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Compactness results for the p-Laplace equation

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Abstract

Given $1 < p < N$ and two measurable functions $V(r) \geq 0$ and $K(r) > 0$, $r > 0$, we define the weighted spaces

$$W = \left\{ u \in D^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(|x|) |u|^p \, dx < \infty \right\}, \quad L_K^q = L^q(\mathbb{R}^N, K(|x|) \, dx)$$

and study the compact embeddings of the radial subspace of W into $L_K^{q_1} + L_K^{q_2}$, and thus into L_K^q $(= L_K^q + L_K^q)$ as a particular case. Both exponents q_1, q_2, q greater and lower than p are considered. Our results do not require any compatibility between how the potentials V and K behave at the origin and at infinity, and essentially rely on power type estimates of their relative growth, not of the potentials separately.

Keywords. Weighted Sobolev spaces, compact embeddings, unbounded or decaying potentials

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1 Introduction

In this paper we pursue the work we made in papers [3, 4, 7], where we studied embedding and compactness results for weighted Sobolev spaces. These results then made possible to get existence and multiplicity results, by variational methods, for semilinear elliptic equations in $\mathbb{R}^N$.

In the present paper we face nonlinear elliptic p-Laplace equations, that is,

$$-\triangle_p u + V(|x|) |u|^{p-1} u = K(|x|) f(u) \quad \text{in } \mathbb{R}^N. \tag{1.1}$$

Here $1 < p < N$, $f : \mathbb{R} \to \mathbb{R}$ is a continuous nonlinearity satisfying $f(0) = 0$ and $V \geq 0, K > 0$ are given potentials.

To study this problem we introduce the space

$$W := \left\{ u \in D^{1,p}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(|x|) |u|^p dx < \infty \right\}$$

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equipped with the standard norm

$$\|u\|^p := \int_{\mathbb{R}^N} \left(|\nabla u|^p + V(|x|)\, |u|^p \right) dx,$$

and say that $u \in W$ is a *weak solution* to (1.1) if

$$\int_{\mathbb{R}^N} |\nabla u|^{p-2} \nabla u \cdot \nabla h\, dx + \int_{\mathbb{R}^N} V(|x|)\, |u|^{p-2} u h\, dx = \int_{\mathbb{R}^N} K(|x|)\, f(u)\, h\, dx \quad \text{for all } h \in W. \tag{1.2}$$

The natural approach in studying weak solutions to equation (1.1) is variational, since these solutions are (at least formally) critical points of the Euler functional

$$I(u) := \frac{1}{p} \|u\|^p - \int_{\mathbb{R}^N} K(|x|)\, F(u)\, dx, \tag{1.3}$$

where $F(t) := \int_0^t f(s)\, ds$. Then the problem of existence is easily solved if V does not vanish at infinity and K is bounded, because standard embeddings theorems of W and its radial subspace into the weighted Lebesgue space

$$L_K^q := L_K^q(\mathbb{R}^N) := L^q(\mathbb{R}^N, K(|x|)\, dx)$$

are available (for suitable q's). As we let V and K to vanish, or to go to infinity, as $|x| \to 0$ or $|x| \to +\infty$, the usual embeddings theorems for Sobolev spaces are not available anymore, and new embedding theorems need to be proved. This has been done in several papers: see e.g. the references in [3, 4, 7] for a bibliography concerning the usual Laplace equation, and [1, 5, 6, 8–12] for equations involving the p-laplacian.

The main novelty of our approach (in [3,4] and in the present paper) is two-fold. First, we look for embeddings of W_r (the radial subspace of W) not into a single Lebesgue space L_K^q but into a sum of Lebesgue spaces $L_K^{q_1} + L_K^{q_2}$. This allows to study separately the behaviour of the potentials V, K at 0 and ∞, and to assume different set of hypotheses about these behaviours. Second, we assume hypotheses not on V and K separately but on their ratio, so allowing asymptotic behaviours of general kind for the two potentials.

Thanks to these novelties, our embedding results yield existence of solutions for (1.1) in cases which are not covered by the previous literature. Moreover, one can check that our embeddings are also new in some of the cases already treated in previous papers, thus giving existence results which improve some well-known theorems in the literature.

In the present paper we limit ourselves to the proof of the compact embeddings, which is the hardest part of the arguments. In a forthcoming paper [2] we will apply these results to get existence and multiplicity results for p-laplacian equations like (1.1).

This paper is organized as follows. In Section 2 we state our main results: a general result concerning the embedding properties of W_r into $L_K^{q_1} + L_K^{q_2}$ (Theorem 2.1) and some explicit conditions ensuring that the embedding is compact (Theorems 2.2, 2.3, 2.5 and 2.7). The general result is proved in Section 3, the explicit conditions in Section 4. The Appendix is devoted to some detailed computations, displaced from Section 4 for sake of clarity.

Notations. We end this introductory section by collecting some notations used in the paper.

• For every $R>0$, we set $B_R := \left\{x \in \mathbb{R}^N : |x| < r\right\}$.

• For any subset $A \subseteq \mathbb{R}^N$, we denote $A^c := \mathbb{R}^N \setminus A$. If A is Lebesgue measurable, $|A|$ stands for its measure.

• By $\rightarrow$ and $\rightharpoonup$ we respectively mean *strong* and *weak* convergence.

• $\hookrightarrow$ denotes *continuous* embeddings.

• $C_{\mathrm{c}}^{\infty}(\Omega)$ is the space of the infinitely differentiable real functions with compact support in the open set $\Omega \subseteq \mathbb{R}^d$.

• If $1 \leq p \leq \infty$ then $L^p(A)$ and $L^p_{\mathrm{loc}}(A)$ are the usual real Lebesgue spaces (for any measurable set $A \subseteq \mathbb{R}^d$). If $\rho : A \rightarrow (0,+\infty)$ is a measurable function, then $L^p(A, \rho\left(z\right) dz)$ is the real Lebesgue space with respect to the measure $\rho\left(z\right) dz$ (dz stands for the Lebesgue measure on $\mathbb{R}^d$).

• $p' := p/(p-1)$ is the Hölder-conjugate exponent of p.

• For $1<p<N$, $D^{1,p}(\mathbb{R}^N) = \{u \in L^{p^*}(\mathbb{R}^N) : \nabla u \in L^2(\mathbb{R}^N)\}$ is the usual Sobolev space, which identifies with the completion of $C_{\mathrm{c}}^{\infty}(\mathbb{R}^N)$ with respect to the norm of the gradient; $D^{1,p}_{\mathrm{rad}}(\mathbb{R}^N)$ is the radial subspace of $D^{1,p}(\mathbb{R}^N)$; $D_0^{1,p}\left(B_R\right)$ is closure of $C_{\mathrm{c}}^{\infty}\left(B_R\right)$ in $D^{1,p}(\mathbb{R}^N)$.

• For $1<p<N$, $p^* := pN/\left(N-p\right)$ is the critical exponent for the Sobolev embedding in dimension N.

2 Main results

We consider $1<p<N$ and we assume the following hypotheses on V, K:

($\mathbf{V}$) $V : (0,+\infty) \rightarrow [0,+\infty]$ is a measurable function such that $V \in L^1\left((r_1,r_2)\right)$ for some $r_2 > r_1 > 0$;

($\mathbf{K}$) $K : (0,+\infty) \rightarrow (0,+\infty)$ is a measurable function such that $K \in L^s_{\mathrm{loc}}\left((0,+\infty)\right)$ for some $s>1$.

Let us define the following function spaces

$$W := D^{1,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, V(|x|)dx), \quad W_r := D^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, V(|x|)dx) \tag{2.1}$$

and let $||u||$ be the standard norm in W (and W_r). Assumption ($\mathbf{V}$) implies that the spaces W and W_r are nontrivial, while hypothesis ($\mathbf{K}$) ensures that W_r is compactly embedded into the weighted Lebesgue space $L^q_K(B_R \setminus B_r)$ for every $1<q<\infty$ and $R>r>0$ (cf. Lemma 3.1 below). In what follows, the summability assumptions in ($\mathbf{V}$) and ($\mathbf{K}$) will not play any other role than this.

Given V and K, we define the following functions of $R>0$ and $q>1$:

$$\mathcal{S}_0\left(q,R\right) := \sup_{u \in W_r,\, \|u\|=1} \int_{B_R} K\left(|x|\right) |u|^q \, dx, \tag{2.2}$$

$$\mathcal{S}_{\infty}\left(q,R\right) := \sup_{u \in W_r,\, \|u\|=1} \int_{\mathbb{R}^N \setminus B_R} K\left(|x|\right) |u|^q \, dx. \tag{2.3}$$

Clearly $\mathcal{S}_0(q,\cdot)$ is nondecreasing, $\mathcal{S}_\infty(q,\cdot)$ is nonincreasing and both of them can be infinite at some R.

Our first result concerns the embedding properties of W_r into $L_K^{q_1}+L_K^{q_2}$ and relies on assumptions which are quite general, sometimes also sharp (see claim (iii)), but not so easy to check. More handy conditions ensuring these general assumptions will be provided by the next results. Some reference on the space $L_K^{q_1}+L_K^{q_2}$ will be given in Section 3.

Theorem 2.1. *Let $1<p<N$, let V, K be as in $(\mathbf{V})$, $(\mathbf{K})$ and let $q_1,q_2>1$.*

(i) If

$$\mathcal{S}_0(q_1,R_1)<\infty \quad \textit{and} \quad \mathcal{S}_\infty(q_2,R_2)<\infty \quad \textit{for some } R_1,R_2>0, \tag{$\mathcal{S}'_{q_1,q_2}$}$$

then W_r is continuously embedded into $L_K^{q_1}(\mathbb{R}^N)+L_K^{q_2}(\mathbb{R}^N)$.

(ii) If

$$\lim_{R\to0^+}\mathcal{S}_0(q_1,R)=\lim_{R\to+\infty}\mathcal{S}_\infty(q_2,R)=0, \tag{$\mathcal{S}''_{q_1,q_2}$}$$

then W_r is compactly embedded into $L_K^{q_1}(\mathbb{R}^N)+L_K^{q_2}(\mathbb{R}^N)$.

(iii) If $K(|\cdot|)\in L^1(B_1)$ and $q_1\leq q_2$, then conditions $\left(\mathcal{S}'_{q_1,q_2}\right)$ and $\left(\mathcal{S}''_{q_1,q_2}\right)$ are also necessary to the above embeddings.

Observe that, of course, $(\mathcal{S}''_{q_1,q_2})$ implies $(\mathcal{S}'_{q_1,q_2})$. Moreover, these assumptions can hold with $q_1=q_2=q$ and therefore Theorem 2.1 also concerns the embedding properties of W_r into L_K^q, $1<q<\infty$.

We now look for explicit conditions on V and K implying $(\mathcal{S}''_{q_1,q_2})$ for some q_1 and q_2. More precisely, we will ensure $(\mathcal{S}''_{q_1,q_2})$ through a more stringent condition involving the following functions of $R>0$ and $q>1$:

$$\mathcal{R}_0(q,R):=\sup_{u\in H^1_{V,\mathrm{r}},\,h\in H^1_V,\,\|u\|=\|h\|=1}\int_{B_R}K(|x|)\,|u|^{q-1}\,|h|\,dx, \tag{2.4}$$

$$\mathcal{R}_\infty(q,R):=\sup_{u\in H^1_{V,\mathrm{r}},\,h\in H^1_V,\,\|u\|=\|h\|=1}\int_{\mathbb{R}^N\setminus B_R}K(|x|)\,|u|^{q-1}\,|h|\,dx. \tag{2.5}$$

Note that $\mathcal{R}_0(q,\cdot)$ is nondecreasing, $\mathcal{R}_\infty(q,\cdot)$ is nonincreasing and both can be infinite at some R. Moreover, for every (q,R) one has $\mathcal{S}_0(q,R)\leq\mathcal{R}_0(q,R)$ and $\mathcal{S}_\infty(q,R)\leq\mathcal{R}_\infty(q,R)$, so that $(\mathcal{S}''_{q_1,q_2})$ is a consequence of the following, stronger condition:

$$\lim_{R\to0^+}\mathcal{R}_0(q_1,R)=\lim_{R\to+\infty}\mathcal{R}_\infty(q_2,R)=0. \tag{$\mathcal{R}''_{q_1,q_2}$}$$

In Theorems 2.2 and 2.7 we will find ranges of exponents q_1 such that $\lim_{R\to0^+}\mathcal{R}_0(q_1,R)=0$. In Theorems 2.3 and 2.5 we will do the same for exponents q_2 such that $\lim_{R\to+\infty}\mathcal{R}_\infty(q_2,R)=0$. Condition $(\mathcal{R}''_{q_1,q_2})$ then follows by joining Theorem 2.2 or 2.7 with Theorem 2.3 or 2.5.

For $\alpha \in \mathbb{R}$ and $\beta \in [0,1]$, define two functions $\alpha^*(\beta)$ and $q^*(\alpha,\beta)$ by setting

$$\alpha^*(\beta) := \max\left\{p\beta - 1 - \frac{p-1}{p}N, -(1-\beta)N\right\} = \begin{cases} p\beta - 1 - \frac{p-1}{p}N & \text{if } 0 \leq \beta \leq \frac{1}{p} \\ -(1-\beta)N & \text{if } \frac{1}{p} \leq \beta \leq 1 \end{cases}$$

and

$$q^*(\alpha,\beta) := p\frac{\alpha - p\beta + N}{N-p}.$$

Note that $\alpha^*(\beta) \leq 0$ and $\alpha^*(\beta) = 0$ if and only if $\beta = 1$.

The first two Theorems 2.2 and 2.3 only rely on a power type estimate of the relative growth of the potentials and do not require any other separate assumption on V and K than $(\mathbf{V})$ and $(\mathbf{K})$, including the case $V(r) \equiv 0$ (see Remark 2.4.1).

