SCUOLA DI DOTTORATO
UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA

Department of

Mathematics and its Applications

PhD program in Pure and Applied Mathematics Cycle: XXVIII

ANALYTICAL PROPERTIES OF FLOWS OF SECOND-GRADIENT FLUIDS

Surname: Mastaglio Name: Sara

Registration number: 775118

Tutor: prof. Alfredo Marzocchi

Supervisor: prof. Alfredo Marzocchi

Coordinator: Roberto Paoletti

ACADEMIC YEAR 2014/2015

Contents

Introduction

Navier-Stokes equations are since many years the most important tool for studying viscous fluids. They are quite well established under a physical point of view, providing anyway one of the most challenging problems in Analysis.

During the last century a number of variants of the Navier-Stokes equations have been proposed, mainly with the goal of describing some nonlinear phenomena like, *e.g.* shear thinning, and they got a considerable success in describing features of some biological fluids, like blood.

Yet, all those fluids, including of course Navier-Stokes, share one common property: the work expended by the inner forces depends only on first derivatives of the velocity, as it has to be, at least for simple fluids.

However, starting first from a very general point of view with the pioneering work of Germain [1], who introduced in a systematic way the concept of virtual power and its use in the foundations of Continuum Mechanics, and subsequently with many others, it became clearer and clearer that another possibile generalization was available, *i.e.* the second-gradient (not to be confused with second-grade) fluid. In these fluids the working done by the inner forces depends also on the second derivatives of the velocity field and includes the possibility of a "hyperviscosity" analogous to hyperstress coefficients which appears in the corresponding solid mechanics theories.

These fluids have been considered only as an exercise, or an analytical variation of the problem, until the work of Fried and Gurtin [4], Giusteri and Fried [6] and Giusteri-Marzocchi-Musesti [16], [7] and [8], in which some convincing features of physical materials of this type were described and used. Subsequently, the very important case of isotropic fluids showed that this generalization leads to what we will call *Hyperviscous Navier-Stokes problem*.

In this Thesis we will deal with the general initial and boundary value problem of such a fluid in a bounded or unbounded domain in three spatial dimensions, which is still open (even for bounded domains) for Navier-Stokes, with natural homogeneous boundary conditions.

The plan of the Thesis is the following: in the first chapter we introduce the second-order fluids through their derivation from the theory of Virtual Powers. In this chapter the advantages of considering such fluids are presented, for example the possibility to treat slender bodies moving in viscous fluids. In the second chapter it is studied the initial boundary value problem for the hyperviscous Navier-Stokes system, that describes the special class of fluids derived in such a theory; in particular, we will consider the flow of these fluids at first in bounded domains and then in exterior domains.

In order to do this, the existence of a solution is proved through the construction of a suitable Galerkin approximated solution that passes to the limit thanks to suitable *a priori* energy estimates which are independent on the size of the bounded domains, thus allowing also existence in unbounded domains. In these estimates, the terms which do not appear in the Navier-Stokes problem will play a crucial role.

The solutions are then proved to be regular, both in time and space, and unique in their functional spaces.

Chapter 1

Second-gradient fluids

1.1 Virtual Powers

The study of the motion of a deformable body consists in the analysis of the configurations that such a body assumes in different instants. Thanks to Continuum Mechanics, we can describe the possible configurations, the kinematics and dynamics of the bodies in a macroscopical way using a continuum approach. What happens to an object in motion can be insight by the mechanical interactions between subbodies of the body we consider or between the external enviroment due to the forces. A possible approach, once established the configurations and the kinematics, is to exploit the Principle of the Virtual Powers, instead of studying forces, in order to write balance equations in integral form. This kind of approach is not a mathematical trick, but instead it is the most natural way to write the mechanical laws directly in weak form, without invoking regularity for setting the balance laws (*e.g.* the balance of momentum) and then relaxing it once the problem is written as a differential problem.

A virtual power has been introduced by many authors. The first who make this systematically was Germain [1], but we will refere to the definition by Degiovanni, Marzocchi and Musesti [17], which also allows measures and sets of finite perimeter as subbodies.

Definition 1.1.1. *Let $\Omega \subseteq \mathbb{R}^n$ be an open set. We define the collection of* diffused subbodies *of Ω as*

$$\Theta(\Omega) = \{\vartheta \in C_c(\Omega) : 0 \leq \vartheta \leq 1 \text{ on } \Omega\}.$$

Definition 1.1.2. *A* power of order $k \in \mathbb{N}$ *is a function*

$$P : \Theta(\Omega) \times C^\infty(\Omega; \mathbb{R}^N) \to \mathbb{R}$$

such that

1. *for every $\boldsymbol{v} \in C^\infty(\Omega; \mathbb{R}^N)$, $P(\vartheta, \boldsymbol{v}) = P(\vartheta_1, \boldsymbol{v}) + P(\vartheta_2, \boldsymbol{v})$ whenever $\vartheta, \vartheta_1, \vartheta_2 \in \Theta(\Omega)$ satisfy $\vartheta = \vartheta_1 + \vartheta_2$;*

2. *for every $\vartheta \in \Theta(\Omega)$, $P(\vartheta, \cdot)$ is linear;*

3. *for every compact set $K \subseteq \Omega$ there exists $c_K \geq 0$ such that for every $\vartheta \in \Theta(\Omega)$ with $\operatorname{supt}\vartheta \subseteq K$ and for every $\boldsymbol{v} \in C^\infty(\Omega, \mathbb{R}^N)$,*

$$|P(\vartheta, \boldsymbol{v})| \leq c_K \sum_{j=0}^{k} ||\nabla^{(j)}\boldsymbol{v}||_{\infty, \operatorname{supt}\vartheta},$$

where $||\nabla^{(j)}\boldsymbol{v}||_{\infty,S} := \sup\left\{|\nabla^{(j)}\boldsymbol{v}(x)| : x \in S\right\}$.

It can be proved also the following theorem from [17] that provides us of an integral representation for the power P. Preliminarly we denote $\mathfrak{M}(\Omega)$ the set of positive Borel measures finite on compact subsets of Ω. Given an integer $N \geq 1$, we define, for $j \geq 1$

$$\operatorname{Sym}_j := \left\{\boldsymbol{f} : (\mathbb{R}^n)^j \to \mathbb{R}^N : \boldsymbol{f} \text{ is } j-\text{linear and symmetric}\right\}.$$

We denote with Sym_j^* the dual space of Sym_j.

Theorem 1.1.3. *For every power P of order k there exist $k+1$ measures $\mu_j \in \mathfrak{M}(\Omega)$ and $k+1$ Borel maps $T_j : \Omega \to \operatorname{Sym}_j^*$ such that $|T_j| = 1$ $\mu_j-a.e.$ and*

$$\forall\vartheta \in \Theta(\Omega), \forall\boldsymbol{v} \in C^\infty(\Omega, \mathbb{R}^N) : P(\vartheta, \boldsymbol{v}) = \sum_{j=0}^{k} \int_\Omega \vartheta \langle T_j, \nabla^{(j)}\boldsymbol{v}\rangle d\mu_j.$$

Moreover, the tensor-valued measures $T_j d\mu_j$ are uniquely determined.

For example, let us suppose that a velocity field and a volumetric density of forces is defined on a body or on one of its subbodies M, then, by simplifying the notation, the quantity

$$P(M,\boldsymbol{v}) = \int_M \boldsymbol{f}\cdot\boldsymbol{v}\, d\mathscr{L}^n$$

defines a virtual power of zero order, that is the power of the applied forces. But if we consider a deformable body, considering only the velocity field is not enough and this is why we need also gradients of higher order of the velocity field, as in Theorem 1.1.3.

Having in mind various virtual powers, the inner one (often also called *inner working*), the external one, a contact one, which is essential and responsible for contact interactions, and finally an inertial power which represents *à la d'Alembert* the contribution of inertia to the lost forces, we can set the main balance assumption as follows:

Axiom 1.1.4 (**D'Alembert Principle**). *Let be given a mechanical system. In every inertial reference we have an admissible motion if and only if the resultant of all the stresses on the system is null.*

At this point we can consider an equilibrium problem through balance equations in integral form, thanks to

Axiom 1.1.5 (**Principle of Virtual Powers**). *For all instants t and for all subbody M, the motion of a continuous body is such that the total virtual power of the strain applied to the subbody, both internal and external, vanishes for every virtual velocity field considered.*

where a virtual velocity is a kinematic admissible field with the geometric constraint on the subbody. The Principle of the Virtual Powers has been introduced and employed for the first times in [1], [2], where it is proved that the integral laws of motion for continua is equivalent to the Principle of Virtual Powers.

Moreover it is important to require also that the model we consider does not depend on the reference system and hence

Axiom 1.1.6 (Material invariance). *The power spent by the internal forces on every rigid velocity field is null at each instant.*

With the cited principles we have a wide range of possible fluids and hence, in order to simplify the handling, we have to add some constitutive axioms, namely some constraints on the state of motion. Moreover it is important to remark that the setting of the order of the gradient is nothing but another modelling choice that allows us to explain the forces that are internal to the material. Moreover, with this kind of approach we can also analyze different physical situations that otherwise could not be modelled with a different choice of the order k of the velocity test in the formulation of the power. In this thesis our attention will be focused on second gradient fluids.

1.2 Second-gradient fluids

The aim of classical studies of Newtonian fluids is to model viscosity; we can observe that during a rigid motion we cannot distinguish the behavior of viscous and ideal fluids. Hence, it is necessary to link the viscous interactions with the shear between adjacent fluid layers. Such interaction opposes to the shear and this suggests the idea that the viscous interactions should be proportional to the symmetric part of the gradient of the velocity field that is forced to be modified by such interactions. Therefore it is natural to assume internal power expenditures of first order and hence the velocity field as a linear form on the Sobolev space $H^1(\Omega; \mathbb{R}^3)$, where Ω is the domain in $\mathbb{R}^3$.

But in this thesis we choose fields in $H^2(\Omega; \mathbb{R}^3)$ and this choice is supported by some argumentations. In this way the velocity field results continuous and it is possible to consider adherence to one-dimensional structures, that otherwise could not be done; furthermore it is possible to consider new boundary interaction with respect to the classical that are assumed; moreover corresponding PDEs are well posed. For example with fluids of second gradient we can study a slender body falling in a fluid: without that hypothesis the problem may have no sense. It is of interest, in this way, to study second-order fluids in $L^2([0,T];H^2(\Omega;\mathbb{R}^3))$, namely also square-summable in time (see [7], [8]).

When we want to model a fluid, the motion is described by a vector field in eulerian coordinates, that is the velocity field, and a scalar quantity, that represents the pressure field; these two fields are the unknown of the motion. With our choice of modeling the power of internal stresses is of second gradient, namely

$$P_{int}(M,\boldsymbol{v}) = \int_M \left(\mathsf{T} : \nabla\boldsymbol{v} + \mathbb{G}\,\vdots\,\nabla\nabla\boldsymbol{v}\right)\, d\mathscr{L}^n \tag{1.2.1}$$

for every subbody M of the body and for every velocity field $\boldsymbol{v}$. Let us obseve that the classical choice expects only the first term. With this formulation we have the Cauchy stress tensor T for the part of first gradient, whereas a third-order tensor $\mathbb{G}$, called *hyperstress*, takes in accounting for the second gradient. It is clear that the only difference with models of first-gradient fluids is the additional term $\mathbb{G}\,\vdots\,\nabla\nabla\boldsymbol{v}$, that turns out to be a generalization.

For first-gradient fluids, in correspondence with the internal expediture of forces, it is defined also the power of external stresses, that is the volumetric one

$$P_{ext}(M,\boldsymbol{v}) = \int_M \rho\boldsymbol{b}\cdot\boldsymbol{v}, \tag{1.2.2}$$

the power of inertial forces,

$$P_{iner}(M,\boldsymbol{v}) = -\int_M \rho\boldsymbol{a}\cdot\boldsymbol{v} \tag{1.2.3}$$

and the contact power

$$P_c(M,\boldsymbol{v}) = \int_{\partial M} \boldsymbol{t}\cdot\boldsymbol{v}. \tag{1.2.4}$$

Hence, for second-gradient fluids to be balanced (see Germain [1]) it is natural to add to contact powers another term, called *hypertraction*

$$P_c(M,\boldsymbol{v}) = \int_{\partial M} \boldsymbol{m}\cdot\frac{\partial\boldsymbol{v}}{\partial n}; \tag{1.2.5}$$

vectors $\boldsymbol{t}$ and $\boldsymbol{m}$ hence represent the traction and hypertraction on the boundary surface.

1.2.1 Balance equation of virtual powers and boundary conditions

At this point, if we apply the principle 1.1.5, we get that for all virtual velocity $\boldsymbol{v}$ and for all subbody M of the body, for a second-gradient fluid the balance

$$P_{int}(M, \boldsymbol{v}) = P_{ext}(M, \boldsymbol{v}) + P_{iner}(M, \boldsymbol{v}) + P_c(M, \boldsymbol{v})$$

holds, whence we deduce the integral form

$$\int_M \mathsf{T} : \nabla \boldsymbol{v} + \int_M \mathbb{G} \vdots \nabla\nabla \boldsymbol{v} = \int_M \rho(\boldsymbol{b} - \boldsymbol{a}) \cdot \boldsymbol{v} + \int_{\partial M} \left(\boldsymbol{t} \cdot \boldsymbol{v} + \boldsymbol{m} \cdot \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \right). \quad (1.2.6)$$

By this new balance equation we can deduce a new macroscopic balance law, namely

$$\operatorname{div} (\mathsf{T} - \operatorname{div} \mathbb{G}) + \rho \boldsymbol{b} = \rho \boldsymbol{a} \quad (1.2.7)$$

instead of $\operatorname{div} \mathsf{T} + \rho \boldsymbol{b} = \rho \boldsymbol{a}$. Moreover, the term at the boundary is no longer $\boldsymbol{t}(\boldsymbol{n}) = \mathsf{T}\boldsymbol{n}$, but now it is replaced by the conditions

$$\boldsymbol{t} = \mathsf{T}\boldsymbol{n} - (\operatorname{div} \mathbb{G})\boldsymbol{n} - \operatorname{div} (\mathbb{G}\boldsymbol{n}) - 2K\,(\mathbb{G}\boldsymbol{n})\,\boldsymbol{n}, \quad \boldsymbol{m} = (\mathbb{G}\boldsymbol{n})\,\boldsymbol{n}, \quad (1.2.8)$$

where K is the mean curvature of the boundary ∂M.

It is worth remarking that this theory, as stressed by Fried and Gurtin in [4], is independent of constitutive relations and hance we can use it both for fluid and solid materials. Actually, the theory we are interested in is that of incompressible fluids, expecially at small-length scales, so the density of the fluid is constant and we can give a costitutive equation, namely the solenoidality of the velocity field; therefore, we can express the stess in the form

$$\mathsf{T} = \mathsf{T}_0 - P\mathsf{I},$$

where T_0 is the residual stress and P represents the pressure; moreover the hyperstress has the form

$$\mathbb{G} = \mathbb{G}_0 - \mathsf{I} \otimes \boldsymbol{\pi},$$

where $\mathbb{G}_0$ is the residual hyperstress in its first two indices and $\boldsymbol{\pi}$ is the *hyperpressure*.

As in [4], the free energy imbalance can be written as a dissipation inequality

$$\mathsf{T}_0 : \mathsf{D} + \mathbb{G}_0 \vdots \nabla\nabla \boldsymbol{v} \geq 0,$$

where D is the symmetric part of the gradient of velocity, and if considered linear isotropic relations, Fried and Gurtin assumed

$$\begin{cases} T_{0ij} = 2\mu D_{ij} = \mu(v_{i,j} + v_{j,i}), \\ G_{0ijk} = \eta_1 v_{i,jk} + \eta_2(v_{k,ij} + v_{j,ik} - v_{i,rr}\delta_{jk}), \end{cases} \tag{1.2.9}$$

where we notice that in addition to the viscosity μ, there are other two coefficients η_1 and η_2.

In this way we get a new flow equation, that for second-gradient fluids, that is

$$\rho \boldsymbol{a} = \rho \boldsymbol{b} - \nabla p + \mu \Delta \boldsymbol{v} - \zeta \Delta\Delta \boldsymbol{v} \tag{1.2.10}$$

with

$$p := P - \operatorname{div} \boldsymbol{\pi} \quad \text{and} \quad \zeta := \eta_1 - \eta_2.$$

We can observe that (1.2.10) is the classical Navier-Stokes equation for incompressible material with an additional term proportional to $\Delta\Delta\boldsymbol{v}$ with the coefficient ζ, called *hyperviscosity*. The dissipation inequality gives us two conditions on the viscosity and the hyperviscosity, namely $\mu > 0$ and $\zeta > 0$; we can also introduce the length

$$L := \sqrt{\frac{\zeta}{\mu}}, \tag{1.2.11}$$

called *effective thickness* of the lower-dimensional objects.

Additionally, we have to provide the problem with the boundary conditions; a common choice for this problem (see [4]) is

$$\boldsymbol{v} = 0 \quad \text{and} \quad \boldsymbol{m} = -\mu\ell \frac{\partial \boldsymbol{v}}{\partial \boldsymbol{n}} \text{ on } \partial M,$$

where $\ell \geq 0$ is a material length that takes into account the adherence of the fluid to the boundary.

In [5] is presented a particular model of viscous, incompressible and isotropic fluids. As already mentioned, the constraint of incompressibility, namely of constant mass density, leads to the constraint $\operatorname{div}\boldsymbol{v} = 0$. In this model the main assumption is that T_0 and $\mathbb{G}_0$ are linear and isotropic functions of respectively D and $\nabla\nabla\boldsymbol{v}$. As a consequence, we have that such a model is quite general, since these are the only two assumptions. These hypothesis leads to $\mathsf{T}_0 = 2\mu\mathsf{D}$, that is the well-known form for the residual stress, whereas for $i, j, k = 1, \dots, N$, where N is the dimension of the space, and

$$G_{0ijk} = \eta_1 v_{i,jk} + \eta_2(v_{k,ij} + v_{j,ik} - v_{i,ss}\delta_{jk}) + \eta_3(v_{k,ss}\delta_{ij} + v_{j,ss}\delta_{ik} - 4v_{i,ss}\delta_{jk}),$$

that contains also the form assumed by Fried and Gurtin in [4] by setting $\eta_3 = 0$. From the dissipation inequality it is possible to obtain

$$\mathsf{T}_0 : \nabla\boldsymbol{v} = 2\mu\mathsf{D} : \nabla\boldsymbol{v} \geq 0$$

and

$$\mathbb{G}_0 \vdots \nabla\nabla\boldsymbol{v} = \eta_1 v_{i,jk}v_{i,jk} + \eta_2(v_{k,ij}v_{i,jk} + v_{j,ik}v_{i,jk} - v_{i,ss}v_{i,jj}) - 4\eta_3 v_{i,ss}v_{i,jj} \geq 0$$

whence the inequalities $\mu \geq 0$ and $\zeta = \eta_1 - \eta_2 - 4\eta_3 \geq 0$ are deduced.

What is the main reason to considering this special class of fluids in addition to the classical models?

