LOCALIZATION FOR RIESZ MEANS ON COMPACT RANK ONE SYMMETRIC SPACES

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Abstract. The classical Riemann localization principle states that if an integrable function of one variable vanishes in an open set, then its trigonometric Fourier expansion converges to zero in this set. This pointwise localization principle fails in higher dimensions. Here we study the Hausdorff dimension of the sets of point where localization for Riesz means for eigenfunction expansions of the Laplace-Beltrami operator on compact rank one symmetric spaces may fail.

Introduction

This article deals with spherical harmonic expansions on spheres and projective spaces. Let $\mathcal{M}$ be a d-dimensional compact rank one symmetric space and denote by $0 = \lambda_0^2 < \lambda_1^2 < \lambda_2^2 < \dots$ the eigenvalues and by $\mathcal{H}_n$ the corresponding eigenspaces of the Laplace-Beltrami operator Δ, the spherical harmonics of degree n. To every square integrable function, and more generally tempered distribution, one can associate a Fourier series:

$$f(x) = \sum_{n=0}^{+\infty} Y_n f(x) ,$$

where $Y_n f(x)$ is the orthogonal projection of $f(x)$ on $\mathcal{H}_n$. These Fourier series converge in the metric of $L^2(\mathcal{M})$ and in the topology of distributions, but in general one cannot ensure the pointwise convergence. For this reason a number of summation methods have been introduced. One of these are the Cesàro means

$$C_N^\alpha f(x) = \frac{1}{A_N^\alpha} \sum_{n=0}^{N} A_{N-n}^\alpha \, Y_n f(x) ,$$

where A_n^α are the binomial coefficients $(\alpha+1)(\alpha+2)\dots(\alpha+n)/n!$. Another one are the Bochner-Riesz means

$$S_R^\alpha f(x) = \sum_{\lambda_n < R} \left(1 - \frac{\lambda_n^2}{R^2}\right)^\alpha Y_n f(x) .$$

In particular, in both cases when $\alpha = 0$ one obtains the so-called spherical partial sums. The Cesàro and the Bochner-Riesz means of the same order give rise to equivalent summability methods. In what follows we shall concentrate on the latter. The classical Riemann localization principle states that if an integrable function of one variable vanishes in an open set, then the partial sums of its trigonometric Fourier expansion converge uniformly to zero in every compact subset of the open set. More generally, if a function is smooth in an open set, then the associate Fourier series converges to the function in this open set. It has been known that there is no direct analogue of the Riemann localization principle in higher dimensions, and there are examples of the failure of localization in Hölder, Lebesgue and Sobolev spaces. See [1], [2], [4], and [5], [20], [25] for the role of antipodal points for spherical harmonic expansions. See also the examples below. Despite the negative results, it has been proved in [3] and [26] that there is an almost everywhere localization principle for square integrable functions on $\mathcal{M}$; that is, if a function in $L^2(\mathcal{M})$ vanishes almost everywhere in an open set, then $\sum_{n=0}^{+\infty} Y_n f(x) = 0$ for almost every x in this open set. See also [8], [9], [13] and [31] for the corresponding result on the Euclidean spaces. The works of Bastis and Meaney deal with almost everywhere localization of Bochner-Riesz means of order $\alpha = 0$ for square integrable functions. On the other hand, it is known that for square integrable functions localization holds everywhere above the so-called critical index $\alpha = (d-1)/2$, while for integrable functions the critical index is $\alpha = d-1$. See [5] and Theorem 2 below. Finally, in [10], [11] and [15] the dimension of sets for which localization fails is studied.

In this paper we continue this line of research in the area of exceptional sets in harmonic analysis. In particular we prove that for Bochner-Riesz means of order α of p integrable functions on compact rank one symmetric spaces localization holds, with a possible exception in a set of point of suitable Hausdorff dimension. More generally we consider localization for distributions in Sobolev spaces. The Bessel potential $G^\gamma f(x)$, $-\infty < \gamma < +\infty$, of a tempered distribution $f(x) = \sum_{n=0}^{+\infty} Y_n f(x)$ is the tempered distribution defined by

$$G^\gamma f(x) \;=\; \sum_{n=0}^{+\infty} \left(1+\lambda_n^2\right)^{-\gamma/2} \; Y_n f(x)\,.$$

Our Sobolev spaces are the spaces of potentials of functions in $L^p(\mathcal{M})$.

Our first result is an exact analogue for Bochner-Riesz means of the result of Meaney in [26] for spherical sums.

Theorem 1. *Assume that $f(x)$ is a tempered distribution on $\mathcal{M}$, with spherical harmonic expansion $\sum_{n=0}^{+\infty} Y_n f(x)$. Also assume that $f(x) = 0$ for all x in a ball $\{|x-\mathbf{o}| < \varepsilon\}$, with radius $\varepsilon > 0$ and centre $\mathbf{o}$. Then the following are equivalent:*

(1) $\lim_{R\to+\infty} \{S_R^\alpha f(\mathbf{o})\} = 0$,
(2) $\lim_{n\to+\infty} \{n^{-\alpha}\, Y_n f(\mathbf{o})\} = 0$.

As shown by Bochner in [4], see also [31], the critical index for pointwise localization of Bochner-Riesz means on Euclidean spaces of dimension d is $(d-1)/2$. On the other hand, as shown by Kogbetlianz in [20] and Bonami and Clerc in [5], spheres and projective spaces are different, since antipodal points come into play. See also the paper of Hörmander [19] for the study of asymptotic properties of the spectral functions and summability of eigenfunction expansions for elliptic differential operators. In our second result we revisit this problem of pointwise localization.

Theorem 2. *Assume that:*

$$-\infty < \gamma < +\infty, \quad \alpha \geq 0, \quad 1 \leq p \leq +\infty\,.$$

Also assume that

(1) $\mathcal{M}$ *is the sphere* $\mathbb{S}^d$ *and*

$$\begin{cases} \alpha+\gamma \,\geq\, d/p \,-\, 1 & \text{if} \quad p < 2d/(d+1)\,, \\ \alpha+\gamma \,>\, (d-1)/2 & \text{if} \quad p = 2d/(d+1)\,, \\ \alpha+\gamma \,\geq\, (d-1)/2 & \text{if} \quad p > 2d/(d+1)\,. \end{cases}$$

(2) $\mathcal{M}$ *is the real projective space* $P^d(\mathbb{R})$ *and, for every* p,

$$\alpha+\gamma \,\geq\, (d-1)/2\,.$$

(3) $\mathcal{M}$ *is the complex projective space* $P^d(\mathbb{C})$ *and*

$$\begin{cases} \alpha+\gamma \,\geq\, (d-4)/2 \,+\, 2/p & \text{if} \quad p < 4/3\,, \\ \alpha+\gamma \,>\, (d-1)/2 & \text{if} \quad p = 4/3\,, \\ \alpha+\gamma \,\geq\, (d-1)/2 & \text{if} \quad p > 4/3\,. \end{cases}$$

(4) $\mathcal{M}$ *is the quaternionic projective space* $P^d(\mathbb{H})$ *and*

$$\begin{cases} \alpha+\gamma \,\geq\, (d-6)/2 \,+\, 4/p & \text{if} \quad p < 8/5\,, \\ \alpha+\gamma \,>\, (d-1)/2 & \text{if} \quad p = 8/5\,, \\ \alpha+\gamma \,\geq\, (d-1)/2 & \text{if} \quad p > 8/5\,. \end{cases}$$

(5) $\mathcal{M}$ *is the Cayley projective space* $P^{16}(Cay)$ *and*

$$\begin{cases} \alpha+\gamma \,\geq\, 3 \,+\, 8/p & \text{if} \quad p < 16/9\,, \\ \alpha+\gamma \,>\, 15/2 & \text{if} \quad p = 16/9\,, \\ \alpha+\gamma \,\geq\, 15/2 & \text{if} \quad p > 16/9\,. \end{cases}$$

Under the above assumptions, if $f(x)$ *is in* $L^p(\mathcal{M})$ *and if* $G^\gamma f(x) = 0$ *in an open set* Ω*, then for every* $x \in \Omega$

$$\lim_{R\to+\infty} \{S_R^\alpha G^\gamma f(x)\} \;=\; 0\,.$$

The result in the above theorem for the sphere $\mathbb{S}^d$ is slightly better than the corresponding in [5], where only the case $\gamma = 0$ is considered and the condition for localization is $\alpha \geq (d-1)/2$ and $\alpha > d/p - 1$. When $p = 1$ the critical index for $\alpha+\gamma$ has a geometric interpretation:

$$\{dimension\, of\, the\, space\} - \frac{1}{2}\,\{dimension\, of\, the\, antipodal\, manifold\} - 1\,.$$

Our third result revisits and extends the almost everywhere localization result of Bastis [3] and Meaney [26].

Theorem 3. *Assume that:*

$$\varepsilon > 0, \quad -\infty < \gamma < +\infty, \quad \alpha \geq 0, \quad 1 \leq p \leq 2, \quad 0 \leq \beta \leq \alpha+\gamma-(d-1)\left(\frac{1}{p}-\frac{1}{2}\right).$$

Then there exists a positive constant C *with the following property: If* $f(x)$ *is in* $L^p(\mathcal{M})$ *and if* $G^\gamma f(x) = 0$ *in an open set* Ω*, then there exists* $F(x)$ *with* $\|F\|_{L^p} \leq C\|f\|_{L^p}$ *and such that for all* $x \in \Omega$ *with distance* $\{x, \partial\Omega\} > \varepsilon$,

$$\sup_{R>0} \{|S_R^\alpha G^\gamma f(x)|\} \;\leq\; G^\beta F(x)\,.$$

By this theorem, since the set where a potential $G^{\beta}F(x) = +\infty$ has Hausdorff dimension at most $d - \beta p$, the set of point where localization fails has a small dimension.

Corollary 4. *Under the above assumptions on p, α and γ, if $G^{\gamma}f(x) = 0$ in an open set Ω, then the following hold:*

(1) *If $\alpha + \gamma = (d-1)\left(\frac{1}{p} - \frac{1}{2}\right)$, then for almost every point in Ω,*

$$\lim_{R \to +\infty} \{S_R^{\alpha} G^{\gamma} f(x)\} \ = \ 0\,;$$

(2) *If $\alpha + \gamma > (d-1)\left(\frac{1}{p} - \frac{1}{2}\right)$, then*

$$\lim_{R \to +\infty} \{S_R^{\alpha} G^{\gamma} f(x)\} \ = \ 0$$

at all points in this open set Ω, with possible exceptions in a set with Hausdorff dimension at most $\delta = d - p\left(\alpha + \gamma - (d-1)\left(\frac{1}{p} - \frac{1}{2}\right)\right)$.

The case $\alpha = \gamma = 0$ and $p = 2$ of the corollary is the above quoted result on the almost everywhere localization for spherical partial sums of square integrable functions. Indeed when $p = 2$ a more precise result holds.