Theorem 2.2. *Let $1 < p < N$ and let V, K be as in $(\mathbf{V})$, $(\mathbf{K})$. Assume that there exists $R_1 > 0$ such that $V(r) < +\infty$ almost everywhere in $(0,R_1)$ and*

$$\operatorname*{ess\,sup}_{r\in(0,R_1)} \frac{K(r)}{r^{\alpha_0}V(r)^{\beta_0}} < +\infty \quad \text{for some } 0 \leq \beta_0 \leq 1 \text{ and } \alpha_0 > \alpha^*(\beta_0). \tag{2.6}$$

Then $\lim_{R\to 0^+} \mathcal{R}_0(q_1,R) = 0$ for every $q_1 \in \mathbb{R}$ such that

$$\max\{1, p\beta_0\} < q_1 < q^*(\alpha_0,\beta_0). \tag{2.7}$$

Theorem 2.3. *Let $1 < p < N$ and let V, K be as in $(\mathbf{V})$, $(\mathbf{K})$. Assume that there exists $R_2 > 0$ such that $V(r) < +\infty$ for almost every $r > R_2$ and*

$$\operatorname*{ess\,sup}_{r>R_2} \frac{K(r)}{r^{\alpha_\infty}V(r)^{\beta_\infty}} < +\infty \quad \text{for some } 0 \leq \beta_\infty \leq 1 \text{ and } \alpha_\infty \in \mathbb{R}. \tag{2.8}$$

Then $\lim_{R\to+\infty} \mathcal{R}_\infty(q_2,R) = 0$ for every $q_2 \in \mathbb{R}$ such that

$$q_2 > \max\{1, p\beta_\infty, q^*(\alpha_\infty,\beta_\infty)\}. \tag{2.9}$$

We observe explicitly that for every $(\alpha,\beta) \in \mathbb{R}\times[0,1]$ one has

$$\max\{1, p\beta, q^*(\alpha,\beta)\} = \begin{cases} q^*(\alpha,\beta) & \text{if } \alpha \geq \alpha^*(\beta) \\ \max\{1,p\beta\} & \text{if } \alpha \leq \alpha^*(\beta) \end{cases}.$$

Remark 2.4.

1. We mean $V(r)^0 = 1$ for every r (even if $V(r) = 0$). In particular, if $V(r) = 0$ for almost every $r > R_2$, then Theorem 2.3 can be applied with $\beta_\infty = 0$ and assumption (2.8) means

$$\operatorname*{ess\,sup}_{r>R_2} \frac{K(r)}{r^{\alpha_\infty}} < +\infty \quad \text{for some } \alpha_\infty \in \mathbb{R}.$$

Similarly for Theorem 2.2 and assumption (2.6), if $V(r) = 0$ for almost every $r \in (0,R_1)$.

2. The inequality $\max\{1, p\beta_0\} < q^*(\alpha_0, \beta_0)$ is equivalent to $\alpha_0 > \alpha^*(\beta_0)$. Then, in (2.7), such inequality is automatically true and does not ask for further conditions on α_0 and β_0.

3. The assumptions of Theorems 2.2 and 2.3 may hold for different pairs (α_0, β_0), $(\alpha_\infty, \beta_\infty)$. In this case, of course, one chooses them in order to get the ranges for q_1, q_2 as large as possible. For instance, if V is not singular at the origin, i.e., V is essentially bounded in a neighbourhood of 0, and condition (2.6) holds true for a pair (α_0, β_0), then (2.6) also holds for all pairs (α_0', β_0') such that $\alpha_0' < \alpha_0$ and $\beta_0' < \beta_0$. Therefore, since $\max\{1, p\beta\}$ is nondecreasing in β and $q^*(\alpha, \beta)$ is increasing in α and decreasing in β, it is convenient to choose $\beta_0 = 0$ and the best interval where one can take q_1 is $1 < q_1 < q^*(\overline{\alpha}, 0)$ with $\overline{\alpha} := \sup\left\{\alpha_0 : \operatorname{ess\,sup}_{r\in(0,R_1)} \frac{K(r)}{r^{\alpha_0}} < +\infty\right\}$ (we mean $q^*(+\infty, 0) = +\infty$).

For any $\alpha \in \mathbb{R}$, $\beta \leq 1$ and $\gamma \in \mathbb{R}$, define

$$q_*(\alpha, \beta, \gamma) := p\frac{\alpha - \gamma\beta + N}{N - \gamma} \quad \text{and} \quad q_{**}(\alpha, \beta, \gamma) := p\frac{p\alpha + (1 - p\beta)\gamma + p(N-1)}{p(N-1) - \gamma(p-1)}. \tag{2.10}$$

Of course q_* and q_{**} are undefined if $\gamma = N$ and $\gamma = \frac{p}{p-1}(N-1)$, respectively.

The next Theorems 2.5 and 2.7 improve the results of Theorems 2.2 and 2.3 by exploiting further informations on the growth of V (see Remarks 2.6.2 and 2.8.3).

Theorem 2.5. *Let $1 < p < N$ and let V, K be as in $(\mathbf{V})$, $(\mathbf{K})$. Assume that there exists $R_2 > 0$ such that $V(r) < +\infty$ for almost every $r > R_2$ and*

$$\operatorname*{ess\,sup}_{r>R_2} \frac{K(r)}{r^{\alpha_\infty} V(r)^{\beta_\infty}} < +\infty \quad \text{for some } 0 \leq \beta_\infty \leq 1 \text{ and } \alpha_\infty \in \mathbb{R} \tag{2.11}$$

and

$$\operatorname*{ess\,inf}_{r>R_2} r^{\gamma_\infty} V(r) > 0 \quad \text{for some } \gamma_\infty \leq p. \tag{2.12}$$

Then $\lim_{R\to+\infty} \mathcal{R}_\infty(q_2, R) = 0$ for every $q_2 \in \mathbb{R}$ such that

$$q_2 > \max\{1, p\beta_\infty, q_*, q_{**}\}, \tag{2.13}$$

where $q_* = q_*(\alpha_\infty, \beta_\infty, \gamma_\infty)$ and $q_{**} = q_{**}(\alpha_\infty, \beta_\infty, \gamma_\infty)$.

For future convenience, we define three functions $\alpha_1 := \alpha_1(\beta, \gamma)$, $\alpha_2 := \alpha_2(\beta)$ and $\alpha_3 := \alpha_3(\beta, \gamma)$ by setting

$$\alpha_1 := -(1-\beta)\gamma, \quad \alpha_2 := -(1-\beta)N, \quad \alpha_3 := -\frac{(p-1)N + (1-p\beta)\gamma}{p}. \tag{2.14}$$

Then an explicit description of $\max\{1, p\beta, q_*, q_{**}\}$ is the following: for every $(\alpha, \beta, \gamma) \in \mathbb{R} \times (-\infty, 1] \times (-\infty, N)$ we have

$$\max\{1, p\beta, q_*, q_{**}\} = \begin{cases} q_{**}(\alpha, \beta, \gamma) & \text{if } \alpha \geq \alpha_1 \\ q_*(\alpha, \beta, \gamma) & \text{if } \max\{\alpha_2, \alpha_3\} \leq \alpha \leq \alpha_1 \\ \max\{1, p\beta\} & \text{if } \alpha \leq \max\{\alpha_2, \alpha_3\} \end{cases}, \tag{2.15}$$

where $\max\{\alpha_2, \alpha_3\} < \alpha_1$ for every $\beta < 1$ and $\max\{\alpha_2, \alpha_3\} = \alpha_1 = 0$ if $\beta = 1$.

Remark 2.6.

1. The proof of Theorem 2.5 does not require $\beta_\infty \geq 0$, but this condition is not a restriction of generality in stating the theorem. Indeed, under assumption (2.12), if (2.11) holds with $\beta_\infty < 0$, then it also holds with α_∞ and β_∞ replaced by $\alpha_\infty - \beta_\infty\gamma_\infty$ and 0 respectively, and this does not change the thesis (2.13), because $q_* (\alpha_\infty - \beta_\infty\gamma_\infty, 0, \gamma_\infty) = q_* (\alpha_\infty, \beta_\infty, \gamma_\infty)$ and $q_{**} (\alpha_\infty - \beta_\infty\gamma_\infty, 0, \gamma_\infty) = q_{**} (\alpha_\infty, \beta_\infty, \gamma_\infty)$.

2. Denote $q^* = q^* (\alpha_\infty, \beta_\infty)$ for brevity. If $\gamma_\infty < p$, then one has

$$\max\{1, p\beta_\infty, q^*\} = \begin{cases} \max\{1, p\beta_\infty\} = \max\{1, p\beta_\infty, q_*, q_{**}\} & \text{if } \alpha_\infty \leq \alpha^* (\beta_\infty) \\ q^* > \max\{1, p\beta_\infty, q_*, q_{**}\} & \text{if } \alpha_\infty > \alpha^* (\beta_\infty) \end{cases},$$

   so that, under assumption (2.12), Theorem 2.5 improves Theorem 2.3. Otherwise, if $\gamma_\infty = p$, we have $q_* = q_{**} = q^*$ and Theorems 2.5 and 2.3 give the same result. This is not surprising, since, by Hardy inequality, the space W coincides with $D^{1,p}(\mathbb{R}^N)$ if $V(r) = r^{-p}$ and thus, for $\gamma_\infty = p$, we cannot expect a better result than the one of Theorem 2.3, which covers the case of $V(r) \equiv 0$, i.e., of $D^{1,p}(\mathbb{R}^N)$.

3. Description (2.15) shows that q_* and q_{**} are not relevant in inequality (2.13) if $\alpha_\infty \leq \alpha_2 (\beta_\infty)$. On the other hand, if $\alpha_\infty > \alpha_2 (\beta_\infty)$, both q_* and q_{**} turn out to be increasing in γ and hence it is convenient to apply Theorem 2.5 with the smallest γ_∞ for which (2.12) holds. This is consistent with the fact that, if (2.12) holds with γ_∞, then it also holds with every γ'_∞ such that $\gamma_\infty \leq \gamma'_\infty \leq p$.

In order to state our last result, we introduce, by the following definitions, an open region $\mathcal{A}_{\beta,\gamma}$ of the αq-plane, depending on $\beta \in [0,1]$ and $\gamma \geq p$. Recall the definitions (2.10) of the functions $q_* = q_* (\alpha, \beta, \gamma)$ and $q_{**} = q_{**} (\alpha, \beta, \gamma)$. We set

$$\begin{aligned}
&\mathcal{A}_{\beta,\gamma} := \{(\alpha, q) : \max\{1, p\beta\} < q < \min\{q_*, q_{**}\}\} && \text{if } p \leq \gamma < N, \\
&\mathcal{A}_{\beta,\gamma} := \{(\alpha, q) : \max\{1, p\beta\} < q < q_{**},\ \alpha > -(1-\beta) N\} && \text{if } \gamma = N, \\
&\mathcal{A}_{\beta,\gamma} := \{(\alpha, q) : \max\{1, p\beta, q_*\} < q < q_{**}\} && \text{if } N < \gamma < \tfrac{p}{p-1}(N-1), \\
&\mathcal{A}_{\beta,\gamma} := \{(\alpha, q) : \max\{1, p\beta, q_*\} < q,\ \alpha > -(1-\beta)\gamma\} && \text{if } \gamma = \tfrac{p}{p-1}(N-1), \\
&\mathcal{A}_{\beta,\gamma} := \{(\alpha, q) : \max\{1, p\beta, q_*, q_{**}\} < q\} && \text{if } \gamma > \tfrac{p}{p-1}(N-1).
\end{aligned} \tag{2.16}$$

Notice that $\frac{p}{p-1}(N-1) > N$ because $p < N$. For more clarity, $\mathcal{A}_{\beta,\gamma}$ is sketched in the following five pictures, according to the five cases above. Recall the definitions (2.14) of the functions $\alpha_1 = \alpha_1 (\beta, \gamma)$, $\alpha_2 = \alpha_2 (\beta)$ and $\alpha_3 = \alpha_3 (\beta, \gamma)$.

Fig.1: $\mathcal{A}_{\beta,\gamma}$ for $p \le \gamma < N$.

- If $\gamma = p$, the two straight lines above are the same.
- If $\beta < 1$ we have $\max\{\alpha_2, \alpha_3\} < \alpha_1 < 0$.
If $\beta = 1$ we have $\alpha_3 < \alpha_2 = \alpha_1 = 0$ and $\mathcal{A}_{1,\gamma}$ reduces to the angle $p < q < q_{**}$.

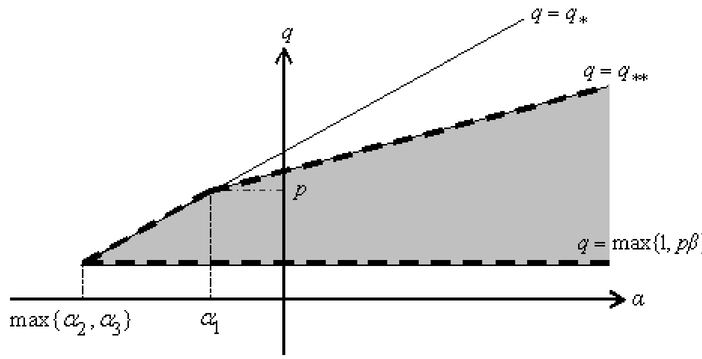


Fig.2: $\mathcal{A}_{\beta,\gamma}$ $\gamma = N$.

- If $\beta < 1$ we have $\alpha_1 = \alpha_2 = \alpha_3 < 0$.
If $\beta = 1$ we have $\alpha_1 = \alpha_2 = \alpha_3 = 0$ and $\mathcal{A}_{1,\gamma}$ reduces to the angle $p < q < q_{**}$.

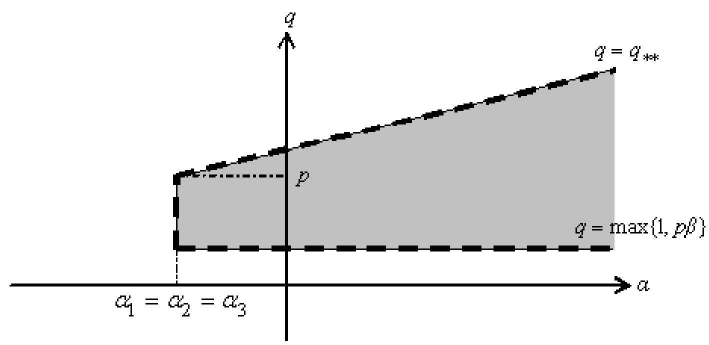


Fig.3: $\mathcal{A}_{\beta,\gamma}$ for $N < \gamma < \frac{p}{p-1}(N-1)$.

- If $\beta < 1$ we have $\alpha_1 < \min\{\alpha_2, \alpha_3\} < 0$.
If $\beta = 1$ we have $0 = \alpha_1 = \alpha_2 < \alpha_3$ and $\mathcal{A}_{1,\gamma}$ reduces to the angle $p < q < q_{**}$.

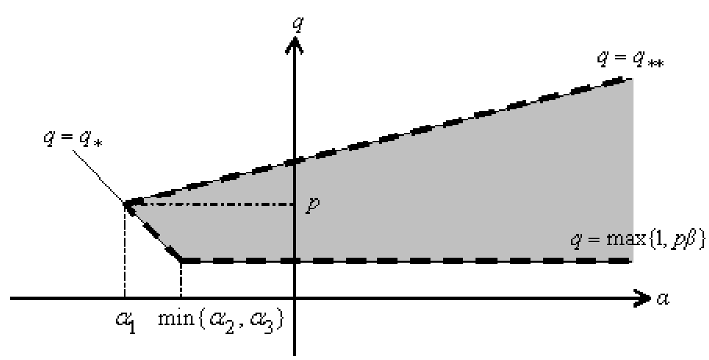


Fig.4: $\mathcal{A}_{\beta,\gamma}$ for $\gamma = \frac{p}{p-1}(N-1)$.