1.2.2 Applications of second-gradient fluids

A first application we mention is that proposed by Giusteri and Fried in [6]. Here a new slender-body theory is introduced and it can be applied to flat bodies, elongated bodies or point-like spherical particles. In particular, the hyperviscous regularization presented above is a tool to find a solution for the flow past a translating particle and such a solution well approximates the classical solution for a point-like sphere.

In [7] and [8] Giusteri, Marzocchi and Musesti analyze the free fall problem. The model consists in the free fall, due to gravity, of a slender rigid body Σ, in a viscous fluid that occupies all the space; the initial state of both the fluid and the rigid body is quiet.

An application in which the advantage of using the second-gradient theory is apparent is that of fluid in a cylinder dragged by the motion of a wire (see [16]).

If the fluid flows between two coaxial cylinders of radii $R_1 < R_2$ and the velocity of the inner cylinder is U along the axis $\boldsymbol{e}_z$, with the outer at rest, then the classical theory for viscous fluids (with viscosity μ) with such boundary conditions, gives the solution

$$u(r) = U \frac{\log R_2 - \log r}{\log R_2 - \log R_1}. \tag{1.2.12}$$

If we want to model a wire, namely setting $R_1 = 0$, this has no limit. In this case the second-gradient theory helps us providing the solution

$$u(r) = \alpha_1 + \alpha_2 I_0\left(\frac{r}{L}\right) + \alpha_3 \log\left(\frac{r}{L}\right) + \alpha_4 K_0\left(\frac{r}{L}\right) \tag{1.2.13}$$

where $\alpha_i, i = 1, \ldots, 4$ are constants depending on the radii R_1 and R_2, the constant L is given by the definition (1.2.11), revised in the light of Musesti's arguments in [5], and I_0 and K_0 are the Bessel functions. Now we can compute the solution in $R_1 = 0$, since, under condition $\alpha_3 = \alpha_4$, the quantity

$$\lim_{r\to 0}\left(\log\left(\frac{r}{L}\right) + K_0\left(\frac{r}{L}\right)\right)$$

is finite.

Chapter 2

The Hyperviscous Navier-Stokes IBVP

2.1 Preliminaries

Definition 2.1.1. *Let K a bounded domain. We define* exterior domain *the interior of the complement of K.*

The following inequality is proved in Crispo and Maremonti [9] and it is similar to Gagliardo-Nirenberg inequality. We recall the result below.

Let first

$$\widehat{W}^{m,p}(\Omega) = \left\{ u \in L^1_{loc}(\Omega) : D^a u \in L^p(\Omega), |a| = m \right\}.$$

Theorem 2.1.2. *Let Ω be an exterior domain of $\mathbb{R}^n$ and w be in $\widehat{W}^{m,p}(\Omega) \cap L^q(\Omega)$, $p \in [1, +\infty]$, $q \geq 1$. Then, for $k \in \{0, 1, \ldots, m-1\}$*

$$||D^k w||_r \leq c_1 ||D^m w||_p^a ||w||_q^{1-a}, \tag{2.1.1}$$

where the constant c_1 is independent of w and

$$\frac{1}{r} = \frac{k}{n} + \left(\frac{1}{p} - \frac{m}{n} \right) a + \frac{1-a}{q},$$

with $a \in [\frac{k}{m}, 1]$ either if $p = 1$ or if $p > 1$ and $m - k - \frac{n}{p} \notin \mathbb{N} \cup \{0\}$, while $a \in [\frac{k}{m}, 1)$ if $p > 1$ and $m - j - \frac{n}{p} \in \mathbb{N} \cup \{0\}$. The result also holds if $q = +\infty$;

however, in the case $k = 0$ and $mp < n$ the following additional condition is required: w tends to zero at infinity or $w \in L^{q'}(\Omega)$ for some finite $q' \geq 1$.

A similar result holds for bounded domains:

Theorem 2.1.3. *Let Ω be a bounded domain of R^n and w be in $\widehat{W}^{m,p}(\Omega) \cap L^q(\Omega)$, $p, q \in [1, +\infty]$, $q \geq 1$. Then, for $k \in \{0, 1, \ldots, m-1\}$,*

$$||D^k w||_r \leq c_1 ||D^m w||_p^a ||w||_q^{1-a} + c_2 ||w||_q, \tag{2.1.2}$$

where the constants c_1 and c_2 are independent of w and

$$\frac{1}{r} = \frac{k}{n} + \left(\frac{1}{p} - \frac{m}{n}\right) a + \frac{1-a}{q},$$

with $a \in [\frac{k}{m}, 1]$ either if $p = 1$ or if $p > 1$ and $m - k - \frac{n}{p} \notin \mathbb{N} \cup \{0\}$, while $a \in [\frac{k}{m}, 1)$ if $p > 1$ and $m - j - \frac{n}{p} \in \mathbb{N} \cup \{0\}$. The result also holds for $\Omega = \mathbb{R}^n$, with $c_2 = 0$. In this case if $q = +\infty$, $k = 0$ and $mp < n$ the following additional condition is required: w tends to zero at infinity or $w \in L^{q'}(\Omega)$ for some finite $q' \geq 1$.

Theorem 2.1.4 (Poincaré inequality). *For all $p \in [1, +\infty)$ and for every bounded domain Ω we have*

$$\forall u \in W_0^{1,p}(\Omega) : ||u||_p \leq \max\left\{1, \frac{(n-1)p}{n}\right\} (\mathcal{L}^n(\Omega))^{\frac{1}{n}} \prod_{j=1}^{n} ||D_j u||_p^{\frac{1}{n}}. \tag{2.1.3}$$

The following theorem from [15] is a foundamental property about reflexive spaces:

Theorem 2.1.5 (Kakutani's Theorem). *Let X be a reflexive space and $\{x_h\}_{h\in\mathbb{N}}$ be a bounded sequence in X. Then there exist $x \in X$ and a subsequence $\{x_{h_k}\}_{k\in\mathbb{N}}$ of $\{x_h\}_{h\in\mathbb{N}}$ such that $x_{h_k} \rightharpoonup x$.*

2.2 Formulation of the problem

Let $\Omega \subseteq \mathbb{R}^3$ be a possibly unbounded domain of class C^4. Let be given $f \in C([0,T]; L^2(\Omega))$ and the solenoidal initial datum $u_0 \in H_0^2(\Omega)$.

The problem we want to take into consideration is described by the following system, where u is a time-dependent vector field, the velocity, and p is a time-dependent scalar field, the pressure:

$$\begin{cases} \dfrac{\partial u}{\partial t} + (\nabla u)\, u + \nabla p = \nu \Delta u - \tau \Delta\Delta u + f & \text{in } (0,T) \times \Omega \\ \operatorname{div} u = 0 & \text{in } (0,T) \times \Omega, \\ u(0,x) = u_0(x) & \text{in } \Omega \\ u(t,x) = \dfrac{\partial u}{\partial n}(t,x) = 0 & \text{on } \partial\Omega \\ \lim\limits_{|x| \to \infty} u(t,x) = 0 & \end{cases} \tag{2.2.1}$$

which is the classical incompressible Navier-Stokes equations with kinematical viscosity ν and with the additional hyperviscous term $\tau\Delta\Delta u$, where $\tau > 0$.

We want to study the existence and uniqueness of solutions for this system, precisely regular solutions that we will define soon, first in a bounded domain Ω and then to extend the argument also in the case in which Ω is an exterior domain.

2.2.1 Hydrodynamic spaces

Let Ω be a domain in $\mathbb{R}^3$. We will denote with $||\cdot||_p$ the classical L^p-norm and with $||\cdot||_{m,p}$ the $W^{m,p}(\Omega)$-norm for $m > 0$, where it is understood that $W^{0,p}(\Omega) = L^p(\Omega)$ and where we set as usual $W^{m,2}(\Omega) := H^m(\Omega)$. We know that

$$||u||_{m,p}^p = \sum_{i=0}^{m} ||\nabla^i u||_p^p.$$

Definition 2.2.1. *For all $p \in [1,\infty)$ we denote with $W_0^{m,p}(\Omega)$ the closure in $W^{m,p}(\Omega)$ of $C_c^\infty(\Omega)$. We also set $W_0^{m,2}(\Omega) := H_0^m(\Omega)$.*

In particular we know from [12] that, if the domain is bounded, then the norm $||u||_{m,p}$ is equivalent to the norm $||\nabla^m u||_p$.

Definition 2.2.2. *Let us introduce the sets*

$$\begin{aligned} G^q(\Omega) &:= \left\{ v = \nabla\pi \in L^q(\Omega),\ \textit{for some } \pi \in W^{1,q}_{loc}(\Omega) \right\}, \\ \mathscr{C}_{div}(\Omega) &:= \left\{ \varphi \in C^\infty_c(\Omega) \textit{ and } \operatorname{div}\varphi = 0 \right\}. \end{aligned} \tag{2.2.2}$$

Then we define $J^q(\Omega)$ as the completion of $\mathscr{C}_{div}(\Omega)$ with respect to the $L^q(\Omega)$-norm and the space $J^{m,q}(\Omega)$ as the completion of $\mathscr{C}_{div}(\Omega)$ with respect to the $W^{m,q}(\Omega)$-norm.

Let us also observe that $J^{0,q}(\Omega)$ is nothing but the space $J^q(\Omega)$. In [10] it is proved that $J^q(\Omega)$ and $J^{m,q}(\Omega)$ coincide with the following sets

$$\begin{aligned} J^q(\Omega) &:= \left\{ v \in L^q(\Omega) \textit{ and } (v,\nabla\pi) = 0,\ \textit{for all } \pi \in W^{1,q'}_{loc}(\Omega) \textit{ with } \nabla\pi \in L^{q'}(\Omega) \right\}, \\ J^{m,q}(\Omega) &:= \left\{ v \in W^{m,q}(\Omega) \textit{ and } (v,\nabla\pi) = 0,\ \textit{for all } \pi \in W^{1,q'}_{loc}(\Omega) \textit{ with } \nabla\pi \in L^{q'}(\Omega) \right\}, \end{aligned} \tag{2.2.3}$$

where $\dfrac{1}{q} + \dfrac{1}{q'} = 1$ with $q > 1$.

In the same work [10], it is proved proved that

$$L^2(\Omega) = J^2(\Omega) \oplus G^2(\Omega). \tag{2.2.4}$$

Now we can state the definition of regular solution.

Definition 2.2.3. *We say that (u,p) is a regular solution to problem* (2.2.1) *if, for all $\eta \in (0,T)$, we have*

- $u \in C([0,T]; J^2(\Omega)) \cap L^2((\eta,T); J^{2,2}(\Omega) \cap H^4(\Omega))$;
- $u_t, \nabla p \in L^2((\eta,T); L^2(\Omega))$;
- *u satisfies equation* (2.2.1) *a.e. in (t,x).*

2.3 The Hyperviscous Stokes problem

In order to prove the existence of the solutions of our problem we need to introduce the analogous of the Stokes problem, that is the one with the additional bilaplacian of u; we will call it the *Hyperviscous Stokes problem.* As in the Stokes problem, we drop the non-linear term of the Navier-Stokes system and moreover we consider the stationary case. Under these assumptions the system becomes

$$\begin{cases} \Delta\Delta u - \alpha\Delta u + \nabla p = f \\ \operatorname{div} u = 0 \\ u(x) = 0, \text{ on } \partial\Omega \\ \dfrac{\partial u}{\partial n}(x) = 0, \text{ on } \partial\Omega \end{cases} \tag{2.3.1}$$

for some $u \in W_0^{2,2}(\Omega)$, $p \in L^2(\Omega)$, $f \in L^2(\Omega)$ and where we put $\alpha := \nu/\tau \geq 0$ and with p and f we mean the original ones divided by τ. Let us notice that α can vanish: indeed, this is the particular case in which in the equation the laplacian disappears, whereas the term of the fourth order, i.e. the bilaplacian, that characterize our problem, is always present.

2.3.1 Sobolev spaces

Equivalent norms

Before dealing with our problem we show the equivalence between some norms. Indeed, since we are concerned with $m = 2$ and $p = 2$ with regard to (2.2.3), we will need to know that

Theorem 2.3.1. *Given a bounded domain Ω and a vector field $v \in H^2(\Omega)$ with $v = \dfrac{\partial v}{\partial n} = 0$ on $\partial\Omega$, then*

$$||v||_{2,2}^2 = ||v||_2^2 + ||\nabla v||_2^2 + ||\Delta v||_2^2. \tag{2.3.2}$$

Proof. The first step is to prove that the norms $||\nabla\nabla v||_2$ and $||\Delta v||_2$ coincide. Suppose first that $v \in C_c^\infty(\Omega)$. By the boundary conditions we can deduce that

$$\int_\Omega |\Delta v_k|^2 = \int_\Omega \sum_{i,j} \frac{\partial^2 v_k}{\partial x_i^2}\frac{\partial^2 v_k}{\partial x_j^2} = -\int_\Omega \sum_{i,j} \frac{\partial^3 v_k}{\partial x_i^2 \partial x_j}\frac{\partial v_k}{\partial x_j} =$$
$$= -\int_\Omega \sum_{i,j} \frac{\partial^3 v_k}{\partial x_j \partial x_i^2}\frac{\partial v_k}{\partial x_j} = \int_\Omega \sum_{i,j} \frac{\partial^2 v_k}{\partial x_j \partial x_i}\frac{\partial^2 v_k}{\partial x_j \partial x_i} = \int_\Omega |\nabla\nabla v_k|^2 \tag{2.3.3}$$

where v_k is the k-th component of v, therfore

$$\int_\Omega |\Delta v|^2 = \int_\Omega |\nabla\nabla v|^2.$$

Then, we show the result for a function $v \in H_0^2(\Omega)$. Since $H_0^2(\Omega)$ is the completion of $C_c^\infty(\Omega)$ with respect to the $H^2(\Omega)$-norm, there exists a sequence $(v_h)_{h\in\mathbb{N}} \subseteq C_c^\infty(\Omega)$ such that $v_h \longrightarrow v$ in $H_0^2(\Omega)$; this fact implies $||v - v_h||_{2,2} \to 0$ and in turn also $||\nabla\nabla v - \nabla\nabla v_h||_2 \to 0$; hence, from (2.3.3), we get also the convergence $||\Delta v - \Delta v_h||_2 \to 0$ and, since $||\Delta v_h||_2 \to ||\Delta v||_2$ and $||\nabla\nabla v_h||_2 \to ||\nabla\nabla v||_2$, we finally get $||\Delta v||_2 = ||\nabla\nabla v||_2$ for a vector field $v \in H_0^2(\Omega)$. From the definition of H^2-norm, we get the claim. □

Definition 2.3.2. *Let Ω be a domain and u, v two vector fields in $H_0^2(\Omega)$. Let us denote by $D(u,v)$ the bilinear form*

$$D(u,v) := \int_\Omega \Delta u \cdot \Delta v + \alpha \int_\Omega \nabla u : \nabla v. \tag{2.3.4}$$

Remark 2.3.3. Given a bounded domain Ω, the norm induced by $D(u,v)$, namely

$$D(u,u) = \int_\Omega |\Delta u|^2 + \alpha \int_\Omega |\nabla u|^2 = ||\Delta u||_2^2 + \alpha ||\nabla u||_2^2, \tag{2.3.5}$$

is equivalent to the $H_0^2(\Omega)$-norm.

Indeed, by the previous argument it follows

$$D(u,u) = ||\Delta u||_2^2 + \alpha ||\nabla u||_2^2 \leq C\left(||u||_2^2 + ||\nabla u||_2^2 + ||\nabla\nabla u||_2^2\right) = C||u||_{2,2}^2. \tag{2.3.6}$$

Then, by the Poincaré inequality 2.1.4, we obtain

$$||u||_{2,2}^2 = ||u||_2^2 + ||\nabla u||_2^2 + ||\nabla\nabla u||_2^2 \leq C_\Omega\left(||\nabla u||_2^2 + \alpha ||\Delta u||_2^2\right) = C_\Omega\left[D(u,u)\right] \tag{2.3.7}$$

for some constant C_Ω.

2.3.2 Existence, uniqueness and regularity of solutions for the Hyperviscous Stokes problem

In order to study the existence of solutions for the Hyperviscous Stokes problem, first of all let us notice that its weak formulation is

$$D(u,v) = (f,v) \tag{2.3.8}$$

for all $v \in J^{2,2}(\Omega)$. In fact, the weak formulation is

$$\int_\Omega \Delta u \cdot \Delta v + \alpha \int_\Omega \nabla u : \nabla v - \int_\Omega p \operatorname{div} v = \int_\Omega f \cdot v \tag{2.3.9}$$

and since the vector field v is solenoidal, it coincides with (2.3.8).

Theorem 2.3.4. *Let Ω be a bounded domain of class C^4 and let $f \in J^2(\Omega)$. Then, there exists a solution $u \in J^{2,2}(\Omega)$ of the problem*

$$D(u,v) = (f,v) \text{ for every } v \in J^{2,2}(\Omega).$$

Proof. The linear functional $v \mapsto \mathcal{L}v = (f,v)$ is continuous in $J^{2,2}(\Omega)$; indeed, using Hölder inequality and the definition of H_0^2-norm, we have

$$|\mathcal{L}v| = |(f,v)| = \left| \int_\Omega f \cdot v \right| \leq ||f||_2 ||v||_2 \leq ||f||_2 ||v||_{2,2}, \tag{2.3.10}$$

hence the functional is also continuous in the space $J^{2,2}(\Omega)$. From Remark 2.3.3 it is clear that $\mathcal{L}$ is also continuous with respect to the $D(\cdot,\cdot)$-norm. Then by the Riesz' representation theorem we have that there exists one and only one element $u \in J^{2,2}(\Omega)$ such as $\mathcal{L}u = D(u,v)$, for every $v \in J^{2,2}(\Omega)$. □

Remark 2.3.5. The same result can be proved for a datum $f \in L^2(\Omega)$. In fact, from (2.2.4), we can decompose f as $(f - \nabla\pi) + \nabla\pi$ where $f - \nabla\pi \in J^2(\Omega)$ and $\nabla\pi \in G^2(\Omega)$. In this way, replacing ∇p with $\nabla\widetilde{p} = \nabla(p - \pi)$, we are in the same situation as in the previous theorem.

Now we recall an important result by Amrouche and Girault in [11] that can be summarized in the following

Theorem 2.3.6. *Given Ω a bounded domain of class $C^{3,1}$ and $f \in L^2(\Omega)$. Then, there exists a unique solution $u \in H^4(\Omega)$ and $p \in H^1(\Omega)$ (p up to an additive constant) of the problem* (2.3.1) *with $\alpha = 0$ and the following inequality*

$$||u||_{4,2} + ||p||_{1,2} \leq C||f||_2, \tag{2.3.11}$$

holds, where C is a constant depending on Ω.

Following the reasoning of the above paper, it is not difficult to show that a similar result holds also for $\Delta\Delta - \alpha\Delta$ instead of $\Delta\Delta$. It is worth noticing that $\Delta u \in L^2(\Omega)$ in this case and is therefore homogeneous to f. We will therefore assume that (2.3.11) holds also for $\alpha \geq 0$.

At this point, we need to recover the pressure field. Let us recall the following result [10]:

Lemma 2.3.1. *Let $v \in L^2_{loc}(\Omega)$ such that*

$$\int_\Omega v \cdot \varphi = 0, \text{ for all } \varphi \in \mathscr{C}_{div}(\Omega). \tag{2.3.12}$$

Then there exists a function $p \in W^{1,2}_{loc}(\Omega)$ such that $v = \nabla p$.

