SHARP BOUNDARY BEHAVIOR OF EIGENVALUES FOR AHARONOV-BOHM OPERATORS WITH VARYING POLES

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Abstract. In this paper, we investigate the behavior of the eigenvalues of a magnetic Aharonov-Bohm operator with half-integer circulation and Dirichlet boundary conditions in a bounded planar domain. We establish a sharp relation between the rate of convergence of the eigenvalues as the singular pole is approaching a boundary point and the number of nodal lines of the eigenfunction of the limiting problem, i.e. of the Dirichlet Laplacian, ending at that point. The proof relies on the construction of a limit profile depending on the direction along which the pole is moving, and on an Almgren-type monotonicity argument for magnetic operators.

1. Introduction

This paper is concerned with the behavior of the eigenvalues of Aharonov-Bohm operators in a planar domain with poles approaching the boundary. For $a = (a_1, a_2) \in \mathbb{R}^2$, we consider the so-called Aharonov-Bohm magnetic potential with pole a and circulation $1/2$

$$A_a(x) = \frac{1}{2}\left(\frac{-(x_2-a_2)}{(x_1-a_1)^2+(x_2-a_2)^2}, \frac{x_1-a_1}{(x_1-a_1)^2+(x_2-a_2)^2}\right), \quad x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{a\},$$

which gives rise to the singular magnetic field $B_a = \operatorname{curl} A_a = \pi\delta_a\mathbf{k}$, where $\mathbf{k}$ is the unit vector orthogonal to the x_1x_2-plane and δ_a is the Dirac delta centered at a. Such a magnetic field is generated by an infinitely long and infinitely thin solenoid intersecting the plane x_1x_2 perpendicularly at a. By Stokes' Theorem, the flux of the magnetic field through the solenoid cross section is equal (up to the normalization factor 2π) to the circulation of the vector potential A_a around the pole a, which remains identically equal to $1/2$.

We consider the magnetic Schrödinger operator $(i\nabla + A_a)^2$ with Aharonov-Bohm vector potential A_a which acts on functions $u : \mathbb{R}^2 \to \mathbb{C}$ as

$$(i\nabla + A_a)^2 u := -\Delta u + 2iA_a \cdot \nabla u + |A_a|^2 u, \tag{1.1}$$

and study the properties of the function mapping the position of the pole a to the eigenvalues of the operator (1.1) on a bounded domain with homogeneous Dirichlet boundary conditions.

As highlighted in [7], the case of half-integer circulation features a relation between critical positions of the moving pole and spectral minimal partitions of the Dirichlet Laplacian. It was proved in [14] that the optimal partition (i.e. the partition of the domain minimizing the largest of the first eigenvalues on the components) corresponds to the nodal domain of an eigenfunction of the Dirichlet Laplacian if it has only points of even multiplicity; the optimal partitions with points of odd multiplicity are instead related to the eigenfunctions of the Aharonov-Bohm operator, in the sense that they can be obtained as nodal domains by minimizing a certain eigenvalue

Date: May 31, 2016.

2010 *Mathematics Subject Classification.* 35P15, 35J10, 35J75, 35B40, 35B44,

Key words and phrases. Aharonov-Bohm operators, Almgren monotonicity formula, spectral theory.

The authors are partially supported by the project ERC Advanced Grant 2013 n. 339958: "Complex Patterns for Strongly Interacting Dynamical Systems – COMPAT". L. Abatangelo and V. Felli are partially supported by the 2015 INdAM-GNAMPA research project "Operatori di Schrödinger con potenziali elettromagnetici singolari: stabilità spettrale e stime di decadimento". V. Felli is partially supported by PRIN-2012-grant "Variational and perturbative aspects of nonlinear differential problems".

of an Aharonov-Bohm Hamiltonian with respect to the number and the position of poles, see [13]. We also refer to [4, 5, 6, 11, 12, 22] for the study of the eigenfunctions, their nodal domains and spectral minimal partitions.

The present paper focuses on the behavior of the eigenvalues of the operator (1.1) when the pole a is moving in the domain reaching a point on the boundary. Our analysis proceeds by the papers [1, 2, 7], which provide the asymptotic expansion of the eigenvalue function as the pole is moving in the interior of the domain. On the other hand, the study of the case of a pole approaching the boundary was initiated in [21]. In this case the limit operator is no more singular and the magnetic eigenvalues converge to those of the standard Laplacian. In [21] the authors predict the rate of this convergence in relation with the number of nodal lines that the limit eigenfunction possesses at the limit point. More precisely, let us denote as λ_N^a the N-th eigenvalue of the operator (1.1) in a planar domain Ω with Dirichlet boundary conditions and as λ_N the N-th eigenvalue of the Dirichlet Laplacian on the same domain; in [21] it is proved that if λ_N is simple and the corresponding eigenfunction φ_N has at a point $b \in \partial\Omega$ a zero of order $j \geq 2$ (so that φ_N has $j-1$ nodal lines ending at b) then

$$\lambda_N^a - \lambda_N \leq -C|a-b|^{2j} \tag{1.2}$$

for a moving on a nodal line approaching b, where $C > 0$ is a positive constant. In particular, estimate (1.2) implies that, if the pole stays on a nodal line, then the magnetic eigenvalue is strictly smaller than the standard Laplacian's one, thus showing that a diamagnetic-type inequality is not necessarily true for eigenvalues higher than the first one. In the case of the pole approaching a boundary point b where no nodal lines of φ_N end, in [21] it is proved that

$$\lambda_N^a - \lambda_N \geq C(\mathrm{dist}(a, \partial\Omega))^2 \tag{1.3}$$

as $a \to b$, where C is a positive constant. Estimate (1.3) was shown to be sharp in [23, Theorem 2.1.15], where the following exact asymptotics was obtained:

$$\frac{\lambda_N^a - \lambda_N}{(\mathrm{dist}(a, \partial\Omega))^2} \to c(\nabla\varphi_N(b) \cdot \nu)^2 \tag{1.4}$$

as a converges to some $b \in \partial\Omega$ where no nodal lines end, where c is a positive constant.

In the present paper, we describe the asymptotic behavior of the eigenvalue λ_N^a as the pole a approaches a point on the boundary of Ω moving on straight lines (not necessarily tangent to nodal lines of the limit eigenfunction), with the aim of sharpening and generalizing the results in [21]. Our main theorem states that, if $\partial\Omega$ is sufficiently smooth, λ_N is simple, and φ_N has $j-1$ ($j \in \mathbb{N}$, $j \geq 1$) nodal lines ending at $b \in \partial\Omega$, then the limit of the quotient

$$\frac{\lambda_N - \lambda_N^a}{|a-b|^{2j}}, \tag{1.5}$$

as a approches b on a straight line, exists, is finite and depends continuously on the line direction; furthermore such a limit is strictly positive if the line is tangent to a nodal line of φ_N, while it is strictly negative if the moving pole direction is in the middle of the tangents to two nodal lines (Theorem 2.1). This establishes, in particular, that a diamagnetic-type inequality $\lambda_N^a > \lambda_N$ holds for eigenvalues higher than the first one, when a lies in the middle of the tangents to two nodal lines of φ_N (or in the middle between a tangent and the boundary). The opposite inequality $\lambda_N^a < \lambda_N$ holds when a belongs to the tangent to a nodal line of φ_N. Thus, the diamagnetic inequality for this specific operator can be seen as a particular case of Theorem 2.1, due to the fact that φ_1 does not have nodal lines.

Furthermore, we provide a variational characterization of the limit of the quotient (1.5), by relating it to the minimum of an energy functional associated to an elliptic problem with a crack sloping at the moving pole direction (Theorem 2.2).

Theorem 2.1 implies that estimate (1.2) is optimal, thus generalizing the sharp estimate (1.4) to any order of vanishing of the limit eigenfunction. Furthermore, our result answers a question left open in [21, Remark 1.9] about the exact behavior of the eigenvalue variation $\lambda_N^a - \lambda_N$ as the pole a approaches a boundary point b, being b the endpoint of one or more nodal lines of the limit eigenfunction and a not belonging to any such nodal line; indeed, as a byproduct of Theorem 2.1, we have that λ_N^a increases as a is moving from a boundary point on the bisector of two nodal lines of the Dirichlet-Laplacian, or on the bisector of one nodal line and the boundary, as conjectured in [21, 23].

2. Statement of the main results

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open and simply connected domain. We assume that $\Omega \in C^{2,\gamma}$ for some $0 < \gamma < 1$, and that

$$0 \in \partial\Omega.$$

Furthermore, it is convenient to suppose that there exists $\bar{R} > 0$ such that

$$\Omega \cap D_{\bar{R}} = D_{\bar{R}}^+, \tag{2.1}$$

where $D_{\bar{R}}^+$ is defined as

$$D_{\bar{R}}^+ := D_{\bar{R}} \cap \mathbb{R}_+^2,$$

being $D_{\bar{R}}$ the open ball of radius $\bar{R}$ centered at 0 and

$$\mathbb{R}_+^2 := \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}.$$

We stress that this assumption is not restrictive provided that a weight is considered in the eigenvalue problem. Starting from a general domain of class $C^{2,\gamma}$, we can indeed perform a conformal transformation in order to obtain a new domain satisfying (2.1): the counterpart is the appearance of a conformal weight (real valued) in the new problem, whose regularity is $C^1(\overline{\Omega})$ thanks to the regularity assumptions on the domain (see [15, Theorem 5.2.4]). More specifically, the weight verifies

$$q(x) \in C^1(\overline{\Omega}), \quad q(x) > 0 \text{ for } x \in \Omega. \tag{2.2}$$

For more details, we refer the [21, Section 3].

For every $a \in \overline{\Omega}$, we introduce the space $H^{1,a}(\Omega, \mathbb{C})$ as the completion of

$$\Big\{u \in H^1(\Omega, \mathbb{C}) \cap C^\infty(\Omega, \mathbb{C}) : u \text{ vanishes in a neighborhood of } a\Big\}$$

with respect to the norm

$$\|u\|_{H^{1,a}(\Omega,\mathbb{C})} = \left(\|\nabla u\|^2_{L^2(\Omega,\mathbb{C}^2)} + \|u\|^2_{L^2(\Omega,\mathbb{C})} + \left\|\frac{u}{|x-a|}\right\|^2_{L^2(\Omega,\mathbb{C})}\right)^{1/2}. \tag{2.3}$$

For every $a \in \overline{\Omega}$, we also introduce the space $H^{1,a}_0(\Omega, \mathbb{C})$ as the completion of $C_c^\infty(\Omega \setminus \{a\})$ with respect to the norm $\|\cdot\|_{H^{1,a}(\Omega,\mathbb{C})}$. In view of the Hardy-type inequality proved in [17] (see (A.1)) and of the Poincaré-type inequality (A.3), an equivalent norm in $H^{1,a}_0(\Omega, \mathbb{C})$ is given by

$$\|u\|_{H^{1,a}_0(\Omega,\mathbb{C})} = \Big(\|(i\nabla + A_a)u\|^2_{L^2(\Omega,\mathbb{C}^2)}\Big)^{1/2}. \tag{2.4}$$

As a consequence of the equivalence between norms (2.3) and (2.4), by gauge invariance it follows that

$$\begin{aligned}&\text{if } a \in \partial\Omega, \text{ then the space } H^{1,a}_0(\Omega, \mathbb{C}) \text{ coincides with the standard } H^1_0(\Omega, \mathbb{C})\\ &\text{and the norms (2.3), (2.4) are therein equivalent to the Dirichlet norm } \|\nabla u\|_{L^2(\Omega,\mathbb{C}^2)}.\end{aligned} \tag{2.5}$$

For every $a \in \overline{\Omega}$ and any weight $q(x)$ verifying (2.2), we consider the weighted eigenvalue problem

$$\begin{cases} (i\nabla + A_a)^2 u = \lambda\, q(x) u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{E_a}$$

in a weak sense, i.e. we say that λ is an eigenvalue of (E_a) if there exists an eigenfunction $u \in H^{1,a}_0(\Omega,\mathbb{C}) \setminus \{0\}$ such that

$$\int_\Omega (i\nabla + A_a)u \cdot \overline{(i\nabla + A_a)v}\, dx = \lambda \int_\Omega q(x) u\bar{v}\, dx, \qquad \text{for all } v \in H^{1,a}_0(\Omega,\mathbb{C}).$$

From classical spectral theory, (E_a) admits a diverging sequence of real eigenvalues $\{\lambda_k^a\}_{k\geq 1}$ with finite multiplicity (being each eigenvalue repeated according to its own multiplicity). To each eigenvalue λ_k^a we associate an eigenfunction φ_k^a suitably normalized (see (2.23) and (5.2)). When $a \in \partial\Omega$, hence in particular when $a = 0$, $\lambda_k^a = \lambda_k$, being λ_k the k-th weighted eigenvalue of the Dirichlet Laplacian (with the same weight $q(x)$); moreover, if

$$\tilde{\theta}_0 : \mathbb{R}^2 \setminus \{0\} \to [-\pi,\pi), \quad \tilde{\theta}_0(r\cos t, r\sin t) = t \quad \text{if } t \in [-\pi,\pi),$$

is the polar angle centered at 0 and discontinuous on the half-line $\{(x_1,0) : x_1 < 0\}$, we have that $e^{-\frac{i}{2}\tilde{\theta}_0}\varphi_k^0 = \varphi_k$ is a weighted eigenfunction of the Laplacian associated to λ_k, i.e.

$$\begin{cases} -\Delta\varphi_k = \lambda_k q(x)\varphi_k, & \text{in } \Omega, \\ \varphi_k = 0, & \text{on } \partial\Omega. \end{cases} \tag{2.6}$$

From [7, Theorem 1.1] and [18, Theorem 1.2] it is known that, for every $k \in \mathbb{N}\setminus\{0\}$, there holds

$$\lambda_k^a \to \lambda_k \quad \text{as } a \to 0. \tag{2.7}$$

Let us assume that there exists $N \geq 1$ such that

$$\lambda_N \quad \text{is simple.} \tag{2.8}$$

We observe that, in view of [20], assumption (2.8) holds generically with respect to domain (and weight) variations. Let $\varphi_N \in H^1_0(\Omega,\mathbb{C}) \setminus \{0\}$ be an eigenfunction of problem (2.6) associated to the eigenvalue λ_N such that

$$\int_\Omega q(x)|\varphi_N(x)|^2\, dx = 1. \tag{2.9}$$

From [10] and [14] (see also [8]) it is known that

$$\varphi_N \text{ has at 0 a zero of order } j \text{ for some } j \in \mathbb{N}\setminus\{0\}; \tag{2.10}$$

more precisely, there exists $\beta \in \mathbb{C}\setminus\{0\}$ such that

$$r^{-j}\varphi_N(r(\cos t, \sin t)) \to \beta\psi_j(\cos t, \sin t) = \beta \sin\left(j\left(\tfrac{\pi}{2} - t\right)\right), \tag{2.11}$$

in $C^{1,\tau}([-\frac{\pi}{2}, \frac{\pi}{2}], \mathbb{C})$ as $r \to 0^+$ for any $\tau \in (0,1)$. Here, for every $j \in \mathbb{N}\setminus\{0\}$, ψ_j is the unique function (up to a multiplicative constant) which is harmonic in $\mathbb{R}^2_+$, homogeneous of degree j and vanishing on $\partial\mathbb{R}^2_+$, more explicitly

$$\psi_j(r\cos t, r\sin t) = r^j \sin\left(j\left(\tfrac{\pi}{2} - t\right)\right), \quad r \geq 0, \quad t \in \left[-\tfrac{\pi}{2}, \tfrac{\pi}{2}\right]. \tag{2.12}$$

We notice that ψ_j has exactly $j-1$ nodal lines (except for the boundary) dividing the π-angle in equal parts. Moreover, via a change of gauge,

$$\text{the function } e^{\frac{i}{2}\tilde{\theta}_0}\psi_j \text{ is a distributional solution to } (i\nabla + A_0)^2(e^{\frac{i}{2}\tilde{\theta}_0}\psi_j) = 0 \text{ in } \mathbb{R}^2_+.$$

Let

$$\varphi_N^0 = \varphi_N e^{\frac{i}{2}\tilde{\theta}_0},$$

so that φ_N^0 is an eigenfunction of problem (E_0) associated to the eigenvalue λ_N.

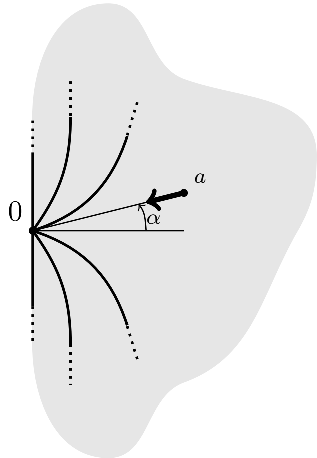

FIGURE 1. The $j-1$ nodal lines of φ_N ending at 0 dividing the π-angle into j equal parts; a approaches 0 along the straight line $a = |a|p$, $p = (\cos\alpha, \sin\alpha)$.

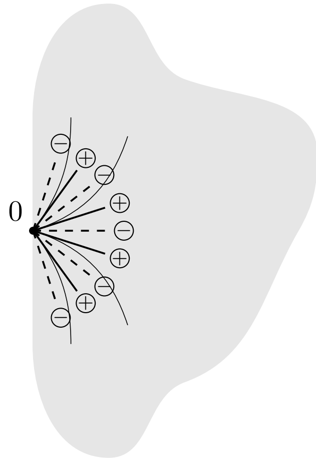

FIGURE 2. The sign of the eigenvalue variation $\lambda_N - \lambda_N^a$: positive tangentially to nodal lines, negative on bisectors of nodal lines.

As already mentioned, we aim at proving sharp asymptotics for the convergence (2.7) as the pole a moves along a straight line up to the origin, see Figure 1. More precisely, we fix

$$p \in \mathbb{S}^1_+ := \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1 \text{ and } x_1 > 0\},$$

and study the limit of the quotient (1.5) as $a = |a|p \to 0$, giving a characterization of such a limit in terms of the direction p, which allows recognizing directions for which it is nonzero (and possibly positive or negative).

We are now in position to state our first main result.

Theorem 2.1. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, open and simply connected domain of class $C^{2,\gamma}$ for some $0 < \gamma < 1$, such that $0 \in \partial\Omega$ and* (2.1) *holds. Let q satisfy* (2.2)*. Let $N \geq 1$ be such that the N-th eigenvalue λ_N of problem* (2.6) *is simple and let $\varphi_N \in H^1_0(\Omega, \mathbb{C}) \setminus \{0\}$ be an eigenfunction of* (2.6) *associated to λ_N satisfying* (2.9)*. Let $j \in \mathbb{N} \setminus \{0\}$ be the order of vanishing of φ_N at 0 as in* (2.10)*–*(2.11)*. For $a \in \Omega$, let λ_N^a be the N-th eigenvalue of problem (E_a).*

Then, for every $p \in \mathbb{S}^1_+$, there exists $\mathfrak{c}_p \in \mathbb{R}$ such that

$$\frac{\lambda_N - \lambda_N^a}{|a|^{2j}} \to |\beta|^2\, \mathfrak{c}_p, \quad \text{as } a = |a|p \to 0, \tag{2.13}$$

with $\beta \neq 0$ being as in (2.11)*. Moreover*

(i) *the function $p \mapsto \mathfrak{c}_p$ is continuous on $\mathbb{S}^1_+$ and tends to 0 as $p \to (0, \pm 1)$;*
(ii) *$\mathfrak{c}_p > 0$ if the half-line $\{tp : t \geq 0\}$ is tangent to a nodal line of φ_N in 0, i.e. if, for some $k = 1, \dots, j-1$, $p = \big(\cos(\frac{\pi}{2} - k\frac{\pi}{j}), \sin(\frac{\pi}{2} - k\frac{\pi}{j})\big)$;*
(iii) *$\mathfrak{c}_p < 0$ if the half-line $\{tp : t \geq 0\}$ is tangent to the bisector of two nodal lines of φ_N or to the bisector of one nodal line and the boundary, i.e. if, for some $k = 0, \dots, j-1$, $p = \big(\cos(\frac{\pi}{2} - \frac{\pi}{2j}(1+2k)), \sin(\frac{\pi}{2} - \frac{\pi}{2j}(1+2k))\big)$.*

The sign properties of $\mathfrak{c}_p$ imply in particular that, as $|a|$ is sufficiently small,

$\lambda_N - \lambda_N^a > 0$ if a is tangent to a nodal line of φ_N in 0,

$\lambda_N - \lambda_N^a < 0$ if a lies in the middle of the tangents to two nodal lines of φ_N in 0, see Figure 2, in agreement with the preexisting results (1.2) and (1.3). This fact, together with the continuity property of $\mathfrak{c}_p$, implies that $\mathfrak{c}_p$ vanishes at least two times between two nodal lines of φ_N in 0, and then $\lambda_N - \lambda_N^a = o(|a|^{2j})$ as $a \to 0$ straightly at least along $2(j-1)$ directions.

