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F Magri

Dipartimento di Matematica ed Applicazioni, Università di Milano Bicocca, 20125 Milano, Italy

E-mail: franco.magri@unimib.it

Abstract. The aim of this paper is to introduce a new category of manifolds, called Haantjes manifolds, and to show their interest by a few selected examples.

1. Introduction

Many years ago, at the first conference in this series held in Lecce in 1979, I had the occasion to present a simple model of integrable systems, henceforth called bihamiltonian systems. They were defined on manifolds called bihamiltonian manifolds. In this conference I wish to present an evolution of the concept of bihamiltonian manifold called Haantjes manifold. I believe that the study of the new class of manifolds may enlight a range of topics even outsides the domain of the theory of integrable systems. Among them I list:

- Topological Field Theories
- Frobenius manifolds
- Singularity Theory
- Quantum cohomology
- Coisotropic deformations of associative algebras
- Quasi-Hamiltonian systems of hydrodynamic type
- Whitham equations
- Orthogonal coordinates in $\mathbf{R^n}$.

In this paper the focus is on the relation between Haantjes manifolds and the theory of singularities.

To introduce the concept of Haantjes manifold, I need to recall that a bihamiltonian system is a vector field X on a bihamiltonian manifold M which admits two compatible Hamiltonian decompositions

$$X = Pdh = Qdk.$$

If one of the Poisson bivectors P and Q is invertible, say Q, this decomposition endows the manifold M with three distinguished geometrical objects: the vector field X, the 1-form $\theta = dh$, and the recursion operator $K = PQ^{-1}$. These geometrical objects satisfy five basic conditions

- $Torsion(K) = 0$
- $d\theta = 0$

- $dK\theta = 0$
- $Lie_X(K) = 0$
- $Lie_X(\theta) = 0.$

Acting on the vector field X and the 1-form θ, the tensor field K generate two infinite chains of vector fields and 1-forms respectively :

$$\begin{aligned} \theta_{k+1} &= K\theta_k, & \theta_0 &= \theta, \\ X_{k+1} &= KX_k, & X_0 &= X. \end{aligned}$$

They are called the *Lenard chains* associated with the given bihamiltonian system. Owing to the five conditions listed above, these chains enjoy the remarkable property that the 1-forms θ_k are closed, and that the vector fields X_k commute. Locally the 1-forms θ_k are the differentials of certain functions I_k. These functions turn out to be in involution with respect to both the Poisson brackets associated with the Poisson bivectors P and Q. Consequently the vector fields X_k and the functions I_k are symmetries and integrals of motion of X. They are the additional informations which permit to integrate the given bihamiltonian vector field X .

The concept of Haantjes manifold arises when one accepts to slighty modify the previous scheme, allowing the recursion operator to *possess torsion* but in a controlled manner. This idea has been presented two years ago at the conference on *Geometrical Methods in Mathematical Physics* organized by Boris Dubrovin in Moscow. In that occasion I suggested to replace the conditions of the classical theory of bihamiltonian manifolds by the new conditions

- $Haantjes(K) = 0$
- $d\theta = 0$
- $dK\theta = 0$
- $\theta(Torsion(K)) = 0$
- $Lie_X(K) = 0$
- $Lie_X(\theta) = 0,$

on the basis of their significance for the theory of Frobenius manifolds and $WDVV$ equations [1]. In this conference I wish to continue the exploration of the new geometrical setting. In particular, I am interested to describe the changes which are undergone by the theory of Lenard chains .

First I collect the definitions which are relevant to the present discussion.

Definition 1. *(***Haantjes manifold***) A manifold M is a <u>Haantjes manifold</u> if it is endowed with a distinguished 1-form θ and with a distinguished vector-valued 1-form K which satisfy the four conditions*

- $Haantjes(K) = 0$
- $d\theta = 0$
- $dK\theta = 0$
- $\theta(Torsion(K)) = 0.$

If moreover there exists a vector field X obeying the conditions

- $Lie_X(K) = 0$
- $Lie_X(\theta) = 0.$

the manifold is said to be a Haantjes manifold with symmetry.