Theorem 5. *Assume that one of the following conditions holds:*

(1) $\alpha \geq 0\,,\qquad 0 \leq \alpha + \gamma < d/2\,,\qquad \delta = d - 2(\alpha+\gamma)$;
(2) $\alpha \geq 0\,,\qquad (d-1)/4 \leq \alpha + \gamma \leq (d-1)/2\,,\qquad \delta = d - 2(\alpha+\gamma) - 1$;
(3) $\alpha \geq 0\,,\qquad \alpha + \gamma \geq (d-1)/2\,,\qquad \delta = 0$.

Assume that $f(x)$ is in $L^2(\mathcal{M})$ and that $G^{\gamma}f(x) = 0$ in an open set Ω, and let $d\upsilon(x)$ a non-negative Borel measure with support in $\Omega \cap \{distance\,\{x, \partial\Omega\} > \varepsilon\}$ for some $\varepsilon > 0$. Then there exists a positive constant C such that

$$\int_{\mathcal{M}} \sup_{R>0} \{|S_R^{\alpha} G^{\gamma} f(x)|\}\ d\upsilon(x) \ \leq \ C\, \|f\|_{L^2} \left\{ \int_{\mathcal{M}} \int_{\mathcal{M}} \frac{d\upsilon(x)\, d\upsilon(y)}{|x-y|^{\delta}} \right\}^{\frac{1}{2}}.$$

By this theorem, the maximal function $\sup_{R>0} \{|S_R^{\alpha} G^{\gamma} f(x)|\}$ cannot be infinite on the support of a measure of finite energy, and this implies the following.

Corollary 6. *Under the above assumptions on p, α, γ and δ, if $G^{\gamma}f(x) = 0$ in an open set Ω, then*

$$\lim_{R \to +\infty} \{S_R^{\alpha} G^{\gamma} f(x)\} \ = \ 0$$

at all points in the open set Ω, with possible exceptions in a set with Hausdorff dimension at most δ.

The above Theorem 5 and Corollary 6 extend to compact rank one symmetric spaces the results in [15] for the Euclidean spaces $\mathbb{R}^d$. It is likely that some of the above results can be further extended to eigenfunction expansions of elliptic differential operators on Riemannian manifolds.

Finally, we want to point out that while the first two theorems on pointwise localization are sharp, we do not expect that in the other theorems the indexes on the dimension of sets where localization may fail are best possible.

1. Harmonic analysis on compact rank one symmetric spaces

In what follows we shall denote by $|x-y|$ the geodesic distance between the two points x and y in a Riemmanian Manifold $\mathcal{M}$. A two-point homogeneous space is a Riemannian manifold with the property that for every two pairs of points x_1, x_2 and y_1, y_2 with $|x_1 - x_2| = |y_1 - y_2|$, there is an isometry g of $\mathcal{M}$ such that $x_1 = gy_1$ and $x_2 = gy_2$. Birkhoff has called this property two-point homogeneity. Wang in [34] has shown that any compact two-point homogeneous space is isometric to a compact rank one symmetric space, that is:

i) the sphere $\mathbb{S}^d = SO(d+1)/SO(d)$
$d = 1, 2, 3, \ldots$;
ii) the real projective space $P^d(\mathbb{R}) = SO(d+1)/O(d)$
$d = 2, 3, 4, \ldots$;
iii) the complex projective space $P^d(\mathbb{C}) = SU(l+1)/S(U(l) \times U(1))$
$d = 4, 6, 8, \ldots$ and $l = d/2$;
iv) the quaternionic projective space $P^d(\mathbb{H}) = Sp(l+1)/Sp(l) \times Sp(1)$
$d = 8, 12, 16, \ldots$ and $l = d/4$;
v) the Cayley projective plane $P^{16}(\mathrm{Cay})$.

Here d denotes the real dimension of any one of these spaces. Without loss of generality, one can renormalise the metric and the measure so that the total measure of $\mathcal{M}$ is 1 and the diameter of $\mathcal{M}$ is π. If $\mathbf{o}$ is a fixed point in $\mathcal{M}$, then $\mathcal{M}$ can be identified with the homogeneous space G/K, where G is the maximal connected group of isometries of $\mathcal{M}$ and K is the stabilizer of $\mathbf{o}$ in G. The measure dx is induced by the normalised left Haar measure dg on G: For any fixed point $\mathbf{o}$ in $\mathcal{M}$ and any function $f(x)$ integrable on $\mathcal{M}$,

$$\int_{\mathcal{M}} f(x)\, dx \;=\; \int_G f(g\mathbf{o})\, dg\,.$$

In particular, the convolution on the group G induced a convolution on the manifold $\mathcal{M}$. A function $f(g)$ on the isometry group G is *right K-invariant* if, for every g in G and k in K,

$$f(kg) \;=\; f(g)\,.$$

A function $f(g)$ on G is *bi-K-invariant* if, for every g in G and k in K,

$$f(kg) \;=\; f(gk) \;=\; f(g)\,.$$

Functions and distributions on $\mathcal{M} = G/K$ can be identify with right K-invariant functions and distributions on G. It suffices to put $f(g) = f(x)$ whenever $g\mathbf{o} = x$. A function is radial around $\mathbf{o}$ if $f(x)$ only depends on $|x - \mathbf{o}|$. Radial functions on $\mathcal{M}$ correspond to bi-K-invariant functions on G. Indeed, by the two-point homogeneity, K fixes $\mathbf{o}$ and acts transitively on the set of points at a given distance from $\mathbf{o}$. The points with distance from $\mathbf{o}$ equal to the diameter of $\mathcal{M}$ are the antipodal manifold of $\mathbf{o}$. If $\mathcal{A}(t)$, $0 \le t \le \pi$, is the surface measure of a sphere of radius t in $\mathcal{M}$, then

$$\mathcal{A}(t) \;=\; C\left(\sin\frac{t}{2}\right)^{M} (\sin t)^{N}\,.$$

The constant $C > 0$ is chosen so that $\int_0^\pi \mathcal{A}(t)dt = 1$, $M + N + 1$ is the dimension of the manifold, M is the dimension of the antipodal manifold, and these parameters are as follows:

$\mathcal{M}$	M	N	$\mathcal{M}$	M	N
$\mathbb{S}^d$	0	$d-1$	$P^d(\mathbb{H})$	$d-4$	3
$P^d(\mathbb{R})$	$d-1$	0	$P^{16}(\mathrm{Cay})$	8	7
$P^d(\mathbb{C})$	$d-2$	1			

If $f(t)$ is integrable on $[0, \pi]$ with respect to the measure $\mathcal{A}(t)dt$ then, for any $\mathbf{o} \in \mathcal{M}$,

$$\int_{\mathcal{M}} f(|x - \mathbf{o}|)\, dx = \int_0^\pi f(t)\, \mathcal{A}(t)\, dt .$$

The compact rank one symmetric spaces admit an isometry invariant second order differential operator, the Laplace-Beltrami operator Δ. The spectrum of this operator is discrete, real and non-negative. One can arrange the eigenvalues in increasing order: $0 = \lambda_0^2 < \lambda_1^2 < \lambda_2^2 < \ldots$. More precisely:

$$\lambda_n^2 = sn\,(sn + a + b + 1) .$$

The index n ranges over all non-negative integers and $s = 1$ if $\mathcal{M} = \mathbb{S}^d$, $P^d(\mathbb{C})$, $P^d(\mathbb{H})$ or $P^{16}(\mathrm{Cay})$, while $s = 2$ if $\mathcal{M} = P^d(\mathbb{R})$. The parameters a and b and the eigenvalues λ_n^2 are given by the following table:

$\mathcal{M}$	a	b	λ_n^2
$\mathbb{S}^d$	$\frac{d-2}{2}$	$\frac{d-2}{2}$	$n(n+d-1)$
$P^d(\mathbb{R})$	$\frac{d-2}{2}$	$\frac{d-2}{2}$	$2n(2n+d-1)$
$P^d(\mathbb{C})$	$\frac{d-2}{2}$	0	$n(n+\frac{d}{2})$
$P^d(\mathbb{H})$	$\frac{d-2}{2}$	1	$n(n+1+\frac{d}{2})$
$P^{16}(\mathrm{Cay})$	7	3	$n(n+11)$

The eigenspaces $\mathcal{H}_n$ corresponding to the eigenvalues λ_n^2 are finite-dimensional, invariant and irreducible under the group action, and they are mutually orthogonal with respect to the inner product

$$\langle f, g \rangle = \int_{\mathcal{M}} f(x)\, \overline{g(x)}\, dx .$$

Moreover $L^2(\mathcal{M}) = \bigoplus_{n=1}^{+\infty} \mathcal{H}_n$. For each integer $n \geq 0$ let $d_n = dimension\{\mathcal{H}_n\}$ and $\{Y_{n,j}(x)\}_{j=1}^{d_n}$ be an orthonormal basis of $\mathcal{H}_n$. The dimensions of these eigenspaces can be computed explicitly, but here it suffices to say that there exists two positive constants c and C such that, for every n,

$$c\,(1+n)^{d-1} \;\le\; d_n \;\le\; C\,(1+n)^{d-1}\;.$$

The Fourier expansion of a square integrable function, and more generally of a tempered distribution, is given by

$$f(x) \;=\; \sum_{n=0}^{+\infty} Y_n f(x) \;=\; \sum_{n=0}^{+\infty} \left\{ \sum_{j=1}^{d_n} \widehat{f}(n,j)\, Y_{n,j}(x) \right\}\;,$$

with

$$\widehat{f}(n,j) \;=\; \int_{\mathcal{M}} f(x)\,\overline{Y_{n,j}(x)}\,dx\;.$$

It is convenient to rewrite the orthogonal projection $Y_n f(x)$ of $f(x)$ onto $\mathcal{H}_n$ as a convolution:

$$\begin{aligned} Y_n f(x) \;&=\; \sum_{j=1}^{d_n} \widehat{f}(n,j)\, Y_{n,j}(x) \\ &=\; \int_{\mathcal{M}} f(y) \left\{ \sum_{j=1}^{d_n} Y_{n,j}(x)\,\overline{Y_{n,j}(y)} \right\} dy \\ &=\; \int_{\mathcal{M}} f(y)\, Z_n(x,y)\,dy\;. \end{aligned}$$

The functions $Z_n(x,y)$ are the so called zonal spherical functions of degree n with pole x, and they are related to a system of Jacobi polynomials: If $t = |x-y|$, then

$$Z_n(x,y) \;=\; \sum_{j=1}^{d_n} Y_{n,j}(x)\,\overline{Y_{n,j}(y)} \;=\; d_n\, \frac{P_n^{(a,b)}(\cos(t/s))}{P_n^{(a,b)}(1)}\;. \tag{1.1}$$

For all these properties of symmetric spaces and Jacobi polynomials see, for example, [18], [21] and [31].

The following lemma gives an estimate for the size of these zonal spherical functions, and play a crucial role in the problem of localization.

