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PLÜCKER FORMS AND THE THETA MAP

By SONIA BRIVIO and ALESSANDRO VERRA

Abstract. Let $\mathrm{SU}_X(r,0)$ be the moduli space of semistable vector bundles of rank r and trivial determinant over a smooth, irreducible, complex projective curve X. The theta map $\theta_r : \mathrm{SU}_X(r,0) \to \mathbb{P}^N$ is the rational map defined by the ample generator of $\mathrm{Pic}\,\mathrm{SU}_X(r,0)$. The main result of the paper is that θ_r is generically injective if $g \gg r$ and X is general. This partially answers the following conjecture proposed by Beauville: θ_r is generically injective if X is not hyperelliptic. The proof relies on the study of the injectivity of the determinant map $d_E : \wedge^r H^0(E) \to H^0(\det E)$, for a vector bundle E on X, and on the reconstruction of the Grassmannian $G(r,rm)$ from a natural multilinear form associated to it, defined in the paper as the Plücker form. The method applies to other moduli spaces of vector bundles on a projective variety X.

1. Introduction. In this paper we introduce the elementary notion of Plücker form of a pair (E,S), where E is a vector bundle of rank r on a smooth, irreducible, complex projective variety X and $S \subset H^0(E)$ is a subspace of dimension rm. Then we apply this notion to the study of the moduli space $\mathrm{SU}_X(r,0)$ of semistable vector bundles of rank r and trivial determinant on a curve X. Let

$$\theta_r : \mathrm{SU}_X(r,0) \longrightarrow \mathbb{P}\big(H^0(\mathcal{L})^*\big)$$

be the so called theta map, defined by the ample generator $\mathcal{L}$ of $\mathrm{Pic}\,\mathrm{SU}_X(r,0)$, [DN]. Assume X has genus g, we prove the following main result:

MAIN THEOREM. *θ_r is generically injective if X is general and $g \gg r$.*

The theorem gives a partial answer to the following conjecture, or optimistic speculation, proposed by Beauville in [B3] 6.1:

SPECULATION. *θ_r is generically injective if X is not hyperelliptic.*

To put in perspective our result we briefly recall some open problems on θ_r and some known results, see [B3]. A serious difficulty in the study of θ_r is represented by its indeterminacy locus, which is quite unknown. Raynaud bundles and few more constructions provide examples of points in this locus when $r \gg 0$, cf. [CGT, R]. In particular, there exists an integer $r(X) > 0$ such that θ_r is not a morphism as soon as $r > r(X)$. As a matter of fact related to this situation, some basic questions are still unsolved. For instance:

- *is θ_r generically finite onto its image for any curve X?*

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- *is θ_r an embedding if r is very low and X is general?*
- *compute $r(g) := \min\{r(X),\ X$ curve of genus $g\}$.*

On the side of known results only the case $r = 2$ is well understood: θ_2 is an embedding unless X is hyperelliptic of genus $g \geq 3$, see [B1, BV1, vGI]. Otherwise θ_2 is a finite 2:1 cover of its image, [DR]. For $r = 3$ it is conjectured that θ_3 is a morphism and this is proved for $g \leq 3$, see [B3, 6.2] and [B2]. To complete the picture of known results we have to mention the case of genus two. In this case θ_r is generically finite, see [B2, BV2]. Moreover it is a morphism iff $r \leq 3$, [Pa].

To prove our main theorem we apply a more general method, working in principle for more moduli spaces of vector bundles over a variety X of arbitrary dimension. Let us briefly describe it.

Assume X is embedded in $\mathbb{P}^n$ and consider a pair (E,S) such that: (i) E is a vector bundle of rank r on X, (ii) S is a subspace of dimension rm of $H^0(E)$, (iii) $\det E \cong \mathcal{O}_X(1)$. Under suitable stability conditions there exists a coarse moduli space $\mathcal{S}$ for (E,S), see for instance [L] for an account of this theory. Let $p_i : X^m \to X$ be the ith projection and let

$$e_{S,E} : S \otimes \mathcal{O}_{X^m} \longrightarrow \bigoplus_{i=1,\dots,m} {p_i}^* E$$

be the natural map induced by evaluating global sections. We will assume that $e_{E,S}$ is generically an isomorphism for general pairs (E,S). For such a pair the degeneracy scheme $\mathbb{D}_{E,S}$ of $e_{E,S}$ is a divisor in X^m, moreover

$$\mathbb{D}_{E,S} \in \left|\mathcal{O}_{X^m}(1,\dots,1)\right|,$$

where $\mathcal{O}_{X^m}(1,\dots,1) := p_1^*\mathcal{O}_X(1) \otimes \cdots \otimes p_m^*\mathcal{O}_X(1)$. In this paper $\mathbb{D}_{E,S}$ is defined as *the Plücker form of* (E,S). The construction of the Plücker form of (E,S) defines a rational map

$$\theta_{r,m} : \mathcal{S} \longrightarrow \left|\mathcal{O}_{X^m}(1,\dots,1)\right|,$$

sending the moduli point of (E,S) to $\mathbb{D}_{E,S}$. Assume $X = \mathbb{G}$, where $\mathbb{G}$ is the Plücker embedding of the Grassmannian $G(r,rm)$. Then consider the pair $(\mathcal{U}^*, H)$, where $\mathcal{U}$ is the universal bundle of $\mathbb{G}$ and $H = H^0(\mathcal{U}^*)$. In this case the Plücker form of $(\mathcal{U}^*, H)$ is the zero locus

$$\mathbb{D}_{\mathbb{G}} \in \left|\mathcal{O}_{\mathbb{G}^m}(1,\dots,1)\right|$$

of a natural multilinear form related to $\mathbb{G}$. More precisely $\mathbb{G}$ is embedded in $\mathbb{P}(\wedge^r V)$, where $V = H^*$, and $\mathbb{D}_{\mathbb{G}}$ is the zero locus of the map

$$d_{r,m} : \left(\wedge^r V\right)^m \longrightarrow \wedge^{rm} V \cong \mathbf{C},$$

induced by the wedge product. In the first part of the paper we prove that $\mathbb{G}$ is uniquely reconstructed from $\mathbb{D}_{\mathbb{G}}$ as soon as $m \geq 3$. We prove the following:

THEOREM. *Let $m \geq 3$ and let $x \in \mathbb{P}(\wedge^r V)$, then $x \in \mathbb{G}$ iff the following conditions hold true:*

(1) $(x,\dots,x) \in (\mathbb{P}(\wedge^r V))^m$ *is a point of multiplicity* $m-1$ *for* $\mathbb{D}_{\mathbb{G}}$,

(2) $\mathrm{Sing}_{m-1}(\mathbb{D}_{\mathbb{G}})$ *has tangent space of maximal dimension at* $(x,\dots,x)$.

It follows essentially from this result that the previous map $\theta_{r,m}$ is generically injective, provided some suitable conditions are satisfied.

Indeed let (E,S) be a pair as above and let $g_{E,S}: X \to \mathbb{G}_{E,S}$ be the classifying map in the Grassmannian $\mathbb{G}_{E,S}$ of r dimensional subspaces of S^*. In Section 4 we use the previous theorem to prove the following:

THEOREM. *$\theta_{r,m}$ is generically injective under the following assumptions:*

(1) $\mathrm{Aut}(X)$ *is trivial and* $m \geq 3$,

(2) $g_{E,S}$ *is a morphism birational onto its image,*

(3) *the determinant map* $d_{E,S}: \wedge^r S \to H^0(\mathcal{O}_X(1))$ *is injective.*

However the main emphasis of this paper is on the case where $X \subset \mathbb{P}^n$ is a general curve of genus g and $\mathcal{O}_X(1)$ has degree $r(m+g-1)$. Assuming this, we consider the moduli space $\mathcal{S}_r$ of pairs $(E, H^0(E))$, where E is a stable vector bundle of determinant $\mathcal{O}_X(1)$ and $h^1(E) = 0$. Let t be an r-root of $\mathcal{O}_X(1)$, then $\mathcal{S}_r$ is birational to $\mathrm{SU}_X(r,0)$ via the map

$$\alpha : \mathcal{S}_r \longrightarrow \mathrm{SU}_X(r,0),$$

sending the moduli point of $(E, H^0(E))$ to the moduli point of $E(-t)$. In the second half of the paper we prove that

$$\theta_{r,m} \circ \alpha^{-1} = \beta \circ \theta_r,$$

where θ_r is the theta map of $\mathrm{SU}_X(r,0)$ and β is a rational map. Moreover we prove that the assumptions of the latter theorem are satisfied if X is general of genus $g \gg r$. *Then it follows that θ_r is generically injective as soon as X is general of genus $g \gg r$.*

This completes the description of the proof of the main theorem of this paper. It seems interesting to use Plücker forms for further applications.

2. Plücker forms. Let V be a complex vector space of positive dimension rm and let $\wedge^r V$ be the r-exterior power of V. On $\wedge^r V$ we consider the multilinear form

$$d_{r,m} : (\wedge^r V)^m \longrightarrow \wedge^{rm} V \simeq \mathbb{C}, \tag{1}$$

such that

$$d_{r,m}(w_1,\dots,w_m) := w_1 \wedge \cdots \wedge w_m.$$

Notice that $d_{r,m}$ is symmetric if r is even and skew symmetric if r is odd. We fix m copies $V_1,\dots,V_m$ of V and the spaces $\mathbb{P}_s := \mathbb{P}(\wedge^r V_s)$, $s=1,\dots,m$, of dimension $N := \binom{rm}{r} - 1$. Then we consider the Segre embedding

$$\mathbb{P}_1 \times \cdots \times \mathbb{P}_m \hookrightarrow \mathbb{P}^{(N+1)m-1}$$

and its projections $\pi_s : \mathbb{P}_1 \times \cdots \times \mathbb{P}_m \to \mathbb{P}_s$, $s=1,\dots,m$. The form $d_{r,m}$ defines the following hyperplane section of $\mathbb{P}_1 \times \cdots \times \mathbb{P}_m$:

$$\text{(2)} \qquad \mathbb{D}_{r,m} := \big\{ \big(w_1,\dots,w_m\big) \in \mathbb{P}_1 \times \cdots \times \mathbb{P}_m \mid d_{r,m}\big(w_1,\dots,w_m\big) = 0 \big\}.$$

Definition 2.1. $\mathbb{D}_{r,m}$ is the Plücker form of $\mathbb{P}(\wedge^r V)^m$.

