. Recall that the tensor product $\ell^2(\Gamma) \otimes V$ is generated (as a vector space) by the pure tensors $\varphi \otimes v$, where $\varphi \in \ell^2(\Gamma)$, $v \in V$: therefore, it is sufficient to define the action of a bounded function on the pure tensors. For $f \in \ell^\infty(\Gamma)$, $\varphi \in \ell^2(\Gamma)$, $v \in V$, define

$$f \cdot (\varphi \otimes v)(z) := (f(z)\varphi(z)) \otimes v, \quad z \in \Gamma.$$

4.2.2 Some covariant representations

We define now a net of covariant representations of the C^* -dynamical system $(\Gamma, C(X))$, where X is as in equation 4.2.

Let \mathcal{C} be the set of cone types in Γ . Fix a scalar system $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$ associated to the standard set of generators and let V be as in Definition 3.4. Consider the space $\mathcal{H}^\infty = \mathcal{H}^\infty(V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s})$ of V -valued multiplicative functions defined using the scalar system.

Fix $\epsilon > 0$ and $\mathbf{m} \in \mathcal{H}^\infty$, consider $\mathbf{m}_\epsilon(z) = \sqrt{\epsilon}e^{-\frac{\epsilon}{2}|z|}\mathbf{m}(z)$ as in equation 3.11. Recall that $\mathbf{m}_\epsilon \in \ell^2(\Gamma, V)$ (see Lemma 3.58 in Chapter 3).

Define

$$\mathcal{H}_\epsilon^\infty := \text{span}_{\mathbb{C}} \{\mathbf{m}_\epsilon \mid \mathbf{m} \in \mathcal{H}^\infty\}.$$

We have the natural inclusion $\mathcal{H}_\epsilon^\infty \hookrightarrow \ell^2(\Gamma, V)$. Hence, we can act on $\mathcal{H}_\epsilon^\infty$ with $\ell^\infty(\Gamma)$. The closure in the ℓ^2 -norm of the image of $\mathcal{H}_\epsilon^\infty$ under this action is denoted by \mathcal{H}_ϵ and it is a subspace of $\ell^2(\Gamma, V)$. Thus, \mathcal{H}_ϵ is the smallest $\ell^\infty(\Gamma)$ -invariant subspace of $\ell^2(\Gamma, V)$ containing $\mathcal{H}_\epsilon^\infty$.

The family of covariant representations that we are going to define will act on the space \mathcal{H}_ϵ .

4. BOUNDARY REPRESENTATIONS

The unitary action of the group: recall that Γ acts on $\ell^2(\Gamma, V)$ by the regular representation. It is then possible to consider the restriction of this action to the subspace \mathcal{H}_ϵ , which yields the unitary representation

$$\begin{aligned} \pi_\epsilon: \Gamma &\rightarrow \mathfrak{U}(\mathcal{H}_\epsilon), \\ (\pi_\epsilon(g)\varphi)(z) &:= \varphi(g^{-1}z), \end{aligned} \tag{4.3}$$

where $g, z \in \Gamma$ and $\varphi \in \mathcal{H}_\epsilon$.

The action of the C*-algebra: let $X = \Gamma \cup \partial\Gamma$. We define the action of the C*-algebra $C(X)$ on the space \mathcal{H}_ϵ restricting the functions to $\partial\Gamma$ and then using the action $\ell^\infty(\Gamma) \curvearrowright \ell^2(\Gamma) \otimes V$.

Fix $f \in C(X)$. Since X is compact, f is bounded, so the restriction $f|_\Gamma$ belongs to $\ell^\infty(\Gamma)$. In particular, for $\mathbf{m}_\epsilon \in \mathcal{H}_\epsilon^\infty$ we have: $f|_\Gamma \cdot \mathbf{m}_\epsilon \in \ell^2(\Gamma) \otimes V$, where the dot denotes the action of $\ell^\infty(\Gamma)$ on $\ell^2(\Gamma, V)$ as above. We define the operator $\rho_\epsilon(f)$ on $\mathcal{H}_\epsilon^\infty$ as follows: for $\mathbf{m}_\epsilon \in \mathcal{H}_\epsilon^\infty$

$$(\rho_\epsilon(f)\mathbf{m}_\epsilon)(z) := f|_\Gamma(z)\mathbf{m}_\epsilon(z), \quad z \in \Gamma. \tag{4.4}$$

Notice that

$$\|\rho_\epsilon(f)\mathbf{m}_\epsilon\|_2^2 = \sum_{z \in \Gamma} \|f(z)\mathbf{m}_\epsilon(z)\|_B^2 \leq \|f\|_\infty^2 \|\mathbf{m}_\epsilon\|_2^2 < \infty,$$

so that the action is well defined. Since \mathcal{H}_ϵ is invariant under the action of $\ell^\infty(\Gamma)$, the action of $C(X)$ defined above extends by continuity to all of \mathcal{H}_ϵ .

In particular, for every $\epsilon > 0$ we have a C*-algebra representation

$$\rho_\epsilon: C(X) \rightarrow \mathfrak{B}(\mathcal{H}_\epsilon).$$

It is easy to verify that ρ_ϵ is a C*-algebra homomorphism:

$$\begin{aligned} [\rho_\epsilon(f_1 + f_2)\mathbf{m}_\epsilon](z) &= f_1|_\Gamma(z)f_2|_\Gamma(z)\mathbf{m}_\epsilon(z) = [(\rho_\epsilon(f_1)\rho_\epsilon(f_2))\mathbf{m}_\epsilon](z), \\ [\rho_\epsilon(\alpha f)\mathbf{m}_\epsilon](z) &= \alpha f|_\Gamma(z)\mathbf{m}_\epsilon(z) = [(\alpha\rho_\epsilon(f))\mathbf{m}_\epsilon](z), \end{aligned}$$

where $f, f_1, f_2 \in C(X)$, $\alpha \in \mathbb{C}$ and $\mathbf{m}_\epsilon \in \mathcal{H}_\epsilon^\infty$; this holds, by continuity, for every element of \mathcal{H}_ϵ .

Theorem 4.8. *The pair $(\pi_\epsilon, \rho_\epsilon)$ as in equations 4.3,4.4 gives a covariant representation on the Hilbert space \mathcal{H}_ϵ of the C*-dynamical system $(\Gamma, C(X))$, where Γ acts on $C(X)$ by the natural left-translation action.*

Proof. We have to prove that the action of $C(X)$ on the functions of $\mathcal{H}_\epsilon^\infty$ satisfies the covariant condition

$$\pi_\epsilon(g)\rho_\epsilon(f)\pi_\epsilon(g)^{-1} = \rho_\epsilon(g.f),$$

where $g \in \Gamma$, $f \in C(X)$, $\pi_\epsilon(g)$ is the left-translation operator acting on \mathcal{H}_ϵ , $\rho_\epsilon(f)$ is the operator associated to f and $g.f$ is the left translate of f . Fix $\mathbf{m}_\epsilon \in \mathcal{H}_\epsilon^\infty$. For all $z \in \Gamma$ we have:

$$\begin{aligned} (\pi_\epsilon(g)\rho_\epsilon(f)\pi_\epsilon(g)^{-1}) \mathbf{m}_\epsilon(z) &= \pi_\epsilon(g) (\rho_\epsilon(f)\pi_\epsilon(g)^{-1}\mathbf{m}_\epsilon) (z) \\ &= (\rho_\epsilon(f)\pi_\epsilon(g)^{-1}\mathbf{m}_\epsilon) (g^{-1}z) \\ &= f(g^{-1}z) (\pi_\epsilon(g)^{-1}\mathbf{m}_\epsilon) (g^{-1}z) \\ &= f(g^{-1}z) \mathbf{m}_\epsilon(g(g^{-1}z)) \\ &= f(g^{-1}z) \mathbf{m}_\epsilon((gg^{-1})z) \\ &= f(g^{-1}z) \mathbf{m}_\epsilon(z) \\ &= (\rho_\epsilon(g.f) \mathbf{m}_\epsilon) (z). \end{aligned}$$

The covariant condition holds on all of \mathcal{H}_ϵ by continuity. \square

We can associate to the covariant representation $(\pi_\epsilon, \rho_\epsilon)$ as above a unique C*-representation $\pi_\epsilon \times \rho_\epsilon$ of the crossed product C*-algebra $\Gamma \rtimes C(X)$ on the space \mathcal{H}_ϵ .

In particular, consider $\mathbf{m}_\epsilon \in \mathcal{H}_\epsilon^\infty$ having norm one: this is always possible up to normalizing \mathbf{m}_ϵ itself, i.e. substituting \mathbf{m}_ϵ by $\frac{\mathbf{m}_\epsilon}{\|\mathbf{m}_\epsilon\|}$. The function $\Phi_\epsilon : \Gamma \rtimes C(X) \rightarrow \mathbb{C}$ whose value on $\mathfrak{F} \in C_c(\Gamma, C(X))$ is given by

$$\Phi_\epsilon(\mathfrak{F}) := \langle (\pi_\epsilon \times \rho_\epsilon)(\mathfrak{F}) \mathbf{m}_\epsilon, \mathbf{m}_\epsilon \rangle_2 \quad (4.5)$$

is the state (positive linear functional of norm one) of the C*-algebra $\Gamma \rtimes C(X)$ associated to \mathbf{m}_ϵ (cf. [Con13]).

Letting $\epsilon > 0$ vary, we get a net of states $\{\Phi_\epsilon\}_\epsilon$ defined on the crossed product C*-algebra.

4.2.3 Limit state

It is well-known that the state space of a C*-algebra with identity is weakly-* compact: see, for example, [Con13, Chap. VIII, Prop. 5.15].

Proposition 4.9 ([Con13]). *Let \mathcal{A} be a C*-algebra with identity, let $\mathcal{S}_\mathcal{A}$ be the state space of \mathcal{A} . Then $\mathcal{S}_\mathcal{A}$ is a weak-* compact convex subset of \mathcal{A}^* (the space of continuous complex-valued linear functionals on \mathcal{A}).*

Consider the C^* -algebra $\mathcal{A} = \Gamma \rtimes C(X)$; our net is $\{\Phi_{\epsilon_j}\}_{\epsilon_j}$, as defined in equation 4.5. By weak- $*$ compactness of the state space of $\Gamma \rtimes C(X)$, we can find a subnet $\epsilon_j \downarrow 0$ and a state Φ of $\Gamma \rtimes C(X)$ such that

$$\Phi = (w^*) \lim_{\epsilon_j \downarrow 0} \Phi_{\epsilon_j}. \quad (4.6)$$

By the GNS construction (see [Con13, Chap. VIII, Sect. 5]), the state Φ corresponds to a crossed product representation on a Hilbert space \mathcal{H} (with a cyclic vector $\xi \in \mathcal{H}$)

$$\pi \rtimes \rho: \Gamma \rtimes C(X) \rightarrow \mathfrak{B}(\mathcal{H}).$$

Thus, we get a pair of representations

$$\pi: \Gamma \rightarrow \mathfrak{U}(\mathcal{H}) \quad (4.7)$$

$$\rho: C(X) \rightarrow \mathfrak{B}(\mathcal{H}) \quad (4.8)$$

satisfying the covariance condition as in Definition 4.1.

Remark 4.10. At this point, it is not clear whether the limit Φ is unique or it depends on the subnet ϵ_j converging to zero. We demand the proof of the uniqueness of the limit to the later Section 4.4.

4.3 FACTORING THROUGH

$$\Gamma \rtimes C(X) \rightarrow \Gamma \rtimes C(\partial\Gamma)$$

In this Section, we show that the limit Φ (which in principle depends on the subnet converging to zero) defines a boundary representation.

Consider the compact space $X = \Gamma \cup \partial\Gamma$: we have the restriction map $C(X) \rightarrow C(\partial\Gamma)$ sending a continuous function $f \in C(X)$ to its restriction on the boundary, $f|_{\partial\Gamma}$. The crossed product $\Gamma \rtimes C(X)$ is defined as a norm completion of the algebra $C_c(\Gamma, C(X))$, which is generated by the finite sums $\sum_{i=1}^n f_i \delta_{x_i}$, where $f_i \in C(X)$, $x_i \in \Gamma$ for every $i = 1, \dots, n$. Given such $\mathfrak{F} = \sum_{i=1}^n f_i \delta_{x_i}$, for every $i = 1, \dots, n$ we have the restriction $f_i|_{\partial\Gamma}$ and we can define the map

$$\sum_{i=1}^n f_i \delta_{x_i} \mapsto \sum_{i=1}^n f_i|_{\partial\Gamma} \delta_{x_i},$$

where the latter is in the algebra $C_c(\Gamma, C(\partial\Gamma))$. We can extend this map to

$$\Gamma \times C(X) \rightarrow \Gamma \times C(\partial\Gamma).$$

In this section, we prove the following.

Theorem 4.11. *Let $(\pi_\epsilon, \rho_\epsilon)$ be the family of covariant representations of $(\Gamma, C(X))$ as in Thm. 4.8, let Φ_ϵ be the correspondent state of $\Gamma \times C(X)$. Let Φ be a limit state obtained as in Sec. 4.2.3, let (π, ρ) be the correspondent covariant representation. Then $\pi \times \rho$ factors through the map $\Gamma \times C(X) \rightarrow \Gamma \times C(\partial\Gamma)$. In particular, there exists a unique representation*

$$\pi' \times \rho': \Gamma \times C(\partial\Gamma) \rightarrow \mathfrak{A}(\mathcal{H}'),$$

such that $(\pi' \times \rho')(f\delta_g) = (\pi \times \rho)(f|_{\partial\Gamma}\delta_g)$ for each $f \in C(X)$ and $g \in \Gamma$.

In a sense, this means that the representation ρ of $C(X)$ detects only the $\partial\Gamma$ -part of the functions.

4.3.1 Proof of Theorem 4.11

We define $\mathcal{H}' := \mathcal{H}$ and $\pi' := \pi$.

We now want to define a C^* -algebra representation $\rho': C(\partial\Gamma) \rightarrow \mathfrak{B}(\mathcal{H})$ that must satisfy $\rho(f) = \rho'(f|_{\partial\Gamma})$ for every $f \in C(X)$.

Lemma 4.12. *Let Φ be a state of $\Gamma \times C(X)$ such that $\Phi_{\epsilon_j} \xrightarrow{w^*} \Phi$ when $\epsilon_j \downarrow 0$. Let (π, ρ, \mathcal{H}) be the representation of $\Gamma \times C(X)$ corresponding to Φ , consider the associated representation ρ of $C(X)$ acting by multiplication. Then $\rho(f) = 0$ for every $f \in C(X)$ with compact support contained in Γ .*

Proof. With a small abuse, by the notation $\Phi(f)$ for $f \in C(X)$ we mean $\Phi(f\delta_e)$, where $f\delta_e \in \Gamma \times C(X)$.

Consider a function $f \in C(X)$ with finite support contained in Γ . In particular, $f|_{\partial\Gamma} = 0$. Recall that the action of f on the space generated by \mathbf{m}_ϵ is given by multiplication. Compute

$$\begin{aligned} \Phi(f) &= \lim_{\epsilon \downarrow 0} \Phi_\epsilon(f) \\ &= \lim_{\epsilon \downarrow 0} \langle \rho_\epsilon(f) \mathbf{m}_\epsilon, \mathbf{m}_\epsilon \rangle_2 \\ &= \lim_{\epsilon \downarrow 0} \sum_{z \in \Gamma} B(f(z) \mathbf{m}_\epsilon(z), \mathbf{m}_\epsilon(z)) \\ &= \lim_{\epsilon \downarrow 0} \epsilon \sum_{z \in \Gamma} f(z) B(\mathbf{m}(z), \mathbf{m}(z)) e^{-\epsilon|z|}. \end{aligned}$$

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Since $\#\{z \in \Gamma \mid f(z) \neq 0\} < \infty$ by the assumption that f has compact support contained in Γ , the sum

$$K = \sum_{z \in \Gamma} f(z) B(\mathbf{m}(z), \mathbf{m}(z)) e^{-\epsilon|z|}$$

has a finite number of nonzero terms, so that

$$\Phi(f) = \lim_{\epsilon \downarrow 0} \epsilon K = 0.$$

Since $\Phi(f) = 0$ for every f with finite support contained in Γ and $\Phi(f) = \langle \rho(f)\xi, \xi \rangle$ where $\xi \in \mathcal{H}$ is a cyclic vector, we get that $\rho(f) = 0$ for every f with finite support contained in Γ . \square

Now we want to approximate continuous functions on $X = \Gamma \cup \partial\Gamma$ that are zero on $\partial\Gamma$ using functions with compact support in Γ .

Lemma 4.13. *Let $h \in C(X)$ be such that $h|_{\partial\Gamma} = 0$. Then h is the pointwise limit of a sequence $\{h_r\}_{r>0} \subseteq C(X)$ of functions with compact support in Γ .*

Proof. Fix $r > 0$. The topology on X is generated by the family of sets

$$B_r(x) := \{z \in \Gamma \mid d(x, z) < r\}, \text{ with } x \in \Gamma,$$

$$B_{\omega, r} := \{z \in X \mid (z|\omega) > r\}, \text{ with } \omega \in \partial\Gamma,$$

where $(\cdot|\cdot)$ is the extended Gromov product. Consider the intersections $\{B_{\omega, r} \cap \partial\Gamma\}_{\omega \in \partial\Gamma}$: these form an open cover of $\partial\Gamma$. By compactness of $\partial\Gamma$, there exists a finite subcover, i.e. there are $N \in \mathbb{N}$ and $\omega_1, \dots, \omega_N \in \partial\Gamma$ such that

$$\partial\Gamma = (B_{\omega_1, r} \cap \partial\Gamma) \cup \dots \cup (B_{\omega_N, r} \cap \partial\Gamma).$$

Consider now

$$U_r := B_{\omega_1, r} \cup \dots \cup B_{\omega_N, r} \subseteq X. \tag{4.9}$$

This set contains $\partial\Gamma$ and it is open in X , since it is union of open subsets; thus $U_r^c := X \setminus U_r$ is contained in Γ and it is a closed set. Now, $U_r^c \subseteq X$ is a closed subset of a compact space, thus it is itself compact; moreover $U_r^c \subseteq \Gamma$ and Γ is discrete, hence U_r^c is a finite subset of Γ . It follows that there is $M \in \mathbb{N}$ such that for all $x \in U_r^c$ it holds $|x| \leq M$ (i.e. U_r^c is contained in a ball centered at the identity).

Given a function $h \in C(X)$ which is zero on the boundary (i.e. $h|_{\partial\Gamma} = 0$), we define for $r > 0$

$$h_r(x) = \begin{cases} h(x) & \text{if } x \in V_r, \\ 0 & \text{if } x \in U_r. \end{cases}$$

Each h_r is continuous on X (its support lies in Γ , which is a discrete space) and it has finite support in Γ . Moreover $\{h_r\}_r$ converge pointwise to h : for all $x \in X$

$$\lim_{r \rightarrow \infty} h_r(x) = h(x).$$

This is evident if $x \in \Gamma$: choose r big enough so that $x \in U_r^c$ and $h_r(x) = h(x)$. This implies the convergence at every point $x \in \Gamma$. On the other hand, if $\omega \in \partial\Gamma$, then both functions are zero: $h(\omega) = 0$ by hypothesis, while $h_r(\omega) = 0$ by definition. \square

We are now ready to define $\rho': C(\partial\Gamma) \rightarrow \mathfrak{B}(\mathcal{H})$. Let $f \in C(\partial\Gamma)$: by an application of Tietze's Theorem, there exists an extension of $\tilde{f} \in C(X)$ such that $\tilde{f}|_{\partial\Gamma} = f$.