- If $\beta < 1$ we have $\alpha_1 < \min\{\alpha_2, \alpha_3\} < 0$.
If $\beta = 1$ we have $0 = \alpha_1 = \alpha_2 < \alpha_3$ and $\mathcal{A}_{1,\gamma}$ reduces to the angle $\alpha > 0$, $q > p$.

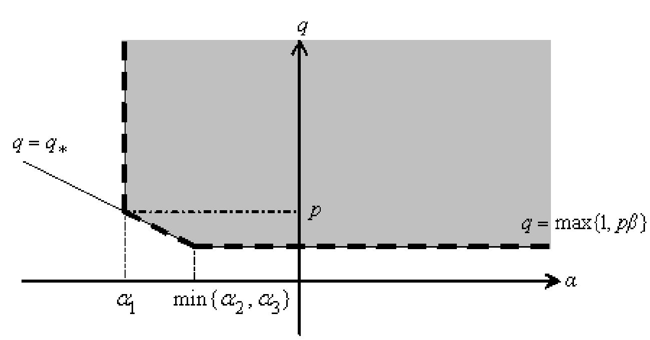
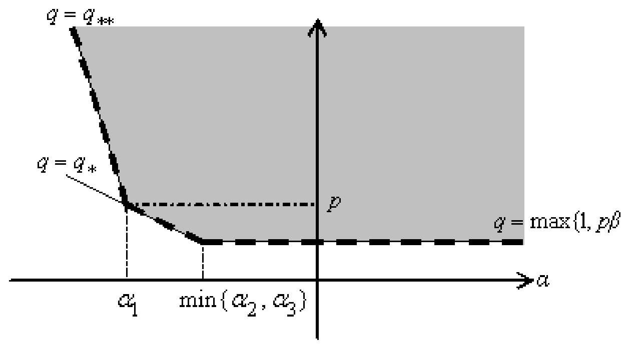


Fig.5: $\mathcal{A}_{\beta,\gamma}$ for $\gamma > \frac{p}{p-1}(N-1)$.
• If $\beta < 1$ we have
$\alpha_1 < \min\{\alpha_2, \alpha_3\} < 0$.
If $\beta = 1$ we have
$0 = \alpha_1 = \alpha_2 < \alpha_3$
and $\mathcal{A}_{1,\gamma}$ reduces to the angle
$q > \max\{p, q_{**}\}$.



Theorem 2.7. *Let $N \geq 3$ and let V, K be as in $(\mathbf{V})$, $(\mathbf{K})$. Assume that there exists $R_1 > 0$ such that $V(r) < +\infty$ almost everywhere in $(0, R_1)$ and*

$$\operatorname*{ess\,sup}_{r\in(0,R_1)} \frac{K(r)}{r^{\alpha_0} V(r)^{\beta_0}} < +\infty \quad \textit{for some } 0 \leq \beta_0 \leq 1 \textit{ and } \alpha_0 \in \mathbb{R} \tag{2.17}$$

and

$$\operatorname*{ess\,inf}_{r\in(0,R_1)} r^{\gamma_0} V(r) > 0 \quad \textit{for some } \gamma_0 \geq p. \tag{2.18}$$

Then $\lim_{R\to 0^+} \mathcal{R}_0(q_1, R) = 0$ for every $q_1 \in \mathbb{R}$ such that

$$(\alpha_0, q_1) \in \mathcal{A}_{\beta_0,\gamma_0}. \tag{2.19}$$

Remark 2.8.

1. Condition (2.19) also asks for a lower bound on α_0, except for the case $\gamma_0 > \frac{p}{p-1}(N-1)$, as it is clear from Figures 1-5.

2. The proof of Theorem 2.7 does not require $\beta_0 \geq 0$, but this is not a restriction of generality in stating the theorem (cf. Remark 2.6.1). Indeed, under assumption (2.18), if (2.17) holds with $\beta_0 < 0$, then it also holds with α_0 and β_0 replaced by $\alpha_0 - \beta_0\gamma_0$ and 0 respectively, and one has that $(\alpha_0, q_1) \in \mathcal{A}_{\beta_0,\gamma_0}$ if and only if $(\alpha_0 - \beta_0\gamma_0, q_1) \in \mathcal{A}_{0,\gamma_0}$.

3. If (2.18) holds with $\gamma_0 > p$, then Theorem 2.7 improves Theorem 2.2. Otherwise, if $\gamma_0 = p$, then one has $\max\{\alpha_2, \alpha_3\} = \alpha^*(\beta_0)$ and $(\alpha_0, q_1) \in \mathcal{A}_{\beta_0,\gamma_0}$ is equivalent to $\max\{1, p\beta_0\} < q_1 < q^*(\alpha_0, \beta_0)$, i.e., Theorems 2.7 and 2.2 give the same result, which is consistent with Hardy inequality (cf. Remark 2.6.2).

4. Given $\beta \leq 1$, one can check that $\mathcal{A}_{\beta,\gamma_1} \subseteq \mathcal{A}_{\beta,\gamma_2}$ for every $p \leq \gamma_1 < \gamma_2$, so that, in applying Theorem 2.7, it is convenient to choose the largest γ_0 for which (2.18) holds. This is consistent with the fact that, if (2.18) holds with γ_0, then it also holds with every γ_0' such that $p \leq \gamma_0' \leq \gamma_0$.

3 Proof of Theorem 2.1

Assume $1 < p < N$ and let V and K be as in $(\mathbf{V})$ and $(\mathbf{K})$.

Recall the definitions (2.1) of the Banach spaces W and W_r. Using the results of [10, Lemma 1], fix two constants $S_{N,p} > 0$ and $C_{N,p} > 0$, only dependent on N and p, such that

$$\forall u \in W, \quad \|u\|_{L^{p^*}(\mathbb{R}^N)} \leq S_{N,p} \|u\| \tag{3.1}$$

and

$$\forall u \in W_r, \quad |u(x)| \leq C_{N,p} \|u\| \frac{1}{|x|^{\frac{N-p}{p}}} \quad \text{almost everywhere on } \mathbb{R}^N. \tag{3.2}$$

Recall from assumption $(\mathbf{K})$ that $K \in L^s_{\text{loc}}((0,+\infty))$ for some $s > 1$.

Lemma 3.1. *Let $R > r > 0$ and $1 < q < \infty$. Then there exist $\tilde{C} = \tilde{C}(N,p,r,R,q,s) > 0$ and $l = l(p,q,s) > 0$ such that $\forall u \in W_r$ one has*

$$\int_{B_R \setminus B_r} K(|x|)|u|^q \, dx \leq \tilde{C} \|K(|\cdot|)\|_{L^s(B_R \setminus B_r)} \|u\|^{q-lp} \left(\int_{B_R \setminus B_r} |u|^p \, dx \right)^l. \tag{3.3}$$

Moreover, if

$$s > \frac{p^*}{p^*-1} = \frac{Np}{N(p-1)+p} \qquad \left(p^* = \frac{Np}{N-p} \right)$$

in assumption $(\mathbf{K})$, then there exists $\tilde{C}_1 = \tilde{C}_1(N,p,r,R,q,s) > 0$ such that $\forall u \in W_r$ and $\forall h \in W$ one has

$$\frac{\int_{B_R \setminus B_r} K(|x|)|u|^{q-1}|h| \, dx}{\tilde{C}_1 \|K(|\cdot|)\|_{L^s(B_R \setminus B_r)}} \leq \begin{cases} \left(\int_{B_R \setminus B_r} |u|^p \, dx \right)^{\frac{q-1}{p}} \|h\| & \text{if } q \leq \tilde{q} \\ \left(\int_{B_R \setminus B_r} |u|^p \, dx \right)^{\frac{\tilde{q}-1}{p}} \|u\|^{q-\tilde{q}} \|h\| & \text{if } q > \tilde{q} \end{cases}$$

where $\tilde{q} := p\left(1 + \frac{1}{N} - \frac{1}{s}\right)$ (note that $s > \frac{Np}{N(p-1)+p}$ implies $\tilde{q} > 1$).

Proof. Let $u \in W_r$ and fix $t \in (1,s)$ such that $t'q > p$ (where $t' = t/(t-1)$). Then, by Hölder inequality and (3.2), we have

$$\begin{aligned} &\int_{B_R \setminus B_r} K(|x|)|u|^q \, dx \\ &\leq \left(\int_{B_R \setminus B_r} K(|x|)^t \, dx \right)^{\frac{1}{t}} \left(\int_{B_R \setminus B_r} |u|^{t'q} \, dx \right)^{\frac{1}{t'}} \\ &\leq |B_R \setminus B_r|^{\frac{1}{t}-\frac{1}{s}} \|K(|\cdot|)\|_{L^s(B_R \setminus B_r)} \left(\int_{B_R \setminus B_r} |u|^{t'q-p}|u|^p \, dx \right)^{\frac{1}{t'}} \\ &\leq |B_R \setminus B_r|^{\frac{1}{t}-\frac{1}{s}} \|K(|\cdot|)\|_{L^s(B_R \setminus B_r)} \left(\frac{C_{N,p}\|u\|}{r^{\frac{N-p}{p}}} \right)^{q-p/t'} \left(\int_{B_R \setminus B_r} |u|^p \, dx \right)^{\frac{1}{t'}}. \end{aligned}$$

This proves (3.3). As to the second part of the lemma, let $u \in W_r$ and $h \in W$. For simplicity, we denote by σ the Hölder-conjugate exponent of p^*, i.e., $\sigma := \frac{Np}{N(p-1)+p}$. By Hölder inequality (note that $\frac{s}{\sigma} > 1$), we have

$$\begin{aligned}
&\int_{B_R \setminus B_r} K(|x|) |u|^{q-1} |h| \, dx \\
&\leq \left(\int_{B_R \setminus B_r} K(|x|)^{\sigma} |u|^{(q-1)\sigma} \, dx \right)^{\frac{1}{\sigma}} \left(\int_{B_R \setminus B_r} |h|^{p^*} \, dx \right)^{\frac{1}{p^*}} \\
&\leq \left(\left(\int_{B_R \setminus B_r} K(|x|)^{s} \, dx \right)^{\frac{\sigma}{s}} \left(\int_{B_R \setminus B_r} |u|^{(q-1)\sigma \left(\frac{s}{\sigma}\right)'} \, dx \right)^{\frac{1}{\left(\frac{s}{\sigma}\right)'}} \right)^{\frac{1}{\sigma}} S_{N,p} \|h\| \\
&\leq S_{N,p} \|K(|\cdot|)\|_{L^s(B_R \setminus B_r)} \|h\| \left(\int_{B_R \setminus B_r} |u|^{p \frac{q-1}{\tilde{q}-1}} \, dx \right)^{\frac{\tilde{q}-1}{p}},
\end{aligned}$$

where we computed $\sigma \left(\frac{s}{\sigma}\right)' = \frac{pNs}{sN(p-1)+ps-pN} = \frac{p}{\tilde{q}-1}$. If $q < \tilde{q}$, then we get

$$\begin{aligned}
&\int_{B_R \setminus B_r} K(|x|) |u|^{q-1} |h| \, dx \\
&\leq S_{N,p} \|K(|\cdot|)\|_{L^s(B_R \setminus B_r)} \|h\| \left(|B_R \setminus B_r|^{1-\frac{q-1}{\tilde{q}-1}} \left(\int_{B_R \setminus B_r} |u|^{p} \, dx \right)^{\frac{q-1}{\tilde{q}-1}} \right)^{\frac{\tilde{q}-1}{p}} \\
&= S_{N,p} \|K(|\cdot|)\|_{L^s(B_R \setminus B_r)} \|h\| \, |B_R \setminus B_r|^{\frac{\tilde{q}-q}{p}} \left(\int_{B_R \setminus B_r} |u|^{p} \, dx \right)^{\frac{q-1}{p}}.
\end{aligned}$$

If $q = \tilde{q}$ the thesis plainly follows. Otherwise, if $q > \tilde{q}$, then by (3.2) we obtain

$$\begin{aligned}
&\int_{B_R \setminus B_r} K(|x|) |u|^{q-1} |h| \, dx \\
&\leq S_{N,p} \|K(|\cdot|)\|_{L^s(B_R \setminus B_r)} \|h\| \left(\int_{B_R \setminus B_r} |u|^{p \frac{q-1}{\tilde{q}-1} - p} |u|^{p} \, dx \right)^{\frac{\tilde{q}-1}{p}} \\
&\leq S_{N,p} \|K(|\cdot|)\|_{L^s(B_R \setminus B_r)} \|h\| \left(\left(\frac{C_{N,p} \|u\|}{r^{\frac{N-p}{p}}} \right)^{p \frac{q-\tilde{q}}{\tilde{q}-1}} \int_{B_R \setminus B_r} |u|^{p} \, dx \right)^{\frac{\tilde{q}-1}{p}} \\
&= S_{N,p} \|K(|\cdot|)\|_{L^s(B_R \setminus B_r)} \|h\| \left(\frac{C_{N,p} \|u\|}{r^{\frac{N-p}{p}}} \right)^{q-\tilde{q}} \left(\int_{B_R \setminus B_r} |u|^{p} \, dx \right)^{\frac{\tilde{q}-1}{p}}.
\end{aligned}$$

This concludes the proof. □

For future reference, we recall here some results from [5] concerning the sum space

$$L_K^{p_1} + L_K^{p_2} := L_K^{p_1}\left(\mathbb{R}^N\right) + L_K^{p_2}\left(\mathbb{R}^N\right) := \left\{ u_1 + u_2 : u_1 \in L_K^{p_1}\left(\mathbb{R}^N\right),\, u_2 \in L_K^{p_2}\left(\mathbb{R}^N\right) \right\},$$

where we assume $1 < p_1 \leq p_2 < \infty$. Such a space can be characterized as the set of the measurable mappings $u : \mathbb{R}^N \to \mathbb{R}$ for which there exists a measurable set $E \subseteq \mathbb{R}^N$ such that $u \in L_K^{p_1}(E) \cap L_K^{p_2}(E^c)$ (of course $L_K^{p_1}(E) := L^{p_1}(E, K(|x|)\,dx)$, and so for $L_K^{p_2}(E^c)$). It is a Banach space with respect to the norm

$$\|u\|_{L_K^{p_1}+L_K^{p_2}} := \inf_{u_1+u_2=u} \max\left\{\|u_1\|_{L_K^{p_1}}, \|u_2\|_{L_K^{p_2}}\right\}$$

and the continuous embedding $L_K^p \hookrightarrow L_K^{p_1} + L_K^{p_2}$ holds for all $p \in [p_1, p_2]$.