On a Haantjes manifold there are two distinguished tensor fields of type $(1, 1)$: the identity maps Id and K. They form the core of a Lenard chain. Any family of tensor fields which prolong this elementary core and enjoy the same properties of Id and K is a Lenard chain of the Haantjes manifold.

Definition 2. *(Lenard chain) A Lenard chain of lenght p on a Haantjes manifold of dimension n is a family of p vector-valued 1-forms K_j which contains $K_0 = Id$ and $K_1 = K$, and satisfies the conditions*

- $Haantjes(K_j) = 0$
- $[K_j, K_l] = 0$
- $dK_j\theta = 0$
- $\theta(Torsion(K_j)) = 0.$

This extension of the concept of Lenard chain, from the case of bihamiltonian manifolds to the case of Haantjes manifolds, is of great use, since it enlarges significantly the field of application of these chains, but it has a cost. In the case $Torsion(K) = 0$ there was a simple and general rule to construct the family of tensor fields K_j: to take the powers of K. This rule is lost on Haantjes manifolds. In the new setting, the family of tensor fields K_j must be constructed in each example according to a rule which accounts for the specific form of the torsion of K. Despite this difficulty, Lenard chains on Haantjes manifolds are still quite worth of interest. This is due to a subtle property enjoyed by them, which I will describe next. I need two preliminary remarks.

Lemma 1. *Consider two tensor fields K_1 and K_2 on a manifold M (not necessarily a Haantjes manifold), and assume that they commute, that their Haantjes tensors vanish, and that their eigenvalues α_j and β_j are real and distinct. Then any 1-form θ which annihilates the torsion of K_1 and K_2 separately, annihilates the torsion of $K_1 + K_2$ as well (or of any linear combination $\lambda_1 K_1 + \lambda_2 K_2$ with real coefficients).*

Lemma 2. *On a manifold M (not necessarily a Haantjes manifold) consider a closed 1-form θ and a tensor field K, possibly with non vanishing Haantjes tensor. Assume that θ annihilates the torsion of K, and that K maps θ into a closed 1-form. Then, the 1-form $K^2\theta$ is closed as well.*

Let make a comment on the last Lemma. By assumption the 1-forms θ and $K\theta$ are closed , as always in the theory of Lenard chains. The condition $\theta(Torsion(K)) = 0$ then entails that the 1-form $K^2\theta$ is closed as well. But if one try to go a step further, one fails. The next 1-form $K^3\theta$ is, usually, not longer closed. So the Lenard chain breaks at the third iteration , and cannot be further prolonged. This is the difference between the case of bihamiltonian manifolds with respect to the case of Haantjes manifolds. The triple of 1-forms $(\theta, K\theta, K^2\theta)$ will be referred to as the *short Lenard chain* generated by θ and K.

Proof. I will omit the proof of the second Lemma, which is a rephrasing of the concept of Nijenhuis torsion. To prove the first Lemma, I make use of the concept of canonical coordinates. It is well-known that the vanishing of the Haantjes torsion of K_1, combined with the assumption that its eigenvalues α_j are real and distinct , entails the existence of a special system of coordinates $(u_1, u_2, \ldots, u_n)$ which diagonalize K_1 [2]. In these coordinates the torsion of K_1 has the following simple form

$$T_{K_1}\left(\frac{\partial}{\partial u_j}, \frac{\partial}{\partial u_k}\right) = (\alpha_j - \alpha_k)\left(\frac{\partial \alpha_k}{\partial u_j}\frac{\partial}{\partial u_k} - \frac{\partial \alpha_j}{\partial u_k}\frac{\partial}{\partial u_j}\right).$$