$\mathbb{D}_{r,m}$ is an element of the linear system $|\mathcal{O}_{\mathbb{P}_1\times\cdots\times\mathbb{P}_m}(1,\dots,1)|$, where

$$\mathcal{O}_{\mathbb{P}_1\times\cdots\times\mathbb{P}_m}(1,\dots,1) = \pi_1^* O_{\mathbb{P}_1}(1) \otimes \ldots \otimes \pi_m^* O_{\mathbb{P}_m}(1).$$

Let $e_1,\dots,e_{rm}$ be a basis of V and let $\mathcal{I}$ be the set of all naturally ordered sets $I := i_1 < \cdots < i_r$ of integers in $[1, rm]$. We fix in $\wedge^r V_s$ the basis

$$e_I^{(s)} := e_{i_1} \wedge \cdots \wedge e_{i_r}, \;\; I = i_1 < \cdots < i_r \in \mathcal{I}.$$

Then any vector of $\wedge^r V_s$ is of the form $\sum p_I^{(s)} e_I^{(s)}$, where the coefficients $p_I^{(s)}$ are the standard Plücker coordinates on $\mathbb{P}_s$. This implies that

$$d_{r,m}(w_1,\dots,w_m) = \sum_{I_1\cup\cdots\cup I_m = \{1,\dots,rm\}} p_{I_1}^{(1)} \cdots p_{I_m}^{(m)} e_{I_1}^{(1)} \wedge \cdots \wedge e_{I_m}^{(m)}$$

for each $(w_1,\dots,w_m) \in (\wedge^r V)^m$. Note that, to give a decomposition

$$I_1 \cup \cdots \cup I_m = \{1,\dots,rm\}$$

as above, is equivalent to give a permutation $\sigma : \{1,\dots,rm\} \to \{1,\dots,rm\}$ which is strictly increasing on each of the intervals

$$\mathbb{U}_1 := [1,r],\; \mathbb{U}_2 := [r+1, 2r],\; \ldots,\; \mathbb{U}_m := \big[(m-1)r+1, mr\big].$$

Let $\mathcal{P}$ be the set of these permutations, then we conclude that

$$d_{r,m}\big(w_1,\dots,w_m\big) = \sum_{\sigma\in\mathcal{P}} \operatorname{sgn}(\sigma) p^{(1)}_{\sigma(\mathbb{U}_1)} \cdots p^{(m)}_{\sigma(\mathbb{U}_m)} e_1 \wedge \cdots \wedge e_{rm}.$$

Assume that $w := (w_1,\dots,w_m) \in (\wedge^r V)^m$ is a vector defining the point $o \in \mathbb{P}_1 \times \cdots \times \mathbb{P}_m$, we want to compute the Taylor series of $\mathbb{D}_{r,m}$ at o. Let $t := (t_1,\dots,t_m) \in (\wedge^r V)^m$, then we have the identity

$$d_{r,m}\big(w_1 + \epsilon t_1, \dots, w_m + \epsilon t_m\big) = \sum_{k=0,\dots,m} \partial_w^{m-k} d_{r,m}(t) \epsilon^k.$$

We will say that the function

$$\partial_w^{m-k} d_{r,m} : (\wedge^r V)^m \longrightarrow \mathbf{C},$$

sending t to the coefficient $\partial_w^{m-k} d_{r,m}(t)$ of ϵ^k, is the *kth polar of* $d_{r,m}$ *at* w, cf. [D]. Let $S := s_1 < \cdots < s_k$ be a strictly increasing sequence of k elements of $M := \{1, \ldots, m\}$. We will put $k := |S|$. Moreover, for $w = (w_1, \ldots, w_m) \in (\wedge^r V)^m$, we define $w_S := w_{s_1} \wedge \cdots \wedge w_{s_k}$. Note that $\partial_w^0(t) = d(w_1, \ldots, w_m)$ for each t. If $m - k \geq 1$ it turns out that

$$\partial_w^{m-k} d_{r,m}(t) = \sum_{|S|=k} \operatorname{sgn}(\sigma_S) w_{M-S} \wedge t_S, \tag{3}$$

where $\sigma_S : M \to M$ is the permutation $(1, \ldots, m) \to (j_1, \ldots, j_{m-k}, s_1, \ldots, s_k)$ such that $S = s_1 < \cdots < s_k$ and $j_1 < \cdots < j_{m-k}$.

Definition 2.2. Let $W := \wedge^r V$ then

$$q : \mathbb{P}(W^m) \longrightarrow \mathbb{P}_1 \times \cdots \times \mathbb{P}_m$$

is the rational map sending the point defined by the vector $w = (w_1, \ldots, w_m)$ of W^m to the m-tuple of points defined by the vectors $w_1, \ldots, w_m$.

Note that the pull-back of $d_{r,m}$ by q is a homogeneous polynomial

$$q^* d_{r,m} \in \operatorname{Sym}^m W^* = H^0\big(\mathcal{O}_{\mathbb{P}(W)}(m)\big).$$

We mention, without its non difficult proof, the following result

PROPOSITION 2.3. $\partial_w^{m-k}(d_{r,m})$ *is the kth polar form at* w *of* $q^* d_{r,m}$.

Let $\hat{o} \in \mathbb{P}(W^m)$ be the point defined by $w = (w_1, \ldots, w_m)$ and let $o = q(\hat{o})$. For the tangent spaces to $\mathbb{P}(W^m)$ at $\hat{o}$ and to $\mathbb{P}_1 \times \cdots \times \mathbb{P}_m$ at o one has

- $T_{\mathbb{P}(W^m), \hat{o}} = W^m / \langle w \rangle$
- $T_{\mathbb{P}_1 \times \cdots \times \mathbb{P}_m, o} = W/\langle w_1 \rangle \oplus \cdots \oplus W/\langle w_m \rangle$.

Moreover the tangent map

$$dq_{\hat{o}} : W^m / \langle w \rangle \longrightarrow W/\langle w_1 \rangle \oplus \cdots \oplus W/\langle w_m \rangle$$

is exactly the map sending

$$(t_1, \ldots, t_m) \bmod \langle w \rangle \longrightarrow (t_1 \bmod \langle w_1 \rangle, \ldots, t_m \bmod \langle w_m \rangle).$$

In particular we have

$$\operatorname{Ker} dq_{\hat{o}} = \{(c_1 w_1, \ldots, c_m w_m), (c_1, \ldots, c_m) \in \mathbb{C}^m\} / \langle w \rangle.$$

We can now use $dq_{\hat{o}}$ to study some properties of $\operatorname{Sing}(\mathbb{D}_{r,m})$. We consider the k-osculating tangent cone $\mathcal{C}_o^k \subset T_{\mathbb{P}_1 \times \cdots \times \mathbb{P}_m, o}$ to $\mathbb{D}_{r,m}$ at o.

LEMMA 2.4. *Keeping the above notations one has:*
(1) $\mathrm{Sing}_k(\mathbb{D}_{r,m}) = \{o \in \mathbb{D}_{r,m} \mid \partial_w^{m-i}(d_{r,m}) = 0,\ i \le k-1\}$.
(2) $\mathcal{C}_o^k = dq_{\hat{o}}(\{t \in W^m \bmod \langle w \rangle \mid \partial_w^{m-i}(d_{r,m})(t) = 0,\ i \le k\})$.

Proof. By the previous description of $dq_{\hat{o}}$ any one dimensional subspace l of $T_{\mathbb{P}_1 \times \cdots \times \mathbb{P}_m, o}$ is the isomorphic image by $dq_{\hat{o}}$ of the tangent space at $\hat{o}$ to an affine line

$$L_t := \{w + \epsilon t \mid \epsilon \in \mathbb{C}\} \subset \mathbb{P}(W^m),$$

for some $t = (t_1, \ldots, t_m) \in W^m$. On the other hand the pull-back of the Taylor series of $\mathbb{D}_{r,m}$ to L_t is

$$d_{r,m}(w + \epsilon t) = \sum_{i=0,\ldots,m} \partial_w^{m-i}(d_{r,m})(t)\epsilon^i,$$

this implies (1) and (2). □

Let $o \in \mathbb{P}_1 \times \cdots \times \mathbb{P}_m$ be the point defined by the vector $(w_1, \ldots, w_m)$ and let $v \in T_{\mathbb{P}_1 \times \cdots \times \mathbb{P}_m, o}$ be a tangent vector to an arc of curve

$$\{w_1 + \epsilon t_1, \ldots, w_m + \epsilon t_m,\ \epsilon \in \mathbf{C}\}.$$

Applying the lemma and the equality (3), it follows:

THEOREM 2.5. (i) $o \in \mathrm{Sing}_k(\mathbb{D}_{r,m}) \Leftrightarrow w_S = 0$, $\forall S \in \mathcal{I}$, $|S| = m-k+1$.
(ii) v *is tangent to* $\mathrm{Sing}_k(\mathbb{D}_{r,m})$ *at* o *iff*

$$\sum_{s \in S} \mathrm{sgn}(\sigma_s) w_{S-\{s\}} \wedge t_s = 0,\ \forall S \in \mathcal{I},\ |S| = m-k+1,$$

where σ_s *is the permutation of* S *shifting* s *to the bottom and keeping the natural order in* $S - s$.