Proposition 3.2 ([5, Proposition 2.7])**.** *Let* $\{u_n\} \subseteq L_K^{p_1} + L_K^{p_2}$ *be a sequence such that* $\forall \varepsilon > 0$ *there exist* $n_\varepsilon > 0$ *and a sequence of measurable sets* $E_{\varepsilon,n} \subseteq \mathbb{R}^N$ *satisfying*

$$\forall n > n_\varepsilon, \quad \int_{E_{\varepsilon,n}} K(|x|)\,|u_n|^{p_1}\,dx + \int_{E_{\varepsilon,n}^c} K(|x|)\,|u_n|^{p_2}\,dx < \varepsilon. \tag{3.4}$$

Then $u_n \to 0$ *in* $L_K^{p_1} + L_K^{p_2}$.

Proposition 3.3 ([5, Propositions 2.17 and 2.14, Corollary 2.19])**.** *Let* $E \subseteq \mathbb{R}^N$ *be a measurable set.*

(i) If $\int_E K(|x|)\,dx < \infty$, *then* $L_K^{p_1} + L_K^{p_2}$ *is continuously embedded into* $L_K^{p_1}(E)$.

(ii) Every $u \in (L_K^{p_1} + L_K^{p_2}) \cap L^\infty(E)$ *satisfies*

$$\|u\|_{L_K^{p_2}(E)}^{p_2/p_1} \leq \left(\|u\|_{L^\infty(E)}^{p_2/p_1-1} + \|u\|_{L_K^{p_2}(E)}^{p_2/p_1-1}\right)\|u\|_{L_K^{p_1}+L_K^{p_2}}. \tag{3.5}$$

If moreover $\|u\|_{L^\infty(E)} \leq 1$, *then*

$$\|u\|_{L_K^{p_2}(E)} \leq 2\,\|u\|_{L_K^{p_1}+L_K^{p_2}} + 1. \tag{3.6}$$

Recall the definitions (2.2)-(2.3) of the functions $\mathcal{S}_0$ and $\mathcal{S}_\infty$.

Proof of Theorem 2.1. We prove each part of the theorem separately.

(i) By the monotonicity of $\mathcal{S}_0$ and $\mathcal{S}_\infty$, it is not restrictive to assume $R_1 < R_2$ in hypothesis $\left(\mathcal{S}'_{q_1,q_2}\right)$. In order to prove the continuous embedding, let $u \in W_r$, $u \neq 0$. Then we have

$$\int_{B_{R_1}} K(|x|)\,|u|^{q_1}\,dx = \|u\|^{q_1} \int_{B_{R_1}} K(|x|)\,\frac{|u|^{q_1}}{\|u\|^{q_1}}dx \leq \|u\|^{q_1}\,\mathcal{S}_0(q_1, R_1) \tag{3.7}$$

and, similarly,

$$\int_{B_{R_2}^c} K(|x|)\,|u|^{q_2}\,dx \leq \|u\|^{q_2}\,\mathcal{S}_\infty(q_2, R_2). \tag{3.8}$$

We now use (3.3) of Lemma 3.1 and the continuous embedding

$$W_r = D_{\mathrm{rad}}^{1,p}(\mathbb{R}^N) \cap L^p(\mathbb{R}^N, V(|x|)dx) \hookrightarrow L_{\mathrm{loc}}^p(\mathbb{R}^N)$$

to deduce that there exists a constant $C_1 > 0$, independent from u, such that

$$\int_{B_{R_2}\setminus B_{R_1}} K(|x|)\,|u|^{q_1}\,dx \leq C_1 \|u\|^{q_1}. \tag{3.9}$$

Hence $u \in L_K^{q_1}(B_{R_2}) \cap L_K^{q_2}(B_{R_2}^c)$ and thus $u \in L_K^{q_1} + L_K^{q_2}$. Moreover, if $u_n \to 0$ in W_r, then, using (3.7), (3.8) and (3.9), we get

$$\int_{B_{R_2}} K(|x|)\,|u_n|^{q_1}\,dx + \int_{B_{R_2}^c} K(|x|)\,|u_n|^{q_2}\,dx = o(1)_{n\to\infty},$$

which means $u_n \to 0$ in $L_K^{q_1} + L_K^{q_2}$ by Proposition 3.2.

(ii) Assume hypothesis $\left(\mathcal{S}''_{q_1,q_2}\right)$. Let $\varepsilon > 0$ and let $u_n \rightharpoonup 0$ in W_r. Then $\{\|u_n\|\}$ is bounded and, arguing as for (3.7) and (3.8), we can take $r_\varepsilon > 0$ and $R_\varepsilon > r_\varepsilon$ such that for all n one has

$$\int_{B_{r_\varepsilon}} K(|x|)\,|u_n|^{q_1}\,dx \leq \|u_n\|^{q_1}\,\mathcal{S}_0(q_1, r_\varepsilon) \leq \sup_n \|u_n\|^{q_1}\,\mathcal{S}_0(q_1, r_\varepsilon) < \frac{\varepsilon}{3}$$

and

$$\int_{B_{R_\varepsilon}^c} K(|x|)\,|u_n|^{q_2}\,dx \leq \sup_n \|u_n\|^{q_2}\,\mathcal{S}_\infty(q_2, R_\varepsilon) < \frac{\varepsilon}{3}.$$

Using (3.3) of Lemma 3.1 and the boundedness of $\{\|u_n\|\}$ again, we infer that there exist two constants $C_2, l > 0$, independent from n, such that

$$\int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} K(|x|)\,|u_n|^{q_1}\,dx \leq C_2 \left(\int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} |u_n|^p\,dx\right)^l,$$

where

$$\int_{B_{R_\varepsilon}\setminus B_{r_\varepsilon}} |u_n|^p\,dx \to 0 \quad \text{as } n\to\infty \quad (\varepsilon \text{ fixed})$$

thanks to the compactness of the embedding $D^{1,p}_{\mathrm{rad}}(\mathbb{R}^N) \hookrightarrow L^p_{\mathrm{loc}}(\mathbb{R}^N)$. Therefore we obtain

$$\int_{B_{R_\varepsilon}} K(|x|)\,|u_n|^{q_1}\,dx + \int_{B_{R_\varepsilon}^c} K(|x|)\,|u_n|^{q_2}\,dx < \varepsilon$$

for all n sufficiently large, which means $u_n \to 0$ in $L_K^{q_1} + L_K^{q_2}$ (Proposition 3.2). This concludes the proof of part (ii).

(iii) First we observe that $K(|\cdot|) \in L^1(B_1)$ and assumption $(\mathbf{K})$ imply $K(|\cdot|) \in L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Assume $W_r \hookrightarrow L_K^{q_1} + L_K^{q_2}$ with $q_1 \leq q_2$. Fix $R_1 > 0$. Then, by (i) of Proposition 3.3, there exist two constants $c_1, c_2 > 0$ such that $\forall u \in W_r$ we have

$$\int_{B_{R_1}} K(|x|)\,|u|^{q_1}\,dx \leq c_1 \|u\|^{q_1}_{L_K^{q_1}+L_K^{q_2}} \leq c_2 \|u\|^{q_1},$$

which implies $\mathcal{S}_0(q_1, R_1) \leq c_2$. By (3.2), fix $R_2 > 0$ such that every $u \in W_r$ with $\|u\| = 1$ satisfies $|u(x)| \leq 1$ almost everywhere on $B_{R_2}^c$. Then, by (3.6), we have

$$\int_{B_{R_2}^c} K(|x|)\,|u|^{q_2}\,dx \leq \left(2\|u\|_{L_K^{q_1}+L_K^{q_2}} + 1\right)^{q_2} \leq (c_3\|u\| + 1)^{q_2} = (c_3 + 1)^{q_2}$$

for some constant $c_3 > 0$. This gives $\mathcal{S}_\infty(q_2, R_2) < \infty$ and thus $\left(\mathcal{S}'_{q_1,q_2}\right)$ holds (with $R_1 > 0$ arbitrary and R_2 large enough). Now assume that the embedding $H^1_{V,\mathrm{r}} \hookrightarrow L^{q_1}_K + L^{q_2}_K$ is compact and, by contradiction, that $\lim_{R\to 0^+} \mathcal{S}_0(q_1, R) > \varepsilon_1 > 0$ (the limit exists by monotonicity). Then for every $n \in \mathbb{N} \setminus \{0\}$ we have $\mathcal{S}_0(q_1, 1/n) > \varepsilon_1$ and thus there exists $u_n \in W_r$ such that $\|u_n\| = 1$ and

$$\int_{B_{1/n}} K(|x|)\,|u_n|^{q_1}\,dx > \varepsilon_1.$$

Since $\{u_n\}$ is bounded in W_r, by the compactness assumption together with the continuous embedding $L^{q_1}_K + L^{q_2}_K \hookrightarrow L^{q_1}_K(B_1)$ ((i) of Proposition 3.3), we get that there exists $u \in W_r$ such that, up to a subsequence, $u_n \to u$ in $L^{q_1}_K(B_1)$. This implies

$$\int_{B_{1/n}} K(|x|)\,|u_n|^{q_1}\,dx \to 0 \quad \text{as } n \to \infty,$$

which is a contradiction. Similarly, if $\lim_{R\to +\infty} \mathcal{S}_\infty(q_2, R) > \varepsilon_2 > 0$, then there exists a sequence $\{u_n\} \subset W_r$ such that $\|u_n\| = 1$ and

$$\int_{B_n^c} K(|x|)\,|u_n|^{q_2}\,dx > \varepsilon_2. \tag{3.10}$$

Moreover, we can assume that $\exists u \in W_r$ such that $u_n \rightharpoonup u$ in W_r, $u_n \to u$ in $L^{q_1}_K + L^{q_2}_K$ and

$$\|u_n - u\| \le \|u_n\| + \|u\| \le 1 + \liminf_{n\to\infty} \|u_n\| = 2. \tag{3.11}$$

Now, by (3.11) and (3.2), fix $R_2 > 0$ such that $|u_n(x) - u(x)| \le 1$ almost everywhere on $B^c_{R_2}$. Then $\{u_n - u\}$ is bounded in $L^{q_2}_K(B^c_{R_2})$ by (3.6) and therefore (3.5) gives

$$\int_{B^c_{R_2}} K(|x|)\,|u_n - u|^{q_2}\,dx \le c_4 \left(\|u_n - u\|_{L^{q_1}_K + L^{q_2}_K}\right)^{q_1} \to 0 \quad \text{as } n \to \infty$$

for some constant $c_4 > 0$. Since $u \in L^{q_2}_K(B^c_{R_2})$ by (3.2) and (3.5), this implies

$$\int_{B_n^c} K(|x|)\,|u_n|^{q_2}\,dx \to 0 \quad \text{as } n \to \infty,$$

which contradicts (3.10). Hence we conclude $\lim_{R\to 0^+} \mathcal{S}_0(q_1, R) = \lim_{R\to +\infty} \mathcal{S}_\infty(q_2, R) = 0$, which completes the proof. □

4 Proof of Theorems 2.2 - 2.7

Assume $1 < p < N$ and let V and K be as in $(\mathbf{V})$ and $(\mathbf{K})$. As in the previous section, we fix a constant $S_{N,p} > 0$ such that (3.1) holds.

Lemma 4.1. *Let $\Omega \subseteq \mathbb{R}^N$ be a nonempty measurable set and assume that $V(r) < +\infty$ almost everywhere in Ω and*

$$\Lambda := \operatorname*{ess\,sup}_{x\in\Omega} \frac{K(|x|)}{|x|^\alpha V(|x|)^\beta} < +\infty \quad \textit{for some } 0 \le \beta \le 1 \textit{ and } \alpha \in \mathbb{R}.$$

Let $u \in W$ and assume that there exist $\nu \in \mathbb{R}$ and $m > 0$ such that

$$|u(x)| \leq \frac{m}{|x|^{\nu}} \quad \text{almost everywhere on } \Omega.$$

Then $\forall h \in W$ and $\forall q > \max\{1, p\beta\}$, one has

$$\int_{\Omega} K(|x|)\,|u|^{q-1}\,|h|\,dx$$

$$\leq \begin{cases} \Lambda m^{q-1} S_{N,p}^{1-p\beta} \left(\int_{\Omega} |x|^{\frac{\alpha-\nu(q-1)}{N(p-1)+p(1-p\beta)}pN}\,dx \right)^{\frac{N(p-1)+p(1-p\beta)}{pN}} \|h\| & \text{if } 0 \leq \beta \leq \frac{1}{p} \\ \Lambda m^{q-p\beta} \left(\int_{\Omega} |x|^{\frac{\alpha-\nu(q-p\beta)}{1-\beta}}\,dx \right)^{1-\beta} \|u\|^{p\beta-1}\,\|h\| & \text{if } \frac{1}{p} < \beta < 1 \\ \Lambda m^{q-p} \left(\int_{\Omega} |x|^{\frac{p}{p-1}(\alpha-\nu(q-p))}\,V(|x|)\,|u|^{p}\,dx \right)^{\frac{p-1}{p}} \|h\| & \text{if } \beta = 1. \end{cases}$$

Proof. We distinguish several cases, where we will use Hölder inequality many times, without explicitly noting it.

Case $\beta = 0$.

We have

$$\begin{aligned} \frac{1}{\Lambda}\int_{\Omega} K(|x|)\,|u|^{q-1}\,|h|\,dx &\leq \int_{\Omega} |x|^{\alpha}\,|u|^{q-1}\,|h|\,dx \\ &\leq \left(\int_{\Omega} \left(|x|^{\alpha}\,|u|^{q-1} \right)^{\frac{pN}{N(p-1)+p}} dx \right)^{\frac{N(p-1)+p}{pN}} \left(\int_{\Omega} |h|^{p^*}\,dx \right)^{\frac{1}{p^*}} \\ &\leq m^{q-1} S_{N,p} \left(\int_{\Omega} |x|^{\frac{\alpha-\nu(q-1)}{N(p-1)+p}pN}\,dx \right)^{\frac{N(p-1)+p}{pN}} \|h\|. \end{aligned}$$

Case $0 < \beta < 1/p$.