Theorem 4.14 (Consequence of Tietze's Theorem). *Given a function $f \in C(\partial\Gamma)$, there exists a function $\tilde{f} \in C(X)$ such that $\tilde{f}|_{\partial\Gamma} = f$ and $\sup_{x \in X} |\tilde{f}(x)| = \sup_{\omega \in \partial\Gamma} |f(\omega)|$.*

More details can be found in Appendix A.2.

Define

$$\rho'(f) := \rho(\tilde{f}). \tag{4.10}$$

First, we must verify the well-posedness of the definition.

Proposition 4.15. *Let $f_1, f_2 \in C(X)$ extend the same $f \in C(\partial\Gamma)$ as in Theorem 4.14. Then $\rho(f_1) = \rho(f_2)$, where ρ is the representation of $C(X)$ as in equation 4.8 (obtained from the limit state).*

Proof. We define $h := f_1 - f_2$. Then, by the hypothesis, $h|_{\partial\Gamma} = 0$. Lemma 4.13 guarantees that h is the pointwise limit of a sequence $\{h_r\}_{r>0}$ of functions with compact support contained in Γ . Moreover, by Lemma 4.12, $\rho(h_r) = 0$ for every $r > 0$. Therefore, since ρ is continuous:

$$\rho(h) = \rho\left(\lim_{r \rightarrow \infty} h_r\right) = \lim_{r \rightarrow \infty} \rho(h_r) = 0.$$

This yields: $\rho(f_1 - f_2) = 0$, which, by linearity, gives: $\rho(f_1) = \rho(f_2)$. \square

We claim that ρ' defined in equation 4.10 is a C^* -representation of $C(\partial\Gamma)$ on \mathcal{H} and that it satisfies the covariance condition with the representation $\pi' = \pi$ as in 4.7, thus giving a covariant representation (π, ρ') of the C^* -dynamical system $(\Gamma, C(\partial\Gamma))$.

Proposition 4.16. *The map $\rho': C(\partial\Gamma) \rightarrow \mathfrak{B}(\mathcal{H})$ is a morphism of C^* -algebras.*

Proof. We must show that, for $f, f_1, f_2 \in C(\partial\Gamma)$ and $\alpha \in \mathbb{C}$ it holds: $\rho'(f_1 + f_2) = \rho'(f_1) + \rho'(f_2)$, $\rho'(f_1 f_2) = \rho'(f_1)\rho'(f_2)$, $\rho'(\alpha f) = \alpha\rho'(f)$, $\rho'(f^*) = \rho'(f)^*$. Let $\tilde{f}_1, \tilde{f}_2 \in C(X)$ be extensions of $f_1, f_2 \in C(\partial\Gamma)$. Then $\tilde{f}_1 + \tilde{f}_2$ is an extension of the function $f_1 + f_2$ to X . Hence, we can choose to compute $\rho'(f_1 + f_2)$ as $\rho(\tilde{f}_1 + \tilde{f}_2)$. Thus:

$$\rho'(f_1 + f_2) = \rho(\tilde{f}_1 + \tilde{f}_2) = \rho(\tilde{f}_1) + \rho(\tilde{f}_2) = \rho'(f_1) + \rho'(f_2).$$

Similar considerations hold for the other operations. □

Proposition 4.17. *The pair (π, ρ') is a covariant representation of $(\Gamma, C(\partial\Gamma))$.*

Proof. The proof is immediate: ρ' is defined through ρ , which satisfies covariance as in 4.1 with the unitary representation π . Thus, ρ' itself satisfies covariance with π . □

The proof of Theorem 4.11 is complete.

Notation 4.18. In what follows we will write, with an abuse, ρ meaning the representation $\rho': C(\partial\Gamma) \rightarrow \mathfrak{B}(\mathcal{H})$. In a sense, in fact, the present section shows that ρ itself depends only on the $\partial\Gamma$ -part of the functions in $C(X)$.

4.4 WELL-POSEDNESS OF THE LIMIT

In this Section, we prove that the limit state Φ obtained in 4.6 is actually unique, i.e. it does not depend on the subnet converging to zero. This means that the family of covariant representations $(\pi_\epsilon, \rho_\epsilon)$ as in Thm. 4.8 has a well-defined limit for $\epsilon \downarrow 0$, which, through the GNS construction and Section 4.3, gives a boundary representation of $\Gamma \rtimes C(\partial\Gamma)$.

Given $\mathbf{m}_\epsilon \in \mathcal{H}_\epsilon^\infty$ of norm one, we have the state

$$\Phi_\epsilon(\mathfrak{F}) := \langle (\pi_\epsilon \rtimes \rho_\epsilon)(\mathfrak{F}) \mathbf{m}_\epsilon, \mathbf{m}_\epsilon \rangle_2, \quad \mathfrak{F} \in \Gamma \rtimes C(X).$$

Without loss, assume that \mathfrak{F} belongs to the subalgebra $C_c(\Gamma, C(X))$, i.e. $\mathfrak{F}: \Gamma \rightarrow C(X)$ is a function with compact support in Γ : in fact, the algebra $C_c(\Gamma, C(X))$ is dense in the crossed product with respect to the universal norm.

By linearity of Φ_ϵ , it is sufficient to consider the elements $\mathfrak{F} = f\delta_g$, where $f \in C(X)$ and $g \in \Gamma$.

The main theorem of this Section is the following.

Theorem 4.19. *Let $(\pi_\epsilon, \rho_\epsilon)$ be the family of covariant representations of $(\Gamma, C(X))$ as in Thm. 4.8, let Φ_ϵ be the correspondent state of $\Gamma \rtimes C(X)$. Let Φ be a limit state obtained as in Sec. 4.2.3 which, by Theorem 4.11, depends only on $C(\partial\Gamma)$. Given $g \in \Gamma$ and $f \in C(\partial\Gamma)$, given nets $\epsilon_j, \epsilon_i \downarrow 0$, it holds that*

$$\lim_{j \rightarrow \infty} \Phi_{\epsilon_j}(f\delta_g) = \lim_{i \rightarrow \infty} \Phi_{\epsilon_i}(f\delta_g).$$

This implies that the limit state is unique and well-defined for all of $\Gamma \rtimes C(X)$.

The crucial point concerns the representations ρ_ϵ of $C(X)$. To get our claim, it will be useful to approximate (uniformly, i.e. in the supremum norm) a function $f \in C(X)$ with step functions. Hence, we need to extend the representation of $C(X)$ to a representation of all bounded functions on X (since X is compact, the latter algebra contains the former).

Notation 4.20. In what follows, denote by $B(X)$ the algebra of all complex-valued bounded Borel-measurable functions on X .

By the general theory of C*-algebras (cf. [Con13]), each state $\Phi_\epsilon|_{C(X)}$ of $C(X)$ can be extended to a state $\tilde{\Phi}_\epsilon$ of the C*-algebra $B(X)$.

The following result is an immediate consequence of [Con13, Chap. VIII, Prop. 5.16].

Proposition 4.21. *The state Φ of $C(X)$ has an extension to a state $\tilde{\Phi}$ of $B(X)$.*

We recall the construction, since it will be needed later. The representation $\rho_\epsilon: C(X) \rightarrow \mathfrak{B}(\mathcal{H}_\epsilon)$ can be associated to a unique projection-valued measure P_ϵ on $(X, \text{Borel}(X), \mathcal{H}_\epsilon)$ such that $\rho_\epsilon(f) = \int_X f dP_\epsilon$, for all $f \in C(X)$ (cf. [Con13, Chap. IX, Thm. 1.14]). For $\epsilon > 0$ and a bounded Borel-measurable function $u \in B(X)$, the operator-valued integral $\tilde{\rho}_\epsilon(u) := \int_X u dP_\epsilon$ is well-defined, so we get a *-representation $\tilde{\rho}_\epsilon$ of $B(X)$ on the space \mathcal{H}_ϵ (cf. [Con13, Chap. IX, Prop. 1.12]). For every $f \in C(X)$ we have $\tilde{\rho}_\epsilon(f) = \rho_\epsilon(f)$. See the proof of [Con13, Chap. IX, Thm. 1.14] for more details.

4.4.1 Proof of Theorem 4.19

We divide the limit functional in two “parts”: the part on the group $\Phi|_\Gamma$ corresponding to the inclusion $\Gamma \hookrightarrow \Gamma \rtimes C(X)$, and the part on the algebra $\Phi|_{C(X)}$, corresponding to the inclusion $C(X) \hookrightarrow \Gamma \rtimes C(X)$.

Since $\Gamma \hookrightarrow \Gamma \times C(X)$ is given by $g \mapsto \mathbf{1}_{\partial\Gamma}\delta_g$ ($\mathbf{1}_{\partial\Gamma}$ is the identically 1 function defined on $\partial\Gamma$), the part on the group is given by the evaluation of Φ on the elements $\mathfrak{F} = \mathbf{1}_{\partial\Gamma}\delta_g$, where $g \in \Gamma$. Since $\rho_\epsilon(\mathbf{1}_{\partial\Gamma}) = I$ (the identity operator), we write with an abuse:

$$\Phi(g) = \Phi(\mathbf{1}_{\partial\Gamma}\delta_g) = \lim_{\epsilon \downarrow 0} \langle \pi_\epsilon(g)\mathbf{m}_\epsilon, \mathbf{m}_\epsilon \rangle.$$

This corresponds to the Γ -part π of the limit representation.

On the other hand, the part on the algebra comes from the evaluation of Φ on the elements of the form $\mathfrak{F} = f\delta_e$, where $f \in C(\partial\Gamma)$ and $e \in \Gamma$ is the identity: in fact, $C(X) \hookrightarrow \Gamma \times C(X)$ by $f \mapsto f\delta_e$. Since $\pi_\epsilon(\delta_e) = I$, we write again with an abuse:

$$\Phi(f) = \Phi(f\delta_e) = \lim_{\epsilon \downarrow 0} \langle \rho_\epsilon(f)\mathbf{m}_\epsilon, \mathbf{m}_\epsilon \rangle.$$

This corresponds to the $C(\partial\Gamma)$ -part ρ of the limit representation.

Part on the group

Let us start with the study of $\Phi(g)$, $g \in \Gamma$. We have:

$$\begin{aligned} \Phi(g) &= \lim_{\epsilon \downarrow 0} \langle \pi_\epsilon(g)\mathbf{m}_\epsilon, \mathbf{m}_\epsilon \rangle \\ &= \lim_{\epsilon \downarrow 0} \sum_{z \in \Gamma} B(\pi_\epsilon(g)\mathbf{m}_\epsilon(z), \mathbf{m}_\epsilon(z)) \\ &= \lim_{\epsilon \downarrow 0} \sum_{z \in \Gamma} B(\mathbf{m}_\epsilon(g^{-1}z), \mathbf{m}_\epsilon(z)) \\ &= \lim_{\epsilon \downarrow 0} \epsilon \sum_{z \in \Gamma} B(\mathbf{m}(g^{-1}z), \mathbf{m}(z)) e^{-\frac{\epsilon}{2}[|g^{-1}z|+|z|]} \\ &= \langle \pi_{\mathbf{m}}(g)\mathbf{m}, \mathbf{m} \rangle_{\mathcal{H}^\infty}, \end{aligned}$$

as we computed in the proof of Theorem 3.59.

This proves that:

- The limit $\Phi(g)$ does not depend on the subnet ϵ_j tending to zero.
- The representation π corresponding to the restriction of the limit Φ to Γ (i.e. to $\Phi(g)$ for $g \in \Gamma$) is unitarily equivalent to the multiplicative representation $\pi_{\mathbf{m}}$ defined in Chapter 3. This will be treated in the later Section 4.5.

Part on the algebra: some step functions on X

Consider now $u \in B(X)$. Let $\tilde{\Phi}$ be the extension to $B(X)$ of Φ corresponding to the extension $\tilde{\rho}$ of ρ obtained through the projection-valued measure. Then:

$$\tilde{\Phi}(u) = \lim_{\epsilon \downarrow 0} \langle u|_{\Gamma} \cdot \mathbf{m}_{\epsilon}, \mathbf{m}_{\epsilon} \rangle.$$

Fix arbitrary $x, y \in \Gamma$, consider $\mathcal{S}(x, y) := C(x, y) \cup \mathcal{O}(x, y) \subseteq X$: the set $C(x, y)$ is the cone based at x with vertex y , while

$$\mathcal{O}(x, y) := \{\omega \in \partial\Gamma \mid \exists \gamma \in [x, \omega] \text{ passing through } y\}. \quad (4.11)$$

The “shadow” $\mathcal{S}(x, y)$ is a closed subset of X , we have $\mathcal{S}(x, y) \cap \Gamma = C(x, y)$ and $\mathcal{S}(x, y) \cap \partial\Gamma = \mathcal{O}(x, y)$. The function $\mathbf{1}_{\mathcal{S}(x, y)}$ is bounded (and Borel-measurable) on $\partial\Gamma$.

Lemma 4.22. *Fix $x, y \in \Gamma$. The limit $\lim_{\epsilon \downarrow 0} \tilde{\Phi}_{\epsilon}(\mathbf{1}_{\mathcal{S}(x, y)})$ does not depend on the particular subnet converging to zero.*

Proof of Lemma 4.22. Let $\mathbf{m}_{\epsilon} \in \mathcal{H}_{\epsilon}^{\infty}$ be the function giving $\tilde{\Phi}_{\epsilon}$. Assume that $\mathbf{1}_{C(x, y)} \mathbf{m} = \sum_{i=1}^N \mathbf{m}_i$ as in Lemma 3.23 for $\mathbf{m}_i \in \mathcal{H}^{\infty}$, $i = 1, \dots, N$. We compute

$$\begin{aligned} \tilde{\Phi}(\mathbf{1}_{\mathcal{S}(x, y)}) &= \lim_{\epsilon \downarrow 0} \tilde{\Phi}_{\epsilon}(\mathbf{1}_{\mathcal{S}(x, y)}) \\ &= \lim_{\epsilon \downarrow 0} \langle \mathbf{1}_{\mathcal{S}(x, y) \cap \Gamma} \mathbf{m}_{\epsilon}, \mathbf{m}_{\epsilon} \rangle \\ &= \lim_{\epsilon \downarrow 0} \langle \mathbf{1}_{C(x, y)} \mathbf{m}_{\epsilon}, \mathbf{m}_{\epsilon} \rangle \\ &= \lim_{\epsilon \downarrow 0} \epsilon \sum_{z \in \Gamma} B(\mathbf{1}_{C(x, y)}(z) \mathbf{m}(z), \mathbf{m}(z)) e^{-\epsilon|z|} \\ &= \lim_{\epsilon \downarrow 0} \epsilon \sum_{z \in \Gamma} B\left(\sum_{i=1}^N \mathbf{m}_i(z), \mathbf{m}(z)\right) e^{-\epsilon|z|} \\ &= \left\langle \sum_{i=1}^N \mathbf{m}_i, \mathbf{m} \right\rangle_{\mathcal{H}^{\infty}}. \end{aligned}$$

The inner product $\langle \sum_{i=1}^N \mathbf{m}_i, \mathbf{m} \rangle_{\mathcal{H}^{\infty}}$, defined as a limit in Chapter 3, does not depend on the subnet $\epsilon_j \downarrow 0$. \square

As a consequence of Lemma 4.22, the limit of $\tilde{\Phi}_{\epsilon}$ does not depend on the subnet when evaluated on finite linear combinations of $\mathbf{1}_{\mathcal{S}(x, y)}$.

Part on the algebra: continuous functions on $\partial\Gamma$

Consider an element $\mathfrak{F} = f\delta_e$ where $f \in C(\partial\Gamma)$ and e is the identity in Γ .

We can uniformly approximate the continuous function f on $\partial\Gamma$ using step functions on $\partial\Gamma$ (this is a consequence of [Rud87, Theorem 1.17] and, since the function f on $\partial\Gamma$ is bounded, the convergence is uniform); let then $\{\mathcal{S}_n\}_n \subseteq B(\partial\Gamma)$ be a sequence of step functions uniformly converging to f as $n \rightarrow \infty$.

We can assume that \mathcal{S}_n are linear combinations of characteristic functions of sets like $\mathcal{O}(x, y)$ for some $x, y \in \Gamma$. In fact, as it is shown in Theorem A.2 in Appendix A, the sets $\{\mathcal{O}(x, y) \mid x, y \in \Gamma\}$ generate the topology on $\partial\Gamma$.

The positive functionals associated to a C*-algebra are bounded, thus continuous (cf. [Con13, Chap. VIII, Cor. 5.12]). So, if $\mathcal{S}_n \rightarrow f$ uniformly, then $\Phi(f) = \lim_{n \rightarrow \infty} \Phi(\mathcal{S}_n)$.

We can assume that \mathcal{S}_n are actually defined on X and are linear combination of characteristic functions of shadows, i.e. of the form

$$\mathcal{S}_n = \sum_{i=1}^N \alpha_i^{(n)} \mathbf{1}_{\mathcal{S}_i^n},$$

for some $N_n \in \mathbb{N}$, $x_i^{(n)}, y_i^{(n)} \in \Gamma$, $i = 1, \dots, N_n$ and where we denote $\mathcal{S}_i^n := \mathcal{S}(x_i^{(n)}, y_i^{(n)})$. We have that $\mathcal{S}(x_i^{(n)}, y_i^{(n)}) \cap \partial\Gamma = \mathcal{O}(x_i^{(n)}, y_i^{(n)})$. Assume then that:

$$\mathcal{S}_n|_{\partial\Gamma} \rightarrow f \quad \text{as } n \rightarrow \infty.$$

We know that $\lim_{\epsilon \downarrow 0} \tilde{\Phi}_\epsilon(\mathcal{S}_n)$ does not depend on the subnet ϵ tending to zero. Thus:

$$\Phi(f) = \tilde{\Phi}(f) = \tilde{\Phi} \left(\lim_{n \rightarrow \infty} \mathcal{S}_n|_{\partial\Gamma} \right) = \lim_{n \rightarrow \infty} \tilde{\Phi}(\mathcal{S}_n|_{\partial\Gamma}),$$

where each $\tilde{\Phi}(\mathcal{S}_n|_{\partial\Gamma}) = \lim_{\epsilon \downarrow 0} \tilde{\Phi}_\epsilon(\mathcal{S}_n|_{\partial\Gamma})$ is well-defined independently on the subnet $\epsilon_j \downarrow 0$ (Lemma 4.22). So, $\Phi(f)$, too, does not depend on the subnet chosen.

This ends the proof of Theorem 4.19: the limit Φ is independent of the subnet tending to zero, so is the associated covariant representation (π, ρ) of $(\Gamma, C(\partial\Gamma))$.

4.5 EQUIVALENCE WITH MULTIPLICATIVE REPRESENTATIONS

Let σ, σ' be two unitary representations of a group G on the Hilbert spaces $\mathcal{K}, \mathcal{K}'$. Then we say that σ and σ' are unitarily equivalent if there exists a unitary operator $U: \mathcal{K} \rightarrow \mathcal{K}'$ such that $U\sigma(g) = \sigma'(g)U$ for every $g \in G$.

In this Section, we show that the Γ -part of the boundary representation (π, ρ) obtained in the previous Sections (namely, the unitary representation π) is unitarily equivalent to the multiplicative representation π_m defined in Chapter 3. This aim is achieved by showing that some matrix coefficients of the representations π_ϵ as in 4.3 converge pointwise to some matrix coefficients of π_m .