Therefore, the 2-form $\theta(Torsion(K_1))$ has components

$$\theta(Torsion(K_1))_{j,k} = (\alpha_j - \alpha_k)\left(\frac{\partial \alpha_k}{\partial u_j}\theta_k - \frac{\partial \alpha_j}{\partial u_k}\theta_j\right),$$

where θ_k are the components of the 1-form θ. Since the eigenvalues of K_1 are distinct, the 1-form θ verifies the condition $\theta(Torsion(K_1)) = 0$ iff

$$\frac{\partial \alpha_k}{\partial u_j}\theta_k - \frac{\partial \alpha_j}{\partial u_k}\theta_j = 0.$$

The same condition holds for K_2:

$$\frac{\partial \beta_k}{\partial u_j}\theta_k - \frac{\partial \beta_j}{\partial u_k}\theta_j = 0.$$

Let us now compute the 2-form $\theta(Torsion(K_1 + K_2))$. It vanishes iff

$$(\beta_j - \beta_k)\left(\frac{\partial \alpha_k}{\partial u_j}\theta_k - \frac{\partial \alpha_j}{\partial u_k}\theta_j\right) + (\alpha_j - \alpha_k)\left(\frac{\partial \beta_k}{\partial u_j}\theta_k - \frac{\partial \beta_j}{\partial u_k}\theta_j\right) = 0.$$

This condition is manifestly a linear combination of the conditions verified separately by K_1 and K_2. Therefore, if θ annihilates the torsion of K_1 and K_2, and if K_1 and K_2 have distinct eigenvalues and satisfy the Haantjes condition, θ annihilates also the torsion of $K_1 + K_2$. □

The important consequence of this result comes to light when, on a Haantjes manifold endowed with a Lenard chain K_j of lenght p, one considers the second generation of iterared 1-forms

$$\theta_{jl} = K_j K_l \theta.$$

This family is referred to as the *square* associated to the Lenard chain of 1-forms.

Proposition 1. *All the forms of the square are closed . So, any Lenard chain of lenght p on a Haantjes manifold of dimension n generates a square of $1/2p(p+1)$ locally exact 1-forms.*

Proof. Consider the linear pencil $K(\lambda) = \lambda_1 K_1 + \cdots + \lambda_p K_p$ of all the tensor fields of the Lenard chain. By the first Lemma its torsion is annihilated by the 1-form θ. By the second Lemma the 1-form $K(\lambda)^2\theta$ is closed for any possible choice of the parameters of the pencil. This form depends quadratically on the parameters. Hence the coefficient of the product of λ_j and λ_k must vanish. This coefficient is the exterior differential of the form θ_{jl} of the square. Therefore, under the assumptions listed above, all the 1-forms of the square are closed. □

The case of major interest is, of course, the case when p equals the dimension of the manifold M. One can show that this case is related to $WDVV$ equations [1] and to the concept of (dispersionless) Hirota *tau*-function. The study of this case is, however, outside the scopes of this note. In this paper I will limit myself to show, in a concrete example, that Lenard chains are related to the theory of singularities of type E_6.

2. Two examples of Haantjes manifolds

In 1951 Albert Nijenhuis [3] noticed that the eigenvalues α_j of a tensor field K with vanishing torsion

$$Torsion(K) = 0$$

may be used as canonical coordinates for K in the domain where they are real, distinct, and functionally independent. Thus

$$K\frac{\partial}{\partial \alpha_j} = \alpha_j \frac{\partial}{\partial \alpha_j}$$

in this domain. If instead of the eigenvalues α_j one uses the coefficients of the characteristic polynomial

$$Q(\alpha) = \alpha^n + a_n \alpha^{n-1} + \cdots + a_1$$

as coordinates, one finds for K the cyclic representation

$$\begin{aligned} K\frac{\partial}{\partial a_1} &= \frac{\partial}{\partial a_2}, \\ K\frac{\partial}{\partial a_2} &= \frac{\partial}{\partial a_3}, \\ &\vdots \\ K\frac{\partial}{\partial a_{n-1}} &= \frac{\partial}{\partial a_n}, \\ K\frac{\partial}{\partial a_n} &= -\left(a_1\frac{\partial}{\partial a_1} + a_2\frac{\partial}{\partial a_2} + \cdots + a_n\frac{\partial}{\partial a_n}\right). \end{aligned}$$