One has $\frac{1}{\beta} > 1$ and $\frac{1-\beta}{1-p\beta}p^* > 1$, with Hölder conjugate exponents $\left(\frac{1}{\beta}\right)' = \frac{1}{1-\beta}$ and $\left(\frac{1-\beta}{1-p\beta}p^*\right)' = \frac{pN(1-\beta)}{N(p-1)+p(1-p\beta)}$. Then we get

$$\begin{aligned} &\frac{1}{\Lambda}\int_{\Omega} K(|x|)\,|u|^{q-1}\,|h|\,dx \\ &\leq \int_{\Omega} |x|^{\alpha}\,V(|x|)^{\beta}\,|u|^{q-1}\,|h|\,dx = \int_{\Omega} |x|^{\alpha}\,|u|^{q-1}\,|h|^{1-p\beta}\,V(|x|)^{\beta}\,|h|^{p\beta}\,dx \\ &\leq \left(\int_{\Omega} \left(|x|^{\alpha}\,|u|^{q-1}\,|h|^{1-p\beta} \right)^{\frac{1}{1-\beta}} dx \right)^{1-\beta} \left(\int_{\Omega} V(|x|)\,|h|^{p}\,dx \right)^{\beta} \\ &\leq \left(\left(\int_{\Omega} \left(|x|^{\frac{\alpha}{1-\beta}}\,|u|^{\frac{q-1}{1-\beta}} \right)^{\left(\frac{1-\beta}{1-p\beta}p^*\right)'} dx \right)^{\frac{1}{\left(\frac{1-\beta}{1-p\beta}p^*\right)'}} \left(\int_{\Omega} |h|^{p^*}\,dx \right)^{\frac{1-p\beta}{(1-\beta)p^*}} \right)^{1-\beta} \|h\|^{p\beta} \\ &\leq m^{q-1} \left(\left(\int_{\Omega} \left(|x|^{\frac{\alpha}{1-\beta}-\nu\frac{q-1}{1-\beta}} \right)^{\left(\frac{1-\beta}{1-p\beta}p^*\right)'} dx \right)^{\frac{1}{\left(\frac{1-\beta}{1-\beta}p^*\right)'}} S_{N,p}^{\frac{1-p\beta}{1-\beta}}\,\|h\|^{\frac{1-p\beta}{1-\beta}} \right)^{1-\beta} \|h\|^{p\beta} \end{aligned}$$

$$= m^{q-1}\left(\int_\Omega |x|^{\frac{\alpha-\nu(q-1)}{N(p-1)+p(1-p\beta)}pN}\,dx\right)^{\frac{N(p-1)+p(1-p\beta)}{pN}} S_{N,p}^{1-p\beta}\left\|h\right\|.$$

Case $\beta = \frac{1}{p}$.

We have

$$\begin{aligned}\frac{1}{\Lambda}\int_\Omega K\left(|x|\right)|u|^{q-1}|h|\,dx &\le \int_\Omega |x|^\alpha |u|^{q-1} V\left(|x|\right)^{\frac{1}{p}}|h|\,dx \\ &\le \left(\int_\Omega |x|^{\alpha\frac{p}{p-1}}|u|^{(q-1)\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}}\left(\int_\Omega V\left(|x|\right)|h|^p\,dx\right)^{\frac{1}{p}} \\ &\le m^{q-1}\left(\int_\Omega |x|^{(\alpha-\nu(q-1))\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}}\left\|h\right\|.\end{aligned}$$

Case $1/p < \beta < 1$.

One has $\frac{p-1}{p\beta-1} > 1$, with Hölder conjugate exponent $\left(\frac{p-1}{p\beta-1}\right)' = \frac{p-1}{p(1-\beta)}$. Then

$$\begin{aligned}&\frac{1}{\Lambda}\int_\Omega K\left(|x|\right)|u|^{q-1}|h|\,dx \\ &\le \int_\Omega |x|^\alpha V\left(|x|\right)^\beta |u|^{q-1}|h|\,dx = \int_\Omega |x|^\alpha V\left(|x|\right)^{\frac{p\beta-1}{p}}|u|^{q-1}V\left(|x|\right)^{\frac{1}{p}}|h|\,dx \\ &\le \left(\int_\Omega |x|^{\alpha\frac{p}{p-1}} V\left(|x|\right)^{\frac{p\beta-1}{p-1}}|u|^{(q-1)\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}}\left(\int_\Omega V\left(|x|\right)|h|^p\,dx\right)^{\frac{1}{p}} \\ &\le \left(\int_\Omega |x|^{\alpha\frac{p}{p-1}}|u|^{(q-1)\frac{p}{p-1}-p\frac{p\beta-1}{p-1}} V\left(|x|\right)^{\frac{p\beta-1}{p-1}}|u|^{p\frac{p\beta-1}{p-1}}\,dx\right)^{\frac{p-1}{p}}\left\|h\right\| \\ &\le \left(\left(\int_\Omega |x|^{\frac{\alpha}{1-\beta}}|u|^{\frac{q-p\beta}{1-\beta}}\,dx\right)^{\frac{p}{p-1}(1-\beta)}\left(\int_\Omega V\left(|x|\right)|u|^p\,dx\right)^{\frac{p\beta-1}{p-1}}\right)^{\frac{p-1}{p}}\left\|h\right\| \\ &\le m^{q-p\beta}\left(\int_\Omega |x|^{\frac{\alpha}{1-\beta}-\nu\frac{q-p\beta}{1-\beta}}\,dx\right)^{1-\beta}\left(\int_\Omega V\left(|x|\right)|u|^p\,dx\right)^{\frac{p\beta-1}{p}}\left\|h\right\| \\ &\le m^{q-p\beta}\left(\int_\Omega |x|^{\frac{\alpha-\nu(q-p\beta)}{1-\beta}}\,dx\right)^{1-\beta}\left\|u\right\|^{p\beta-1}\left\|h\right\|.\end{aligned}$$

Case $\beta = 1$.

Assumption $q > \max\left\{1, p\beta\right\}$ means $q > p$ and thus we have

$$\begin{aligned}&\frac{1}{\Lambda}\int_\Omega K\left(|x|\right)|u|^{q-1}|h|\,dx \\ &\le \int_\Omega |x|^\alpha V\left(|x|\right)|u|^{q-1}|h|\,dx = \int_\Omega |x|^\alpha V\left(|x|\right)^{\frac{p-1}{p}}|u|^{q-1}V\left(|x|\right)^{\frac{1}{p}}|h|\,dx \\ &\le \left(\int_\Omega |x|^{\alpha\frac{p}{p-1}} V\left(|x|\right)|u|^{(q-1)\frac{p}{p-1}}\,dx\right)^{\frac{p-1}{p}}\left(\int_\Omega V\left(|x|\right)|h|^p\,dx\right)^{\frac{1}{p}}\end{aligned}$$

$$\leq \left(\int_{\Omega} |x|^{\alpha \frac{p}{p-1}} |u|^{(q-1)\frac{p}{p-1}-p} V(|x|) |u|^p \, dx\right)^{\frac{p-1}{p}} \|h\|$$

$$\leq m^{q-p} \left(\int_{\Omega} |x|^{\frac{p}{p-1}(\alpha-\nu(q-p))} V(|x|) |u|^p \, dx\right)^{\frac{p-1}{p}} \|h\| .$$

□

As in the previous section, we fix a constant $C_{N,p} > 0$ such that (3.2) holds. Recall the definitions (2.4)-(2.5) of the functions $\mathcal{R}_0$ and $\mathcal{R}_\infty$.

Proof of Theorem 2.2. Assume the hypotheses of the theorem and let $u \in W_r$ and $h \in W$ be such that $\|u\| = \|h\| = 1$. Let $0 < R \leq R_1$. We will denote by C any positive constant which does not depend on u, h and R.

By (3.2) and the fact that

$$\operatorname*{ess\,sup}_{x \in B_R} \frac{K(|x|)}{|x|^{\alpha_0} V(|x|)^{\beta_0}} \leq \operatorname*{ess\,sup}_{r \in (0,R_1)} \frac{K(r)}{r^{\alpha_0} V(r)^{\beta_0}} < +\infty,$$

we can apply Lemma 4.1 with $\Omega = B_R$, $\alpha = \alpha_0$, $\beta = \beta_0$, $m = C_{N,p} \|u\| = C_{N,p}$ and $\nu = \frac{N-p}{p}$. If $0 \leq \beta_0 \leq 1/p$ we get

$$\begin{aligned}
\int_{B_R} K(|x|) |u|^{q_1-1} |h| \, dx &\leq C \left(\int_{B_R} |x|^{\frac{\alpha_0 - \frac{N-p}{p}(q_1-1)}{N(p-1)+p(1-p\beta_0)} pN} dx\right)^{\frac{N(p-1)+p(1-p\beta_0)}{pN}} \\
&\leq C \left(\int_0^R r^{\frac{p\alpha_0-(N-p)(q_1-1)}{N(p-1)+p(1-p\beta_0)} N + N-1} dr\right)^{\frac{N(p-1)+p(1-p\beta_0)}{pN}} \\
&= C \left(R^{\frac{p\alpha_0 - p^2\beta_0 + pN - (N-p)q_1}{N(p-1)+p(1-p\beta_0)} N}\right)^{\frac{N(p-1)+p(1-p\beta_0)}{pN}},
\end{aligned}$$

since

$$p\alpha_0 - p^2\beta_0 + pN - (N-p) q_1 = (N-p) \left(q^*(\alpha_0, \beta_0) - q_1\right) > 0.$$

On the other hand, if $1/p < \beta_0 < 1$ we have

$$\begin{aligned}
\int_{B_R} K(|x|) |u|^{q_1-1} |h| \, dx &\leq C \left(\int_{B_R} |x|^{\frac{\alpha_0 - \frac{N-p}{p}(q_1-p\beta_0)}{1-\beta_0}} dx\right)^{1-\beta_0} \\
&\leq C \left(\int_0^R r^{\frac{\alpha_0 - \frac{N-p}{p}(q_1-p\beta_0)}{1-\beta_0} + N-1} dr\right)^{1-\beta_0} \\
&= C \left(R^{\frac{p\alpha_0-(N-p)(q_1-p\beta_0)}{p(1-\beta_0)} + N}\right)^{1-\beta_0},
\end{aligned}$$

since

$$\frac{p\alpha_0 - (N-p)(q_1 - p\beta_0)}{p(1-\beta_0)} + N = \frac{N-p}{p(1-\beta_0)} \left(q^*(\alpha_0, \beta_0) - q_1\right) > 0.$$

Finally, if $\beta_0 = 1$, we obtain

$$\begin{aligned}\int_{B_R} K(|x|)\,|u|^{q_1-1}\,|h|\,dx &\leq C\left(\int_{B_R} |x|^{\frac{1}{p-1}(p\alpha_0-(N-p)(q_1-p))}\,V(|x|)\,|u|^p\,dx\right)^{\frac{p-1}{p}}\\ &\leq C\left(R^{\frac{1}{p-1}(p\alpha_0-(N-p)(q_1-p))}\int_{B_R} V(|x|)\,|u|^p\,dx\right)^{\frac{p-1}{p}}\\ &\leq CR^{\frac{p\alpha_0-(N-p)(q_1-p)}{p}},\end{aligned}$$

since

$$p\alpha_0 - (N-p)(q_1-p) = (N-p)(q^*(\alpha_0,1) - q_1) > 0.$$

So, in any case, we deduce $\mathcal{R}_0(q_1,R) \leq CR^\delta$ for some $\delta = \delta(N,p,\alpha_0,\beta_0,q_1) > 0$ and this concludes the proof. □

Proof of Theorem 2.3. Assume the hypotheses of the theorem and let $u \in W_r$ and $h \in W$ be such that $\|u\| = \|h\| = 1$. Let $R \geq R_2$. We will denote by C any positive constant which does not depend on u, h and R.

By (3.2) and the fact that

$$\operatorname*{ess\,sup}_{x\in B_R^c} \frac{K(|x|)}{|x|^{\alpha_\infty}\,V(|x|)^{\beta_\infty}} \leq \operatorname*{ess\,sup}_{r>R_2} \frac{K(r)}{r^{\alpha_\infty}V(r)^{\beta_\infty}} < +\infty,$$

we can apply Lemma 4.1 with $\Omega = B_R^c$, $\alpha = \alpha_\infty$, $\beta = \beta_\infty$, $m = C_{N,p}\|u\| = C_{N,p}$ and $\nu = \frac{N-p}{p}$. If $0 \leq \beta_\infty \leq 1/p$ we get

$$\begin{aligned}\int_{B_R^c} K(|x|)\,|u|^{q_2-1}\,|h|\,dx &\leq C\left(\int_{B_R^c} |x|^{\frac{\alpha_\infty-\frac{N-p}{p}(q_2-1)}{N(p-1)+p(1-p\beta_\infty)}pN}\,dx\right)^{\frac{N(p-1)+p(1-p\beta_\infty)}{pN}}\\ &\leq C\left(\int_R^{+\infty} r^{\frac{p\alpha_\infty-(N-p)(q_2-1)}{N(p-1)+p(1-p\beta_\infty)}N+N-1}dr\right)^{\frac{N(p-1)+p(1-p\beta_\infty)}{pN}}\\ &= C\left(R^{\frac{p\alpha_\infty-p^2\beta_\infty+pN-(N-p)q_2}{N(p-1)+p(1-p\beta_\infty)}N}\right)^{\frac{N(p-1)+p(1-p\beta_\infty)}{pN}},\end{aligned}$$

since

$$p\alpha_\infty - p^2\beta_\infty + pN - (N-p)\,q_2 = (N-p)(q^*(\alpha_\infty,\beta_\infty) - q_2) < 0.$$

On the other hand, if $1/p < \beta_\infty < 1$ we have

$$\begin{aligned}\int_{B_R^c} K(|x|)\,|u|^{q_2-1}\,|h|\,dx &\leq C\left(\int_{B_R^c} |x|^{\frac{\alpha_\infty-\frac{N-p}{p}(q_2-p\beta_\infty)}{1-\beta_\infty}}\,dx\right)^{1-\beta_\infty}\\ &\leq C\left(\int_R^{+\infty} r^{\frac{\alpha_\infty-\frac{N-p}{p}(q_2-p\beta_\infty)}{1-\beta_\infty}+N-1}dr\right)^{1-\beta_\infty}\\ &= C\left(R^{\frac{p\alpha_\infty-(N-p)(q_2-p\beta_\infty)}{p(1-\beta_\infty)}+N}\right)^{1-\beta_\infty},\end{aligned}$$

since

$$\frac{p\alpha_\infty - (N-p)(q_2 - p\beta_\infty)}{p(1-\beta_\infty)} + N = \frac{N-p}{p(1-\beta_\infty)} \left(q^*(\alpha_\infty, \beta_\infty) - q_2\right) < 0.$$

Finally, if $\beta_\infty = 1$, we obtain

$$\begin{aligned} \int_{B_R^c} K(|x|)\, |u|^{q_2-1} |h|\, dx &\leq C \left(\int_{B_R^c} |x|^{\frac{1}{p-1}((p\alpha_\infty-(N-p)(q_2-p))} V(|x|)\, |u|^p\, dx \right)^{\frac{p-11}{p}} \\ &\leq C \left(R^{\frac{1}{p-1}(p\alpha_\infty-(N-p)(q_2-p))} \int_{B_R^c} V(|x|)\, |u|^p\, dx \right)^{\frac{p-1}{p}} \\ &\leq C R^{\frac{p\alpha_\infty-(N-p)(q_2-p)}{p}}, \end{aligned}$$

since

$$p\alpha_\infty - (N-p)(q_2-p) = (N-p)\left(q^*(\alpha_\infty, 1) - q_2\right) < 0.$$

So, in any case, we get $\mathcal{R}_\infty(q_2, R) \leq CR^\delta$ for some $\delta = \delta(N, p, \alpha_\infty, \beta_\infty, q_2) < 0$, which completes the proof. □

In proving Theorem 2.5, we will need the following lemma.