We recall the following definitions from [Dix77, Chap. 13]. Given a unitary representation (σ, \mathcal{K}) of a topological group G and vectors $\xi, \eta \in \mathcal{K}$, we recall that the **matrix coefficient** of σ associated to ξ, η is the function $G \rightarrow \mathbb{C}$

$$g \in \Gamma \mapsto \langle \sigma(g)\xi, \eta \rangle_{\mathcal{K}}.$$

For $\xi \in \mathcal{K}$ fixed, the matrix coefficient $g \mapsto \langle \sigma(g)\xi, \xi \rangle_{\mathcal{K}}$ is called the **function of positive type** associated to ξ . It is well known that if the functions of positive type associated to cyclic vectors of two unitary representations coincide, then the two representations are unitarily equivalent (cf. [Dix77, Theorem 13.4.5(iii)]).

Theorem 4.23. *Fix the surface group $\Gamma = \Gamma_2$ with the standard symmetric set of generators S , choose a scalar system $\{V_c, H_{c,c',s}\}$ and define the multiplicative representation π_m (Sect. 3.4 in Chap. 3). Consider the crossed product representation $\pi \times \rho: \Gamma \times C(\partial\Gamma) \rightarrow \mathcal{B}(\mathcal{H})$ associated with the state $\Phi = (w^*) \lim_{\epsilon \downarrow 0} \Phi_\epsilon$ as in Sect. 4.2.3. Then the unitary representation π of Γ is equivalent to the multiplicative representation π_m .*

Proof. The thesis is verified by a direct calculation: it is sufficient to show that the functions of positive type of the two representations coincide.

The functions of positive type of the representation π are given precisely by the values of the state Φ on the elements of Γ , i.e. on the elements of the form $\mathbf{1}_{\partial\Gamma}\delta_g$, where $\mathbf{1}_{\partial\Gamma}(\omega) = 1$ for all $\omega \in \partial\Gamma$.

Fix a multiplicative function $\mathbf{m} \in \mathcal{H}^\infty$, let $\mathbf{m}_\epsilon: x \mapsto \sqrt{\epsilon}e^{-\frac{\epsilon}{2}|x|}\mathbf{m}(x)$ as in equation 3.11 of Chapter 3.

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We remind that $\rho_\epsilon(\mathbf{1}_{\partial\Gamma}) = I$, the identity operator, and that π_ϵ acts on \mathcal{H}_ϵ by left translations. So, we can compute:

$$\begin{aligned}
\Phi(g) &= \Phi(\mathbf{1}_{\partial\Gamma}\delta_g) = \lim_{\epsilon \downarrow 0} \Phi_\epsilon(\mathbf{1}_{\partial\Gamma}\delta_g) \\
&= \lim_{\epsilon \downarrow 0} \langle \pi_\epsilon(g) \mathbf{m}_\epsilon, \mathbf{m}_\epsilon \rangle_2 \\
&= \lim_{\epsilon \downarrow 0} \sum_{z \in \Gamma} B\left(\sqrt{\epsilon} e^{-\frac{\epsilon}{2}|g^{-1}z|} \mathbf{m}(g^{-1}z), \sqrt{\epsilon} e^{-\frac{\epsilon}{2}|z|} \mathbf{m}(z)\right) \\
&= \lim_{\epsilon \downarrow 0} \epsilon \sum_{z \in \Gamma} B(\mathbf{m}(g^{-1}z), \mathbf{m}(z)) e^{-\frac{\epsilon}{2}[|g^{-1}z|+|z|]} \\
&= \lim_{\epsilon \downarrow 0} \epsilon \sum_{z \in \Gamma} B(\mathbf{m}(g^{-1}z), \mathbf{m}(z)) e^{-\epsilon|z|} \\
&= \langle \pi_{\mathbf{m}}(g) \mathbf{m}, \mathbf{m} \rangle_{\mathcal{H}^\infty},
\end{aligned}$$

where in the fifth equality we used (as in Thm. 3.59)

$$e^{-\epsilon(|g|+|z|)} \leq e^{-\epsilon|g^{-1}z|} \leq e^{\epsilon(|g|-|z|)}$$

to estimate

$$\begin{aligned}
e^{-\epsilon|g|} \epsilon \sum_{z \in \Gamma} B(\mathbf{m}(g^{-1}z), \mathbf{m}(z)) e^{-\epsilon|z|} &\leq \\
&\epsilon \sum_{z \in \Gamma} B(\mathbf{m}(g^{-1}z), \mathbf{m}(z)) e^{-\frac{\epsilon}{2}[|g^{-1}z|+|z|]} \\
&\leq e^{\epsilon|g|} \epsilon \sum_{z \in \Gamma} B(\mathbf{m}(g^{-1}z), \mathbf{m}(z)) e^{-\epsilon|z|}.
\end{aligned}$$

This concludes the proof. \square

Consider the representation $\pi \times \tilde{\rho}: \Gamma \times B(\partial\Gamma) \rightarrow \mathfrak{B}(\mathcal{H})$, where $\tilde{\rho}$ is the extension of the limit representation ρ of $C(\partial\Gamma)$ to all of $B(\partial\Gamma)$ obtained as in Proposition 4.21. We can assume that the unitary representation π acts on multiplicative functions, since it is equivalent to the multiplicative representation $\pi_{\mathbf{m}}$. In particular, we have an isometric embedding $J: \mathcal{H}_{\mathbf{m}} \rightarrow \mathcal{H}$. The representation $\tilde{\rho}$ itself can be thought to act on (the isometric image of) multiplicative functions: we can therefore write $\tilde{\rho}(f)\mathbf{m}$ for $f \in B(\partial\Gamma)$ and $\mathbf{m} \in \mathcal{H}^\infty$ to mean $\tilde{\rho}(f)J(\mathbf{m})$.

Lemma 4.24. *In the above hypotheses, if we consider the bounded function $\mathbf{1}_{\mathcal{O}(x,y)} \in B(\partial\Gamma)$, then the operator $\tilde{\rho}(\mathbf{1}_{\mathcal{O}(x,y)})$, where $\mathcal{O}(x,y)$ is as in eq. 4.11, acting on the space $\mathcal{H}_{\mathbf{m}}$ is a projection.*

Proof. If $\mathcal{S}(x, y) = C(x, y) \cup \mathcal{O}(x, y)$ and $\mathbf{m} \in \mathcal{H}^\infty$, we can write:

$$\tilde{\rho}(1_{\mathcal{O}(x,y)}) \mathbf{m} = \mathbf{1}_{\mathcal{S}(x,y) \cap \Gamma} \cdot \mathbf{m} = \mathbf{1}_{C(x,y)} \cdot \mathbf{m}.$$

Since $\text{supp}(\mathbf{m})$ is a cone in Γ , if $z \in \Gamma$:

$$(\mathbf{1}_{C(x,y)} \cdot \mathbf{m})(z) = \begin{cases} \mathbf{m}(z) & \text{if } z \in C(x, y) \cap \text{supp}(\mathbf{m}), \\ 0 & \text{otherwise.} \end{cases}$$

We proved in Theorem 1.34 that the intersection of two cones in the Cayley graph is a finite union of subcones. The function $\mathbf{1}_{C(x,y)} \cdot \mathbf{m}$ is then equivalent (in the sense of Def. 3.18) to a finite sum of multiplicative functions, hence it is itself a multiplicative function, see Lemma 3.23. It holds that:

$$\tilde{\rho}(1_{\mathcal{O}(x,y)})^2 = \tilde{\rho}(1_{\mathcal{O}(x,y)} 1_{\mathcal{O}(x,y)}) = \tilde{\rho}(1_{\mathcal{O}(x,y)}). \quad (4.12)$$

Moreover,

$$\|\tilde{\rho}(1_{\mathcal{O}(x,y)})\| = \sup_{\|\xi\|=1} \|\tilde{\rho}(1_{\mathcal{O}(x,y)}) \xi\| = 1, \quad (4.13)$$

where the maximum is attained for $\mathbf{m} \in \mathcal{H}^\infty$ with support $C(x, y)$ and $\|\mathbf{m}\| = 1$, so that: $\tilde{\rho}(1_{\mathcal{O}(x,y)}) \mathbf{m} = \mathbf{m}$. By [Con13, Chap. II, Prop. 3.3], equations 4.12 and 4.13 are enough to guarantee that $\tilde{\rho}(1_{\mathcal{O}(x,y)})$ is a projection. \square

CHAPTER 5

IRREDUCIBILITY

A representation Π of a C^* -algebra \mathcal{A} on a Hilbert space \mathcal{K} is irreducible if the only subspaces of \mathcal{K} which are stable under the action of $\Pi(\mathcal{A})$ are trivial, i.e. the null space 0 and the whole space \mathcal{K} .

In particular, a representation of the crossed product $\Gamma \rtimes C(\partial\Gamma)$ is irreducible if the only subspaces of the representation space which are invariant both under the action of Γ and under that of $C(\partial\Gamma)$ are trivial.

In Chapter 5 we defined the representation $\pi \times \rho$ of the crossed product C^* -algebra $\Gamma \rtimes C(\partial\Gamma)$, which corresponds to a pair of representations (π, ρ) of $(\Gamma, C(\partial\Gamma))$ that satisfy the covariance condition of Definition 4.1.

In particular, the representation ρ can be extended to the whole C^* -algebra of bounded functions on the boundary $B(\partial\Gamma)$ (cf. Prop. 4.21 and below). We denote with $\tilde{\rho}$ this extension. If $\mathcal{O}(x, y)$ is as in equation 4.11, then $\tilde{\rho}(\mathbf{1}_{\mathcal{O}(x, y)})$ is a projection on \mathcal{H}_m , the Hilbert space of multiplicative functions (Lemma 4.24).

This fact will be important in the proof of the main result of this Chapter.

Theorem 5.1. *Let $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$ be a normalized scalar system associated to the surface group Γ_2 and the standard set of generators S_2 . Then the crossed product representation $\pi \times \rho$ constructed in Chapter 4 is irreducible.*

Remark 5.2. All the metric notions in the present Chapter are intended with respect to the word metric associated to a symmetric set of generators.

5.1 TECHNICAL LEMMATA

The following Lemma will be used in the proof of Lemma 5.12 and in Claim 5.13. These two provide some steps in the proof of Theorem 5.1.

Lemma 5.3. *Let X be a compact Hausdorff space, let $\rho: C(X) \rightarrow \mathfrak{B}(\mathcal{H})$ be a C^* -representation. Consider an orthogonal projection $P: \mathcal{H} \rightarrow \mathcal{H}$ commuting with ρ , i.e. $P\rho(f) = \rho(f)P$ for all $f \in C(X)$. Then $P\tilde{\rho}(u) = \tilde{\rho}(u)P$ for all $u \in B(X)$, where $\tilde{\rho}: B(X) \rightarrow \mathfrak{B}(\mathcal{H})$ is the extension of ρ to all of $B(X)$.*

Proof. Fix $\xi \in \mathcal{H}$ and consider the positive measure defined by $E_{\xi, \xi}(B) := \langle \tilde{\rho}(\mathbf{1}_B)\xi, \xi \rangle$, where $B \subseteq X$ is a Borel subset. Fix a bounded Borel function $u \in B(X)$. It is well known that the space $C_c(X) = C(X)$ is dense in $L^p(X, dE_{\xi, \xi})$, for all $p \in [1, \infty)$ (cf. [Rud87, Theorem 3.14]). Thus, in particular, $C(X)$ is dense in $L^2(X, dE_{\xi, \xi})$ with respect to the norm $\|f\|_2^2 = \int_X |f|^2 dE_{\xi, \xi}$, where $f \in C(X)$. Since X is compact, then $u \in L^2(X, dE_{\xi, \xi})$. Therefore, we can find a sequence $\{f_n\}_n \subseteq C(X)$ such that $f_n \rightarrow u$ as $n \rightarrow \infty$ and the convergence is in norm of $L^2(X, dE_{\xi, \xi})$.

Compute now

$$\begin{aligned}
\|(P\tilde{\rho}(u) - \tilde{\rho}(u)P)\xi\| &= \|(P\tilde{\rho}(u) - P\rho(f_n) + P\rho(f_n) - \tilde{\rho}(u)P)\xi\| \\
&= \|(P\tilde{\rho}(u) - P\rho(f_n) + \rho(f_n)P - \tilde{\rho}(u)P)\xi\| \\
&\leq \|(P\tilde{\rho}(u) - P\rho(f_n))\xi\| + \|(\rho(f_n)P - \tilde{\rho}(u)P)\xi\| \\
&= \|P\tilde{\rho}(u - f_n)\xi\| + \|\tilde{\rho}(f_n - u)P\xi\| \\
&\leq \|P\| \|\tilde{\rho}(u - f_n)\xi\| + \|\tilde{\rho}(f_n - u)(P\xi)\|,
\end{aligned}$$

where we used that $P\rho(f_n) = \rho(f_n)P$, $\tilde{\rho}(f_n) = \rho(f_n)$ for $n \in \mathbb{N}$ and $\|P\| = 1$. By the Functional Calculus, for every bounded Borel function ϕ on X we have that $\|(\int_X \phi dE)\xi\|^2 = \int_X |\phi|^2 dE_{\xi, \xi}$. In particular, since $f_n \rightarrow u$ in $L^2(X, dE_{\xi, \xi})$:

$$\|\tilde{\rho}(u - f_n)\xi\|^2 = \int_X |u - f_n|^2 dE_{\xi, \xi} \rightarrow 0 \quad (5.1)$$

as $n \rightarrow \infty$. We claim that for $\xi \in \mathcal{H}$ and P projection on \mathcal{H} the positive measures $E_{\xi, P\xi}$ and $E_{P\xi, P\xi}$ coincide. It is indeed sufficient to show that $\int_X f dE_{\xi, P\xi} = \int_X f dE_{P\xi, P\xi}$ for every $f \in C(X)$. Since $P = P^2 = P^*$, by definition of the measures: $\int_X f dE_{\xi, P\xi} = \langle \rho(f)\xi, P\xi \rangle = \langle P\rho(f)\xi, P\xi \rangle =$

$\langle \rho(f)P\xi, P\xi \rangle = \int_X f dE_{P\xi, P\xi}$. Moreover, we claim that the positive measure $E_{\xi, P\xi}$ is absolutely continuous with respect to $E_{\xi, \xi}$. In fact, if B is any Borel set in X , then $E_{\xi, \xi}(B) = 0$ means that $\langle \tilde{\rho}(\mathbf{1}_B)\xi, \xi \rangle = 0$, which, since $\tilde{\rho}(\mathbf{1}_B)$ is a positive idempotent, yields $\tilde{\rho}(\mathbf{1}_B)\xi = 0$, so that $E_{\xi, P\xi}(B) = \langle \tilde{\rho}(\mathbf{1}_B)\xi, P\xi \rangle = 0$. Therefore, by equation 5.1 we also get:

$$\|\tilde{\rho}(f_n - u)(P\xi)\|^2 = \int_X |f_n - u|^2 dE_{P\xi, P\xi} \rightarrow 0.$$

Thus we can conclude:

$$\|(P\tilde{\rho}(u) - \tilde{\rho}(u)P)\xi\| \leq \|\tilde{\rho}(u - f_n)\xi\| + \|\tilde{\rho}(f_n - u)(P\xi)\| \xrightarrow{n \rightarrow \infty} 0,$$

which tells us that $\|(P\tilde{\rho}(u) - \tilde{\rho}(u)P)\xi\| = 0$. Since $\xi \in \mathcal{H}$ is arbitrary, we get that $P\tilde{\rho}(u) = \tilde{\rho}(u)P$. \square

The second Lemma we need is about geodesics in the Cayley graph \mathcal{G} having the same limit point. This is an “infinite version” of Lemma 1.24.

Lemma 5.4. *Given two geodesics γ_1, γ_2 in the graph $\mathcal{G}_k = \mathcal{G}(\Gamma_k, S_k)$, if $\gamma_1(0) = \gamma_2(0)$ and $\gamma_1(\infty) = \gamma_2(\infty)$, then, for every $i \in \mathbb{N}$ there exists a 2-cell P in the Cayley complex such that both $\gamma_1(i)$ and $\gamma_2(i)$ belong to the boundary of P (write $\gamma_1(i) \in \partial P$, $\gamma_2(i) \in \partial P$). In particular, $\sup_{t \in [0, \infty)} d_{\mathcal{G}_k}(\gamma_1(t), \gamma_2(t)) \leq 2k$.*

The proof of Lemma 5.4 was suggested by A. Sisto.

Proof. By the term “polygon” we mean a 2-cell in the Cayley complex associated to \mathcal{G}_k (cf. Section 1.2).

We can assume without loss that $\gamma_1(t) \neq \gamma_2(t)$ for $0 < t < \infty$. Otherwise, let t_0 be such that $\gamma_1(t_0) = \gamma_2(t_0)$: then we divide the geodesic digon in two parts: the first one formed by the geodesic segments $\gamma_1[0, t_0]$ and $\gamma_2[0, t_0]$, the second one given by $\gamma_1[t_0, \infty)$ and $\gamma_2[t_0, \infty)$. The first part of the digon satisfies the thesis by the classification of hyperbolic digons and triangles in $C'(\frac{1}{6})$ -small cancellation complexes: it is just a consequence of Lemma 1.24. We deal then with the second part after reparameterizing $\gamma_1[t_0, \infty)$ and $\gamma_2[t_0, \infty)$.

Given a polygon P in the Cayley complex, if we denote by ∂P the boundary of the 2-cell, we define

$$d(e, P) = \min_{x \in \partial P} d(e, x).$$

By contradiction, assume that there exists a finite number of polygons intersecting both γ_1 and γ_2 (i.e. such that at every step both the geodesics belong to the boundary of a common cell).

Let then P_0 be the polygon intersecting both γ_1 and γ_2 which maximizes the value of $d(e, P)$ (which is a non-negative integer and is bounded by the contradiction hypothesis). There exists $n_0 \in \mathbb{N}$ with the following two properties:

- for every $i \leq n_0$ there exists a 2-cell of the complex P_i such that both $\gamma_1(i) \in \partial P_i$ and $\gamma_2(i) \in \partial P_i$;
- for every $i > n_0$ there is no 2-cell of the complex P such that $\gamma_1(i) \in \partial P$ and $\gamma_2(i) \in \partial P$.

Consider now $m \in \mathbb{N}$, $m > n_0$. So, the vertices $\gamma_1(m), \gamma_2(m)$ do not lie on (the boundary of) a common polygon. Fix a geodesic triangle having vertices $e, \gamma_1(m), \gamma_2(m)$, let η be a geodesic joining $\gamma_1(m)$ and $\gamma_2(m)$.

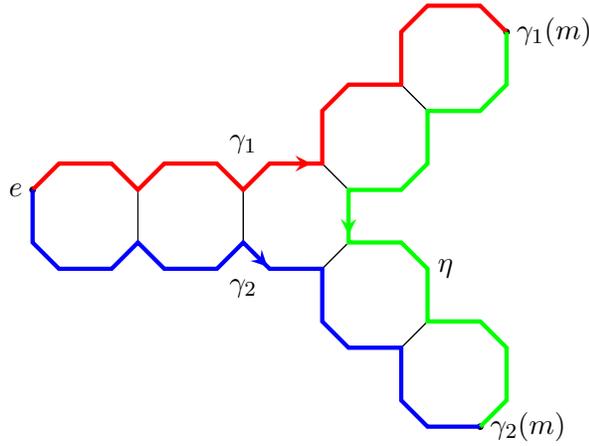


Figure 5.1: A geodesic triangle in the Cayley complex $\mathcal{C}_{S,R}(\Gamma)$ for genus $k = 2$.