The idea pursued in this section is to look for tensor fields K verifying the weaker condition

$$Haantjes(K) = 0$$

inside the class of tensor fields defined by the equations

$$\begin{aligned} K\frac{\partial}{\partial a_1} &= \frac{\partial}{\partial a_2}, \\ K\frac{\partial}{\partial a_2} &= \frac{\partial}{\partial a_3}, \\ &\vdots \\ K\frac{\partial}{\partial a_{n-1}} &= \frac{\partial}{\partial a_n}, \\ K\frac{\partial}{\partial a_n} &= -\left(A_1\frac{\partial}{\partial a_1} + A_2\frac{\partial}{\partial a_2} + \cdots + A_n\frac{\partial}{\partial a_n}\right), \end{aligned}$$

inspired by the example studied by Nijenhuis. This class depends on the choice of n arbitrary functions $(A_1, A_2, \cdots, A_n)$ of the cyclic coordinates $(a_1, a_2, \cdots, a_n)$. Since these functions are the coefficients of the characteristic polynomial of K

$$Q(\alpha) = \alpha^n + A_n \alpha^{n-1} + \cdots + A_1,$$

the new class may be tought of as a gentle deformation of the tensor field of Nijenhuis. The deformation process changes the characteristic polynomial of K, but keeps unchanged the cyclic coordinates. The vector field

$$A = A_1\frac{\partial}{\partial a_1} + \cdots + A_n\frac{\partial}{\partial a_n}$$

is accordingly referred to as the *characteristic vector field* of K. In the example of Nijenhuis the characteristic vector field is the Euler vector field

$$A = -\left(a_1\frac{\partial}{\partial a_1} + a_2\frac{\partial}{\partial a_2} + \cdots + a_n\frac{\partial}{\partial a_n}\right).$$

The study of the Haantjes torsion of K shows that there are other two elementary solutions of the Haantjes condition besides that found by Nijenhuis. They are selected by demanding that the components of the characteristic vector field A depend *linearly* on the cyclic coordinates. These new solutions are :

$$A = -\frac{1}{n+1}\left[a_2\frac{\partial}{\partial a_1} + 2a_3\frac{\partial}{\partial a_2} + \cdots + (n-1)a_n\frac{\partial}{\partial a_{n-1}}\right],$$
$$B = -\frac{1}{n-1}\left[-a_1\frac{\partial}{\partial a_1} + a_3\frac{\partial}{\partial a_3} + \cdots + (n-3)a_{n-1}\frac{\partial}{\partial a_{n-1}} + (n-2)a_n\frac{\partial}{\partial a_n}\right].$$

The corresponding tensor fields will be denoted by L and M respectively. They inherit several interesting properties from the tensor field of Nijenhuis. A few of these properties will be presented below.

Eigencovectors. I notice first that the 1-form

$$\eta = \lambda(da_1 + \alpha da_2 + \alpha^2 da_3 + \cdots + \alpha^{n-1}da_n)$$

is an eigencovector of K, corresponding to the eigenvalue α, for any choice of the arbitrary function λ . This property follows from the cyclic form of K, and holds for any choice of the arbitrary functions A_j.The function λ serves to normalize the eigencovector . When the Haantjes torsion of K vanishes, and when the eigenvalues α_j are real and distinct, the function λ may be chosen so that the 1-form η is exact :

$$\eta = du.$$

The function u is the canonical coordinate associated with the eigenvalue α. The construction of the canonical coordinates is usually a difficult problem. It becomes simpler when the tensor field K has symmetries. The following result is stated without proof.