Proof. (i) By Lemma 2.4(1), $o \in \mathrm{Sing}_k(\mathbb{D}_{r,m})$ iff the ith polar $\partial_w^i(d_{r,m})$ is zero for $i \le k-1$. This is equivalent to $w_S = 0$ for $\mid S \mid = m-k+1$. (ii) As above, consider a tangent vector v at o to the arc of curve $\{w_1 + \epsilon t_1, \ldots, w_m + \epsilon t_m,\ \epsilon \in \mathbf{C}\}$. By Lemma 2.4(2), v is tangent to $\mathrm{Sing}_k(\mathbb{D}_{r,m})$ at o iff the coefficient of ϵ in $(w + \epsilon t)_S$ is zero, $\forall |S| = m-k+1$. This is equivalent to the condition $\sum_{s \in S} \mathrm{sgn}(\sigma_s) w_{S-\{s\}} \wedge t_s = 0$, $\forall S \in \mathcal{I}$, $|S| = m-k+1$. □

COROLLARY 2.6. *The Plücker form* $\mathbb{D}_{r,m}$ *has no point of multiplicity* $\ge m$.

Proof. Assume $\mathbb{D}_{r,m}$ has multiplicity $\ge m$ at o. Then $w_S = 0$, $\forall S$ with $|S| = 1$. This means $w_1 = \cdots = w_m = 0$, which is impossible. □

We are especially interested to the behavior of $\mathbb{D}_{r,m}$ along its intersection with the diagonal

$$\Delta \subset \mathbb{P}_1 \times \cdots \times \mathbb{P}_m \subset \mathbb{P}^{(N+1)m-1}. \tag{4}$$

We recall that Δ spans the projectivized space of the symmetric tensors of $(\wedge^r V)^{\otimes m}$. Moreover, Δ is the m-Veronese embedding of $\mathbb{P}(\wedge^r V)$. If r is odd $d_{r,m}$ is skew symmetric and $\mathbb{D}_{r,m}$ contains Δ. If r is even then

$$\mathbb{D}_{r,m} \cdot \Delta$$

is an interesting hypersurface of degree m in the projective space Δ.

Applying Theorem 2.5 to a point o in the diagonal, we have:

COROLLARY 2.7. *Let* $o \in \Delta$. *Then:*
(i) $o \in \mathrm{Sing}_k(\mathbb{D}_{r,m}) \Leftrightarrow w^{\wedge m-k+1} = 0$;
(ii) $v \in T_{\mathrm{Sing}_k(\mathbb{D}_{r,m}),o}$ *if and only if*

$$\sum_{j\in S} \mathrm{sgn}(\sigma_s) w^{\wedge(m-k)} \wedge t_j = 0, \quad \forall S \in \mathcal{I},\ |S| = m-k+1.$$

Remark 2.8. Let $o \in \Delta$ be as above, it follows from the corollary that:

$$o \in \Delta \cap \mathrm{Sing}_{m-1}(\mathbb{D}_{r,m}) \Longleftrightarrow w \wedge w = 0.$$

It is easy to see that $\Delta \subset \mathrm{Sing}_{m-1}(\mathbb{D}_{r,m})$ if r is odd. Let r be even then

$$\mathbb{G} \subset \Delta \cap \mathrm{Sing}_{m-1}\big(\mathbb{D}_{r,m}\big),$$

where $\mathbb{G}$ is the Plücker embedding in $\Delta = \mathbb{P}(\wedge^r V)$ of the Grassmannian $G(r,V)$. However it is not true that the equality holds in the latter case. In fact the equation $w \wedge w = 0$ defines $\mathbb{G}$ if and only if $r = 2$, see [Ha2].

3. Plücker forms and Grassmannians. In this section we will keep the notation $\mathbb{G}$ for the Plücker embedding of $G(r,V)$. Our purpose is now to show that $\mathbb{G}$ is uniquely reconstructed from $\mathbb{D}_{r,m}$ and the diagonal Δ. More precisely we will show the following:

THEOREM 3.1. *Let* $m \geq 3$, *then*

$$\mathbb{G} = \left\{o \in \Delta \cap \mathrm{Sing}_{m-1}\left(\mathbb{D}_{r,m}\right) \mid \dim T_{\mathrm{Sing}_{m-1}(\mathbb{D}_{r,m}),o} \text{ is maximal}\right\}.$$

For the proof we need some preparation. The following result of linear algebra will be useful: let E be a vector space of dimension d and let $w \in \wedge^r E$ be a non zero vector. Consider the linear map

$$\mu_w^s : \wedge^s E \longrightarrow \wedge^{r+s} E$$

sending t to $w \wedge t$. We have:

PROPOSITION 3.2. *Let $d-2r \geq s$, then μ_w^s has rank $\geq \binom{d-r}{s}$ and the equality holds if and only if the vector w is decomposable.*

Proof. We fix, with the previous notations, a basis $\{e_1,\dots,e_d\}$ of E and the corresponding basis $\{e_I, I = i_1 < \cdots < i_r\}$ of $\wedge^r E$. Let $e_{I_0} := e_1 \wedge \cdots \wedge e_r$ so that $I_0 = 1 < 2 < \cdots < r$. Since w is non zero we can assume that $w = e_{I_0} + \sum_{I \neq I_0} a_I e_I$. Let W^-, W^+ be the subspaces of E respectively generated by $\{e_1,\dots,e_r\}$ and $\{e_{r+1},\dots,e_d\}$. Then we have the direct sum decomposition

$$\wedge^{r+s} E = E^+ \oplus E^-,$$

where E^+ and E^- are defined as follows:

$$E^+ = \left\{ e_{I_0} \wedge u,\ u \in \wedge^s W^+ \right\} \quad \text{and} \quad E^- = \left\{ \sum_{i=1,\dots,r} e_i \wedge v_i,\ v_i \in \wedge^{r+s-1} E \right\}.$$

Let $p^+ : \wedge^{r+s} E \to E^+$ be the projection map. Since $w = e_{I_0} + \sum_{I \neq I_0} a_I e_I$, the map

$$(p^+ \circ \mu_w^s)_{|_{\wedge^s W^+}} : \wedge^s W^+ \longrightarrow E^+$$

is just the map $u \to e_{I_0} \wedge u$, in particular it is an isomorphism. This implies that

$$\operatorname{rank} \mu_w^s \geq \operatorname{rank}\left(p_r \circ \mu_w^s\right) = \dim \wedge^s W^+ = \binom{d-s}{r}.$$

Let w be *decomposable*, then there is no restriction to assume $w = e_{I_o}$ and it follows $\dim \operatorname{Im} \mu_w^s = \binom{d-r}{s}$. Now let us assume that w is *not decomposable*. To complete the proof it suffices to show that, in this case,

$$\dim \operatorname{Im} \mu_w^s > \binom{d-s}{r}. \tag{5}$$

By the above remarks μ_w^s is injective on $\wedge^s W^+$. Hence the inequality (5) holds iff

$$\mu_w^s\left(\wedge^s W^+\right) \neq \operatorname{Im} \mu_w^s. \tag{6}$$

On the other hand $p^+ \circ \mu_w^s : \wedge^r W^+ \to E^+$ is an isomorphism and $\dim \wedge^s W^+ = \binom{d-r}{s}$. Therefore inequality (6) is satisfied iff there exists a vector $\tau \in \wedge^{r+s} E$ such that

$$0 \neq \tau \in \operatorname{Im} \mu_w^s \cap \operatorname{Ker} p^+. \tag{7}$$

So, to complete the proof, it remains to show the following:

CLAIM. *Let $d-2r \geq s$ and w be not decomposable. Then there exists a vector τ as above.*

Proof. By induction on s. If $s=1$ we have $\dim \operatorname{Im} \mu_w^1 \geq d-r$. It is proved in [G] Prop. 6.27, that the strict inequality holds iff w is not decomposable. Hence we have $\dim \operatorname{Im} \mu_w^1 > d-r$ and there exists a non zero $\tau \in \operatorname{Im} \mu_w^1 \cap \operatorname{Ker} p^+$.