The existence of a solution u for the Hyperviscous Stokes problem can be obtained by choosing

$$v = f - \Delta\Delta u + \alpha\Delta u$$

and using Lemma 2.3.1. In this way we recover our equation

$$\Delta\Delta u - \alpha\Delta u + \nabla p = f \tag{2.3.13}$$

that has a unique solution $u \in H^4(\Omega)$, $p \in H^1(\Omega)$ (p up to an additive constant).

Definition 2.3.7. *Let $P : L^2(\Omega) \to J^2(\Omega)$ and $P^\perp : L^2(\Omega) \to G^2(\Omega)$ denote respectively the projection of $L^2(\Omega)$ on $J^2(\Omega)$ and the projection of $L^2(\Omega)$ on $G^2(\Omega)$, so that*

$$w = Pw + P^\perp w$$

and

$$\forall\ w \in L^2(\Omega) : (Pw, P^\perp w) = (Pw, \nabla\pi_w) = 0$$

hold.

Let us observe that we can decompose $f \in L^2(\Omega)$ as $Pf + P^\perp f$ and that we can write $P^\perp f = \nabla p_f$; hence, if we set $\pi = p - p_f$, we finally obtain

$$\Delta\Delta u - \alpha\Delta u + \nabla\pi = Pf. \tag{2.3.14}$$

Since all the terms are square-summable (we just proved with Theorem 2.3.6 that $u \in H^4(\Omega)$ and so $\Delta\Delta u \in L^2(\Omega)$) and and since by definition of $J^2(\Omega)$ we have $P(\nabla\pi) = 0$, we can apply the operator P to the latter equation in order to get

$$P\Delta\Delta u - \alpha P\Delta u = P(Pf). \tag{2.3.15}$$

Let us observe that we do not lose generality if we consider $f \in J^2(\Omega)$, by redefining the pressure, in the regularity results due to Amrouche and Girault and so, in this case, $P(Pf) = f$.

2.3.3 Eigenvalue problem for the Hyperviscous Stokes problem

Now we want to consider the weak form of the eigenvalue problem for the Hyperviscous Stokes operator. It consists in finding functions $u \in J^{2,2}(\Omega)$, with $u \neq 0$, and numbers λ such that

$$\forall v \in J^{2,2}(\Omega): \; D(u,v) = \lambda (u,v). \tag{2.3.16}$$

The set of the eigenfunctions correspondent to an eigenvalue λ constitutes a linear subspace of the principal space $J^{2,2}(\Omega)$. So, we can introduce a subspace of $J^{2,2}(\Omega)$ linked to the differential operator

$$\mathscr{L}u = P\Delta\Delta u - \alpha P\Delta u, \tag{2.3.17}$$

restricted to the fields where $u = \dfrac{\partial u}{\partial n} = 0$ on $\partial\Omega$. Then, let us introduce the subspace of $J^{2,2}(\Omega)$ given by the domain of $\mathscr{L}$

$$D_{\mathscr{L}}(\Omega) = \left\{u \in J^{2,2}(\Omega) : P\Delta\Delta u - \alpha P\Delta u \in J^2(\Omega)\right\}; \tag{2.3.18}$$

clearly, the constraint of solenoidality on u is preserved also after applying the operator $\mathscr{L}$; this subspace is endowed with the following norm and scalar product

$$||u||_{\mathscr{L}} = \sqrt{D(u,u)}, \quad (u,v)_{\mathscr{L}} = D(u,v). \tag{2.3.19}$$

Proposition 2.3.8. *The operator* $\mathscr{L} : D_{\mathscr{L}}(\Omega) \to J^2(\Omega)$ *verifies*

$$\forall u, v \in D_{\mathscr{L}}(\Omega) : (\mathscr{L}u, v) = (u, \mathscr{L}v) = D(u, v). \tag{2.3.20}$$

Proof. Since u, v are solenoidal and vanish at the boundary and considering the Helmholtz decomposition of the bilaplacian of w

$$\Delta\Delta w = P\Delta\Delta w + \nabla\pi_{\Delta\Delta w}, \tag{2.3.21}$$

we have

$$\begin{aligned} \int_\Omega P\Delta\Delta u \cdot v = \int_\Omega (\Delta\Delta u - \nabla\pi_{\Delta\Delta u}) \cdot v = \int_\Omega \Delta\Delta u \cdot v = \\ = \int_\Omega u \cdot \Delta\Delta v = \int_\Omega u \cdot (\Delta\Delta v - \nabla\pi_{\Delta\Delta v}) = \int_\Omega u \cdot P\Delta\Delta v \end{aligned} \tag{2.3.22}$$

whence, since the laplacian is self-adjoint, $u, v \in H_0^2(\Omega)$ and since a similar result holds for the term $P\Delta u$, we deduce that the operator $\mathscr{L}$ is self-adjoint too, namely

$$(\mathscr{L}u, v) = \int_\Omega (P\Delta\Delta u - \alpha P\Delta u) \cdot v = \int_\Omega u \cdot (P\Delta\Delta v - \alpha P\Delta v) = (u, \mathscr{L}v). \tag{2.3.23}$$

Moreover

$$\begin{aligned} (\mathscr{L}u, v) = \int_\Omega (P\Delta\Delta u - \alpha P\Delta u) \cdot v = \\ = \int_\Omega (\Delta\Delta u - \nabla\pi_{\Delta\Delta u}) \cdot v - \alpha \int_\Omega (\Delta u - \nabla\pi_{\Delta u}) \cdot v = \\ = \int_\Omega \Delta\Delta u \cdot v - \alpha \int_\Omega \Delta u \cdot v = \int_\Omega \Delta u \cdot \Delta v + \alpha \int_\Omega \nabla u : \nabla v = D(u, v), \end{aligned} \tag{2.3.24}$$

that completes the proof. □

Theorem 2.3.9. *The operator* $\mathscr{L} : D_{\mathscr{L}}(\Omega) \to J^2(\Omega)$ *is invertible with continuous inverse* $\mathscr{L}^{-1} : J^2(\Omega) \to J^{2,2}(\Omega)$*. Moreover, the operator* $\mathscr{L}^{-1}$ *is self-adjoint, namely*

$$\forall u, v \in J^2(\Omega) : \left(\mathscr{L}^{-1}u, v\right) = \left(u, \mathscr{L}^{-1}v\right).$$

Proof. For all $f \in J^2(\Omega)$ let us set $\mathscr{L}^{-1}f = u$ where $u \in J^{2,2}(\Omega)$ is the solution of the problem $\mathscr{L}u = f$ with $u = \dfrac{\partial u}{\partial n} = 0$. Then, by using Schwarz inequality and the definition of H^2-norm, we have

$$||u||^2_{\mathscr{L}} = D(u,u) = (\mathscr{L}u, u) = (f, u) \leq ||f||_2||u||_2 \leq ||f||_2||u||_{2,2} \qquad (2.3.25)$$

and remembering that the $D(\cdot,\cdot)$-norm and the H^2-norm are equivalent, there exists a positive constant K such that $||u||_{2,2} \leq K||u||_{\mathscr{L}}$, so that

$$||f||_2||u||_{2,2} \leq K||f||_2||u||_{\mathscr{L}} \qquad (2.3.26)$$

and thus we have

$$||u||_{\mathscr{L}} \leq K||f||_2. \qquad (2.3.27)$$

We can also write this last equation as

$$||\mathscr{L}^{-1}f||_{\mathscr{L}} \leq K||f||_2 \qquad (2.3.28)$$

which gives us the continuity of $\mathscr{L}^{-1}$. Finally, let $f, g \in J^2(\Omega)$; since $\mathscr{L}^{-1}f, \mathscr{L}^{-1}g \in D_{\mathscr{L}}(\Omega)$ and since by Proposition 2.3.8 we know that the operator $\mathscr{L}$ is self-adjoint, we have

$$\left(f, \mathscr{L}^{-1}g\right) = \left(\mathscr{L}\mathscr{L}^{-1}f, \mathscr{L}^{-1}g\right) = \left(\mathscr{L}^{-1}f, \mathscr{L}\mathscr{L}^{-1}g\right) = \left(\mathscr{L}^{-1}f, g\right), \qquad (2.3.29)$$

hence also the operator $\mathscr{L}^{-1}$ is self-adjoint. □

At this point it is useful to recall the Rellich-Kondrachov Theorem from [12] pag. 168.

Theorem 2.3.10. *Let Ω be a bounded domain in $\mathbb{R}^n$ and let Ω^k be the intersection of Ω with a k-dimensional plane in $\mathbb{R}^n$. Let $j \geq 0$ and $m \geq 1$ be integers, and let $1 \leq p < \infty$. If $mp > n$, then the following imbedding is compact:*

$$W^{j+m,p}(\Omega) \to W^{j,q}\left(\Omega^k\right) \;\; if \;\; 1 \leq q < \infty.$$

[1]

[1] The theorem verifies ad additional condition (the cone condition, see [12]) which is verified by our assumption $\partial\Omega \in C^4$.

To fit this theorem to our situation we must choose $k = n = 3$, $j = 0$, $q = m = p = 2$ that is in agreement with the hypothesis $mp > n$. So the imbedding $W^{2,2} \to L^2$ is compact and since $J^{2,2}$ is a closed subspace of $W^{2,2}$ also the imbedding $J^{2,2} \to L^2$ is compact. In this way we proved the theorem

Proposition 2.3.11. *The operator* $\mathscr{L}^{-1} : J^2(\Omega) \to J^{2,2}(\Omega)$ *is compact.*

Then, since the operator $\mathscr{L}$ is compact and self-adjoint we can deduce the classical properties of its eigenvalues and eigenvectors. The study of the eigenvalue problem for the Hyperviscous-Stokes can be reduced to the eigenvalue problem for the compact operator $\mathscr{L}^{-1}$ in J^2. The operator $\mathscr{L}^{-1}$ admits in $J^{2,2}$ a basis of eigenvectors $\{a_k\}$ corresponding to the eigenvalues $\{\mu_k\}$:

$$\mathscr{L}^{-1} a_k = \mu_k a_k \tag{2.3.30}$$

with $\lim\limits_{k\to\infty} \mu_k = 0$.

By applying the operator $\mathscr{L}$ to both members (2.3.30), we get

$$\mathscr{L} a_k = \frac{1}{\mu_k} a_k \tag{2.3.31}$$

where $\dfrac{1}{\mu_k} = \lambda_k$ and $\lim\limits_{k\to\infty} \lambda_k = +\infty$; we can choose the basis in $J^{2,2}$ and orthonormal in J^2, thus $(a_k, a_j) = \delta_{kj}$ and $[(\Delta a_k, \Delta a_j) + \alpha(\nabla a_k, \nabla a_j)] = \lambda_k \delta_{kj}$; obviously, the eigenfunctions $\{a_k\}$ satisfy

$$a_k = \frac{\partial a_k}{\partial n} = 0 \text{ on } \partial\Omega.$$

Remark 2.3.12. For all $k \in \mathbb{N}$ we have $a_k \in H^4(\Omega)$. Indeed, it is enough to apply the Theorem 2.3.6 to show such regularity on the eigenfunctions.

2.4 Existence of regular solutions for the weak Hyperviscous Navier-Stokes problem

In the following section we will present some theorems necessary in order to prove the central theorem of the thesis, which is the following

Theorem 2.4.1. *For all* $u_0 \in J^{2,2}(\Omega)$ *there exists at least one regular solution* (u, p) *to problem* (2.2.1) *on* $(0, T_{u_0})$. *Moreover,*

$$\begin{gathered} u \in C\left([0, T_{u_0}); J^{2,2}(\Omega)\right) \cap L^2\left(0, T_{u_0}; H^4(\Omega)\right); \\ u_t, \nabla p \in L^2\left(0, T_{u_0}; L^2(\Omega)\right). \end{gathered} \tag{2.4.1}$$

We want to apply the method of Galerkin approximations to (2.2.1) and hence we consider the previously found basis of eigenfunctions $\{a_k\}$ in $J^{2,2}(\Omega)$ and orthonormal in $J^2(\Omega)$. We will look for an approximate solution of the form

$$u^m(t, x) := \sum_{k=1}^{m} c_k^m(t) a_k(x) \tag{2.4.2}$$

where the coefficients $c_k^m \in C^1$ depend only on t; so, by deriving with respect to time, we obtain

$$u_t^m(t, x) := \sum_{k=1}^{m} \dot{c}_k^m(t) a_k(x), \tag{2.4.3}$$

where $u_t^m := \dfrac{\partial u^m}{\partial t}$ and the dot is the time derivative.

We impose that for all $m \in \mathbb{N}$ the coefficients $\{c_k^m(t)\}$ are solutions of the ODE Cauchy problem

$$\begin{cases} (u_t^m, a_k) + (P\Delta\Delta u^m - \alpha P\Delta u^m, a_k) + ((\nabla u^m) u^m, a_k) = 0, \quad t > 0 \\ c_k^m(0) := (u_0, a_k), \quad k = 1, \dots, m \end{cases} \tag{2.4.4}$$

that is the weak form of (2.2.1) taking $a_k \in J^{2,2}(\Omega)$ as test function and with $f = 0$. Since λ_k is an eigenvalue of the operator $\mathscr{L} = P\Delta\Delta - \alpha P\Delta$, we have

$$(P\Delta\Delta u^m - \alpha P\Delta u^m, a_k) = c_j^m \left[(\Delta a_j, \Delta a_k) + \alpha\left(\nabla a_j, \nabla a_k\right)\right] = c_j^m \lambda_j \delta_{jk} = \lambda_k c_k^m \tag{2.4.5}$$

and since u^m and u_t^m can be written in the form (2.4.2) and (2.4.3), the system (2.4.4) has the form

$$\begin{cases} \dot{c}_k^m(t) + \lambda_k c_k^m + A_{ijk} c_i^m c_j^m = 0, \\ c_k^m(0) := (u_0, a_k), \quad k = 1, \dots, m \end{cases} \tag{2.4.6}$$

with $A_{ijk} = ((\nabla a_i)a_j, a_k)$.
We notice also that

$$||\Delta a_k||_2^2+\alpha||\nabla a_k||_2^2 = \int_\Omega |\Delta a_k|^2+\alpha \int_\Omega |\nabla a_k|^2 = (\Delta a_k, \Delta a_k)+\alpha\left(\nabla a_k, \nabla a_k\right) = \lambda_k \tag{2.4.7}$$

and that

$$(\Delta u_0, \Delta a_k) + \alpha\left(\nabla u_0, \nabla a_k\right) = \lambda_k\left(u_0, a_k\right).$$

Now we show some estimates on the initial term, namely

Theorem 2.4.2. *The following inequalities hold for some positive constant* C_α

$$||u^m(0)||_2^2 \leq C_\alpha ||u_0||_2^2, \tag{2.4.8}$$

$$\alpha||\nabla u^m(0)||_2^2 + ||\Delta u^m(0)||_2^2 \leq C_\alpha\left(||u_0||_2^2 + ||\nabla u_0||_2^2 + ||\Delta u_0||_2^2\right) \tag{2.4.9}$$

$$||\nabla u^m(0)||_2^2 \leq C_\alpha\left(||u_0||_2^2 + ||\nabla u_0||_2^2 + ||\Delta u_0||_2^2\right) \tag{2.4.10}$$

and

$$||\Delta u^m(0)||_2^2 \leq C_\alpha\left(||u_0||_2^2 + ||\nabla u_0||_2^2 + ||\Delta u_0||_2^2\right) \tag{2.4.11}$$

hold.

Proof. Preliminarly we need to prove that $c_k^m(t) = c_k^k(t)$. Remembering the second equality of (2.4.4), we can observe that if we equate the terms of order m in (2.4.2), we have

$$\begin{array}{ll} m=1 & u^1(0) = c_1^1(0)a_1 = (u_0, a_1)a_1 \\ m=2 & u^2(0) = c_1^2(0)a_1 + c_2^2(0)a_2 = (u_0, a_1)a_1 + (u_0, a_2)a_2 \\ m=3 & u^3(0) = c_1^3(0)a_1 + c_2^3(0)a_2 + c_3^3(0)a_3 = (u_0, a_1)a_1 + (u_0, a_2)a_2 + (u_0, a_3)a_3 \end{array} \tag{2.4.12}$$

so that by induction $c_k^m(0) = c_k^k(0)$.

Now, since

$$u^m(0) = \sum_{k=1}^m c_k^m(0)a_k, \tag{2.4.13}$$

while for the initial datum we have in $L^2(\Omega)$

$$\begin{aligned} u_0 = \sum_{k=1}^{\infty}(u_0, a_k)_{J^{2,2}} a_k = \sum_{k=1}^{\infty}\left[(\Delta u_0, \Delta a_k) + \alpha(\nabla u_0, \nabla a_k)\right] a_k = \\ = \sum_{k=1}^{\infty} \lambda_k (u_0, a_k) a_k = \sum_{k=1}^{\infty} \lambda_k c_k^k(0) a_k, \end{aligned} \tag{2.4.14}$$

where we used the identity $c_k^m(t) = c_k^k(t)$; since the basis $\{a_k\}$ is orthonormal in $J^2(\Omega)$, we get

$$||u^m(0)||^2 = \int_\Omega \sum_{k=1}^{m} c_k^m(0)^2 \leq \int_\Omega \sum_{k=1}^{\infty} c_k^k(0)^2 \leq C_\alpha \int_\Omega \sum_{k=1}^{\infty} \lambda_k^2 c_k^k(0)^2 = C_\alpha ||u_0||^2,$$

that gives (2.4.8).

Now, we can observe that our problem can be rewritten as

$$P\Delta\Delta u - \alpha P \Delta u + \frac{\alpha^2}{4} u = f + \frac{\alpha^2}{4} u \tag{2.4.15}$$

hence, a_k are also the eigenfunctions of the operator

$$P\Delta\Delta - \alpha P\Delta + \frac{\alpha^2}{4}\mathbb{I} := \mathscr{A}^2,$$

and $\lambda_k' = \lambda_k + \dfrac{\alpha^2}{4}$ are the correspondent eigenvalues. Therefore

$$\left(\mathscr{A}^2 u_0, a_k\right) = \lambda_k'(u_0, a_k) \tag{2.4.16}$$

and hence

$$(\mathscr{A} u_0, \mathscr{A} a_k) = \lambda_k'(u_0, a_k). \tag{2.4.17}$$

We deduce also $(\mathscr{A} a_j, \mathscr{A} a_k) = \lambda_j' \delta_{jk}$ and $||\mathscr{A} a_k||_2^2 = \lambda_k'$.