Lemma 3. *Let X be a symmetry of K, and demand that*

$$\eta(X) = 1.$$

Then the 1-form η is exact.

This result is used to select the normalization factor λ.

Canonical coordinates. To find the symmetries of L and M, I look first at the symmetries of the corresponding characteristic vector fields. Since the coordinates a_1 and a_2 are cyclic for A and B respectively, the conclusion is that the vector fields

$$X = \frac{\partial}{\partial a_1}, \qquad Y = \frac{\partial}{\partial a_2}$$

are symmetries of L and M respectively. Let us use these symmetries to normalize the eigenvector forms. The normalization rule leads to

$$\lambda_L = 1 \qquad \text{and} \qquad \lambda_M = \alpha^{-1},$$

so that the normalized eigencovectors are

$$\eta_L = da_1 + \alpha da_2 + \alpha^2 da_3 + \cdots + \alpha^{n-1} da_n,$$
$$\eta_M = \alpha^{-1} da_1 + da_2 + \alpha^1 da_3 + \cdots + \alpha^{n-2} da_n.$$

The canonical coordinates follow then in two steps. A simple integration by parts allows to write immediately η as a total differential up to a multiple of the differential $d\alpha$ of the eigenvalue α. The extra term is subsequently eliminated by means of the characteristic polynomial of K. One arrives in this way to the following expressions

$$u_L = a_1 + a_2\alpha + \cdots + a_n\alpha^{n-1} + \alpha^{n+1},$$
$$u_M = a_1\alpha^{-1} + a_2 + a_3\alpha + \cdots + a_n\alpha^{n-2} + \alpha^{n-1},$$

for the canonical coordinates of L and M respectively. These coordinates are special deformations of the canonical coordinates

$$u = \alpha^n$$

of the tensor field studied by Nijenhuis. They are called miniversal deformations of type A_n and D_n respectively.

Torsion and annihilating 1-form . The tensors fields L and M have torsion, but their torsion is very mild since they are gentle deformations of the torsionless tensor field considered by Nijenhuis. For L one finds that all the components of the torsion vanish except

$$T_L\left(\frac{\partial}{\partial a_j}, \frac{\partial}{\partial a_n}\right) = \frac{\partial}{\partial a_j},$$

for $j = 1, 2, \cdots, n-1$. Similarly , for M one finds that all the components of the torsion vanish except

$$T_M\left(\frac{\partial}{\partial a_j}, \frac{\partial}{\partial a_n}\right) = \frac{\partial}{\partial a_{j+1}}$$

for $j = 1, 2, \cdots, n-1$. These outcomes make it simple to construct the 1-forms annihilating the torsion of L and M respectively. They are:

$$\theta_L = da_n, \qquad \theta_M = da_1.$$

These forms are exact and invariant along the the symmetry vector fields X and Y introduced above.

Haantjes manifolds with symmetry. To check the last condition, that is that the 1-forms $L\theta_L$ and $M\theta_M$ are exact, it is expedient to pass to represent the cyclic tensor fields K on the dual basis da_j. The equations

$$\begin{aligned} Kda_n &= da_{n-1} + A_n da_n, \\ Kda_{n-1} &= da_{n-2} + A_{n-1} da_n, \\ &\vdots \\ Kda_2 &= da_1 + A_2 da_n, \\ Kda_1 &= A_1 da_n \end{aligned}$$

hold for any tensor field of the class considered at the beginning of this section. By inserting in these equations the explicit form of the characteristic vector fields of L and M respectively, one may check that the required property holds in both cases. The conclusion is that among the deformations of the bihamiltonian manifold studied by Nijenhuis in 1951 there are two Haantjes manifolds with symmetry. The canonical coordinates of these manifolds are the miniversal deformations of singularities of type A_n and D_n respectively [4].