Now assume that $\tau \in \operatorname{Im} \mu_w^{s-1}$ is a non zero vector satisfying the induction hypothesis. Let $N = \{v \in E \mid \tau \wedge v = 0\}$. Then N is the Kernel of the map $\mu_\tau^1 : E \to \wedge^{r+s-1} E$ and, by the first part of the proof, $\dim N \leq r+s-1$. Since we are assuming $s+r \leq d-r$, it follows that we can find a vector $e_k \in \{e_1, \dots, e_d\}$ such that

$$e_k \wedge e_1 \wedge \cdots \wedge e_r \neq 0 \quad \text{and} \quad e_k \notin N. \tag{8}$$

Then for such a vector we have

$$0 \neq e_k \wedge \tau = \sum b_J e_k \wedge e_J, \quad |J| = r+s-1,\ I_0 \not\subset \{J \cup k\}$$

and, moreover, $e_k \wedge \tau \in \operatorname{Im} \mu_w^s$. Hence the claim follows. □

From now on we will assume $m \geq 3$. Moreover we identify $\wedge^r V$ to its image via the diagonal embedding

$$\delta : \wedge^r V \longrightarrow (\wedge^r V)^m,$$

sending w to $\delta(w) := (w, \dots, w)$. Let $o \in \Delta$ be the point defined by $w = (w, \dots, w)$. From Corollary 2.7(i), we have that

$$\Delta \cap \operatorname{Sing}_{m-1}(\mathbb{D}_{r,m}) = \{o \in \Delta \mid w \wedge w = 0\}.$$

Moreover let $(t_1, \dots, t_m) \in (\wedge^r V)^m$, and let v be a tangent vector at o to

$$\{(w+\epsilon t_1, \dots, w+\epsilon t_m),\ \epsilon \in \mathbf{C}\} \subset \mathbb{P}_1 \times \cdots \times \mathbb{P}_m,$$

it follows from Corollary 2.7 that v is tangent to $\operatorname{Sing}_{m-1}(\mathbb{D}_{r,m})$ at o iff

$$w \wedge t_j + t_i \wedge w = 0, \quad 1 \leq i < j \leq m,$$

in the vector space $\wedge^{2r} V$. Let

$$\vartheta : (\wedge^r V)^m \longrightarrow (\wedge^r V / \langle w \rangle)^m$$

be the natural quotient map, where $(\wedge^r V/\langle w \rangle)^m = T_{\mathbb{P}_1 \times \cdots \times \mathbb{P}_m, o}$. Consider

$$T_o := \{(t_1, \dots, t_m) \in (\wedge^r V)^m \mid w \wedge t_j + t_i \wedge w = 0,\ 1 \leq i < j \leq m\}$$

and note that, by the latter remark, one has

$$\vartheta^{-1}\big(T_{\operatorname{Sing}_{m-1}(\mathbb{D}_{r,m}), o}\big) = T_o.$$

For any point $o \in \Delta \cap \mathrm{Sing}_{m-1}(\mathbb{D}_{r,m})$ we define

$$c_o = \text{codimension of } T_{\mathrm{Sing}_{m-1}(\mathbb{D}_{r,m}),o} \text{ in } T_{\mathbb{P}_1\times\cdots\times\mathbb{P}_m,o}, \tag{9}$$

Since ϑ is surjective, it is clear that c_o is the codimension of T_o in $(\wedge^r V)^m$.

LEMMA 3.3. *Let c_o be as above and let $B := \binom{(m-1)r}{r}$, then*
(i) *$c_o \geq mB$ if r is even and $m \geq 3$,*
(ii) *$c_o \geq (m-1)B$ if r is odd and $m \geq 3$,*
(iii) *$c_o = m-1$ if $m \leq 2$.*
Moreover the equality holds in (i) and (ii) iff w is a decomposable vector.

Proof. Let $w^{\perp} \subset \wedge^r V$ be the orthogonal space of $w = (w, \dots, w)$ with respect to the bilinear form

$$\wedge : \wedge^r V \times \wedge^r V \longrightarrow \wedge^{2r} V.$$

Moreover let $N \subset (\wedge^r V)^m$ be the subspace defined by the equations

$$(-1)^r t_i + t_j = 0,\ 1 \leq i < j \leq m.$$

It is easy to check that

$$T_o = N + \left(w^{\perp}\right)^m.$$

Let $m \geq 3$ then N is the diagonal subspace if r is odd and $N = (0)$ if r is even. By Proposition 3.2, we have that $\operatorname{codim} w^{\perp} \geq B$ and moreover the equality holds iff w is a decomposable vector. This implies (i), (ii) and the latter statement. Let $m \leq 2$ then N is either the diagonal subspace or the space of pairs $(t, -t)$, $t \in \wedge^r V$. Arguing as above it follows that $c_o = (m-1)B$, i.e. $c_o = m-1$. This completes the proof. □

Proof of Theorem 3.1. The proof is now immediate: let $o \in \Delta \cap \mathrm{Sing}_{m-1}(\mathbb{D}_{r,m})$. It is obvious that the codimension c_o is minimal iff $\dim T_{\mathrm{Sing}_{m-1}(\mathbb{D}_{r,m}),o}$ is maximal. Assume $m \geq 3$, by Lemma 3.3 c_o is minimal iff $o \in \mathbb{G}$. □

Keeping our usual notations we have

$$\mathbb{G}^m \subset \mathbb{P}_1 \times \cdots \times \mathbb{P}_m \subset \mathbb{P}^{(N+1)^m - 1},$$

where the latter inclusion is the Segre embedding and $\mathbb{G}$ is the previous Plücker embedding. The restriction of $\mathbb{D}_{r,m}$ to $\mathbb{G}^m$ has a geometric interpretation given in the next lemma.

Let $o = (w_1, \dots, w_m) \in \mathbb{G}^m$. Then we have $w_s := v_1^{(s)} \wedge \cdots \wedge v_r^{(s)}$, where $v_1^{(s)}, \dots, v_r^{(s)} \in V_s$ and $s = 1, \dots, m$. In particular w_s is a decomposable vector, so it defines a point l_s in $\mathbb{G}$. The vector space corresponding to l_s is generated by the basis $v_1^s, \dots, v_r^s$. We will denote its projectivization by L_s.

LEMMA 3.4. *The following conditions are equivalent:*
(i) $o \in \mathbb{D}_{r,m}$,
(ii) $w_1 \wedge \cdots \wedge w_m = 0$,
(iii) $\{v_i^j\}$, $1 \le i \le r$, $1 \le j \le m$, *is not a basis of* V,
(iv) *there exists a hyperplane in* $\mathbb{P}(V)$ *containing* $L_1 \cup \cdots \cup L_m$.

Proof. Immediate. □

LEMMA 3.5. $\mathbb{D}_{r,m}$ *cuts on* $\mathbb{G}^m$ *an integral hyperplane section.*

Proof. Consider the correspondence

$$I = \{(l_1, \ldots, l_m, H) \in \mathbb{G}^m \times \mathbb{P}(V^*) \mid L_1 \cup \cdots \cup L_m \subset H\},$$

and its projections $p_1 \colon I \to \mathbb{G}^m$ and $p_2 \colon I \to \mathbb{P}(V^*)$. Note that the fibre of p_2 at any H is the product of Grassmannians of $r-1$ spaces in H, which is irreducible. Hence I is irreducible. On the other hand we have $p_1(I) = \mathbb{D}_{r,m} \cap \mathbb{G}^m$ by Lemma 3.4(iv). Hence the latter intersection is irreducible. Since $O_{\mathbb{G}^m}(1)$ is not divisible in Pic$(\mathbb{G}^m)$, it follows that $\mathbb{D}_{r,m} \cdot \mathbb{G}^m$ is integral. □

On $\mathbb{G}$ we consider the universal bundle $\mathcal{U}_r$. We recall that $\mathcal{U}_r$ *is uniquely defined by its Chern classes, unless* $m = 2$. Let $l \in \mathbb{G}$ and let $L \subset \mathbb{P}(V)$ be the space corresponding to l. Then the fibre of $\mathcal{U}_r^*$ at l is $H^0(\mathcal{O}_L(1))$, moreover $H^0(\mathcal{U}_r{}^*) = V^* = H^0(O_{\mathbb{P}(V)}(1))$. Let $\pi_s : \mathbb{G}^m \to \mathbb{G}$ be the projection onto the sth factor. On $\mathbb{G}^m$ we consider the vector bundle of rank rm

$$\mathcal{F} \colon = \bigoplus_{s=1,\ldots,m} \pi_s^* \mathcal{U}_r^*.$$

For any point $o = (l_1, \ldots, l_m) \in \mathbb{G}^m$, we have

$$\mathcal{F}_o = (\mathcal{U}_r^*)_{l_1} \oplus \cdots \oplus (\mathcal{U}_r^*)_{l_m} = H^0(O_{L_1}(1)) \oplus \cdots \oplus H^0(O_{L_m}(1)).$$

In particular the natural evaluation map

$$ev^m : V^* \otimes \mathcal{O}_{\mathbb{G}^m} \longrightarrow \mathcal{F}, \tag{10}$$

is a morphism of vector bundles of the same rank rm.

Definition 3.6. $\mathbb{D}_{\mathbb{G}}$ is the degeneracy locus of ev^m.

THEOREM 3.7. $\mathbb{D}_{\mathbb{G}} = \mathbb{D}_{r,m} \cdot \mathbb{G}^m$.

Proof. Let $o = (l_1, \ldots, l_m) \in \mathbb{G}^m$, then ev_o^m is the natural restriction map

$$H^0(O_{\mathbb{P}(V)}(1)) \longrightarrow H^0(O_{L_1}(1)) \oplus \ldots \oplus H^0(O_{L_m}(1)).$$

Note that ev_o^m is an isomorphism iff $L_1 \cup \cdots \cup L_m$ is not in a hyperplane of $\mathbb{P}(V)$. This implies that $\mathbb{D}_{\mathbb{G}}$ is a divisor. Moreover $\mathbb{D}_{\mathbb{G}} = \mathbb{D}_{r,m} \cap \mathbb{G}^m$ by Lemma 3.4 and $\mathbb{D}_{\mathbb{G}} \in |O_{\mathbb{G}^m}(1,\ldots,1)|$. Hence $\mathbb{D}_{\mathbb{G}} = \mathbb{D}_{r,m} \cdot \mathbb{G}^m$. □

4. Plücker forms and moduli of vector bundles. In this section we consider any integral, smooth projective variety $X \subset \mathbb{P}^n$ of dimension $d \geq 1$. We assume that X is linearly normal and not degenerate.