Lemma 7. *With the notation* $Z_n(x,y) = Z_n(\cos t)$*, if* $|x-y| = t$*, the following estimates hold:*

(1) *For every* x *and* y*,*

$$|Z_n(x,y)| \;\le\; Z_n(x,x) \;=\; d_n \;\le\; C\,(1+n)^{d-1}\;;$$

(2) *If* $0 \le t \le \pi/2$*, then*

$$|Z_n(x,y)| \;\le\; C\,(1+n)^{d-1}\,(1+nt)^{-(d-1)/2}\;;$$

(3) *If* $\pi/2 \le t \le \pi$, *then*

$$|Z_n(x,y)| \le \begin{cases} C\,(1+n)^{d-1}\,(1+n(\pi-t))^{-(d-1)/2} & \textit{if} \quad \mathcal{M} = \mathbb{S}^d\,, \\ C\,(1+n)^{(d-1)/2} & \textit{if} \quad \mathcal{M} = P^d(\mathbb{R})\,, \\ C\,(1+n)^{d/2}\,(1+n(\pi-t))^{-1/2} & \textit{if} \quad \mathcal{M} = P^d(\mathbb{C})\,, \\ C\,(1+n)^{(d+2)/2}\,(1+n(\pi-t))^{-3/2} & \textit{if} \quad \mathcal{M} = P^d(\mathbb{H})\,, \\ C\,(1+n)^{11}\,(1+n(\pi-t))^{-7/2} & \textit{if} \quad \mathcal{M} = P^{16}(Cay)\,; \end{cases}$$

Proof. The proof of (1) follows from (1.1).
The Jacobi polynomials $P_n^{(a,b)}(\cos t)$ have an asymptotic expansion, as $n \to \infty$, in terms of the Bessel functions:

$$\left(\sin\frac{t}{2}\right)^a \left(\cos\frac{t}{2}\right)^b \frac{P_n^{(a,b)}(\cos t)}{P_n^{(a,b)}(1)}$$
$$= \frac{\Gamma(a+1)}{(n+(a+b+1)/2)^a} \left(\frac{t}{\sin t}\right)^{\frac{1}{2}} J_a\left((n+(a+b+1)/2)\,t\right)$$
$$+ \begin{cases} t^{a+2}\,O\,(1) & \text{if} \quad 0 < t \le cn^{-1}\,, \\ t^{1/2}\,O\left(n^{-a-3/2}\right) & \text{if} \quad cn^{-1} \le t \le \pi - \epsilon\,. \end{cases}$$

Here $c, \epsilon > 0$ are fixed and $J_a(x)$ is the Bessel function of order a. See [32] p.197. On the other hand, the Bessel functions (see [22]) satisfy the estimate:

$$|J_a(x)| \le C \min\left\{|x|^a,\ |x|^{-1/2}\right\}.$$

In particular, if $0 \le t \le \pi/2$,

$$\left|\frac{P_n^{(a,b)}(\cos t)}{P_n^{(a,b)}(1)}\right| \le C\ (1+nt)^{-a-1/2}\,.$$

This gives the estimate for $0 \le t \le \pi/2$ in (2) . Observe that this estimate depends only on the dimension of the symmetric space.
The estimates when $\pi/2 \le t \le \pi$ in (3) are similar and they follow from the symmetry relation

$$P_n^{(a,b)}(-x) = (-1)^n\, P_n^{(b,a)}(x)$$

and the estimate

$$\left|\frac{P_n^{(a,b)}(\cos t)}{P_n^{(a,b)}(1)}\right| \le C\,(1+n)^{b-a}\ (1+n\,(\pi-t))^{-b-1/2}\,.$$

□

Observe that the exponent of n in the estimate of $Z_n(x,y)$ when $x = y$ is

$$\{\textit{dimension of the space}\} - 1\,,$$

while the exponent of n when y is in the antipodal manifold of x is

$$\{\textit{dimension of the space}\} - \frac{1}{2}\{\textit{dimension of the antipodal manifold}\} - 1\,.$$

We conclude this section by recalling the definition and some properties of the Bessel potential and the associated Bessel kernel,

$$G^\gamma f(x) = \int_{\mathcal{M}} G^\gamma(x,y)\, f(y)\, dy\,,$$

$$G^\gamma(x,y) = \sum_{n=0}^{+\infty} \left(1+\lambda_n^2\right)^{-\gamma/2} Z_n(x,y)\,.$$

The Bessel potentials on Euclidean spaces are presented in [30]. The properties on a manifold are essentially the same.

Lemma 8. *If $\gamma > 0$, then the Bessel kernel $G^\gamma(x,y)$ is positive and integrable, and it is smooth in $\{|x-y| \neq 0\}$. Moreover, if $0 < \gamma < d$, then $G^\gamma(x,y) \approx |x-y|^{\gamma-d}$ when $|x-y| \to 0$.*

Proof. It follows from the definition of the Gamma function that, for $\gamma > 0$,

$$\left(1+\lambda^2\right)^{-\gamma/2} = \frac{1}{\Gamma(\gamma/2)} \int_0^{+\infty} t^{\frac{\gamma}{2}-1}\, e^{-t(1+\lambda^2)}\, dt\,.$$

Therefore $G^\gamma(x,y)$ can be subordinated to the heat kernel:

$$G^\gamma(x,y) = \sum_{n=0}^{+\infty} \left(1+\lambda_n^2\right)^{-\gamma/2} Z_n(x,y)$$

$$= \frac{1}{\Gamma(\gamma/2)} \int_0^{+\infty} t^{\frac{\gamma}{2}-1}\, e^{-t} \left(\sum_{n=0}^{+\infty} e^{-\lambda_n^2 t}\, Z_n(x,y) \right) dt\,.$$

The heat kernel is smooth and positive and it satisfies some Gaussian estimates. More precisely, there exists smooth functions $u_k(x,y)$ such that, if t is small,

$$0 < \sum_{n=0}^{+\infty} e^{-\lambda_n^2 t} Z_n(x,y) = (4\pi t)^{-d/2} e^{-|x-y|^2/(4t)} \left(\sum_{k=0}^{n} t^k\, u_k(x,y) + O(t^{n+1}) \right).$$

See [12]. The estimates for the Bessel kernel follows by integrating these estimates. □

2. Proof of Theorem 1

In what follows we consider operators of the form

$$Tf(x) = \sum_{n=0}^{+\infty} \sum_{j=1}^{d_n} m(\lambda_n)\, \widehat{f}(n,j)\, Y_{n,j}(x)\,.$$

We shall always assume that $m(\lambda)$ is an even function on $-\infty < \lambda < +\infty$ with tempered growth, i.e. $|m(\lambda)| \leq C(1+\lambda)^k$ for some k. The multiplier $m(\lambda)$ is the Fourier transform of a tempered distribution and, formally,

$$Tf(x) = \int_{\mathcal{M}} \left(\sum_{n=0}^{+\infty} m(\lambda_n)\, Z_n(x,y) \right) f(y)\, dy\,.$$

Hence, the convolution of a zonal kernel $T(x,y)=\sum_{n=0}^{+\infty} m(\lambda_n) Z_n(x,y)$ with a tempered distribution $f(x)$ is that tempered distribution whose Fourier transform is the pointwise product between the Fourier transform of $T(x,y)$ and $f(x)$. An example of such operators are the Bochner-Riesz means of complex order α:

$$S_R^\alpha f(x) = \sum_{\lambda_n<R} \left(1-\frac{\lambda_n^2}{R^2}\right)^\alpha \left(\sum_{j=1}^{d_n} \widehat{f}(n,j)\, Y_{n,j}(x)\right) = \int_{\mathcal{M}} S_R^\alpha(x,y) f(y)\, dy,$$

where

$$S_R^\alpha(x,y) \;=\; \sum_{\lambda_n<R} \left(1-\frac{\lambda_n^2}{R^2}\right)^\alpha Z_n(x,y)\,.$$

Then, with the above notations,

$$m_R(\lambda_n) \;=\; \left(1-\frac{\lambda_n^2}{R^2}\right)_+^\alpha\,.$$

The main tool in our localization result is a decomposition of $S_R^\alpha(x,y)$ into a kernel with small support plus a remainder. A natural decomposition is

$$S_R^\alpha(x,y) \;=\; S_R^\alpha(x,y)\chi_{\{|x-y|<\varepsilon\}}(x,y) \;+\; S_R^\alpha(x,y)\left(1-\chi_{\{|x-y|<\varepsilon\}}(x,y)\right)\,.$$

This decomposition has been exploited for example in [17]. Here we exploit a sort of smoothed version of the one above, and we decompose $S_R^\alpha(x,y)$ into a kernel with small support $\{|x-y|\le\varepsilon\}$ and a kernel with small Fourier transform. Let $\varepsilon>0$ and let $\psi(\lambda)$ be an even test function with cosine Fourier transform

$$\begin{cases} \widehat{\psi}(\tau) \;=\; 1 & \text{if} \quad |\tau|\le\varepsilon/2\,, \\ \widehat{\psi}(\tau) \;=\; 0 & \text{if} \quad |\tau|\ge\varepsilon\,. \end{cases}$$

Denote by $m_R*\psi(\lambda)$ the convolution on $\mathbb{R}$, i.e.

$$m_R*\psi(\lambda) \;=\; \int_{\mathbb{R}} m_R(\lambda-\tau)\psi(\tau)\, d\tau\,.$$

Then

$$\begin{aligned} &\sum_{n=0}^{+\infty}\sum_{j=1}^{d_n} m_R(\lambda_n)\,\widehat{f}(n,j)\, Y_{n,j}(x) \\ &= \sum_{n=0}^{+\infty} m_R*\psi(\lambda_n) \left(\sum_{j=1}^{d_n}\widehat{f}(n,j)\, Y_{n,j}(x)\right) \\ &+ \sum_{n=0}^{+\infty} \left(m_R(\lambda_n)-m_R*\psi(\lambda_n)\right) \left(\sum_{j=1}^{d_n}\widehat{f}(n,j)\, Y_{n,j}(x)\right) \\ &= A_R f(x) \;+\; B_R f(x)\,. \end{aligned}$$

The operators A_R and B_R are associated to the kernels

$$A_R(x,y) \;=\; \sum_{n=0}^{+\infty} m_R * \psi(\lambda_n)\, Z_n(x,y)\,,$$

$$B_R(x,y) \;=\; \sum_{n=0}^{+\infty} \left(m_R(\lambda_n) - m_R * \psi(\lambda_n)\right)\, Z_n(x,y)\,.$$

The kernel $A_R(x,y)$ has small support.