2.1. **Variational characterization of the function $p \mapsto \mathfrak{c}_p$ and of the limit profile.** Our second main result is a variational characterization of the function $p \mapsto \mathfrak{c}_p$ appearing in Theorem 2.1, for which the following additional notation is needed.

Let us fix $\alpha \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right)$ and $p = (\cos\alpha, \sin\alpha) \in \mathbb{S}^1_+$. We denote by Γ_p the segment joining 0 to p, that is to say

$$\Gamma_p = \{(r\cos\alpha, r\sin\alpha) : r \in (0,1)\},$$

and define the space $\mathcal{H}_p$ as the completion of

$$\left\{u \in H^1(\mathbb{R}^2_+ \setminus \Gamma_p) : u = 0 \text{ on } \partial\mathbb{R}^2_+ \text{ and } u = 0 \text{ in a neighborhood of } \infty\right\}$$

with respect to the Dirichlet norm

$$\|u\|_{\mathcal{H}_p} := \|\nabla u\|_{L^2(\mathbb{R}^2_+ \setminus \Gamma_p)}. \tag{2.14}$$

From the Hardy-type inequality for magnetic Sobolev spaces proved in [17] (see (A.2)) and a change of gauge, it follows that functions in $\mathcal{H}_p$ also satisfy a Hardy-type inequality, so that $\mathcal{H}_p$ can be characterized as

$$\mathcal{H}_p = \left\{u \in L^1_{\rm loc}(\mathbb{R}^2_+) : \nabla_{\mathbb{R}^2_+ \setminus \Gamma_p} u \in L^2(\mathbb{R}^2_+),\ \tfrac{u}{|x|} \in L^2(\mathbb{R}^2_+), \text{ and } u = 0 \text{ on } \partial\mathbb{R}^2_+\right\},$$

where $\nabla_{\mathbb{R}^2_+ \setminus \Gamma_p} u$ denotes the distributional gradient of u in $\mathbb{R}^2_+ \setminus \Gamma_p$.

The functions in $\mathcal{H}_p$ may clearly be discontinuous on Γ_p. For this reason, we introduce two trace operators. Let us consider the sets $U_p^+ = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_2 > x_1 \tan\alpha\} \cap D_1^+$ and $U_p^- = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_2 < x_1 \tan\alpha\} \cap D_1^+$. First, for any function u defined in a neighborhood of U_p^+, respectively U_p^-, we define the restriction

$$\mathcal{R}_p^+(u) = u|_{U_p^+}, \quad \text{respectively} \quad \mathcal{R}_p^-(u) = u|_{U_p^-}. \tag{2.15}$$

We observe that, since $\mathcal{R}_p^\pm$ maps $\mathcal{H}_p$ into $H^1(U_p^\pm)$ continuously, the trace operators

$$\gamma_p^\pm : \quad \mathcal{H}_p \longrightarrow H^{1/2}(\Gamma_p), \ \ u \longmapsto \gamma_p^\pm(u) := \mathcal{R}_p^\pm(u)|_{\Gamma_p} \tag{2.16}$$

are well defined and continuous from $\mathcal{H}_p$ to $H^{1/2}(\Gamma_p)$. Furthermore, by Poincaré and Sobolev trace inequalities, it is easy to verify that the operator norm of $\gamma_p^\pm$ is bounded uniformly with respect to $p \in \mathbb{S}^1_+$, in the sense that there exists a constant $L > 0$ independent of p such that, recalling (2.14),

$$\|\gamma_p^\pm(u)\|_{H^{1/2}(\Gamma_p)} \le L\|u\|_{\mathcal{H}_p} \quad \text{for all } u \in \mathcal{H}_p. \tag{2.17}$$

Clearly, for a continuous function u, $\gamma_p^+(u) = \gamma_p^-(u)$.

We will give a variational characterization of the limit of the quotient (1.5) by relating it to the minimum of the functional $J_p : \mathcal{H}_p \to \mathbb{R}$ defined as

$$J_p(u) = \frac{1}{2}\int_{\mathbb{R}^2_+ \setminus \Gamma_p} |\nabla u|^2\,dx + j\cos\left(j\left(\tfrac{\pi}{2} - \alpha\right)\right)\int_{\Gamma_p} |x|^{j-1}(\gamma_p^+(u) - \gamma_p^-(u))\,ds \tag{2.18}$$

on the set

$$\mathcal{K}_p := \{u \in \mathcal{H}_p : \ \gamma_p^+(u + \psi_j) + \gamma_p^-(u + \psi_j) = 0\}. \tag{2.19}$$

The following theorem relates the value $\mathfrak{c}_p$ appearing in the limit (2.13) with the minimum of J_p over $\mathcal{K}_p$.

Theorem 2.2. *The minimum of J_p over $\mathcal{K}_p$ is uniquely achieved at a function $w_p \in \mathcal{K}_p$. Furthermore, letting*

$$\mathfrak{m}_p := \min_{u\in\mathcal{K}_p} J_p(u) = J_p(w_p), \tag{2.20}$$

we have that

$$\mathfrak{c}_p = -2\mathfrak{m}_p,$$

with $\mathfrak{c}_p$ being as in Theorem 2.1.

The proofs of Theorems 2.1 and 2.2 rely on the exact determination of the limit of a suitable blow-up sequence of the eigenfunctions φ_N^a, in the spirit of [1, 2]. We emphasize that the boundary case presents some significant additional difficulties, due to lack of local symmetry and unavailability of regularity results of the function $a \mapsto \lambda_N^a$ up to the boundary. The overcoming of these difficulties requires a nontrivial adaptation of the techniques developed in [1, 2] for interior poles. Being this blow-up result of independent interest, it is worthwhile to be stated precisely. To this aim, let us define, for every $\alpha \in [0, 2\pi)$ and $b = (b_1, b_2) = |b|(\cos\alpha, \sin\alpha) \in \mathbb{R}^2 \setminus \{0\}$,

$$\theta_b : \mathbb{R}^2 \setminus \{b\} \to [\alpha, \alpha + 2\pi) \quad \text{and} \quad \theta_0^b : \mathbb{R}^2 \setminus \{0\} \to [\alpha, \alpha + 2\pi)$$

such that

$$\begin{aligned} &\theta_b(b + r(\cos t, \sin t)) = t \quad \text{for all } r > 0 \text{ and } t \in [\alpha, \alpha + 2\pi), \\ &\theta_0^b(r(\cos t, \sin t)) = t \quad \text{for all } r > 0 \text{ and } t \in [\alpha, \alpha + 2\pi). \end{aligned} \tag{2.21}$$

We observe that the difference function $\theta_0^b - \theta_b$ is regular except for the segment $\{tb : \ t \in [0, 1]\}$. Moreover, we also define $\theta_0 : \mathbb{R}^2 \setminus \{0\} \to [0, 2\pi)$ as

$$\theta_0(\cos t, \sin t) = t \quad \text{for all } t \in [0, 2\pi).$$

For $a \in \Omega$, let $\varphi_N^a \in H_0^{1,a}(\Omega, \mathbb{C})$ be an eigenfunction of (E_a) related to the weighted eigenvalue λ_N^a, i.e. solving

$$\begin{cases} (i\nabla + A_a)^2 \varphi_N^a = \lambda_N^a q(x)\, \varphi_N^a, & \text{in } \Omega, \\ \varphi_N^a = 0, & \text{on } \partial\Omega, \end{cases} \tag{2.22}$$

and satisfying the normalization conditions

$$\int_\Omega q(x)|\varphi_N^a(x)|^2\, dx = 1 \quad \text{and} \quad \int_\Omega e^{\frac{i}{2}(\theta_0^a - \theta_a)(x)} q(x)\varphi_N^a(x)\overline{\varphi_N^0(x)}\, dx \in \mathbb{R}^+. \tag{2.23}$$

The following theorem gives us the behavior of the eigenfunction φ_N^a for a close to the boundary point 0; more precisely, it shows that a homogeneous scaling of order j of φ_N^a along a fixed direction associated to $p \in \mathbb{S}^1_+$ converges to the limit profile $\Psi_p \in \bigcup_{r>1} H^{1,p}(D_r^+, \mathbb{C})$ given by

$$\Psi_p := e^{\frac{i}{2}(\theta_p - \theta_0^p + \tilde\theta_0)}(w_p + \psi_j), \tag{2.24}$$

with w_p as in (2.20) and ψ_j as in (2.12).

Theorem 2.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, open and simply connected domain of class $C^{2,\gamma}$ for some $0 < \gamma < 1$, such that $0 \in \partial\Omega$ and (2.1) holds. Let q satisfy (2.2), $N \geq 1$ be such that (2.8) holds, and $j \in \mathbb{N} \setminus \{0\}$ be the order of vanishing of a N-th eigenfunction φ_N^0 of (E_0) satisfying (2.9). Let $\varphi_N^a \in H_0^{1,a}(\Omega, \mathbb{C})$ solve (2.22)–(2.23). Then, for every $p \in \mathbb{S}^1_+$,*

$$\frac{\varphi_N^a(|a|x)}{|a|^j} \to \beta\Psi_p \qquad \text{as } a = |a|p \to 0,$$

in $H^{1,p}(D_R^+, \mathbb{C})$ for every $R > 1$, almost everywhere in $\mathbb{R}^2_+$ and in $C^2_{\rm loc}(\overline{\mathbb{R}^2_+} \setminus \{p\}, \mathbb{C})$, with $\beta \neq 0$ as in (2.11).

We notice that the rate of the convergences in Theorems 2.1 and 2.3 is related to the nodal properties of the limit eigenfunction, see (2.11), as already highlighted in [2, 7, 21]. From the results in [8, Theorem 1.4] we know that the asymptotic behavior in (2.11) is in turn related to the so-called Almgren quotient (for a precise definition see §5). More precisely,

$$\lim_{r\to 0^+} \frac{r\int_{D_r^+}\left(|(i\nabla + A_0)\varphi_N^0|^2 - \lambda_N q(x)|\varphi_N^0|^2\right)\,dx}{\int_{\partial D_r^+}|\varphi_N^0|^2\,ds} = j. \tag{2.25}$$

2.2. **Organization of the paper and main ideas.** In §3 we treat the variational characterization of the limit profile described above. This extends the one obtained in [21, Proposition 1.6] for the case $j = 1$ and the one constructed in [2, Proposition 4.2] for a general j when the pole a approaches a fixed point (which in this case lays in the interior of the domain) tangentially to a nodal line of the limit eigenfunction.

On one hand, the case $j = 1$ is considerably easier because the growth at infinite of the limit profile is the least possible: this allows characterizing immediately the limit profile through its Almgren frequency, since the $\liminf$ and the $\limsup$ of the Almgren quotient at infinity are the same. On the other hand, the construction presented in [2] holds for general j, but only for a moving tangentially to a nodal line of the limit eigenfunction: this restriction forces the limit profile to vanish on a half-line, so that the authors are able to construct the limit profile first on a half-plane solving a minimization problem, then reflecting and multiplying by a suitable phase jumping on the half-line. Finally, we remark that the sharp estimates obtained in [1] for a approaching an interior point along a general direction don't make use of an explicit construction of the limit profile: in that case, the sharp estimate on nodal lines is enough to compute the leading term of the Taylor expansion of the eigenvalue variation, thanks to symmetry and periodicity properties of the Fourier coefficients of the limit profile with respect to the direction.

In the present paper we are dealing with general j as a approaches a boundary point along a general direction (not even perpendicular to the boundary of Ω), so that we cannot take advantage of any remarkable bound for the Almgren quotient nor of any symmetry property. This requires a completely new approach, based on the construction of the limit profile by solving an elliptic crack problem prescribing the jump of the solution along the crack Γ_p, rather than its value, see (3.13)–(3.15).

In §4 we describe the properties of the function $\mathfrak{m}_p$ defined in (2.20).

Next we turn to study a suitable blow-up of the eigenfunctions φ_N^a. Due to the difficulties in proving a priori energy bounds for the blow-up sequence

$$\frac{\varphi_N^a(|a|x)}{|a|^j}, \tag{2.26}$$

we introduce the following auxiliary blow-up sequence

$$\tilde{\varphi}_a(x) = \sqrt{\frac{\bar{K}|a|}{\int_{\partial D_{\bar{K}|a|}}|\varphi_N^a|^2\,ds}}\,\varphi_N^a(|a|x), \tag{2.27}$$

for a suitable $\bar{K} > 0$. In §5 we take advantage of the Almgren's frequency function to obtain a priori bounds on (2.27), see (5.13). We recall that the frequency function in the context of magnetic operators was first introduced in [16] for magnetic potentials in the Kato class and then extended to Aharonov-Bohm type potentials in [9].

§6 and §7 provide preliminary upper and lower bounds for the difference $\lambda_N - \lambda_N^a$, which are then summarized in Corollary 7.3. These preliminary estimates are obtained by considering suitable competitor functions, and by plugging them into the Courant-Fisher minimax characterization of eigenvalues. More precisely, to obtain an upper bound for $\lambda_N - \lambda_N^a$ we use the

Rayleigh quotient for λ_N, and to get a lower bound for $\lambda_N - \lambda_N^a$ we use the Rayleigh quotient for λ_N^a.

At this first stage, the estimate from above of $\lambda_N - \lambda_N^a$ is given in terms of the normalization factor appearing in (2.27); in order to determine the exact asymptotic behavior of such normalization term, in §8 we obtain some energy estimates of the difference between approximating and limit eigenfunctions after blow-up, exploiting the invertibility of the differential of the function F defined in (8.1). As a consequence, in §9 we succeed in proving that

$$|a|^{-2j-1}\int_{\partial D_{\tilde K|a|}} |\varphi_N^a|^2\,ds$$

tends to a positive finite limit depending on $p \in \mathbb{S}^1_+$ as $a = |a|p \to 0$, and in turn the equivalence of the two blow-up sequences (2.26) and (2.27). This allows us to conclude the proofs of Theorem 2.3 in §9 and those of Theorems 2.1, 2.2 in §10.

Finally, in the appendix, we recall a Hardy-type inequality for Aharonov-Bohm operators and some Poincaré-type inequalities used throughout the paper.

2.3. **Notation.**

- For $r > 0$ and $a \in \mathbb{R}^2$, $D_r(a) = \{x \in \mathbb{R}^2 : |x - a| < r\}$ denotes the disk of center a and radius r.
- For all $r > 0$, $D_r = D_r(0)$ denotes the disk of center 0 and radius r.
- $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0\}$ and $\mathbb{R}^2_- = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 < 0\}$.
- For all $r > 0$, $D_r^+ = D_r \cap \mathbb{R}^2_+$ denotes the right half-disk of center 0 and radius r.
- For $f \in L^\infty(\Omega)$, $\|f\|_\infty = \|f\|_{L^\infty(\Omega)}$.

3. Limit profile

Keeping in mind the definitions of $\mathcal{R}_p^\pm$ (2.15) and of $\gamma_p^\pm$ (2.16) given in the §2.1, we introduce the following further notation. For $p = (\cos\alpha, \sin\alpha) \in \mathbb{S}^1_+$, let

$$\nu_p^+ = (\sin\alpha, -\cos\alpha) \quad \text{and} \quad \nu_p^- = -\nu_p^+$$

be the normal unit vectors to Γ_p. For every $u \in C^1(D_1^+ \setminus \Gamma_p)$ with $\mathcal{R}_p^+(u) \in C^1(\overline{U_p^+})$ and $\mathcal{R}_p^-(u) \in C^1(\overline{U_p^-})$, we define the normal derivatives $\frac{\partial^\pm u}{\partial \nu_p^\pm}$ on Γ_p respectively as

$$\frac{\partial^+ u}{\partial \nu_p^+} := \nabla \mathcal{R}_p^+(u) \cdot \nu_p^+\Big|_{\Gamma_p}, \quad \text{and} \quad \frac{\partial^- u}{\partial \nu_p^-} := \nabla \mathcal{R}_p^-(u) \cdot \nu_p^-\Big|_{\Gamma_p}.$$

For a function u differentiable in a neighborhood of Γ_p, we get

$$\frac{\partial^+ u}{\partial \nu_p^+} = -\frac{\partial^- u}{\partial \nu_p^-} \quad \text{on } \Gamma_p. \tag{3.1}$$

We remark that since ψ_j is differentiable, it verifies (3.1), so that

$$\frac{\partial^+ \psi_j}{\partial \nu_p^+}(r\cos\alpha, r\sin\alpha) = -\frac{\partial^- \psi_j}{\partial \nu_p^-}(r\cos\alpha, r\sin\alpha) = j r^{j-1}\cos\left(j\left(\tfrac{\pi}{2} - \alpha\right)\right).$$

Hence the functional $J_p : \mathcal{H}_p \to \mathbb{R}$ defined in (2.18) can be equivalently written as

$$\begin{aligned} J_p(u) &= \frac{1}{2}\int_{\mathbb{R}^2_+ \setminus \Gamma_p} |\nabla u|^2\,dx + \int_{\Gamma_p} \frac{\partial^+ \psi_j}{\partial \nu_p^+}(\gamma_p^+(u) - \gamma_p^-(u))\,ds \\ &= \frac{1}{2}\int_{\mathbb{R}^2_+ \setminus \Gamma_p} |\nabla u|^2\,dx + \int_{\Gamma_p} \gamma_p^+(u)\frac{\partial^+ \psi_j}{\partial \nu_p^+}\,ds + \int_{\Gamma_p} \gamma_p^-(u)\frac{\partial^- \psi_j}{\partial \nu_p^-}\,ds. \end{aligned}$$

In the following lemma we prove that J_p admits a unique minimum point in the set $\mathcal{K}_p$ defined in (2.19).

Lemma 3.1. *The minimum* $\mathfrak{m}_p = \min_{\mathcal{K}_p} J_p$ *is uniquely achieved at a function* $w_p \in \mathcal{K}_p$. *Furthermore,* w_p *is the unique solution to the variational problem*

$$\begin{cases} w_p \in \mathcal{K}_p, \\ \displaystyle\int_{\mathbb{R}^2_+ \setminus \Gamma_p} \nabla w_p \cdot \nabla \varphi \, dx + 2 \int_{\Gamma_p} \frac{\partial^+ \psi_j}{\partial \nu_p^+} \gamma_p^+(\varphi) \, ds = 0, \quad \text{for every } \varphi \in \mathcal{K}_p^0, \end{cases} \tag{3.2}$$

where

$$\mathcal{K}_p^0 := \{u \in \mathcal{H}_p : \ \gamma_p^+(u) + \gamma_p^-(u) = 0\}. \tag{3.3}$$

Proof. From (2.17) and the continuity of the embedding $H^{1/2}(\Gamma_p) \hookrightarrow L^2(\Gamma_p)$, we have that there exists $C > 0$ independent of $p \in \mathbb{S}^1_+$ such that, for all $u \in \mathcal{H}_p$,

$$\begin{aligned} \left| \int_{\Gamma_p} \frac{\partial^\pm \psi_j}{\partial \nu_p^\pm} \gamma_p^\pm(u) \, ds \right| &= \left| j \cos\left(j\left(\tfrac{\pi}{2} - \alpha\right)\right) \int_{\Gamma_p} |x|^{j-1} \gamma_p^\pm(u) \, ds \right| \\ &\le j \int_{\Gamma_p} |\gamma_p^\pm(u)| \, ds \le j \|\gamma_p^\pm(u)\|_{L^2(\Gamma_p)} \le C \|\gamma_p^\pm(u)\|_{H^{1/2}(\Gamma_p)} \le CL \|u\|_{\mathcal{H}_p} \end{aligned}$$

and then, from the elementary inequality $ab \le \frac{a^2}{4\varepsilon} + \varepsilon b^2$, we deduce that, for every $\varepsilon > 0$, there exists a constant $C_\varepsilon > 0$ (depending on ε but independent of p) such that, for every $u \in \mathcal{H}_p$,

$$\left| \int_{\Gamma_p} \frac{\partial^\pm \psi_j}{\partial \nu_p^\pm} \gamma_p^\pm(u) \, ds \right| \le \varepsilon \|u\|^2_{\mathcal{H}_p} + C_\varepsilon. \tag{3.4}$$

This implies that J_p is coercive in $\mathcal{H}_p$. Furthermore $\mathcal{K}_p$ is convex and closed by the continuity of the trace operators. Hence, via standard minimization methods, J_p achieves its minimum over $\mathcal{K}_p$ at some function $w_p \in \mathcal{K}_p$. The Euler-Lagrange equation for w_p is (3.2).