Definition 4.1. (E,S) is a good pair on X if
(i) *E is a vector bundle of rank r on X,*
(ii) $\det E \cong \mathcal{O}_X(1)$,
(iii) *$S \subset H^0(E)$ is a subspace of dimension rm,*
(iv) *E is globally generated by S,*
(v) *the classifying map of (E,S) is a morphism birational onto its image.*

Given (E,S) we have the dual space $V := S^*$ and its Plücker form

$$\mathbb{D}_{r,m} \subset \mathbb{P}(\wedge^r V)^m.$$

We want to use it. Let us fix preliminarily some further notations:

Definition 4.2.
(i) $\mathbb{G}_{E,S}$ is the Plücker embedding of the Grassmannian $G(r,V)$,
(ii) $\mathcal{U}_{E,S}$ is the universal bundle of $\mathbb{G}_{E,S}$,
(iii) $d_{E,S} : \wedge^r S \to H^0(\mathcal{O}_X(1))$ is the standard determinant map,
(iv) $\lambda_{E,S} : \mathbb{P}^n \to \mathbb{P}(\wedge^r V)$ is the projectivized dual of $d_{E,S}$,
(v) $g_{E,S} : X \to \mathbb{G}_{E,S}$ is the classifying map defined by S.

We recall that $g_{E,S}$ associates to $x \in X$ the parameter point of the space $\operatorname{Im} ev_x^*$, where $ev : S \otimes \mathcal{O}_X \to E$ is the evaluation map. It is well known that $g_{E,S}$ is defined by the subspace $\operatorname{Im} d_{E,S}$ of $H^0(\mathcal{O}_X(1))$, in particular

$$g_{E,S} = \lambda_{E,S_{|X}}.$$

Since E is globally generated by S and $g_{E,S}$ is a birational morphism, the next three lemmas describe standard properties.

LEMMA 4.3. *One has $E \cong \lambda_{E,S}^* \mathcal{U}_{E,S}^*$ and $S = \lambda_{E,S}^* H^0(\mathcal{U}_{E,S}^*)$ for any good pair (E,S).*

We say that the good pairs (E_1,S_1), (E_2,S_2) are *isomorphic* if there exists an isomorphism $u : E_1 \to E_2$ such that $u^* S_1 = S_2$.

LEMMA 4.4. *Let (E_1,S_1) and (E_2,S_2) be good pairs. Then the following conditions are equivalent:*
(i) *$d_{E_1,S_1} = d_{E_2,S_2} \circ (\wedge^r \alpha)$ for some isomorphism $\alpha : S_1 \to S_2$.*

(ii) $f^*E_1 \cong E_2$ *and* $f^*S_1 = S_2$ *for some automorphism* $f \in \mathrm{Aut}(X)$.

Proof. (i)⇒(ii). The projectivized dual of $\wedge^r\alpha$ induces an isomorphism $a : \mathbb{G}_{E_2,S_2} \to \mathbb{G}_{E_1,S_1}$ such that $g_{E_1,S_1} = a \circ g_{E_2,S_2}$. On the other hand, $g_{E_i,S_i} : X \to \mathbb{G}_{E_i,S_i}$ is a morphism birational onto its image for $i = 1,2$. Hence a lifts to an automorphism $f : X \to X$ with the required properties. (ii)⇒(i). It suffices to put $\alpha = f^*$. □

Let $\rho_i : X^m \to X$ be the projection onto the ith factor of X^m. Then

$$ev_{E,S}\colon S \otimes O_{X^m} \longrightarrow \bigoplus_{i=1,\dots,m} \rho_i^* E := \mathcal{E}$$

is the morphism defined as follows. Let $U \subset X^m$ be open, we observe that $\mathcal{E}(U) = E(U)^m$. Then we define the map $ev_{E,S}(U) : S \to E(U)^m$ as the natural restriction map. Since $ev_{E,S}$ is a morphism of vector bundles of the same rank, its degeneracy locus is either X^m or a divisor

$$\mathbb{D}_{E,S} \in \left|O_{X^m}(1,\dots,1)\right|.$$

Definition 4.5. We will say that the divisor $\mathbb{D}_{E,S}$ is the determinant divisor, or the Plücker form, of the pair (E,S).

If the previous locus is X^m we will say that (E,S) has no Plücker form.

LEMMA 4.6. *Let* (E_1,S_1) *and* (E_2,S_2) *be isomorphic good pairs. Then* $\mathbb{D}_{E_2,S_2} = \mathbb{D}_{E_1,S_1}$.

Proof. Let $u : E_1 \to E_2$ be an isomorphism such that $u^*S_2 = S_1$. Then, by taking the pull back of u to $ev_{E_1,S_1} : S_1 \otimes \mathcal{O}_X \to \mathcal{E}_1$, we obtain ev_{E_2,S_2}. This implies that $\mathbb{D}_{E_1,S_1} = \mathbb{D}_{E_2,S_2}$. □

Remark 4.7. Note that $\mathbb{D}_{E,S}$ contains the multidiagonal Δ_m, i.e. the set of all the points $(x_1,\dots,x_m) \in X^m$ such that $x_i = x_j$ for some distinct $i,j \in \{1,\dots,m\}$. Moreover, Δ_m is a divisor in X^m iff $\dim X = 1$. In this case $\mathbb{D}_{E,S}$ is reducible:

PROPOSITION 4.8. *Assume that* X *is a curve, then*

$$\mathbb{D}_{E,S} = (r+\epsilon)\Delta_m + \mathbb{D}^*_{E,S}.$$

where $\epsilon \geq 0$ *and the support of the divisor* $\mathbb{D}^*_{E,S}$ *is the Zariski closure of the set*

$$\left\{(x_1,\dots,x_m) \in X^m - \Delta_m \mid \exists s \in S,\ s(x_i) = 0,\ i = 1,\dots,m\right\}.$$

Proof. Let $x = (x_1,\dots,x_m) \in \Delta_m$. Then $ev_{E,S}$ has rank $\leq rm - r$ at x. This implies that x is a point of multiplicity $\geq r$ of the determinant divisor $\mathbb{D}_{E,S}$. Hence Δ_m is a component of $\mathbb{D}_{E,S}$ of multiplicity $\geq r$. This implies the statement. □

Actually, $\epsilon = 0$ if E is a general semistable vector bundle on the curve X. It is enough to verify this property in the case $E = L^{\oplus r}$ and $S = H^0(E)$, where L is a general line bundle on X of degree $m+g-1$. In this case the Plücker form of (E,S) is indeed r times the Plücker form of $(L, H^0(L))$.

It is also non difficult to compute that $\mathbb{D}_{E,S} - r\Delta_m$ is numerically equivalent to $a^* r\Theta$, where $a : X^m \to \mathrm{Pic}^m(X)$ is the natural Abel map and $\Theta \subset \mathrm{Pic}^m(X)$ is a theta divisor. Finally we consider the commutative diagram

$$\begin{array}{ccc} X^m & \xrightarrow{g^m_{E,S}} & \mathbb{G}^m_{E,S} \\ \downarrow & & \downarrow \\ (\mathbb{P}^n)^m & \xrightarrow{\lambda^m_{E,S}} & (\mathbb{P}^N)^m \end{array}$$

where the vertical arrows are the inclusion maps.

LEMMA 4.9. *Let $\mathbb{D}_{E,S}$ be the Plücker form of a good pair (E,S), then*

$$\mathbb{D}_{E,S} = \left(\lambda^m_{E,S}\right)^* \mathbb{D}_{r,m}.$$

Proof. Lifting by $g^m_{E,S}$ the map $ev^m : V \otimes \mathcal{O}_{\mathbb{G}_{E,S}} \to \bigoplus_{i=1,\dots,m} \pi_s^* \mathcal{U}^*_{E,S}$, one obtains the map $ev_{E,S} \colon S \otimes O_{X^m} \to \bigoplus_{i=1,\dots,m} \rho_i^* E$. From the commutativity of the above diagram it follows that $\mathbb{D}_{E,S} = (\lambda^m_{E,S})^* \mathbb{D}_{r,m} = (g^m_{E,S})^* \mathbb{D}_{\mathbb{G}_{E,S}}$. □

To a good pair (E,S) we have associated its Plücker form $\mathbb{D}_{E,S}$. Now we want to prove that, under suitable assumptions, a good pair (E,S) is uniquely reconstructed from $\mathbb{D}_{E,S}$. To this purpose we define the following projective variety in the ambient space $\mathbb{P}^n$ of X.

Definition 4.10. $\Gamma_{E,S}$ is the closure of the set of points $x \in \mathbb{P}^n$ such that:
(i) $\mathbb{D}_{E,S}$ has multiplicity $m-1$ at the point $o = (x,\dots,x) \in (\mathbb{P}^n)^m$,
(ii) the tangent space to $\mathrm{Sing}(\mathbb{D}_{E,S})$ at o has maximal dimension.