Lemma 9. *The kernel $A_R(x,y)$ has support in $\{|x-y| \le \varepsilon\}$. In particular, if a tempered distribution $f(x)$ vanishes in an open set Ω then, for all $x \in \Omega$ with distance $\{x, \partial\Omega\} > \varepsilon$,*

$$A_R f(x) \;=\; 0\,.$$

Proof. Let $\cos(\tau\sqrt{\Delta})f(x)$ be the solution of the Cauchy problem for the wave equation in $\mathbb{R} \times \mathcal{M}$,

$$\begin{cases} \dfrac{\partial^2}{\partial \tau^2}\, u(\tau,x) \;+\; \Delta u(\tau,x) \;=\; 0\,, \\ u(0,x) \;=\; f(x)\,, \quad \dfrac{\partial}{\partial \tau}\, u(0,x) \;=\; 0\,. \end{cases}$$

Solving the wave equation by separation of variables, one obtains

$$\cos\left(\tau\sqrt{\Delta}\right) f(x) \;=\; \sum_{n=0}^{+\infty} \sum_{j=1}^{d_n} \cos(\tau\lambda_n)\, \widehat{f}(n,j)\, Y_{n,j}(x)\,.$$

Hence, in the distribution sense,

$$A_R f(x) \;=\; \sum_{n=0}^{+\infty} \sum_{j=1}^{d_n} m_R * \psi(\lambda_n)\, \widehat{f}(n,j)\, Y_{n,j}(x)$$

$$= \sum_{n=0}^{+\infty} \sum_{j=1}^{d_n} \left(\int_0^{+\infty} \widehat{m_R * \psi}(\tau)\, \cos(\tau\lambda_n)\, d\tau \right) \widehat{f}(n,j)\, Y_{n,j}(x)$$

$$= \int_0^{+\infty} \widehat{m_R * \psi}(\tau) \left(\sum_{n=0}^{+\infty} \sum_{j=1}^{d_n} \cos(\tau\lambda_n)\, \widehat{f}(n,j)\, Y_{n,j}(x) \right) d\tau$$

$$= \int_0^{+\infty} \widehat{m_R}(\tau)\, \widehat{\psi}(\tau)\, \cos\left(\tau\sqrt{\Delta}\right) f(x)\, d\tau\,.$$

By assumption $\widehat{\psi}(\tau) = 0$ if $|\tau| \ge \varepsilon$. Moreover, by the finite propagation of waves, if $f(x) = 0$ in Ω, then also $\cos(\tau\sqrt{\Delta})f(x) = 0$ for every $x \in \Omega$ and $\tau <$ *distance* $\{x, \partial\Omega\}$. Then the lemma follows. □

The Fourier transform of the kernel $B_R(x,y)$ is small.

Lemma 10. *If $m_R(\lambda) = (1-\lambda^2/R^2)_+^\alpha$, then for every $k > 0$ and $A > 0$ there exist $C > 0$ and $h > 0$ such that for every complex α with $0 \le \mathrm{Re}(\alpha) \le A$ and every $R > 1$,*

$$|m_R(\lambda) - m_R * \psi(\lambda)| \;\le\; C\,(1+|\alpha|)^h\, R^{-\mathrm{Re}(\alpha)}\, (1+|R-\lambda|)^{-k}\,.$$

Proof. First observe that for every τ,

$$R \;-\; |\lambda-\tau| \;\le\; |R-\lambda| \;+\; |\tau|\,.$$

So we can write

$$\begin{aligned}|m_R(\lambda-\tau)| \;&=\; R^{-2\,\mathrm{Re}(\alpha)}\;(R+|\lambda-\tau|)^{\mathrm{Re}(\alpha)}\;(R-|\lambda-\tau|)_+^{\mathrm{Re}(\alpha)}\\ &\le\; 2^{\mathrm{Re}(\alpha)}\;R^{-\,\mathrm{Re}(\alpha)}\;(|R-\lambda|+|\tau|)^{\mathrm{Re}(\alpha)}\;.\end{aligned}$$

Since $\psi(\lambda)$ is a test function, if $|R-\lambda|\le 1$ then

$$\begin{aligned}&|m_R(\lambda)-m_R*\psi(\lambda)| \;\le\; |m_R(\lambda)| \;+\; |m_R*\psi(\lambda)|\\ &\le\; 2^{\mathrm{Re}(\alpha)}\;R^{-\,\mathrm{Re}(\alpha)}\;\left\{\;|R-\lambda|^{\mathrm{Re}(\alpha)}\;+\;\int_{\mathbb{R}}\;(|R-\lambda|\;+\;|\tau|)^{\mathrm{Re}(\alpha)}\;|\psi(\tau)|\,d\tau\;\right\}\\ &\le\; 2^{\mathrm{Re}(\alpha)}\;R^{-\,\mathrm{Re}(\alpha)}\;\left\{\;1\;+\;\int_{\mathbb{R}}\;(1+|\tau|)^{\mathrm{Re}(\alpha)}\;|\psi(\tau)|\,d\tau\;\right\}\;=\;C\,R^{-\,\mathrm{Re}(\alpha)}\,.\end{aligned}$$

Now consider the case $|R-\lambda|\ge 1$. Observe that for every $l\in\mathbb{N}$ there exists a polynomial $P_l(\lambda)$ of degree l such that

$$\frac{\partial^l}{\partial\lambda^l}\left(1-\frac{\lambda^2}{R^2}\right)_+^{\alpha} \;=\; R^{-l}\,P_l\left(\frac{\lambda}{R}\right)\;\left(1-\frac{\lambda^2}{R^2}\right)_+^{\alpha-l}\;.$$

The coefficients of $P_l(\lambda)$ are dominated by $(1+|\alpha|)^l$, therefore

$$\left|\;\frac{\partial^l}{\partial\lambda^l}\,m_R(\lambda)\;\right|\;\le\;C\;\;(1+|\alpha|)^l\;\,R^{-\,\mathrm{Re}(\alpha)}\;\;|R-\lambda|^{\mathrm{Re}(\alpha)-l}\;.$$

By assumption $\psi(\lambda)$ has mean one, so

$$|m_R(\lambda)-m_R*\psi(\lambda)|\;=\;\left|\;\int_{\mathbb{R}}\;(m_R(\lambda-\tau)-m_R(\lambda))\;\,\psi(\tau)\,d\tau\;\right|\;.$$

Since the positive moments of $\psi(\lambda)$ are zero, for this last integral we can write

$$\begin{aligned}&\int_{\mathbb{R}}\;(m_R(\lambda-\tau)-m_R(\lambda))\;\,\psi(\tau)\,d\tau\\ &=\;\int_{\mathbb{R}}\left(m_R(\lambda-\tau)\;-\;\sum_{l=0}^{L-1}\;\frac{(-\tau)^l}{l!}\;\frac{\partial^l}{\partial\lambda^l}m_R(\lambda)\right)\;\psi(\tau)\,d\tau\;.\end{aligned}$$

Splitting the integration on $\mathbb{R}$ into $\{|\tau|\le|R-\lambda|/2\}$ and $\{|\tau|\ge|R-\lambda|/2\}$, one gets

$$\begin{aligned}|m_R(\lambda)-m_R*\psi(\lambda)|\;&=\;\int_{\{|\tau|\ge|R-\lambda|/2\}}\;|m_R(\lambda-\tau)|\;\,|\psi(\tau)|\;\,d\tau\\ &+\;\sum_{l=0}^{L-1}\;\frac{1}{l!}\;\left|\frac{\partial^l}{\partial\lambda^l}\;m_R(\lambda)\right|\;\int_{\{|\tau|\ge|R-\lambda|/2\}}\;\left|\tau^l\right|\;\,|\psi(\tau)|\;\,d\tau\\ &+\;\frac{1}{L!}\;\sup_{\{|\tau|\le|R-\lambda|/2\}}\;\left\{\;\left|\frac{\partial^L}{\partial\lambda^L}\;m_R(\lambda-\tau)\right|\;\right\}\;\int_{\{|\tau|\le|R-\lambda|/2\}}\;\left|\tau^L\right|\;\,|\psi(\tau)|\;\,d\tau\;.\end{aligned}$$

We have the following estimates:

$$
\begin{aligned}
&\int_{\{|\tau|\geq|R-\lambda|/2\}} |m_R(\lambda-\tau)|\ |\psi(\tau)|\ d\tau \\
&\leq\ 2^{\mathrm{Re}(\alpha)}\ R^{-\mathrm{Re}(\alpha)} \int_{\{|\tau|\geq|R-\lambda|/2\}} (\ |R-\lambda|+|\tau|\ )^{\mathrm{Re}(\alpha)}\ |\psi(\tau)|\ d\tau \\
&\leq\ 6^{\mathrm{Re}(\alpha)}\ R^{-\mathrm{Re}(\alpha)} \int_{\{|\tau|\geq|R-\lambda|/2\}} |\tau|^{\mathrm{Re}(\alpha)}\ |\psi(\tau)|\ d\tau \\
&\leq\ C\, R^{-\mathrm{Re}(\alpha)}\ |R-\lambda|^{-k}\ ,
\end{aligned}
$$

$$
\begin{aligned}
&\left|\frac{\partial^l}{\partial\lambda^l}\, m_R(\lambda)\right| \int_{\{|\tau|\geq|R-\lambda|/2\}} \left|\tau^l\right|\ |\psi(\tau)|\ d\tau \\
&\leq\ C\ (1+|\alpha|)^l\ R^{-\mathrm{Re}(\alpha)}\ |R-\lambda|^{\mathrm{Re}(\alpha)-l} \int_{\{|\tau|\geq|R-\lambda|/2\}} |\tau^l|\ |\psi(\tau)|\ d\tau \\
&\leq\ C\ (1+|\alpha|)^l\ R^{-\mathrm{Re}(\alpha)}\ |R-\lambda|^{-k}\ ,
\end{aligned}
$$

$$
\begin{aligned}
&\sup_{\{|\tau|\leq|R-\lambda|/2\}} \left\{ \left|\frac{\partial^L}{\partial\lambda^L}\, m_R(\lambda-\tau)\right| \right\} \int_{\{|\tau|\leq|R-\lambda|/2\}} \left|\tau^L\right|\ |\psi(\tau)|\ d\tau \\
&\leq\ C\ (1+|\alpha|)^L\ R^{-\mathrm{Re}(\alpha)}\ |R-\lambda|^{\mathrm{Re}(\alpha)-L} \int_{\mathbb{R}} \left|\tau^L\right|\ |\psi(\tau)|\ d\tau \\
&\leq\ C\ (1+|\alpha|)^L\ R^{-\mathrm{Re}(\alpha)}\ |R-\lambda|^{\mathrm{Re}(\alpha)-L}\ .
\end{aligned}
$$

The thesis follows by taking $L \geq \mathrm{Re}(\alpha)+k$ and $h = L+1$. □

We are now ready to prove the theorem.

A necessary condition for the pointwise Bochner-Riesz summability of

$$
S_R^\alpha f(\mathbf{o})\ =\ \sum_{\lambda_n<R} \left(1-\frac{\lambda_n^2}{R^2}\right)^\alpha\ Y_n f(\mathbf{o})
$$

is that $\{\lambda_n^{-\alpha} Y_n f(\mathbf{o})\} \to 0$ when $n\to\infty$. See [36] for the corresponding result for Cesàro means. Hence (1) implies (2).

Conversely, assume that $f(x) = 0$ in $\{|x-\mathbf{o}| < \varepsilon\}$. By Lemma 9, $A_R f(\mathbf{o}) = 0$. By Lemma 10,

$$
|B_R f(\mathbf{o})|\ \leq\ C \sum_{n=0}^{+\infty} R^{-\alpha}\ (1+\lambda_n)^\alpha\ (1+|R-\lambda_n|)^{-k}\ \left|(1+\lambda_n)^{-\alpha}\, Y_n f(\mathbf{o})\right|\ .
$$

Observe that $\sum_{n=0}^{+\infty} R^{-\alpha}\,(1+\lambda_n)^\alpha\,(1+|R-\lambda_n|)^{-k}\ <\ C\ <\ +\infty$, with C independent on R. If $\{(1+\lambda_n)^{-\alpha} Y_n f(\mathbf{o})\} \to 0$, then also $\{B_R f(\mathbf{o})\} \to 0$. Hence (2) implies (1).