Then, by applying the operator $\mathscr{A}$ to both sides of (2.4.13), we get

$$\mathscr{A} u^m(0) = \sum_{k=1}^{m} c_k^m(0) \mathscr{A} a_k = \sum_{k=1}^{m} (u_0, a_k) \frac{\lambda_k'}{\lambda_k'} \mathscr{A} a_k = \sum_{k=1}^{m} (\mathscr{A} u_0, \mathscr{A} a_k) \frac{\mathscr{A} a_k}{||\mathscr{A} a_k||_2^2} \tag{2.4.18}$$

and then

$$\begin{aligned} ||\mathscr{A} u^m(0)||_2^2 = \sum_{k=1}^{m} (\mathscr{A} u_0, \mathscr{A} a_k)^2 \frac{||\mathscr{A} a_k||_2^2}{||\mathscr{A} a_k||_2^4} = \sum_{k=1}^{m} \left(\mathscr{A} u_0, \frac{\mathscr{A} a_k}{||\mathscr{A} a_k||_2}\right)^2 \leq \\ \leq \sum_{k=1}^{\infty} \left(\mathscr{A} u_0, \frac{\mathscr{A} a_k}{||\mathscr{A} a_k||_2}\right)^2 = ||\mathscr{A} u_0||_2^2. \end{aligned} \tag{2.4.19}$$

Rewriting the last inequality we find

$$||\Delta u^m(0)||_2^2 + \alpha||\nabla u^m(0)||_2^2 + \frac{\alpha^2}{4}||u^m(0)||_2^2 \leq ||\Delta u_0||_2^2 + \alpha||\nabla u_0||_2^2 + \frac{\alpha^2}{4}||u_0||_2^2 \tag{2.4.20}$$

whence we deduce (2.4.9), (2.4.10) and (2.4.11). □

2.4.1 Energy relations

Theorem 2.4.3. *Let* $\{a_k\} \in J^{2,2}(\Omega)$ *so that* $u^m \in J^{2,2}(\Omega)$. *Then for all* $m \in \mathbb{N}$ *the following equalities*

$$a)\quad \frac{1}{2}\frac{d}{dt}||u^m||_2^2 + ||\Delta u^m||_2^2 + \alpha||\nabla u^m||_2^2 = 0,\ t > 0, \tag{2.4.21}$$

$$b)\quad \frac{1}{2}\frac{d}{dt}\left(||\Delta u^m||_2^2 + \alpha||\nabla u^m||_2^2\right) + ||P\Delta\Delta u^m||_2^2 + \alpha^2||P\Delta u^m||_2^2 + \\ -2\alpha\left(P\Delta\Delta u^m, P\Delta u^m\right) + \left(\left(\nabla u^m\right)u^m, P\Delta\Delta u^m\right) - \alpha\left(\left(\nabla u^m\right)u^m, P\Delta u^m\right) = 0, \tag{2.4.22}$$

$$c)\quad \frac{1}{2}\frac{d}{dt}\left(||\Delta u^m||_2^2 + \alpha||\nabla u^m||_2^2\right) + ||u_t^m||_2^2 + \left(\left(\nabla u^m\right)u^m, u_t^m\right) = 0. \tag{2.4.23}$$

hold. Moreover, $||u^m||_2^2$ *is not increasing in time.*

Proof. a) If we multiply (2.4.4) for $c_k^m(t)$ and sum on k, we have

$$\sum_{k=1}^m c_k^m\left(u_t^m, a_k\right) + \sum_{k=1}^m c_k^m\left(P\Delta\Delta u^m - \alpha P\Delta u^m, a_k\right) + \sum_{k=1}^m c_k^m\left(\left(\nabla u^m\right)u^m, a_k\right) = 0$$

and by linearity we can also write

$$\left(u_t^m, \sum_{k=1}^m c_k^m a_k\right) + \left(P\Delta\Delta u^m - \alpha P\Delta u^m, \sum_{k=1}^m c_k^m a_k\right) + \left(\left(\nabla u^m\right)u^m, \sum_{k=1}^m c_k^m a_k\right) = 0$$

that is, remembering (2.4.2),

$$\left(u_t^m, u^m\right) + \left(P\Delta\Delta u^m - \alpha P\Delta u^m, u^m\right) + \left(\left(\nabla u^m\right)u^m, u^m\right) = 0. \tag{2.4.24}$$

The first term is

$$\left(u_t^m, u^m\right) = \left(\frac{d}{dt}u^m, u^m\right) = \frac{1}{2}\frac{d}{dt}||u^m||_2^2,$$

while the second one is, after some easy calculation and for (2.4.7),

$$(P\Delta\Delta u^m - \alpha P\Delta u^m, u^m) = (\Delta u^m, \Delta u^m) + \alpha\,(\nabla u^m, \nabla u^m) = \\ = (c_k^m)^2\lambda_k = ||\Delta u^m||_2^2 + \alpha||\nabla u^m||_2^2.$$

Finally, the last term of (2.4.24) vanishes: indeed, for $u \in J^{1,2}(\Omega)$ with $u = 0$ on $\partial\Omega$,

$$\operatorname{div}(u \otimes u) = (\operatorname{div} u)u + (\operatorname{grad} u)u \tag{2.4.25}$$

holds, hence we get

$$\int_\Omega (\nabla u)u \cdot u = \int_\Omega \operatorname{div}((u \otimes u)u) - \int (u \otimes u) : \nabla u = -\int_\Omega (\nabla u)u \cdot u, \tag{2.4.26}$$

whence

$$\int_\Omega (\nabla u)u \cdot u = 0. \tag{2.4.27}$$

Hence we obtain

$$\frac{1}{2}\frac{d}{dt}||u^m||_2^2 + ||\Delta u^m||_2^2 + \alpha||\nabla u^m||_2^2 = 0,\ t > 0. \tag{2.4.28}$$

By this identity, since

$$\frac{1}{2}\frac{d}{dt}||u^m||_2^2 = -\left(||\Delta u^m||_2^2 + \alpha||\nabla u^m||_2^2\right), \tag{2.4.29}$$

we see that the kinetic energy of the approximating term $||u^m||_2^2$ is not increasing in time.

b) Now, if we multiply (2.4.4) by $\lambda_k c_k^m(t)$ and if we sum on k, we have, as in the previous case,

$$\sum_{k=1}^m \lambda_k c_k^m\,(u_t^m, a_k) + \sum_{k=1}^m \lambda_k c_k^m\,(P\Delta\Delta u^m - \alpha P\Delta u^m, a_k) + \sum_{k=1}^m \lambda_k c_k^m\,((\nabla u^m)\,u^m, a_k) = 0.$$

The first term, remembering (2.4.7), is given by

$$\sum_{k=1}^m \lambda_k c_k^m\,(u_t^m, a_k) = \\ = \sum_{k=1}^m (\Delta a_k, \Delta a_k)\,c_k^m\,(u_t^m, a_k) + \alpha\sum_{k=1}^m (\nabla a_k, \nabla a_k)\,c_k^m\,(u_t^m, a_k) = \\ = \left(\frac{d}{dt}\Delta u^m, \Delta u^m\right) + \alpha\left(\frac{d}{dt}\nabla u^m, \nabla u^m\right) = \frac{1}{2}\frac{d}{dt}||\Delta u^m||_2^2 + \frac{\alpha}{2}\frac{d}{dt}||\nabla u^m||_2^2;$$

while the second term, since $P\Delta\Delta a_k - \alpha P\Delta a_k = \lambda_k a_k$, by (2.3.31) is

$$\begin{aligned}
&\sum_{k=1}^{m} \lambda_k c_k^m \left(P\Delta\Delta u^m - \alpha P\Delta u^m, a_k\right) = \\
&= \left(P\Delta\Delta u^m - \alpha P\Delta u^m, \sum_{k=1}^{m} \left(\lambda_k a_k\right) c_k^m\right) = \\
&= \left(P\Delta\Delta u^m - \alpha P\Delta u^m, \sum_{k=1}^{m} c_k^m \left(P\Delta\Delta a_k - \alpha P\Delta a_k\right)\right) = \\
&= \left(P\Delta\Delta u^m - \alpha P\Delta u^m, P\Delta\Delta u^m - \alpha P\Delta u^m\right) = \\
&= ||P\Delta\Delta u^m||_2^2 + \alpha^2 ||P\Delta u^m||_2^2 - 2\alpha \left(P\Delta\Delta u^m, P\Delta u^m\right);
\end{aligned}$$

the last term, since again $P\Delta\Delta a_k - \alpha P\Delta a_k = \lambda_k a_k$, becomes

$$\sum_{k=1}^{m} \lambda_k c_k^m \left(\left(\nabla u^m\right) u^m, a_k\right) = \left(\left(\nabla u^m\right) u^m, P\Delta\Delta u^m - \alpha P\Delta u^m\right).$$

Collecting all terms, we obtain the second identity

$$\begin{aligned}
&\frac{1}{2}\frac{d}{dt}\left(||\Delta u^m||_2^2 + \alpha ||\nabla u^m||_2^2\right) + ||P\Delta\Delta u^m||_2^2 + \alpha^2 ||P\Delta u^m||_2^2 + \\
&-2\alpha \left(P\Delta\Delta u^m, P\Delta u^m\right) + \left(\left(\nabla u^m\right) u^m, P\Delta\Delta u^m\right) - \alpha \left(\left(\nabla u^m\right) u^m, P\Delta u^m\right) = 0.
\end{aligned} \tag{2.4.22}$$

c) Finally, if we multiply (2.4.4) for $\dot{c}_k^m(t)$ and if we sum on k, we have, remembering (2.4.3),

$$\left(u_t^m, u_t^m\right) + \left(P\Delta\Delta u^m - \alpha P\Delta u^m, u_t^m\right) + \left(\left(\nabla u^m\right) u^m, u_t^m\right) = 0.$$

The first term is simply

$$\left(u_t^m, u_t^m\right) = ||u_t^m||_2^2;$$

while the second is

$$\begin{aligned}
&\left(P\Delta\Delta u^m - \alpha P\Delta u^m, u_t^m\right) = \left(P\Delta\Delta u^m, u_t^m\right) - \alpha \left(P\Delta u^m, u_t^m\right) = \\
&= \left(\Delta u^m, \Delta u_t^m\right) + \alpha \left(\nabla u^m, \nabla u_t^m\right) = \frac{1}{2}\frac{d}{dt}||\Delta u^m||_2^2 + \frac{\alpha}{2}\frac{d}{dt}||\nabla u^m||_2^2
\end{aligned}$$

and the third remains unchanged. Hence, we get

$$\frac{1}{2}\frac{d}{dt}\left(||\Delta u^m||_2^2 + \alpha ||\nabla u^m||_2^2\right) + ||u_t^m||_2^2 + \left(\left(\nabla u^m\right) u^m, u_t^m\right) = 0, \tag{2.4.23}$$

which completes the proof. □

Remark 2.4.4. We can observe that in the case with f not null such relations are different, but this is not a problem for our aim. Following the calculations above we can find the following energy relations. The first one reads

$$\frac{1}{2}\frac{d}{dt}||u^m||_2^2 + ||\Delta u^m||_2^2 + \alpha||\nabla u^m||_2^2 = (f, u^m) \tag{2.4.30}$$

and considering that $(f, u^m) \leq \frac{1}{2}||f||_2^2 + \frac{1}{2}||u^m||_2^2$, we can also obtain

$$\frac{d}{dt}||u^m||_2^2 + 2||\Delta u^m||_2^2 + 2\alpha||\nabla u^m||_2^2 \leq ||f||_2^2 + ||u^m||_2^2 \tag{2.4.31}$$

and moreover

$$\frac{d}{dt}||u^m||_2^2 \leq ||f||_2^2 + ||u^m||_2^2. \tag{2.4.32}$$

Hence, integrating it from 0 to t, with $t \in [0, T]$, and using (2.4.8) and (2.4.64) that will be proved next, we get

$$\begin{aligned} ||u^m(t)||_2^2 &\leq ||u^m(0)||_2^2 + \int_0^t ||f||_2^2 + \int_0^t ||u^m||_2^2 \leq \\ &\leq C_\alpha||u_0||_2^2 + \big|\big|\, ||f||_2 \big|\big|_{L^2(0,T)}^2 + C_\alpha T||u_0||_2^2 \end{aligned} \tag{2.4.33}$$

whence

$$||u^m(t)||_2^2 \leq C_\alpha(1 + T)||u_0||_2^2 + \big|\big|\, ||f||_2 \big|\big|_{L^2(0,T)}^2, \tag{2.4.34}$$

that replaces the calculations in which we will use $||u^m||_2^2$ not increasing.

The second energy relation becomes

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(||\Delta u^m||_2^2 + \alpha||\nabla u^m||_2^2\right) + ||P\Delta\Delta u^m||_2^2 + \alpha^2||P\Delta u^m||_2^2 + \\ &-2\alpha\left(P\Delta\Delta u^m, P\Delta u^m\right) + \left(\left(\nabla u^m\right)u^m, P\Delta\Delta u^m\right) - \alpha\left(\left(\nabla u^m\right)u^m, P\Delta u^m\right) = \\ &= (f, P\Delta\Delta u^m - \alpha P\Delta u^m). \end{aligned} \tag{2.4.35}$$

Considering that, by Young's inequality,

$$(f, P\Delta\Delta u^m - \alpha P\Delta u^m) \leq c_\varepsilon||f||_2^2 + \varepsilon||P\Delta\Delta u^m||_2^2 + \varepsilon\alpha||P\Delta u^m||_2^2$$

with $\varepsilon < \alpha$, we come up also to

$$\frac{1}{2}\frac{d}{dt}\left(||\Delta u^m||_2^2 + \alpha||\nabla u^m||_2^2\right) + (1-\varepsilon)||P\Delta\Delta u^m||_2^2 + \alpha(\alpha-\varepsilon)||P\Delta u^m||_2^2 + \\ -2\alpha\left(P\Delta\Delta u^m, P\Delta u^m\right) + \left(\left(\nabla u^m\right)u^m, P\Delta\Delta u^m\right) - \alpha\left(\left(\nabla u^m\right)u^m, P\Delta u^m\right) \le c_\varepsilon||f||_2^2. \tag{2.4.36}$$

The third energy relation is

$$\frac{1}{2}\frac{d}{dt}\left(||\Delta u^m||_2^2 + \alpha||\nabla u^m||_2^2\right) + ||u_t^m||_2^2 + \left(\left(\nabla u^m\right)u^m, u_t^m\right) = (f, u_t^m). \tag{2.4.37}$$

whence

$$\frac{d}{dt}\left(||\Delta u^m||_2^2 + \alpha||\nabla u^m||_2^2\right) + ||u_t^m||_2^2 + 2\left(\left(\nabla u^m\right)u^m, u_t^m\right) \le ||f||_2^2. \tag{2.4.38}$$

Anyway, for simplicity, from now on we will consider the case with vanishing f.

2.4.2 Functional estimates

Let us prove now some estimates which we will need later. The first inequality we want to prove is given by the

Proposition 2.4.5. *Let Ω be a Lipschitz domain in $\mathbb{R}^3$ and $u \in H_0^2(\Omega)$. Then, there exists a constant $\widetilde{C}$ independent of the domain such that*

$$||\nabla u||_3 \le \widetilde{C}\left(||\nabla u||_2^{\frac{1}{2}}||\Delta u||_2^{\frac{1}{2}} + ||\nabla u||_2\right). \tag{2.4.39}$$

Proof. In order to prove (2.4.39) we choose the inequality given in theorem 2.1.3, since it is valid for bounded domains and it is enough to put $c_2 = 0$ to recover the case of exterior domains; then we take $w = D_i u_k$ (for all $i, k = 1, 2, 3$, namely $D_i u_k$ is each component of the gradient ∇u_k), $n = 3$, $k = 0$, $r = 3$, $m = 1$, $p = 2$, $a = 1/2$, $q = 2$. So, we obtain

$$\begin{aligned} ||D_i u_k||_3 &\le c_1\left(||D_i u_k||_2^{\frac{1}{2}}||D_j(D_i u_k)||_2^{\frac{1}{2}}\right) + c_2||D_i u_k||_2 \le \\ &\le c_1\left(||\nabla u_k||_2^{\frac{1}{2}}||\nabla\nabla u_k||_2^{\frac{1}{2}}\right) + c_2||\nabla u_k||_2 \end{aligned} \tag{2.4.40}$$

and hence, summing all the components that form the gradient and by redefining the constants c_1 and c_2, we get

$$||\nabla u||_3 \le c_1\left(||\nabla u||_2^{\frac{1}{2}}||\nabla\nabla u||_2^{\frac{1}{2}}\right) + c_2||\nabla u||_2 \tag{2.4.41}$$

Finally, in theorem 2.3.1 we proved that $||\nabla\nabla u||_2 = ||\Delta u||_2$ and thus, setting $\widetilde{C} := \max\{c_1, c_2\}$, we get (2.4.39). □

Theorem 2.4.6. *Let* Ω *be a possibly unbounded domain and let be given* $u \in J^{2,2}(\Omega)$ *and* $w \in J^2(\Omega)$.

Then, an estimate on the generic trilinear term $((\nabla u)\, u, w)$ *is*

$$|((\nabla u)\, u, w)| \leq \overline{c}\left(||\nabla u||_2^6 + ||\nabla u||_2^4\right) + \frac{1}{2}\left(||\Delta u||_2^2 + ||w||_2^2\right). \tag{2.4.42}$$

Proof. Applying Young's inequality, Hölder's inequality, Sobolev's inequality and (2.4.39), we get

$$\begin{aligned}
|((\nabla u)\, u, w)| &\leq ||(\nabla u)u||_2||w||_2 \leq \frac{1}{2}||(\nabla u)u||_2^2 + \frac{1}{2}||w||_2^2 \leq \\
&\leq \frac{1}{2}||\nabla u||_3^2||u||_6^2 + \frac{1}{2}||w||_2^2 \leq \frac{c}{2}||\nabla u||_3^2||\nabla u||_2^2 + \frac{1}{2}||w||_2^2 \leq \\
&\leq \frac{c}{2}\left[\widetilde{C}\left(||\nabla u||_2^{\frac{1}{2}}||\Delta u||_2^{\frac{1}{2}} + ||\nabla u||_2\right)\right]^2 ||\nabla u||_2^2 + \frac{1}{2}||w||_2^2 \leq \\
&\leq c\widetilde{C}^2\left(||\nabla u||_2||\Delta u||_2 + ||\nabla u||_2^2\right)||\nabla u||_2^2 + \frac{1}{2}||w||_2^2 = \\
&= k||\nabla u||_2^3||\Delta u||_2 + k||\nabla u||_2^4 + \frac{1}{2}||w||_2^2 \leq \\
&\leq \frac{k}{2}||\nabla u||_2^6 + \frac{1}{2}||\Delta u||_2^2 + k||\nabla u||_2^4 + \frac{1}{2}||w||_2^2 \leq \\
&\leq \overline{c}\left(||\nabla u||_2^4 + ||\nabla u||_2^6\right) + \frac{1}{2}\left(||\Delta u||_2^2 + ||w||_2^2\right),
\end{aligned} \tag{2.4.43}$$

having set the constants $\overline{c} := \max\{k/2, k\}$ that is independent of the Lebesgue measure of the domain Ω. The proof is complete. □

Theorem 2.4.7. *The inequality*

$$||(\nabla u)u||_2^2 \leq \widetilde{c}\left(||\Delta u||_2^2 + ||\nabla u||_2^2\right)^2 + c||u||_2^4 \tag{2.4.44}$$

holds.