This is a first instance of the connection between Haantjes manifolds and the Singularity Theory. The outcome of this example may be deepened by pursuing the study of Lenard chains. The second example dealt with in the next section goes one step in this direction. It is meant to suggest that the geometrical framework of Haantjes manifolds is sufficiently wide to encompass the Frobenius structure of the orbit spaces of Coxeter groups [5].

3. An example of Lenard chain

The class of tensor fields studied in the previous section depended on a single characteristic vector field. The obvious extension is to consider now a class of tensor fields depending on two characteristic vector fields. This class is defined by the equations

$$\begin{aligned}
L\frac{\partial}{\partial a_1} &= \frac{\partial}{\partial a_2},\\
L\frac{\partial}{\partial a_2} &= \frac{\partial}{\partial a_3},\\
&\vdots\\
L\frac{\partial}{\partial a_{n-1}} &= \frac{\partial}{\partial a_n},\\
L\frac{\partial}{\partial a_n} &= A,\\
L\frac{\partial}{\partial a_{n+1}} &= \frac{\partial}{\partial a_{n+2}},\\
L\frac{\partial}{\partial a_{n+2}} &= \frac{\partial}{\partial a_{n+3}},\\
&\vdots\\
L\frac{\partial}{\partial a_{n+m-1}} &= \frac{\partial}{\partial a_{n+m}},\\
L\frac{\partial}{\partial a_{n+m}} &= B,
\end{aligned}$$

where A and B are unspecified vector fields to be properly determined afterward. The new class will be referred to as the *gluing* of two Nijenhuis operators (for a different but related notion of gluing see [6]).

In this section I sketch the study of this type of tensor fields on a manifold of low dimension. Specifically, I consider the class of tensor fields defined by the equations

$$\begin{aligned}
L\frac{\partial}{\partial a_1} &= \frac{\partial}{\partial a_2}, & L\frac{\partial}{\partial a_2} &= \frac{\partial}{\partial a_3}, & L\frac{\partial}{\partial a_3} &= A,\\
L\frac{\partial}{\partial a_4} &= \frac{\partial}{\partial a_5}, & L\frac{\partial}{\partial a_5} &= \frac{\partial}{\partial a_6}, & L\frac{\partial}{\partial a_6} &= B,
\end{aligned}$$

on a manifold of dimension six. The aim is to construct a concrete example of Lenard chain. The process is rather articulate, and I split it in seven steps for convenience.

The intertwining tensor M. In view of the study of the spectral properties of the tensor field L it is quite useful to introduce a second tensor field M, called the intertwining tensor field associated with L. It is defined as the unique tensor field which commute with L, and that intertwines the generator $\frac{\partial}{\partial a_1}$ of the first cyclic chain of L to the generator $\frac{\partial}{\partial a_4}$ of the second cyclic chain. On the basis associated with the cyclic coordinates a_j, this tensor field admits the following representation:

$$
\begin{aligned}
&M\frac{\partial}{\partial a_1} = \frac{\partial}{\partial a_4}, &\quad &M\frac{\partial}{\partial a_2} = \frac{\partial}{\partial a_5}, &\quad &M\frac{\partial}{\partial a_3} = \frac{\partial}{\partial a_6},\\
&M\frac{\partial}{\partial a_4} = C, & &M\frac{\partial}{\partial a_5} = D, & &M\frac{\partial}{\partial a_6} = E.
\end{aligned}
$$

The three vector fields (C, D, E), referred to as the characteristic vector fields of M, are well determined rational functions of the characteristic vector fields (A, B) of L. I omit to give their explicit representation here, since later on we shall have the occasion to characterize them in a more effective way.