By the classification of geodesic triangles in the graph \mathcal{G}_k , the triangle with vertices $e, \gamma_1(m), \gamma_2(m)$ has one of the forms II, III₁ or III₂ in Figure 5.2: the polygon P_0 is a common central 2-cell intersecting all the three geodesic segments γ_1, γ_2 and η ; moreover, after P_0 the two geodesics γ_1, γ_2 “diverge”.

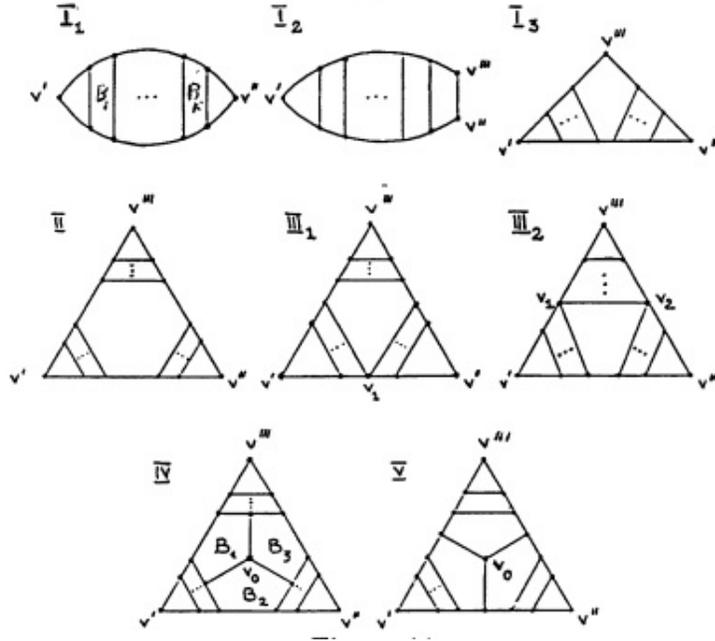


Figure 5.2: The classification of geodesic triangles in small cancellation complexes, cf. [Str90, Fig. 11].

In particular, we claim that we can estimate the length of the geodesic η : in fact, the length of the geodesic segment of γ_1 joining $\gamma_1(n_0)$ and $\gamma_1(m)$ is $m - n_0$. The same holds for the part of γ_2 joining $\gamma_2(n_0)$ and $\gamma_2(m)$. Hence, the part of η between $\gamma_1(m)$ and the polygon P_0 has length comparable (up to a constant) with $m - n_0$: it intersects the boundaries of the same 2-cells as the segment $[\gamma_1(n_0), \gamma_1(m)]$ of γ_1 in not less than 1 edge and not more than $2k$ edges (see Figure 5.1). Similar considerations hold for the part of η between P_0 and $\gamma_2(m)$. Therefore, $\ell(\eta) \approx 2(m - n_0)$, i.e. there exist constants $c_1, c_2, C_1, C_2 > 0$ depending on the graph \mathcal{G}_k such that

$$2c_1(m - n_0) - c_2 \leq \ell(\eta) \leq 2C_1(m - n_0) - C_2.$$

An example in Figure 5.3.

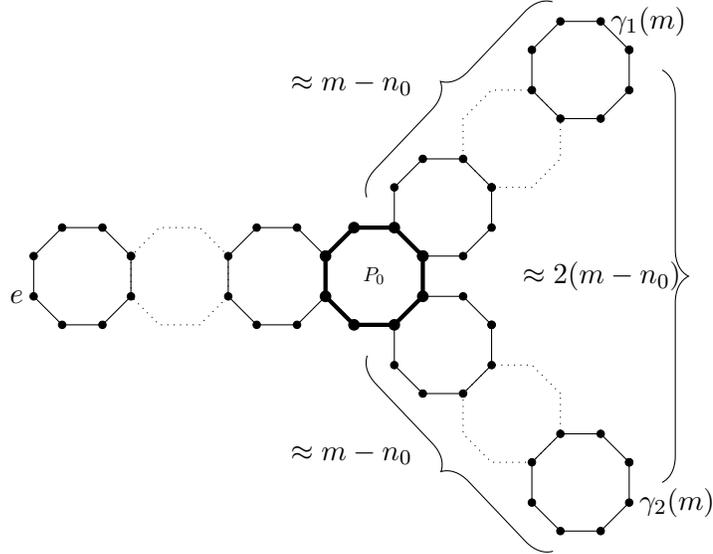


Figure 5.3: The geodesic joining $\gamma_1(m)$ and $\gamma_2(m)$ has length comparable with $(m - n_0)$.

Since η is a geodesic joining $\gamma_1(m)$ and $\gamma_2(m)$, for $m > n_0$ large we can estimate:

$$d(\gamma_1(m), \gamma_2(m)) = \ell(\eta) \approx 2(m - n_0),$$

which, for m large enough, grows linearly in m , contradicting the hypothesis that

$$\sup_{t \in [0, \infty)} d(\gamma_1(t), \gamma_2(t)) < \infty,$$

i.e. the assumption that $\gamma_1(\infty) = \gamma_2(\infty)$. □

Remark 5.5. We recall that we identify each cone type \mathbf{c} in Γ with the unique cone $C(z, e)$ having vertex in the identity and belonging to the orbit (cf. Remark 1.29); we remind, moreover, that given $x \in \Gamma$ and \mathbf{c} a cone type, the symbol $x.\mathbf{c}$ denotes the action of x on the cone with vertex in the identity given by \mathbf{c} , i.e. $x.\mathbf{c} = x.C(z, e) = C(xz, z)$ (cf. Notation 3.35).

Recall that, for $g \in \Gamma$ and $\pi_{\mathfrak{m}}$ the multiplicative representation of Chapter 3, the operator $\pi_{\mathfrak{m}}(g)$ acts on \mathcal{H}^∞ by left translation: if $\mathfrak{m} \in \mathcal{H}^\infty$, then

$$\pi_{\mathfrak{m}}(g)\mathfrak{m}(z) = \mathfrak{m}(g^{-1}z) \quad \text{for } z \in \Gamma.$$

If (π, ρ, \mathcal{H}) is the boundary representation constructed in Chapter 4, then the unitary representation π is equivalent to $\pi_{\mathfrak{m}}$ and we identify the two actions on $\mathcal{H}_{\mathfrak{m}}$ (cf. Section 4.5).

Lemma 5.6. *Let $x, y \in \Gamma$, let $\mathfrak{c} = C(y^{-1}x, e)$, let $v \in V_{\mathfrak{c}}$, consider the elementary multiplicative function $\mathfrak{m}[x, y, v]$ supported on $C(x, y)$. Then:*

$$\mathfrak{m}[x, y, v] = \mathfrak{m}[y^{-1}x, e, v](y^{-1} \cdot) = \pi(y)\mathfrak{m}[y^{-1}x, e, v].$$

Proof. The support of $\mathfrak{m}[x, y, v]$ is the cone $C(x, y)$, which is a translate of $C(y^{-1}x, e)$: in fact, $y.C(y^{-1}x, e) = C(x, y)$ (by definition of the action of Γ on the set of cones, cf. Sect. 1.7 in Chap. 1). The cone $C(y^{-1}x, e)$ is the support of $\mathfrak{m}[y^{-1}x, e, v]$. Let $z \in C(x, y)$: then $y^{-1}z \in C(y^{-1}x, e)$. So, if $\mathfrak{m}[x, y, v]$ is non-zero on z , then the same holds for $\mathfrak{m}[y^{-1}x, e, v]$ on $y^{-1}z$. If $z = y$, then $\mathfrak{m}[x, y, v](y) = v$; moreover, $\mathfrak{m}[y^{-1}x, e, v](y^{-1}y) = v$, so the functions coincide on the vertices of their supports.

If $\gamma = \{z_0, \dots, z_n\}$ is a geodesic, then the action $y.\gamma$ denotes the correspondent actions of y on the vertices and edges of γ : $y.\gamma = \{yz_0, \dots, yz_n\}$. Let $z \in C(x, y) \setminus \{y\}$. Then $\gamma \in [e, y^{-1}z]$ if and only if $y.\gamma \in [y, z]$ and the maps H_{γ} and $H_{y.\gamma}$ coincide by the definition in Sect. 3.1.2. So:

$$\begin{aligned} \pi(y)\mathfrak{m}[y^{-1}x, e, v](z) &= \mathfrak{m}[y^{-1}x, e, v](y^{-1}z) \\ &= \sum_{\gamma \in [e, y^{-1}z]} H_{\gamma}v \\ &= \sum_{y.\gamma \in [y, z]} H_{y.\gamma}v \\ &= \mathfrak{m}[x, y, v](z). \end{aligned}$$

This proves the claim. □

5.2 INTERTWINING PROJECTIONS

Consider now the representation $\pi \times \rho$ of the crossed product $\Gamma \times C(\partial\Gamma)$ which we defined in Chapter 4.

An intertwiner for the representation $\pi \times \rho$ is an operator $P: \mathcal{H} \rightarrow \mathcal{H}$ which commutes both with π and with ρ . By Schur's Lemma, the representation $\pi \times \rho$ is irreducible if and only if every intertwiner P for $\pi \times \rho$ is a scalar multiple of the identity operator I . Because of the following general result, we can choose P to be an orthogonal projection of \mathcal{H} .

Theorem 5.7 ([Con13]). *Let (Π, \mathcal{K}) be a C^* -representation of a C^* -algebra \mathcal{A} . Then Π is irreducible if and only if every projection on \mathcal{K} commuting with Π is trivial (i.e. 0 or I itself).*

We apply this Theorem to $\mathcal{A} = \Gamma \rtimes C(\partial\Gamma)$ and $\Pi = \pi \rtimes \rho$.

By Lemma 5.3, if P commutes with ρ , then P commutes with the representation $\tilde{\rho}$ or $B(\partial\Gamma)$. We can think of P as acting on the multiplicative functions in \mathcal{H}^∞ : in fact, the limit representation π is equivalent to the multiplicative representation π_m (see Thm. 4.23). Therefore, P satisfies

$$\begin{aligned}\pi(g)P &= P\pi(g) \quad \text{for all } g \in \Gamma, \\ \tilde{\rho}(f)P &= P\tilde{\rho}(f) \quad \text{for all } f \in B(\partial\Gamma).\end{aligned}$$

Remark 5.8. We recall again Notation 3.35: we identify each cone type (which by definition is an orbit) with its unique cone representative having vertex in the identity. Hence, the notation $x.\mathbf{c}$, where $x \in \Gamma$ and $\mathbf{c} \in \mathcal{C}$, means the action of the group element x on the cone with vertex in the identity representing \mathbf{c} : this yields again a cone in the orbit of \mathbf{c} .

Notation 5.9. Let $z \in \Gamma$, let $\mathbf{c} = C(z, e)$, let $v \in V_{\mathbf{c}}$. We denote by $\mathbf{m}[e, \mathbf{c}, v]$ the elementary function $\mathbf{m}[z, e, v]$. Similarly, if $x \in \Gamma$ and $x.\mathbf{c} = C(xz, x)$, we denote by $\mathbf{m}[x, \mathbf{c}, v]$ the function $\mathbf{m}[xz, x, v]$.

Choose an arbitrary $x \in \Gamma$, let $(\mathbf{c}_1, \mathbf{c}_2, y)$ be a compatible triple (cf. Def. 3.42), assume that

$$\begin{aligned}\mathbf{c}_1 &= C(z_1, e), \\ \mathbf{c}_2 &= C(z_2, e).\end{aligned}$$

Then $\mathbf{c}_1 \cap y.\mathbf{c}_2 \neq \emptyset$. Consider the cones $x.\mathbf{c}_1$ and $xy.\mathbf{c}_2$, i.e.

$$\begin{aligned}x.\mathbf{c}_1 &= x.C(z_1, e) = C(xz_1, x), \\ xy.\mathbf{c}_2 &= xy.C(z_2, e) = C(xy z_2, xy),\end{aligned}$$

and consider the multiplicative functions supported on these cones $\mathbf{m}[x, \mathbf{c}_1, v_1] = \mathbf{m}[xz_1, x, v_1]$ and $\mathbf{m}[xy, \mathbf{c}_2, v_2] = \mathbf{m}[xy z_2, xy, v_2]$, where $v_1 \in V_{\mathbf{c}_1}$ and $v_2 \in \overline{V_{\mathbf{c}_2}}$ are arbitrary (cf. Notation 5.9). Define B' as the linear form on $V_{\mathbf{c}_1} \otimes \overline{V_{\mathbf{c}_2}}$ whose value on $v_1 \otimes \overline{v_2}$ is

$$B'_{\mathbf{c}_1, \mathbf{c}_2, y}(v_1 \otimes \overline{v_2}) := \langle P\mathbf{m}[x, \mathbf{c}_1, v_1], \mathbf{m}[xy, \mathbf{c}_2, v_2] \rangle. \quad (5.2)$$

When we range over $v_1 \in V_{\mathbf{c}_1}$ and $v_2 \in V_{\mathbf{c}_2}$, we change the arguments in B' . Varying $(\mathbf{c}_1, \mathbf{c}_2, y)$ among the compatible triples, we get a linear

form $B' = (B'_{\mathbf{c}_1, \mathbf{c}_2, y})$ on the space $\bigoplus_{\mathbf{c}_1, \mathbf{c}_2, y} (V_{\mathbf{c}_1} \otimes \overline{V_{\mathbf{c}_2}})_y$ (the sum is over all compatible triples and each $(V_{\mathbf{c}_1} \otimes \overline{V_{\mathbf{c}_2}})_y$ appearing is just a copy of $V_{\mathbf{c}_1} \otimes \overline{V_{\mathbf{c}_2}}$). Thus, B' belongs to the dual space

$$B' \in \left[\bigoplus_{\mathbf{c}_1, \mathbf{c}_2, y} (V_{\mathbf{c}_1} \otimes \overline{V_{\mathbf{c}_2}})_y \right]^*.$$

Remark 5.10. Recall that the matrix \mathfrak{T}^* as in Def. 3.38 acts on this same space.

We claim that, as long as P commutes with the unitary representation π , the definition of B' is well posed, i.e. it does not depend on the choice of $x \in \Gamma$, but only on the triple $(\mathbf{c}_1, \mathbf{c}_2, y)$.

Lemma 5.11. *If the projection P commutes with the representation π , then the definition of $B'_{\mathbf{c}_1, \mathbf{c}_2, y}$ does not depend on $x \in \Gamma$, but only on $(\mathbf{c}_1, \mathbf{c}_2, y)$.*

Proof. By Lemma 5.6 (adapted to Notation 5.9):

$$\mathbf{m}[x, \mathbf{c}_1, v_1] = \pi(x)\mathbf{m}[e, \mathbf{c}_1, v_1],$$

similarly, $\mathbf{m}[xy, \mathbf{c}_2, v_2] = \pi(x)\mathbf{m}[y, \mathbf{c}_2, v_2]$. Since π is a unitary representation, for every $x \in \Gamma$ it holds that $\pi(x)^*\pi(x) = I$ (the identity operator). By the hypothesis, $\pi(x)P = P\pi(x)$ for all $x \in \Gamma$. Then we have:

$$\begin{aligned} B'_{\mathbf{c}_1, \mathbf{c}_2, y}(v_1 \otimes \bar{v}_2) &= \langle P\mathbf{m}[x, \mathbf{c}_1, v_1], \mathbf{m}[xy, \mathbf{c}_2, v_2] \rangle \\ &= \langle P\pi(x)\mathbf{m}[e, \mathbf{c}_1, v_1], \pi(x)\mathbf{m}[y, \mathbf{c}_2, v_2] \rangle \\ &= \langle \pi(x)P\mathbf{m}[e, \mathbf{c}_1, v_1], \pi(x)\mathbf{m}[y, \mathbf{c}_2, v_2] \rangle \quad (5.3) \\ &= \langle \pi(x)^*\pi(x)P\mathbf{m}[e, \mathbf{c}_1, v_1], \mathbf{m}[y, \mathbf{c}_2, v_2] \rangle \\ &= \langle P\mathbf{m}[e, \mathbf{c}_1, v_1], \mathbf{m}[y, \mathbf{c}_2, v_2] \rangle. \end{aligned}$$

In particular, the definition of $B'_{\mathbf{c}_1, \mathbf{c}_2, y}$ does not depend on the choice of $x \in \Gamma$, but only on the triple $(\mathbf{c}_1, \mathbf{c}_2, y)$. \square

Lemma 5.12. *Assume that the projection P commutes with the representation $\tilde{\rho}$ of $B(\partial\Gamma)$ and $B' = (B'_{\mathbf{c}_1, \mathbf{c}_2, y})$ is as in equation 5.2. Then $\mathfrak{T}^*B' = B'$, i.e. the linear form B' is an eigenvector with eigenvalue one for the linear map corresponding to the matrix \mathfrak{T}^* .*

Lemma 5.12 requires some preliminary analysis, which we include in Claim 5.13.

Recall that by definition the matrix \mathfrak{D} as in eq. 3.10 has scalar entries as follows:

$$\mathfrak{D}_{\mathbf{c}'_1, \mathbf{c}'_2, s^{-1}yt}^{\mathbf{c}_1, \mathbf{c}_2, y} = \begin{cases} 0 & \text{if } \mathfrak{T}_{\mathbf{c}'_1, \mathbf{c}'_2, s^{-1}yt}^{\mathbf{c}_1, \mathbf{c}_2, y} = 0, \\ 1 & \text{if } \mathfrak{T}_{\mathbf{c}'_1, \mathbf{c}'_2, s^{-1}yt}^{\mathbf{c}_1, \mathbf{c}_2, y} \neq 0, \end{cases}$$

and that we have

$$\mathfrak{D}_{\mathbf{c}'_1, \mathbf{c}'_2, s^{-1}yt}^{\mathbf{c}_1, \mathbf{c}_2, y} \left(H_{\mathbf{c}_1, \mathbf{c}'_1, s} \otimes \overline{H}_{\mathbf{c}_2, \mathbf{c}'_2, t} \right) = \mathfrak{T}_{\mathbf{c}'_1, \mathbf{c}'_2, s^{-1}yt}^{\mathbf{c}_1, \mathbf{c}_2, y}. \quad (5.4)$$

Claim 5.13. *Let $x \in \Gamma$, let $(\mathbf{c}_1, \mathbf{c}_2, y)$ be a compatible triple. Let $s, t \in S$ such that $\mathbf{c}_1 \xrightarrow{s} \mathbf{c}'_1$ and $\mathbf{c}_2 \xrightarrow{t} \mathbf{c}'_2$. Fix $z_1, z_2 \in \Gamma$ such that the cones $C(xz_1, x)$, $C(xyz_2, xy)$, $C(xz_1, xs)$ and $C(xyz_2, xyt)$ have respectively cone types \mathbf{c}_1 , \mathbf{c}_2 , \mathbf{c}'_1 , \mathbf{c}'_2 . Then $\mathbf{1}_{C(xz_1, xs)} \mathbf{1}_{C(xyz_2, xyt)} \neq 0$ (not identically the zero function) if and only if $(\mathbf{c}'_1, \mathbf{c}'_2, s^{-1}yt)$ is a compatible triple and $\mathfrak{D}_{\mathbf{c}'_1, \mathbf{c}'_2, s^{-1}yt}^{\mathbf{c}_1, \mathbf{c}_2, y} = 1$.*

Proof of Claim 5.13. Following Not. 3.35, we write $xs.\mathbf{c}'_1 := C(xz_1, xs)$ and $xyt.\mathbf{c}'_2 := C(xyz_2, xyt)$.