Proof. By using Hölder's inequality, we have

$$\int |u|^3 = \left(\int |u|^6\right)^{\frac{1}{4}} \left(\int |u|^2\right)^{\frac{3}{4}} \tag{2.4.45}$$

whence we easy get

$$||u||_3 \leq ||u||_6^{\frac{1}{2}} ||u||_2^{\frac{1}{2}}. \tag{2.4.46}$$

Then, by using the previous inequality, once again Hölder's inequality, Young's inequality and the following Sobolev Embedding Theorem in the forms $H_0^2(\Omega) \subseteq W_0^{1,6}(\Omega)$ and $H_0^1(\Omega) \subseteq L^6(\Omega)$, we get

$$\begin{gathered} ||(\nabla u)u||_2^2 \leq ||\nabla u||_6^2 ||u||_3^2 \leq ||\nabla u||_6^2 ||u||_6 ||u||_2 \leq \\ \leq k(||u||_2^2 + ||\Delta u||_2^2)||\nabla u||_2 ||u||_2 \leq \frac{k}{2}(||u||_2^2 + ||\Delta u||_2^2)\,(||\nabla u||_2^2 + ||u||_2^2) \leq \\ \leq \widetilde{c}\,(||\Delta u||_2^2 + ||\nabla u||_2^2)^2 + c||u||_2^4, \end{gathered} \tag{2.4.47}$$

that is the claim.

□

2.4.3 A priori estimates and inequalities for solutions

Let u^m be defined by (2.4.2). We list and prove some *a priori* estimates for the approximating solutions. But before we need the following

Remark 2.4.8.

$$||P\Delta v||_2^2 \leq ||\Delta v||_2^2. \tag{2.4.48}$$

Proof. Since by definition $\Delta v = P\Delta v + \nabla \pi_{\Delta v}$ holds with $P\Delta v \in J^2(\Omega)$ and $\nabla \pi_{\Delta v} \in G^2(\Omega)$, we get

$$\begin{gathered} ||\Delta v||_2^2 = \int_\Omega |\Delta v|^2 = \int_\Omega |P\Delta v|^2 + \int_\Omega |\nabla \pi_{\Delta v}|^2 + 2\int_\Omega P\Delta v \cdot \nabla \pi_{\Delta v} = \\ = ||P\Delta v||_2^2 + ||\nabla \pi_{\Delta v}||_2^2 \geq ||P\Delta v||_2^2. \end{gathered} \tag{2.4.49}$$

□

Theorem 2.4.9. *The inequality*

$$\begin{gathered} \frac{d}{dt}\,(||u^m||_2^2 + ||\nabla u^m||_2^2 + ||\Delta u^m||_2^2) + \\ + ||\nabla u^m||_2^2 + ||\Delta u^m||_2^2 + ||P\Delta\Delta u^m||_2^2 + ||u_t^m||_2^2 \leq \\ \leq c_\alpha\,(||\nabla u^m||_2^6 + ||\nabla u^m||_2^4) \end{gathered} \tag{2.4.50}$$

holds and in particular also

$$\frac{d}{dt}\left(||u^m||_2^2 + ||\nabla u^m||_2^2 + ||\Delta u^m||_2^2\right) \leq c_\alpha \left(||\nabla u^m||_2^6 + ||\nabla u^m||_2^4\right), \tag{2.4.51}$$

where $c_\alpha > 0$ and is independent of the size of Ω.

Proof. Let us consider a positive integer $M > 1 + \alpha + 4\alpha^2$; if we sum M times (2.4.21) with (2.4.22) and (2.4.23), we obtain

$$\begin{gathered}\frac{1}{2}\frac{d}{dt}\big(M||u^m||_2^2 + 2\alpha||\nabla u^m||_2^2 + 2||\Delta u^m||_2^2\big) + \\ +\alpha M||\nabla u^m||_2^2 + M||\Delta u^m||_2^2 + \alpha^2||P\Delta u^m||_2^2 + ||P\Delta\Delta u^m||_2^2 + ||u_t^m||_2^2 = \\ = 2\alpha\left(P\Delta\Delta u^m, P\Delta u^m\right) - \left((\nabla u^m)\,u^m, P\Delta\Delta u^m\right) + \\ +\alpha\left((\nabla u^m)\,u^m, P\Delta u^m\right) - \left((\nabla u^m)\,u^m, u_t^m\right).\end{gathered} \tag{2.4.52}$$

By estimating each term of the right-hand side with its absolute value and neglecting the term $||P\Delta u^m||_2^2$, we have

$$\begin{gathered}\frac{d}{dt}\left(M||u^m||_2^2 + 2\alpha||\nabla u^m||_2^2 + 2||\Delta u^m||_2^2\right) + \\ +2\alpha M||\nabla u^m||_2^2 + 2M||\Delta u^m||_2^2 + 2||P\Delta\Delta u^m||_2^2 + 2||u_t^m||_2^2 \leq \\ \leq 4\alpha\left|\left(P\Delta\Delta u^m, P\Delta u^m\right)\right| + 2\left|\left((\nabla u^m)\,u^m, P\Delta\Delta u^m\right)\right| + \\ +2\alpha\left|\left((\nabla u^m)\,u^m, P\Delta u^m\right)\right| + 2\left|\left((\nabla u^m)\,u^m, u_t^m\right)\right|.\end{gathered} \tag{2.4.53}$$

Now we also have that, using Cauchy-Schwarz inequality and Young inequality,

$$\begin{gathered}4\alpha\left|\left(P\Delta\Delta u^m, P\Delta u^m\right)\right| \leq 4\alpha||P\Delta\Delta u^m||_2||P\Delta u^m||_2 \leq \\ \leq 4\alpha\left(\frac{1}{8\alpha}||P\Delta\Delta u^m||_2^2 + 2\alpha||P\Delta u^m||_2^2\right),\end{gathered} \tag{2.4.54}$$

whence, using (2.4.48),

$$4\alpha\left|\left(P\Delta\Delta u^m, P\Delta u^m\right)\right| \leq \frac{1}{2}||P\Delta\Delta u^m||_2^2 + 8\alpha^2||\Delta u^m||_2^2. \tag{2.4.55}$$

Then, estimating each term at the right-hand side using (2.4.42) and (2.4.48), we have

$$\left|\left((\nabla u^m)\,u^m, P\Delta u^m\right)\right| \leq \overline{c}\left(||\nabla u^m||_2^6 + ||\nabla u^m||_2^4\right) + ||\Delta u^m||_2^2, \tag{2.4.56}$$

$$|((\nabla u^m)\, u^m, P\Delta\Delta u^m)| \leq \overline{c}\left(||\nabla u^m||_2^6 + ||\nabla u^m||_2^4\right) + \frac{1}{2}\left(||\Delta u^m||_2^2 + ||P\Delta\Delta u^m||_2^2\right) \tag{2.4.57}$$

and

$$|((\nabla u^m)\, u^m, u_t^m)| \leq \overline{c}\left(||\nabla u^m||_2^6 + ||\nabla u^m||_2^4\right) + \frac{1}{2}\left(||\Delta u^m||_2^2 + ||u_t^m||_2^2\right). \tag{2.4.58}$$

Hence,

$$\begin{aligned} &\frac{d}{dt}\left(M||u^m||_2^2 + 2\alpha||\nabla u^m||_2^2 + 2||\Delta u^m||_2^2\right) + \\ &+2\alpha M||\nabla u^m||_2^2 + 2M||\Delta u^m||_2^2 + 2||P\Delta\Delta u^m||_2^2 + 2||u_t^m||_2^2 \leq \\ &\leq 2\overline{c}(2+\alpha)\left(||\nabla u^m||_2^6 + ||\nabla u^m||_2^4\right) + (2 + 2\alpha + 8\alpha^2)||\Delta u^m||_2^2 + \\ &+\frac{3}{2}||P\Delta\Delta u^m||_2^2 + ||u_t^m||_2^2 \end{aligned} \tag{2.4.59}$$

and therefore

$$\begin{aligned} &\frac{d}{dt}\left(M||u^m||_2^2 + 2\alpha||\nabla u^m||_2^2 + 2||\Delta u^m||_2^2\right) + \\ &+2\alpha M||\nabla u^m||_2^2 + 2(M - 1 - \alpha - 4\alpha^2)||\Delta u^m||_2^2 + \frac{1}{2}||P\Delta\Delta u^m||_2^2 + ||u_t^m||_2^2 \leq \\ &\leq 2\overline{c}(2+\alpha)\left(||\nabla u^m||_2^6 + ||\nabla u^m||_2^4\right). \end{aligned} \tag{2.4.60}$$

Therefore, we can conclude that there exists $C > 0$ such that

$$\begin{aligned} &\frac{d}{dt}\left(M||u^m||_2^2 + 2\alpha||\nabla u^m||_2^2 + 2||\Delta u^m||_2^2\right) + \\ &+2\alpha M||\nabla u^m||_2^2 + 2(M - 1 - \alpha - 4\alpha^2)||\Delta u^m||_2^2 + \frac{1}{2}||P\Delta\Delta u^m||_2^2 + ||u_t^m||_2^2 \leq \\ &\leq C\left(||\nabla u^m||_2^6 + ||\nabla u^m||_2^4\right). \end{aligned} \tag{2.4.61}$$

holds with $C = 2\overline{c}(2+\alpha)$. Then, defining k as the minimum of all the coefficients in the left-hand side of the previous inequality, we get

$$\begin{aligned} &\frac{d}{dt}\left(||u^m||_2^2 + ||\nabla u^m||_2^2 + ||\Delta u^m||_2^2\right) + \\ &+||\nabla u^m||_2^2 + ||\Delta u^m||_2^2 + ||P\Delta\Delta u^m||_2^2 + ||u_t^m||_2^2 \leq \\ &\leq \frac{C}{k}\left(||\nabla u^m||_2^6 + ||\nabla u^m||_2^4\right). \end{aligned} \tag{2.4.62}$$

Finally, setting $c_\alpha := \dfrac{C}{k}$, we have (2.4.50) and, in particular, (2.4.51). □

Theorem 2.4.10. *The inequalities*

$$||u^m(t)||_2^2 + 2\int_0^t \left(||\Delta u^m(\tau)||_2^2 + \alpha||\nabla u^m(\tau)||_2^2\right) d\tau \leq C_\alpha ||u_0||_2^2, \qquad (2.4.63)$$

$$||u^m(t)||_2^2 \leq C_\alpha ||u_0||_2^2, \qquad (2.4.64)$$

$$\int_0^t \left(||\Delta u^m(\tau)||_2^2 + \alpha||\nabla u^m(\tau)||_2^2\right) d\tau \leq \frac{C_\alpha}{2}||u_0||_2^2, \qquad (2.4.65)$$

hold; in particular also

$$\int_0^t ||\nabla u^m(\tau)||_2^2 d\tau \leq \frac{C_\alpha}{2\alpha}||u_0||_2^2, \qquad (2.4.66)$$

$$\int_0^t ||\Delta u^m(\tau)||_2^2 d\tau \leq \frac{C_\alpha}{2}||u_0||_2^2 \qquad (2.4.67)$$

hold.

Proof. Integrating (2.4.21) from 0 to t, we have

$$\frac{1}{2}\left(||u^m(t)||_2^2 - ||u^m(0)||_2^2\right) + \int_0^t \left(||\Delta u^m(\tau)||_2^2 + \alpha||\nabla u^m(\tau)||_2^2\right) d\tau = 0 \qquad (2.4.68)$$

and by (2.4.8), we obtain the inequality

$$||u^m(t)||_2^2 + 2\int_0^t \left(||\Delta u^m(\tau)||_2^2 + \alpha||\nabla u^m(\tau)||_2^2\right) d\tau = ||u^m(0)||_2^2 \leq C_\alpha ||u_0||_2^2. \qquad (2.4.63)$$

Moreover, from (2.4.63) we get (2.4.64), (2.4.65), (2.4.66) and (2.4.67). □

As a direct consequence we can estimate the term

$$\int_0^t \left(||\nabla u^m(\tau)||_2^4 + ||\nabla u^m(\tau)||_2^2\right) d\tau \qquad (2.4.69)$$

with a bound independent of m.

Theorem 2.4.11. *The following estimate*

$$\int_0^t \left(||u^m(\tau)||_2^2 + ||u^m(\tau)||_2^4\right) d\,\tau \leq \frac{C_\alpha}{2\alpha}||u_0||_2^2 \left(1 + \alpha C_\alpha ||u_0||_2^2\right), \qquad (2.4.70)$$

holds for every $t > 0$, where C_α is a positive constant.

Proof. By integrating by parts, we have

$$\int_\Omega |\nabla u^m|^2 = -\int_\Omega u^m \cdot \Delta u^m \leq ||u^m||_2||\Delta u^m||_2 \tag{2.4.71}$$

and taking its square, we find

$$||\nabla u^m||_2^4 \leq ||u^m||_2^2||\Delta u^m||_2^2. \tag{2.4.72}$$

Now, by integrating it in time and by using (2.4.64) and (2.4.67), we get

$$\begin{aligned}\int_0^t ||\nabla u^m(\tau)||_2^4 d\tau \leq \int_0^t ||u^m(\tau)||_2^2||\Delta u^m(\tau)||_2^2 d\tau \leq \\ \leq C_\alpha||u_0||_2^2 \int_0^t ||\Delta u^m(\tau)||_2^2 d\tau \leq \frac{C_\alpha^2}{2}||u_0||_2^4.\end{aligned} \tag{2.4.73}$$

Then, using such estimate and (2.4.66), we obtain

$$\begin{aligned}\int_0^t \left(||\nabla u^m(\tau)||_2^2 + ||\nabla u^m(\tau)||_2^4\right) d\tau \leq \int_0^t ||\nabla u^m(\tau)||_2^2 d\tau + \int_0^t ||\nabla u^m(\tau)||_2^4 d\tau \leq \\ \leq \frac{C_\alpha}{2\alpha}||u_0||_2^2\left(1 + \alpha C_\alpha||u_0||_2^2\right),\end{aligned} \tag{2.4.74}$$

that is the claim. □

Theorem 2.4.12. *The nonlinear term satisfies*

$$(\nabla u^m)u^m \in L^2(0,T;L^1(\Omega)). \tag{2.4.75}$$

Proof. By using Hölder's inequality, we have

$$||(\nabla u^m)u^m||_1^2 = \left(\int_\Omega |\nabla u^m||u^m|\right)^2 \leq ||u^m||_2^2||\nabla u^m||_2^2 \tag{2.4.76}$$

that integrated in time and by using (2.4.64) and (2.4.66) is

$$\int_0^t |(\nabla u^m(\tau))u^m(\tau)||_1^2 d\tau \leq \int_0^t ||u^m(\tau)||_2^2||\nabla u^m(\tau)||_2^2 d\tau \leq \frac{C_\alpha^2}{2\alpha}||u_0||_2^4 \tag{2.4.77}$$

whence we have the claim. □

Theorem 2.4.13. *For all* $u_0 \in J^{2,2}(\Omega)$ *there exists* $T_{u_0} > 0$ *such that the quantities* $||\Delta u^m||_2^2 + ||\nabla u^m||_2^2$ *and* $||(\nabla u^m)u^m||_2^2$ *exist and are bounded in* $[0, T_{u_0}]$. *Conversely, we can also say that for all* $T > 0$ *there exists* $u_0 \in J^{2,2}(\Omega)$ *with* $||\Delta u^m||_2^2 + ||\nabla u^m||_2^2$ *small enough such that there exists the solution* u^m *in* $[0, T]$.

Moreover

$$u^m \in L^\infty([0, T_{u_0}]; J^{2,2}(\Omega)) \cap L^\infty([0, T_{u_0}]; J^{1,2}(\Omega)), \tag{2.4.78}$$

whence

$$u^m \in L^\infty([0, T_{u_0}]; W_0^{1,6}(\Omega)), \tag{2.4.79}$$

and

$$(\nabla u^m)u^m \in L^2(0, T_{u_0}; L^2(\Omega)). \tag{2.4.80}$$

Proof. Now, let us take (2.4.23) and using Hölder, Cauchy-Schwarz, Young inequality and (2.4.44), we have

$$\begin{aligned} ||u_t^m||_2^2 + \frac{1}{2}\frac{d}{dt}\left(||\Delta u^m||_2^2 + \alpha ||\nabla u^m||_2^2\right) &= -\left(\left(\nabla u^m\right)u^m, u_t^m\right) \leq \\ \leq |\left(\left(\nabla u^m\right)u^m, u_t^m\right)| &\leq ||\left(\nabla u^m\right)u^m||_2 ||u_t^m||_2 \leq \\ \leq \frac{||(\nabla u^m)u^m||_2^2}{2} + \frac{||u_t^m||_2^2}{2} &\leq \\ \leq \frac{\widetilde{c}}{2}\left(||\Delta u^m||_2^2 + ||\nabla u^m||_2^2\right)^2 + \frac{c}{2}||u^m||_2^4 + ||u_t^m||_2^2 & \end{aligned} \tag{2.4.81}$$

whence, by redefining the constants, we deduce

$$\frac{d}{dt}\left(||\Delta u^m||_2^2 + ||\nabla u^m||_2^2\right) \leq \overline{k}\left[\left(||\Delta u^m||_2^2 + ||\nabla u^m||_2^2\right)^2 + ||u^m||_2^4\right] \tag{2.4.82}$$

for a positive constant $\overline{k}$ and by (2.4.64), we have

$$\frac{d}{dt}\left(||\Delta u^m||_2^2 + ||\nabla u^m||_2^2\right) \leq \overline{k}\left(||\Delta u^m||_2^2 + ||\nabla u^m||_2^2\right)^2 + k_\alpha ||u_0||_2^4 \tag{2.4.83}$$

for some positive constants k_α and $\overline{k}$.

In this way, setting $\varphi(t) := ||\Delta u^m||_2^2 + ||\nabla u^m||_2^2$, we get

$$\frac{d\varphi(t)}{dt} \leq \widetilde{k}_\alpha\left(\varphi(t)^2 + 1\right) \tag{2.4.84}$$

with $\widetilde{k}_\alpha := \max\left\{\overline{k}, k_\alpha ||u_0||_2^4\right\}$. By solving it, we can deduce

$$\operatorname{arctg}\varphi(t) \le \operatorname{arctg}\varphi(0) + \widetilde{k}_\alpha t \tag{2.4.85}$$

which leads to

$$\varphi(t) \le \operatorname{tg}\left(\operatorname{arctg}\varphi(0) + \widetilde{k}_\alpha t\right), \tag{2.4.86}$$

that, remembering (2.4.9), is

$$||\Delta u^m||_2^2 + ||\nabla u^m||_2^2 \le \operatorname{tg}\left(\operatorname{arctg}(C_\alpha(||u_0||_2^2 + ||\nabla u_0||_2^2 + ||\Delta u_0||_2^2)) + \widetilde{k}_\alpha t\right), \tag{2.4.87}$$

whence we get (2.4.78) and as a consequence also (2.4.79).