Eigencovectors of L. Since the characteristic vector fields (A, B) determine completely the tensor fields L and M, they fix their eigenvalues α_j and β_j as well. The study of the eigenvalue problems

$$L\eta = \alpha\eta, \qquad M\eta = \beta\eta,$$

leads to two important conclusions. The first is that the eigenvalues of L and M are related to their characteristic vector fields through the equations

$$
\begin{aligned}
(A_1 + \alpha A_2 + \alpha^2 A_3) + \beta(A_4 + \alpha A_5 + \alpha^2 A_6) &= \alpha^3,\\
(B_1 + \alpha B_2 + \alpha^2 B_3) + \beta(B_4 + \alpha B_5 + \alpha^2 B_6) &= \alpha^3\beta,\\
(C_1 + \alpha C_2 + \alpha^2 C_3) + \beta(C_4 + \alpha C_5 + \alpha^2 C_6) &= \beta^2,\\
(D_1 + \alpha D_2 + \alpha^2 D_3) + \beta(D_4 + \alpha D_5 + \alpha^2 D_6) &= \alpha\beta^2,\\
(E_1 + \alpha E_2 + \alpha^2 E_3) + \beta(E_4 + \alpha E_5 + \alpha^2 E_6) &= \alpha^2\beta^2.
\end{aligned}
$$

The second is that the eigencovector η related to the generic pair of eigenvalues α and β is given by

$$\eta = \lambda(da_1 + \alpha da_2 + \alpha^2 da_3 + \beta da_4 + \alpha\beta da_5 + \alpha^2\beta da_6).$$

It is defined, as usual, up to the arbitrary scaling factor λ. Having in mind to study the Haantjes condition for the tensor fields L and M, it is worth to notice that the correspondance between characteristic vector fields and eigenvalues is bijective , at least generically in an open dense domain. This property allows to introduce a second representation of L and M in terms of their eigenvalues, conceived as arbitrarily functions of the cyclic coordinates.

Haantjes conditions on the eigenvalues. To impose the Haantjes conditions upon the tensor fields L and M amounts to choose the normalizing factor λ and the eigenvalues α and β in such a way to make the 1-form η exact. This problem is easily solved by an integration by parts. Choose $\lambda = 1$ and write η in the form

$$\eta = d(a_1 + \alpha a_2 + \alpha^2 a_3 + \beta a_4 + \alpha\beta a_5 + \alpha^2\beta a_6) - (a_2 + 2\alpha a_3 + \beta a_5 + 2\alpha\beta a_6)d\alpha - (a_4 + \alpha a_5 + \alpha^2 a_6)d\beta.$$

To make this form exact it is sufficient to impose the constraints

$$
\begin{aligned}
-(a_2 + 2\alpha a_3 + \beta a_5 + 2\alpha\beta a_6) &= \alpha^3,\\
-(a_4 + \alpha a_5 + \alpha^2 a_6) &= \beta^2.
\end{aligned}
$$

on the eigenvalues of L and M. This is the form of the Haantjes conditions on the eigenvalues.

Haantjes conditions on the characteristic vector fields. The final form of the Haantjes conditions is on the characteristic vector fields , seen as functions of the cyclic coordinate instead than as functions of the eigenvalues. This final form is attained by a process of elimination. One has to eliminate the eigenvalues α and β among the five equations which give the characteristic vector fields as functions of the eigenvalues , and the two equations which give the eigenvalues as functions of the cyclic coordinates. It is convenient to split the process of elimination in two parts. First I give the explicit representation of the characteristic vector fields A and C, which depend quite simply from the cyclic coordinates. Their form is:

- *Field* A :

$$A_1 = -a_2, \qquad A_2 = -2a_3, \qquad A_3 = 0,$$
$$A_4 = -a_5, \qquad A_5 = -2a_6, \qquad A_6 = 0;$$

- *Field* C :

$$C_1 = -a_2, \qquad C_2 = -2a_3, \qquad C_3 = 0,$$
$$C_4 = 0, \qquad C_5 = 0, \qquad C_6 = 0.$$

Then I give the expression of the components of the other characteristic vector fields as functions of the components of A and C. The expression are a little more cumbersome:

- *Field* B :

$$B_1 = A_4C_1 + A_5(0 + A_1C_3), \qquad B_2 = A_4C_2 + A_5(C_1 + A_2C_3),$$
$$B_3 = A_4C_3 + A_5(C_2 + A_3C_3), \qquad B_4 = A_1 + A_4A_5C_3,$$
$$B_5 = A_2 + A_5A_5C_3, \qquad B_6 = A_3;$$

- *Field* D :

$$D_1 = 0 + A_1C_3, \qquad D_2 = C_1 + A_2C_3,$$
$$D_3 = C_2 + A_3C_3, \qquad D_4 = A_4C_3,$$
$$D_5 = A_5C_3, \qquad D_6 = A_6C_3;$$

- *Field* E :

$$E_1 = A_1C_2 + C_3(0 + A_1A_3), \qquad E_2 = A_2C_2 + C_3(A_1 + A_2A_3),$$
$$E_3 = A_3C_2 + C_3(A_2 + A_3A_3) + C_1, \qquad E_4 = A_4C_2 + C_3(A_4 + 0),$$
$$E_5 = A_5C_2 + C_3(A_4 + A_5), \qquad E_6 = A_6C_2 + C_3(A_5 + 0).$$

Canonical coordinates. At this point we dispose of all the informations concerning the tensor fields L and M. This allows to compute, for instance, their canonical coordinates. From the final form of the eigencovector η one may see that the canonical coordinate associated with the pair of eigenvalues α and β of L and M is

$$u = a_1 + \alpha a_2 + \alpha^2 a_3 + \beta a_4 + \alpha\beta a_5 + \alpha^2\beta a_6 + 1/4\alpha^4 + 1/3\beta^3.$$

This formula shows that the tensor field L is related to the theory of singularities of type E_6 [4].

The Haantjes manifold. To complete the construction of the Haantjes manifold it remains to find a 1-form θ which is exact and annihilates the torsion of L, and such that the iterated 1-form $L\theta$ be furthermore a second exact 1-form. A look at the action of L on the dual basis da_j provides quickly this form. Indeed the 1-form

$$\theta = da_6$$

verifies all the required conditions. So the pair (L, θ) defines a Haantjes manifold of dimension six.

The Lenard chain. Let us set, as usual, $K_0 = Id$ and $K_1 = L$. To construct a Lenard chain one has to find, inside the ring of tensor fields commuting with L and M, other four tensor fields (K_2, K_3, K_4, K_5) satisfying the same conditions verified by K_0 and K_1. One looks for polynomials in L and M, and one selects these polynomials by imposing the conditions $dK\theta = 0$ and $dK^2\theta = 0$ characteristic of the theory of short Lenard chains. I omit the details of the computation, giving only the final result. The Haantjes manifold defined by the pair (L, θ) is endowed with a Lenard chain formed by the tensor fields:

$$\begin{aligned}
&K_0 = Id,\\
&K_1 = L,\\
&K_2 = M - 1/24a_6^2 Id,\\
&K_3 = L^2 + 1/4a_6 M + (1/4a_3 + 1/144a_6^3)Id,\\
&K_4 = LM + 1/12a_6^2 L - 1/12a_5a_6 Id,\\
&K_5 = L^2M - 1/8a_6^2L^2 + (1/4a_3 + 1/288a_6^3)M - 1/2a_5a_6L+\\
&\qquad\qquad + (-1/12a_4a_6 + 1/96a_3a_6^2 - 1/24a_5^2 + 1/3456a_6^5)\, Id.
\end{aligned}$$

It is a special Lenard chain, since it possesses an additional symmetry. Indeed the tensor fields K_j and the 1-form θ are invariant along the vector field $\frac{\partial}{\partial a_1}$ which served as starting point of the present construction. It is therefore a Lenard chain with symmetry. This property has far reaching consequences, but their study go outside the limits of the present paper.

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