THEOREM 4.11. *Assume that $d_{E,S}$ is injective and $m \geq 3$. Then:*
(i) *$\Gamma_{E,S}$ is a cone in $\mathbb{P}^n$ with directrix the Grassmannian $\mathbb{G}_{E,S}$,*
(ii) *the vertex of the cone $\Gamma_{E,S}$ is the center of the projection $\lambda_{E,S}$.*

Proof. Since $\lambda_{E,S}$ is the projective dual of $d_{E,S}$, the tensor product map

$$d^{\otimes m}_{E,S} : (\wedge^r S)^{\otimes m} \longrightarrow H^0\big(\mathcal{O}_X(1)\big)^{\otimes m}$$

is precisely the pull-back map

$$\left(\lambda^m_{E,S}\right)^* : H^0\big(\mathcal{O}_{(\mathbb{P}(\wedge^r V))^m}(1,\dots,1)\big) \longrightarrow H^0\big(\mathcal{O}_{(\mathbb{P}^n)^m}(1,\dots,1)\big).$$

Moreover it is injective. Let $F \in H^0(\mathcal{O}_{(\mathbb{P}(\wedge^r V))^m}(1,\dots,1))$ be the polynomial of multidegree $(1,\dots,1)$ defining $\mathbb{D}_{r,m}$. Then we can choose coordinates on

$(\mathbb{P}(\wedge^r V))^m$ and $(\mathbb{P}^n)^m$ so that $d_{E,S}^{\otimes m}(F) = F$. Assume that $\lambda_{E,S}^m$ is a morphism at the point $o \in (\mathbb{P}^n)^m$, then it follows that:

(a) $\lambda_{E,S}^m(o) \in \mathrm{Sing}_{m-1}(\mathbb{D}_{r,m})$ *iff* $o \in \mathrm{Sing}_{m-1}(\mathbb{D}_{E,S})$,

(b) *the codimension is equal for the tangent spaces to* $\mathrm{Sing}_{m-1}(\mathbb{D}_{r,m})$ *at* $\lambda_{E,S}^m(o)$ *and to* $\mathrm{Sing}_{m-1}(\mathbb{D}_{E,S})$ *at* o.

Assume that $o = (x, \dots, x)$ is a diagonal point in $(\mathbb{P}^n)^m$. Then $x \in \Gamma_{E,S}$ iff o satisfies (i) and (ii) in Definition 4.10. By (a) and (b), conditions (i) and (ii) hold true for o iff they hold true for $\lambda_{E,S}^m(o)$ as a point of $\mathbb{D}_{r,m}$. Finally, by Theorem 3.1, $\lambda_{E,S}(o)$ satisfies (i) and (ii) iff x belongs to the Grassmannian $\mathbb{G}_{E,S}$. Hence $\Gamma_{E,S}$ is a cone over $\mathbb{G}_{E,S}$ with vertex the center of $\lambda_{E,S}$. □

We are now able to show the main result of the current section.

THEOREM 4.12. *Let* (E_1, S_1) *and* (E_2, S_2) *be good pairs defining the same Plücker form* $\mathbb{D} \subset (\mathbb{P}^n)^m$. *Assume that* $m \geq 3$ *and* d_{E_i,S_i} *is injective for any* $i = 1, 2$, *then there exists* $f \in \mathrm{Aut}(X)$ *such that* $f^*E_2 \cong E_1$ *and* $f^*S_2 = S_1$.

Proof. Let Γ be the closure of the set of diagonal points $o = (x, \dots, x) \in \mathbb{D}$ of multiplicity $m-1$ and tangent space $T_{\mathrm{Sing}_{m-1}(\mathbb{D}),o}$ of maximal dimension. By Theorem 4.11, Γ is a cone in $\mathbb{P}^n$: its directrix is the Grassmannian G_{E_i,S_i} and its vertex is the center of the projection λ_{E_i,S_i}, both for $i = 1$ and $i = 2$. Since the projection maps λ_{E_i,S_i} have the same center, there exist an isomorphism $\sigma\colon G_{E_2,S_2} \to G_{E_1,S_1}$ such that $\lambda_{E_1,S_1} = \sigma \circ \lambda_{E_2,S_2}$. Since $m \geq 3$, then $\sigma = \wedge^r \alpha^*$ for an isomorphism $\alpha\colon S_1 \to S_2$, see [Ha2, p. 122]. Then, applying Lemma 4.4, it follows $f^*E_1 \cong E_2$ and $f^*S_1 = S_2$ for some $f \in \mathrm{Aut}(X)$. □

To conclude this section we briefly summarize, in a general statement, how to deduce from the previous results the generic injectivity of some natural maps, defined on a moduli space of good pairs as above. Therefore we assume that a coarse moduli space $\mathcal{S}$ exists for the family of good pairs (E, S) under consideration. This is, for instance the case when E is stable with respect to the polarization $\mathcal{O}_X(1)$ and $S = H^0(E)$. Then there exists a natural map

$$\theta_{r,m} : \mathcal{S} \longrightarrow \left|\mathcal{O}_{X^m}(1, \dots, 1)\right|$$

sending the moduli point of (E, S) to its determinant divisor $\mathbb{D}_{E,S}$. Let (E_1, S_1) and (E_2, S_2) be good pairs as above defining two general points of $\mathcal{S}$. Assume that $\mathbb{D}_{E_1,S_1} = \mathbb{D}_{E_2,S_2}$. Then we know from Theorem 4.12 that then (E_1, S_1) and (E_2, S_2) are isomorphic if $m \geq 3$, $\mathrm{Aut}(X) = 1$ and

$$d_{E_i,S_i} : \wedge^r S_i \longrightarrow H^0\big(\mathcal{O}_X(1)\big).$$

is injective. This implies the next statement:

THEOREM 4.13. *Let $m \geq 3$ and* $\mathrm{Aut}(X) = 1$. *Assume $d_{E,S} : \wedge^r S \to H^0(\mathcal{O}_X(1))$ is injective for good pairs (E,S) with moduli in a dense open subset of $\mathcal{S}$. Then $\theta_{r,m}$ is generically injective.*

5. Plücker forms and the theta map of $\mathrm{SU}_X(r,0)$**.** Now we apply the preceding arguments to study the theta map of the moduli space $\mathrm{SU}_X(r,0)$ of semistable vector bundles of rank r and trivial determinant over a curve X of genus $g \geq 2$. By definition the theta map

$$\theta_r : \mathrm{SU}_X(r,0) \longrightarrow \mathbb{P}\big(H^0(\mathcal{L})^*\big)$$

is just the rational map defined by the ample generator $\mathcal{L}$ of $\mathrm{SU}_X(r,0)$. We prove our main result:

THEOREM 5.1. *Let X be general and $g \gg r$, then θ_r is generically injective.*

To prove the theorem we need some preparation. At first we replace the space $\mathrm{SU}_X(r,0)$ by a suitable translate of it, namely the moduli space

$$\mathcal{S}_r$$

of semistable vector bundles E on X having rank r and fixed determinant $\mathcal{O}_X(1)$ of degree $r(m+g-1)$. We assume that X has general moduli and that $\mathcal{O}_X(1)$ is general in $\mathrm{Pic}^{r(m+g-1)}(X)$, with $m \geq 3$ and $r \geq 2$. In particular $\mathcal{O}_X(1)$ is very ample: we also assume that X is embedded in $\mathbb{P}^n$ by $\mathcal{O}_X(1)$.

We recall that $\mathcal{S}_r$ is biregular to $\mathrm{SU}_X(r,0)$, the biregular map being induced by tensor product with an rth root of $\mathcal{O}_X(-1)$.

PROPOSITION 5.2. *Let E be a semistable vector bundle on X with general moduli in $\mathcal{S}_r$. Then:*

(i) *$h^0(E) = rm$ and $(E, H^0(E))$ is a good pair,*

(ii) *the Plücker form of $(E, H^0(E))$ exists.*

Proof. (i) It suffices to produce *one* semistable vector bundle E on X, of degree $r(m+g-1)$ and rank r, such that $h^0(E) = rm$ and $(E, H^0(E))$ is a good pair in the sense of Definition 4.1. Then the statement follows because the conditions defining a good pair are open. Let $L \in \mathrm{Pic}^{m+g-1}(X)$ be general, then $h^0(L) = m$ and L is globally generated. Since $m \geq 3$, L defines a morphism birational onto its image

$$f : X \longrightarrow \mathbb{P}(H^0(L)^*).$$

Putting $E := L^{\oplus r}$ we have a globally generated, semistable vector bundle such that $h^0(E) = rm$. Hence, to prove that $(E, H^0(E))$ is a good pair, it remains to show that its classifying map

$$g_E : X \longrightarrow \mathbb{G}_E :\ = G\big(r, H^0(E)^*\big)$$

is birational onto its image. We observe that $H^0(E) = H_1 \oplus \cdots \oplus H_r$, where H_i is just a copy of $H^0(L)$, $i = 1, \dots, r$. Let $f_i : X \to \mathbb{P}(H_i^*)$ be the corresponding copy of f, for any $i = 1, \dots, r$. Then $g_E : X \to \mathbb{G}_E$ can be described as follows: let $\mathbb{P}(E_x^*) \subset \mathbb{P}(H^0(E)^*)$ be the linear embedding induced by the evaluation map, it turns out that $\mathbb{P}(E_x^*)$ is the linear span of $f_1(x), \dots, f_r(x)$. This implies that $g_E = u \circ (f_1 \times \cdots \times f_r)$, where

$$u : \mathbb{P}\big(H_1^*\big) \times \cdots \times \mathbb{P}\big(H_r^*\big) \longrightarrow \mathbb{G}_E$$

is the rational map sending $(y_1, \dots, y_r)$ to the linear span of the points $y_i \in \mathbb{P}(H_i^*) \subset \mathbb{P}(H^0(E)^*)$, $i = 1, \dots, r$. Since f is birational onto its image, the same is true for the map $f_1 \times \cdots \times f_r$. Moreover u is clearly birational onto its image. Hence g_E is birational onto its image. Finally g_E is a morphism, since $L^{\oplus r}$ is globally generated. This completes the proof of (i).