In order to prove uniqueness, let us assume that w_p and v_p solve (3.2). Then $w_p - v_p \in \mathcal{K}_p^0$ and, taking the difference between the equations (3.2) for w_p and v_p, we have that $w_p - v_p$ satisfies

$$\int_{\mathbb{R}^2_+ \setminus \Gamma_p} \nabla(w_p - v_p) \cdot \nabla \varphi \, dx = 0, \quad \text{for every } \varphi \in \mathcal{K}_p^0,$$

which, choosing $\varphi = w_p - v_p$ yields that $\int_{\mathbb{R}^2_+ \setminus \Gamma_p} |\nabla(w_p - v_p)|^2 \, dx = 0$ so that $w_p \equiv v_p$. ☐

Proposition 3.2. (i) *For every* $p \in \mathbb{S}^1_+$, *the function* Ψ_p *defined in* (2.24) *satisfies the following properties:*

$$\Psi_p \in H^{1,p}(D_r^+, \mathbb{C}) \text{ for all } r > 1; \tag{3.5}$$

$$\begin{cases} (i\nabla + A_p)^2 \Psi_p = 0, & \text{in } \mathbb{R}^2_+ \text{ in a weak } H^{1,p} - \text{sense}, \\ \Psi_p = 0, & \text{on } \partial \mathbb{R}^2_+; \end{cases} \tag{3.6}$$

$$\int_{\mathbb{R}^2_+ \setminus \Gamma_p} \left| (i\nabla + A_p)(\Psi_p - e^{\frac{i}{2}(\theta_p - \theta_0^p + \tilde\theta_0)} \psi_j) \right|^2 dx < +\infty; \tag{3.7}$$

$$e^{\frac{i}{2}(\theta_p - \theta_0^p + \tilde\theta_0)} w_p = \Psi_p(x) - e^{\frac{i}{2}(\theta_p - \theta_0^p + \tilde\theta_0)} \psi_j(x) = O(|x|^{-1}), \quad \text{as } |x| \to +\infty. \tag{3.8}$$

(ii) *The function* Ψ_p *defined in* (2.24) *is the unique function satisfying* (3.5), (3.6) *and* (3.7).

Proof. The fact that $w_p \in \mathcal{K}_p$ and the relation

$$\mathcal{R}_p^\pm(\theta_p - \theta_0^p)\Big|_{\Gamma_p} = \pm\pi$$

imply that

$$\gamma_p^+(\Psi_p) = \gamma_p^-(\Psi_p).$$

As a consequence we have that $(i\nabla + A_p)\Psi_p$ (meant as a distribution in $\mathbb{R}^2_+$) is equal to the $L^2_{\rm loc}(\mathbb{R}^2_+, \mathbb{C})$-function $ie^{\frac{i}{2}(\theta_p - \theta_0^p + \tilde\theta_0)}\nabla_{\mathbb{R}^2_+\setminus\Gamma_p}(w_p + \psi_j)$, thus yielding (3.5).

In order to prove (3.6), we observe that, for any $\varphi \in C^\infty_{\rm c}(\mathbb{R}^2_+ \setminus \{p\})$, we have that $\tilde\varphi := e^{-\frac{i}{2}(\theta_p - \theta_0^p + \tilde\theta_0)}\varphi \in \mathcal{K}_p^0$ (as defined in (3.3)). Hence, by (3.2),

$$\int_{\mathbb{R}^2_+} (i\nabla + A_p)\Psi_p \cdot \overline{(i\nabla + A_p)\varphi}\, dx = \int_{\mathbb{R}^2_+\setminus\Gamma_p} i\, e^{\frac{i}{2}(\theta_p - \theta_0^p + \tilde\theta_0)}\nabla(w_p + \psi_j)\cdot\Big(-ie^{-\frac{i}{2}(\theta_p - \theta_0^p + \tilde\theta_0)}\nabla\tilde\varphi\Big)\, dx$$

$$= \int_{\mathbb{R}^2_+\setminus\Gamma_p} \nabla(w_p + \psi_j)\cdot\nabla\tilde\varphi\, dx = -2\int_{\Gamma_p} \frac{\partial^+\psi_j}{\partial\nu_p^+}\gamma_p^+(\tilde\varphi)\, ds + \int_{\mathbb{R}^2_+\setminus\Gamma_p} \nabla\psi_j\cdot\nabla\tilde\varphi\, dx. \tag{3.9}$$

Testing the equation $-\Delta\psi_j = 0$ by $\tilde\varphi$ and integrating by parts in $\{(x_1, x_2) \in \mathbb{R}^2_+ : x_2 < x_1 \tan\alpha\}$ and in $\{(x_1, x_2) \in \mathbb{R}^2_+ : x_2 > x_1 \tan\alpha\}$ respectively, we obtain that the right hand side of (3.9) is equal to zero. This proves (3.6).

Property (3.7) is a straightforward consequence of the fact that $w_p \in \mathcal{H}_p$. To prove (3.8), we observe that the Kelvin transform of w_p, i.e. the function $\widetilde{w}_p(x) = w(\frac{x}{|x|^2})$ belongs to $H^1(D_1^+)$, vanishes in $\partial\mathbb{R}^2_+ \cap D_1$, and weakly satisfies $-\Delta\widetilde{w}_p = 0$ in D_1^+. Then from [10] and [14] (see also [8]) we deduce that $\widetilde{w}_p = O(|x|)$ as $|x| \to 0$ and hence $w_p = O(|x|^{-1})$ as $|x| \to +\infty$.

Finally, to prove (ii), let us consider some $\Psi \in \bigcup_{r>1} H^{1,p}(D_r^+, \mathbb{C})$ weakly satisfying

$$\begin{cases} (i\nabla + A_p)^2\Psi = 0, & \text{in } \mathbb{R}^2_+, \\ \Psi = 0, & \text{on } \partial\mathbb{R}^2_+, \end{cases}$$

and

$$\int_{\mathbb{R}^2_+\setminus\Gamma_p} |(i\nabla + A_p)(\Psi - e^{\frac{i}{2}(\theta_p - \theta_0^p + \tilde\theta_0)}\psi_j)|^2 < +\infty. \tag{3.10}$$

Then the difference $\Phi = \Psi - \Psi_p$ weakly solves $(i\nabla + A_p)^2\Phi = 0$ in $\mathbb{R}^2_+$ and $\Phi = 0$ on $\partial\mathbb{R}^2_+$. Moreover from (3.7) and (3.10) it follows that

$$\int_{\mathbb{R}^2_+} |(i\nabla + A_p)\Phi(x)|^2 dx < +\infty,$$

which, in view of (3.6) and (A.2), implies that $\int_{\mathbb{R}^2_+} |x - p|^{-2}|\Phi(x)|^2\, dx = 0$. Hence $\Phi \equiv 0$ in $\mathbb{R}^2_+$ and $\Psi = \Psi_p$. □

Remark 3.3. *Since Ψ_p solves (3.6), classical regularity theory yields that $\Psi_p \in C^\infty(\overline{\mathbb{R}^2_+}\setminus\{p\}, \mathbb{C})$, whereas from [9] it follows that $\Psi_p(x) = O(|x - p|^{1/2})$ and $\nabla\Psi_p(x) = O(|x - p|^{-1/2})$ as $x \to p$. Therefore we have that $w_p \in C^\infty(\overline{U^\pm} \setminus \{p\})$ with $U^+ = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_2 > x_1 \tan\alpha\}$ and $U^- = \{(x_1, x_2) \in \mathbb{R}^2_+ : x_2 < x_1 \tan\alpha\}$, and that $|\nabla w_p(x)| = O(|x - p|^{-1/2})$. Then*

$$\frac{\partial^\pm w_p}{\partial\nu_p^\pm} \in L^q(\Gamma_p) \quad \text{and} \quad \frac{\partial w_p}{\partial\nu} \in L^q(\partial D_1 \cap \mathbb{R}^2_+) \quad \text{for all } q < 2,$$

where $\nu(x) = \frac{x}{|x|}$ denotes the unit normal vector to ∂D_1. Using a simple approximation argument and recalling that $H^{1/2}(\Gamma_p) \hookrightarrow L^q(\Gamma_p)$ for all $q \geq 1$, we obtain the following formulas for integration by parts:

$$\int_{\mathbb{R}^2_+\setminus\Gamma_p} \nabla w_p\cdot\nabla\varphi\, dx = \int_{\Gamma_p} \frac{\partial^+ w_p}{\partial\nu_p^+}\gamma_p^+(\varphi)\, ds + \int_{\Gamma_p} \frac{\partial^- w_p}{\partial\nu_p^-}\gamma_p^-(\varphi)\, ds, \tag{3.11}$$

for all $\varphi \in \mathcal{H}_p$ *and*

$$\int_{D_1^+\setminus\Gamma_p} \nabla w_p \cdot \nabla \varphi \, dx = \int_{\partial D_1^+} \frac{\partial w_p}{\partial \nu} \varphi \, ds + \int_{\Gamma_p} \frac{\partial^+ w_p}{\partial \nu_p^+} \gamma_p^+(\varphi) \, ds + \int_{\Gamma_p} \frac{\partial^- w_p}{\partial \nu_p^-} \gamma_p^-(\varphi) \, ds, \tag{3.12}$$

for all $\varphi \in H^1(D_1^+ \setminus \Gamma_p)$ *such that* $\varphi = 0$ *on* $\partial\mathbb{R}^2_+$.

Remark 3.4. *In view of* (3.11)*, the weak problem* (3.2) *solved by* w_p *can be reformulated as an elliptic problem with jump conditions on the internal crack* Γ_p *as follows:*

$$\begin{cases} -\Delta w_p = 0, & \text{in } \mathbb{R}^2_+ \setminus \Gamma_p, & (3.13)\\ \gamma_p^+(w_p + \psi_j) + \gamma_p^-(w_p + \psi_j) = 0, & \text{on } \Gamma_p, & (3.14)\\ \dfrac{\partial^+(w_p + \psi_j)}{\partial \nu_p^+} - \dfrac{\partial^-(w_p + \psi_j)}{\partial \nu_p^-} = 0, & \text{on } \Gamma_p, & (3.15) \end{cases}$$

where the equality in (3.15) *is meant in the sense of* $L^q(\Gamma_p)$ *for any* $q < 2$ *(see Remark 3.3) and hence almost everywhere. We refer to* [19] *for elliptic problems in cracked domains with jumps of the unknown function and its normal derivative prescribed on the cracks.*

The following result provides a characterization of $\mathfrak{m}_p$ as a Fourier coefficient of w_p. It will be used to relate $\mathfrak{m}_p$ with the optimal lower/upper bounds for $\lambda_N - \lambda_N^a$, see Lemmas 7.4 and 10.1.

Proposition 3.5. *For every* $p \in \mathbb{S}^1_+$*, let*

$$\omega_p(r) := \int_{-\pi/2}^{\pi/2} w_p(r\cos t, r\sin t) \sin\left(j\left(\tfrac{\pi}{2} - t\right)\right) dt, \quad r \geq 1, \tag{3.16}$$

with w_p *defined in* (2.20)*. Then*

$$\omega_p(r) = \omega_p(1) r^{-j} \quad \text{for all } r \geq 1 \quad \text{and} \quad \mathfrak{m}_p = -j\omega_p(1).$$

Proof. By direct calculations, since $-\Delta w_p = 0$ in $\mathbb{R}^2_+ \setminus D_1^+$, we have that ω_p satisfies

$$-(r^{1+2j}(r^{-j}\omega_p(r))')' = 0, \quad \text{for } r > 1.$$

Hence there exists a constant $C \in \mathbb{R}$ such that

$$r^{-j}\omega_p(r) = \omega_p(1) + \frac{C}{2j}\left(1 - \frac{1}{r^{2j}}\right), \quad \text{for all } r \geq 1.$$

From (3.8) it follows that $\omega_p(r) = O(r^{-1})$ as $r \to +\infty$. Hence, letting $r \to +\infty$ in the previous relation, we find $C = -2j\omega_p(1)$, so that $\omega_p(r) = \omega_p(1)r^{-j}$ for all $r \geq 1$. By taking the derivative in this relation and in the definition of ω_p (3.16), we obtain

$$-j\omega_p(1) = \int_{\partial D_1^+} \frac{\partial w_p}{\partial \nu} \psi_j \, ds. \tag{3.17}$$

Choosing $\varphi = \psi_j$ in (3.12) and then replacing (3.17), we obtain

$$\begin{aligned} \int_{D_1^+\setminus\Gamma_p} \nabla w_p \cdot \nabla \psi_j \, dx &= \int_{\partial D_1^+} \frac{\partial w_p}{\partial \nu} \psi_j \, ds + \int_{\Gamma_p} \left(\frac{\partial^+ w_p}{\partial \nu_p^+} + \frac{\partial^- w_p}{\partial \nu_p^-}\right) \psi_j \, ds \\ &= -j\omega_p(1) + \int_{\Gamma_p} \left(\frac{\partial^+ w_p}{\partial \nu_p^+} + \frac{\partial^- w_p}{\partial \nu_p^-}\right) \psi_j \, ds. \end{aligned} \tag{3.18}$$

Testing the equation $-\Delta\psi_j = 0$ by w_p and integrating by parts in $D_1^+ \setminus \Gamma_p$, we arrive at

$$\begin{aligned} \int_{D_1^+\setminus\Gamma_p} \nabla w_p \cdot \nabla \psi_j \, dx &= \int_{\partial D_1^+} \frac{\partial \psi_j}{\partial \nu} w_p \, ds + \int_{\Gamma_p} \frac{\partial^+ \psi_j}{\partial \nu_p^+} (\gamma_p^+(w_p) - \gamma_p^-(w_p)) \, ds \\ &= j\omega_p(1) + \int_{\Gamma_p} \frac{\partial^+ \psi_j}{\partial \nu_p^+} (\gamma_p^+(w_p) - \gamma_p^-(w_p)) \, ds, \end{aligned} \tag{3.19}$$

where in the last step we used the fact that $\frac{\partial\psi_j}{\partial\nu} = j\psi_j$ on ∂D_1^+. By combining (3.18) and (3.19), we arrive at

$$j\omega_p(1) = \frac{1}{2}\int_{\Gamma_p}\left(\frac{\partial^+ w_p}{\partial\nu_p^+} + \frac{\partial^- w_p}{\partial\nu_p^-}\right)\psi_j\,ds - \frac{1}{2}\int_{\Gamma_p}\frac{\partial^+\psi_j}{\partial\nu_p^+}(\gamma_p^+(w_p) - \gamma_p^-(w_p))\,ds. \tag{3.20}$$

On the other hand, taking $\varphi = w_p$ in (3.11), we obtain

$$\int_{\mathbb{R}^2_+\setminus\Gamma_p}|\nabla w_p|^2\,dx = \int_{\Gamma_p}\frac{\partial^+ w_p}{\partial\nu_p^+}\gamma_p^+(w_p)\,ds + \int_{\Gamma_p}\frac{\partial^- w_p}{\partial\nu_p^-}\gamma_p^-(w_p)\,ds,$$

which, by definition of $\mathfrak{m}_p$, yields

$$\mathfrak{m}_p = J_p(w_p) = \frac{1}{2}\int_{\Gamma_p}\left(\frac{\partial^+(w_p+\psi_j)}{\partial\nu_p^+}\gamma_p^+(w_p) + \frac{\partial^-(w_p+\psi_j)}{\partial\nu_p^-}\gamma_p^-(w_p)\right)ds$$
$$+ \frac{1}{2}\int_{\Gamma_p}\frac{\partial^+\psi_j}{\partial\nu_p^+}(\gamma_p^+(w_p) - \gamma_p^-(w_p))\,ds. \tag{3.21}$$

Moreover (3.14) and (3.15) imply that

$$\frac{\partial^+(w_p+\psi_j)}{\partial\nu_p^+}\gamma_p^+(w_p+\psi_j) + \frac{\partial^-(w_p+\psi_j)}{\partial\nu_p^-}\gamma_p^-(w_p+\psi_j) = 0 \quad \text{on } \Gamma_p. \tag{3.22}$$

Combining (3.21) and (3.22) we obtain

$$\mathfrak{m}_p = -\frac{1}{2}\int_{\Gamma_p}\left(\frac{\partial^+(w_p+\psi_j)}{\partial\nu_p^+} + \frac{\partial^-(w_p+\psi_j)}{\partial\nu_p^-}\right)\psi_j\,ds + \frac{1}{2}\int_{\Gamma_p}\frac{\partial^+\psi_j}{\partial\nu_p^+}(\gamma_p^+(w_p) - \gamma_p^-(w_p))\,ds. \tag{3.23}$$

Since ψ_j is regular, it satisfies (3.1). Then the statement follows by comparing (3.20) with (3.23). □

4. Properties of $\mathfrak{m}_p$

In this section we collect some properties of the map $\mathfrak{m}_p$ defined in (2.20). The next lemma ensures that $p \mapsto \mathfrak{m}_p$ is not the null function, by providing its sign when p belongs either to the bisector of two nodal lines of ψ_j, or to one of the nodal lines of ψ_j.

Lemma 4.1. (i) *If $p = (\cos\alpha, \sin\alpha)$ with $\alpha = \frac{\pi}{2} - (1+2k)\frac{\pi}{2j}$ for some $k = 0,\dots,j-1$, then*

$$\mathfrak{m}_p = \frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Gamma_p}|\nabla w_p|^2\,dx > 0.$$

(ii) *If $p = (\cos\alpha, \sin\alpha)$ with $\alpha = \frac{\pi}{2} - k\frac{\pi}{j}$ for some $k = 1,\dots,j-1$, then*

$$\mathfrak{m}_p = -\frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Gamma_p}|\nabla w_p|^2\,dx < 0.$$

Proof. (i) If $\alpha = \frac{\pi}{2} - (1+2k)\frac{\pi}{2j}$ for some $k = 0,\dots,j-1$, then $\partial^\pm\psi_j/\partial\nu_p^\pm = 0$ on Γ_p, so that $J_p(u) = \frac{1}{2}\|u\|^2_{\mathcal{H}_p}$; since in this case $0 \notin \mathcal{K}_p$ (since $\psi_j \not\equiv 0$ on Γ_p), we conclude that $\mathfrak{m}_p = \min_{\mathcal{K}_p} J_p > 0$.