Lemma 4.2. *Assume that there exists $R_2 > 0$ such that $V(r) < +\infty$ for almost every $r > R_2$ and*

$$\lambda_\infty := \operatorname*{ess\,inf}_{r>R_2} r^{\gamma_\infty} V(r) > 0 \quad \textit{for some } \gamma_\infty \leq p.$$

Then there exists a constant $c_\infty > 0$, only dependent on N and p, such that

$$\forall u \in W_r, \quad |u(x)| \leq c_\infty \lambda_\infty^{-\frac{p-1}{p^2}} \|u\|\, |x|^{-\frac{p(N-1)-\gamma_\infty(p-1)}{p^2}} \quad \textit{almost everywhere in } B_{R_2}^c. \tag{4.1}$$

Proof. The lemma is essentially proved in [10, Lemma 4], but without making explicit the dependence of the constant on λ_∞ and with slightly different assumptions on V: a global decay condition and the continuity on $(0, +\infty)$, which allows a density argument that is not so obvious in our hypotheses. In the form given here, instead, the lemma follows by adapting the proof of [3, Proposition 28], where the result is proved for $p = 2$. □

Proof of Theorem 2.5. Assume the hypotheses of the theorem and denote

$$\Lambda_\infty := \operatorname*{ess\,sup}_{r>R_2} \frac{K(r)}{r^{\alpha_\infty} V(r)^{\beta_\infty}} \quad \text{and} \quad \lambda_\infty := \operatorname*{ess\,inf}_{r>R_2} r^{\gamma_\infty} V(r).$$

Let $u \in W_r$ and $h \in W$ be such that $\|u\| = \|h\| = 1$. Let $R \geq R_2$ and observe that $\forall \xi \geq 0$ one has

$$\operatorname*{ess\,sup}_{r>R} \frac{K(r)}{r^{\alpha_\infty+\xi\gamma_\infty} V(r)^{\beta_\infty+\xi}} \leq \operatorname*{ess\,sup}_{r>R_2} \frac{K(r)}{r^{\alpha_\infty} V(r)^{\beta_\infty} \left(r^{\gamma_\infty} V(r)\right)^\xi} \leq \frac{\Lambda_\infty}{\lambda_\infty^\xi} < +\infty. \tag{4.2}$$

We will denote by C any positive constant which does not depend on u, h or R (such as $\Lambda_\infty / \lambda_\infty^\xi$ if ξ does not depend on u, h or R).

Denoting $\alpha_1 = \alpha_1(\beta_\infty, \gamma_\infty)$, $\alpha_2 = \alpha_2(\beta_\infty)$ and $\alpha_3 = \alpha_3(\beta_\infty, \gamma_\infty)$, as defined in (2.14), we will distinguish several cases, according to the description (2.15). In each of such cases, we will choose a suitable $\xi \geq 0$ and, thanks to (4.2) and (4.1), we will apply Lemma 4.1 with $\Omega = B_R^c$, $\alpha = \alpha_\infty + \xi\gamma_\infty$, $\beta = \beta_\infty + \xi$ (whence Λ will be given by the left hand side of (4.2)), $m = c_\infty \lambda_\infty^{-\frac{p-1}{p^2}} \|u\| = c_\infty \lambda_\infty^{-\frac{p-1}{p^2}}$ and $\nu = \frac{p(N-1)-\gamma_\infty(p-1)}{p^2}$. Recall that we are assuming $\gamma_\infty \leq p < N$, so that $\nu > 0$. We will obtain that

$$\int_{B_R^c} K(|x|)\, |u|^{q_2-1} |h|\, dx \leq C R^\delta$$

for some $\delta < 0$, not dependent on R, so that the result follows.

Case $\alpha_\infty \geq \alpha_1$.

We take $\xi = 1 - \beta_\infty$ and apply Lemma 4.1 with $\beta = \beta_\infty + \xi = 1$ and $\alpha = \alpha_\infty + \xi\gamma_\infty = \alpha_\infty + (1-\beta_\infty)\gamma_\infty$. We get

$$\begin{aligned}\int_{B_R^c} K(|x|)\, |u|^{q_2-1} |h|\, dx &\leq C \left(\int_{B_R^c} |x|^{\frac{p}{p-1}(\alpha-\nu(q_2-p))} V(|x|)\, |u|^p\, dx \right)^{\frac{p-1}{p}} \\ &\leq C \left(R^{\frac{p}{p-1}(\alpha-\nu(q_2-p))} \int_{B_R^c} V(|x|)\, |u|^p\, dx \right)^{\frac{p-1}{p}} \leq C R^{\alpha-\nu(q_2-p)},\end{aligned}$$

since

$$\begin{aligned}\alpha - \nu(q_2-p) &= \alpha_\infty + (1-\beta_\infty)\gamma_\infty - \frac{p(N-1)-\gamma_\infty(p-1)}{p^2}(q_2-p) \\ &= \frac{p(N-1)-\gamma_\infty(p-1)}{p^2}(q_{**}-q_2) < 0.\end{aligned}$$

Case $\max\{\alpha_2, \alpha_3\} < \alpha_\infty < \alpha_1$.

Take $\xi = \frac{\alpha_\infty + (1-\beta_\infty)N}{N-\gamma_\infty} > 0$ and apply Lemma 4.1 with $\beta = \beta_\infty + \xi$ and $\alpha = \alpha_\infty + \xi\gamma_\infty$. For doing this, observe that $\alpha_3 < \alpha_\infty < \alpha_1$ implies

$$\beta = \beta_\infty + \xi = \frac{\alpha_\infty - \gamma_\infty\beta_\infty + N}{N-\gamma_\infty} \in \left(\frac{1}{p}, 1\right).$$

We get

$$\int_{B_R^c} K(|x|)\, |u|^{q_2-1} |h|\, dx \leq C \left(\int_{B_R^c} |x|^{\frac{\alpha-\nu(q_2-p\beta)}{1-\beta}}\, dx \right)^{1-\beta} \leq C \left(R^{\frac{\alpha-\nu(q_2-p\beta)}{1-\beta}+N} \right)^{1-\beta},$$

since

$$\frac{\alpha-\nu(q_2-p\beta)}{1-\beta} + N = \frac{\nu}{1-\beta}\left(p\frac{\alpha_\infty - \beta_\infty\gamma_\infty + N}{N-\gamma_\infty} - q_2 \right) = \frac{\nu}{1-\beta}(q_* - q_2) < 0.$$

Case $\beta_\infty = 1$ *and* $\alpha_\infty \leq 0 = \alpha_2$ $(= \max\{\alpha_2, \alpha_3\})$.

Take $\xi = 0$ and apply Lemma 4.1 with $\beta = \beta_\infty + \xi = 1$ and $\alpha = \alpha_\infty + \xi\gamma_\infty = \alpha_\infty$. We get

$$\int_{B_R^c} K(|x|)\,|u|^{q_2-1}\,|h|\,dx \le C\left(\int_{B_R^c} |x|^{\frac{p}{p-1}(\alpha_\infty-\nu(q_2-p))}\,V(|x|)\,|u|^p\,dx\right)^{\frac{p-1}{p}} \le CR^{\alpha_\infty-\nu(q_2-p)},$$

since $\alpha_\infty - \nu(q_2-p) \le -\nu(q_2-p) < 0$.

Case $\frac{1}{p} < \beta_\infty < 1$ *and* $\alpha_\infty \le \alpha_2$ $(= \max\{\alpha_2, \alpha_3\})$.

Take $\xi = 0$ again and apply Lemma 4.1 with $\beta = \beta_\infty \in \left(\frac{1}{p}, 1\right)$ and $\alpha = \alpha_\infty$. We get

$$\begin{aligned}\int_{B_R^c} K(|x|)\,|u|^{q_2-1}\,|h|\,dx &\le C\left(\int_{B_R^c} |x|^{\frac{\alpha_\infty-\nu(q_2-p\beta_\infty)}{1-\beta_\infty}}\,dx\right)^{1-\beta_\infty}\\ &\le C\left(R^{\frac{\alpha_\infty-\nu(q_2-p\beta_\infty)}{1-\beta_\infty}+N}\right)^{1-\beta_\infty},\end{aligned}$$

since

$$\frac{\alpha_\infty-\nu(q_2-p\beta_\infty)}{1-\beta_\infty}+N = \frac{\alpha_\infty-\alpha_2-\nu(q_2-p\beta_\infty)}{1-\beta_\infty} < 0$$

Case $\beta_\infty \le \frac{1}{p}$ *and* $\alpha_\infty \le \alpha_3$ $(= \max\{\alpha_2, \alpha_3\})$.

Take $\xi = \frac{1-p\beta_\infty}{p} \ge 0$, we can apply Lemma 4.1 with $\beta = \beta_\infty + \xi = \frac{1}{p}$ and $\alpha = \alpha_\infty + \xi\gamma_\infty$. We get

$$\int_{B_R^c} K(|x|)\,|u|^{q_2-1}\,|h|\,dx \le C\left(\int_{B_R^c} |x|^{\frac{(\alpha-\nu(q_2-1))p}{p-1}}\,dx\right)^{\frac{p-1}{p}} \le CR^{\alpha-\nu(q_2-1)+\frac{N(p-1)}{p}},$$

since

$$\alpha-\nu(q_2-1)+\frac{N(p-1)}{p} = \alpha_\infty+\frac{1-p\beta_\infty}{p}\gamma_\infty+\frac{N(p-1)}{p}-\nu(q_2-1) = \alpha_\infty-\alpha_3-\nu(q_2-1) < 0.$$

□

The proof of Theorem 2.7 will be achieved by the following lemmas.

Lemma 4.3. *Assume that there exists* $R > 0$ *such that* $V(r) < +\infty$ *almost everywhere in* $(0, R)$ *and*

$$\lambda(R) := \operatorname*{ess\,inf}_{r\in(0,R)} r^{\gamma_0} V(r) > 0 \quad \textit{for some } \gamma_0 \ge p.$$

Then there exists a constant $c_0 > 0$, *only dependent on* N *and* p, *such that* $\forall u \in W_r$ *one has*

$$|u(x)| \le c_0\left(\left(\frac{1}{\lambda(R)}\right)^{\frac{p-1}{p}} + \frac{R^{\frac{\gamma_0-p}{p}}}{\lambda(R)}\right)^{\frac{1}{p}} \|u\|\,|x|^{-\frac{p(N-1)-(p-1)\gamma_0}{p^2}} \quad \textit{almost everywhere in } B_R. \tag{4.3}$$

Proof. The estimate (4.3) is essentially proved in [10, Lemma 5], but without expliciting the dependence of the constants on R and with slightly different assumptions on V (a global decay condition and the continuity on $(0,+\infty)$) and u (which is taken in $W_r \cap D_0^{1,p}(B_R)$). In the form given here, the lemma follows by adapting the proof of [3, Proposition 29], where the result is proved for $p = 2$. ☐

Lemma 4.4. *Assume that there exists $R > 0$ such that $V(r) < +\infty$ almost everywhere in $(0,R)$ and*

$$\Lambda_{\alpha,\beta}(R) := \operatorname*{ess\,sup}_{r\in(0,R)} \frac{K(r)}{r^\alpha V(r)^\beta} < +\infty \quad \text{for some } \frac{1}{p} \leq \beta \leq 1 \text{ and } \alpha \in \mathbb{R} \tag{4.4}$$

and

$$\lambda(R) := \operatorname*{ess\,inf}_{r\in(0,R)} r^{\gamma_0} V(r) > 0 \quad \text{for some } \gamma_0 > p.$$

Assume also that there exists $q > p\beta$ such that

$$(p(N-1)-(p-1)\gamma_0)\, q < p^2(\alpha+N) - p\left((p-1)\gamma_0 + p\right)\beta.$$

Then $\forall u \in W_r$ and $\forall h \in W$ one has

$$\int_{B_R} K(|x|)\,|u|^{q-1}\,|h|\,dx \leq c_0^{q-p\beta} a(R)\, R^{\frac{p^2(\alpha+N)-p((p-1)\gamma_0+p)\beta-(p(N-1)-(p-1)\gamma_0)q}{p^2}} \|u\|^{q-1}\|h\|,$$

where c_0 is the constant of Lemma 4.3 and $a(R) := \Lambda_{\alpha,\beta}(R)\left(\left(\frac{1}{\lambda(R)}\right)^{\frac{p-1}{p}} + \frac{R^{\frac{\gamma_0-p}{p}}}{\lambda(R)}\right)^{\frac{q-p\beta}{p}}$.