We can notice that the solution exists for

$$\operatorname{arctg}(C_\alpha(||u_0||_2^2 + ||\nabla u_0||_2^2 + ||\Delta u_0||_2^2)) + \widetilde{k}_\alpha t < \frac{\pi}{2} \tag{2.4.88}$$

whence we have

$$T_{u_0} < \frac{1}{\widetilde{k}_\alpha}\left(\frac{\pi}{2} - \operatorname{arctg}\left(C_\alpha(||u_0||_2^2 + ||\nabla u_0||_2^2 + ||\Delta u_0||_2^2)\right)\right). \tag{2.4.89}$$

Finally, by recovering (2.4.44), (2.4.64) and by using (2.4.87), we get

$$\begin{aligned} ||(\nabla u^m)u^m||_2^2 \le \widetilde{c}(||\Delta u^m||_2^2 + ||\nabla u^m||_2^2)^2 + c||u^m||_2^4 \le \\ \le \widetilde{c}\left(\operatorname{tg}\left(\operatorname{arctg}\left(C_\alpha\left(||u_0||_2^2 + ||\nabla u_0||_2^2 + ||\Delta u_0||_2^2\right)\right) + \widetilde{k}_\alpha T_{u_0}\right)\right)^2 + k_\alpha ||u_0||_2^4 \end{aligned} \tag{2.4.90}$$

that, integrated in time, gives, for some constant $h(u_0)$,

$$\int_0^t ||(\nabla u^m)u^m||_2^2 \le T_{u_0} h(u_0), \tag{2.4.91}$$

which is (2.4.80) and completes the proof.

Let us remark that we can also obtain

$$\int_0^t ||(\nabla u^m)u^m||_2 \le T_{u_0} \overline{h}(u_0), \tag{2.4.92}$$

for a positive constant $\overline{h}(u_0)$. □

As a consequence of (2.4.87), we have

$$||\nabla u^m(t)||_2^2 \leq c_1(t) \tag{2.4.93}$$

and

$$||P\Delta u^m(t)||_2^2 \leq ||\Delta u^m(\tau)||_2^2 \leq c_1(t) \tag{2.4.94}$$

in $[0, T_{u_0})$ for a positive time-dependent function $c_1(t)$, whose expression is given by

$$c_1(t) := \operatorname{tg}\left(\operatorname{arctg}(C_\alpha(||u_0||_2^2 + ||\nabla u_0||_2^2 + ||\Delta u_0||_2^2)) + \widetilde{k}_\alpha t\right). \tag{2.4.95}$$

Theorem 2.4.14. *The inequality*

$$\int_0^t \left(||\nabla u^m(\tau)||_2^2 + ||\Delta u^m(\tau)||_2^2 + ||P\Delta\Delta u^m(\tau)||_2^2 + ||u_\tau^m(\tau)||_2^2\right) d\tau \leq c_2(t) \tag{2.4.96}$$

holds. In particular

$$\int_0^t ||P\Delta\Delta u^m(\tau)||_2^2 d\tau \leq c_2(t) \tag{2.4.97}$$

and

$$\int_0^t ||u_\tau^m(\tau)||_2^2 d\tau \leq c_2(t) \tag{2.4.98}$$

hold in $[0, T_{u_0})$ *for a time-dependent positive function* $c_2(t)$*, whose expression is given in* (2.4.102)*.*

Proof. Let us integrate (2.4.50) from 0 to t, namely

$$\begin{aligned} &||u^m(t)||_2^2 + ||\nabla u^m(t)||_2^2 + ||\Delta u^m(t)||_2^2 + \\ &+ \int_0^t \left(||\nabla u^m(\tau)||_2^2 + ||\Delta u^m(\tau)||_2^2 + ||P\Delta\Delta u^m(\tau)||_2^2 + ||u_\tau^m(\tau)||_2^2\right) d\tau \leq \\ &\leq ||u^m(0)||_2^2 + ||\nabla u^m(0)||_2^2 + ||\Delta u^m(0)||_2^2 + c_\alpha \int_0^t \left(||\nabla u^m(\tau)||_2^6 + ||\nabla u^m(\tau)||_2^4\right) d\tau. \end{aligned} \tag{2.4.99}$$

By (2.4.8), (2.4.10) and (2.4.11), we have

$$\begin{aligned} &\int_0^t \left(||\nabla u^m(\tau)||_2^2 + ||\Delta u^m(\tau)||_2^2 + ||P\Delta\Delta u^m(\tau)||_2^2 + ||u_\tau^m(\tau)||_2^2\right) d\tau \leq \\ &\leq \widetilde{C}_\alpha \left(||u_0||_2^2 + ||\nabla u_0||_2^2 + ||\Delta u_0||_2^2\right) + c_\alpha \int_0^t \left(||\nabla u^m(\tau)||_2^6 + ||\nabla u^m(\tau)||_2^4\right) d\tau. \end{aligned} \tag{2.4.100}$$

Now, by applying a direct consequence of (2.4.87), we obtain

$$\int_0^t \left(||\nabla u^m(\tau)||_2^2 + ||\Delta u^m(\tau)||_2^2 + ||P\Delta\Delta u^m(\tau)||_2^2 + ||u_\tau^m(\tau)||_2^2\right) d\tau \leq$$
$$\leq \widetilde{C}_\alpha \left(||u_0||_2^2 + ||\nabla u_0||_2^2 + ||\Delta u_0||_2^2\right) +$$
$$+c_\alpha \int_0^t \operatorname{tg}^3 \left(\operatorname{arctg}(C_\alpha(||u_0||_2^2 + ||\nabla u_0||_2^2 + ||\Delta u_0||_2^2)) + \widetilde{k}_\alpha T_{u_0}\right) d\tau +$$
$$+c_\alpha \int_0^t \operatorname{tg}^2 \left(\operatorname{arctg}(C_\alpha(||u_0||_2^2 + ||\nabla u_0||_2^2 + ||\Delta u_0||_2^2)) + \widetilde{k}_\alpha T_{u_0}\right) d\tau \leq c_2(t), \tag{2.4.101}$$

where we set

$$c_2(t) := \widetilde{C}_\alpha \left(||u_0||_2^2 + ||\nabla u_0||_2^2 + ||\Delta u_0||_2^2\right) +$$
$$+c_\alpha \left[\operatorname{tg}^3 \left(\operatorname{arctg}(C_\alpha(||u_0||_2^2 + ||\nabla u_0||_2^2 + ||\Delta u_0||_2^2)) + \widetilde{k}_\alpha T_{u_0}\right) + \right.$$
$$\left. + \operatorname{tg}^2 \left(\operatorname{arctg}(C_\alpha(||u_0||_2^2 + ||\nabla u_0||_2^2 + ||\Delta u_0||_2^2)) + \widetilde{k}_\alpha T_{u_0}\right)\right] t; \tag{2.4.102}$$

in this way we get, for $t \in [0, T_{u_0})$, inequalities (2.4.96), (2.4.97) and (2.4.98). □

Let us observe that (2.4.63), (2.4.93), (2.4.94), (2.4.96) estimate the quantities

$$||u^m(t)||_2^2, \quad ||\nabla u^m(t)||_2^2, \quad ||P\Delta u^m(t)||_2^2,$$
$$||u^m(t)||_2^2 + 2\int_0^t \left(||\Delta u^m(\tau)||_2^2 + \alpha||\nabla u^m(\tau)||_2^2\right) d\tau,$$
$$\int_0^t \left(||\nabla u^m(\tau)||_2^2 + ||\Delta u^m(\tau)||_2^2 + ||P\Delta\Delta u^m(\tau)||_2^2 + ||u_\tau^m(\tau)||_2^2\right) d\tau$$

with $||u_0||_2, ||\nabla u_0||_2, ||\Delta u_0||_2$ that are quantities independent of m and of the size of the domain Ω.

2.4.4 Equicontinuity of the sequence $\{\mathbf{u^m}\}$

Theorem 2.4.15. *The sequence*

$$\{u^m\} \subset C\left([0, T_{u_0}); L^2(\Omega)\right)$$

is equicontinuous.

Proof. We consider the Fourier series of a generic $\psi \in J^2(\Omega)$, that is

$$\psi(x) = \sum_{k=0}^{\infty} \psi_k a_k(x) \tag{2.4.103}$$

where $\{a_k\}_{k\in\mathbb{N}}$ is the orthonormal basis already defined in (2.3.30) and the ψ_k are its Fourier coefficients.

Then, if we multiply equation (2.4.4) by ψ_k and we sum on $k = 0, \ldots, \mu$, we get

$$\sum_{k=0}^{\mu} \psi_k (u_t^m, a_k) + \sum_{k=0}^{\mu} \psi_k (P\Delta\Delta u^m - \alpha P\Delta u^m, a_k) + \sum_{k=0}^{\mu} \psi_k ((\nabla u^m) u^m, a_k) = 0, \tag{2.4.104}$$

that is

$$(u_t^m, \psi^\mu) + (P\Delta\Delta u^m - \alpha P\Delta u^m, \psi^\mu) + ((\nabla u^m) u^m, \psi^\mu) = 0, \tag{2.4.105}$$

since we put $\psi^\mu = \sum_{k=0}^{\mu} \psi_k a_k$, the partial sum of the Fourier expansion.

Then, integrating it from s to t and applying the Foundamental Theorem of Calculus, we obtain

$$(u^m(t) - u^m(s), \psi^\mu) = \int_s^t [(\alpha P\Delta u^m - P\Delta\Delta u^m, \psi^\mu) - ((\nabla u^m) u^m, \psi^\mu)]\, d\tau \tag{2.4.106}$$

for all $\mu \in \mathbb{N}$ and $s, t \geq 0$.

At this point, considering the absolute value of (2.4.106) and using Cauchy inequality, Hölder inequality and (2.4.92), we get

$$
\begin{aligned}
&|(u^m(t) - u^m(s), \psi^\mu)| \leq \\
&\leq \int_s^t |(\alpha P\Delta u^m - P\Delta\Delta u^m, \psi^\mu) - ((\nabla u^m)\, u^m, \psi^\mu)|\, d\tau \leq \\
&\leq ||\psi^\mu||_2 \int_s^t (||\alpha P\Delta u^m - P\Delta\Delta u^m - (\nabla u^m)\, u^m||_2)\, d\tau \leq \\
&\leq ||\psi^\mu||_2 \int_s^t (||\alpha P\Delta u^m||_2 + ||P\Delta\Delta u^m||_2 + ||\,(\nabla u^m)\, u^m||_2)\, d\tau \leq \\
&\leq \overline{C}||\psi^\mu||_2 \left[\int_s^t ||P\Delta u^m||_2 d\tau + \int_s^t ||P\Delta\Delta u^m||_2 d\tau + \int_s^t T_{u_0}\overline{h}(u_0) d\tau\right] \leq \\
&\leq \overline{C}||\psi^\mu||_2 \Bigg[(t-s)^{\frac{1}{2}} \left(\int_s^t ||P\Delta u^m||_2^2 d\tau\right)^{\frac{1}{2}} + (t-s)^{\frac{1}{2}} \left(\int_s^t ||P\Delta\Delta u^m||_2^2 d\tau\right)^{\frac{1}{2}} + \\
&\qquad + (t-s) T_{u_0}\overline{h}(u_0)\Bigg]
\end{aligned}
\tag{2.4.107}
$$

where $\overline{C} := \max\{1, \alpha\}$. At this point it is appropriate to observe that we can deduce an inequality similar to (2.4.96) also by the integration from s to t; in particular we can say that, integrating (2.4.50) from s to t for all $s, t > 0$, we have

$$
\begin{aligned}
&\int_s^t \frac{d}{dt}\left(||u^m||_2^2 + ||\nabla u^m||_2^2 + ||\Delta u^m||_2^2\right) d\tau + \\
&+ \int_s^t \left(||\nabla u^m||_2^2 + ||\Delta u^m||_2^2 + ||P\Delta\Delta u^m||_2^2 + ||u^m_\tau||_2^2\right) d\tau \leq \\
&\leq \int_s^t c_\alpha \left(||\nabla u^m||_2^6 + ||\nabla u^m||_2^4\right) d\tau \leq \\
&\leq \int_0^t c_\alpha \left(||\nabla u^m||_2^6 + ||\nabla u^m||_2^4\right) d\tau
\end{aligned}
\tag{2.4.108}
$$

and so, since $||u^m(t)||_2^2$ is not increasing in time for (2.4.29),

$$
\begin{aligned}
&||u^m(t)||_2^2 + ||\nabla u^m(t)||_2^2 + ||\Delta u^m(t)||_2^2 + \\
&+ \int_s^t \left(||\nabla u^m||_2^2 + ||\Delta u^m||_2^2 + ||P\Delta\Delta u^m||_2^2 + ||u^m_\tau||_2^2\right) d\tau \leq \\
&\leq ||u^m(s)||_2^2 + ||\nabla u^m(s)||_2^2 + ||\Delta u^m(s)||_2^2 + \int_0^t c_\alpha \left(||\nabla u^m||_2^6 + ||\nabla u^m||_2^4\right) d\tau \leq \\
&\leq \widetilde{C}_\alpha \left(||u_0||_2^2 + ||\nabla u_0||_2^2 + ||\Delta u_0||_2^2\right) + \int_0^t c_\alpha \left(||\nabla u^m||_2^6 + ||\nabla u^m||_2^4\right) d\tau
\end{aligned}
\tag{2.4.109}
$$

whence we can find an estimate similar to (2.4.100), that is

$$\int_s^t ||P\Delta\Delta u^m||_2^2 d\tau \leq$$
$$\leq \int_0^t c_\alpha \left(||\nabla u^m||_2^6 + ||\nabla u^m||_2^4\right) d\tau + \widetilde{C}_\alpha \left(||u_0||_2^2 + ||\nabla u_0||_2^2 + ||\Delta u_0||_2^2\right) \tag{2.4.110}$$

leading to an analogous of inequality (2.4.96), whence

$$\int_s^t ||P\Delta\Delta u^m||_2^2 d\tau \leq c_2(t). \tag{2.4.111}$$

Now, applying (2.4.64) and (2.4.94) integrated from s to t and (2.4.111), we have

$$|(u^m(t) - u^m(s), \psi^\mu)| \leq$$
$$\leq \overline{C}||\psi^\mu||_2 \left[(t-s)^{\frac{1}{2}} (t-s)^{\frac{1}{2}} \left(\int_s^t c_1(\tau)^2 d\tau\right)^{\frac{1}{2}} + (t-s)^{\frac{1}{2}} c_2(t)^{\frac{1}{2}} + (t-s)T_{u_0}\overline{h}(u_0)\right]. \tag{2.4.112}$$

Dividing by $||\psi^\mu||_2$ and taking the sup on the space $\mathcal{S}$ of functions ψ^μ, we get

$$\sup_{\psi^\mu \in \mathcal{S}} \frac{|(u^m(t) - u^m(s), \psi^\mu)|}{||\psi^\mu||_2} \leq$$
$$\leq \overline{C} \left[(t-s)^{\frac{1}{2}} (t-s)^{\frac{1}{2}} \left(\int_s^t c_1(\tau)^2 d\tau\right)^{\frac{1}{2}} + (t-s)^{\frac{1}{2}} c_2(t)^{\frac{1}{2}} + (t-s)T_{u_0}\overline{h}(u_0)\right]. \tag{2.4.113}$$

We can now observe that the right-hand side does not depend on m but only by constant terms like $||u_0||_2^2, ||\nabla u_0||_2^2, ||\Delta u_0||_2^2$; furthermore, each term is multiplied by $(t-s)$ which tends to 0 when $t \to s$ and so all the right-hand side tends to 0 when $t \to s$:

$$||u^m(t) - u^m(s)||_2 \leq g(t-s) \to 0, \quad \text{as } t \to s. \tag{2.4.114}$$

In this way we proved that the sequence

$$\{u^m\} \subset C\left([0, T_{u_0}); L^2(\Omega)\right)$$

is equicontinuous. □

2.4.5 Existence of a weak limit of $\{\mathbf{u^m}\}$

In this section we want to prove the existnce of a weak limit of the approximating sequence u^m.

With Theorem 2.1.5 we will be able to find a subsequence of the approximated solution and its time-derivative, in order to end up to their convergence in $L^2(\Omega)$. But before

Theorem 2.4.16. *Let $v \in W^{4,p}(\Omega)$ with $\Omega \subseteq \mathbb{R}^n$ with $n \geq 3$ an exterior domain of class C^4. If $v = \dfrac{\partial v}{\partial n} = 0$ on $\partial\Omega$ and $p \in (1, +\infty)$, then there exist two constants c_0 and c_1 independent of v such that*

$$||D^4 v||_p \leq c_0 ||\Delta\Delta v||_p + c_1 ||v||_p. \tag{2.4.115}$$

Moreover, $c_1 = 0$ if the domain Ω is $\mathbb{R}^n$ or if it is bounded.

Proof. Let us consider a regular cutoff function h with support in the sphere of radius $R > 2\,\mathrm{diam}(\mathbb{R}^n \setminus \Omega)$ such that $h = 1$ for $|x| \leq R/2$. Set $H := 1 - h$,

$$\varphi := \Delta v \quad \text{and} \quad \phi := H\Delta v.$$

Clearly, ϕ satisfies

$$\Delta\phi = H\Delta\Delta v - \varphi\Delta h - 2\nabla\varphi \cdot \nabla h, \text{ in } \mathbb{R}^n.$$

Setting

$$F := H\Delta\Delta v - \varphi\Delta h - 2\nabla\varphi \cdot \nabla h \tag{2.4.116}$$

we get the equation $\Delta\psi = F$ and for the Calderón-Zygmund Theorem we have the bound

$$||D^2\psi||_p \leq c||F||_p. \tag{2.4.117}$$

Now, since $D^2\phi = D^2\psi$ and since

$$\begin{aligned} D^2\phi = HD^2\Delta v + D^2 H\Delta v + 2\nabla H \cdot \nabla\Delta v = \\ = HD^2\Delta v - \varphi D^2 h - 2\nabla\varphi \cdot \nabla h \end{aligned} \tag{2.4.118}$$

we obtain

$$\begin{aligned}&||HD^2\Delta v||_p - ||D^2H\Delta v + 2\nabla H\cdot\nabla\Delta v||_p \le \\ &\qquad \le ||HD^2\Delta v + D^2H\Delta v + 2\nabla H\cdot\nabla\Delta v||_p \le c||F||_p; \quad (2.4.119)\end{aligned}$$

therefore

$$\begin{aligned}&||HD^2\Delta v||_p = ||H\Delta D^2 v||_p \le \\ &\qquad \le c||F||_p + ||D^2H\Delta v + 2\nabla H\cdot\nabla\Delta v||_p \le \\ &\qquad\qquad \le c||F||_p + ||D^2H\Delta v||_p + 2||\nabla H\cdot\nabla\Delta v||_p. \quad (2.4.120)\end{aligned}$$

Then, setting

$$\widehat{\varphi} := D^2 v \text{ and } \widehat{\phi} := H\widehat{\varphi},$$

we have that $\widehat{\phi}$ satisfies the identity

$$\begin{aligned}&\Delta\widehat{\phi} = \widehat{\varphi}\Delta H + H\Delta\widehat{\varphi} + 2\nabla H\cdot\nabla\widehat{\varphi} = \\ &\qquad = H\Delta D^2 v - \widehat{\varphi}\Delta h - 2\nabla\widehat{\varphi}\cdot\nabla h := \widehat{F}, \text{ in } \mathbb{R}^n \quad (2.4.121)\end{aligned}$$