3. Proof of Theorem 2

Fix $x \in \mathcal{M}$, $\varepsilon > 0$ and assume that $G^\gamma f(y) = 0$ if $|x-y| \leq \varepsilon$, and decompose $S_R^\alpha G^\gamma f(x)$ into $A_R G^\gamma f(x) + B_R G^\gamma f(x)$ as in the previous section. By Lemma 9, $A_R G^\gamma f(x) = 0$. Since $B_R G^\gamma f(x)$ converges to zero when $f(x)$

is a test function, in order to prove the theorem it then suffices to show that these linear functionals are uniformly bounded on $L^p(\mathcal{M})$. Recall that

$$B_R G^\gamma f(x) = \int_{\mathcal{M}} f(y) \left(\sum_{n=0}^{+\infty} \left(m_R(\lambda_n) - m_R * \psi(\lambda_n) \right) \left(1+\lambda_n^2\right)^{-\gamma/2} Z_n(x,y) \right) dy.$$

The norms of the functionals on $L^p(\mathcal{M})$ are the norms on $L^q(\mathcal{M})$, $1/p+1/q = 1$, of the associate kernels. In particular, if $G^\gamma f(y) = 0$ when $|x-y| \le \varepsilon$,

$$|B_R G^\gamma f(x)|$$

$$\le \left\{ \int_{|x-y|\ge\varepsilon} \left| \sum_{n=0}^{+\infty} \frac{(m_R(\lambda_n) - m_R * \psi(\lambda_n))}{(1+\lambda_n^2)^{\gamma/2}} Z_n(x,y) \right|^q dy \right\}^{\frac{1}{q}} \left\{ \int_{\mathcal{M}} |f(y)|^p \, dy \right\}^{\frac{1}{p}}$$

$$\le \sum_{n=0}^{+\infty} \frac{|m_R(\lambda_n) - m_R * \psi(\lambda_n)|}{(1+\lambda_n^2)^{\gamma/2}} \left\{ \int_{|x-y|\ge\varepsilon} |Z_n(x,y)|^q \, dy \right\}^{\frac{1}{q}} \left\{ \int_{\mathcal{M}} |f(y)|^p \, dy \right\}^{\frac{1}{p}} .$$

By Lemma 10,

$$|m_R(\lambda_n) - m_R * \psi(\lambda_n)| \left(1+\lambda_n^2\right)^{-\gamma/2} \le C R^{-\alpha} \left(1 + |R-\lambda_n|\right)^{-k} \left(1+\lambda_n\right)^{-\gamma} .$$

This implies that in the above sum only a finite number of terms come into play, the ones with $\lambda_n \approx R$. Then the theorem follows from the following estimates for $\left\{ \int_{|x-y|\ge\varepsilon} |Z_n(x,y)|^q \, dy \right\}^{1/q}$.

Lemma 11. *Let $\varepsilon > 0$.*

(1) *If $\mathcal{M} = \mathbb{S}^d$, then*

$$\left\{ \int_{|x-y|\ge\varepsilon} |Z_n(x,y)|^q \, dy \right\}^{\frac{1}{q}}$$

$$\le C \begin{cases} (1+n)^{d-1-d/q} & \text{if } \; q > 2d/(d-1) , \\ (1+n)^{(d-1)/2} \left(\log(2+n)\right)^{(d-1)/(2d)} & \text{if } \; q = 2d/(d-1) , \\ (1+n)^{(d-1)/2} & \text{if } \; q < 2d/(d-1) . \end{cases}$$

(2) *If $\mathcal{M} = P^d(\mathbb{R})$ then, for every q,*

$$\left\{ \int_{|x-y|\ge\varepsilon} |Z_n(x,y)|^q \, dy \right\}^{\frac{1}{q}} \le C \; (1+n)^{(d-1)/2} .$$

(3) *If $\mathcal{M} = P^d(\mathbb{C})$, then*

$$\left\{ \int_{|x-y|\ge\varepsilon} |Z_n(x,y)|^q \, dy \right\}^{\frac{1}{q}} \le C \begin{cases} (1+n)^{d/2-2/q} & \text{if } \; q > 4 , \\ (1+n)^{(d-1)/2} \left(\log(2+n)\right)^{1/4} & \text{if } \; q = 4 , \\ (1+n)^{(d-1)/2} & \text{if } \; q < 4 . \end{cases}$$

(4) *If $\mathcal{M} = P^d(\mathbb{H})$, then*

$$\left\{ \int_{|x-y|\ge\varepsilon} |Z_n(x,y)|^q \, dy \right\}^{\frac{1}{q}} \le C \begin{cases} (1+n)^{(d+2)/2-4/q} & \text{if } \; q > 8/3 , \\ (1+n)^{(d-1)/2} \left(\log(2+n)\right)^{3/8} & \text{if } \; q = 8/3 , \\ (1+n)^{(d-1)/2} & \text{if } \; q < 8/3 . \end{cases}$$

(5) *If* $\mathcal{M} = P^{16}(Cay)$, *then*

$$\left\{ \int_{|x-y|\geq\varepsilon} |Z_n(x,y)|^q \, dy \right\}^{\frac{1}{q}} \leq C \begin{cases} (1+n)^{11-8/q} & \text{if } \; q > 16/7\,, \\ (1+n)^{15/2} \, (\log(2+n))^{7/16} & \text{if } \; q = 16/7\,, \\ (1+n)^{15/2} & \text{if } \; q < 16/7\,. \end{cases}$$

Proof. The zonal spherical functions $Z_n(x,y)$ are radial around x and, with the notation $|x-y| = t$ and $Z_n(x,y) = Z_n(\cos t)$, an integration in polar coordinates gives

$$\left\{ \int_{|x-y|\geq\varepsilon} |Z_n(x,y)|^q \, dy \right\}^{\frac{1}{q}} = \left\{ \int_{\varepsilon}^{\pi} |Z_n(\cos t)|^q \, \mathcal{A}(t)\, dt \right\}^{\frac{1}{q}} .$$

If $\mathcal{M} = \mathbb{S}^d$, then $\mathcal{A}(t) = C\,(\sin t)^{d-1}$. Moreover, if $0 < \varepsilon \leq t \leq \pi$, by parts (2) and (3) of Lemma 7,

$$|Z_n(\cos t)| \;\leq\; C\,(1+n)^{d-1}\;(1+n(\pi-t))^{-(d-1)/2}\;.$$

Hence

$$\left\{ \int_{\varepsilon}^{\pi} |Z_n(\cos t)|^q \, \mathcal{A}(t)\, dt \right\}^{\frac{1}{q}}$$

$$\leq\; C\,(1+n)^{d-1-\frac{d}{q}} \;+\; C\,(1+n)^{\frac{d-1}{2}} \left\{ \int_{1/n}^{\pi-\varepsilon} t^{-\frac{d-1}{2}q+d-1}\, dt \right\}^{\frac{1}{q}}$$

$$\leq\; C \begin{cases} (1+n)^{d-1-d/q} & \text{if } \; q > 2d/(d-1)\,, \\ (1+n)^{(d-1)/2} \, (\log(2+n))^{(d-1)/(2d)} & \text{if } \; q = 2d/(d-1)\,, \\ (1+n)^{(d-1)/2} & \text{if } \; q < 2d/(d-1)\,. \end{cases}$$

This proves the lemma for $\mathbb{S}^d$. The proof for projective spaces is similar. □

Examples. The indices in Theorem 2 are best possible. As shown by Bochner [4], the critical index for pointwise localization of Bochner-Riesz means of functions in $L^p(\mathbb{R}^d)$ is $\alpha = (d-1)/2$ for every $1 \leq p \leq +\infty$. The Bochner-Riesz kernel in $\mathbb{R}^d$ is a Bessel function,

$$S_R^\alpha f(x) \;=\; \int_{\mathbb{R}^d} \pi^{-\alpha}\,\Gamma(\alpha+1)\,R^{d/2-\alpha}\,|y|^{-\alpha-d/2}\,J_{\alpha+d/2}\,(2\pi R|y|)\;f(x-y)\,dy\,.$$

Hence, by the asymptotic expansion of Bessel functions, $S_R^\alpha f(x)$ is approximate by

$$\frac{\Gamma(\alpha+1)\,R^{(d-1)/2-\alpha}}{\pi^{\alpha+1}} \int_{\mathbb{R}^d} |y|^{-\alpha-(d+1)/2} \cos\left(2\pi R|y| - \frac{(2\alpha+d+1)\pi}{4}\right) f(x-y)dy.$$

From this approximation it easily follows that a necessary condition for localization is the boundedness of the term $R^{(d-1)/2-\alpha}$, that is $\alpha \geq (d-1)/2$. This result of Bochner has been extended by Il'in [1] to spectral decompositions of self-adjoint elliptic operators: If $\alpha+\gamma < (d-1)/2$, then for any point $x \in \mathcal{M}$ there exists a function finite and in the Hölder class $\mathcal{C}^\alpha(\mathcal{M})$, which vanishes in a neighbourhood of x, and such that $\limsup_{R\to+\infty} \{|S_R^\alpha G^\gamma f(x)|\} = +\infty$. In particular, for every $1 \leq p \leq +\infty$ the assumption $\alpha + \gamma \geq (d-1)/2$ is necessary for pointwise localization. By Theorem 1, another necessary condition for the pointwise Bochner-Riesz summability of $S_R^\alpha f(x)$ is that $\{\lambda_n^{-\alpha} Y_n f(x)\} \to 0$ when $n \to \infty$. Fix $0 < \varepsilon < d$, $\mathbf{o} \in \mathcal{M}$, and define

$$f(x) \;=\; G^\varepsilon(x,\mathbf{o}) \;=\; \sum_{n=0}^{+\infty} \left(1+\lambda_n^2\right)^{-\varepsilon/2} Z_n(x,\mathbf{o})\,.$$

By Lemma 8, this function is smooth on $\mathcal{M}-\{\mathbf{o}\}$ and it behaves as $|x-\mathbf{o}|^{\varepsilon-d}$ when $x \to \mathbf{o}$. In particular, this function is in $L^p(\mathcal{M})$ for every $p < d/(d-\varepsilon)$ and, by subtracting a suitable smooth function with fast decaying Fourier expansion, it can be put to zero in $\{|x-\mathbf{o}| \geq \pi/2\}$. Finally observe that, when $|x-\mathbf{o}| = \pi$,

$$\left|(1+\lambda_n^2)^{-\varepsilon/2} Z_n(x,\mathbf{o})\right| \;\approx\; \begin{cases} (1+n)^{\,d-1-\varepsilon} & \text{if} \quad \mathcal{M} = \mathbb{S}^d\,, \\ (1+n)^{(d-1)/2-\varepsilon} & \text{if} \quad \mathcal{M} = P^d(\mathbb{R})\,, \\ (1+n)^{d/2-\varepsilon} & \text{if} \quad \mathcal{M} = P^d(\mathbb{C})\,, \\ (1+n)^{(d+2)/2-\varepsilon} & \text{if} \quad \mathcal{M} = P^d(\mathbb{H})\,, \\ (1+n)^{\,11-\varepsilon} & \text{if} \quad \mathcal{M} = P^{16}(\mathrm{Cay})\,. \end{cases}$$

These estimates imply that the indices in Theorem 2 are best possible for every p when $\mathcal{M}$ is the sphere $\mathbb{S}^d$ or the real projective space $P^d(\mathbb{R})$, and also for $p=1$ when $\mathcal{M}$ is the complex projective space $P^d(\mathbb{C})$, or $P^d(\mathbb{H})$, or $P^{16}(\mathrm{Cay})$. Indeed, Gigante and Jotsaroop [16] have shown that the theorem is sharp for every p and every compact rank one symmetric space.