(ii) In the second case we have that $\psi_j \equiv 0$ and $\frac{\partial^+\psi_j}{\partial\nu_p^+}(r\cos\alpha, r\sin\alpha) = j(-1)^k r^{j-1}$ on Γ_p, so that

$$J_p(u) = \frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Gamma_p}|\nabla u|^2\,dx + 2(-1)^k j\int_{\Gamma_p}|x|^{j-1}\gamma_p^+(u), \quad \text{for all } u \in \mathcal{K}_p. \tag{4.1}$$

From (4.1) it follows easily that $\mathfrak{m}_p = \min_{\mathcal{K}_p} J_p < 0$. Furthermore, in this case (3.23) is reduced to

$$\mathfrak{m}_p = \frac{1}{2}\int_{\Gamma_p} \frac{\partial^+ \psi_j}{\partial \nu_p^+}(\gamma_p^+(w_p) - \gamma_p^-(w_p))\,ds,$$

and hence, by definition of J_p and $\mathfrak{m}_p$,

$$\mathfrak{m}_p = \frac{1}{2}\left(\mathfrak{m}_p - \frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Gamma_p} |\nabla w_p|^2\,dx\right)$$

which yields that $\mathfrak{m}_p = -\frac{1}{2}\int_{\mathbb{R}^2_+\setminus\Gamma_p} |\nabla w_p|^2\,dx$.

□

The following proposition establishes the continuity of the map $p \mapsto \mathfrak{m}_p$.

Proposition 4.2. *The map $p \mapsto \mathfrak{m}_p$ is continuous in $\mathbb{S}^1_+$. Moreover, it can be extended continuously at $p = (0,1)$ and at $p = (0,-1)$ by letting $\mathfrak{m}_{(0,1)} = \mathfrak{m}_{(0,-1)} = 0$.*

Proof. First we claim that there exists $C > 0$ independent of p such that

$$\int_{\mathbb{R}^2_+\setminus\Gamma_p} |\nabla w_p|^2\,dx \le C \quad \text{for every } p \in \mathbb{S}^1_+. \tag{4.2}$$

To prove the claim, we consider a regular cut-off function η defined in $\mathbb{R}^2_+$ such that $\eta = 1$ in D_1^+ and $\eta = 0$ in $\mathbb{R}^2_+ \setminus D_2^+$. Then $-\eta\psi_j \in \mathcal{K}_p$ for every $p \in \mathbb{S}^1_+$ and

$$\mathfrak{m}_p \le J_p(-\eta\psi_j) = \frac{1}{2}\int_{\mathbb{R}^2_+} |\nabla(-\eta\psi_j)|^2\,dx.$$

This fact, together with the inequality (3.4) applied with $u = w_p$, provides (4.2).

Let $p_n = (\cos\alpha_n, \sin\alpha_n) \to p = (\cos\alpha, \sin\alpha)$ as $n \to +\infty$, for some $\alpha_n \in (-\pi/2, \pi/2)$, $\alpha \in [-\pi/2, \pi/2]$. We consider the rotation

$$\mathcal{R}_n = \begin{pmatrix} \cos(\alpha-\alpha_n) & -\sin(\alpha-\alpha_n) \\ \sin(\alpha-\alpha_n) & \cos(\alpha-\alpha_n) \end{pmatrix}.$$

With a slight abuse of notation, we denote by w_{p_n} the trivial extension of w_{p_n} in $\mathbb{R}^2$ (extended to 0 in the set $\mathbb{R}^2_- = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 < 0\}$) and we define the rotated functions

$$\tilde{w}_n(\mathcal{R}_n(x)) = w_{p_n}(x), \quad x \in \mathbb{R}^2.$$

We define the space $\tilde{\mathcal{H}}_p$ as the completion of

$$\Big\{u \in H^1(\mathbb{R}^2 \setminus \Gamma_p) : u = 0 \text{ on } (-\infty,0)\times\{0\} \text{ and } u = 0 \text{ in a neighborhood of } \infty\Big\}$$

with respect to the norm $\|u\|_{\tilde{\mathcal{H}}_p} = \|\nabla u\|_{L^2(\mathbb{R}^2\setminus\Gamma_p)}$.

We notice that, for all $p \in \mathbb{S}^1_+$, $\mathcal{H}_p = \{u \in \tilde{\mathcal{H}}_p : u = 0 \text{ a.e. in } \mathbb{R}^2_-\}$. For large n, we also define

$$\tilde{\mathcal{H}}_{p,n} = \{u \in \tilde{\mathcal{H}}_p : u = 0 \text{ a.e. in } H_n^-\},$$

where $H_n^- = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 < -\tan(\alpha-\alpha_n)x_2\}$, and observe that $\tilde{w}_n \in \tilde{\mathcal{H}}_{p,n}$.

Let $\tilde{\psi}_{j,n}(\mathcal{R}_n(x)) = \psi_j(x)$ and $H_n^+ = \{(x_1,x_2) \in \mathbb{R}^2 : x_1 > -\tan(\alpha-\alpha_n)x_2\}$. By (3.14) we have that

$$\gamma_p^+(\tilde{w}_n + \tilde{\psi}_{j,n}) + \gamma_p^-(\tilde{w}_n + \tilde{\psi}_{j,n}) = 0, \tag{4.3}$$

while from (3.2) it follows that

$$\int_{H_n^+\setminus\Gamma_p} \nabla\tilde{w}_n \cdot \nabla\varphi\,dx + 2\int_{\Gamma_p} \frac{\partial^+\tilde{\psi}_{j,n}}{\partial\nu_p^+}\gamma_p^+(\varphi)\,ds = 0, \tag{4.4}$$

for every $\varphi \in \tilde{\mathcal{K}}^0_{p,n} = \{u \in \tilde{\mathcal{H}}_{p,n} : \gamma_p^+(u) + \gamma_p^-(u) = 0\}$.

Moreover, from (4.2) it follows that

$$\|\tilde{w}_n\|^2_{\tilde{\mathcal{H}}_p} \le C,$$

hence there exist $\tilde{w}_p \in \tilde{\mathcal{H}}_p$ and a subsequence $\{\tilde{w}_{n_k}\}_k$ such that $\tilde{w}_{n_k} \rightharpoonup \tilde{w}_p$ weakly in $\tilde{\mathcal{H}}_p$ and a.e. in $\mathbb{R}^2$. By a.e. convergence, we have that $\tilde{w}_p = 0$ a.e. in $\mathbb{R}^2_-$, hence

$$\tilde{w}_p \in \mathcal{H}_p \text{ if } p \in \mathbb{S}^1_+ \quad \text{while} \quad \tilde{w}_p \in \mathcal{D}^{1,2}(\mathbb{R}^2_+) \text{ if } p = (0, \pm 1).$$

Moreover, (4.3) and the continuity of the trace embeddings $\gamma_p^\pm$ defined in (2.16) imply that $\gamma_p^+(\tilde{w}_p + \psi_j) + \gamma_p^-(\tilde{w}_p + \psi_j) = 0$, thus yielding

$$\tilde{w}_p \in \mathcal{K}_p.$$

Recall the definition of $\mathcal{K}_p^0$ in (3.3) and let

$$\varphi \in \mathcal{K}_p^0 \cap \{u \in C^\infty(\mathbb{R}^2_+ \setminus \Gamma_p) : \operatorname{supp}(u) \subset\subset \mathbb{R}^2_+\}; \tag{4.5}$$

then, for n sufficiently large, $\varphi \in \tilde{\mathcal{K}}^0_{p,n}$ (extended by 0 in H_n^-), so that (4.4) and the weak $\tilde{\mathcal{H}}_p$-convergence $\tilde{w}_{n_k} \rightharpoonup \tilde{w}_p$ provide

$$\int_{\mathbb{R}^2_+ \setminus \Gamma_p} \nabla \tilde{w}_p \cdot \nabla \varphi \, dx + 2 \int_{\Gamma_p} \frac{\partial^+ \psi_j}{\partial \nu_p^+} \gamma_p^+(\varphi) \, ds = 0.$$

Since the space defined in (4.5) is dense in $\mathcal{K}_p^0$, the previous relation holds for every $\varphi \in \mathcal{K}_p^0$. Hence $\tilde{w}_p$ satisfies (3.2) if $p \in \mathbb{S}^1_+$, while $\tilde{w}_p$ satisfies $-\Delta \tilde{w}_p = 0$ weakly in $\mathbb{R}^2_+$ if $p = (0, \pm 1)$. Then the uniqueness result proved in Lemma 3.1 implies that

$$\tilde{w}_p = w_p \text{ if } p \in \mathbb{S}^1_+ \quad \text{and} \quad \tilde{w}_p = 0 \text{ if } p = (0, \pm 1).$$

From Proposition 3.5 we have that

$$\begin{aligned} \mathfrak{m}_{p_{n_k}} &= -j \int_{-\pi/2}^{\pi/2} w_{p_{n_k}}(\cos t, \sin t) \sin\left(j\left(\tfrac{\pi}{2} - t\right)\right) dt \\ &= -j \int_{-\pi/2+\alpha-\alpha_{n_k}}^{\pi/2+\alpha-\alpha_{n_k}} \tilde{w}_{n_k}(\cos t, \sin t) \sin\left(j\left(\tfrac{\pi}{2} - t + \alpha - \alpha_{n_k}\right)\right) dt. \end{aligned} \tag{4.6}$$

The weak $\tilde{\mathcal{H}}_p$-convergence $\tilde{w}_{n_k} \rightharpoonup \tilde{w}_p$ and continuity of the trace embedding $\tilde{\mathcal{H}}_p \hookrightarrow L^2(\partial D_1)$ allow passing to the limit in (4.6) thus yielding that

$$\lim_{k\to\infty} \mathfrak{m}_{p_{n_k}} = -j \int_{-\pi/2}^{\pi/2} w_p(\cos t, \sin t) \sin\left(j\left(\tfrac{\pi}{2} - t\right)\right) dt = \mathfrak{m}_p \quad \text{if } p \in \mathbb{S}^1_+$$

and

$$\lim_{k\to\infty} \mathfrak{m}_{p_{n_k}} = 0 \quad \text{if } p = (0, \pm 1).$$

By the Urysohn property, we conclude that $\lim_{n\to\infty} \mathfrak{m}_{p_n} = \mathfrak{m}_p$ if $p \in \mathbb{S}^1_+$ and $\lim_{n\to\infty} \mathfrak{m}_{p_n} = 0$ if $p = (0, \pm 1)$. □

5. Monotonicity formula and local energy estimates

For $1 \le k \le N$ and $a \in \Omega$, let φ_k^a be an eigenfunction of problem (E_a) related to the eigenvalue λ_k^a. More precisely, let φ_k^a solve

$$\begin{cases} (i\nabla + A_a)^2 \varphi_k^a = \lambda_k^a q(x) \varphi_k^a, & \text{in } \Omega, \\ \varphi_k^a = 0, & \text{on } \partial\Omega, \end{cases} \tag{5.1}$$

and satisfy the orthonormality conditions

$$\int_\Omega q(x) |\varphi_k^a(x)|^2 \, dx = 1 \quad \text{and} \quad \int_\Omega q(x)\, \varphi_k^a(x) \overline{\varphi_\ell^a(x)} \, dx = 0 \text{ if } k \ne \ell. \tag{5.2}$$

For $k = N$ we choose φ_N^a being as in (2.23). From (2.7), (2.8), (2.9), (2.22), (2.23), (A.1), and standard elliptic estimates, we can deduce that

$$(i\nabla + A_a)\varphi_N^a \to (i\nabla + A_0)\varphi_N^0 \quad \text{in } L^2(\Omega, \mathbb{C}^2) \tag{5.3}$$

and

$$\varphi_N^a \to \varphi_N^0 \quad \text{in } H^1(\Omega, \mathbb{C}) \text{ and in } C^2_{\text{loc}}(\Omega, \mathbb{C}). \tag{5.4}$$

The asymptotic behavior of the eigenfunctions φ_k^a, for $1 \leq k \leq N$, close to the singular point a was studied in [9, Theorem 1.3], [11, Theorem 2.1]; in particular it is known that there exist coefficients $c_{a,k}, d_{a,k} \in \mathbb{C}$ such that

$$\varphi_k^a(a + (r\cos t, r\sin t)) = r^{1/2}\frac{e^{it/2}}{\sqrt{\pi}}\Big(c_{a,k}\cos\big(\tfrac{t}{2}\big) + d_{a,k}\sin\big(\tfrac{t}{2}\big)\Big) + o(r^{1/2}), \quad \text{as } r \to 0.$$

To derive energy estimates for the eigenfunctions φ_k^a in neighborhoods of 0 with size $|a|$, we use a monotonicity argument based on the study of an Almgren-type frequency function in the spirit of [3].

5.1. **Almgren-type frequency function.**

Definition 5.1. *Recall the definition of $\bar{R}$ in* (2.1)*. Let $\lambda \in \mathbb{R}$, $b \in \mathbb{R}^2_+$ and $u \in H^{1,b}(D^+_{\bar{R}}, \mathbb{C})$, with $u = 0$ on $\{x_1 = 0\}$. For any $|b| < r < \bar{R}$, we define the Almgren-type frequency function as*

$$\mathcal{N}(u, r, \lambda, A_b) = \frac{E(u, r, \lambda, A_b)}{H(u, r)},$$

where

$$E(u, r, \lambda, A_b) = \int_{D_r^+} |(i\nabla + A_b)u|^2\,dx - \lambda\int_{D_r^+} q(x)\,|u|^2\,dx, \quad H(u, r) = \frac{1}{r}\int_{\partial D_r^+} |u|^2\,ds. \tag{5.5}$$

We first prove that the frequency function of the eigenfunctions (5.1) is well defined in a suitable interval. To this aim, we observe that, since $a \in \Omega \mapsto \lambda_k^a$ admits a continuous extension on $\overline{\Omega}$ as proved in [21, Theorem 1.1], we have that

$$\Lambda := \sup_{\substack{a\in\overline{\Omega} \\ 1\leq k\leq N}} \lambda_k^a \in (0, +\infty). \tag{5.6}$$

Lemma 5.2. (i) *There exists $0 < R_0 < \min\{\bar{R}, (2\Lambda\|q\|_\infty)^{-1/2})\}$ such that $H(\varphi_k^a, r) > 0$ for all $|a| < R_0$, $r \in (|a|, R_0]$ and $1 \leq k \leq N$.*

(ii) *For every $r \in (0, R_0]$, there exist $C_r > 0$ and $\alpha_r \in (0, r)$ such that $H(\varphi_k^a, r) \geq C_r$ for all $|a| < \alpha_r$ and $1 \leq k \leq N$.*

Proof. We skip the proof of (i), which is very similar to that of [2, Lemma 5.2]. In order to prove (ii), suppose by contradiction that there exist $0 < r \leq R_0$, $a_n \in \Omega$ with $a_n \to 0$, $k_n \in \{1, \ldots, N\}$ such that

$$\lim_{n\to+\infty} H(\varphi_{k_n}^{a_n}, r) = 0.$$

From (5.1), (5.2) and (5.6) we deduce that

$$\int_\Omega |(i\nabla + A_{a_n})\varphi_{k_n}^{a_n}|^2\,dx = \lambda_{k_n}^{a_n} \leq \Lambda,$$

so that, by the Hardy-type inequality (A.1),

$$\|\varphi_{k_n}^{a_n}\|_{H_0^1(\Omega,\mathbb{C})} \leq C,$$

for a constant C independent of n. Then, along a subsequence, $\lambda_{k_n}^{a_n} \to \lambda \in \mathbb{R}$ and $\varphi_{k_n}^{a_n} \to \varphi$ a.e., weakly in $H_0^1(\Omega, \mathbb{C})$ and strongly in $L^2(\Omega, \mathbb{C})$, for some $\varphi \in H_0^1(\Omega, \mathbb{C})$. From (5.2) we have that $\int_\Omega q(x)|\varphi(x)|^2\,dx = 1$ and then $\varphi \not\equiv 0$.

By (2.5), $\varphi \in H_0^{1,0}(\Omega, \mathbb{C})$. We notice that $A_{a_n}\varphi_{k_n}^{a_n} \to A_0\varphi$ a.e. and, in view of (A.1),

$$\|A_{a_n}\varphi_{k_n}^{a_n}\|^2_{L^2(\Omega,\mathbb{C}^2)} \leq 4\int_\Omega |(i\nabla + A_{a_n})\varphi_{k_n}^{a_n}|^2\,dx \leq 4\Lambda.$$

Therefore, up to a subsequence, $A_{a_n}\varphi_{k_n}^{a_n} \rightharpoonup A_0\varphi$ weakly in $L^2(\Omega, \mathbb{C}^2)$. Then we can pass to the limit in (5.1), so that $\lambda = \lambda_{k_0}$ for some $k_0 \in \{1, \dots, N\}$ and

$$(i\nabla + A_0)^2\varphi = \lambda_{k_0} q(x)\varphi \quad \text{in } \Omega. \tag{5.7}$$

Furthermore, by compactness of the trace embedding $H^1(D_r^+, \mathbb{C}) \hookrightarrow L^2(\partial D_r^+, \mathbb{C})$, we have that

$$0 = \lim_{n\to\infty} \frac{1}{r}\int_{\partial D_r^+} |\varphi_{k_n}^{a_n}|^2\,ds = \frac{1}{r}\int_{\partial D_r^+} |\varphi|^2\,ds,$$

which implies that $\varphi = 0$ on ∂D_r^+. By testing (5.7) by φ in D_r^+, in view of Lemma A.1, we obtain that

$$0 = \int_{D_r^+}\Big(|(i\nabla + A_0)\varphi|^2 - \lambda_{k_0} q(x)|\varphi|^2\Big)\,dx \geq (1 - \Lambda\|q\|_\infty r^2)\int_{D_r^+} |(i\nabla + A_0)\varphi|^2\,dx.$$

Since $r \leq R_0 < (2\Lambda\|q\|_\infty)^{-1/2}$, we deduce that $\int_{D_r^+} |(i\nabla + A_0)\varphi|^2\,dx = 0$. Lemma A.1 then implies that $\varphi \equiv 0$ in D_r^+. From the unique continuation principle (see [9, Corollary 1.4]) we conclude that $\varphi \equiv 0$ in Ω, thus giving rise to a contradiction. □

In the following we let

$$0 < R_0 < \min\{\bar{R}, (2\Lambda\|q\|_\infty)^{-1/2})\}$$

be such that Lemma 5.2 (i) holds. As a consequence of Lemma 5.2 we have that the function $r \mapsto \mathcal{N}(\varphi_k^a, r, \lambda_k^a, A_a)$ is well defined in the interval $(|a|, R_0]$ for all $|a| < R_0$ and $1 \leq k \leq N$.

We recall some results proved in [21], which will be used in the sequel.

Lemma 5.3 ([21, Lemma 5.2])**.** *For all $1 \leq k \leq N$ and $a \in \Omega$, let φ_k^a be as in (5.1)–(5.2). Then*

$$\frac{1}{H(\varphi_k^a, r)}\frac{d}{dr}H(\varphi_k^a, r) = \frac{2}{r}\mathcal{N}(\varphi_k^a, r, \lambda_k^a, A_a) \quad \textit{for all } |a| < r < R_0. \tag{5.8}$$

Lemma 5.4 ([21, Lemma 5.3])**.** *Let $1 \leq k \leq N$ and $r_0 \leq R_0$. If $|a| \leq r_1 < r_2 \leq r_0$, then*

$$\frac{H(\varphi_k^a, r_2)}{H(\varphi_k^a, r_1)} \geq e^{-2\Lambda\|q\|_\infty r_0^2}\left(\frac{r_2}{r_1}\right)^2.$$

The formula for the derivative of $E(\varphi_k^a, r, \lambda_k^a, A_a)$ presents some differences with respect to [21], since in [21] the integrals in (5.5) were taken over half-balls centered at the projection of a on $\partial\mathbb{R}^2_+$.