Proof. Let $u \in W_r$ and $h \in W$. By assumption (4.4) and Lemma 4.3, we can apply Lemma 4.1 with $\Omega = B_R$, $\Lambda = \Lambda_{\alpha,\beta}(R)$, $\nu = \frac{(N-1)p-(p-1)\gamma_0}{p^2}$ and

$$m = c_0\left(\left(\frac{1}{\lambda(R)}\right)^{\frac{p-1}{p}} + \frac{R^{\frac{\gamma_0-p}{p}}}{\lambda(R)}\right)^{\frac{1}{p}} \|u\|.$$

If $\frac{1}{p} \leq \beta < 1$, we get

$$\begin{aligned}
\int_{B_R} K(|x|)\,|u|^{q-1}\,|h|\,dx &\leq \Lambda m^{q-p\beta}\left(\int_\Omega |x|^{\frac{\alpha-\nu(q-p\beta)}{1-\beta}}\,dx\right)^{1-\beta} \|u\|^{p\beta-1}\|h\| \\
&= c_0^{q-p\beta} a(R)\left(\int_{B_R} |x|^{\frac{p^2\alpha-(p(N-1)-(p-1)\gamma_0)(q-p\beta)}{p^2(1-\beta)}}\,dx\right)^{1-\beta} \|u\|^{q-1}\|h\| \\
&\leq c_0^{q-p\beta} a(R)\left(R^{\frac{p^2\alpha-(p(N-1)-(p-1)\gamma_0)(q-p\beta)}{p^2(1-\beta)}+N}\right)^{1-\beta} \|u\|^{q-1}\|h\|,
\end{aligned}$$

since

$$\begin{aligned}
&\frac{p^2\alpha-(p(N-1)-(p-1)\gamma_0)(q-p\beta)}{p^2(1-\beta)} + N \\
&= \frac{p^2(\alpha+N)-p((p-1)\gamma_0+p)\beta-(p(N-1)-(p-1)\gamma_0)q}{p^2(1-\beta)} > 0.
\end{aligned}$$

If instead we have $\beta = 1$, we get

$$\begin{aligned}
&\int_{B_R} K(|x|)\,|u|^{q-1}\,|h|\,dx \\
&\leq \Lambda m^{q-p}\left(\int_{\Omega} |x|^{\frac{p}{p-1}(\alpha-\nu(q-p))}\,V(|x|)\,|u|^p\,dx\right)^{\frac{p-1}{p}}\|h\| \\
&= c_0^{q-p}a(R)\left(\int_{B_R}|x|^{\frac{p^2\alpha-(p(N-1)-(p-1)\gamma_0)(q-p)}{p(p-1)}}\,V(|x|)\,|u|^p\,dx\right)^{\frac{p-1}{p}}\|u\|^{q-p}\|h\| \\
&\leq c_0^{q-p}a(R)\left(R^{\frac{p^2\alpha-(p(N-1)-(p-1)\gamma_0)(q-p)}{p(p-1)}}\int_{B_R}V(|x|)\,|u|^p\,dx\right)^{\frac{p-1}{p}}\|u\|^{q-p}\|h\| \\
&\leq c_0^{q-p}a(R)\,R^{\frac{p^2\alpha-(p(N-1)-(p-1)\gamma_0)(q-p)}{p^2}}\|u\|^{q-1}\|h\|,
\end{aligned}$$

since

$$\begin{aligned}
&p^2\alpha-(p(N-1)-(p-1)\gamma_0)(q-p) \\
&= p^2(\alpha+N)-p((p-1)\gamma_0+p)-(p(N-1)-(p-1)\gamma_0)\,q>0.
\end{aligned}$$

□

Proof of Theorem 2.7. Assume the hypotheses of the theorem and denote

$$\Lambda_0 := \operatorname*{ess\,sup}_{r\in(0,R_1)}\frac{K(r)}{r^{\alpha_0}V(r)^{\beta_0}} \quad \text{and} \quad \lambda_0 := \operatorname*{ess\,inf}_{r\in(0,R_1)} r^{\gamma_0}V(r).$$

If $\gamma_0 = p$ the thesis of the theorem is true by Theorem 2.2 (see Remark 2.8.3), whence we can assume $\gamma_0 > p$ without restriction. We claim that for every $0 < R \leq R_1$ there exists $b(R) > 0$ such that $b(R) \to 0$ as $R \to 0^+$ and

$$\int_{B_R} K(|x|)\,|u|^{q_1-1}\,|h|\,dx \leq b(R)\,\|u\|^{q_1-1}\|h\|, \quad \forall u\in W_r,\ \forall h\in W,$$

which clearly gives the result. In order to prove this claim, let $0 < R \leq R_1$. Then one has

$$\lambda(R) := \operatorname*{ess\,inf}_{r\in(0,R)} r^{\gamma_0}V(r) \geq \lambda_0 > 0 \tag{4.5}$$

and for every $\xi \geq 0$ we have

$$\begin{aligned}
\Lambda_{\alpha_0+\xi\gamma_0,\beta_0+\xi}(R) &:= \operatorname*{ess\,sup}_{r\in(0,R)}\frac{K(r)}{r^{\alpha_0+\xi\gamma_0}V(r)^{\beta_0+\xi}} \leq \operatorname*{ess\,sup}_{r\in(0,R_1)}\frac{K(r)}{r^{\alpha_0}V(r)^{\beta_0}\left(r^{\gamma_0}V(r)\right)^{\xi}} \\
&\leq \frac{\Lambda_0}{\lambda_0^{\xi}} < +\infty.
\end{aligned} \tag{4.6}$$

Denoting $\alpha_1 = \alpha_1(\beta_0,\gamma_0)$, $\alpha_2 = \alpha_2(\beta_0)$ and $\alpha_3 = \alpha_3(\beta_0,\gamma_0)$, as defined in (2.14), we will now distinguish five cases, which reflect the five definitions (2.16) of the set $\mathcal{A}_{\beta_0,\gamma_0}$. For the sake of clarity, some computations will be displaced in the Appendix.

Case $p < \gamma_0 < N$.

In this case, $(\alpha_0, q_1) \in \mathcal{A}_{\beta_0,\gamma_0}$ means

$$\alpha_0 > \max\{\alpha_2, \alpha_3\} \quad \text{and}$$
$$\max\{1, p\beta_0\} < q_1 < \min\left\{p\frac{\alpha_0 - \beta_0\gamma_0 + N}{N - \gamma_0}, p\frac{p\alpha_0 + (1 - p\beta_0)\gamma_0 + p(N-1)}{p(N-1) - (p-1)\gamma_0}\right\}$$

and these conditions ensure that we can fix $\xi \geq 0$, independent of R, in such a way that $\alpha = \alpha_0 + \xi\gamma_0$ and $\beta = \beta_0 + \xi$ satisfy

$$\frac{1}{p} \leq \beta \leq 1 \quad \text{and} \quad p\beta < q_1 < \frac{p^2(\alpha + N) - p\left((p-1)\gamma_0 + p\right)\beta}{p(N-1) - (p-1)\gamma_0} \tag{4.7}$$

(see Appendix). Hence, by (4.6) and (4.5), we can apply Lemma 4.4 (with $q = q_1$), so that $\forall u \in W_r$ and $\forall h \in W$ we get

$$\int_{B_R} K(|x|)\,|u|^{q_1-1}\,|h|\,dx \leq c_0^{q_1-p\beta} a(R)\, R^{\frac{p^2(\alpha+N) - p((p-1)\gamma_0+p)\beta - (p(N-1)-(p-1)\gamma_0)q_1}{p^2}} \|u\|^{q_1-1}\|h\|.$$

This gives the result, since $R^{p^2(\alpha+N)-p((p-1)\gamma_0+p)\beta-(p(N-1)-(p-1)\gamma_0)q_1} \to 0$ as $R \to 0^+$ and

$$a(R) = \Lambda_{\alpha_0+\xi\gamma_0,\beta_0+\xi}(R)\left(\left(\frac{1}{\lambda(R)}\right)^{\frac{p-1}{p}} + \frac{R^{\frac{\gamma_0-p}{p}}}{\lambda(R)}\right)^{\frac{q_1-p\beta}{p}} \leq \frac{\Lambda_0}{\lambda_0^{\xi}}\left(\left(\frac{1}{\lambda_0}\right)^{\frac{p-1}{p}} + \frac{R_1^{\frac{\gamma_0-p}{p}}}{\lambda_0}\right)^{\frac{q_1-p\beta}{p}}.$$

Case $\gamma_0 = N$.

In this case, $(\alpha_0, q_1) \in \mathcal{A}_{\beta_0,\gamma_0}$ means

$$\alpha_0 > \alpha_1 \ (= \alpha_2 = \alpha_3) \quad \text{and} \quad \max\{1, p\beta_0\} < q_1 < p\frac{p\alpha_0 + (1 - p\beta_0)\gamma_0 + p(N-1)}{p(N-1) - (p-1)\gamma_0}$$

and these conditions still ensure that we can fix $\xi \geq 0$ in such a way that $\alpha = \alpha_0 + \xi\gamma_0$ and $\beta = \beta_0 + \xi$ satisfy (4.7) (see Appendix), so that the result ensues again by Lemma 4.4.

Case $N < \gamma_0 < \frac{p}{p-1}(N-1)$.

In this case, $(\alpha_0, q_1) \in \mathcal{A}_{\beta_0,\gamma_0}$ means

$$\alpha_0 > \alpha_1 \quad \text{and}$$
$$\max\left\{1, p\beta_0, p\frac{\alpha_0 - \beta_0\gamma_0 + N}{N - \gamma_0}\right\} < q_1 < p\frac{p\alpha_0 + (1 - p\beta_0)\gamma_0 + p(N-1)}{p(N-1) - (p-1)\gamma_0}$$

and the conclusion then follows as in the former cases (see Appendix).

Case $\gamma_0 = \frac{p}{p-1}(N-1)$.

In this case, $(\alpha_0, q_1) \in \mathcal{A}_{\beta_0,\gamma_0}$ means

$$\alpha_0 > \alpha_1 \quad \text{and} \quad \max\left\{1, p\beta_0, p\frac{\alpha_0 - \beta_0\gamma_0 + N}{N - \gamma_0}\right\} < q_1$$

and these conditions ensure that we can fix $\xi \geq 0$ in such a way that $\alpha = \alpha_0 + \xi\gamma_0$ and $\beta = \beta_0 + \xi$ satisfy

$$\frac{1}{p} \leq \beta \leq 1, \quad q_1 > p\beta \quad \text{and} \quad 0 < p^2(\alpha + N) - p\left((p-1)\gamma_0 + p\right)\beta$$

(see Appendix). The result then follows again from Lemma 4.4.

Case $\gamma_0 > \frac{p}{p-1}(N-1)$.

In this case, $(\alpha_0, q_1) \in \mathcal{A}_{\beta_0,\gamma_0}$ means

$$q_1 > \max\left\{1, p\beta_0, p\frac{\alpha_0 - \beta_0\gamma_0 + N}{N - \gamma_0}, p\frac{p\alpha_0 + (1 - p\beta_0)\gamma_0 + p(N-1)}{p(N-1) - (p-1)\gamma_0}\right\}$$

and this condition ensures that we can fix $\xi \geq 0$ in such a way that $\alpha = \alpha_0 + \xi\gamma_0$ and $\beta = \beta_0 + \xi$ satisfy

$$\frac{1}{p} \leq \beta \leq 1 \quad \text{and} \quad q_1 > \max\left\{p\beta, p\frac{p(\alpha + N) - \left((p-1)\gamma_0 + p\right)\beta}{p(N-1) - (p-1)\gamma_0}\right\}$$

(see Appendix). The result still follows from Lemma 4.4. □

5 Appendix

This Appendix is devoted to complete the computations of the proof of Theorem 2.7. We still distinguish the same cases considered there.

Case $p < \gamma_0 < N$.

In this case, $(\alpha_0, q_1) \in \mathcal{A}_{\beta_0,\gamma_0}$ means

$$\begin{aligned} &\alpha_0 > \max\{\alpha_2, \alpha_3\} \quad \text{and} \\ &\max\{1, p\beta_0\} < q_1 < \min\left\{p\frac{\alpha_0 - \beta_0\gamma_0 + N}{N - \gamma_0}, p\frac{p\alpha_0 + (1 - p\beta_0)\gamma_0 + p(N-1)}{p(N-1) - (p-1)\gamma_0}\right\}. \end{aligned}$$

This ensures that we can find $\xi \geq 0$ such that

$$\frac{1}{p} \leq \beta_0 + \xi \leq 1 \quad \text{and} \quad p(\beta_0 + \xi) < q_1 < \frac{p^2(\alpha_0 + \xi\gamma_0) + p^2 N - p\left((p-1)\gamma_0 + p\right)(\beta_0 + \xi)}{p(N-1) - (p-1)\gamma_0},$$

i.e.,

$$\begin{aligned} &\tfrac{1}{p} - \beta_0 \leq \xi \leq 1 - \beta_0 \quad \text{and} \\ &p\beta_0 + p\xi < q_1 < p\tfrac{\gamma_0 - p}{p(N-1) - (p-1)\gamma_0}\xi + \tfrac{p^2\alpha_0 + p^2 N - p((p-1)\gamma_0 + p)\beta_0}{p(N-1) - (p-1)\gamma_0}. \end{aligned}$$

Indeed, this amounts to find ξ such that

$$\begin{cases} \max\left\{0, \frac{1 - p\beta_0}{p}\right\} \leq \xi \leq 1 - \beta_0 \\ \xi < \frac{q_1 - p\beta_0}{p} \\ q_1 - \frac{p^2\alpha_0 + p^2 N - p((p-1)\gamma_0 + p)\beta_0}{p(N-1) - (p-1)\gamma_0} < p\frac{\gamma_0 - p}{p(N-1) - (p-1)\gamma_0}\xi, \end{cases}$$

which, since $\frac{\gamma_0-p}{p(N-1)-(p-1)\gamma_0}>0$, is equivalent to

$$\begin{cases} \max\left\{0,\frac{1-p\beta_0}{p}\right\}\leq\xi\leq 1-\beta_0 \\ q_1\frac{p(N-1)-(p-1)\gamma_0}{p(\gamma_0-p)}-\frac{p\alpha_0+pN-((p-1)\gamma_0+p)\beta_0}{\gamma_0-p}<\xi<\frac{q_1-p\beta_0}{p}. \end{cases}$$

Since $\frac{1}{p}-\beta_0\leq 1-\beta_0$ is obvious (recall that $p>1$) and $1-\beta_0\geq 0$ holds by assumption, such a system has a solution ξ if and only if

$$\begin{cases} \frac{1-p\beta_0}{p}<\frac{q_1-p\beta_0}{p} \\ q_1\frac{p(N-1)-(p-1)\gamma_0}{p(\gamma_0-p)}-\frac{p\alpha_0+pN-((p-1)\gamma_0+p)\beta_0}{\gamma_0-p}<1-\beta_0 \\ q_1\frac{p(N-1)-(p-1)\gamma_0}{p(\gamma_0-p)}-\frac{p\alpha_0+pN-((p-1)\gamma_0+p)\beta_0}{\gamma_0-p}<\frac{q_1-p\beta_0}{p} \\ \frac{q_1-p\beta_0}{p}>0, \end{cases}$$

which is equivalent to

$$\begin{cases} 1<q_1 \\ q_1<p\frac{p\alpha_0+p(N-1)+(1-p\beta_0)\gamma_0}{p(N-1)-(p-1)\gamma_0} \\ q_1<p\frac{\alpha_0+N-\gamma_0\beta_0}{N-\gamma_0} \\ q_1>p\beta_0. \end{cases}$$

Case $\gamma_0=N$.