The problem of convergence of eigenvalue expansions on compact Riemannian manifolds has also been studied in [6], [7], [27], [28], [29] and [33]. In these papers it is proved that localization for spherical sums may fail for piecewise smooth functions on three dimensional manifolds, the so-called Pinsky phenomenon, while it is proved that a sufficient condition for the pointwise Bochner-Riesz summability of order α for piecewise smooth function is $\alpha > (d-3)/2$.

4. Proof of Theorem 3

From Lemma 9 we know that $A_R G^\gamma f(x) = 0$ if *distance* $\{x, \partial\Omega\} > \varepsilon$. Then we only need to control $B_R G^\gamma f(x)$, which can be factorized as

$$S_R^\alpha G^\gamma f(x) \;=\; G^\beta\, S_R^\alpha G^{\gamma-\beta} f(x)\,.$$

When $\beta \geq 0$ the operator G^β is positive, and this gives

$$\sup_{R>0}\{|B_R G^\gamma f(x)|\} \;\leq\; G^\beta \left(\sup_{R>0}\left\{\left|B_R G^{\gamma-\beta} f\right|\right\}\right)(x)\,.$$

Then, in order to prove the theorem, it suffices to prove that the maximal operator $\sup_{R>0}\left\{|B_R G^{\gamma-\beta} f(x)|\right\}$ is bounded on $L^p(\mathcal{M})$. It suffices to consider two cases:

(1) $p=1$ and $\mathrm{Re}(\alpha)-\beta+\gamma \geq (d-1)/2$;
(2) $p=2$ and $\mathrm{Re}(\alpha)-\beta+\gamma \geq 0$.

The intermediate cases will follow by Stein's interpolation theorem for analytic families of operators (see [31], Chapter V).

Lemma 12. *If $f(x)$ is in $L^1(\mathcal{M})$ and $\operatorname{Re}(\alpha)-\beta+\gamma\geq(d-1)/2$, then*

$$\int_{\mathcal{M}}\sup_{R>0}\left\{\left|B_RG^{\gamma-\beta}f(x)\right|\right\}dx\ \leq\ C\,(1+|\alpha|)^h\int_{\mathcal{M}}|f(x)|\,dx\,.$$

Proof. Let $B_RG^{\gamma-\beta}(x,y)$ be the kernel associated to the operator $B_RG^{\gamma-\beta}$,

$$B_RG^{\gamma-\beta}(x,y)\ =\ \sum_{n=0}^{+\infty}\left(1+\lambda_n^2\right)^{(\beta-\gamma)/2}\ \left(m(\lambda_n)-m*\psi(\lambda_n)\right)\ Z_n(x,y)\,.$$

The maximal operator $\sup_{R>0}\left\{\left|B_RG^{\gamma-\beta}f(x)\right|\right\}$ is dominated by

$$\sup_{R>0}\left\{\left|B_RG^{\gamma-\beta}f(x)\right|\right\}\ \leq\ \int_{\mathcal{M}}\sup_{R>0}\left\{\left|B_RG^{\gamma-\beta}(x,y)\right|\right\}\,|f(y)|\,dy\,.$$

By Lemma 10,

$$\left|B_RG^{\gamma-\beta}(x,y)\right|\ \leq\ C\,(1+|\alpha|)^h\,R^{-\operatorname{Re}(\alpha)}\sum_{n=0}^{+\infty}\frac{(1+\lambda_n^2)^{(\beta-\gamma)/2}\ |Z_n(x,y)|}{(1+|R-\lambda_n|)^k}\,.$$

By part (2) of Lemma 7, with $t=|x-y|$ and $0\leq t\leq\pi/2$,

$$\begin{aligned}&\left|B_RG^{\gamma-\beta}(x,y)\right|\\ &\leq\ C\,(1+|\alpha|)^h\,R^{-\operatorname{Re}(\alpha)}\sum_{n=0}^{+\infty}\frac{(1+\lambda_n)^{\beta-\gamma}\,(1+\lambda_n)^{d-1}\,(1+\lambda_nt)^{-(d-1)/2}}{(1+|R-\lambda_n|)^k}\\ &=\ C\,(1+|\alpha|)^h\,R^{-\operatorname{Re}(\alpha)}\,t^{-(d-1)/2}\sum_{n=0}^{+\infty}\left(\frac{t+\lambda_nt}{1+\lambda_nt}\right)^{(d-1)/2}\ \frac{(1+\lambda_n)^{\beta-\gamma+(d-1)/2}}{(1+|R-\lambda_n|)^k}\\ &\leq\ C\,(1+|\alpha|)^h\,R^{-\operatorname{Re}(\alpha)}\,t^{-(d-1)/2}\sum_{n=0}^{+\infty}\frac{(1+\lambda_n)^{\beta-\gamma+(d-1)/2}}{(1+|R-\lambda_n|)^k}\\ &\leq\ C\,(1+|\alpha|)^h\,R^{-\operatorname{Re}(\alpha)+\beta-\gamma+(d-1)/2}\,t^{-(d-1)/2}\,.\end{aligned}$$

This implies that, if $\operatorname{Re}(\alpha)-\beta+\gamma\geq(d-1)/2$ and $0\leq|x-y|\leq\pi/2$,

$$\sup_{R>0}\left\{\left|B_RG^{\gamma-\beta}(x,y)\right|\right\}\ \leq\ C\,(1+|\alpha|)^h\,|x-y|^{-(d-1)/2}\,.$$

Similarly, by part (3) of Lemma 7, if $\operatorname{Re}(\alpha)-\beta+\gamma\geq(d-1)/2$ and $\pi/2\leq|x-y|\leq\pi$,

$$\sup_{R>0}\left\{\left|B_RG^{\gamma-\beta}(x,y)\right|\right\}\leq\begin{cases}C\,(1+|\alpha|)^h\ (\pi-|x-y|)^{-(d-1)/2} & \text{if}\quad\mathcal{M}=\mathbb{S}^d\,,\\ C\,(1+|\alpha|)^h & \text{if}\quad\mathcal{M}=P^d(\mathbb{R})\,,\\ C\,(1+|\alpha|)^h\ (\pi-|x-y|)^{-1/2} & \text{if}\quad\mathcal{M}=P^d(\mathbb{C})\,,\\ C\,(1+|\alpha|)^h\ (\pi-|x-y|)^{-3/2} & \text{if}\quad\mathcal{M}=P^d(\mathbb{H})\,,\\ C\,(1+|\alpha|)^h\ (\pi-|x-y|)^{-7/2} & \text{if}\quad\mathcal{M}=P^{16}(\mathrm{Cay})\,.\end{cases}$$

By these estimates, $\sup_{R>0}\left\{\left|B_R G^{\gamma-\beta}(x,y)\right|\right\}$ is integrable with respect to x for every y, and

$$\begin{aligned}
&\int_{\mathcal{M}} \sup_{R>0}\left\{\left|B_R G^{\gamma-\beta} f(x)\right|\right\} dx \\
&\leq \int_{\mathcal{M}}\left(\int_{\mathcal{M}} \sup_{R>0}\left\{\left|B_R G^{\gamma-\beta}(x,y)\right|\right\} dx\right) |f(y)|\, dy \\
&\leq C \sup_{y\in\mathcal{M}}\left\{\int_{\mathcal{M}} \sup_{R>0}\left\{\left|B_R G^{\gamma-\beta}(x,y)\right| dx\right\}\right\} \int_{\mathcal{M}} |f(y)|\, dy\,.
\end{aligned}$$

Actually this proof shows that $\sup_{R>0}\left\{\left|B_R G^{\gamma-\beta} f(x)\right|\right\}$ can be controlled by a fractional integral of order $(d-1)/2$ of $f(x)$. In particular, if $f(x) \in L^1(\mathcal{M})$, then $\sup_{R>0}\left\{\left|B_R G^{\gamma-\beta} f(x)\right|\right\}$ is in $L^p(\mathcal{M})$ for all $p < 2d/(d+1)$. □

Lemma 13. *If $f(x)$ is in $L^2(\mathcal{M})$ and $\mathrm{Re}(\alpha)-\beta+\gamma \geq 0$, then*

$$\int_{\mathcal{M}} \sup_{R>0}\left\{\left|B_R G^{\gamma-\beta} f(x)\right|^2\right\} dx \;\leq\; C\,(1+|\alpha|)^{2h} \int_{\mathcal{M}} |f(x)|^2\, dx\,.$$

Proof. By Lemma 10,

$$\begin{aligned}
&\left|B_R G^{\gamma-\beta} f(x)\right| \\
&= \left|\sum_{n=0}^{+\infty}\sum_{j=1}^{d_n} (1+\lambda_n^2)^{(\beta-\gamma)/2}\ \left(m_R(\lambda_n) - m_R * \psi(\lambda_n)\right)\ \widehat{f}(n,j)\, Y_{n,j}(x)\right| \\
&\leq\; C\,(1+|\alpha|)^h\, R^{-\mathrm{Re}(\alpha)} \sum_{n=0}^{+\infty} \frac{(1+\lambda_n^2)^{(\beta-\gamma)/2}}{(1+|R-\lambda_n|)^k} \left|\sum_{j=1}^{d_n} \widehat{f}(n,j)\, Y_{n,j}(x)\right| \\
&\leq\; C\,(1+|\alpha|)^h\, R^{-\mathrm{Re}(\alpha)} \left\{\sum_{n=0}^{+\infty} \frac{(1+\lambda_n^2)^{\beta-\gamma}}{(1+|R-\lambda_n|)^{2k}}\right\}^{\frac12} \left\{\sum_{n=0}^{+\infty}\left|\sum_{j=1}^{d_n} \widehat{f}(n,j)\, Y_{n,j}(x)\right|^2\right\}^{\frac12} \\
&\leq\; C\,(1+|\alpha|)^h\, R^{-\mathrm{Re}(\alpha)+\beta-\gamma} \left\{\sum_{n=0}^{+\infty}\left|\sum_{j=1}^{d_n} \widehat{f}(n,j)\, Y_{n,j}(x)\right|^2\right\}^{\frac12}.
\end{aligned}$$