Lemma 5.5. *Let $p \in \mathbb{S}^1_+$, $1 \leq k \leq N$ and $a = |a|p$. Then, for all $|a| < r \leq R_0$,*

$$\frac{d}{dr}E(\varphi_k^a, r, \lambda_k^a, A_a) = 2\int_{\partial D_r^+} |(i\nabla + A_a)\varphi_k^a \cdot \nu|^2\,ds - \frac{\lambda_k^a}{r}\int_{D_r^+} |\varphi_k^a|^2(2q + \nabla q \cdot x)\,dx - \frac{2}{r}M_k^a,$$

where $\nu(x) = \frac{x}{|x|}$ denotes the unit normal vector to ∂D_r and

$$M_k^a = \frac{1}{4}\Big(a_1(c_{a,k}^2 - d_{a,k}^2) + 2a_2 c_{a,k} d_{a,k}\Big).$$

Furthermore, there exists $C > 0$ depending on $p \in \mathbb{S}^1_+$ such that, for all $\mu \geq 2$,

$$\frac{|M_k^a|}{H(\varphi_k^a, \mu|a|)} \leq \frac{C}{\mu^2}. \tag{5.9}$$

Proof. The expression of M_k^a follows by a Pohozaev-type identity in D_r^+, proceeding as in [21, Lemmas 5.5-5.7]. Next, in the same spirit as in [21, Lemmas 5.7-5.8], we can relate the value M_k^a to the function $v(y) = \varphi_k^a(|a|y^2 + a)$ defined in $\tilde{\Omega} := \{y \in \mathbb{C} : |a|y^2 + a \in D_{2|a|}^+\}$. Such a domain is fixed (with respect to $|a|$, but depends on p), since $a = |a|p$ is moving on a straight line. Therefore, we proceed exactly in the same way as in the proofs therein and obtain a bound depending on p

$$\frac{|M_k^a|}{H(\varphi_k^a, 2|a|)} \leq C.$$

Expression (5.9) follows from Lemma 5.4. □

Lemma 5.6 ([21, Lemma 5.11])**.** *Let $1 \leq k \leq N$, $p \in \mathbb{S}^1_+$, and $r_0 \leq R_0$. There exists $c_{r_0,p}$ such that, for all $\mu > 2$, $a = |a|p$ with $|a| < r_0/\mu$, and $\mu|a| \leq r < r_0$,*

$$e^{\frac{\Lambda\|2q+\nabla q\cdot x\|_\infty}{2-2\Lambda r_0{}^2\|q\|_\infty}r^2}\left(\mathcal{N}(\varphi_k^a, r, \lambda_k^a, A_a) + 1\right) \leq e^{\frac{\Lambda\|2q+\nabla q\cdot x\|_\infty}{2-2\Lambda r_0{}^2\|q\|_\infty}r_0^2}\left(\mathcal{N}(\varphi_k^a, r_0, \lambda_k^a, A_a) + 1\right) + \frac{c_{r_0,p}}{\mu^2}.$$

Proof. The proof proceeds as in [21, Lemma 5.11] (see also [2, Lemma 5.6]), where we can replace a_1 with $|a|$ thanks to Lemma 5.5 above. □

Lemma 5.7. *For every $\delta \in (0, 1/4)$ and $p \in \mathbb{S}^1_+$ there exist $r_\delta > 0$ and $K_{\delta,p} > 2$ such that, if $\mu \geq K_{\delta,p}$, $a = |a|p$ with $|a| < r_\delta/\mu$, and $\mu|a| \leq r < r_\delta$, then $\mathcal{N}(\varphi_N^a, r, \lambda_N^a, A_a) \leq j + \delta$.*

Proof. Let $m > 0$ be sufficiently small so that $m(2 + j + m/2) < 1/2$. By assumption (2.10) and by (2.25) we have that

$$\lim_{r\to 0^+} \mathcal{N}(\varphi_N^0, r, \lambda_N^0, A_0) = j,$$

hence we can choose $r_\delta > 0$ sufficiently small so that

$$r_\delta < R_0, \quad e^{\frac{\Lambda\|2q+\nabla q\cdot x\|_\infty}{2-2\Lambda r_\delta{}^2\|q\|_\infty}r_\delta^2} \leq 1 + \delta m, \quad \mathcal{N}(\varphi_N^0, r_\delta, \lambda_N^0, A_0) < j + \delta m.$$

By (5.3)–(5.4) there exists $\alpha_\delta > 0$ such that $\mathcal{N}(\varphi_N^a, r_\delta, \lambda_N^a, A_a) < j + \delta m$ for every a with $|a| < \alpha_\delta$. We apply Lemma 5.6 with $r_0 = r_\delta$ and $k = N$, to deduce that for every $\mu > 2$, $|a| < \min\{\alpha_\delta, \frac{r_\delta}{\mu}\}$ and $\mu|a| < r < r_\delta$ it holds

$$\begin{aligned}\mathcal{N}(\varphi_N^a, r, \lambda_N^a, A_a) + 1 &\leq (1 + \delta m)(1 + j + \delta m) + \frac{c_{r_\delta,p}}{\mu^2} \\ &\leq 1 + j + \delta m(2 + j + \delta m) + \frac{c_{r_\delta,p}}{\mu^2} < 1 + j + \frac{\delta}{2} + \frac{c_{r_\delta,p}}{\mu^2}.\end{aligned}$$

To conclude the proof it is sufficient to choose $K_{\delta,p} > \max\left\{2, \left(2c_{r_\delta,p}/\delta\right)^{1/2}, r_\delta/\alpha_\delta\right\}$. □

5.2. **Local energy estimates.** Let us fix $\delta \in (0, 1/4)$ and $p \in \mathbb{S}^1_+$, and let

$$\bar{r} = r_\delta > 0 \quad \text{and} \quad \bar{K} = K_{\delta,p} > 2 \tag{5.10}$$

be as in Lemma 5.7. For all $a \in \Omega$ such that $a = |a|p$ and $|a| < \bar{r}/\bar{K}$, we denote

$$H_a = H(\varphi_N^a, \bar{K}|a|).$$

As a direct corollary of Lemmas 5.3, 5.4, and 5.7, we obtain the following estimates for H_a.

Corollary 5.8. *There exists $C > 0$ independent of $|a|$ such that*

$$H_a \geq C|a|^{2(j+\delta)}, \quad \textit{if } |a| < \min\left\{\frac{\bar{r}}{\bar{K}}, \alpha_{\bar{r}}\right\}, \tag{5.11}$$

$$H_a = O(|a|^2) \quad \textit{as } |a| \to 0, \tag{5.12}$$

with $\alpha_{\bar{r}}$ being as in Lemma 5.2, part (ii).

Proof. In view of Lemma 5.7, integration of (5.8) over the interval $(\bar K|a|,\bar r)$ yields

$$H_a \geq H(\varphi_N^a,\bar r)\left(\frac{\bar K|a|}{\bar r}\right)^{2(j+\delta)}, \quad \text{if } |a|<\min\left\{\frac{\bar r}{\bar K},\alpha_{\bar r}\right\}.$$

Then Lemma 5.2 (ii) provides (5.11). To prove (5.12) we notice that there exists $C>0$ such that

$$H_a \leq CH(\varphi_N^a,r_0)|a|^2,$$

because of Lemma 5.4, and moreover $\lim_{a\to 0} H(\varphi_N^a,r_0)\leq C$ because of (5.4). □

From the Poincaré type Lemmas A.1 and A.2, the scaling property of the Almgren-type frequency function $\mathcal{N}$, and Lemma 5.7, it follows that, for all $R\geq \bar K$, the family of functions

$$\{\tilde\varphi_a : a=|a|p,\ |a|<\tfrac{\bar r}{R}\} \text{ is bounded in } H^{1,p}(D_R^+,\mathbb{C}) \tag{5.13}$$

where

$$\tilde\varphi_a(x) := \frac{\varphi_N^a(|a|x)}{\sqrt{H_a}}, \tag{5.14}$$

see [2, Theorem 5.9] for details in a similar case. In particular, for all $R\geq\bar K$, we have that

$$\int_{D^+_{R|a|}} |(i\nabla + A_a)\varphi_N^a|^2 dx = O(H_a), \quad \text{as } |a|\to 0^+, \tag{5.15}$$

$$\int_{\partial D^+_{R|a|}} |\varphi_N^a|^2 dx = O(|a|H_a), \quad \text{as } |a|\to 0^+,$$

$$\int_{D^+_{R|a|}} |\varphi_N^a|^2 dx = O(|a|^2H_a), \quad \text{as } |a|\to 0^+. \tag{5.16}$$

Lemmas 5.4 and 5.6 imply the following local energy estimates for the eigenfunctions φ_k^a.

Lemma 5.9. *For $1\leq k\leq N$ and $a=|a|p\in\Omega$, let $\varphi_k^a\in H_0^{1,a}(\Omega,\mathbb{C})$ be a solution to* (5.1)–(5.2)*. Let R_0, α_{R_0} be as in Lemma 5.2. For every $\mu\geq\frac{R_0}{\alpha_{R_0}}$, $a=|a|p\in\Omega$ with $|a|<\frac{R_0}{\mu}$, and $1\leq k\leq N$, we have that*

$$\int_{\partial D^+_{\mu|a|}} |\varphi_k^a|^2\,ds \leq C(\mu|a|)^3, \tag{5.17}$$

$$\int_{D^+_{\mu|a|}} |(i\nabla + A_a)\varphi_k^a|^2\,dx \leq C(\mu|a|)^2, \tag{5.18}$$

$$\int_{D^+_{\mu|a|}} |\varphi_k^a|^2\,dx \leq C(\mu|a|)^4, \tag{5.19}$$

for some $C>0$ (depending on p).

Proof. From Lemma 5.6, it follows that, if $\mu>2$ and $|a|<\frac{R_0}{\mu}$ then, for all $1\leq k\leq N$,

$$\mathcal{N}(\varphi_k^a,\mu|a|,\lambda_k^a,A_a)\leq e^{\frac{\Lambda\|2q+\nabla q\cdot x\|_\infty}{2-2\Lambda R_0{}^2\|q\|_\infty}R_0^2}(\mathcal{N}(\varphi_k^a,R_0,\lambda_k^a,A_a)+1)+\frac{c_{R_0,p}}{\mu^2}-1. \tag{5.20}$$

From (5.1), (5.2), and (5.6) we deduce that

$$\int_{D^+_{R_0}} |(i\nabla + A_a)\varphi_k^a|^2\,dx \leq \int_\Omega |(i\nabla + A_a)\varphi_k^a|^2\,dx = \lambda_k^a\leq\Lambda. \tag{5.21}$$

Therefore, in view of Lemma 5.2, if $|a|<\alpha_{R_0}$,

$$\mathcal{N}(\varphi_k^a,R_0,\lambda_k^a,A_a) = \frac{\int_{D^+_{R_0}}|(i\nabla + A_a)\varphi_k^a|^2\,dx - \lambda_k^a\int_{D^+_{R_0}}q(x)|\varphi_k^a|^2\,dx}{H(\varphi_k^a,R_0)} \leq \frac{\Lambda}{C_{R_0}}. \tag{5.22}$$

Combining (5.20) and (5.22) we obtain that, if $\mu \geq \frac{R_0}{\alpha_{R_0}}$ and $|a| < \frac{R_0}{\mu}$, then

$$\int_{D^+_{\mu|a|}} |(i\nabla + A_a)\varphi_k^a|^2\,dx - \lambda_k^a \int_{D^+_{\mu|a|}} q(x)|\varphi_k^a|^2\,dx \leq \mathrm{const}\, H(\varphi_k^a, \mu|a|)$$

for some positive const > 0. Hence, from Lemmas A.1 and A.2,

$$(1 - 2\Lambda\|q\|_\infty \mu^2|a|^2) \int_{D^+_{\mu|a|}} |(i\nabla + A_a)\varphi_k^a|^2\,dx \leq \mathrm{const}\, H(\varphi_k^a, \mu|a|)$$

which implies

$$\int_{D^+_{\mu|a|}} |(i\nabla + A_a)\varphi_k^a|^2\,dx \leq \frac{\mathrm{const}}{1 - 2\Lambda\|q\|_\infty R_0^2} H(\varphi_k^a, \mu|a|). \tag{5.23}$$

From Lemma 5.4, it follows that, if $\mu \geq \frac{R_0}{\alpha_{R_0}}$ and $|a| < \frac{R_0}{\mu}$,

$$H(\varphi_k^a, \mu|a|) \leq e^{2\Lambda\|q\|_\infty R_0^2} \left(\frac{\mu|a|}{R_0}\right)^2 H(\varphi_k^a, R_0). \tag{5.24}$$

On the other hand, Lemma A.2 and (5.21) yield

$$H(\varphi_k^a, R_0) \leq \int_{D^+_{R_0}} |(i\nabla + A_a)\varphi_k^a|^2\,dx \leq \Lambda. \tag{5.25}$$

Estimate (5.17) follows combining (5.24), and (5.25), whereas estimate (5.18) follows from (5.23), (5.24), and (5.25). Finally, (5.19) can be deduced from (5.17), (5.18) and Lemma A.1. □

6. Upper bound for $\lambda_N - \lambda_N^a$: the Rayleigh quotient for λ_N

Let $R > 2$. For $|a|$ sufficiently small and $1 \leq k \leq N$, we define

$$v_{k,R,a} := \begin{cases} v_{k,R,a}^{ext}, & \text{in } \Omega \setminus D^+_{R|a|}, \\ v_{k,R,a}^{int}, & \text{in } D^+_{R|a|}, \end{cases} \quad k = 1, \dots, N, \tag{6.1}$$

where

$$v_{k,R,a}^{ext} := e^{\frac{i}{2}(\theta_0^a - \theta_a)}\varphi_k^a \quad \text{in } \Omega \setminus D^+_{R|a|},$$

with φ_k^a as in (5.1)–(5.2) and θ_a, θ_0^a as in (2.21), so that it solves

$$\begin{cases} (i\nabla + A_0)^2 v_{k,R,a}^{ext} = \lambda_k^a q\, v_{k,R,a}^{ext}, & \text{in } \Omega \setminus D^+_{R|a|}, \\ v_{k,R,a}^{ext} = e^{\frac{i}{2}(\theta_0^a - \theta_a)}\varphi_k^a & \text{on } \partial(\Omega \setminus D^+_{R|a|}), \end{cases}$$

whereas $v_{k,R,a}^{int}$ is the unique solution to the problem

$$\begin{cases} (i\nabla + A_0)^2 v_{k,R,a}^{int} = 0, & \text{in } D^+_{R|a|}, \\ v_{k,R,a}^{int} = e^{\frac{i}{2}(\theta_0^a - \theta_a)}\varphi_k^a, & \text{on } \partial D^+_{R|a|}. \end{cases}$$

It is easy to verify that $\dim\big(\mathrm{span}\{v_{1,R,a}, \dots, v_{N,R,a}\}\big) = N$.

Arguing as in [2, Theorem 6.1] and using estimates (5.17)–(5.19), we obtain that, for every $R > \max\{2, \frac{R_0}{\alpha_{R_0}}\}$, $a = |a|p \in \Omega$ with $|a| < \frac{R_0}{R}$, and $1 \leq k \leq N$,

$$\int_{D^+_{R|a|}} |(i\nabla + A_0)v^{int}_{k,R,a}|^2\, dx \leq \hat{C}(R|a|)^2, \tag{6.2}$$

$$\int_{\partial D^+_{R|a|}} |v^{int}_{k,R,a}|^2\, ds \leq \hat{C}(R|a|)^3,$$

$$\int_{D^+_{R|a|}} |v^{int}_{k,R,a}|^2\, dx \leq \hat{C}(R|a|)^4, \tag{6.3}$$

for some $\hat{C} > 0$ (depending on p but independent of $|a|$). For all $R > \bar{K}$ and $a = |a|p \in \Omega$ with $|a|$ small, we also define

$$Z^R_a(x) := \frac{v^{int}_{N,R,a}(|a|x)}{\sqrt{H_a}}. \tag{6.4}$$

As a consequence of (5.13) and of the Dirichlet principle, arguing as in [2, Lemma 6.3], we can prove that the family of functions

$$\{Z^R_a : a = |a|p, |a| < \tfrac{\bar{r}}{R}\} \text{ is bounded in } H^{1,0}(D^+_R, \mathbb{C}). \tag{6.5}$$

In particular, for all $R > \bar{K}$,

$$\int_{D^+_{R|a|}} |(i\nabla + A_0)v^{int}_{N,R,a}|^2 dx = O(H_a), \quad \text{as } |a| \to 0^+, \tag{6.6}$$

$$\int_{\partial D^+_{R|a|}} |v^{int}_{N,R,a}|^2 dx = O(|a|H_a), \quad \text{as } |a| \to 0^+,$$

$$\int_{D^+_{R|a|}} |v^{int}_{N,R,a}|^2 dx = O(|a|^2 H_a), \quad \text{as } |a| \to 0^+. \tag{6.7}$$

Lemma 6.1. *Let $p \in \mathbb{S}^1_+$. There exists $\tilde{R} > 2$ such that, for all $R > \tilde{R}$ and $a = |a|p \in \Omega$ with $|a| < \frac{R_0}{R}$,*

$$\frac{\lambda_N - \lambda^a_N}{H_a} \leq f_R(a)$$

where

$$f_R(a) = \int_{D^+_R} |(i\nabla + A_0)Z^R_a|^2\, dx - \int_{D^+_R} |(i\nabla + A_p)\tilde{\varphi}_a|^2\, dx + o(1), \quad \textit{as } |a| \to 0^+,$$

$$f_R(a) = O(1), \quad \textit{as } |a| \to 0^+,$$

with $\tilde{\varphi}_a$ and Z^R_a defined in (5.14) and (6.4) respectively. In particular $\lambda_N - \lambda^a_N \leq \mathrm{const}\ H_a$ as $a = |a|p \to 0$, for some const > 0 *independent of $|a|$.*

Proof. Let us fix $R > \max\{2, \bar{K}, \frac{R_0}{\alpha_{R_0}}\}$. Let us consider the family of functions $\{\tilde{v}_{k,R,a}\}_{k=1,\dots,N}$ resulting from $\{v_{k,R,a}\}_{k=1,\dots,N}$ by a weighted Gram–Schmidt process, that is

$$\tilde{v}_{k,R,a} := \frac{\hat{v}_{k,R,a}}{\sqrt{\int_\Omega q|\hat{v}_{k,R,a}|^2\, dx}}, \quad k = 1, \dots, N,$$

where $\hat{v}_{N,R,a} := v_{N,R,a}$,

$$\hat{v}_{k,R,a} := v_{k,R,a} - \sum_{\ell=k+1}^{N} d^{R,a}_{\ell,k}\hat{v}_{\ell,R,a}, \quad \text{for } k = 1, \dots, N-1,$$

and

$$d_{\ell,k}^{R,a} := \frac{\int_\Omega q\, v_{k,R,a}\overline{\hat{v}_{\ell,R,a}}\,dx}{\int_\Omega q\,|\hat{v}_{\ell,R,a}|^2\,dx}.$$

By constructions, there hold

$$\int_\Omega q|\tilde{v}_{k,R,a}|^2\,dx = 1 \text{ for all } k \quad \text{and} \quad \int_\Omega q\,\tilde{v}_{k,R,a}\overline{\tilde{v}_{\ell,R,a}}\,dx = 0 \text{ for all } k \neq \ell. \tag{6.8}$$

From (5.2), (5.16), (5.19), (6.3), (6.7), and an induction argument, we deduce that

$$\int_\Omega q\,|\hat{v}_{k,R,a}|^2\,dx = 1 + O(|a|^4) \quad \text{and} \quad d_{\ell,k}^{R,a} = O(|a|^4) \text{ for } \ell \neq k \quad \text{as } |a| \to 0^+, \tag{6.9}$$

$$\int_\Omega q\,|\hat{v}_{N,R,a}|^2\,dx = \int_\Omega q\,|v_{N,R,a}|^2\,dx = 1 + O\big(|a|^2 H_a\big) \quad \text{as } |a| \to 0^+, \tag{6.10}$$

$$d_{N,k}^{R,a} = O\big(|a|^3\sqrt{H_a}\big) \quad \text{as } |a| \to 0^+, \quad \text{for all } k < N. \tag{6.11}$$