(ii) Again it suffices to produce *one* good pair $(E, H^0(E))$ with the required property. It is easy to see that this is the case if $E = L^{\oplus r}$ as in (i). □

Now we consider the rational map

$$\theta_{r,m} : \mathcal{S}_r \longrightarrow \big|\mathcal{O}_X(1, \dots, 1)\big|$$

sending the moduli point $[E] \in \mathcal{S}_r$ of a general E to the Plücker form

$$\mathbb{D}_E \in \big|\mathcal{O}_X(1, \dots, 1)\big|$$

of the pair $(E, H^0(E))$. Let $t \in \mathrm{Pic}^{m+g-1}(X)$ be an r-root of $\mathcal{O}_X(1)$, then we have a map

$$a_t : X^m \longrightarrow \mathrm{Pic}^{g-1}(X)$$

sending $(x_1, \dots, x_m)$ to $\mathcal{O}_X(t - x_1 - \cdots - x_m)$. It is just the natural Abel map $a\colon X^m \to \mathrm{Pic}^m(X)$, multiplied by -1 and composed with the tensor product by t. Fixing a Poincaré bundle $\mathcal{P}$ on $X \times \mathrm{Pic}^{g-1}(X)$ we have the sheaf

$$R^1 q_{2*}\big(q_1^* E(-t) \otimes \mathcal{P}\big),$$

where q_1, q_2 are the natural projection maps of $X \times \mathrm{Pic}^{g-1}(X)$. It is well known the support of this sheaf is either $\mathrm{Pic}^{g-1}(X)$ or a Cartier divisor Θ_E, see [BNR]. Moreover, due to the choice of t, one has

$$\Theta_E \in |r\Theta|,$$

where $\Theta := \{N \in \mathrm{Pic}^{g-1}(X) \mid h^0(N) \geq 1\}$ is the natural theta divisor of $\mathrm{Pic}^{g-1}(X)$. In particular, one has $h^0(E \otimes N(-t)) = h^1(E \otimes N(-t))$ so that

$$\mathrm{Supp}\,\Theta_E = \{N \in \mathrm{Pic}^{g-1}(X) \mid h^0\big(E \otimes N(-t)\big) \geq 1\}.$$

Finally, it is well known that there exists a suitable identification

$$| r\Theta | = \mathbb{P}\big(H^0(\mathcal{L})^*\big)$$

such that $\theta_r([E]) = \Theta_E$, [BNR]. Computing Chern classes it follows

$$a_t^* \Theta_E + r\Delta_m \in \big|\mathcal{O}_{X^m}(1,\dots,1)\big|,$$

where $\Delta_m \subset X^m$ is the multidiagonal divisor. On the other hand, $r\Delta_m$ is a component of $\mathbb{D}_E$ by Proposition 4.8. Moreover, it follows from the definition of determinant divisor that $\mathbb{D}_E$ contains $a_t^{-1}(\Theta_E)$. Therefore we have

$$a_t^* \Theta_E + r\Delta_m = \mathbb{D}_E. \tag{11}$$

Let $\alpha : |r\Theta| \to |\mathcal{O}_{X^m}(1,\dots,1)|$ be the linear map sending $D \in | r\Theta |$ to $a_t^* D + r\Delta_m$. We conclude the following from the latter equality:

PROPOSITION 5.3. *$\theta_{r,m}$ factors through the theta map θ_r, that is $\theta_{r,m} = \alpha \circ \theta_r$.*

Proof of Theorem 5.1. Let $\theta_{r,m} : \mathcal{S}_r \to |\mathcal{O}_{X^m}(1,\dots,1)|$ be as above. We have $\mathrm{Aut}(X) = 1$ and $m \geq 3$. We know that $(E, H^0(E))$ is a good pair if $[E] \in \mathcal{S}_r$ is general and that $\theta_{r,m}$ factors through the theta map θ_r. Theorem 4.13 says that $\theta_{r,m}$ is generically injective if $(E, H^0(E))$ is a good pair and the determinant map

$$d_E : \wedge^r H^0(E) \longrightarrow H^0\big(\mathcal{O}_X(1)\big)$$

is injective for a general $[E]$. This is proved in the next section. □

6. The injectivity of the determinant map. Let (X, E) be a pair such that X is a smooth irreducible curve of genus g and E is a semistable vector bundle of rank r on X and degree $r(g-1+m)$, with $m \geq 3$. If E is a general semistable vector bundle on X, it follows that:

(i) *$(E, H^0(E))$ is a good pair,*
(ii) *its Plücker form exists.*

(see Definition 4.1 and Proposition 5.2). It is therefore clear that the previous conditions are satisfied on a dense open set U of the moduli space of (X, E).

ASSUMPTION. *From now on we will assume that (X, E) defines a point of U, so that X is a general curve of genus g and E is semistable and satisfies (i) and (ii).*

In this section we prove the following result:

THEOREM 6.1. *Let X and E be sufficiently general and $g \gg r$, then:*
(i) *the determinant map $d_E : \wedge^r H^0(E) \to H^0(\det E)$ is injective,*
(ii) *the classifying map $g_E : X \to \mathbb{G}_E$ is an embedding.*

Since $m \geq 3$, det $E := \mathcal{O}_X(1)$ is very ample. So we will assume as usual that the curve X is embedded in $\mathbb{P}^n = \mathbb{P}(H^0(\mathcal{O}_X(1))^*)$. Let us also recall that

$$\mathbb{G}_E \subset \mathbb{P}^{\binom{rm}{r}-1}$$

denotes the Plücker embedding of the Grassmannian $G(r, H^0(E)^*)$. Let

$$\lambda_E : \mathbb{P}^n \to \mathbb{P}^{\binom{rm}{r}-1}$$

be the projectivized dual of d_E. We have already remarked in Section 4 that g_E is just the restriction $\lambda_{E|_X}$. This immediately implies that:

LEMMA 6.2. *d_E is injective $\Leftrightarrow$ λ_E is surjective $\Leftrightarrow$ the curve $g_E(X)$ spans the Plücker space $\mathbb{P}^{\binom{rm}{r}-1}$.*

Since $(E, H^0(E))$ is a good pair, $g_E : X \to g_E(X)$ is a birational morphism. Let

$$\langle g_E(X) \rangle \subset \mathbb{P}^{\binom{rm}{r}-1}$$

be the linear span of $g_E(X)$. Then the previous Theorem 6.1 is an immediate consequence of the following one:

THEOREM 6.3. *For a general pair (X, E) as above g_E is an embedding and*

$$\dim\langle g_E(X) \rangle \geq r(m-1) + g.$$

In other words, the statement says that g_E is an embedding and that d_E has rank $> r(m-1) + g$. This theorem and the previous lemma imply the following:

COROLLARY 6.4. *For a general (X, E), d_E is injective if $g \geq \binom{rm}{r} - r(m-1) - 1$.*

Hence the proof of Theorem 6.1 also follows.

Proof of Theorem 6.3. To prove the theorem, hence Theorem 6.1, we observe that the moduli space of all pairs (X, E) is an integral, quasi-projective variety defined over the moduli space $\mathcal{M}_g$ of X. On the other hand, the conditions in the statement of the theorem are open. Therefore, it suffices to construct *one pair* (X, E) such that E is semistable, $h^0(E) = rm$ and these conditions are satisfied. We will construct such a pair *by induction on the genus*

$$g \geq 0$$

of X. For $g = 0$ we have $X = \mathbb{P}^1$ and $E = \mathcal{O}_{\mathbb{P}^1}(m-1)^r$.

LEMMA 6.5. *Let $X = \mathbb{P}^1$ and $E = \mathcal{O}_{\mathbb{P}^1}(m-1)^r$, with $m \geq 2$. Then d_E is surjective and g_E is an embedding.*

Proof. The proof of the surjectivity of d_E is standard. It also follows from the results in [T]. In order to deduce that g_E is an embedding recall that g_E is defined by $\operatorname{Im} d_E$, hence by the complete linear system $|\mathcal{O}_{\mathbb{P}^1}(r(m-1))|$. □

Now we assume by induction that the statement is true for g and prove it for $g+1$.

Let (X,E) be a general pair such that X has genus g. We recall that then X is a general curve of genus g and $(E, H^0(E))$ is a good pair admitting a Plücker form.

By induction g_E is an embedding and $\dim\langle g_E(X)\rangle \geq r(m-1)+g$. We need to prove various lemmas.

LEMMA 6.6. *The evaluation map $ev_{x,y} : H^0(E) \to E_x \oplus E_y$ is surjective for general $x,y \in X$.*

Proof. If not we would have $h^0(E(-x-y)) > h^0(E) - 2r = r(m-2)$, for any pair $(x,y) \in X^2$. This implies that $h^0(E(-x-y-z_1-\cdots-z_{m-2})) \geq 1$, $\forall (x,y,z_1,\ldots,z_{m-2}) \in X^m$ and hence that $(E, H^0(E))$ has no Plücker form. But then, by Proposition 5.2 (ii), (X,E) is not general: a contradiction. □

From now on we put

$$C := g_E(X).$$

Choosing x,y so that $ev_{x,y}$ is surjective, we have a linear embedding

$$E_x^* \oplus E_y^* \subset H^0(E)^*$$

induced by the dual map $ev_{x,y}^*$. This induces an inclusion of Plücker spaces

$$\mathbb{P}^{\binom{2r}{r}-1} := \mathbb{P}\big(\wedge^r \left(E_x^* \oplus E_y^*\right)\big) \subset \mathbb{P}^{\binom{rm}{r}-1} := \mathbb{P}\big(\wedge^r H^0(E)^*\big)$$

and of their corresponding Grassmannians

$$\mathbb{G}_{x,y} := G\big(r, \left(E_x^* \oplus E_y^*\right)\big) \subset \mathbb{G}_E.$$

LEMMA 6.7. *Assume $\langle C\rangle$ is a proper subspace of the Plücker space of $\mathbb{G}_E$. Let x,y be general points of X. Then $\langle \mathbb{G}_{x,y}\rangle$ is not in $\langle C\rangle$.*