Hence, if $\mathrm{Re}(\alpha)-\beta+\gamma \geq 0$,

$$\sup_{R>0}\left\{\left|B_R G^{\gamma-\beta} f(x)\right|\right\} \;\leq\; C\,(1+|\alpha|)^h \left\{\sum_{n=0}^{+\infty}\left|\sum_{j=1}^{d_n} \widehat{f}(n,j)\, Y_{n,j}(x)\right|^2\right\}^{\frac12},$$

and

$$\int_{\mathcal{M}} \sup_{R>0}\left\{\left|B_R G^{\gamma-\beta} f(x)\right|^2\right\} dx \;\leq\; C\,(1+|\alpha|)^{2h} \sum_{n=0}^{+\infty}\sum_{j=1}^{d_n} \left|\widehat{f}(n,j)\right|^2.$$

□

Lemma 14. *If $f(x)$ is in $L^p(\mathcal{M})$, $1 \leq p \leq 2$, and $\alpha-\beta+\gamma \geq (d-1)\left(\frac{1}{p}-\frac{1}{2}\right)$, then*

$$\int_{\mathcal{M}} \sup_{R>0} \left\{ \left| B_R G^{\gamma-\beta} f(x) \right|^p \right\} dx \;\leq\; C\,(1+|\alpha|)^{ph} \int_{\mathcal{M}} |f(x)|^p \; dx\,.$$

Proof. The lemma follows from the previous two lemmas via Stein's interpolation theorem for analytic families of operators. □

5. Proof of Corollary 4

The Bessel (α, p) capacity of a Borel set E in $\mathcal{M}$ is defined as

$$\mathcal{B}_{\beta,p}(E) \;=\; \inf \left\{ \|f\|_{L^p}^p \;:\; G^\beta f(x) \;\geq\; 1 \text{ on } E \right\}.$$

See [35], Chapter 2.6, for the definition on the Euclidean spaces $\mathbb{R}^d$. The definition and properties of Bessel capacity in a manifold are similar. For $\varepsilon > 0$ and $t > 0$, let

$$E \;=\; \left\{ x \in \Omega \cap \{distance\,\{x, \partial\Omega\} > \varepsilon\} \;:\; \limsup_{R\to+\infty} \{|S_R^\alpha G^\gamma f(x)|\} \;>\; t \right\}.$$

Theorem 3 and an approximation of $f(x)$ with test functions show that the (α, p) capacity of E is zero for every $\varepsilon > 0$ and $t > 0$. On the other hand, if a set has (α, p) capacity zero, then it also has $d - \beta p + \eta$ Hausdorff measure zero for every $\eta > 0$. This implies that the $d - \beta p$ Hausdorff dimension is zero.

Examples. In [25] it is shown that there exists radial functions in $L^{2d/(d+1)}(\mathcal{M})$, vanishing on half of $\mathcal{M}$, with spherical harmonic expansions diverging almost everywhere on $\mathcal{M}$. Actually a small modification of the argument gives divergence everywhere. See [14] for the two dimensional case of the expansion in Legendre polynomials. On the other hand, by an application of the Rademacher-Menshov theorem on orthogonal series, in [24] it is shown that the spherical partial sums of functions in L^2 Sobolev spaces of positive order converge almost everywhere.

As mentioned at the end of the proof of Lemma 12, we suspect that some of the indexes in Theorem 3 and Corollary 4 can be improved.

6. Proof of Theorem 5

If $G^\gamma f(x)$ vanishes in an open set Ω and if $\alpha + \gamma \geq 0$, by the positivity of the Bessel kernel, for every x in $\Omega \cap \{distance\,\{x, \partial\Omega\} > \varepsilon\}$ one has

$$\begin{aligned} &\sup_{R>0} \{|S_R^\alpha G^\gamma f(x)|\} \\ &= \sup_{R>0} \{|B_R G^\gamma f(x)|\} \;=\; \sup_{R>0} \left\{ \left| G^{\alpha+\gamma} B_R G^{-\alpha} f(x) \right| \right\} \\ &\leq\; G^{\alpha+\gamma} \left(\sup_{R>0} \left\{ \left| B_R G^{-\alpha} f \right| \right\} \right)(x)\,. \end{aligned}$$

By Lemma 13, if $\alpha \geq 0$, then

$$\int_{\mathcal{M}} \sup_{R>0} \left\{ \left| B_R G^{-\alpha} f(x) \right|^2 \right\} dx \ \leq \ C \int_{\mathcal{M}} |f(x)|^2 \, dx \, .$$

Then part (1) of Theorem 5 follows from the following lemma and the estimate $G^{2(\alpha+\gamma)}(x,y) \leq C\,|x-y|^{2(\alpha+\gamma)-d}$ in Lemma 8.

Lemma 15. *For every $F(x) \in L^2(\mathcal{M})$, for every non-negative finite Borel measure $d\upsilon(x)$, and for every $\eta > 0$,*

$$\int_{\mathcal{M}} |G^{\eta} F(x)| \ d\upsilon(x) \ \leq \ \|F\|_{L^2} \left\{ \int_{\mathcal{M}} \int_{\mathcal{M}} G^{2\eta}(x,y) \, d\upsilon(x) \, d\upsilon(y) \right\}^{\frac{1}{2}} \, .$$

Proof. Since the Bessel kernel is positive, it follows that

$$|G^{\eta} F(x)| \ \leq \ G^{\eta} |F|(x) \, .$$

Then it suffices to assume $F(x) \geq 0$. We get

$$\int_{\mathcal{M}} G^{\eta} F(x) \, d\upsilon(x) \ = \ \int_{\mathcal{M}} \left(\sum_{n=0}^{+\infty} \sum_{j=1}^{d_n} \left(1+\lambda_n^2\right)^{-\eta/2} \, \widehat{F}(n,j) \, Y_{n,j}(x) \right) \, d\upsilon(x)$$

$$\leq \ \left\{ \sum_{n=0}^{+\infty} \sum_{j=1}^{d_n} \left| \widehat{F}(n,j) \right|^2 \right\}^{\frac{1}{2}} \left\{ \sum_{n=0}^{+\infty} \sum_{j=1}^{d_n} \left(1+\lambda_n^2\right)^{-\eta} \left| \int_{\mathcal{M}} Y_{n,j}(x) \, d\upsilon(x) \right|^2 \right\}^{\frac{1}{2}}$$

$$= \ \|F\|_{L^2} \left\{ \int_{\mathcal{M}} \int_{\mathcal{M}} \left(\sum_{n=0}^{+\infty} \left(1+\lambda_n^2\right)^{-\eta} \sum_{j=1}^{d_n} Y_{n,j}(x) \, \overline{Y_{n,j}(y)} \right) \, d\upsilon(x) \, d\upsilon(y) \right\}^{\frac{1}{2}}$$

$$= \ \|F\|_{L^2} \left\{ \int_{\mathcal{M}} \int_{\mathcal{M}} G^{2\eta}(x,y) \, d\upsilon(x) \, d\upsilon(y) \right\}^{\frac{1}{2}} \, .$$

□

To prove part (2) and (3) of Theorem 5 it suffices to replace the maximal operator $\sup_{R>0} \{|S_R^{\alpha} G^{\gamma} f(x)|\}$ with a linearised version $g(x) B_{R(x)} G^{\gamma} f(x)$, where $g(x)$ and $R(x)$ are arbitrary Borel functions with $|g(x)| \leq 1$ and $R(x) \geq 1$. Moreover, possibly splitting the measure $d\nu(x)$ into a finite sum of measures with small support, one can assume that the diameter of the support of the measure is smaller than half of the diameter of the manifold $\mathcal{M}$. In particular, if a point x is in the support of the measure, then the antipodal points are far from this support. Set

$$g(x) \ = \ \frac{\overline{B_{R(x)} G^{\gamma} f(x)}}{\left| B_{R(x)} G^{\gamma} f(x) \right|} \, .$$

Then, with the notation $\widehat{B_R}(\lambda) = m_R(\lambda) - m_R * \psi(\lambda)$,

$$\int_{\mathcal{M}} \left| B_{R(x)} G^\gamma f(x) \right| \, dv(x)$$

$$= \int_{\mathcal{M}} g(x) \left(\sum_{n=0}^{+\infty} \sum_{j=1}^{d_n} \left(1+\lambda_n^2\right)^{-\gamma/2} \; \widehat{B_{R(x)}}(\lambda_n) \, \widehat{f}(n,j) \, Y_{n,j}(x) \right) dv(x)$$

$$\leq \|f\|_{L^2} \left\{ \sum_{n=0}^{+\infty} \sum_{j=1}^{d_n} \left(1+\lambda_n^2\right)^{-\gamma} \left| \int_{\mathcal{M}} g(x) \, \widehat{B_{R(x)}}(\lambda_n) \, Y_{n,j}(x) \, dv(x) \right|^2 \right\}^{\frac{1}{2}} .$$

Lemma 16. *Assume that $dv(x)$ is a nonnegative measure with support smaller than half of the diameter of the manifold $\mathcal{M}$, and that*

$$\delta \;=\; \begin{cases} d - 2(\alpha+\gamma) - 1 & \textit{if} \quad (d-1)/4 \leq \alpha + \gamma \leq (d-1)/2 \, , \\ 0 & \textit{if} \quad \alpha + \gamma \geq (d-1)/2 \, . \end{cases}$$

Also assume that $|g(x)| \leq 1$. Then

$$\sum_{n=0}^{+\infty} \sum_{j=1}^{d_n} \left(1+\lambda_n^2\right)^{-\gamma} \left| \int_{\mathcal{M}} g(x) \, \widehat{B_{R(x)}}(\lambda_n) \, Y_{n,j}(x) \, dv(x) \right|^2 \leq C \int_{\mathcal{M}} \int_{\mathcal{M}} \frac{dv(x) \, dv(y)}{|x-y|^\delta} .$$

Proof. By the addition formula $Z_n(x,y) = \sum_{j=1}^{d_n} Y_{n,j}(x) \overline{Y_{n,j}(y)}$, we can write

$$\sum_{n=0}^{+\infty} \sum_{j=1}^{d_n} \left(1+\lambda_n^2\right)^{-\gamma} \left| \int_{\mathcal{M}} g(x) \, \widehat{B_{R(x)}}(\lambda_n) \, Y_{n,j}(x) \, dv(x) \right|^2 =$$

$$\int_{\mathcal{M}} \int_{\mathcal{M}} \left(\sum_{n=0}^{+\infty} \left(1+\lambda_n^2\right)^{-\gamma} \; \widehat{B_{R(x)}}(\lambda_n) \, \widehat{B_{R(y)}}(\lambda_n) \, Z_n(x,y) \right) g(x) \, \overline{g(y)} \, dv(x) \, dv(y) \, .$$

Define

$$I(x,y) \;=\; \left| \sum_{n=0}^{+\infty} \left(1+\lambda_n^2\right)^{-\gamma} \; \widehat{B_{R(x)}}(\lambda_n) \, \widehat{B_{R(y)}}(\lambda_n) \, Z_n(x,y) \right| .$$