From the classical Courant-Fisher minimax characterization of eigenvalues and (6.8) it follows that

$$\lambda_N \le \max_{\substack{(\alpha_1,\dots,\alpha_N)\in\mathbb{C}^N \\ \sum_{k=1}^N |\alpha_k|^2 = 1}} \int_\Omega \left|(i\nabla + A_0)\Big(\sum_{k=1}^N \alpha_k \tilde{v}_{k,R,a}\Big)\right|^2 dx,$$

so that

$$\lambda_N - \lambda_N^a \le \max_{\substack{(\alpha_1,\dots,\alpha_N)\in\mathbb{C}^N \\ \sum_{k=1}^N |\alpha_k|^2 = 1}} \sum_{k,n=1}^N m_{k,n}^{a,R}\alpha_k\overline{\alpha_n}, \tag{6.12}$$

where

$$m_{k,n}^{a,R} = \int_\Omega (i\nabla + A_0)\tilde{v}_{k,R,a}\cdot\overline{(i\nabla + A_0)\tilde{v}_{n,R,a}}\,dx - \lambda_N^a\delta_{kn},$$

with $\delta_{kn} = 1$ if $k = n$ and $\delta_{kn} = 0$ if $k \neq n$. From (6.10), (6.4), and (5.14) we deduce that

$$\begin{aligned} m_{N,N}^{a,R} &= \frac{\lambda_N^a\big(1 - \int_\Omega q\,|v_{N,R,a}|^2\,dx\big)}{\int_\Omega q\,|v_{N,R,a}|^2\,dx} \\ &\quad + \frac{\Big(\int_{D^+_{R|a|}} |(i\nabla + A_0)v_{N,R,a}^{int}|^2 dx - \int_{D^+_{R|a|}} |(i\nabla + A_a)\varphi_N^a|^2 dx\Big)}{\int_\Omega q\,|v_{N,R,a}|^2\,dx} \\ &= H_a\bigg(\int_{D_R^+} |(i\nabla + A_0)Z_a^R|^2\,dx - \int_{D_R^+} |(i\nabla + A_p)\tilde{\varphi}_a|^2\,dx + o(1)\bigg), \end{aligned}$$

as $|a| \to 0^+$. We observe that, in view of (5.13) and (6.5),

$$\int_{D_R^+} |(i\nabla + A_0)Z_a^R|^2\,dx - \int_{D_R^+} |(i\nabla + A_p)\tilde{\varphi}_a|^2\,dx = O(1) \quad \text{as} \quad |a| \to 0^+. \tag{6.13}$$

From (2.7), (6.9), (6.11), (5.18), and (6.2), we obtain that, if $k < N$,

$$\begin{aligned} m_{k,k}^{a,R} &= -\lambda_N^a + \frac{1}{\int_\Omega q\,|\hat{v}_{k,R,a}|^2\,dx}\bigg(\lambda_k^a - \int_{D^+_{R|a|}} |(i\nabla + A_a)\varphi_k^a|^2 dx + \int_{D^+_{R|a|}} |(i\nabla + A_0)v_{k,R,a}^{int}|^2 dx\bigg) \\ &\quad + \frac{1}{\int_\Omega q\,|\hat{v}_{k,R,a}|^2\,dx}\int_\Omega \left|(i\nabla + A_0)\Big(\sum_{\ell>k} d_{\ell,k}^{R,a}\hat{v}_{\ell,R,a}\Big)\right|^2 dx \\ &\quad - \frac{2}{\int_\Omega q\,|\hat{v}_{k,R,a}|^2\,dx}\,\mathfrak{Re}\bigg(\int_\Omega (i\nabla + A_0)v_{k,R,a}\cdot\overline{(i\nabla + A_0)\Big(\sum_{\ell>k} d_{\ell,k}^{R,a}\hat{v}_{\ell,R,a}\Big)}\,dx\bigg) \\ &= (\lambda_k - \lambda_N) + o(1) \quad \text{as } |a| \to 0. \end{aligned}$$

We observe that from (2.8) it follows that $\lambda_k - \lambda_N < 0$ for all $k < N$.

From (5.15), (5.18), (6.6), and (6.2), we deduce that, for all $k < N$,

$$\begin{aligned}&\left(\int_\Omega q|\hat v_{k,R,a}|^2\,dx\right)^{1/2}\left(\int_\Omega q|\hat v_{N,R,a}|^2\,dx\right)^{1/2} m_{k,N}^{a,R}\\&=\int_{D^+_{R|a|}}\Big((i\nabla+A_0)v^{int}_{k,R,a}\cdot\overline{(i\nabla+A_0)v^{int}_{N,R,a}}-(i\nabla+A_a)\varphi_k^a\cdot\overline{(i\nabla+A_a)\varphi_N^a}\Big)\,dx\\&\quad-\int_\Omega(i\nabla+A_0)\Big(\sum_{\ell>k}d^{R,a}_{\ell,k}\hat v_{\ell,R,a}\Big)\cdot\overline{(i\nabla+A_0)v_{N,R,a}}\,dx=O\Big(|a|\sqrt{H_a}\Big),\end{aligned}$$

so that, by (6.9) and (6.10),

$$m_{k,N}^{a,R}=O\Big(|a|\sqrt{H_a}\Big)\quad\text{and}\quad m_{N,k}^{a,R}=\overline{m_{k,N}^{a,R}}=O\Big(|a|\sqrt{H_a}\Big)$$

as $|a|\to0^+$. In a similar way, from (5.18) and (6.2) we can deduce that, for all $k,n<N$ with $k\neq n$,

$$m_{k,n}^{a,R}=O(|a|^2)\quad\text{as }|a|\to0.$$

Thanks to Corollary 5.8 we can apply [2, Lemma 6.1] to conclude that

$$\max_{\substack{(\alpha_1,\dots,\alpha_N)\in\mathbb{C}^N\\ \sum_{k=1}^N|\alpha_k|^2=1}}\sum_{j,n=1}^N m_{k,n}^{a,R}\alpha_k\overline{\alpha_n}=H_a\bigg(\int_{D_R^+}|(i\nabla+A_0)Z_a^R|^2\,dx-\int_{D_R^+}|(i\nabla+A_p)\tilde\varphi_a|^2\,dx+o(1)\bigg)$$

as $|a|\to0^+$. The conclusion then follows from (6.12) and (6.13). □

7. Lower bound for $\lambda_N-\lambda_N^a$: the Rayleigh quotient for λ_N^a

For $R>2$, $1\le k\le N$, and $|a|$ sufficiently small we define

$$w_{k,R,a}:=\begin{cases}w^{ext}_{k,R,a}, & \text{in }\Omega\setminus D^+_{R|a|},\\ w^{int}_{k,R,a}, & \text{in }D^+_{R|a|},\end{cases}\qquad k=1,\dots,N,$$

where $w^{ext}_{k,R,a}:=e^{\frac i2(\theta_a-\theta_0^a)}\varphi_k^0$ in $\Omega\setminus D^+_{R|a|}$ solves

$$\begin{cases}(i\nabla+A_a)^2w^{ext}_{k,R,a}=\lambda_k q\,w^{ext}_{k,R,a}, & \text{in }\Omega\setminus D^+_{R|a|},\\ w^{ext}_{k,R,a}=e^{\frac i2(\theta_a-\theta_0^a)}\varphi_k^0, & \text{on }\partial(\Omega\setminus D^+_{R|a|}),\end{cases}$$

whereas $w^{int}_{k,R,a}$ is the unique solution to the problem

$$\begin{cases}(i\nabla+A_a)^2w^{int}_{k,R,a}=0, & \text{in }D^+_{R|a|},\\ w^{int}_{k,R,a}=e^{\frac i2(\theta_a-\theta_0^a)}\varphi_k^0, & \text{on }\partial D^+_{R|a|}.\end{cases}$$

From (5.2) it follows easily that $\dim\big(\operatorname{span}\{w_{1,R,a},\dots,w_{N,R,a}\}\big)=N$. From [10] and [14] (see also [8]) we have that

$$\int_{D^+_{R|a|}}|(i\nabla+A_0)\varphi_k^0|^2\,dx=O(|a|^2),\tag{7.1}$$

$$\int_{\partial D^+_{R|a|}}|\varphi_k^0|^2\,ds=O(|a|^3)\quad\text{and}\quad\int_{D^+_{R|a|}}|\varphi_k^0|^2\,dx=O(|a|^4)\quad\text{as }|a|\to0^+.\tag{7.2}$$

From estimates (7.1)–(7.2) and the Dirichlet principle we deduce that

$$\int_{D^+_{R|a|}} |(i\nabla + A_a)w^{int}_{k,R,a}|^2\,dx = O(|a|^2), \tag{7.3}$$

$$\int_{\partial D^+_{R|a|}} |w^{int}_{k,R,a}|^2\,ds = O(|a|^3) \quad \text{and} \quad \int_{D^+_{R|a|}} |w^{int}_{k,R,a}|^2\,dx = O(|a|^4) \quad \text{as } |a|\to 0^+. \tag{7.4}$$

For all $R > 2$ and $a = |a|p \in \Omega$ with $|a|$ small, we define

$$U^R_a(x) := \frac{w^{int}_{N,R,a}(|a|x)}{|a|^j}, \quad W_a(x) := \frac{\varphi^0_N(|a|x)}{|a|^j}. \tag{7.5}$$

From (2.11) we deduce that

$$W_a \to \beta e^{\frac{i}{2}\tilde\theta_0}\psi_j \quad \text{as } |a|\to 0^+ \tag{7.6}$$

in $H^{1,0}(D^+_R,\mathbb{C})$ for every $R > 2$, where ψ_j is given in (2.12) and $\beta\in\mathbb{C}\setminus\{0\}$ is as in (2.11). Let u_R be the unique solution to the problem

$$\begin{cases} (i\nabla + A_p)^2 u_R = 0, & \text{in } D^+_R,\\ u_R = e^{\frac{i}{2}(\theta_p-\theta_0^p)}e^{\frac{i}{2}\tilde\theta_0}\psi_j, & \text{on } \partial D^+_R. \end{cases} \tag{7.7}$$

Using the Dirichlet principle and (7.6), we can prove that, for all $R > 2$,

$$U^R_a \to \beta u_R, \quad \text{in } H^{1,p}(D^+_R,\mathbb{C}), \tag{7.8}$$

as $a = |a|p \to 0$.

Lemma 7.1. *For every $r > 1$, $u_R \to \Psi_p$ in $H^{1,p}(D^+_r,\mathbb{C})$ as $R\to+\infty$.*

Proof. Let $r > 2$. For every $R > r$, let $\eta_R : \mathbb{R}^2\to\mathbb{R}$ be a smooth cut-off function such that $\eta_R\equiv 0$ in $D_{R/2}$, $\eta_R\equiv 1$ on $\mathbb{R}^2\setminus D_R$, $0\le\eta_R\le 1$, and $|\nabla\eta_R|\le 4/R$ in $\mathbb{R}^2$. From the Dirichlet Principle, (3.7), and (3.8) we deduce that

$$\begin{aligned}\int_{D^+_r} |(i\nabla + A_p)(u_R-\Psi_p)|^2\,dx &\le \int_{D^+_R}\Big|(i\nabla + A_p)\Big(\eta_R(e^{\frac{i}{2}(\theta_p-\theta_0^p)}e^{\frac{i}{2}\tilde\theta_0}\psi_j - \Psi_p)\Big)\Big|^2\,dx\\ &\le 2\int_{\mathbb{R}^2_+\setminus D^+_{R/2}}\Big|(i\nabla + A_p)\Big(e^{\frac{i}{2}(\theta_p-\theta_0^p)}e^{\frac{i}{2}\tilde\theta_0}\psi_j - \Psi_p\Big)\Big|^2\,dx\\ &\quad+\frac{32}{R^2}\int_{D^+_R\setminus D^+_{R/2}}\Big|e^{\frac{i}{2}(\theta_p-\theta_0^p)}e^{\frac{i}{2}\tilde\theta_0}\psi_j - \Psi_p\Big|^2\,dx = o(1)\end{aligned}$$

as $R\to+\infty$. □

Lemma 7.2. *Let $p\in\mathbb{S}^1_+$. Let $\tilde R$ be as in Lemma 6.1. For all $R > \tilde R$ and $a = |a|p\in\Omega$ such that $|a| < \frac{R_0}{R}$, there holds*

$$\frac{\lambda_N-\lambda_N^a}{|a|^{2j}} \ge g_R(a)$$

where $\lim_{|a|\to 0^+} g_R(a) = i|\beta|^2\tilde\kappa_R$, *being β as in (2.11) and*

$$\tilde\kappa_R = \int_{\partial D^+_R}\Big(e^{-\frac{i}{2}(\theta_p-\theta_0^p)}e^{-\frac{i}{2}\tilde\theta_0}(i\nabla + A_p)u_R\cdot\nu - (i\nabla)\psi_j\cdot\nu\Big)\psi_j\,ds. \tag{7.9}$$

Proof. Let $\{\tilde w_{k,R,a}\}_{k=1,\dots,N}$ be the family of functions resulting from $\{w_{k,R,a}\}$ by the weighted Gram–Schmidt process

$$\tilde w_{k,R,a} := \frac{\hat w_{k,R,a}}{\sqrt{\int_\Omega q\,|\hat w_{k,R,a}|^2\,dx}}, \quad k = 1,\dots,N,$$

where $\hat{w}_{N,R,a} := w_{N,R,a}$ and, for $k = 1, \dots, N-1$, $\hat{w}_{k,R,a} := w_{k,R,a} - \sum_{\ell=k+1}^{N} c_{\ell,k}^{R,a} \hat{w}_{\ell,R,a}$, with

$$c_{\ell,k}^{R,a} := \frac{\int_\Omega q\, w_{k,R,a} \overline{\hat{w}_{\ell,R,a}}\, dx}{\int_\Omega q\, |\hat{w}_{\ell,R,a}|^2\, dx}.$$

By construction, there hold

$$\int_\Omega q|\tilde{w}_{k,R,a}|^2\, dx = 1 \text{ for all } 1 \le k \le N \quad \text{and} \quad \int_\Omega q\, \tilde{w}_{k,R,a}\overline{\tilde{w}_{\ell,R,a}}\, dx = 0 \text{ for all } k \neq \ell. \tag{7.10}$$

From (5.2), (7.2), (7.4), (7.6), and (7.8), and an induction argument,it follows that

$$\int_\Omega q\, |\hat{w}_{k,R,a}|^2 = 1 + O(|a|^4) \quad \text{and} \quad c_{\ell,k}^{R,a} = O(|a|^4) \text{ for } \ell \neq k \quad \text{as } |a| \to 0^+, \tag{7.11}$$

$$\int_\Omega q\, |\hat{w}_{N,R,a}|^2\, dx = \int_\Omega q\, |w_{N,R,a}|^2\, dx = 1 + O\big(|a|^{2j+2}\big) \quad \text{as } |a| \to 0^+, \tag{7.12}$$

$$c_{N,k}^{R,a} = O\big(|a|^{3+j}\big) \quad \text{as } |a| \to 0^+, \quad \text{for all } k < N. \tag{7.13}$$

From the classical Courant-Fisher minimax characterization of eigenvalues and (7.10) it follows that

$$\lambda_N^a \le \max_{\substack{(\alpha_1,\dots,\alpha_N)\in\mathbb{C}^N \\ \sum_{k=1}^N |\alpha_k|^2 = 1}} \int_\Omega \left| (i\nabla + A_a)\Big(\sum_{k=1}^N \alpha_k \tilde{w}_{k,R,a} \Big) \right|^2 dx,$$

so that

$$\lambda_N^a - \lambda_N \le \max_{\substack{(\alpha_1,\dots,\alpha_N)\in\mathbb{C}^N \\ \sum_{k=1}^N |\alpha_k|^2 = 1}} \sum_{k,n=1}^N h_{k,n}^{a,R} \alpha_k \overline{\alpha_n}, \tag{7.14}$$

where

$$h_{k,n}^{a,R} = \int_\Omega (i\nabla + A_a)\tilde{w}_{k,R,a} \cdot \overline{(i\nabla + A_a)\tilde{w}_{n,R,a}}\, dx - \lambda_N \delta_{kn}.$$

From (7.12), (7.5), (7.6), and (7.8) it follows that

$$\begin{aligned}
h_{N,N}^{a,R} &= \frac{\lambda_N (1 - \int_\Omega q\, |w_{N,R,a}|^2\, dx)}{\int_\Omega q\, |w_{N,R,a}|^2\, dx} \\
&\quad + \frac{\Big(\int_{D^+_{R|a|}} \big|(i\nabla + A_a) w^{int}_{N,R,a}\big|^2 dx - \int_{D^+_{R|a|}} \big|(i\nabla + A_0)\varphi_N^0\big|^2 dx \Big)}{\int_\Omega q\, |w_{N,R,a}|^2\, dx} \\
&= |a|^{2j} \Big(\int_{D_R^+} |(i\nabla + A_p) U_a^R|^2\, dx - \int_{D_R^+} |(i\nabla + A_0) W_a|^2\, dx + o(1) \Big) \\
&= |a|^{2j} |\beta|^2 \Big(\int_{D_R^+} |(i\nabla + A_p) u_R|^2\, dx - \int_{D_R^+} |\nabla \psi_j|^2\, dx + o(1) \Big) \\
&= -i|a|^{2j} |\beta|^2 (\tilde{\kappa}_R + o(1))
\end{aligned}$$

as $|a| \to 0^+$, with $\tilde{\kappa}_R$ as in (7.9). From (7.11), (7.13), (7.3), and (7.1), we obtain that, if $k < N$,

$$h_{k,k}^{a,R} = (\lambda_k - \lambda_N) + o(1) \quad \text{as } |a| \to 0.$$

We observe that from (2.8) it follows that $\lambda_k - \lambda_N < 0$ for all $k < N$.

From (7.6), (7.8), (7.1), (7.3), (7.11), and (7.12) we deduce that, for all $k < N$,

$$h_{k,N}^{a,R} = O(|a|^{1+j}) \quad \text{and} \quad h_{N,k}^{a,R} = \overline{h_{k,N}^{a,R}} = O(|a|^{1+j})$$

as $|a| \to 0^+$. Moreover, from (7.1) and (7.3) we have that, for all $k, n < N$ with $k \neq n$,

$$h_{k,n}^{a,R} = O(|a|^2) \quad \text{as } |a| \to 0.$$

Using [2, Lemma 6.1] we can conclude that

$$\max_{\substack{(\alpha_1,\dots,\alpha_N)\in\mathbb{C}^N\\ \sum_{k=1}^N |\alpha_k|^2=1}} \sum_{k,n=1}^N h_{k,n}^{a,R}\alpha_k\overline{\alpha_n} = |a|^{2j}(-i|\beta|^2\tilde\kappa_R + o(1))$$

as $|a|\to 0^+$. The conclusion then follows from (7.14). ☐

A combination of Lemmas 6.1 and 7.2 with Corollary 5.8 yields the following preliminary estimates of the eigenvalue variation.

Corollary 7.3. *Let $p\in\mathbb{S}^1_+$. Then*

(i) $|\lambda_N - \lambda_N^a| = O(1)\max\{H_a, |a|^{2j}\}$ *as* $a=|a|p\to 0$;

(ii) $|\lambda_N - \lambda_N^a| = O(H_a^{j/(j+\delta)})$ *as* $a=|a|p\to 0$.