Since $(\mathbf{c}_1, \mathbf{c}_2, y)$ is a compatible triple, it holds that $\mathbf{c}_1 \cap y.\mathbf{c}_2 \neq \emptyset$, $|y| \leq 2k$ and x, xy belong to a common 2-cell of the tiling of \mathbb{H}^2 associated with the Cayley graph \mathcal{G}_k .

Assume that $\mathbf{1}_{xs.\mathbf{c}'_1} \mathbf{1}_{xyt.\mathbf{c}'_2}$ is not the null function. Then, there exists $z \in xs.\mathbf{c}'_1 \cap xyt.\mathbf{c}'_2$. In particular, $z \in x.\mathbf{c}_1 \cap xy.\mathbf{c}_2$. Therefore, there exist two geodesics $\gamma_1 \in [x, z]$ passing through xs and $\gamma_2 \in [xy, z]$ passing through xyt . By Lemma 1.23 and $|y| \leq 2k$, the geodesic triangle with vertices x , xyt and z must have the form I_2 of Figure 1.1 of Chapter 1, as reported in Figure 5.4 (see also Rem. 1.22).

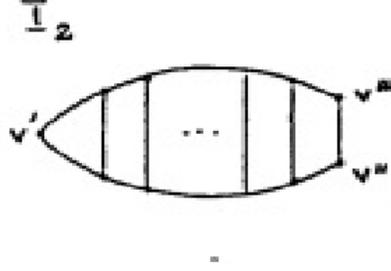


Figure 5.4: Form I_2 of Figure 1.1 in Chap. 1. The vertices v'' , v''' correspond to x and xy , while v' corresponds to z . A 2-cell in the figure is a $2k$ -gon for us.

In other words, since $|y| \leq 2k$ and x , xy belong to the boundary of a common $2k$ -gon, then xs and xyt , too, belong to a common 2-cell and $|s^{-1}yt| \leq 2k$, which guarantees that $(\mathbf{c}'_1, \mathbf{c}'_2, s^{-1}yt)$ is a compatible triple (by the hypothesis we know that $\mathbf{c}_1 \xrightarrow{s} \mathbf{c}'_1$ and $\mathbf{c}_2 \xrightarrow{t} \mathbf{c}'_2$). By definition, then, $\mathfrak{D}_{\mathbf{c}'_1, \mathbf{c}'_2, s^{-1}yt}^{\mathbf{c}_1, \mathbf{c}_2, y} = 1$.

Conversely, if $(\mathbf{c}'_1, \mathbf{c}'_2, s^{-1}yt)$ is a compatible triple and $x \in \Gamma$ is as in the hypothesis, assume $\mathfrak{D}_{\mathbf{c}'_1, \mathbf{c}'_2, s^{-1}yt}^{\mathbf{c}_1, \mathbf{c}_2, y} = 1$. Then we can find $z \in x.\mathbf{c}_1 \cap xy.\mathbf{c}_2$ such that $z \in xs.\mathbf{c}'_1 \cap xyt.\mathbf{c}'_2$. This implies that $\mathbf{1}_{xs.\mathbf{c}'_1} \mathbf{1}_{xyt.\mathbf{c}'_2}$ is not the null function. \square

Now we can prove Lemma 5.12.

Proof of Lemma 5.12. Fix $x \in \Gamma$. If $x.\mathbf{c}_1 = C(xz_1, x)$ and $xy.\mathbf{c}_2 = C(xyz_2, xy)$ are two cones having cone types \mathbf{c}_1 and \mathbf{c}_2 respectively, we define the subset of generators:

$$\begin{aligned} S_1 &= \{s \in S \mid xs \in C(xz_1, x)\}, \\ S_2 &= \{t \in S \mid xyt \in C(xyz_2, xy)\}. \end{aligned}$$

Let $(\mathbf{c}_1, \mathbf{c}_2, y)$ be a compatible triple. If $v_1 \in V_{\mathbf{c}_1}$ and $v_2 \in V_{\mathbf{c}_2}$ are arbitrary,

then, by Lemma 3.21:

$$\begin{aligned} \mathbf{m}[xz_1, x, v_1] &\sim \sum_{s \in S_1} \mathbf{m}[xz_1, xs, H_{\mathbf{c}_1, \mathbf{c}'_1, s} v_1], \\ \mathbf{m}[xyz_2, xy, v_2] &\sim \sum_{t \in S_2} \mathbf{m}[xyz_2, xyt, H_{\mathbf{c}_2, \mathbf{c}'_2, t} v_1]. \end{aligned}$$

Assume that \mathbf{c}'_1 ranges over the first-level subcones of \mathbf{c}_1 and \mathbf{c}'_2 ranges over the first-level subcones of \mathbf{c}_2 : each \mathbf{c}'_1 is determined by $\mathbf{c}_1 \xrightarrow{s} \mathbf{c}'_1$; similarly, \mathbf{c}'_2 is determined by $\mathbf{c}_2 \xrightarrow{t} \mathbf{c}'_2$. Then we can write, following Notation 5.9:

$$\begin{aligned} \mathbf{m}[xz_1, x, v_1] &= \mathbf{m}[x, \mathbf{c}_1, v_1] \sim \sum_{s \in S_1} \mathbf{m}[xs, \mathbf{c}'_1, H_{\mathbf{c}_1, \mathbf{c}'_1, s} v_1], \\ \mathbf{m}[xyz_2, xy, v_2] &= \mathbf{m}[xy, \mathbf{c}_2, v_2] \sim \sum_{t \in S_2} \mathbf{m}[xyt, \mathbf{c}'_2, H_{\mathbf{c}_2, \mathbf{c}'_2, t} v_1]. \end{aligned}$$

Let $xs.\mathbf{c}'_1$ range among the first-level subcones of $C(xz_1, x)$ and $xyt.\mathbf{c}'_2$ range among those of $C(xyz_2, xy)$. It is clear that

$$\mathbf{1}_{xs.\mathbf{c}'_1} \mathbf{m}[xs, \mathbf{c}'_1, H_{\mathbf{c}_1, \mathbf{c}'_1, s} v_1] = \mathbf{m}[xs, \mathbf{c}'_1, H_{\mathbf{c}_1, \mathbf{c}'_1, s} v_1]$$

and

$$\mathbf{1}_{xyt.\mathbf{c}'_2} \mathbf{m}[xyt, \mathbf{c}'_2, H_{\mathbf{c}_2, \mathbf{c}'_2, t} v_1] = \mathbf{m}[xyt, \mathbf{c}'_2, H_{\mathbf{c}_2, \mathbf{c}'_2, t} v_1].$$

In fact, we are just multiplying the elementary functions by the characteristic functions of their supports.

We denote the set $\mathcal{O}(xs.\mathbf{c}'_1) := \mathcal{O}(xz_1, xs)$ (cf. eq. 4.11 in Chap. 4) and similarly $\mathcal{O}(xyt.\mathbf{c}'_2) := \mathcal{O}(xyz_2, xyt)$. Since $v_1 \in V_{\mathbf{c}_1}$ and $v_2 \in V_{\mathbf{c}_2}$:

$$\begin{aligned} B'(v_1 \otimes \bar{v}_2) &= B'_{\mathbf{c}_1, \mathbf{c}_2, y}(v_1 \otimes \bar{v}_2) = \langle P\mathbf{m}[x, \mathbf{c}_1, v_1], \mathbf{m}[xy, \mathbf{c}'_2, v_2] \rangle \\ &= \sum_{s \in S_1} \sum_{t \in S_2} \langle P\mathbf{m}[xs, \mathbf{c}'_1, H_{\mathbf{c}_1, \mathbf{c}'_1, s} v_1], \mathbf{m}[xyt, \mathbf{c}'_2, H_{\mathbf{c}_2, \mathbf{c}'_2, t} v_2] \rangle \\ &= \sum_{s \in S_1} \sum_{t \in S_2} \langle P\mathbf{1}_{xs.\mathbf{c}'_1} \mathbf{m}[xs, \mathbf{c}'_1, H_{\mathbf{c}_1, \mathbf{c}'_1, s} v_1], \mathbf{1}_{xyt.\mathbf{c}'_2} \mathbf{m}[xyt, \mathbf{c}'_2, H_{\mathbf{c}_2, \mathbf{c}'_2, t} v_2] \rangle \\ &= \sum_{s \in S_1} \sum_{t \in S_2} \langle P\tilde{\rho}(\mathbf{1}_{\mathcal{O}(xs.\mathbf{c}'_1)}) \mathbf{m}[xs, \mathbf{c}'_1, H_{\mathbf{c}_1, \mathbf{c}'_1, s} v_1], \tilde{\rho}(\mathbf{1}_{\mathcal{O}(xyt.\mathbf{c}'_2)}) \mathbf{m}[xyt, \mathbf{c}'_2, H_{\mathbf{c}_2, \mathbf{c}'_2, t} v_2] \rangle \\ &= \sum_{s \in S_1} \sum_{t \in S_2} \langle \tilde{\rho}(\mathbf{1}_{\mathcal{O}(xs.\mathbf{c}'_1)}) P\mathbf{m}[xs, \mathbf{c}'_1, H_{\mathbf{c}_1, \mathbf{c}'_1, s} v_1], \tilde{\rho}(\mathbf{1}_{\mathcal{O}(xyt.\mathbf{c}'_2)}) \mathbf{m}[xyt, \mathbf{c}'_2, H_{\mathbf{c}_2, \mathbf{c}'_2, t} v_2] \rangle \\ &= \sum_{s \in S_1} \sum_{t \in S_2} \langle \tilde{\rho}(\mathbf{1}_{\mathcal{O}(xyt.\mathbf{c}'_2)}) \tilde{\rho}(\mathbf{1}_{\mathcal{O}(xs.\mathbf{c}'_1)}) P\mathbf{m}[xs, \mathbf{c}'_1, H_{\mathbf{c}_1, \mathbf{c}'_1, s} v_1], \mathbf{m}[xyt, \mathbf{c}'_2, H_{\mathbf{c}_2, \mathbf{c}'_2, t} v_2] \rangle \\ &= \sum_{s \in S_1} \sum_{t \in S_2} \langle \mathbf{1}_{xyt.\mathbf{c}'_2} \mathbf{1}_{xs.\mathbf{c}'_1} P\mathbf{m}[xs, \mathbf{c}'_1, H_{\mathbf{c}_1, \mathbf{c}'_1, s} v_1], \mathbf{m}[xyt, \mathbf{c}'_2, H_{\mathbf{c}_2, \mathbf{c}'_2, t} v_2] \rangle. \end{aligned}$$

We used the fact that $\tilde{\rho}(\mathbf{1}_{\mathcal{O}(xs.c'_1)})P = P\tilde{\rho}(\mathbf{1}_{\mathcal{O}(xs.c'_1)})$ (similarly for the other) implied by Lemma 5.3 and that $\tilde{\rho}(\mathbf{1}_{\mathcal{O}(xs.c'_1)})$ is a projection (cf. Lemma 4.24).

By Claim 5.13, if $\mathbf{1}_{xs.c'_1}\mathbf{1}_{xyt.c'_2}$ is not identically zero, then $\mathfrak{D}_{c'_1,c'_2,s^{-1}yt}^{c_1,c_2,y}$ is 1. The elements on which the function $\mathbf{1}_{xs.c'_1}\mathbf{1}_{xyt.c'_2}$ is zero are outside the supports of $\mathfrak{m}[xs, c'_1, H_{c_1,c'_1,s}v_1]$ and $\mathfrak{m}[xyt, c'_2, H_{c_2,c'_2,t}v_2]$, so we add no new term by substituting $\mathbf{1}_{xs.c'_1}\mathbf{1}_{xyt.c'_2}$ with $\mathfrak{D}_{c'_1,c'_2,s^{-1}yt}^{c_1,c_2,y}$.

Let ϕ be a sesquilinear form giving an anti-isomorphism between the space $\bigoplus_{c_1,c_2,y} (V_{c_1} \otimes \overline{V_{c_2}})_y$ and its dual, let R' be the unique vector corresponding to the linear form B' (cf. Remark 3.32). Then:

$$B'(v_1 \otimes \overline{v_2}) = \phi(R', v_1 \otimes \overline{v_2})$$

and we conclude the computation:

$$\begin{aligned} B'(v_1 \otimes \overline{v_2}) &= \sum_{s \in S_1} \sum_{t \in S_2} \mathfrak{D}_{c'_1,c'_2,s^{-1}yt}^{c_1,c_2,y} \langle P\mathfrak{m}[xs, c'_1, H_{c_1,c'_1,s}v_1], \mathfrak{m}[xyt, c'_2, H_{c_2,c'_2,t}v_2] \rangle \\ &= \sum_{s \in S_1} \sum_{t \in S_2} \mathfrak{D}_{c'_1,c'_2,s^{-1}yt}^{c_1,c_2,y} B'_{c'_1,c'_2,s^{-1}yt} \left(H_{c_1,c'_1,s}v_1 \otimes \overline{H_{c_2,c'_2,t}v_2} \right) \\ &= \sum_{s \in S_1} \sum_{t \in S_2} B'_{c'_1,c'_2,s^{-1}yt} \left(\mathfrak{D}_{c'_1,c'_2,s^{-1}yt}^{c_1,c_2,y} \left(H_{c_1,c'_1,s}v_1 \otimes \overline{H_{c_2,c'_2,t}v_2} \right) (v_1 \otimes \overline{v_2}) \right) \\ &= \sum_{s \in S_1} \sum_{t \in S_2} B'_{c'_1,c'_2,s^{-1}yt} \left(\mathfrak{T}_{c'_1,c'_2,s^{-1}yt}^{c_1,c_2,y} (v_1 \otimes \overline{v_2}) \right) \\ &= \sum_{s \in S_1} \sum_{t \in S_2} \phi \left(R'_{c'_1,c'_2,s^{-1}yt}, \mathfrak{T}_{c'_1,c'_2,s^{-1}yt}^{c_1,c_2,y} (v_1 \otimes \overline{v_2}) \right) \\ &= \sum_{s \in S_1} \sum_{t \in S_2} \phi \left((\mathfrak{T}^*)_{c_1,c_2,y}^{c'_1,c'_2,s^{-1}yt} R'_{c'_1,c'_2,s^{-1}yt}, v_1 \otimes \overline{v_2} \right) \\ &= \phi \left(\sum_{s \in S_1, t \in S_2} (\mathfrak{T}^*)_{c_1,c_2,y}^{c'_1,c'_2,s^{-1}yt} R'_{c'_1,c'_2,s^{-1}yt}, v_1 \otimes \overline{v_2} \right) \\ &= \phi \left((\mathfrak{T}^* R')_{c_1,c_2,y}, v_1 \otimes \overline{v_2} \right) = (\mathfrak{T}^* B') (v_1 \otimes \overline{v_2}), \end{aligned}$$

where in the fourth equality we used equation 5.4 and by $\mathfrak{T}^* B'$ we mean the action of the linear map corresponding to \mathfrak{T}^* on the linear form B' . Thus, since v_1 and v_2 are arbitrary:

$$\mathfrak{T}^* B' = B',$$

which proves the thesis. \square

Corollary 5.14. *Let $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$ be a normalized scalar system for Γ_2, S_2 and B_0 the sesquilinear form on V for which the inner product $\langle \cdot, \cdot \rangle$ on \mathcal{H}^∞ is well defined as in Thm. 3.30. If B' is as in eq. 5.2, then $B' = \lambda B_0$ for some positive scalar λ .*

Proof. In the case where $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$ is a normalized scalar system for Γ_2, S_2 , the matrix \mathfrak{D} satisfies Perron-Frobenius Theorem and so does the matrix \mathfrak{T} (cf. [Sen06]). Then, in Theorem 3.30 B_0 was chosen as the eigenvector of \mathfrak{T}^* for the Perron-Frobenius eigenvalue, which is 1. Thus, since the 1-eigenspace of \mathfrak{T}^* is one-dimensional, $B' = \lambda B_0$ for some positive scalar λ . \square

Now we can prove Theorem 5.1.

Theorem (5.1). *Let $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$ be a normalized scalar system associated to the surface group Γ_2 and the standard set of generators S_2 . Then the crossed product representation $\pi \times \rho$ constructed in Chapter 4 is irreducible.*

Proof. Let P be an intertwining projection for $\pi \times \rho$, assume $P \neq 0$. Because of Lemma 5.11, B' is well-defined when P commutes with π ; because of Lemma 5.12, it is a 1-eigenvalue of \mathfrak{T} . Corollary 5.14 implies: if B_0 is the eigenvector of \mathfrak{T}^* for the eigenvalue 1 that we chose in Theorem 3.30 and which gives the well-posedness of $\langle \cdot, \cdot \rangle$ on \mathcal{H}^∞ , then:

$$\begin{aligned} \langle P\mathbf{m}[e, \mathbf{c}_1, v_1], \mathbf{m}[y, \mathbf{c}_2, v_2] \rangle &= B'(v_1 \otimes \overline{v_2}) = \lambda B_0(v_1 \otimes \overline{v_2}) \\ &= \lambda \langle \mathbf{m}[e, \mathbf{c}_1, v_1], \mathbf{m}[y, \mathbf{c}_2, v_2] \rangle. \end{aligned}$$

It follows that:

$$\langle (P - \lambda I)\mathbf{m}[e, \mathbf{c}_1, v_1], \mathbf{m}[y, \mathbf{c}_2, v_2] \rangle = 0$$

for every couple of elementary multiplicative functions $\mathbf{m}[e, \mathbf{c}_1, v_1], \mathbf{m}[y, \mathbf{c}_2, v_2]$; the same holds for generic functions $\mathbf{m} \in \mathcal{H}^\infty$ whose supports are just translates of the cones $\mathbf{c}_1, \mathbf{c}_2$ (P intertwines the unitary representation π , cf. Lemma 5.11). Thus, the operator $P - \lambda I$ is zero on the dense subspace \mathcal{H}^∞ , hence it is just zero. This yields: $P = \lambda I$.

Since P is a projection, we get:

$$|\lambda| = |\lambda| \|I\| = \|\lambda I\| = \|P\| = 1,$$

therefore $\lambda = 1$, i.e. $P = I$.

Thus, we have proved that every projection commuting with the representation $\pi \times \rho: \Gamma \times C(X) \rightarrow \mathfrak{B}(\mathcal{H})$ is trivial, so, by Theorem 5.7, the representation $\pi \times \rho$ is irreducible. \square

APPENDIX A

GEOMETRIC RESULTS

Fix a surface group Γ and a symmetric set of generators S (the standard set will do). Let \mathcal{G} denote the Cayley graph of Γ with respect to S and fix once for all the word metric associated to S on Γ and the correspondent edge-path metric on \mathcal{G} . Every metric notion in the present Appendix has to be intended in this context.

A.1 TOPOLOGY ON THE BOUNDARY

Consider the topology \mathcal{T} defined on the boundary $\partial\Gamma$ of a surface group as in [Ohs02] or in [KB02]: for a generic element $\xi \in \partial\Gamma$ we choose as a basis of neighborhoods the family of sets

$$\partial\Gamma_{\xi,r} := \{\omega \in \partial\Gamma \mid (\omega|\xi) > r\}, \quad r > 0,$$

where $(\cdot|\cdot)$ denotes the Gromov product extended to $\partial\Gamma$ with respect to any fixed base point.