Similarly to the previous case, we can say that also the bound

$$||D^2\widehat{\psi}||_p \le c||\widehat{F}||_p \quad (2.4.122)$$

holds. Then, since $D^2\widehat{\phi} = D^2\widehat{\psi}$ and since

$$\begin{aligned}D^2\widehat{\phi} &= \widehat{\varphi}D^2H + HD^2\widehat{\varphi} + 2\nabla H\cdot\nabla\widehat{\varphi} = \\ &= D^2HD^2v + HD^4v + 2\nabla H\cdot\nabla D^2 v = \\ &= HD^4v - D^2vD^2h - 2\nabla D^2v\cdot\nabla h\end{aligned} \quad (2.4.123)$$

we get, analogously,

$$||HD^4v||_p - ||D^2HD^2v + 2\nabla H\cdot\nabla D^2v||_p \le c||\widehat{F}||_p \quad (2.4.124)$$

whence, by using the definition of $\widehat{F}$ and the estimate (2.4.120), after having redefined the constants, it holds

$$\begin{aligned}
&||HD^4v||_p \leq c||\widehat{F}||_p + ||D^2HD^2v + 2\nabla H\cdot\nabla D^2v||_p = \\
&= c||\widehat{F}||_p + ||D^2vD^2h + 2\nabla D^2v\cdot\nabla h||_p \leq \\
&\leq c||H\Delta D^2v - \widehat{\varphi}\Delta h - 2\nabla\widehat{\varphi}\cdot\nabla h||_p + ||D^2vD^2h + 2\nabla D^2v\cdot\nabla h||_p \leq \\
&\leq c||H\Delta D^2v||_p + c||\widehat{\varphi}\Delta h + 2\nabla\widehat{\varphi}\cdot\nabla h||_p + ||D^2vD^2h + 2\nabla D^2v\cdot\nabla h||_p \leq \\
&\leq c^2||F||_p + c||D^2H\Delta v||_p + 2c||\nabla H\cdot\nabla\Delta v||_p + \\
&+c||\widehat{\varphi}\Delta h + 2\nabla\widehat{\varphi}\cdot\nabla h||_p + ||D^2vD^2h + 2\nabla D^2v\cdot\nabla h||_p = \\
&= c^2||F||_p + c||\Delta vD^2h||_p + 2c||\nabla\Delta v\cdot\nabla h||_p + \\
&+c||D^2v\Delta h + 2\nabla D^2v\cdot\nabla h||_p + ||D^2vD^2h + 2\nabla D^2v\cdot\nabla h||_p \leq \\
&\overline{c}||F||_p + c'\left(||\Delta vD^2h||_p + ||\nabla\Delta v\cdot\nabla h||_p + \right. \\
&\left. +||D^2v\Delta h||_p + ||\nabla D^2v\cdot\nabla h||_p + ||D^2vD^2h||_p + ||\nabla D^2v\cdot\nabla h||_p\right)
\end{aligned} \tag{2.4.125}$$

so that

$$\begin{aligned}
&||HD^4v||_p \leq \overline{c}||F||_p + c'\left(||\Delta vD^2h||_p + ||\nabla\Delta v\cdot\nabla h||_p + \right. \\
&\left. +||D^2v\Delta h||_p + ||\nabla D^2v\cdot\nabla h||_p + ||D^2vD^2h||_p + ||\nabla D^2v\cdot\nabla h||_p\right).
\end{aligned} \tag{2.4.126}$$

Let now k be a regular cutoff function with support in the sphere B_{2R} with radius $2R$ and such that $k(x) = 1$ for $|x| > (3/2)R$ and such that $k + H = 1$. Setting $\sigma := kv$, with

$$\sigma = 0 \text{ and } \frac{\partial\sigma}{\partial n} = 0 \text{ on } \partial(\Omega\cap B_{2R}),$$

the function σ satisfies the identity

$$\Delta\Delta\sigma = k\Delta\Delta v + v\Delta\Delta k + 2\Delta v\Delta k + 4\nabla k\cdot\nabla\Delta v + 4\nabla v\cdot\nabla\Delta k + 4\nabla\nabla v\cdot\nabla\nabla k := G, \tag{2.4.127}$$

in Ω.

Let D be a bounded domain of class C^4. Then, for all solutions in $W^{4,p}(D)$ of (2.4.127), for Douglis and Nirenberg (see [18]), the following estimate holds

$$||D^4\sigma||_p \leq c_2||G||_p + c_3||\sigma||_p \leq c_2||G||_p + c_3||v||_p. \tag{2.4.128}$$

It follows that

$$D^4\sigma = kD^4v + vD^4k + 4D^2vD^2k + 4Dk\cdot D^3v + 4Dv\cdot D^3k, \text{ in } \Omega, \tag{2.4.129}$$

and taking into account the properties of the cutoff functions H and k, the estimates (2.4.126) and (2.4.128) and by definition of F, we obtain

$$||D^4 v||_p \leq ||kD^4 v||_p + ||HD^4 v||_p \leq c||\Delta\Delta v||_p +$$
$$+ \sum_{|\alpha|=1}^{4} d_\alpha ||D^\alpha k D^{4-\alpha} v||_p + \sum_{|\alpha|=1}^{4} d_\alpha ||D^\alpha h D^{4-\alpha} v||_p + \sum_{\alpha=1}^{2} e_\alpha ||D^\alpha h D^{2-\alpha} v||_p + c_3 ||v||_p. \tag{2.4.130}$$

At this point we can consider the inequality of the Theorem 2.1.3 (see [9]), valid also in exterior domains, with $r = q = s = p = 2$ and we apply it to the first, second and third derivatives of v in the second member of (2.4.130); then, by applying Young's inequality and bringing the norm of the fourth derivatives of v to the first member with a small coefficient, we get the thesis. □

Theorem 2.4.17. *The inequality*

$$||\Delta\Delta u^m||_2^2 \;\leq\; \widetilde{K}(||P\Delta\Delta u^m||_2^2 + ||P\Delta u^m||_2^2 + ||(\nabla u^m)u^m||_2^2 + ||u_t^m||_2^2) \tag{2.4.131}$$

holds in $[0, T_{u_0}]$.

Proof. Let us consider the Helmholtz decomposition of the bilaplacian of a function u^m, namely

$$\Delta\Delta u^m = P\Delta\Delta u^m + \nabla\pi_{\Delta\Delta}. \tag{2.4.132}$$

By integrating it and testing with $P\Delta\Delta u^m$, we get

$$\int_\Omega \Delta\Delta u^m \cdot P\Delta\Delta u^m = \int_\Omega P\Delta\Delta u^m \cdot P\Delta\Delta u^m + \int_\Omega \nabla\pi_{\Delta\Delta} \cdot P\Delta\Delta u^m; \tag{2.4.133}$$

by using once again (2.4.132) and considering that $P\Delta\Delta u^m \in J^2(\Omega)$ and $\nabla\pi_{\Delta\Delta} \in G^2(\Omega)$ we obtain

$$\int_\Omega \Delta\Delta u^m \cdot (\Delta\Delta u^m - \nabla\pi_{\Delta\Delta}) = ||P\Delta\Delta u^m||_2^2, \tag{2.4.134}$$

hence

$$||\Delta\Delta u^m||_2^2 - \int_\Omega \Delta\Delta u^m \cdot \nabla\pi_{\Delta\Delta} = ||P\Delta\Delta u^m||_2^2 \tag{2.4.135}$$

and then, by applying Cauchy-Schwarz and Young's inequality, we get

$$\begin{aligned} ||\Delta\Delta u^m||_2^2 = ||P\Delta\Delta u^m||_2^2 + \int_\Omega \Delta\Delta u^m \cdot \nabla\pi_{\Delta\Delta} \leq \\ \leq ||P\Delta\Delta u^m||_2^2 + ||\Delta\Delta u^m||_2 ||\nabla\pi_{\Delta\Delta}||_2 \leq \\ \leq ||P\Delta\Delta u^m||_2^2 + \varepsilon||\Delta\Delta u^m||_2^2 + C||\nabla\pi_{\Delta\Delta}||_2^2 \end{aligned} \tag{2.4.136}$$

that leads to

$$(1-\varepsilon)||\Delta\Delta u^m||_2^2 \leq ||P\Delta\Delta u^m||_2^2 + C||\nabla\pi_{\Delta\Delta}||_2^2, \tag{2.4.137}$$

whence

$$||\Delta\Delta u^m||_2^2 \leq K(||P\Delta\Delta u^m||_2^2 + ||\nabla\pi_{\Delta\Delta}||_2^2), \tag{2.4.138}$$

where $K := \max\left\{\frac{1}{1-\varepsilon}, \frac{C}{1-\varepsilon}\right\}$. Finally, by applying the results of Amrouche and Girault (Theorem 2.3.6, see [11]) with $\alpha P\Delta u^m - (\nabla u^m)u^m - u_t^m$ instead of f, we obtain

$$\begin{aligned} ||\Delta\Delta u^m||_2^2 \leq \overline{K}(||P\Delta\Delta u^m||_2^2 + ||\alpha P\Delta u^m - (\nabla u^m)u^m - u_t^m||_2^2) \leq \\ \leq \widetilde{K}(||P\Delta\Delta u^m||_2^2 + ||P\Delta u^m||_2^2 + ||(\nabla u^m)u^m||_2^2 + ||u_t^m||_2^2) \end{aligned} \tag{2.4.139}$$

where $\widetilde{K}$ is a positive constant and where we used Hölder inequality, Young's inequality. □

Let us observe that the inequality just proven depends on the size of the domain Ω. The regularity results of Amrouche and Girault [11] do not avoid the dependence of the constants on the measure of the domain, but it could be possible that such result is still valid, following, for example, the arguments of Heywood [3] also for the fourth derivatives: the norm $||\nabla\nabla u||_2^2$ is bounded by some other terms multiplied by a constant that depends only on the regularity of the domain but not on its measure.

Theorem 2.4.18. *There exists a weak limit u of $\{u^m\}$ such that*

$$u \in L^2(0, T_{u_0}; H^4(\Omega) \cap J^{2,2}(\Omega)); \qquad u_t \in L^2(0, T_{u_0}; L^2(\Omega)).$$

Proof. From Theorem 2.1.3 (in [9]) with the choice $r = p = q = 2,\ k = 3,\ m = 4,\ a = 3/4$, we get for the subsequence $u^m(t)$

$$||D^3u^m||_2 \leq c_1||D^4u^m||_2^{3/4}||u^m||_2^{1/4} + c_2||u^m||_2 \tag{2.4.140}$$

and hence

$$||\nabla\nabla\nabla u^m||_2 \leq c_1||\nabla\nabla\nabla\nabla u^m||_2^{3/4}||u^m||_2^{1/4} + c_2||u^m||_2. \tag{2.4.141}$$

Using Young's inequality we get

$$||\nabla\nabla\nabla u^m||_2^2 \leq \overline{c}_1||\nabla\nabla\nabla\nabla u^m||_2^2 + \overline{c}_2||u^m||_2^2 \tag{2.4.142}$$

and using Theorem 2.4.16 and Theorem 2.4.131 we get

$$\begin{gathered} ||\nabla\nabla\nabla u^m||_2^2 \leq \widehat{c}_1||\Delta\Delta u^m||_2^2 + \widehat{c}_2||u^m||_2^2 \leq \\ \leq k(||P\Delta\Delta u^m||_2^2 + ||P\Delta u^m||_2^2 + ||(\nabla u^m)u^m||_2^2 + ||u^m||_2^2 + ||u_t^m||_2^2), \end{gathered} \tag{2.4.143}$$

for a positive constant k.

Moreover, integrating it from 0 to T_{u_0} and by applying (2.4.97), (2.4.98), (2.4.91) and (2.4.94), (2.4.64) integrated from 0 to T_{u_0} we get

$$\int_0^{T_{u_0}} ||\nabla\nabla\nabla u^m||_2^2 \leq k_0, \tag{2.4.144}$$

for a positive constant k_0 dependent on $T_{u_0},\ ||u_0||_2,\ ||\nabla u_0||_2$ and $||\Delta u_0||_2$.

We can apply Theorem 2.4.16 and Theorem 2.4.131 also to the term $||\nabla\nabla\nabla\nabla u^m||_2^2$ in order to obtain

$$\begin{gathered} \int_0^{T_{u_0}} ||\nabla\nabla\nabla\nabla u^m||_2^2 \leq \int_0^{T_{u_0}} (\widehat{c}_3||\Delta\Delta u^m||_2^2 + \widehat{c}_4||u^m||_2^2) \leq \\ \leq \int_0^{T_{u_0}} k(||P\Delta\Delta u^m||_2^2 + ||P\Delta u^m||_2^2 + ||(\nabla u^m)u^m||_2^2 + ||u^m||_2^2 + ||u_t^m||_2^2) \leq \widetilde{k}_0, \end{gathered} \tag{2.4.145}$$

for a positive constant $\widetilde{k}_0$ dependent on $T_{u_0},\ ||u_0||_2,\ ||\nabla u_0||_2$ and $||\Delta u_0||_2$.

Now, from (2.4.64), the subsequence $\{u^m(t)\}_{m\in\mathbb{N}}$ is bounded because

$$\int_0^{T_{u_0}} ||u^m(\tau)||_2^2 d\tau \leq T_{u_0}C_\alpha||u_0||_2^2, \tag{2.4.146}$$

whence

$$\Big|\Big|\,||u^m(t)||_2\Big|\Big|^2_{L^2(0,T_{u_0})} \le T_{u_0} C_\alpha ||u_0||_2^2 \tag{2.4.147}$$

so that

$$\{u^m(t)\} \subset L^2(0, T_{u_0}; L^2(\Omega)). \tag{2.4.148}$$

Moreover, considering (2.3.3), we get

$$\begin{aligned} \Big|\Big|\,||u^m(t)||_{4,2}\Big|\Big|^2_{L^2(0,T_{u_0})} &= \int_0^{T_{u_0}} ||u^m(\tau)||^2_{4,2}\, d\tau = \\ = \int_0^{T_{u_0}} \big(||u^m(\tau)||_2^2 &+ ||\nabla u^m(\tau)||_2^2 + ||\Delta u^m(\tau)||_2^2 + \\ &+ ||\nabla\nabla\nabla u^m(\tau)||_2^2 + ||\nabla\nabla\nabla\nabla u^m(\tau)||_2^2\big)\, d\tau \end{aligned} \tag{2.4.149}$$

and remembering (2.4.64), (2.4.66), (2.3.3), (2.4.67), (2.4.144) and (2.4.145), we come up to

$$\Big|\Big|\,||u^m(t)||_{4,2}\Big|\Big|^2_{L^2(0,T_{u_0})} \le C_{T_{u_0}}(||u_0||_2, ||\nabla u_0||_2, ||\Delta u_0||_2). \tag{2.4.150}$$

At this point, observing that both u_0 is given and square integrable, we deduce

$$\{u^m(t)\} \subset L^2(0, T_{u_0}; H^4(\Omega)); \tag{2.4.151}$$

we notice also that $J^{2,2}(\Omega) \subset H^2(\Omega)$ and hence

$$\{u^m(t)\} \subset L^2(0, T_{u_0}; H^4(\Omega) \cap J^{2,2}(\Omega)). \tag{2.4.152}$$

Finally, we notice that also $u_t^m(t)$ is bounded in $L^2(\Omega)$; in fact, by (2.4.98)

$$\Big|\Big|\,||u_t^m(t)||_2\Big|\Big|^2_{L^2(0,T_{u_0})} = \int_0^{T_{u_0}} ||u_\tau^m(\tau)||_2^2 d\tau \le c \tag{2.4.153}$$

and hence

$$\{u_t^m(t)\} \subset L^2(0, T_{u_0}; L^2(\Omega)). \tag{2.4.154}$$

Then, by applying theorem 2.1.5 we deduce the existence of $(u_k^m)_{k\in\mathbb{N}}$ and $\big(u_{t_k}^m\big)_{k\in\mathbb{N}}$ such that

$$u_k^m \overset{L^2}{\rightharpoonup} u, \quad u_{t_k}^m \overset{L^2}{\rightharpoonup} u_t,$$

with

$$u \in L^2(0, T_{u_0}; H^4(\Omega) \cap J^{2,2}(\Omega)) \text{ and } u_t \in L^2(0, T_{u_0}; L^2(\Omega)). \tag{2.4.155}$$

□

2.4.6 Compactness properties of $\{\mathbf{u^m}\}$

In this section we want to prove that $\{u^m(t)\}$ is relatively compact in an appropriate space. Before continuing we need the following Lemma proved in [13]

Lemma 2.4.1. *A subset K in $C(0,T;X)$ is relatively compact if and only if*

- *K is equicontinuous;*
- *for all $t \in [0,T]$, the set $K(t) := \{u(t) : u(t) \in K\}$ is relatively compact in X.*

Theorem 2.4.19. *The sequence $\{u^m(t)\}$ is relatively compact in $C(0, T_{u_0}; J^2(\Omega))$.*

Proof. We can apply Lemma 2.4.1 setting $X = J^2(\Omega)$ and $K = \{u^m(t)\}$. The equicontinuity of $\{u^m(t)\}$ was proved in the previous section. We now have to prove that, for all $t \in [0, T_{u_0}]$, the set

$$K(t) := \left\{u^j(t) : u^j(t) \in \{u^m(t)\}\right\}$$

is relatively compact in $J^2(\Omega)$.

Actually, we already know that $u_h^m \overset{L^2}{\rightharpoonup} u$, so that, we have to prove that $u_h^m \overset{L^2}{\longrightarrow} u$.

If we fix t, considering (2.4.64), (2.4.93) and (2.4.94), we get

$$||u^m(t)||_{2,2} \leq c$$

with c independent of m. Therefore the sequence $\{u^m(t)\}$ is bounded in the reflexive space $H^2(\Omega)$ and by applying theorem 2.1.5 we get that $\{u^m(t)\}$ admits a subsequence that converges weakly in $H^2(\Omega)$. Moreover, $\{u^m(t)\}$ converges strongly in $J^2(\Omega) \subseteq L^2(\Omega)$, that is what we wanted to show.

Applying Lemma 2.4.1, we conclude that for all $t \in [0, T_{u_0}]$ the sequence $\{u^m(t)\}$ is relatively compact in $C(0, T_{u_0}; J^2(\Omega))$. Hence $\{u^m(t)\}$ converges strongly in $C(0, T_{u_0}; J^2(\Omega))$. $\square$

2.4.7 Continuity in time

Theorem 2.4.20. *The limit function u of u^m is continuous in $t = 0$, namely*

$$\lim_{t\to 0} ||u(t) - u_0||_2 = 0.$$

Proof. By definition, we know that

$$u^m(0) = \sum_{k=1}^{m} c_k^m(0) a_k$$

and now we also know that the sequence $\{u^m(t)\}$ converges to $u(0)$; moreover, by (2.4.4),

$$\sum_{k=1}^{m} c_k^m(0) a_k = u_0$$

so we get the following equality in $L^2(\Omega)$:

$$u(0) = u_0. \tag{2.4.156}$$

By considering (2.4.114) with $m \to \infty$, $s = 0$ and applying (2.4.156) to it, we get

$$||u(t) - u_0||_2 \xrightarrow{t\to 0} 0. \tag{2.4.157}$$

□

2.4.8 Regularity in space and in time

Theorem 2.4.21. *The solution (u, p) of (2.2.1) verifies*

$$\begin{gathered} u \in C([0, T_{u_0}); J^{2,2}(\Omega)) \cap L^2(0, T_{u_0}; H^4(\Omega)) \\ u_t, \nabla p \in L^2(0, T_{u_0}; L^2(\Omega)). \end{gathered} \tag{2.4.158}$$

Proof. Let us consider again (2.4.4) integrated from 0 to T_{u_0}, we get

$$\int_0^{T_{u_0}} \left(u_t^m + P\Delta\Delta u^m - \alpha P\Delta u^m + (\nabla u^m)\, u^m, a_k\right) dt = 0 \tag{2.4.159}$$

We proved that the sequence $\{u^m(t)\}$ converges strongly to $u(t)$ in $C(0, T_{u_0}; J^2(\Omega))$ and by the fact that

$$\forall \varphi \in J^2(\Omega) : (P\Delta\Delta v, \varphi) = (\Delta\Delta v, \varphi),$$

we get

$$\int_0^{T_{u_0}} (u_t + \Delta\Delta u - \alpha\Delta u + (\nabla u)\, u, a_k)\, dt = 0; \tag{2.4.160}$$

so, u satisfies the following integral equation:

$$\int_0^{T_{u_0}} (u_t + \Delta\Delta u - \alpha\Delta u + (\nabla u)\, u, \psi)\, dt = 0, \tag{2.4.161}$$

for all functions $\psi := h(t)\phi(x)$ with $h(t) \in C_c^\infty([0, T_{u_0}))$ and $\phi \in \mathscr{C}_{div}(\Omega)$. Since $\{a_k\}$ is a basis of $J^{2,2}(\Omega)$, orthonormal in $J^2(\Omega)$, for all $\phi \in \mathscr{C}_{div}(\Omega) \subseteq J^2(\Omega)$ we can write $\phi = \sum_{k=1}^{\infty} c_k(t) a_k(x)$.