Using the estimates for $Z_n(x,y)$ in part (2) of Lemma 7 and the estimate on $\widehat{B_R}(\lambda)$ in Lemma 10, if $t = |x-y|$ with $0 \leq t \leq \pi/2$,

$$I(x,y) \;\leq\; C \sum_{n=0}^{+\infty} \left(1+\lambda_n\right)^{d-2\gamma-1} \; \left(1+\lambda_n t\right)^{-(d-1)/2} \; \left| \widehat{B_{R(x)}}(\lambda_n) \right| \; \left| \widehat{B_{R(y)}}(\lambda_n) \right|$$

$$\leq\; C \, R(x)^{-\alpha} \, R(y)^{-\alpha} \sum_{n=0}^{+\infty} \frac{\left(1+\lambda_n\right)^{d-2\gamma-1} \; \left(1+\lambda_n t\right)^{-(d-1)/2}}{\left(1+|R(x)-\lambda_n|\right)^k \; \left(1+|R(y)-\lambda_n|\right)^k} \, .$$

Observe that in the last sum only a finite number of terms come into play, the ones with $|R(x)-\lambda_n| \lesssim C$ and $|R(y)-\lambda_n| \lesssim C$. This gives

$$I(x,y) \;\leq C \, R(y)^{-\alpha} \, R(x)^{d-\alpha-2\gamma-1} \; \left(1+R(x)t\right)^{-(d-1)/2} \; \left(1+|R(x)-R(y)|\right)^{-k}$$

$$+ \; C \, R(x)^{-\alpha} \, R(y)^{d-\alpha-2\gamma-1} \; \left(1+R(y)t\right)^{-(d-1)/2} \; \left(1+|R(x)-R(y)|\right)^{-k} \, .$$

If k is large and if $R(x)$ and $R(y)$ are close to each other, $R(x) \le R(y) \le 3R(x)$, then $(1 + |R(x) - R(y)|)^{-k}$ can be bounded by one and this gives

$$\begin{aligned} I(x,y) &\le C\,R(y)^{-\alpha}\,R(x)^{d-\alpha-2\gamma-1}\,(1+R(x)t)^{-(d-1)/2} \\ &\quad + C\,R(x)^{-\alpha}\,R(y)^{d-\alpha-2\gamma-1}\,(1+R(y)t)^{-(d-1)/2} \\ &\le C\,R(x)^{d-2\alpha-2\gamma-1}\,(1+R(x)t)^{-(d-1)/2}\,. \end{aligned}$$

If $R(x)$ and $R(y)$ are far from each other, $3R(x) < R(y)$,
then $(1 + |R(x) - R(y)|)^{-k}$ can be bounded by $R(y)^{-k}$ and this gives

$$\begin{aligned} I(x,y) &\le C\,R(x)^{d-\alpha-2\gamma-1}\,R(y)^{-\alpha-k}\,(1+R(x)t)^{-(d-1)/2} \\ &\quad + C\,R(x)^{-\alpha}\,R(y)^{d-\alpha-2\gamma-1-k}\,(1+R(y)t)^{-(d-1)/2} \\ &\le C\,R(x)^{d-2\alpha-2\gamma-1}\,(1+R(x)t)^{-(d-1)/2} \\ &\quad + C\,R(y)^{d-2\alpha-2\gamma-1}\,(1+R(y)t)^{-(d-1)/2}\,. \end{aligned}$$

In both cases, when $(d-1)/4 \le \alpha + \gamma \le (d-1)/2$,

$$\begin{aligned} I(x,y) \;&\le\; C \sup_{R\ge1} \left\{ R^{d-2\alpha-2\gamma-1}\,(1+Rt)^{-(d-1)/2} \right\} \\ \le\; C\,t^{-d+2\alpha+2\gamma+1} &\sup_{R\ge1} \left\{ (Rt)^{d-2\alpha-2\gamma-1}\,(1+Rt)^{-(d-1)/2} \right\} \\ &\le\; C\,t^{-d+2\alpha+2\gamma+1}\,. \end{aligned}$$

A similar computation shows that, if $\alpha+\gamma \ge (d-1)/2$, then $I(x,y) \le C$. □

This concludes the proof of Theorem 5. By this theorem, the Bochner-Riesz means cannot diverge on the supports of measures with finite energy. Hence, by the relation between energy, capacity, and dimension, these means cannot diverge on sets with large dimension.

7. Proof of Corollary 6

It suffices to show that the maximal function $\sup_{R>0}\{|S_R^\alpha G^\gamma f(x)|\}$ cannot be infinite on subset of Ω with Hausdorff dimension greater then δ. Consider $\tau, \sigma, \eta \in \mathbb{R}$ and recall that the τ-energy of a finite Borel measure υ on a metric space $\mathcal{M}$ is defined by

$$\int_{\mathcal{M}}\int_{\mathcal{M}} \frac{d\upsilon(x)\,d\upsilon(y)}{|x-y|^\tau}\,.$$

If $\sigma < \eta$, it follows directly from the properties of the Hausdorff measure that every set of dimension η has infinite σ-dimensional measure. Besides, by Frostman's Lemma (see [23], Theorem 8.17), there is a finite and nontrivial Borel measure supported on one of these sets with $\upsilon\{|x-p| < r\} \le r^\sigma$ for each $p \in \mathcal{M}$ and $r > 0$. In particular, this measure has finite τ-energy for every $\tau < \sigma$. To obtain the corollary it then sufficient to apply Theorem 5 with $\delta < \tau < \sigma < \eta$.

References

[1] Š.A. Alimov, V.A. Il'in, E.M. Nikišin, *Questions on the convergence of multiple trigonometric series and spectral expansions. I.* (Russian), Uspehi Mat. Nauk **31** (1976), no.6 (192), 28–83.

[2] Š.A. Alimov, V.A. Il'in, E.M. Nikišin, *Questions on the convergence of multiple trigonometric series and spectral expansions. II.* (Russian), Uspehi Mat. Nauk **32** (1977), no.1 (193), 107–130, 271.

[3] A.Ĭ. Bastis, *Convergence almost everywhere of expansions in eigenfunctions of the Laplace operator on a sphere* (Russian), Math. Zametki **33** (1983), no.6, 857–863.

[4] S. Bochner, *Summation of multiple Fourier series by spherical means*, Trans. Amer. Math. Soc. **40** (1936), 175–207.

[5] A. Bonami, J-L. Clerc, *Sommes de Cesàro et multiplicateurs des développements en harmoniques sphériques* (French), Trans. Amer. Math. Soc. **183** (1973), 223–263.

[6] L. Brandolini, L. Colzani, *Localization and convergence of eigenfunction expansions*, J. Fourier Anal. Appl. **5** (1999), no.5, 431–447.

[7] L. Brandolini, L. Colzani, *Decay of Fourier transforms and summability of eigenfunction expansions*, Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) **29** (2000), no.3, 611–638.

[8] A. Carbery, J.L. Rubio de Francia, L. Vega, *Almost everywhere summability of Fourier integrals*, J. London Math. Soc. **38** (1988), 513–524.

[9] A. Carbery, F. Soria, *Almost-everywhere convergence of Fourier integrals for functions in Sobolev spaces, and an L^2-localization principle*, Rev. Math. Iberoamericana **4** (1988), no.2, 319–337.

[10] A. Carbery, F. Soria, *Sets of divergence for the localization problem for Fourier integrals*, C. R. Acad. Sci. Paris **325** (1997), Série I, 1283–1286.

[11] A. Carbery, F. Soria, A. Vargas, *Localization and weighted inequalities for spherical Fourier means*, J. Anal. Math. **103** (2007), 133–156.

[12] I. Chavel, *Eigenvalues in Riemannian geometry*, Pure and Applied Mathematics **115**. Academic Press, 1984.

[13] L. Colzani, *Time decay and regularity of solutions to the wave equation and convergence of Fourier expansions*, J. Fourier Anal. Appl. **9** (2003), 49–66.

[14] L. Colzani, *Expansions in Legendre polynomials and Lagrange interpolation*, Acta. Math Hung. **61** (1993), 289–302.

[15] L. Colzani, G. Gigante, A. Vargas, *Localization for Riesz Means of Fourier expansions*, Trans. Amer. Math. Soc. **366** (2014), no.12, 6229–6245.

[16] G. Gigante, K. Jotsaroop, personal communication.

[17] Y. Guang Wang, I.H. Sloan, R.S. Womersley, *Riemann localization on the spheres*, Draft (2014).

[18] S. Helgason, *Groups and geometric analysis: integral geometry, invariant differential operators and spherical functions*, American Mathematical Society, 2000.

[19] L. Hörmander, *On the Riesz means of spectral functions and eigenfunction expansions for elliptic differential operators*, in *Some recent advances in the basic sciences*, Proc. Annual. Sci. Conf. Belfer Grad. School Sci. Yeshiva Univ. New York, (1955-1956), 155–202.

[20] E. Kogbetlianz, *Sur le rôle de l'antipode dans la théorie des séries et sur les phénomènes analogues*, Bull. Sci. Math. (2) **49** (1925), 13–32.

[21] T. Koornwinder, *The addition formula for Jacobi polynomials and spherical harmonics*, SIAM J. Appl. Math. **25** (1973), 236–246.

[22] N.N. Lebedev, *Special functions and their applications*, Dover Publications, 1972.

[23] P. Mattila, *Geometry of sets and measures in Euclidian spaces*, vol. 44 of Cambridge Studies in Advanced Mathematics, Cambridge University Press, 1995.

[24] C. Meaney, *On almost-everywhere convergent eigenfunction expansions of the Laplace-Beltrami operator*, Math. Proc. Cambridge Philos. Soc. **92** (1982), no.1, 129–131.
[25] C. Meaney, *Divergent Jacobi polynomial series*, Proc. Amer. Math. Soc. **87** (1983), no.3, 459–462.
[26] C. Meaney, *Localization of spherical harmonic expansions*, Mh. Math. **98** (1984), 65–74.
[27] M.A. Pinsky, *Fourier inversion for piecewise smooth functions in several variables*, Proc. Amer. Math. Soc. **118** (1993), no.3, 903–910.
[28] M.A. Pinsky, *Pointwise Fourier inversion and related eigenfunction expansions*, Comm. Pure Appl. Math. **47** (1994), no.5, 653–681.
[29] M.A. Pinsky, M.E. Taylor, *Pointwise Fourier inversion: a wave equation approach*, J. Fourier Anal. Appl. **3** (1997), no.6, 647–703.
[30] E.M. Stein, *Singular integrals and differentiability properties of functions*, Princeton University Press, 1970.
[31] E.M. Stein, G. Weiss, *Introduction to Fourier Analysis in Euclidian Spaces*, Princeton University Press, 1971.
[32] G. Szegö, *Orthogonal polynomials*, American Mathematical Society, 1939.
[33] M.E. Taylor, *Pointwise Fourier inversion on tori and other compact manifolds*, J. Fourier Anal. Appl. **5** (1999), no.5, 449–463.
[34] H.C. Wang, *Two-point homogeneous spaces*, Ann. of Math. **55** (1952), 177–191.
[35] W.P. Ziemer, *Weakly differentiable functions. Sobolev spaces and functions of bounded variation*, Springer-Verlag, 1989.
[36] A. Zygmund, *Trigonometric series*, Vol I, II, 3rd ed., Cambridge University Press, 2002.

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