We recall the definition of a **shadow**: for $r > 0$ and $x, y \in \Gamma$,

$$\mathcal{O}_r(x, y) := \{\omega \in \partial\Gamma \mid \exists \text{ geodesic } \gamma \in [x, \omega] \text{ intersecting } B_r(y)\}.$$

Equivalently, a shadow can be defined as

$$\mathcal{O}_r(x, y) := \{\omega \in \partial\Gamma \mid (\omega|y)_x \geq d(x, y) - r\}.$$

Remark A.1. The topology generated by the family of shadows $\{\mathcal{O}_r(x, y) \mid x, y \in \Gamma, r > 0\}$ is equivalent to the topology defined by the family of neighborhoods $\{\partial\Gamma_{\xi, r} \mid \xi \in \partial\Gamma, r > 0\}$.

Proof. Given a generic topological space X and two topologies \mathcal{T}_1 and \mathcal{T}_2 on X , let \mathcal{B}_1^x and \mathcal{B}_2^x be local bases for $\mathcal{T}_1, \mathcal{T}_2$ at $x \in X$. Then $\mathcal{T}_1 \subseteq \mathcal{T}_2$ if and only if for every set $U_1 \in \mathcal{B}_1^x$ there exists a set $U_2 \in \mathcal{B}_2^x$ such that $U_2 \subseteq U_1$.

Let \mathcal{T}_1 be the topology on $X = \Gamma \cup \partial\Gamma$ generated by the sets $\partial\Gamma_{\xi, r}$ for $\xi \in \partial\Gamma$ and $r > 0$, let \mathcal{T}_2 be the topology generated by the shadows. We show that $\mathcal{T}_1 \subseteq \mathcal{T}_2$ and $\mathcal{T}_2 \subseteq \mathcal{T}_1$. Fix $\delta > 0$ such that $\Gamma \cup \partial\Gamma$ is δ -hyperbolic.

Let $x, y \in \Gamma$ and $r > 0$, let $\xi \in \mathcal{O}_r(x, y)$. Consider $r' \geq d(x, y) - r$. Choose $\omega \in \partial\Gamma_{\xi, r'}$. Then $(\omega|\xi)_x > r'$. By hyperbolicity (cf. [Ohs02, Lem. 2.57]):

$$(\omega|y)_x \geq \min\{(\omega|\xi)_x, (\xi|y)_x\} - \delta.$$

Moreover, $\min\{(\omega|\xi)_x, (\xi|y)_x\} \geq d(x, y) - r$, so:

$$(\omega|y)_x \geq d(x, y) - r - \delta.$$

Therefore: $\omega \in \mathcal{O}_{r+\delta}(x, y)$: By arbitrariness of $\omega \in \partial\Gamma_{\xi, r'}$ we get: $\partial\Gamma_{\xi, r'} \subseteq \mathcal{O}_{r+\delta}(x, y)$.

Fix now $\xi \in \partial\Gamma$ and $r > 0$, let $\omega \in \partial\Gamma_{\xi, r}$. Consider $x, y \in \Gamma$ such that $\xi \in \mathcal{O}_{r'}(x, y)$ for $r' > 0$ satisfying $(\omega|\xi)_x \geq d(x, y) - r'$. Then $\min\{(\omega|y)_x, (\xi|y)_x\} \geq d(x, y) - r'$ and by hyperbolicity as above $(\omega|y)_x \geq d(x, y) - r' - \delta$, which yields $\omega \in \mathcal{O}_{r'+\delta}(x, y)$. Thus $\mathcal{O}_{r'+\delta}(x, y) \subseteq \partial\Gamma_{\xi, r}$. \square

Define now a new family of subsets of the boundary (for $x, y \in \Gamma$):

$$\mathcal{O}(x, y) := \{\omega \in \partial\Gamma \mid \exists \text{ geodesic } \gamma \in [x, \omega] \text{ passing through } y\}.$$

We claim that the family of sets $\{\mathcal{O}(x, y) \mid x, y \in \Gamma\}$ generates the usual topology \mathcal{T} on $\partial\Gamma$. This is a consequence of the following.

Proposition A.2. *Let Γ be a surface group, let S be a symmetric set of generators, consider the word metric on Γ . Fix $x, y \in \Gamma$ and $r > 0$. Then, there exist $z_1, \dots, z_N \in \Gamma$ such that*

$$\mathcal{O}_r(x, y) = \bigcup_{i=1}^N \mathcal{O}(x, z_i).$$

Proof. Fix $x, y \in \Gamma$ and $r > 0$. The ball

$$B_r(y) := \{z \in \Gamma \mid d(y, z) \leq r\}$$

has finitely many elements, say z_1, \dots, z_N for some $N \in \mathbb{N}$.

Consider

$$\mathcal{O}_r(x, y) = \{\omega \in \partial\Gamma \mid \exists \text{ geodesic } \gamma \in [x, \omega) \text{ intersecting } B_r(y)\}.$$

For every $\omega \in \mathcal{O}_r(x, y)$ there exists $i_0 \in \{1, \dots, N\}$ such that z_{i_0} lies on a geodesic starting from x and having ω as a limit point. In particular, then: $\omega \in \mathcal{O}(x, z_{i_0})$. This yields

$$\mathcal{O}_r(x, y) \subseteq \bigcup_{i=1}^N \mathcal{O}(x, z_i).$$

Conversely, if we fix $i_0 \in \{1, \dots, N\}$ for $N \in \mathbb{N}$ as above, let $\omega \in \mathcal{O}(x, z_{i_0})$: then each geodesic starting from x and having ω as a limit point intersects every ball $B_\epsilon(z_{i_0})$, for every $\epsilon > 0$: the geodesic passes exactly through the center z_{i_0} . We have then $\omega \in \mathcal{O}_\epsilon(x, z_{i_0})$. If now $r > 0$ is fixed and y is such that $d(y, z_{i_0}) \leq r$, then there exists a geodesic in $[x, \omega)$ which intersects the ball $B_r(y)$ in z_{i_0} : so, $\omega \in \mathcal{O}_r(x, y)$. It follows that $\mathcal{O}(x, z_{i_0}) \subseteq \mathcal{O}_r(x, y)$ whenever $d(y, z_{i_0}) \leq r$. Hence, if

$$B_r(y) = \{z_1, \dots, z_N\},$$

then we get

$$\mathcal{O}_r(x, y) = \bigcup_{i=1}^N \mathcal{O}(x, z_i).$$

This concludes the proof. \square

Proposition A.3. *Let Γ be a surface group, let S be a symmetric set of generators, consider the word metric on Γ . Fix $x, y \in \Gamma$. The subset $\mathcal{O}(x, y) \subseteq \partial\Gamma$ is compact and connected in $\partial\Gamma$.*

The proof of this Proposition was suggested by M. Kapovich.

Proof. Recall that every metric notion is intended with respect to the word metric associated to S . Moreover, \mathcal{G} denotes the Cayley graph of Γ with respect to S equipped with the natural edge-path metric.

Let us start with compactness. Consider a sequence of geodesic rays $\gamma_n: [0, \infty) \rightarrow \mathcal{G}$ such that for all $n \in \mathbb{N}$: $\gamma_n(0) = x$ and $\gamma_n(\ell) = y$, where

$\ell = d(x, y)$. Then, by Ascoli-Arzelà Theorem, there exists a subsequence γ_{n_j} converging to a geodesic ray γ such that $\gamma(0) = x$ and $\gamma(\ell) = y$: in fact, each element of the sequence satisfy the two properties. Consider the sequence $\{\omega_n\} \subseteq \partial\Gamma$ defined by $\omega_n = \lim_{t \rightarrow \infty} \gamma_n(t) = \gamma_n(\infty)$ for each $n \in \mathbb{N}$. If $\gamma_{n_j} \rightarrow \gamma$ as above and $\omega = \gamma(\infty)$, then $\omega_n \rightarrow \omega$ as $n \rightarrow \infty$.

Let us now address connectedness. The Cayley graph \mathcal{G} can be embedded in the hyperbolic plane \mathbb{H}^2 . Fix an embedding and consider \mathcal{G} as a subset of \mathbb{H}^2 . Then, the ideal boundary of \mathcal{G} is \mathbb{S}^1 , the boundary of \mathbb{H}^2 . Consider the usual path metric on \mathcal{G} . If $x \neq y$, pick two distinct points $\omega_1, \omega_2 \in \mathcal{O}(x, y)$. Then there exist two geodesic rays γ_1, γ_2 starting from x , passing through y and having (respectively) ω_1, ω_2 as limit points. Since $\omega_1 \neq \omega_2$, there exists $T \geq \ell = d(x, y)$ such that $\gamma_1(T) = \gamma_2(T)$ and $\gamma_1(t) \neq \gamma_2(t)$ for all $t > T$. The set $A := \gamma_1([T, \infty)) \cup \gamma_2([T, \infty))$ is homeomorphic to a line in \mathbb{H}^2 and it intersects the boundary \mathbb{S}^1 in the two points ω_1, ω_2 : then, it splits \mathbb{H}^2 into two connected components. Consider the component C not containing x : the ideal boundary ∂C of C is an arc in \mathbb{S}^1 with endpoints ω_1, ω_2 . We claim that for every $\omega \in \partial C$ it holds $\omega \in \mathcal{O}(x, y)$. Let γ be a geodesic ray issuing from x and such that $\gamma(\infty) = \omega$. Then the image in \mathbb{H}^2 of γ intersects the line A at some point $z = \gamma(t_0)$ for some $t_0 > 0$. Assume without loss of generality that $z \in \gamma_1([0, \infty))$. We define a new geodesic ray $\tilde{\gamma}$ by:

$$\tilde{\gamma}(t) = \begin{cases} \gamma_1(t) & \text{if } t \in [0, t_0], \\ \gamma(t) & \text{if } t \in [t_0, \infty). \end{cases}$$

This is indeed a geodesic and it satisfies $\tilde{\gamma}(0) = x$, $\tilde{\gamma}(\ell) = y$ and $\tilde{\gamma}(\infty) = \omega$. Hence, $\omega \in \mathcal{O}(x, y)$. \square

A.2 TIETZE'S EXTENSION THEOREM

A topological space X is **normal** if for every pair of closed disjoint sets $C_1, C_2 \subseteq X$ there exists a pair of open disjoint sets $U_1, U_2 \subseteq X$ such that $U_1 \supseteq C_1$ e $U_2 \supseteq C_2$.

Tietze's Theorem ([Rud87]). *Let X be a normal topological space and $A \subseteq X$ a closed subset. Let $f: A \rightarrow \mathbb{R}$ be a continuous function. Then, there exists a continuous function $\tilde{f}: X \rightarrow \mathbb{R}$ such that $\tilde{f}(a) = f(a)$ for all $a \in A$. Moreover, $\sup_{x \in X} |\tilde{f}(x)| = \sup_{a \in A} |f(a)|$.*

A proof of Tietze's Theorem can be found in [Rud87, Thm. 20.4].

Remark A.4. Consider X, A as above. Given a continuous function $f: A \subseteq X \rightarrow \mathbb{C}$ (assuming complex values), Tietze's Theorem guarantees that there is a continuous extension \tilde{f} of f to all X .

Thus, in our setting, we have the following.

Theorem A.5. *Given a function $f \in C(\partial\Gamma)$, there exists a function $\tilde{f} \in C(X)$ such that $\tilde{f}|_{\partial\Gamma} = f$ and $\sup_{x \in X} |\tilde{f}(x)| = \sup_{\omega \in \partial\Gamma} |f(\omega)|$.*

Proof. The space $X = \Gamma \cup \partial\Gamma$ is Hausdorff and compact, thus it is a normal space. The boundary $\partial\Gamma$ is homeomorphic to a circle and it is compact: the boundary $\partial\Gamma$ is a closed subset of X . We apply Tietze's Theorem considering $A = \partial\Gamma \subseteq X$. \square

APPENDIX B

COMPATIBLE TRIPLES

In this Appendix, we list the compatible triples for the surface group Γ_2 . In particular, we change the names of the standard generators from a_1, b_1, a_2, b_2 to a, b, c, d and we let:

$$\Gamma_2 = \langle a, b, c, d \mid [a, b][c, d] = e \rangle.$$

Remark B.1. All the metric notions in the present Chapter are intended with respect to the word metric associated with the standard symmetric set of generators. We fix once for all a scalar system $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$ and we construct the space of multiplicative functions on it (see Chap. 3).

We denote by x, y, z, ζ group elements, by s, t generators and by w, u, v geodesic words on the alphabet S given by the generators.

The letters a, b, c, d are reserved for the explicit generators of Γ_2 : we use them for some “concrete” examples.

When referring to the geodesic word w representing a cone type $C(w)$, we always choose a geodesic word of minimal length such that $C(w^{-1}, e)$ represents the cone type: thus, it goes without saying that when we write the cone type $C(w)$ for a geodesic word w we assume that $\ell(w) \in \{1, 2, 3, 4\}$ (for genus $k = 2$).

When we write that a group element equals a word on the alphabet S (e.g. $x = w$ for $x \in \Gamma$ and $w \in S^*$), we actually mean that the word represents the element on the other side of the equal symbol (i.e. $x = \bar{w}$). This abuse is meant to avoid the use of the symbol $\bar{\bar{\cdot}}$, which would be formally precise.

B.1 COMPATIBLE TRIPLES AND THE INNER PRODUCT

We refer to the notations and the definitions of Section 3.3.

Fix a scalar system $\{V_{\mathbf{c}}, H_{\mathbf{c}, \mathbf{c}', s}\}$, let $V = \bigoplus_{\mathbf{c}} V_{\mathbf{c}}$ as in Def. 3.4. Recall that, chosen a positive-definite sesquilinear form B on the vector space V , we define for $f_1, f_2 \in \mathcal{H}^\infty$

$$\langle f_1, f_2 \rangle = \lim_{\epsilon \downarrow 0} \epsilon \sum_{z \in \Gamma} B(f_1(z), f_2(z)) e^{-\epsilon|z|}$$

Let $\mathbf{c}_0 = C(x_1, e)$ and $\mathbf{d}_0 = C(x_2, e)$ be two cones with non-empty intersection, consider two elementary multiplicative functions $f_1 = \mathbf{m}[x_1, e, v_1]$ and $f_2 = \mathbf{m}[x_2, e, v_2]$ supported on \mathbf{c}_0 and \mathbf{d}_0 respectively.

Fix $M \in \mathbb{N}$ as in Lemma 3.31. Consider an arbitrary $z \in C(x_1, e) \cap C(x_2, e)$ with $n = |z| > M$, let $\gamma_1, \gamma_2 \in [e, z]$ be two geodesics joining e and z . Then, for each $i = 0, \dots, n$ we have: $d(\gamma_1(i), \gamma_2(i)) \leq 4$ and $d(e, \gamma_1(i)) = d(e, \gamma_2(i))$.

Denote $\zeta = \gamma_1(i)$, let $y \in \Gamma$ be such that $\gamma_2(i) = \zeta y$. Then

$$|y| = d(\gamma_1(i), \gamma_2(i)) \leq 4.$$

Let \mathbf{c} be the cone type of $C(x_1, \zeta)$ and \mathbf{d} be the cone type of $C(x_2, \zeta y)$.

Let $S^{(n)} = \left(S_{\mathbf{c}, \mathbf{d}, y}^{(n)} \right)$ be as in equation 3.8:

$$S_{\mathbf{c}, \mathbf{d}, y}^{(n)} = \sum_{|\zeta| = |\zeta y| = n} f_1(\zeta)_{\mathbf{c}} \otimes \overline{f_2(\zeta y)}_{\mathbf{d}}.$$

Since f_1, f_2 are elementary multiplicative, we have $f_1(\zeta)_{\mathbf{c}} \neq 0$ if and only if $\zeta \cdot \mathbf{c}$ is an n -th level subcone of \mathbf{c}_0 ; similarly, $\overline{f_2(\zeta y)}_{\mathbf{d}} \neq 0$ if and only if $\zeta y \cdot \mathbf{d}$ is an n -th level subcone of \mathbf{d}_0 .

If $(\mathbf{c}, \mathbf{d}, y)$ corresponds to a non-zero term $f_1(\zeta)_{\mathbf{c}} \otimes \overline{f_2(\zeta y)}_{\mathbf{d}}$ in the sum above, we call $(\mathbf{c}, \mathbf{d}, y)$ a **compatible triple**. We admit in the number of the compatible triples the cone types of $C(x_1, z)$ and $C(x_2, z)$ (i.e. $\zeta_1 = \zeta_2 = z$ and $y = e$).

For each compatible triple $(\mathbf{c}, \mathbf{d}, y)$, the symmetric triple $(\mathbf{d}, \mathbf{c}, y^{-1})$ is compatible, too.

We are interested only in $S^{(n)}$ with $n \geq M$, where $M \in \mathbb{N}$ is fixed as in Lemma 3.31. Thus, when we consider the digons with vertices in e and

z (for $|z| > M$) and we determine the triples $(\mathbf{c}, \mathbf{d}, y)$, we can neglect the “first part” of the digons.

The geodesic digon given by γ_1, γ_2 is simple if the geodesics γ_1 and γ_2 intersect only in the endpoints. If the “final part” (the last edges near the vertex z) of the geodesic digon is not simple, then, at a certain vertex the two geodesics intersect: there exists j_0 such that $\gamma_1(j_0) = \gamma_2(j_0)$. Consider $\zeta_1 = \gamma_1(j_0 + 1)$ and $\zeta_2 = \gamma_2(j_0 + 1)$: then, the cone types of the subcones $C(x_1, \zeta_1)$ and $C(x_2, \zeta_2)$ can be either $C(s)$ and $C(t)$, for s, t generators (as in first Figure B.1), or $C(s_1 s_2)$ and $C(t)$, for s_1, s_2 and t generators (as in the second Figure B.1). In this latter case, the consecutive edges $e_1 = (\gamma_1(j_0 - 1), \gamma_1(j_0))$ and $e_2 = (\gamma_1(j_0), \gamma_1(j_0 + 1))$ (on the blue geodesic in Figure B.1) belong to a common 8-gon and no other edges emanate from the common vertex of e_1, e_2 between the two of them.

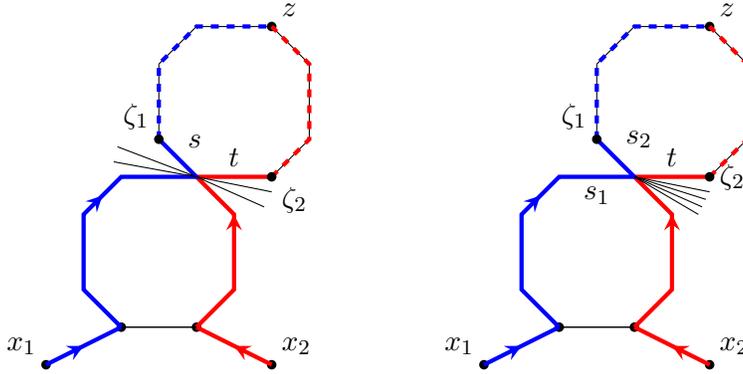


Figure B.1: The two geodesic segments intersect at a vertex. Here we get the compatible triples $(C(s), C(t), s^{-1}t)$ (left) and $(C(s_1 s_2), C(t), s_2^{-1}t)$ (right).

Assume now that the final part of the geodesic digon is simple (or at least that it has a connected topological interior). The possible configurations are reported in the following Figure B.2.