In order to recover the pressure, we apply Lemma 2.3.1 to $v = u_t + \Delta\Delta u - \alpha\Delta u + (\nabla u)\, u$; the hypotheses are satisfied, since Ω is bounded and for regularity results,

$$u_t + \Delta\Delta u - \alpha\Delta u + (\nabla u)\, u \in L^2_{loc}(\Omega).$$

Moreover, from (2.4.161), we have

$$\int_0^{T_{u_0}} \int_\Omega (u_t + \Delta\Delta u - \alpha\Delta u + (\nabla u)\, u) \cdot \psi\, dx\, dt = 0, \tag{2.4.162}$$

that is, also

$$\int_0^{T_{u_0}} \left[\int_\Omega (u_t + \Delta\Delta u - \alpha\Delta u + (\nabla u)\, u) \cdot \phi(x)\, dx \right] h(t)\, dt = 0. \tag{2.4.163}$$

By the arbitrariness of h, we get

$$\int_\Omega (u_t + \Delta\Delta u - \alpha\Delta u + (\nabla u)\, u) \cdot \phi(x)\, dx = 0 \tag{2.4.164}$$

for all $t > 0$. Applying Lemma 2.3.1, there exists a function $p \in W^{1,2}_{loc}(\Omega)$ such that

$$u_t + \Delta\Delta u - \alpha\Delta u + (\nabla u)\, u = \nabla p$$

for all $t > 0$ and for a.e. $x \in \Omega$ such that (u, p) is a solution to problem (2.2.1).

At this point we need the following result (see [14]):

Lemma 2.4.2. *Suppose that*

$$\overline{u} \in L^2(0, T_{u_0}; H^4(\Omega) \cap H_0^2(\Omega)) \quad \textit{and} \quad \overline{u}_t \in L^2(0, T_{u_0}; L^2(\Omega)).$$

Then, $\overline{u}$ *coincides a.e. in* $t \in [0, T_{u_0})$ *with a function* $u \in C([0, T_{u_0}); H_0^2(\Omega))$.

Hence, by applying this Lemma with $J^{2,2}(\Omega)$ instead of $H_0^2(\Omega)$, we obtain that our solution u coincides a.e. in $t \in [0, T_{u_0})$ with a function in $C([0, T_{u_0}); J^{2,2}(\Omega))$. So

$$\begin{gathered} u \in C([0, T_{u_0}); J^{2,2}(\Omega)) \cap L^2(0, T_{u_0}; H^4(\Omega)) \\ u_t, \in L^2(0, T_{u_0}; L^2(\Omega)). \end{gathered} \tag{2.4.165}$$

Moreover, also

$$\nabla p \in L^2(0, T_{u_0}; L^2(\Omega)). \tag{2.4.166}$$

Indeed, since

$$\nabla p = u_t + \Delta\Delta u - \alpha \Delta u + (\nabla u)\, u,$$

and we already proved in (2.4.155) that $u, u_t, \nabla u, \Delta u, \Delta\Delta u \in L^2(0, T_{u_0})$ with values in their respective spaces. We have also to prove that $(\nabla u)u \in L^2(0, T_{u_0}; L^2(\Omega))$. Indeed, it is a direct consequence of (2.4.80). Thus, we obtain (2.4.166). □

In this way the of the existence of a regular solution for the Hyperviscous Navier-Stokes initial boundary value problem is completed.

2.5 Exterior domains

In this section we want to establish the existence of regular solutions also for exterior domains. The argument will use what has been done in the previous sections for bounded domains. It is important to observe that all the estimates found in the previous sections do not depend on the measure of the domain, so they are still valid also for this case, with the exception of (2.4.131). In this case, even if theorem 2.3.6 is proven only for bounded domains, a similarity argument shows that the estimate holds for a solution in an arbitrary large domain with the same constant if the claim is replaced by the weaker form

$$||\Delta\Delta u||_2 + ||\nabla p||_2 \leq C||f||_2.$$

Therefore this estimate is also independent of the size of the domain.

To recover (2.4.131) it is sufficient to observe that $||u||_2$, $||\nabla u||_2$, $||\Delta u||_2$ have been already estimated independently of the domain, while $||\nabla\nabla\nabla u||_2$ can be estimated by $||\Delta\Delta u||_2$ and $||u||_2$ by means of (2.4.142) and Theorem 2.4.16.

In the passage to exterior domains we follow the arguments of Heywood in [3].

Let Ω be an exterior domain with enough regular boundary. Let us define the bounded domains

$$\Omega_h := \Omega \cap B_h(0)$$

where $B_h(0)$ is the ball centered in 0 and with radius h, such that

$$\overline{\Omega}_h \subset \Omega_{h+1} \text{ and } \Omega = \bigcup_{h\in\mathbb{N}} \Omega_h.$$

Now we take the initial datum u_0 in $J^{2,2}(\Omega)$ and a sequence of initial velocity functions $\{\alpha_h\}$ in $J^{2,2}(\Omega)$ with $\operatorname{supt}\alpha_h \subset \Omega_h$, with

$$||\nabla\alpha_h||_2 \leq ||\nabla u_0||_2 \quad \text{and} \quad ||\Delta\alpha_h||_2 \leq ||\Delta u_0||_2$$

and $||\nabla\alpha_h - \nabla u_0||_2 \to 0$ and $||\Delta\alpha_h - \Delta u_0||_2 \to 0$ as $h \to \infty$.

At this point let us consider a solution u^h of the initial boundary value problem (2.4.4) in $[0, T_{\alpha_h}] \times \Omega_h$ with the initial velocity α_h. It is not restrictive to consider the boundary conditions $u^h = \dfrac{\partial u^h}{\partial n} = 0$ on $\partial\Omega_h$ for some h big enough.

Since $||\nabla\alpha_h||_2 + ||\Delta\alpha_h||_2 \leq ||\nabla u_0||_2 + ||\Delta u_0||_2$, all the solutions of the sequence $\{u^h\}$ exist on the interval $[0, T_{\alpha_h})$, verify the inequalities (2.4.93), (2.4.94), 2.4.96 in $(0, T_{\alpha_h}) \times \Omega_h$ with $T_{u_0} \leq T_{\alpha_h}$ and hence the intervals do not tend to zero as $h \to +\infty$.

Now, let us consider a subsequence $\{u^{hk}\}$ weakly convergent in $L^2(0, T'_{u_0}; H^2(\Omega_l))$ and its derivatives $\{u_t^{hk}\}$ weakly convergent in $L^2(0, T'_{u_0}; L^2(\Omega_l))$ for every l and every $T'_{u_0} < T_{u_0}$. Then, (2.4.93), (2.4.94), (2.4.96) hold for the limit u and all its derivatives.

Hence we can deduce all the facts proved in the case of bounded domain, as an analogous result of Theorem 2.4.20 and Theorem 2.4.21 for some instant $T'_{u_0} < T_{u_0}$, in particular $u_t + \Delta\Delta u - \alpha\Delta u + (\nabla u)\, u \in L^2(0, T'; L^2(\Omega))$ and once again, as in Theorem 2.4.21, it can be proved the existence of a function $p(t, x)$ with $\nabla p \in L^2(0, T'_{u_0}; L^2(\Omega))$ such that $u_t + \Delta\Delta u - \alpha\Delta u + (\nabla u)\, u = \nabla p$.

Finally, following the reasoning of Heywood [3] it can be shown that $\lim_{t\to 0} u(t) = u_0$ strongly in $J^{2,2}(\Omega)$.

In this way we proved the existence of a regular solution in exterior domains on an interval $(0, T_{u_0})$.

2.6 Uniqueness of the regular solutions

In this section we want to prove the uniqueness of the regular solution for the problem (2.2.1), set either in bounded domains or in exterior domains. The existence of solutions for such problem is ensured by the former sections.

First of all we will need to prove the following

Theorem 2.6.1. *Let Ω be an open (bounded or exterior) domain of $\mathbb{R}^3$. Let $u_0 \in J^{2,2}(\Omega)$. Then, for some $T_{u_0} > 0$ we have*

$$u \in L^\infty\left(0, T_{u_0}; L^3(\Omega)\right) \cap L^\infty\left(0, T_{u_0}; J^{1,2}(\Omega)\right).$$

Proof. At first we can deduce from the hypothesis that $u \in J^{1,2}(\Omega)$ too. Let us prove that it is also in $L^3(\Omega)$.

We need to use once again the interpolation inequality from Crispo and Maremonti ([9]) with the choice $k = 0$, $r = 3$, $p = q = 2$, $m = 2$ and $a = 1/4$ and hence we obtain

$$||u||_3 \leq c_1 ||\nabla\nabla u||_2^{\frac{1}{4}} ||u||_2^{\frac{3}{4}} + c_2 ||u||_2 \tag{2.6.1}$$

for some constants c_1 and c_2, independent of u and with c_2 that vanishes for bounded domains. Then, by applying (2.4.78) we have the claim. □

Now, we can state the proper uniqueness theorem, that is the following

Theorem 2.6.2. *Given Ω a (bounded or exterior) domain of $\mathbb{R}^3$. For all $u_0 \in J^{2,2}(\Omega)$ and $f \in C([0,T_{u_0}];L^2(\Omega))$, the regular solution (u,p) of problem (2.2.1) on $(0,T_{u_0})$ is unique.*

Proof. Let both (u_1,p_1) and (u_2,p_2) be regular solutions of the problem (2.2.1); hence, both of them verify equations:

$$\begin{aligned} &\frac{\partial u_1}{\partial t} + (\nabla u_1)\, u_1 + \nabla p_1 = \nu\Delta u_1 - \tau\Delta\Delta u_1 + f \\ &\frac{\partial u_2}{\partial t} + (\nabla u_2)\, u_2 + \nabla p_2 = \nu\Delta u_2 - \tau\Delta\Delta u_2 + f. \end{aligned} \tag{2.6.2}$$

Taking the difference between the two and setting $u := u_1 - u_2$ and $p := p_1 - p_2$, we get

$$\frac{\partial u}{\partial t} + (\nabla u_1)\, u_1 - (\nabla u_2)\, u_2 + \nabla p = \nu\Delta u - \tau\Delta\Delta u \tag{2.6.3}$$

where u is still in $J^{2,2}(\Omega)$. Then, by adding and subtracting the term $(\nabla u_1)u_2$, taking the scalar product of the whole equation with u and integrating it in Ω, we obtain

$$\int_\Omega \frac{\partial u}{\partial t}\cdot u + \int_\Omega (\nabla u_1)\, u\cdot u - \int_\Omega (\nabla u)\, u_2\cdot u + \int_\Omega \nabla p\cdot u = \nu\int_\Omega \Delta u\cdot u - \tau\int_\Omega \Delta\Delta u\cdot u \tag{2.6.4}$$

that, since $\operatorname{div} u = 0$ and for an argumentation similar to (2.4.26), leads to

$$\frac{1}{2}\frac{d}{dt}||u||_2^2 + \int_\Omega (\nabla u_1)\, u\cdot u = -\nu||\nabla u||_2^2 - \tau||\Delta u||_2^2 \tag{2.6.5}$$

and, taking to the right the nonlinear term, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}||u||_2^2 + \nu||\nabla u||_2^2 + \tau||\Delta u||_2^2 &= -\int_\Omega (\nabla u_1)\, u\cdot u \le \\ &\le \int_\Omega |(\nabla u_1)\, u\cdot u| \le \Big|\Big| |\nabla u_1|\, |u|^2 \Big|\Big|_1. \end{aligned} \tag{2.6.6}$$

Let us estimate term on the right

$$\left|\left||\nabla u_1|\,|u|^2\right|\right|_1 \leq (c_1||\nabla\nabla u||_2 + c_2||u||_2)\,||u||_2||\nabla u_1||_2 \leq \\ \leq \widetilde{c}\,(c_1||\nabla\nabla u||_2 + c_2||u||_2)\,||u||_2 \tag{2.6.7}$$

where, to estimate $|\nabla u_1|$ we used Theorem 2.6.1 and to estimate one of the two $|u|$, we used the fact that for Morrey's Theorem

$$W_0^{2,2} \hookrightarrow L^\infty \\ ||u||_\infty \leq c_1||\nabla\nabla u||_2 + c_2||u||_2. \tag{2.6.8}$$

Let us observe that $c_2 = 0$ in the case of bounded domains.

Then, setting $c := \widetilde{c}c_1$ and $\overline{c} := \widetilde{c}c_2$ and using (2.6.6), we get

$$\frac{1}{2}\frac{d}{dt}||u||_2^2 + \nu||\nabla u||_2^2 + \tau||\Delta u||_2^2 \leq \\ \leq c||\nabla\nabla u||_2||u||_2 + \overline{c}||u||_2^2 \leq \\ \leq c\,\varepsilon||\nabla\nabla u||_2^2 + cM_\varepsilon||u||_2^2 + \overline{c}||u||_2^2 = \\ = c\,\varepsilon||\Delta u||_2^2 + cM_\varepsilon||u||_2^2 + \overline{c}||u||_2^2 \tag{2.6.9}$$

where we used Young's inequality with a coefficient ε such that $\overline{\varepsilon} := c\,\varepsilon \ll 1$ and (2.3.3).

Finally, we have

$$\frac{1}{2}\frac{d}{dt}||u||_2^2 + \nu||\nabla u||_2^2 + (\tau - \overline{\varepsilon})||\Delta u||_2^2 \leq (cM_\varepsilon + \overline{c})\,||u||_2^2 \tag{2.6.10}$$

whence, setting $C := cM_\varepsilon + \overline{c}$, we obtain

$$\frac{1}{2}\frac{d}{dt}||u||_2^2 \leq C||u||_2^2; \tag{2.6.11}$$

by applying Gronwall's Lemma we get $u = 0$, namely $u_1 = u_2$ and hence, since $\nabla(p_1 - p_2) = 0$, also $p_1 = p_2$ up to a constant. $\square$

Conclusions

In conclusion we proved existence, uniqueness and regularity of the solution for the hyperviscous Navier-Stokes problem that models the motion of a fluid at first in a bounded domain and then in an exterior domain. Our results show that the regularity of the solutions imply continuity in space up to second derivatives. In particular, the continuity of the velocity is suitable for pointwise conditions. This may allow future applications of this second-gradient model to problems where the velocity is prescribed on thin structures like in the case of bodies that fall in a viscous fluid.

Bibliography

[1] P. Germain. *La méthode des puissances virtuelles en mécanique des milieux continus. I. Théorie du second gradient.* Mécanique, 12 (1973) 235-274.

[2] S.S. Antman and J.E. Osborn. *The principle of virtual work and integral laws of motion.* Arch. Rational Mech. Anal., 69(3) (1979) 231-262.

[3] J.G. Heywood. *The Navier-Stokes Equations: On the Existence, Regularity and Decay of solutions.* Indiana Univ. Math. J., 29 (1980) 639-681.

[4] E. Fried, M.E. Gurtin. *Tractions, Balances, and Boundary Conditions for Nonsimple Materials with Application to Liquid Flow at Small-Length Scales.* Arch. Rational Mech. Anal., 182 (2006) 513-554.

[5] A. Musesti. *Isotropic linear constitutive relations for nonsimple fluids.* Acta Mech., 204 (2009) 81-88.

[6] G.G. Giusteri, E. Fried. *Slender-body theory for viscous flow via dimensional reduction and hyperviscous regularization.* Meccanica, 49 (2014) 2153-2167.

[7] G.G. Giusteri, A. Marzocchi, A. Musesti. *Steady free fall of one-dimensional bodies in a hyperviscous fluid at low Reynolds number*, Evol. Equat. Control Theory, 3(3) (2014), 429-445.

[8] G.G. Giusteri, A. Marzocchi, A. Musesti. *Nonlinear free fall of one-dimensional rigid bodies in hyperviscous fluids*, Discret Contin. Dyn. Syst. Ser. B, 19(7) (2014), 2145-2157.

[9] F. Crispo, P. Maremonti. *An Interpolation Inequality in Exterior Domains*. Rend. Sem. Mat. Univ. Padova, 112 (2004).

[10] P. Maremonti. *Navier-Stokes initial boundary value problem: a short review of the chief results*. Unpublished.

[11] C. Amrouche, V. Girault. *Problèmes généralisés de Stokes*. Potugal. Math. 49(4) (1992), 463-503.

[12] R.A. Adams, J.J.F. Fournier. *Sobolev Spaces*. Academic Press (2003).

[13] A. Kufner, O. John, S. Fučík, *Function Spaces*. Noordhoff Int. Publishing Leyden (1977).

[14] R. Temam. *Navier-Stokes Equations: Theory and Numerical Analysis*. North-Holland Pub. Co., Amsterdam-New York-Tokyo.

[15] H. Brezis. *Analisi funzionale. Teoria e applicazioni*. Liguori (1986).

[16] G.G. Giusteri, A. Marzocchi, A. Musesti. *Three-dimensional nonsimple viscous liquids dragged by one-dimensional immersed bodies*. Mechanics Research Communications, 37 (2010), 642-646.

[17] M. Degiovanni, A. Marzocchi, A. Musesti. *Virtual powers on diffused subbodies and normal traces of tensor-valued measures*. M. Šilhavý ed., Mathematical modeling of bodies with complicated bulk and boundary behavior, Quaderni di Matematica, 20, 21-53.

[18] S. Agmon, A. Douglis, L. Nirenberg. *Estimates near the boundary for solutions of elliptic partial differential equations satisfying general boundary conditions*. Communications on pure and applied mathematics, 12 (1959), 623-727.

Acknowledgements

I wish to express my sincere thanks to prof. Alfredo Marzocchi for his help, for the useful time shared with me on this topic and for giving me support in this path; this work would not have been possible without him.

A special thanks goes also to proff. Paolo Maremonti, Marco Degiovanni, Alessandro Musesti and Luca Lussardi for their help, advice and discussions.