Proof. As a direct consequence of Lemmas 6.1 and 7.2, we obtain that there exist $c_p, d_p\in\mathbb{R}$ such that, if $a=|a|p$ with $|a|$ sufficiently small, then

$$c_p|a|^{2j} \le \lambda_N - \lambda_N^a \le d_p H_a. \tag{7.15}$$

We notice that, up to now, we still do not have any indication of the sign of the constants c_p, d_p. Estimate (i) follows directly from (7.15). Estimate (ii) follows combining (i) with (5.11). ☐

Lemma 7.4. *Let $\tilde\kappa_R$ be as in* (7.9)*. Then,*

$$\lim_{R\to+\infty}\tilde\kappa_R = 2i\mathfrak{m}_p,$$

with $\mathfrak{m}_p$ as in (2.20)*.*

Proof. First, for simplicity, we rename

$$v_R = e^{-\frac{i}{2}(\theta_p-\theta_0^p)}e^{-\frac{i}{2}\tilde\theta_0}u_R,$$

where u_R is the unique solution of (7.7). Let's introduce the function

$$\varphi_R(r) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} v_R(r\cos t, r\sin t)\sin\left(j\left(\tfrac{\pi}{2}-t\right)\right)dt, \quad r>1.$$

By direct calculations, it is easy to verify that, since $-\Delta v_R = 0$ in $D_R^+\setminus D_1^+$, φ_R satisfies

$$-\left(r^{1+2j}\left(r^{-j}\varphi_R(r)\right)'\right)' = 0, \quad \text{for } r\in(1,R]. \tag{7.16}$$

Since $v_R=\psi_j$ on ∂D_R^+, we have that

$$\varphi_R(R) = \frac{\pi}{2}R^j.$$

Hence, by integrating (7.16) over $(1,r)$, we get

$$\varphi_R(r) = \frac{\frac{\pi}{2}-\varphi_R(1)R^{-2j}}{1-R^{-2j}}r^j + \frac{\varphi_R(1)-\frac{\pi}{2}}{1-R^{-2j}}r^{-j}, \quad r\in(1,R].$$

By differentiation of the previous identity, we obtain that

$$\varphi_R'(R) = \frac{jR^{j-1}}{1-R^{-2j}}\left(\frac{\pi}{2}(1+R^{-2j}) - 2\varphi_R(1)R^{-2j}\right). \tag{7.17}$$

On the other hand

$$i\varphi_R'(R) = \frac{i}{R^{j+1}}\int_{\partial D_R^+}\frac{\partial v_R}{\partial\nu}\psi_j\,ds. \tag{7.18}$$

By combining (7.17) and (7.18) we get

$$i\int_{\partial D_R^+}\frac{\partial v_R}{\partial\nu}\psi_j\,ds=\frac{ij}{1-R^{-2j}}\left(\frac{\pi}{2}R^{2j}+\frac{\pi}{2}-2\varphi_R(1)\right). \tag{7.19}$$

The second term of the right hand side of (7.9) can be calculated explicitly:

$$i\int_{\partial D_R^+}\frac{\partial\psi_j}{\partial\nu}\psi_j\,ds=ij\frac{\pi}{2}R^{2j}. \tag{7.20}$$

From (7.19), (7.20) and (7.9) it follows that

$$\tilde{\kappa}_R=\frac{ij}{1-R^{-2j}}\left(-2\varphi_R(1)+\pi\right). \tag{7.21}$$

Finally, Lemma 7.1 and Proposition 3.5 imply that

$$\lim_{R\to+\infty}\varphi_R(1)=\omega_p(1)+\frac{\pi}{2}=-\frac{\mathfrak{m}_p}{j}+\frac{\pi}{2}.$$

This allows passing to the limit in (7.21) thus getting the conclusion. □

8. Energy estimates for the eigenfunction variation

This section aims at providing some energy estimates for the function $v_{N,R,a}$ defined in (6.1), in order to improve the estimates on H_a collected in Lemma 5.8.

Throughout this section, we will regard the space $H^1_0(\Omega,\mathbb{C})$ (which coincides with $H^{1,0}_0(\Omega,\mathbb{C})$, see (2.5)) as a real Hilbert space endowed with the scalar product

$$(u,v)_{H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C})}=\mathfrak{Re}\left(\int_\Omega(i\nabla+A_0)u\cdot\overline{(i\nabla+A_0)v}\,dx\right),$$

which induces on $H^1_0(\Omega,\mathbb{C})$ the norm (2.4) (with $a=0$), which is equivalent to the Dirichlet norm, as observed in (2.5). To take in mind that here $H^1_0(\Omega,\mathbb{C})$ is treated as a vector space over $\mathbb{R}$, we denote it as $H^1_{0,\mathbb{R}}(\Omega,\mathbb{C})$ and its real dual space as $(H^1_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star$.

Let us consider the function

$$\begin{aligned}&F:\mathbb{C}\times H^1_{0,\mathbb{R}}(\Omega,\mathbb{C})\to\mathbb{R}\times\mathbb{R}\times(H^1_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star\\&F(\lambda,\varphi)=\Big(\|u\|^2_{H^{1,0}_0(\Omega,\mathbb{C})}-\lambda_N,\ \mathfrak{Im}\big(\textstyle\int_\Omega q(x)\varphi\overline{\varphi^0_N}\,dx\big),\ (i\nabla+A_0)^2\varphi-\lambda q\varphi\Big),\end{aligned} \tag{8.1}$$

where $(i\nabla+A_0)^2\varphi-\lambda\varphi\in(H^1_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star$ acts as

$${}_{(H^1_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star}\Big\langle(i\nabla+A_0)^2\varphi-\lambda q\varphi,u\Big\rangle_{H^1_{0,\mathbb{R}}(\Omega,\mathbb{C})}=\mathfrak{Re}\left(\textstyle\int_\Omega(i\nabla+A_0)\varphi\cdot\overline{(i\nabla+A_0)u}\,dx-\lambda\int_\Omega q\varphi\overline{u}\,dx\right)$$

for all $\varphi\in H^1_{0,\mathbb{R}}(\Omega,\mathbb{C})$. In (8.1) $\mathbb{C}$ is also meant as a vector space over $\mathbb{R}$. From (E_0) and (2.9), we have that $F(\lambda_N,\varphi^0_N)=(0,0,0)$.

Lemma 8.1. *The function F defined in* (8.1) *is Fréchet-differentiable at (λ_N,φ^0_N) and its Fréchet-differential $dF(\lambda_N,\varphi^0_N)\in\mathcal{L}(\mathbb{C}\times H^1_{0,\mathbb{R}}(\Omega,\mathbb{C}),\mathbb{R}\times\mathbb{R}\times(H^1_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star)$ is invertible.*

Proof. The proof follows from the Fredholm alternative and assumption (2.8) by quite standard arguments, see [2, Lemma 7.1] for details for a similar operator. □

Theorem 8.2. *Let $p\in\mathbb{S}^1_+$ and $R>\tilde{R}$, being $\tilde{R}$ as in Lemma 6.1. For $a=|a|p$ with $|a|<\frac{\bar{r}}{R}$, let $v_{N,R,a}$ be as defined in* (6.1)*. Then $\|v_{N,R,a}-\varphi^0_N\|_{H^{1,0}_0(\Omega,\mathbb{C})}=O\left(\sqrt{H_a}\right)$ as $|a|\to0^+$.*

Proof. From (6.1), (5.14), (6.4), (7.5), we have that

$$\begin{aligned}\int_\Omega \big|(i\nabla + A_0)(v_{N,R,a} - \varphi_N^0)\big|^2\,dx &= \int_\Omega |e^{\frac{i}{2}(\theta_0^a-\theta_a)}(i\nabla + A_a)\varphi_N^a - (i\nabla + A_0)\varphi_N^0|^2\,dx \\ &\quad + H_a \int_{D_R^+} \Big|(i\nabla + A_0)\Big(Z_a^R - \tfrac{|a|^j}{\sqrt{H_a}} W_a\Big)\Big|^2\,dx \\ &\quad - H_a \int_{D_R^+} \Big|e^{\frac{i}{2}(\theta_0^p-\theta_p)}(i\nabla + A_p)\tilde\varphi_a - \tfrac{|a|^j}{\sqrt{H_a}}(i\nabla + A_0)W_a\Big)\Big|^2\,dx.\end{aligned}$$

We can estimate the second term at the right hand side in the following way

$$\begin{aligned}&H_a \int_{D_R^+} \Big|(i\nabla + A_0)\Big(Z_a^R - \tfrac{|a|^j}{\sqrt{H_a}} W_a\Big)\Big|^2\,dx \\ &\qquad \le 2H_a \int_{D_R^+} \Big|(i\nabla + A_0)Z_a^R\Big|^2 + 2|a|^{2j} \int_{D_R^+} \Big|(i\nabla + A_0)W_a\Big|^2 = O(|a|^2)\end{aligned}$$

as $|a| \to 0$, via (5.12), (6.5), (7.6). The estimate of the third term is analogous recalling (5.13) in addition. In view of (5.3), we thus conclude that $v_{N,R,a} \to \varphi_N^0$ in $H_0^1(\Omega,\mathbb{C})$ as $|a| \to 0^+$. Therefore, we take advantage from Lemma 8.1 and expand

$$F(\lambda_N^a, v_{N,R,a}) = dF(\lambda_N,\varphi_N^0)(\lambda_N^a - \lambda_N, v_{N,R,a} - \varphi_N^0) + o\big(|\lambda_N^a - \lambda_N| + \|v_{N,R,a} - \varphi_N^0\|_{H_0^{1,0}(\Omega,\mathbb{C})}\big) \tag{8.2}$$

as $|a| \to 0$. In view of Lemma 8.1, the operator $dF(\lambda_N,\varphi_N^0)$ is invertible (and its inverse is continuous by the Open Mapping Theorem), then from (8.2) it follows that

$$\begin{aligned}&|\lambda_N^a - \lambda_N| + \|v_{N,R,a} - \varphi_N^0\|_{H_0^1(\Omega,\mathbb{C})} \\ &\le \|(dF(\lambda_N,\varphi_N^0))^{-1}\|_{\mathcal{L}(\mathbb{R}\times\mathbb{R}\times(H^1_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star,\mathbb{C}\times H^1_{0,\mathbb{R}}(\Omega,\mathbb{C}))} \|F(\lambda_N^a, v_{N,R,a})\|_{\mathbb{R}\times\mathbb{R}\times(H^1_{0,\mathbb{R}}(\Omega))^\star}(1+o(1))\end{aligned}$$

as $|a| \to 0^+$. It remains to estimate the norm of

$$\begin{aligned}&F(\lambda_N^a, v_{N,R,a}) = (\alpha_a, \beta_a, w_a) \\ &\quad = \left(\|v_{N,R,a}\|^2_{H_0^{1,0}(\Omega,\mathbb{C})} - \lambda_N, \mathfrak{Im}\left(\textstyle\int_\Omega q v_{N,R,a}\overline{\varphi_N^0}\,dx\right), (i\nabla + A_0)^2 v_{N,R,a} - \lambda_N^a\, q\, v_{N,R,a}\right)\end{aligned}$$

in $\mathbb{R}\times\mathbb{R}\times(H^1_{0,\mathbb{R}}(\Omega))^\star$. As far as α_a is concerned, using (6.5), (5.13), and Corollary 7.3 (part (ii)), since $\delta < 1 \le j$ we have that

$$\begin{aligned}\alpha_a &= \left(\int_{D^+_{R|a|}} |(i\nabla + A_0)v^{int}_{N,R,a}|^2\,dx - \int_{D^+_{R|a|}} |(i\nabla + A_a)\varphi_N^a|^2\,dx\right) + (\lambda_N^a - \lambda_N) \\ &= H_a\left(\int_{D_R^+} |(i\nabla + A_0)Z_a^R|^2\,dx - \int_{D_R^+} |(i\nabla + A_p)\tilde\varphi_a|^2\,dx\right) + (\lambda_N^a - \lambda_N) \\ &= O(H_a^{j/(j+\delta)}) = O(\sqrt{H_a}), \quad \text{as } |a| \to 0^+.\end{aligned}$$

As far as β_a is concerned, by the normalization in (2.23), (2.2), (6.7), (5.16), and (2.11), we have that

$$\begin{aligned}\beta_a &= \mathfrak{Im}\left(\int_{D^+_{R|a|}} q\, v^{int}_{N,R,a}\overline{\varphi_N^0}\,dx - \int_{D^+_{R|a|}} q\, e^{\frac{i}{2}(\theta_0^a-\theta_a)}\varphi_N^a\overline{\varphi_N^0}\,dx + \int_\Omega q\, e^{\frac{i}{2}(\theta_0^a-\theta_a)}\varphi_N^a\overline{\varphi_N^0}\,dx\right) \\ &= O(\sqrt{H_a}|a|^{j+2}) = o(\sqrt{H_a}), \quad \text{as } |a| \to 0^+.\end{aligned}$$

Let $\varphi \in C_c^\infty(\Omega,\mathbb{C})$. Then, if $|a|$ is sufficiently small, $e^{\frac{i}{2}(\theta_a-\theta_0^a)}\varphi \in H_0^{1,a}(\Omega,\mathbb{C})$ and then, in view of (5.1),

$$0=\int_\Omega e^{\frac{i}{2}(\theta_0^a-\theta_a)}(i\nabla+A_a)\varphi_N^a\cdot\overline{(i\nabla+A_0)\varphi}\,dx-\lambda_N^a\int_\Omega q\,e^{\frac{i}{2}(\theta_0^a-\theta_a)}\varphi_N^a\overline{\varphi}\,dx.$$

Hence, by (6.1),

$$\begin{aligned}\int_\Omega&(i\nabla+A_0)v_{N,R,a}\cdot\overline{(i\nabla+A_0)\varphi}\,dx-\lambda_N^a\int_\Omega q\,v_{N,R,a}\overline{\varphi}\,dx\\&=\int_{D^+_{R|a|}}(i\nabla+A_0)v^{int}_{N,R,a}\cdot\overline{(i\nabla+A_0)\varphi}\,dx-\lambda_N^a\int_{D^+_{R|a|}}qv^{int}_{N,R,a}\overline{\varphi}\,dx\\&-\int_{D^+_{R|a|}}e^{\frac{i}{2}(\theta_0^a-\theta_a)}(i\nabla+A_a)\varphi_N^a\cdot\overline{(i\nabla+A_0)\varphi}\,dx+\lambda_N^a\int_{D^+_{R|a|}}q\,e^{\frac{i}{2}(\theta_0^a-\theta_a)}\varphi_N^a\overline{\varphi}\,dx,\end{aligned}$$

which, in view of (5.15), (5.16), (6.6), (6.7), yields

$$\sup_{\varphi\in C_c^\infty(\Omega,\mathbb{C})\setminus\{0\}}\frac{{}_{(H^1_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star}\Big\langle(i\nabla+A_0)^2v_{N,R,a}-\lambda_N^aqv_{N,R,a},\varphi\Big\rangle_{H^1_{0,\mathbb{R}}(\Omega,\mathbb{C})}}{\|\varphi\|_{H_0^{1,0}(\Omega,\mathbb{C})}}=O(\sqrt{H_a}),$$

as $|a|\to0^+$. By density of $C_c^\infty(\Omega,\mathbb{C})$ in $H_0^{1,0}(\Omega,\mathbb{C})$ we conclude that

$$\|w_a\|_{(H^1_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star}=O(\sqrt{H_a}),\quad\text{as }|a|\to0^+,$$

thus completing the proof. □

As a consequence of Theorem 8.2, we obtain the following improvement of Corollary 5.8.

Theorem 8.3. *We have that* $|a|^{2j}=O(H_a)$ *as* $a=|a|p\to0$.

Proof. Directly from scaling and Theorem 8.2, we obtain that, for every $R>\tilde R$,

$$\left(\int_{\left(\frac{1}{|a|}\Omega\right)\setminus D_R^+}\left|(i\nabla+A_p)\Big(\tilde\varphi_a(x)-e^{\frac{i}{2}(\theta_p-\theta_0^p)}\frac{|a|^j}{\sqrt{H_a}}W_a\Big)\right|^2dx\right)^{1/2}=O(1),\quad\text{as }a=|a|p\to0,\tag{8.3}$$

from which it follows that

$$\frac{|a|^j}{\sqrt{H_a}}\left(\int_{D^+_{2R}\setminus D^+_R}\left|(i\nabla+A_p)\Big(e^{\frac{i}{2}(\theta_p-\theta_0^p)}W_a\Big)\right|^2dx\right)^{1/2}\le O(1)+\left(\int_{D^+_{2R}\setminus D^+_R}\left|(i\nabla+A_p)\tilde\varphi_a(x)\right|^2dx\right)^{1/2}$$

as $a=|a|p\to0$. Via (7.6) and (5.13), this reads $\frac{|a|^j}{\sqrt{H_a}}=O(1)$ as $|a|\to0^+$, thus concluding the proof. □

9. Blow-up analysis

Theorem 9.1. *For* $p\in\mathbb{S}^1_+$ *and* $a=|a|p\in\Omega$, *let* φ_N^a *solve* (2.22). *Let* $\tilde\varphi_a$ *be as in* (5.14), $\bar K$ *be as in* (5.10), β *be as in* (2.11) *and* Ψ_p *be the function defined in* (2.24). *Then*

$$\lim_{a=|a|p\to0}\frac{|a|^j}{\sqrt{H_a}}=\frac{1}{|\beta|}\sqrt{\frac{\bar K}{\int_{\partial D^+_{\bar K}}|\Psi_p|^2\,ds}}\tag{9.1}$$

and

$$\tilde\varphi_a\to\frac{\beta}{|\beta|}\sqrt{\frac{\bar K}{\int_{\partial D^+_{\bar K}}|\Psi_p|^2\,ds}}\Psi_p,\qquad\text{as }a=|a|p\to0,\tag{9.2}$$

in $H^{1,p}(D_R^+,\mathbb{C})$ *for every* $R>1$, *almost everywhere in* $\mathbb{R}^2_+$, *and in* $C^2_{\rm loc}(\overline{\mathbb{R}^2_+}\setminus\{p\},\mathbb{C})$.

Proof. From Theorem 8.3, we know that $\frac{|a|^j}{\sqrt{H_a}} = O(1)$ as $a = |a|p \to 0$. Furthermore, we have that the family $\{\tilde{\varphi}_a : a = |a|p, |a| < \frac{\bar{r}}{R}\}$ is bounded in $H^{1,p}(D_R^+, \mathbb{C})$ for all $R \geq \bar{K}$, see (5.13). Then, by a diagonal process, for every sequence $a_n = |a_n|p$ with $|a_n| \to 0$, there exist $c \in [0, +\infty)$, $\tilde{\Phi} \in \bigcup_{R>1} H^{1,p}(D_R^+, \mathbb{C})$, and a subsequence a_{n_ℓ} such that

$$\lim_{\ell \to +\infty} \frac{|a_{n_\ell}|^j}{\sqrt{H_{a_{n_\ell}}}} = c$$

and

$$\tilde{\varphi}_{a_{n_\ell}} \rightharpoonup \tilde{\Phi} \quad \text{weakly in } H^{1,p}(D_R^+, \mathbb{C}) \text{ for every } R > 1 \text{ and almost everywhere.}$$

By (5.14) and compactness of the trace embedding, we have that

$$\frac{1}{\bar{K}} \int_{\partial D_{\bar{K}}^+} |\tilde{\Phi}|^2 \, ds = 1; \tag{9.3}$$

in particular $\tilde{\Phi} \not\equiv 0$. Passing to the weak limit in the equation satisfied by $\tilde{\varphi}_a$, i.e. in equation

$$(i\nabla + A_p)^2 \tilde{\varphi}_a = \lambda_N^a |a|^2 q(|a|x) \tilde{\varphi}_a, \quad \text{in } \tfrac{1}{|a|}\Omega = \{x \in \mathbb{R}^2 : |a|x \in \Omega\}, \tag{9.4}$$

we obtain that $\tilde{\Phi}$ weakly solves

$$(i\nabla + A_p)^2 \tilde{\Phi} = 0, \quad \text{in } \mathbb{R}^2_+. \tag{9.5}$$

By continuity of the trace operator $H^{1,p}(D_R^+, \mathbb{C}) \to L^2(\{0\} \times (-R, R), \mathbb{C})$ and vanishing of $\tilde{\varphi}_{a_{n_\ell}}$ on $\{0\} \times (-R, R)$ for large ℓ, we also have that

$$\tilde{\Phi} = 0, \quad \text{on } \partial\mathbb{R}^2_+. \tag{9.6}$$

By elliptic estimates, we can prove that $\tilde{\varphi}_{a_{n_\ell}} \to \tilde{\Phi}$ in $C^2_{\rm loc}(\overline{\mathbb{R}^2_+} \setminus \{p\}, \mathbb{C})$. Therefore, for every $R > 1$, $\int_{\partial D_R^+} |\tilde{\varphi}_{a_{n_\ell}}|^2 \, ds \to \int_{\partial D_R^+} |\tilde{\Phi}|^2 \, ds$ as $\ell \to +\infty$ and, passing to the limit in (9.4) tested by $\tilde{\varphi}_{a_{n_\ell}}$, we obtain that

$$\int_{D_R^+} |(i\nabla + A_p)\tilde{\varphi}_{a_{n_\ell}}|^2 \, dx \to \int_{D_R^+} |(i\nabla + A_p)\tilde{\Phi}|^2 \, dx, \quad \text{as } \ell \to +\infty.$$

Therefore, in view of the Poincaré inequality (A.3), we deduce the convergence of norms $\|\tilde{\varphi}_{a_{n_\ell}}\|_{H^{1,p}(D_R^+,\mathbb{C})} \to \|\tilde{\Phi}\|_{H^{1,p}(D_R^+,\mathbb{C})}$ as $\ell \to +\infty$ and then conclude that the convergence $\tilde{\varphi}_{a_{n_\ell}} \to \tilde{\Phi}$ is actually strong in $H^{1,p}(D_R^+, \mathbb{C})$ for every $R > 1$.