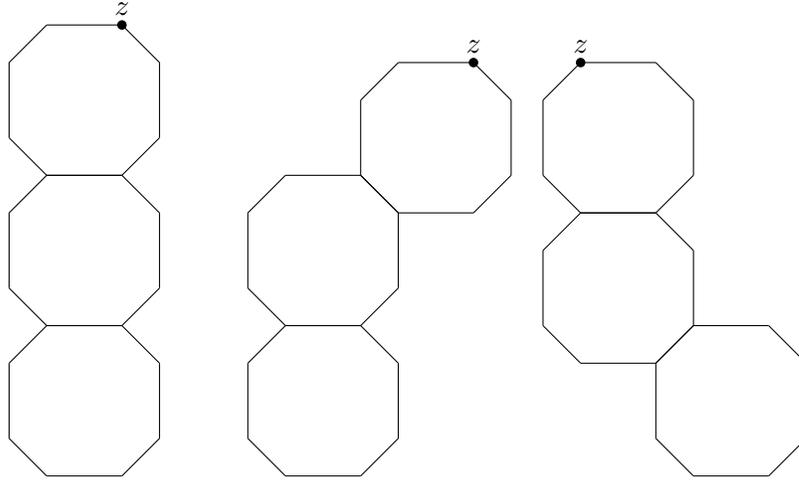


Figure B.2: Final part of a simple geodesic digon: the 8-gons can be attached in these three fashions (or some symmetric case of these).

B.2 COMPATIBLE TRIPLES WITH $y = e$

We determine the possible triples $(\mathbf{c}, \mathbf{d}, e)$ that can appear in the sum

$$S_{\mathbf{c}, \mathbf{d}, e}^{(n)} = \sum_{|z|=n} f_1(z)_{\mathbf{c}} \otimes \overline{f_2(z)}_{\mathbf{d}},$$

where n is greater than M as in Lemma 3.31. Notice that here $y = e$, so the two cones have the same vertex: we are considering the subcones $z.\mathbf{c}$ and $z.\mathbf{d}$.

B.2.1 Case 0

The very first case arises when at the point z the subcones have the same cone type: thus we have the compatible triples

$$(C(w), C(w), e), \tag{B.1}$$

where w is any geodesic word representing a cone type (excluding the trivial one).

Since we have 48 possible choices for the cone type $C(w)$, we get 48 such compatible triples, which we call **triples of type 0**.

B.2.2 Case 1

Let us analyze the cases in Figure B.2.

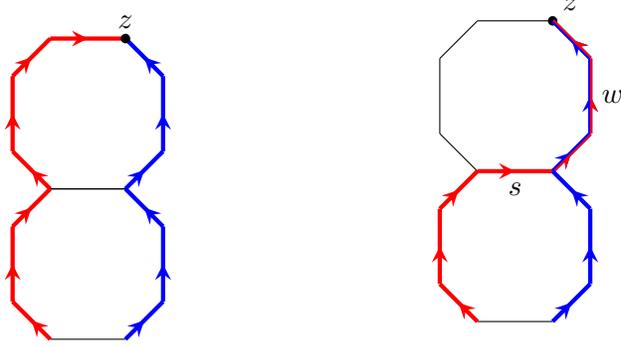


Figure B.3: The red geodesic in the figure on the left gives the same cone type as the red geodesic in the figure on the right.

In the case represented in the left Figure B.3, one of the two geodesics (the blue one) crosses four edges on the last octagon. We can rewrite the blue geodesic as in the right Figure B.3. This case yields the compatible triple

$$(C(sw), C(w), e), \tag{B.2}$$

where s is a generator and sw is a subword of a fundamental relator. The symmetric triple $(C(w), C(sw), e)$ is compatible, too. These are called **triples of type 1**.

Concretely, we have all these possible cases, depending on the length of the geodesic word w (it must be $\ell(w) \in \{1, 2, 3\}$ and so $w = s_1$, $w = s_1s_2$ or $w = s_1s_2s_3$):

Triples	Number
$(C(ss_1), C(s_1), e)$ and $(C(s_1), C(ss_1), e)$	16+16
$(C(ss_1s_2), C(s_1s_2), e)$ and $(C(s_1s_2), C(ss_1s_2), e)$	16+16
$(C(ss_1s_2s_3), C(s_1s_2s_3), e)$ and $(C(s_1s_2s_3), C(ss_1s_2s_3), e)$	16+16

The number of triples of type 1 is $48 + 48$; let us enumerate the possibilities for $(C(sw), C(w), e)$ (the other is symmetric): the choice of the cone types $C(sw)$ has 16 possibilities for each choice of the length $\ell(sw) \in \{2, 3, 4\}$ (16 choices if $\ell(sw) = 2$, 16 for $\ell(sw) = 3$ and 16 for $\ell(sw) = 4$); e.g. notice that the triples

$$(C(dcd^{-1}c^{-1}), C(cd^{-1}c^{-1}), e) \quad \text{and} \quad (C(aba^{-1}b^{-1}), C(ba^{-1}b^{-1}), e)$$

are distinct, so we have to consider all the geodesic subwords of the fundamental relator having length 4 ($aba^{-1}b^{-1}$ and $dcd^{-1}c^{-1}$ produce different triples, even if they give the same cone type: in fact, when we “erase” the first letter, the geodesic words that we get $ba^{-1}b^{-1}$ and $cd^{-1}c^{-1}$, which represent two group elements having different cone types).

In the left and central Figures B.4 we represent the cases where w has length 1, 2 (in this situation, though, the digon with vertex z is no longer simple, but it has a connected topological interior: in fact, the two geodesics “share” some final edges).

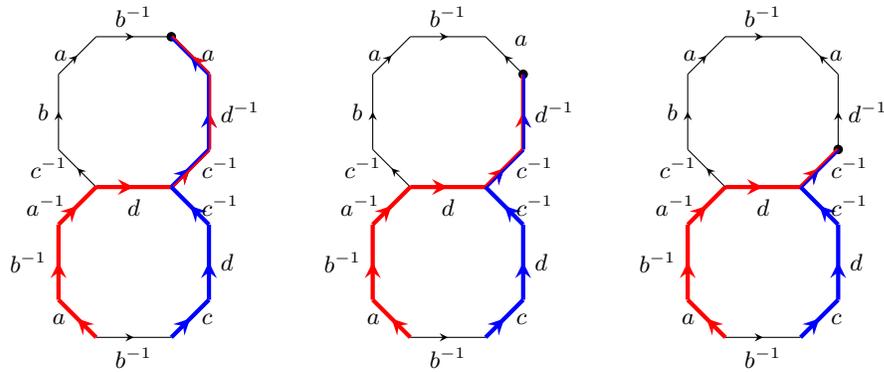


Figure B.4: From the left to the right, we represent the triples $(C(dc^{-1}d^{-1}a), C(c^{-1}d^{-1}a), e)$, $(C(dc^{-1}d^{-1}), C(c^{-1}d^{-1}), e)$ and $(C(dc^{-1}), C(c^{-1}), e)$.

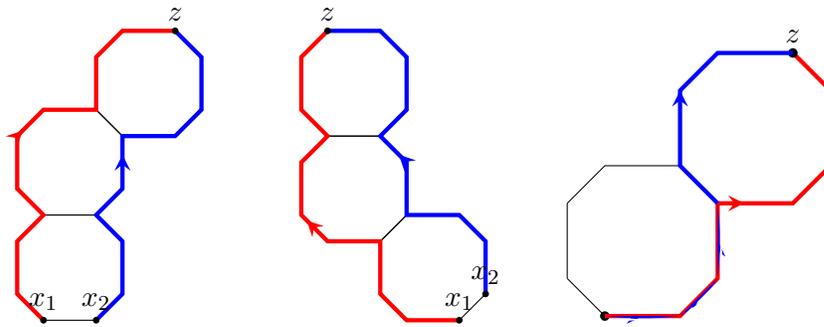


Figure B.5: The left and central case can be rewritten to get the figure on the right.

In the left and central cases of Figures B.5, the cone type determined

by the blue geodesic is of the special kind having two predecessors, i.e. if we choose a representative geodesic word of minimal length for $C(w)$, we will have $\ell(w) = 4$. We need to modify the red geodesic in order to get a geodesic crossing the maximal number of edges on the final octagon (which determines the cone type, cf. Remark 1.30 and Section 2.3.2). Thus, we reach the situation represented in the right Figure B.5 and we get a compatible triple of type 0 like in the previous equation B.1,

$$(C(w), C(w), e).$$

B.3 COMPATIBLE TRIPLES WITH $y \neq e$

Let us analyze the possible compatible triples with $y \neq e$: these appear in the sums

$$S_{\mathbf{c}, \mathbf{d}, y}^{(n)} = \sum_{|z|=|zy|=n} f_1(z)_{\mathbf{c}} \otimes \overline{f_2(zy)}_{\mathbf{d}}.$$

The points z and zy have the same distance from e ; moreover, the word length $|y|$ can be at most $2k = 4$ (by Lemma 3.40 and Lemma 1.24), thus, z and zy must have even distance equal to 2 or to 4. Then, y must be a subword of a fundamental relator of length 2 or 4. The cone types \mathbf{c}, \mathbf{d} must satisfy the condition $\mathbf{c} \cap y.\mathbf{d} \neq \emptyset$.

B.3.1 Case 2

The **triples of type 2** are of the form

$$(C(w), C(w'), w^{-1}w') \tag{B.3}$$

where w, w' are geodesic words of minimal length representing two different cone types satisfying $\ell(w) = \ell(w')$ and $y = w^{-1}w'$ is a subword of a fundamental relator.

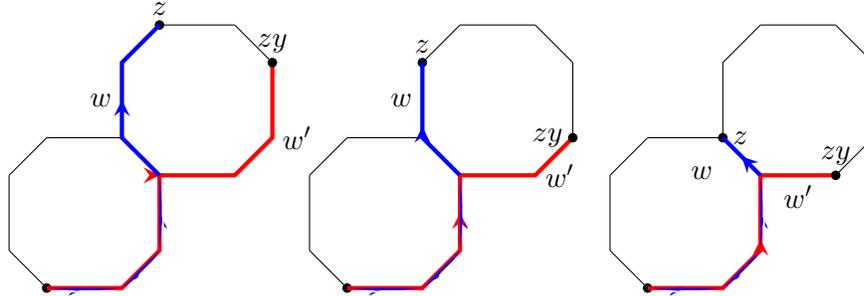


Figure B.6: The possible triples of type 2 depending on the length of w .

We have 48 possible choices for such triples: 16 come from the case $\ell(w) = \ell(w') = 1$ (choose a generator s : then there are two possible choices for the second generator t so that $s^{-1}t$ is a subword of a fundamental relator), other 16 arise when $\ell(w) = \ell(w') = 2$ (once the word w of length 2 is chosen, the triple is determined) and 16 for $\ell(w) = \ell(w') = 3$ (once the word w of length 3 is chosen, the triple is determined). [This enumeration already includes the symmetric triple of $(\mathbf{c}, \mathbf{d}, y)$, namely $(\mathbf{d}, \mathbf{c}, y^{-1})$.]

Triples	Number
$(C(s), C(t), s^{-1}t)$	16
$(C(s_1s_2), C(t_1t_2), s_2^{-1}s_1^{-1}t_1t_2)$	16
$(C(s_1s_2s_3), C(t_1t_2t_3), s_3^{-1}s_2^{-1}s_1^{-1}t_1t_2t_3)$	16

B.3.2 Case 3

The **triples of type 3** are of the form

$$(C(w), C(t), s^{-1}t) \tag{B.4}$$

where w is a subword of a fundamental relator representing a cone type with $\ell(w) \geq 2$, s is the last letter of the word w , t is a generator and $s^{-1}t$ is a subword of a fundamental relator.

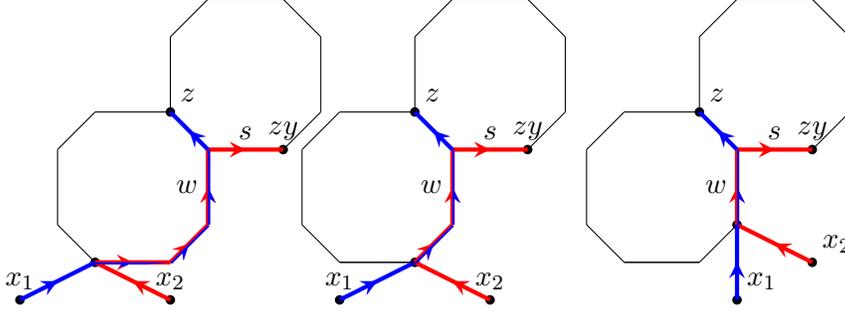


Figure B.7: The possible triples of type 3 depending on the length of w .

There are 48 triples corresponding to this case: once that the representative w of the cone type $C(w)$ is chosen, the triple is determined: there is a unique t satisfying the requirement. The length $\ell(w)$ can be 2, 3 or 4, we have 16 choices of w for each given length. In the Figure B.7 we represent the situation giving rise to triples of type 3.

Triples	Number
$(C(s), C(t_1 t_2), s^{-1} t_2)$ and $(C(t_1 t_2), C(s), t_2^{-1} s)$	16+16
$(C(s), C(t_1 t_2 t_3), s^{-1} t_3)$ and $(C(t_1 t_2 t_3), C(s) t_3^{-1} s)$	16+16
$(C(s), C(t_1 t_2 t_3 t_4), s^{-1} t_4)$ and $(C(t_1 t_2 t_3 t_4), C(s), t_4^{-1} s)$	16+16

B.3.3 Case 4

The **triples of type 4** are of the form

$$(C(w), C(w'), w^{-1} s w') \tag{B.5}$$

where w, w' are subwords of a fundamental relator representing two different cone types, $\ell(w') = \ell(w) - 1$, s is a generator and $y = w^{-1} s w'$ is a subword of a fundamental relator ($w^{-1} s w'$ determined here may not be a geodesic word - remember that we are interested in *the group element* represented by it).

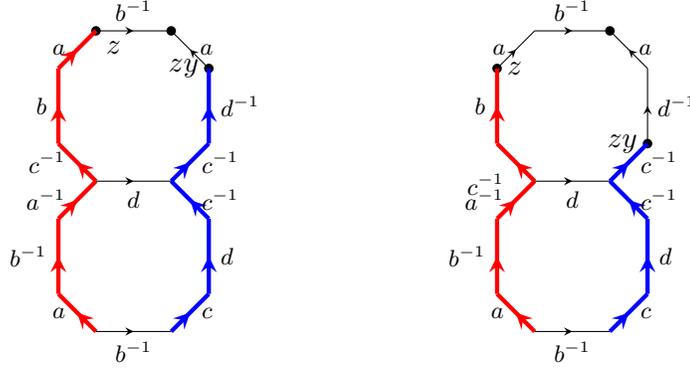


Figure B.8: The possible triples of type 4 depending on the length of w . We represent here two concrete examples with the labels given by the generators a, b, c, d of Γ_2 .

The possible cases are when w has length 3 and 2, as in Figure B.8: there are 16 possibilities for the choice of each. In Figures B.8 we represent the triples

$$(C(c^{-1}ba), C(c^{-1}d^{-1}), a^{-1}b^{-1}cdc^{-1}d^{-1}) \text{ and } (C(c^{-1}ba), C(c^{-1}d^{-1}), b^{-1}cdc^{-1}).$$

Thus, in the first triple: $w = c^{-1}ba$, $w' = c^{-1}d^{-1}$, $s = d$. We remark that $a^{-1}b^{-1}cdc^{-1}d^{-1}$ represents the same group element as $b^{-1}a^{-1}$, which has word length 2, so it does not violate $|y| \leq 4$.

Triples	Number
$(C(s_1s_2), C(t_1), s_2^{-1}s_1^{-1}pt_1)$ and $(C(t_1), C(s_1s_2), t_1^{-1}p^{-1}s_1s_2)$	16+16
$(C(s_1s_2s_3), C(t_1t_2), s_3^{-1}s_2^{-1}s_1^{-1}pt_1t_2)$ and $(C(t_1t_2), C(s_1s_2s_3), t_2^{-1}t_1^{-1}p^{-1}s_1s_2s_3)$	16+16

B.3.4 Case 5

The **triples of type 5** are of the form

$$(C(s), C(w), s^{-1}t) \tag{B.6}$$

where w is a geodesic word representing a cone type with $\ell(w) = 3$, s is a generator, t is a generator (not the last letter of w , but the one such that wt^{-1} is a subword of a fundamental relator) and $y = t^{-1}s$ is a subword of a fundamental relator.

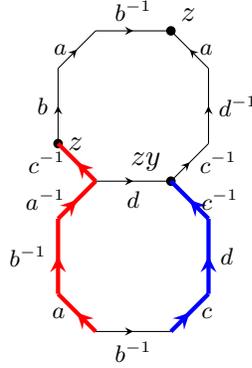


Figure B.9: The triple $(C(c^{-1}), C(cdc^{-1}), cd)$ is a concrete example of triple of type 5.

In Figure B.9 the triple $(C(c^{-1}), C(cdc^{-1}), cd)$ is represented: $s = c^{-1}$, $w = cdc^{-1}$ and $t = d$.

There are 16 possible choices for this kind of triple, as the possible choices of a geodesic word of length 3.

Triples	Number
$(C(s), C(t_1t_2t_3), s^{-1}p)$ and $(C(t_1t_2t_3), C(s), p^{-1}s)$	16+16

B.3.5 Some remarks

The analogs of case 5 with $\ell(w) = 2$ or $\ell(w) = 4$ are not significant (i.e. they do not give new triples): in fact, the red geodesic in the left Figure B.10 can be modified to the red one in the right Figure B.10: in this way, in correspondence of the vertices z and zy we get the triple $(C(w), C(t), s^{-1}t)$ as in equation B.4, where $\ell(w) = 3$, s is the last letter of w , t is a generator and $s^{-1}t$ is a subword of a fundamental relator.

B. COMPATIBLE TRIPLES

Triples	Number
$(C(w), C(w), e)$	48
$(C(ss_1s_2s_3), C(s_1s_2s_3), e)$ and $(C(s_1s_2s_3), C(ss_1s_2s_3), e)$	16+16
$(C(ss_1s_2), C(s_1s_2), e)$ and $(C(s_1s_2), C(ss_1s_2), e)$	16+16
$(C(ss_1), C(s_1), e)$ and $(C(s_1), C(ss_1), e)$	16+16
$(C(s), C(t), s^{-1}t)$	16
$(C(s_1s_2), C(t_1t_2), s_2^{-1}s_1^{-1}t_1t_2)$	16
$(C(s_1s_2s_3), C(t_1t_2t_3), s_3^{-1}s_2^{-1}s_1^{-1}t_1t_2t_3)$	16
$(C(s), C(t_1t_2), s^{-1}t_2)$ and $(C(t_1t_2), C(s), t_2^{-1}s)$	16+16
$(C(s), C(t_1t_2t_3), s^{-1}t_3)$ and $(C(t_1t_2t_3), C(s)t_3^{-1}s)$	16+16
$(C(s), C(t_1t_2t_3t_4), s^{-1}t_4)$ and $(C(t_1t_2t_3t_4), C(s), t_4^{-1}s)$	16+16
$(C(s_1s_2s_3), C(t_1t_2), s_3^{-1}s_2^{-1}s_1^{-1}pt_1t_2)$ and $(C(t_1t_2), C(s_1s_2s_3), t_2^{-1}t_1^{-1}p^{-1}s_1s_2s_3)$	16+16
$(C(s_1s_2), C(t_1), s_2^{-1}s_1^{-1}pt_1)$ and $(C(t_1), C(s_1s_2), t_1^{-1}p^{-1}s_1s_2)$	16+16
$(C(s), C(t_1, t_2t_3), s^{-1}p)$ and $(C(t_1t_2t_3), C(s), p^{-1}s)$	16+16

The total number of compatible triples for $\Gamma = \Gamma_2$ is 384.