In this case, $(\alpha_0,q_1)\in\mathcal{A}_{\beta_0,\gamma_0}$ means

$$\alpha_0>\alpha_1\ (=\alpha_2=\alpha_3)\quad\text{and}$$
$$\max\{1,p\beta_0\}<q_1<p\frac{p\alpha_0+(1-p\beta_0)\gamma_0+p(N-1)}{p(N-1)-(p-1)\gamma_0}=p\frac{p\alpha_0+(p+1)N-p\beta_0N-p}{N-p}$$

and this ensures that we can find $\xi\geq 0$ such that

$$\frac{1}{p}\leq\beta_0+\xi\leq 1\quad\text{and}\quad p(\beta_0+\xi)<q_1<\frac{p^2(\alpha_0+\xi\gamma_0)+p^2N-p((p-1)\gamma_0+p)(\beta_0+\xi)}{p(N-1)-(p-1)\gamma_0},$$

i.e.,

$$\frac{1}{p}-\beta_0\leq\xi\leq 1-\beta_0\quad\text{and}\quad p\beta_0+p\xi<q_1<p\xi+\frac{p^2\alpha_0+p^2N-p(N(p-1)+p)\beta_0}{N-p}.$$

Indeed, this amounts to find ξ such that

$$\begin{cases} \max\left\{0,\frac{1-p\beta_0}{p}\right\}\leq\xi\leq 1-\beta_0 \\ \frac{q_1}{p}-\frac{p\alpha_0+pN-((p-1)N+p)\beta_0}{N-p}<\xi<\frac{q_1-p\beta_0}{p}, \end{cases}$$

which has a solution ξ if and only if

$$\begin{cases} \frac{1-p\beta_0}{p}<\frac{q_1-p\beta_0}{p} \\ \frac{q_1}{p}-\frac{p\alpha_0+pN-(N(p-1)+p)\beta_0}{N-p}<1-\beta_0 \\ \frac{q_1}{p}-\frac{p\alpha_0+pN-(N(p-1)+p)\beta_0}{N-p}<\frac{q_1-p\beta_0}{p} \\ 0<\frac{q_1-\beta_0}{p}. \end{cases}$$

These conditions are equivalent to

$$\begin{cases} 1 < q_1 \\ q_1 < p\frac{p\alpha_0+(p+1)N-p\beta_0 N-p}{N-p} \\ 0 < \frac{p\alpha_0+pN-(N(p-1)+p)\beta_0}{N-p} - \beta_0 = p\frac{\alpha_0+N(1-\beta_0)}{N-p} = p\frac{\alpha_0-\alpha_1}{N-p} \\ p\beta_0 < q_1. \end{cases}$$

Case $N < \gamma_0 < \frac{p}{p-1}(N-1)$.

In this case, $(\alpha_0, q_1) \in \mathcal{A}_{\beta_0,\gamma_0}$ means

$$\alpha_0 > \alpha_1 \quad \text{and}$$
$$\max\left\{1, p\beta_0, p\frac{\alpha_0-\beta_0\gamma_0+N}{N-\gamma_0}\right\} < q_1 < p\frac{p\alpha_0+(1-p\beta_0)\gamma_0+p(N-1)}{p(N-1)-(p-1)\gamma_0}$$

and these conditions ensure that we can find $\xi \geq 0$ such that

$$\frac{1}{p} \leq \beta_0+\xi \leq 1 \quad \text{and} \quad p(\beta_0+\xi) < q_1 < \frac{p^2(\alpha_0+\xi\gamma_0)+p^2N-p((p-1)\gamma_0+p)(\beta_0+\xi)}{(N-1)p-(p-1)\gamma_0},$$

i.e.,

$$\frac{1}{p}-\beta_0 \leq \xi \leq 1-\beta_0 \quad \text{and}$$
$$p\beta_0+p\xi < q_1 < p\frac{\gamma_0-p}{p(N-1)-(p-1)\gamma_0}\xi + \frac{p^2\alpha_0+p^2N-p((p-1)\gamma_0+p)\beta_0}{p(N-1)-(p-1)\gamma_0}.$$

Indeed, this is equivalent to find ξ such that

$$\begin{cases} \max\left\{0, \frac{1-p\beta_0}{p}\right\} \leq \xi \leq 1-\beta_0 \\ \xi < \frac{q_1-p\beta_0}{p} \\ q_1 - \frac{p^2\alpha_0+p^2N-p((p-1)\gamma_0+p)\beta_0}{p(N-1)-(p-1)\gamma_0} < p\frac{\gamma_0-p}{p(N-1)-(p-1)\gamma_0}\xi, \end{cases}$$

which, since $\frac{\gamma_0-p}{p(N-1)-(p-1)\gamma_0} > 0$, amounts to

$$\begin{cases} \max\left\{0, \frac{1-p\beta_0}{p}\right\} \leq \xi \leq 1-\beta_0 \\ \xi < \frac{q_1-p\beta_0}{p} \\ q_1\frac{p(N-1)-(p-1)\gamma_0}{p(\gamma_0-p)} - \frac{p\alpha_0+pN-((p-1)\gamma_0+p)\beta_0}{\gamma_0-p} < \xi. \end{cases}$$

Such a system has a solution ξ if and only if

$$\begin{cases} 0 < \frac{q_1-p\beta_0}{p} \\ \frac{1-p\beta_0}{p} < \frac{q_1-p\beta_0}{p} \\ q_1\frac{p(N-1)-(p-1)\gamma_0}{p(\gamma_0-p)} - \frac{p\alpha_0+pN-((p-1)\gamma_0+p)\beta_0}{\gamma_0-p} < 1-\beta_0 \\ q_1\frac{p(N-1)-(p-1)\gamma_0}{p(\gamma_0-p)} - \frac{p\alpha_0+pN-((p-1)\gamma_0+p)\beta_0}{\gamma_0-p} < \frac{q_1-p\beta_0}{p}, \end{cases}$$

which is equivalent to

$$\begin{cases} q_1 > p\beta_0 \\ q_1 > 1 \\ q_1 < p\frac{p\alpha_0+pN-p\beta_0\gamma_0+\gamma_0-p}{p(N-1)-(p-1)\gamma_0} \\ -p\frac{\alpha_0+N-\beta_0\gamma_0}{\gamma_0-p} < q_1\frac{\gamma_0-N}{\gamma_0-p}. \end{cases}$$

Case $\gamma_0 = \frac{p}{p-1}(N-1)$.

In this case, $(\alpha_0, q_1) \in \mathcal{A}_{\beta_0,\gamma_0}$ means

$$\alpha_0 > \alpha_1 \quad \text{and} \quad q_1 > \max\left\{1, p\beta_0, -p\frac{(\alpha_0+N)(p-1)-p(N-1)\beta_0}{N-p}\right\}.$$

This ensures that we can find $\xi \geq 0$ such that

$$\frac{1}{p} \leq \beta_0+\xi \leq 1, \quad q_1 > p(\beta_0+\xi) \quad \text{and} \quad p(\alpha_0+\xi\gamma_0)+pN-((p-1)\gamma_0+p)(\beta_0+\xi) > 0,$$

i.e.,

$$\frac{1}{p}-\beta_0 \leq \xi \leq 1-\beta_0, \quad q_1 > p\beta_0+p\xi \quad \text{and} \quad \alpha_0+N(1-\beta_0)+\frac{N-p}{p-1}\xi > 0.$$

Indeed, this amounts to find ξ such that

$$\begin{cases} \max\left\{0, \frac{1-p\beta_0}{p}\right\} \leq \xi \leq 1-\beta_0 \\ -\frac{(p-1)(\alpha_0+N(1-\beta_0))}{N-p} < \xi < \frac{q_1-p\beta_0}{p} \end{cases}$$

and such a system has a solution ξ if and only if

$$\begin{cases} \frac{1-p\beta_0}{p} < \frac{q_1-p\beta_0}{p} \\ -\frac{(p-1)(\alpha_0+N(1-\beta_0))}{N-p} < 1-\beta_0 \\ -\frac{(p-1)(\alpha_0+N(1-\beta_0))}{N-p} < \frac{q_1-p\beta_0}{p} \\ 0 < \frac{q_1-p\beta_0}{p}, \end{cases}$$

which means

$$\begin{cases} 1 < q_1 \\ \alpha_0 > -(N-1)\frac{p}{p-1}(1-\beta_0) = \alpha_1 \\ \frac{q_1}{p} > \beta_0 - \frac{(p-1)(\alpha_0+N(1-\beta_0))}{N-p} = -\frac{(p-1)n(\alpha_0+N)-p(N-1)\beta_0}{N-p} \\ p\beta_0 < q_1. \end{cases}$$

Case $\gamma_0 > \frac{p}{p-1}(N-1)$.

In this case, $(\alpha_0, q_1) \in \mathcal{A}_{\beta_0,\gamma_0}$ means

$$q_1 > \max\left\{1, p\beta_0, p\frac{\alpha_0-\beta_0\gamma_0+N}{N-\gamma_0}, p\frac{p\alpha_0+(1-p\beta_0)\gamma_0+pN-p}{p(N-1)-(p-1)\gamma_0}\right\}$$

and this condition ensures that we can find $\xi \geq 0$ such that

$$\frac{1}{p} \leq \beta_0 + \xi \leq 1 \quad \text{and} \quad q_1 > p \max\left\{\beta_0 + \xi, \frac{p\left(\alpha_0 + \xi\gamma_0\right) + pN - \left((p-1)\gamma_0 + p\right)\left(\beta_0 + \xi\right)}{p(N-1) - (p-1)\gamma_0}\right\},$$

which amounts to find ξ such that

$$\begin{cases} \max\left\{0, \frac{1-p\beta_0}{p}\right\} \leq \xi \leq 1 - \beta_0 \\ \frac{q_1}{p} > \max\left\{\beta_0 + \xi, \frac{\gamma_0 - p}{p(N-1)-(p-1)\gamma_0}\xi + \frac{p\alpha_0 + pN - ((p-1)\gamma_0 + p)\beta_0}{p(N-1)-(p-1)\gamma_0}\right\}. \end{cases} \tag{5.1}$$

In order to check this, we take into account that $\gamma_0 > \frac{p}{p-1}(N-1)$ implies $\gamma_0 > N$, and observe that

$$\beta_0 + \xi = \frac{\gamma_0 - p}{p(N-1) - (p-1)\gamma_0}\xi + \frac{p\alpha_0 + pN - \left((p-1)\gamma_0 + p\right)\beta_0}{p(N-1) - (p-1)\gamma_0} \Longleftrightarrow \xi = \frac{\alpha_0 + \left(1 - \beta_0\right)N}{N - \gamma_0}.$$

Accordingly, we distinguish three subcases:

(I) $\frac{\alpha_0 + (1-\beta_0)N}{N-\gamma_0} \geq 1 - \beta_0$, i.e., $\alpha_0 \leq -\gamma_0\left(1 - \beta_0\right) = \alpha_1$;

(II) $\frac{\alpha_0 + (1-\beta_0)N}{N-\gamma_0} \leq \max\left\{0, \frac{1-p\beta_0}{p}\right\}$, i.e.,

$$\alpha_0 + \left(1 - \beta_0\right)N \geq \left(N - \gamma_0\right)\max\left\{0, \frac{1 - p\beta_0}{p}\right\} = \min\left\{0, \left(N - \gamma_0\right)\frac{1 - p\beta_0}{p}\right\},$$

i.e.,

$$\alpha_0 \geq \min\left\{0, \left(N - \gamma_0\right)\frac{1 - p\beta_0}{p}\right\} - \left(1 - \beta_0\right)N = \min\left\{\alpha_2, \alpha_3\right\};$$

(III) $\max\left\{0, \frac{1-p\beta_0}{p}\right\} < \frac{\alpha_0 + (1-\beta_0)N}{N-\gamma_0} < 1 - \beta_0$, i.e., $\alpha_1 < \alpha_0 < \min\left\{\alpha_2, \alpha_3\right\}$.

Subcase (I).

Since $\xi \leq 1 - \beta_0$ implies

$$\begin{aligned} &\max\left\{\beta_0 + \xi, \frac{\gamma_0 - p}{p(N-1) - (p-1)\gamma_0}\xi + \frac{p\alpha_0 + pN - \left((p-1)\gamma_0 + p\right)\beta_0}{p(N-1) - (p-1)\gamma_0}\right\} \\ &= \frac{\gamma_0 - p}{p(N-1) - (p-1)\gamma_0}\xi + \frac{p\alpha_0 + pN - \left((p-1)\gamma_0 + p\right)\beta_0}{p(N-1) - (p-1)\gamma_0}, \end{aligned}$$

the inequalities (5.1) become

$$\begin{cases} \max\left\{0, \frac{1-p\beta_0}{p}\right\} \leq \xi \leq 1 - \beta_0 \\ \frac{q_1}{p} > \frac{\gamma_0 - p}{p(N-1)-(p-1)\gamma_0}\xi + \frac{p\alpha_0 + pN - ((p-1)\gamma_0 + p)\beta_0}{p(N-1)-(p-1)\gamma_0}, \end{cases}$$

i.e.,

$$\begin{cases} \max\left\{0, \frac{1-p\beta_0}{p}\right\} \leq \xi \leq 1 - \beta_0 \\ q_1\frac{p(N-1)-(p-1)\gamma_0}{p(\gamma_0 - p)} - \frac{p^2\alpha_0 + p^2N - p((p-1)\gamma_0 + p)\beta_0}{p(\gamma_0 - p)} < \xi, \end{cases}$$

which, since $\max\left\{0, \frac{1-p\beta_0}{p}\right\} \leq 1-\beta_0$ is clearly true, has a solution ξ if and only if

$$q_1 \frac{p(N-1)-(p-1)\gamma_0}{p\left(\gamma_0-p\right)} - \frac{p^2\alpha_0 + p^2 N - p\left((p-1)\gamma_0 + p\right)\beta_0}{p\left(\gamma_0-p\right)} < 1-\beta_0,$$

i.e.,

$$\begin{aligned} q_1 &> \frac{p^2\alpha_0 + p^2 N - p\left((p-1)\gamma_0 + p\right)\beta_0 + p\left(\gamma_0-p\right)\left(1-\beta_0\right)}{p(N-1)-(p-1)\gamma_0} \\ &= p\frac{p\alpha_0 + pN - p + \gamma_0\left(1-p\beta_0\right)}{p(N-1)-(p-1)\gamma_0}. \end{aligned}$$

Subcase (II).

Since $\xi \geq \max\left\{0, \frac{1-p\beta_0}{p}\right\}$ implies $\max\left\{\beta_0+\xi, \frac{\gamma_0-p}{p(N-1)-(p-1)\gamma_0}\xi + \frac{p\alpha_0+pN-((p-1)\gamma_0+p)\beta_0}{p(N-1)-(p-1)\gamma_0}\right\} = \beta_0+\xi$, the inequalities (5.1) become

$$\begin{cases} \max\left\{0, \frac{1-p\beta_0}{p}\right\} \leq \xi \leq 1-\beta_0 \\ \xi < \frac{q_1}{p} - \beta_0, \end{cases}$$

which has a solution ξ if and only if $\max\left\{0, \frac{1-p\beta_0}{p}\right\} \leq \frac{q_1}{p} - \beta_0$, i.e., $q_1 > \max\left\{1, p\beta_0\right\}$.

Subcase (III).

We take $\xi = \frac{\alpha_0+(1-\beta_0)N}{N-\gamma_0}$ and thus the inequalities (5.1) are equivalent just to

$$\frac{q_1}{p} > \beta_0 + \frac{\alpha_0 + \left(1-\beta_0\right)N}{N-\gamma_0} = \frac{\alpha_0 + N - \gamma_0\beta_0}{N-\gamma_0}.$$

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