Therefore we can pass to the limit along a_{n_ℓ} in (8.3) and, recalling (7.6), we obtain that

$$\int_{\mathbb{R}^2_+ \setminus D_R^+} \left| (i\nabla + A_p) \left(\tilde{\Phi} - e^{\frac{i}{2}(\theta_p - \theta_0^p + \tilde{\theta}_0)} c\beta\psi_j \right) \right|^2 dx < +\infty,$$

for every $R > \tilde{R}$.

This implies that $c > 0$; indeed, otherwise, $c = 0$ would imply that $\int_{\mathbb{R}^2_+} |(i\nabla + A_p)\tilde{\Phi}|^2 \, dx < +\infty$, which, in view of (9.5)-(9.6) and (A.2), would yield $\tilde{\Phi} \equiv 0$, thus contradicting (9.3). Therefore, from (9.5), (9.6) and Proposition 3.2 we have necessarily that

$$\tilde{\Phi} = c\beta\Psi_p. \tag{9.7}$$

From (9.7), (9.3) and the fact that $c > 0$, we have that

$$c = \frac{1}{|\beta|} \sqrt{\frac{\bar{K}}{\int_{\partial D_{\bar{K}}^+} |\Psi_p|^2 \, ds}},$$

so that the convergences (9.1)–(9.2) hold along the subsequence $\{a_{n_\ell}\}_\ell$. Since the limits in (9.1)–(9.2) depend neither on the sequence $\{a_n\}_n$ nor the subsequence $\{a_{n_\ell}\}_\ell$, we conclude that the convergences holds for $|a| \to 0^+$. □

Proof of Theorem 2.3. It follows directly from (9.1), (9.2). □

From Theorem 9.1 it follows that the blow-up family of functions Z_a^R introduced in (6.4) converges to a multiple of the unique solution z_R to

$$\begin{cases} (i\nabla + A_0)^2 z_R = 0, & \text{in } D_R^+, \\ z_R = e^{\frac{i}{2}(\theta_0^p - \theta_p)}\Psi_p, & \text{on } \partial D_R^+. \end{cases} \tag{9.8}$$

Lemma 9.2. *Under the same assumptions as in Theorem 9.1, let Z_a^R be as in (6.4). Then, for all $R > \tilde{R}$,*

$$Z_a^R \to \frac{\beta}{|\beta|}\sqrt{\frac{\bar{K}}{\int_{\partial D_{\bar{K}}^+} |\Psi_p|^2\, ds}}\, z_R, \qquad \text{in } H^{1,0}(D_R^+, \mathbb{C}),$$

as $a = |a|p \to 0$.

Proof. Once the convergence (9.2) is established, it follows from a standard Dirichlet principle, see [2, Lemma 8.3] for details. □

10. Sharp asymptotics for convergence of eigenvalues: $f_R(a)$

In view of Lemmas 6.1 and 7.2 and of the asymptotics of H_a given by (9.1), to compute the limit of $\frac{\lambda_N^a - \lambda_N}{|a|^{2j}}$ it remains to compute the limit of $f_R(a)$ as $a = |a|p \to 0$ and $R \to +\infty$.

Lemma 10.1. *For all $R > \tilde{R}$ (where $\tilde{R}$ is given in Lemma 6.1) and $a = |a|p \in \Omega$ with $|a| < \frac{R_0}{R}$, let $f_R(a)$ be as in Lemma 6.1. Then,*

$$\lim_{|a|\to 0^+} f_R(a) = -i\frac{\bar{K}}{\int_{\partial D_{\bar{K}}^+} |\Psi_p|^2\, ds}\kappa_R,$$

where

$$\kappa_R = \int_{\partial D_R^+} \Big((i\nabla + A_0) z_R \cdot \nu\, \overline{z_R} - (i\nabla + A_p)\Psi_p \cdot \nu\, \overline{\Psi_p} \Big)\, ds. \tag{10.1}$$

Furthermore, $\lim_{R\to+\infty} \kappa_R = -2i\mathfrak{m}_p$, where $\mathfrak{m}_p$ is defined in (2.20).

Proof. First, we observe that, by Theorem 9.1, Lemma 9.2, and the equations of z_R (9.8) and Ψ_p (3.6),

$$\begin{aligned} \lim_{|a|\to 0^+} f_R(a) &= \lim_{|a|\to 0^+} \left(\int_{D_R^+} |(i\nabla + A_0) Z_a^R|^2\, dx - \int_{D_R^+} |(i\nabla + A_p)\tilde{\varphi}_a|^2\, dx \right) + o(1) \\ &= \frac{\bar{K}}{\int_{\partial D_{\bar{K}}^+} |\Psi_p|^2\, ds} \left(\int_{D_R^+} |(i\nabla + A_0) z_R|^2\, dx - \int_{D_R^+} |(i\nabla + A_p)\Psi_p|^2\, dx \right) = -i\frac{\bar{K}}{\int_{\partial D_{\bar{K}}^+} |\Psi_p|^2\, ds}\kappa_R, \end{aligned}$$

with κ_R from (10.1). We divide the computation of the limit $\lim_{R\to+\infty}\kappa_R$ in two steps.

Step 1. We claim that

$$\kappa_R = \int_{\partial D_R^+} \left(e^{\frac{i}{2}(\theta_p - \theta_0^p)} (i\nabla + A_0) z_R - (i\nabla + A_p)\Psi_p \right) \cdot \nu\, e^{-\frac{i}{2}(\theta_p - \theta_0^p + \tilde{\theta}_0)} \psi_j\, ds + o(1), \tag{10.2}$$

as $R \to +\infty$. Indeed, we observe that κ_R can be written as

$$\kappa_R = \int_{\partial D_R^+} \left(e^{\frac{i}{2}(\theta_p - \theta_0^p)} (i\nabla + A_0) z_R - (i\nabla + A_p)\Psi_p \right) \cdot \nu\, e^{-\frac{i}{2}(\theta_p - \theta_0^p + \tilde{\theta}_0)} \psi_j\, ds + I_1(R) + I_2(R),$$

where

$$I_1(R) = \int_{\partial D_R^+} (i\nabla + A_0)\left(z_R - e^{\frac{i}{2}\tilde\theta_0}\psi_j\right)\cdot\nu\,\left(\overline{e^{\frac{i}{2}(\theta_0^p-\theta_p)}\Psi_p} - e^{-\frac{i}{2}\tilde\theta_0}\psi_j\right)ds,$$

$$I_2(R) = -\int_{\partial D_R^+} (i\nabla + A_p)\left(\Psi_p - e^{\frac{i}{2}(\tilde\theta_0+\theta_p-\theta_0^p)}\psi_j\right)\cdot\nu\,\left(\overline{\Psi_p} - e^{-\frac{i}{2}(\theta_p-\theta_0^p+\tilde\theta_0)}\psi_j\right)ds.$$

Let η_R be a smooth cut-off function satisfying

$$\eta_R \equiv 0 \text{ in } D_{R/2}, \quad \eta_R \equiv 1 \text{ on } \mathbb{R}^2\setminus D_R, \quad 0\le\eta_R\le 1 \quad\text{and}\quad |\nabla\eta_R|\le 4/R \text{ in } \mathbb{R}^2.$$

By testing the equation

$$(i\nabla + A_p)^2\left(\Psi_p - e^{\frac{i}{2}(\theta_p-\theta_0^p+\tilde\theta_0)}\psi_j\right) = 0,$$

which is satisfied in $\mathbb{R}^2_+\setminus D_R^+$, on $(\Psi_p - e^{\frac{i}{2}(\theta_p-\theta_0^p+\tilde\theta_0)}\psi_j)(1-\eta_{2R})^2$, we obtain that

$$\begin{aligned} I_2(R) &= i\int_{\mathbb{R}^2_+\setminus D_R^+} |(i\nabla + A_p)\left(\Psi_p - e^{\frac{i}{2}(\theta_p-\theta_0^p+\tilde\theta_0)}\psi_j\right)|^2(1-\eta_{2R})^2\,dx \\ &\quad + 2\int_{\mathbb{R}^2_+\setminus D_R^+}(1-\eta_{2R})\left(\overline{\Psi_p} - e^{-\frac{i}{2}(\theta_p-\theta_0^p+\tilde\theta_0)}\psi_j\right)(i\nabla + A_p)\left(\Psi_p - e^{\frac{i}{2}(\theta_p-\theta_0^p+\tilde\theta_0)}\psi_j\right)\cdot\nabla\eta_{2R}\,dx. \end{aligned}$$

Hence,

$$\begin{aligned} |I_2(R)| &\le 2\int_{\mathbb{R}^2_+\setminus D_R^+}\left|(i\nabla + A_p)\left(\Psi_p - e^{\frac{i}{2}(\theta_p-\theta_0^p+\tilde\theta_0)}\psi_j\right)\right|^2 dx \\ &\qquad + \frac{4}{R^2}\int_{D_{2R}^+\setminus D_R^+}\left|\Psi_p - e^{\frac{i}{2}(\theta_p-\theta_0^p+\tilde\theta_0)}\psi_j\right|^2 dx \to 0, \quad \text{as } R\to+\infty, \end{aligned}$$

thanks to (3.7), (3.8). On the other hand, by testing the equation $(i\nabla + A_0)^2(z_R - e^{\frac{i}{2}\tilde\theta_0}\psi_j) = 0$ in D_R^+ on $\eta_R(e^{\frac{i}{2}(\theta_0^p-\theta_p)}\Psi_p - e^{\frac{i}{2}\tilde\theta_0}\psi_j)$, the Dirichlet principle yields that

$$\begin{aligned} |I_1(R)| &= \left|i\int_{D_R^+}(i\nabla + A_0)\left(z_R - e^{\frac{i}{2}\tilde\theta_0}\psi_j\right)\cdot\overline{(i\nabla + A_0)\left(\eta_R\left(e^{\frac{i}{2}(\theta_0^p-\theta_p)}\Psi_p - e^{\frac{i}{2}\tilde\theta_0}\psi_j\right)\right)}\,dx\right| \\ &\le \int_{D_R^+}\left|(i\nabla + A_0)\left(\eta_R\left(e^{\frac{i}{2}(\theta_0^p-\theta_p)}\Psi_p - e^{\frac{i}{2}\tilde\theta_0}\psi_j\right)\right)\right|^2 dx \\ &\le 2\int_{D_R^+\setminus D_{R/2}^+}\left|(i\nabla + A_0)\left(e^{\frac{i}{2}(\theta_0^p-\theta_p)}\Psi_p - e^{\frac{i}{2}\tilde\theta_0}\psi_j\right)\right|^2 dx \\ &\quad + \frac{32}{R^2}\int_{D_R^+\setminus D_{R/2}^+}\left|e^{\frac{i}{2}(\theta_0^p-\theta_p)}\Psi_p - e^{\frac{i}{2}\tilde\theta_0}\psi_j\right|^2 dx. \end{aligned}$$

Hence $\lim_{R\to+\infty} I_1(R) = 0$ thanks to (3.7) and (3.8). The proof of (10.2) is thereby complete.

Step 2. We now compute $\lim_{R\to+\infty}\kappa_R$. First, we define

$$\zeta_R(r) = \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{-\frac{i}{2}\tilde\theta_0(r\cos t, r\sin t)} z_R(r\cos t, r\sin t)\sin\left(j\left(\tfrac{\pi}{2}-t\right)\right)dt.$$

Thanks to the equation satisfied by z_R (9.8), we have that

$$\left(r^{1+2j}\left(r^{-j}\zeta_R(r)\right)'\right)' = 0, \quad \text{in } (0,R].$$

Therefore, by integrating over (r,R), we obtain, for some $B\in\mathbb{C}$,

$$\zeta_R(r) = \frac{\zeta_R(R)}{R^j}r^j - \frac{B}{R^{2j}}r^j + Br^{-j}, \quad \text{in } (0,R].$$

Next, we note that the function $z_R^0 := e^{-\frac{i}{2}\tilde\theta_0} z_R$ is a solution to $-\Delta z_R^0 = 0$ in D_R^+ and $z_R^0 = 0$ on $\partial\mathbb{R}^2_+ \cap D_R$, so $z_R^0 = O(|x|)$ as $|x| \to 0$ (see e.g. [10]). This implies that $B = 0$ and

$$\zeta_R(r) = \frac{\zeta_R(R)}{R^j} r^j \quad \text{and} \quad \zeta_R'(r) = \frac{j\zeta_R(R)}{R^j} r^{j-1}, \qquad \text{in } (0, R].$$

On the other hand, we can compute

$$\zeta_R'(R) = \frac{1}{R^{j+1}} \int_{\partial D_R^+} \nabla\left(e^{-\frac{i}{2}\tilde\theta_0} z_R\right) \cdot \nu\, \psi_j \, ds = -\frac{i}{R^{j+1}} \int_{\partial D_R^+} (i\nabla + A_0) z_R \cdot \nu\, e^{-\frac{i}{2}\tilde\theta_0} \psi_j \, ds.$$

Hence, by combining the two previous equations, we have that

$$\int_{\partial D_R^+} (i\nabla + A_0) z_R \cdot \nu\, e^{-\frac{i}{2}\tilde\theta_0} \psi_j \, ds = ijR^j \zeta_R(R). \tag{10.3}$$

To compute explicitly $\zeta_R(R)$, we can use the boundary conditions of z_R on ∂D_R^+, Proposition 3.5 and (2.12) to obtain

$$\begin{aligned} \zeta_R(R) &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} e^{\frac{i}{2}(\theta_0^p - \theta_p - \tilde\theta_0)(R\cos t, R\sin t)} \Psi_p(R\cos t, R\sin t) \sin\left(j\left(\tfrac{\pi}{2} - t\right)\right) dt \\ &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} (w_p + \psi_j)(R\cos t, R\sin t) \sin\left(j\left(\tfrac{\pi}{2} - t\right)\right) dt = -\frac{\mathfrak{m}_p}{jR^j} + R^j \frac{\pi}{2}. \end{aligned} \tag{10.4}$$

By combining (10.3) and (10.4), we get

$$\int_{\partial D_R^+} (i\nabla + A_0) z_R \cdot \nu\, e^{-\frac{i}{2}\tilde\theta_0} \psi_j \, ds = -i\mathfrak{m}_p + ijR^{2j}\frac{\pi}{2}. \tag{10.5}$$

Next, in view of (2.24) we rewrite

$$\int_{\partial D_R^+} (i\nabla + A_p)\Psi_p \cdot \nu\, e^{\frac{i}{2}(\theta_0^p - \theta_p - \tilde\theta_0)} \psi_j \, ds = i \int_{\partial D_R^+} \nabla(w_p + \psi_j) \cdot \nu\, \psi_j \, ds.$$

By using Proposition 3.5 and (2.12), we immediately obtain that

$$i \int_{\partial D_R^+} \nabla(w_p + \psi_j) \cdot \nu\, \psi_j \, ds = i\mathfrak{m}_p + ijR^{2j}\frac{\pi}{2}. \tag{10.6}$$

Finally, by combining (10.2), (10.5) and (10.6) we obtain that $\lim_{R\to+\infty} \kappa_R = -2i\mathfrak{m}_p$, thus concluding the proof. □

Proof of Theorems 2.1 and 2.2. From Lemmas 7.2, 6.1, Theorem 9.1 and Lemma 10.1, it follows that, for all $R > \tilde R$,

$$\begin{aligned} i|\beta|^2 \tilde\kappa_R + o(1) &\le \frac{\lambda_N - \lambda_N^a}{|a|^{2j}} \le f_R(a) \frac{H_a}{|a|^{2j}} \\ &= \left(-i \frac{\bar K}{\int_{\partial D_{\bar K}^+} |\Psi_p|^2 \, ds} \kappa_R + o(1)\right) \left(|\beta|^2 \frac{\int_{\partial D_{\bar K}^+} |\Psi_p|^2 \, ds}{\bar K} + o(1)\right), \end{aligned}$$

as $a = |a|p \to 0$. Hence,

$$i|\beta|^2 \tilde\kappa_R \le \liminf_{a=|a|p\to 0} \frac{\lambda_N - \lambda_N^a}{|a|^{2j}} \le \limsup_{a=|a|p\to 0} \frac{\lambda_N - \lambda_N^a}{|a|^{2j}} \le -i|\beta|^2 \kappa_R,$$

for every $R > \tilde R$. From Lemmas 7.4 and 10.1, by letting $R \to +\infty$, we obtain that

$$-2|\beta|^2 \mathfrak{m}_p \le \liminf_{|a|\to 0^+} \frac{\lambda_N - \lambda_N^a}{|a|^{2j}} \le \limsup_{|a|\to 0^+} \frac{\lambda_N - \lambda_N^a}{|a|^{2j}} \le -2|\beta|^2 \mathfrak{m}_p,$$

which yields that

$$\lim_{a=|a|p\to 0} \frac{\lambda_N - \lambda_N^a}{|a|^{2j}} = -2|\beta|^2 \mathfrak{m}_p,$$

thus proving (2.13) together with Theorem 2.2. Statements (i),(ii), and (iii) of Theorem 2.1 follow from combination of Theorem 2.2, Lemma 4.1, and Proposition 4.2. ☐

Appendix: Hardy & Poincaré inequalities

In this appendix we recall some well-known Hardy and Poincaré-type inequalities used throughout the paper.

In [17] the following Hardy-type inequalities were proved:

$$\int_{\mathbb{R}^2} |(i\nabla + A_a)u|^2 \, dx \geq \frac{1}{4} \int_{\mathbb{R}^2} \frac{|u(x)|^2}{|x-a|^2} \, dx, \tag{A.1}$$

which holds for all functions $u \in \mathcal{D}^{1,2}_a(\mathbb{R}^2)$, being $\mathcal{D}^{1,2}_a(\mathbb{R}^2)$ the completion of $C^\infty_c(\mathbb{R}^2 \setminus \{a\}, \mathbb{C})$ with respect to the norm $\|(i\nabla + A_a)u\|_{L^2(\mathbb{R}^2,\mathbb{C}^2)}$, and

$$\int_{D_r(a)} |(i\nabla + A_a)u|^2 \, dx \geq \frac{1}{4} \int_{D_r(a)} \frac{|u(x)|^2}{|x-a|^2} \, dx, \tag{A.2}$$

which holds for all $r > 0$, $a \in \mathbb{R}^2$ and $u \in H^{1,a}(D_r(a), \mathbb{C})$, see also [9, Lemma 3.1 and Remark 3.2].

We also recall from [21] two Poincaré-type inequalities in half-balls.

Lemma A.1 ([21, Lemma 3.3]). *Let $r > 0$ and $a \in D_r^+$. For all $u \in H^{1,a}(D_r^+, \mathbb{C})$, with $u = 0$ on $\{x_1 = 0\}$, we have*

$$\frac{1}{r^2} \int_{D_r^+} |u|^2 \, dx \leq \frac{1}{r} \int_{\partial D_r^+} |u|^2 \, ds + \int_{D_r^+} |(i\nabla + A_a)u|^2 \, dx. \tag{A.3}$$

Lemma A.2 ([21, Lemma 3.4]). *Let $r > 0$ and $a \in D_r^+$. For all $u \in H^{1,a}(D_r^+, \mathbb{C})$, with $u = 0$ on $\{x_1 = 0\}$, we have*

$$\frac{1}{r} \int_{\partial D_r^+} |u|^2 \, ds \leq \int_{D_r^+} |(i\nabla + A_a)u|^2 \, dx.$$

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