Considering the different types which we used to present the triples we have:

- type 0: triples $(C(w), C(w), e)$, where w is a representative of a cone type;
- type 1: triples $(C(sw), C(w), e)$, where s is a generator and sw is a geodesic word;
- type 2: triples $(C(w), C(w'), w^{-1}w')$, where w, w' are representatives of cone types having the same length $\ell(w) = \ell(w')$ and $w^{-1}w'$ is a subword of the relator or its inverse;
- type 3: triples $(C(w), C(s), t^{-1}s)$ and $(C(s), C(w), s^{-1}t)$, where s is a generator, w is a geodesic representative of a cone type, t is the last letter of w and $s^{-1}t$ is a subword of the relator or its inverse;
- type 4: triples $(C(w), C(w'), w^{-1}sw')$, where w, w' are geodesic representative of cone types, $s \in S$ and they satisfy: $\ell(w) \in \{2, 3\}$, $\ell(w') = \ell(w) - 1$ and $w^{-1}sw'$ is a reduced subword of a fundamental relator (not necessarily a geodesic word!);
- type 5: triples $(C(s), C(w), s^{-1}t)$, where s, t are generators, w is a geodesic word of length 3 representing a cone type and wt^{-1} is a subword of a fundamental relator (t is not the final letter of w !).

B.5 THE MATRIX

The matrix \mathfrak{T} has rows and columns indexed by the compatible triples. In fact, all other possible triples give rise to a null contribution when we calculate the recursion (cf. Lemmata 3.39 and 3.40).

B.5.1 Condition characterizing \mathfrak{D}

Consider now two compatible triples $(\mathbf{c}, \mathbf{d}, y)$ and $(\mathbf{c}', \mathbf{d}', y')$. Then, the entry $\mathfrak{T}_{\mathbf{c}', \mathbf{d}', y'}^{\mathbf{c}, \mathbf{d}, y}$ is non-zero if and only if there exist two generators s, t such that $\mathbf{c} \xrightarrow{s} \mathbf{c}'$ and $\mathbf{d} \xrightarrow{t} \mathbf{d}'$ and $y' = s^{-1}yt$ as an element of the group.

B.5.2 Definition of the matrices \mathfrak{D} and \mathfrak{T} .

We define the matrix (with scalar entries) \mathfrak{D} as in equation 3.10 (Chap. 3) having only the compatible triples as row and column index and a 1 entry corresponding to the row $(\mathbf{c}, \mathbf{d}, y)$ and to the column $(\mathbf{c}', \mathbf{d}', y')$ if and only if $\mathfrak{T}_{\mathbf{c}', \mathbf{d}', y'}^{\mathbf{c}, \mathbf{d}, y}$ is non-zero, as explained above, while $\mathfrak{D}_{\mathbf{c}', \mathbf{d}', y'}^{\mathbf{c}, \mathbf{d}, y} = 0$ if the correspondent one in \mathfrak{T} is zero. When we consider a scalar matrix system, the matrix \mathfrak{T} has scalar non-negative entries and it is clear that \mathfrak{T} is irreducible if and only if \mathfrak{D} is irreducible (cf. [Sen06]). In fact,

$$\mathfrak{T}_{\mathbf{c}', \mathbf{d}', s^{-1}yt}^{\mathbf{c}, \mathbf{d}, y} = \mathfrak{D}_{\mathbf{c}', \mathbf{d}', s^{-1}yt}^{\mathbf{c}, \mathbf{d}, y} (H_{\mathbf{c}, \mathbf{c}', s} \otimes \overline{H_{\mathbf{d}, \mathbf{d}', t}}),$$

where the 1's in \mathfrak{D} are substituted with the map $H_{\mathbf{c}, \mathbf{c}', s} \otimes \overline{H_{\mathbf{d}, \mathbf{d}', t}}$ (which in the case of a scalar system are positive scalars) corresponding to the row and column index in the order, while the 0's are identified with the null map.

B.5.3 Computing

The matrix \mathfrak{D} has been explicitly calculated for the case $k = 2$: the compatible triples are known.

We implemented few modules in PythonTM for representing the compatible triples and we used the code to produce the matrix following the condition as above. A double `for` cycle running over the set of compatible triples tests the condition characterizing the non-zero entries of \mathfrak{T} as in B.5.1 above.

The module implements some functions to calculate the compatible triples starting from a set of representatives for the cone types of the group and

few basic rules to calculate products and reductions in a surface group of genus 2.

The output, i.e. the matrix, is saved in a file, which then we imported in MATLAB (MathWorks®). We used the function `perron` (whose code is available at <https://it.mathworks.com/matlabcentral/fileexchange/22763-perron-root-computation>) to calculate the Perron-Frobenius eigenvalue of the matrix. The function gives an error message if the matrix is not irreducible (in this case, Perron-Frobenius Theorem does not apply). Since the script runs without errors, the matrix is irreducible and its spectral radius is a simple eigenvalue.

Here follows the output we get when we launch the command `perron` in MATLAB.

```
>> perron(T. ')\n18.1446
```

It is then possible to choose the linear maps $H_{\mathbf{c},\mathbf{c}',s}$ as scalars so that the Perron-Frobenius eigenvalue of \mathfrak{T} is 1 (cf. 3.45) and Theorem 3.30 holds.

APPENDIX C

PYTHONTM CODE

The file `FundRel.txt` containing the fundamental relators.

```
abABcdCD
bABcdCDa
ABcdCDab
BcdCDabA
cdCDabAB
dCDabABc
CDabABcd
DabABcdC
dcDCbaBA
cDCbaBA d
DCbaBA d c
CbaBA d c D
baBA d c DC
aBA d c DC b
BA d c DC ba
AdcDCbaB
```

The file `Representatives.txt`, produced by a script, contains a set of geodesic words representing the 48 non-trivial cone types in Γ_2 .

```
a, b, c, d, A, B, C, D, ab, aB, ba, bA,
cd, cD, dc, dC, Ad, AB, BA, Cb, CD, Da,
DC, abA, aBA,baB, bAB, cdC, cDC, dcD, dCD,
Adc, ABc, Bcd, BAd, Cba, CDa, Dab, DCb, abAB,
aBA d, baBA, bABc, cDCb, dCDa, ABcd, CDab.
```

Here follow the three scripts used to calculate the transition matrix \mathfrak{D} .

The first one implements some operations in the group Γ_2 , the second produces the compatible triples, the third computes the matrix.

```
import string

#####

#OPERATIONS ON WORDS IN THE FREE GROUP

def Invert(Word):
    Inverse = ''
    for i in range(len(Word)):
        if Word[-i-1] > 'Z':
            Inverse += Word[-i-1].upper()
        else:
            Inverse += Word[-i-1].lower()
    return Inverse

def OneStepReduction(Word):
    i = 0
    n = len(Word)
    while i < n-1:
        if Word[i] != Word[i+1] and \
            Word[i].upper() == Word[i+1].upper():
            Word = Word[:i]+Word[i+2:]
            n = len(Word)
        else:
            i += 1
    return Word

def Reduce(Word):
    for i in range((len(Word)+1)/2):
        Word = OneStepReduction(Word)
    return Word

def Multiply(Word1, Word2):
    Juxtaposition = Word1 + Word2
    Product = Reduce(Juxtaposition)
    return Product

#####

#FUNDAMENTAL RELATORS AND THEIR SUBWORDS

f = open("FundRel.txt", "r")
```

```

#Fund is the list of fundamental relators
Fund = f.readlines()

#Get rid of the escape sequence \n
for i in range(len(Fund)):
    Fund[i] = Fund[i][:len(Fund[i])-1]
f.close()

#List of subwords of the
#fundamental relators of length 4
Half = []

#List of equivalent words of length 4
Equiv = []
for i in range(len(Fund)):
    Half += [Fund[i][:len(Fund[i])/2], Fund[i][len(Fund[i])/2:]]
    Equiv += [[Fund[i][:len(Fund[i])/2], \
                Invert(Fund[i][len(Fund[i])/2:]])]

#Prefixes of a fundamental relator of length 5.
#If a word contains one of these, then it
#can be rewritten in a shorter way
Ref1 = []
Eq1 = []
for i in range(len(Fund)):
    Ref1 += [Fund[i][:5]]
    Eq1 += [[Fund[i][:5], Invert(Fund[i][5:]])]

Ref2 = []
Eq2 = []
for i in range(len(Fund)):
    Ref2 += [Fund[i][:6]]
    Eq2 += [[Fund[i][:6], Invert(Fund[i][6:]])]

Ref3 = []
Eq3 = []
for i in range(len(Fund)):
    Ref3 += [Fund[i][:7]]
    Eq3 += [[Fund[i][:7], Invert(Fund[i][7:]])]

#####

#OPERATIONS ON WORDS IN THE SURFACE GROUP

```

```

def NumberOfShortenings(Word):
    Word = Reduce(Word)
    i = 0
    for j in range(len(Ref1)):
        if string.find(Word, Ref1[j]) >= 0:
            i += 1
        else:
            pass
    return i

def Shorten(Word):
    Word = Reduce(Word)
    for i in range(len(Fund)):
        while string.find(Word, Fund[i]) >= 0:
            k = string.find(Word, Fund[i])
            Word = Word[:k]+Word[k+8:]
        while string.find(Word, Ref3[i]) >= 0:
            k = string.find(Word, Ref3[i])
            Word = Word[:k]+Eq3[i][1]+Word[k+7:]
        while string.find(Word, Ref2[i]) >= 0:
            k = string.find(Word, Ref2[i])
            Word = Word[:k]+Eq2[i][1]+Word[k+6:]
        while string.find(Word, Ref1[i]) >= 0:
            k = string.find(Word, Ref1[i])
            Word = Word[:k]+Eq1[i][1]+Word[k+5:]
    return Word

def IsGeodesic(Word):
    if Word != Reduce(Word) || NumberOfShortenings(Word) != 0:
        return False
    else:
        return True

def Rewrite(Word):
    if IsGeodesic(Word) != True:
        print 'Error: non-geodesic word.'
        return None
    else:
        Rewritings = [Word]
        #count = 0
        for i in range(len(Equiv)):
            while string.find(Word, Equiv[i][0]) >= 0:
                k = string.find(Word, Equiv[i][0])
                Word = Word[:k]+Equiv[i][1]+Word[k+4:]
            if Word in Rewritings:

```

```

        pass
    else:
        Rewritings += [Word]
        #count += 1
    #print count
    return Rewritings

def GetCannonSuffix(Word):
    Rewritings = Rewrite(Word)
    MaximalSuffix = GetSuffix(Word)
    for i in range(len(Rewritings)):
        Suff = GetSuffix(Rewritings[i])
        if len(Suff) > len(MaximalSuffix):
            MaximalSuffix = Suff
    return MaximalSuffix

def GetSuffix(Word):
    Suff = None
    if IsGeodesic(Word) != True:
        print 'Error: non-geodesic word.'
    else:
        for j in range(1,5):
            for i in range(len(Fund)):
                if string.find(Fund[i][:4], \
                    Word[-j:]) >= 0:
                    Suff = Word[-j:]
            else:
                pass
    return Suff

def GetEquivSuff(Suff):
    EqSuff = None
    if (Suff in Half) != True:
        print 'Error: not a suffix of length 4.'
    else:
        for k in range(len(Equiv)):
            if Equiv[k][0] == Suff:
                EqSuff = Equiv[k][1]
            else:
                pass
    return EqSuff

def GetLonger(Suff1, Suff2):
    if len(Suff2) > len(Suff1):
        return Suff2

```

```

        else:
            return Suff1

def GetSuffixSuccessor(Word, Letter):
    Suff1 = GetCannonSuffix(Word)
    Suff2 = GetEquivSuff(Suff1)
    Prod1 = Multiply(Suff1, Letter)
    Prod2 = Multiply(Suff2, Letter)
    if len(Prod1) < len(Suff1) or \
        len (Prod2) < len(Suff2):
        return None
    else:
        NewSuff1 = GetCannonSuffix(Prod1)
        NewSuff2 = GetCannonSuffix(Prod2)
        return GetLonger(NewSuff1, NewSuff2)

#####

def CyclicallyPermute(Tuple):
    Permuted = ()
    for i in range(len(Tuple)):
        Permuted += (Tuple[i:]+Tuple[:i],)
    return Permuted

FundRel = ()
Relator = 'abABcdCD'
RInv = Invert(Relator)
F1 = CyclicallyPermute(Relator)
F2 = CyclicallyPermute(RInv)
FundRel = F1 + F2

f = open("FundRel.txt", "w")
for i in range(len(FundRel)):
    f.write(FundRel[i]+'\\n')
f.close()

Eq = []
for i in range(len(F1)):
    Eq += [[F1[i][:4], Invert(F1[i][4:])]

#####

def GetSuccessors(Word):
    Successors = []
    for s in ('a', 'A', 'b', 'B', 'c', 'C', 'd', 'D'):

```

```

        if len(Word) < 4:
            Prod = Multiply(Word,s)
            if len(Prod) < len(Word):
                continue
            else:
                Successors += [GetCannonSuffix(Prod)]
        else:
            Successors += [GetSuffixSuccessor(Word,s)]
    return Successors

#####

f = open("Representatives.txt","r")
rep = f.readlines()
#Get rid of the escape sequence \n
for i in range(len(rep)):
    rep[i] = rep[i][:len(rep[i])-1]
f.close()

def PrefixRelator(Word):
    Boolean = False
    for i in range(len(Fund)):
        if string.find(Fund[i], Word) >= 0:
            Boolean = True
        else:
            pass
    return Boolean

def SameOctagon(Word1, Word2):
    Prod = Multiply(Invert(Word1),Word2)
    return PrefixRelator(Prod)

```

```

from Operations import *

##### SOME ADDITIONAL FUNCTIONS #####

def FindTheRel(s):
    "Given a word, finds the relator having it as a \
    prefix if it exists. If it does not, the function \
    returns the empty string."
    Relator = ''
    for f in Fund:
        if f[:len(s)] == s:
            Relator = f
    return Relator

```

```

def TheOtherCone(s):
    "To get the 'associated word', we need to eliminate \
    the last two letters and to eliminate the first \
    len(s)+1 - for TripleFourthKindA"
    if len(s) != 2 and len(s) != 3:
        print "Error in the length!"
        return ''
    w = FindTheRel(Invert(s))
    m = -1
    n = -1
    if len(s) == 3:
        m = len(w) - 2
    else:
        m = len(w) - 4
    y = w[:m]
    w = w[:m]
    w = w[len(s)+1:]
    return [w, y]

def FindGenerator(i):
    "For TripleFourthKindB"
    ind = -1
    for j in range(len(Fund)):
        if Fund[j][:len(i)] == i:
            ind = j
    x = Fund[ind][len(i)]
    f = Invert(Fund[ind])
    p = f.find(x)
    k = f[p+1]
    return [ i, k, Multiply(Invert(k),Invert(x))]

def SymmList(T):
    S = []
    for x in T:
        try:
            S += [[x[1], x[0], Invert(x[2])]]
        except:
            print "Error!!!"
    return S

def PrintList(l):
    for i in range(len(l)):
        print l[i]

```

```

##### TRIPLES OF TYPE 0 #####

T0 = []
for i in range(len(rep)):
    T0 += [[rep[i], rep[i], '']]

##### TRIPLES OF TYPE 1 #####

def TripleFirstKind(n):
    T = []
    r = []
    for i in range(len(rep)):
        if len(rep[i]) == n: r += [rep[i]]

    for i in range(len(r)):
        for j in range(len(r)):
            if SameOctagon(r[i],r[j])==True and \
                r[i]!=r[j]:
                T += [[r[i],r[j],Multiply(Invert(r[i]),\
                    r[j])]]

    return T

##### TRIPLES OF TYPE 2 #####

r1 = []
for i in range(len(rep)):
    if len(rep[i]) == 1: r1 += [rep[i]]

def TripleSecondKind(n):
    T = []
    r = []
    for i in range(len(rep)):
        if len(rep[i]) == n: r += [rep[i]]

    if n == 4:
        for x in Half:
            if not(x in r):
                r.append(x)

    for i in range(len(r)):
        for j in range(len(r1)):
            prod=Multiply(Invert(r[i][len(r[i])-1:]),r1[j])
            if PrefixRelator(prod)==True and prod!='' \
                and PrefixRelator(Multiply(r[i],r1[j]))!=True \
                and Multiply(r[i][len(r[i])-2], r1[j])!='':

```

```

        T += [ [r[i], r1[j], prod] ]

    return T

##### TRIPLES OF TYPE 3 #####

def TripleThirdKind(n):
    T = []
    r = []
    for i in range(len(rep)):
        if len(rep[i]) == n: r += [rep[i]]

    if n <= 1:
        return T

    if n == 4:
        for x in Half:
            if not(x in r):
                r.append(x)

    for i in range(len(r)):
        T += [ [r[i], r[i][1:], '' ] ]
    return T

##### TRIPLES OF TYPE 4 #####

def TripleFourthKindA(n):
    if n != 2 and n != 3:
        print "Error in the length!"
        return []

    T = []
    r = []
    for p in rep:
        if len(p) == n: r += [p]

    for x in r:
        ret = TheOtherCone(x)
        ret = [x] + ret
        T += [ret]
    return T

##### TRIPLES OF TYPE 5 #####

def TripleFourthKindB(n):
    r = []
    if n == 4:

```

```

        for x in Half:
            if not(x in r):
                r.append(x)
    for p in rep:
        if len(p) == n:
            r += [p]
    T = []
    for x in r:
        T += [FindGenerator(x)]
    return T

#####Da Successors#####

ListOfSucc = []

Rewritings = []

for i in range(len(rep)):
    if len(rep[i]) == 4:
        Rewritings = Rewrite(rep[i])
        for j in range(len(Rewritings)):
            if Rewritings[j] not in rep:
                rep += [Rewritings[j]]

for x in rep:
    ListOfSucc += [[x, GetSuccessors(x)]]

#####

##### SET OF ALL TRIPLES #####

T = T0

for i in [1, 2, 3]:
    T += TripleFirstKind(i)

for i in [2, 3, 4]:
    T += TripleSecondKind(i)
    T += SymmList(TripleSecondKind(i))

for i in [2, 3, 4]:
    T += TripleThirdKind(i)
    T += SymmList(TripleThirdKind(i))

for i in [2, 3]:

```

```

    T += TripleFourthKindA(i)
    T += SymmList(TripleFourthKindA(i))

T += TripleFourthKindB(3)
T += SymmList(TripleFourthKindB(3))

if __name__ == '__main__':

    g = open("Triples.txt", "w")
    for i in range(len(T)):
        g.write("C("+T[i][0]+"),C("+T[i][1]+"), \
            "+T[i][2]+");\n")
    g.close()

    l = len(T)
    print "%d triples" % l

```

```

from Operations import *
from Triples import *

def MatrixEntry(t1,t2):
    wA1=t1[0]
    wA2=t1[1]
    sA=t1[2]
    wB1=t2[0]
    wB2=t2[1]
    sB=t2[2]

    ind1 = rep.index(wA1)
    ind2 = rep.index(wA2)

    if (wB1 in ListOfSucc[ind1][1]) and \
        (wB2 in ListOfSucc[ind2][1]):
        return True
    else:
        return False

h = open("MatrixD.txt", "w")
MatrixD = []
for x in T:
    NewLine = []
    for y in T:
        if MatrixEntry(x,y):
            NewLine += [1]

```

```
                h.write("1,")
            else:
                NewLine += [0]
                h.write("0,")
            MatrixD += [NewLine]
            h.write("\n")
h.close()

l = len(MatrixD[0])
print "%d x %d matrix" %(l, l)
```

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