Proof. For a general $x \in X$ consider the linear map $\pi : H^0(E)^* \to H^0(E(-x))^*$ dual to the inclusion $H^0(E(-x)) \subset H^0(E)$. It induces a surjective linear projection

$$\wedge^r \pi : \mathbb{P}\big(\wedge^r H^0(E)^*\big) \longrightarrow \mathbb{P}\big(\wedge^r H^0\big(E(-x)\big)^*\big),$$

with center the linear span $\langle\sigma\rangle$ of $\sigma := \{L \in \mathbb{G}_E \mid \dim(L \cap E_x^*) \geq 1\}$. In particular $\wedge^r \pi$ restricts to a rational map between Grassmannians

$$f : \mathbb{G}_E \longrightarrow \mathbb{G}_{E(-x)},$$

where $\mathbb{G}_{E(-x)} := G(r, H^0(E(-x))^*) \simeq G(r,(m-1)r)$. Let $l \in \mathbb{G}_E$ be the parameter point of the space L, then $f(l)$ is the parameter point of $\pi(L)$. Clearly f is defined at l iff $L \cap E_x^* = 0$. Moreover, the closure of the fibre of f at $f(l)$ is the Grassmannian $G(r, L \oplus E_x^*)$. In particular, the closure of the fibre at $f(y)$ is $\mathbb{G}_{x,y}$, for a general $y \in X$. We distinguish two cases:

(1) *$f(C)$ spans the Plücker space of* $\mathbb{G}_{E(-x)}$. Since $f = \wedge^r \pi_{|\mathbb{G}_E}$ and $\wedge^r \pi$ is linear, it follows that $\bigcup_{y \in C} \langle \mathbb{G}_{x,y} \rangle$ spans the Plücker space of $\mathbb{G}_E$. Since $\langle C \rangle$ is proper in it, we conclude that $\langle \mathbb{G}_{x,y} \rangle$ is not in $\langle C \rangle$ for some y, hence for general points $x, y \in X$.

(2) *$f(C)$ does not span the Plücker space of* $\mathbb{G}_{E(-x)}$. Since the Plücker form of $(E, H^0(E))$ exists and $m \geq 3$, we can fix $x, y, z_1, \ldots, z_{m-2} \in X$ so that $h^0(E(-x-y-z)) = 0$, where $z := z_1 + \cdots + z_{m-2}$. Then we have $H^0(E(-x)) \cap H^0(E(-y-z)) = 0$ in $H^0(E)$. Putting $E_z^* := E_{z_1}^* \oplus \cdots \oplus E_{z_i}^*$, it follows that

$$\pi_{|(E_z^* \oplus E_y^*)} : E_y^* \oplus E_z^* \longrightarrow H^0\big(E(-x)\big)^*$$

is an isomorphism, that is, $\wedge^r \pi$ induces the following isomorphism of projective spaces:

$$i_{y,z} \colon \mathbb{P}\big(\wedge^r \big(E_y^* \oplus E_z^*\big)\big) \longrightarrow \mathbb{P}\big(\wedge^r H^0\big(E(-x)\big)^*\big).$$

On the other hand, $\mathbb{P}(\wedge^r(E_y^* \oplus E_z^*))$ is spanned by the union of its natural linear subspaces $\langle \mathbb{G}_{y,z_i} \rangle = \mathbb{P}(\wedge^r(E_y^* \oplus E_{z_i}^*))$, $i = 1, \ldots, m-2$. Since $\langle f(C) \rangle$ is a proper subspace of $\mathbb{P}(\wedge^r H^0(E(-x))^*)$, it follows that $\langle \mathbb{G}_{y,z_i} \rangle$ is not in $\langle C \rangle$, for some $i = 1, \ldots, m-2$. □

Now we assume that $\langle C \rangle$ is a proper subspace of the Plücker space of $\mathbb{G}_E$ and fix general points $x, y \in X$ so that the conditions of the previous lemma are satisfied. Keeping the previous notations let $P \subset \mathbb{P}^{rm-1}$ be the tautological image of $\mathbb{P}(E^*)$ and let $P_z := \mathbb{P}(E_z^*)$, $z \in X$. We observe that the Grassmannian $\mathbb{G}_{x,y}$ is ruled by smooth rational normal curves of degree r passing through x and y. More precisely, let

$$\mathbb{P}^{2r-1} := \mathbb{P}\big(E_x^* \oplus E_y^*\big)$$

and for $t \in \mathbb{G}_{x,y}$ let

$$P_t \subset \mathbb{P}^{2r-1} \subset \mathbb{P}^{rm-1}$$

be the projectivized space corresponding to t. We have:

LEMMA 6.8. *For a general $t \in \mathbb{G}_{x,y}$ there exists a unique Segre product $S := \mathbb{P}^1 \times \mathbb{P}^{r-1}$ such that $P_x \cup P_y \cup P_t \subset S \subset \mathbb{P}^{2r-1}$. Moreover:*

(i) *the ruling of S is parametrized by a degree r rational normal curve*

$$R \subset \mathbb{G}_{x,y} \subset \mathbb{G}_E \subset \mathbb{P}^{\binom{rm}{r}-1},$$

(ii) *the universal bundle $\mathcal{U}_r$ of $\mathbb{G}_E$ restricts to $\mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus r}$ on R,*

(iii) *the restriction map $H^0(\mathcal{U}^*) \to H^0(\mathcal{O}_{\mathbb{P}^1}(1))^{\oplus r}$ is surjective.*

Proof. Since x, y are general in X, Lemma 6.6 implies that $P_x \cap P_y = \emptyset$. Since t is general in $\mathbb{G}_{x,y}$, we have $P_t \cap P_x = P_t \cap P_y = \emptyset$. It is a standard fact that the union of all lines in $\mathbb{P}^{2r-1}$ meeting P_x, P_y and P_t is the Segre embedding $S \subset \mathbb{P}^{2r-1}$ of the product $\mathbb{P}^1 \times \mathbb{P}^{r-1}$, which is actually the unique Segre variety containing the above linear spaces, see [Ha2, p. 26, 2.12]. It is also well known that S is the tautological image of the projective bundle associated to $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$, see [Ha1]. Therefore, the map assigning to each point $p \in \mathbb{P}^1$ the fiber of S over p is the classifying map of $\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$. So it defines an embedding of $\mathbb{P}^1$ into the Grassmannian $\mathbb{G}_{x,y}$, whose image is a rational normal curve R. This implies (ii) and (iii). □

Let $t \in \mathbb{G}_{x,y}$ be a sufficiently general point, where x, y are general in X. Then, by Lemma 6.7, t is not in the linear space $\langle C \rangle$. Since $\mathbb{G}_{x,y}$ is ruled by the family of curves R, we can also assume that $C \cup R$ is a nodal curve with exactly two nodes in x and y. So far we have constructed a nodal curve

$$\Gamma := C \cup R \tag{12}$$

such that

(i) *Γ has arithmetic genus $g+1$ and degree $r(m+g)$,*

(ii) $\dim\langle\Gamma\rangle \geq \dim\langle C \rangle + 1 = r(m-1) + g + 1$.

LEMMA 6.9.

(i) *The curve Γ is smoothable in $\mathbb{G}_E$,*

(ii) *$h^1(\mathcal{O}_\Gamma(1)) = 0$ and $h^0(\mathcal{O}_\Gamma(1)) = r(m+g) - g$.*

Let $\mathcal{U}_r$ be the universal bundle on $\mathbb{G}_E$, we have also the vector bundle on Γ:

$$F := \mathcal{U}_r^* \otimes \mathcal{O}_\Gamma. \tag{13}$$

LEMMA 6.10.

(i) *The restriction map $H^0(\mathcal{U}_r^*) \to H^0(F)$ is an isomorphism,*

(ii) *$h^1(F) = 0$ and $h^0(F) = rm$.*

LEMMA 6.11. *Let $x_1, \ldots, x_m$ be general points on C. Then $h^0(F(-x_1 - \cdots - x_m)) = 0$.*

Proof. Let us recall that $C = g_E(X)$ and that $E \cong \mathcal{U}_r^* \otimes \mathcal{O}_C$. Under the assumptions made at the beginning of this section, X is a general curve of genus g, $(E, H^0(E))$ is a good pair admitting a Plücker form. This implies that $h^0(E(-x_1 - \cdots - x_m)) = 0$, where $x_1, \ldots, x_m$ are general points on X. Notice also that $F \otimes \mathcal{O}_C \cong E$ and that, by the previous lemma, the restriction map $H^0(F) \to H^0(E)$ is an isomorphism.

Let $d := x_1 + \cdots + x_m$ and let $s \in H^0(F(-d))$. Then s is zero on X because $h^0(E(-d)) = 0$. In particular s is zero on $\{x, y\} = C \cap R$. Hence its restriction on R is a global section $s_{|R}$ of $\mathcal{O}_R(-x-y)$. But $F \otimes \mathcal{O}_R(-x-y)$ is $\mathcal{O}_{\mathbf{P}^1}(-1)^{\oplus r}$ so that $s_{|_R} = 0$. Hence s is zero on Γ and $h^0(F(-d)) = 0$. $\square$

We are now able to complete the proof of Theorem 6.3, postponing the proofs of Lemmas 6.9 and 6.10.

Completion of the proof of Theorem 6.3. We start from a curve $\Gamma = C \cup R$ as above. Therefore the component $C = g_E(X)$ is the embedding in $\mathbb{G}_E$ of a curve X with general moduli and, by the previous lemma, there exists $(x_1, \ldots, x_m) \in C^m$ such that $h^0(F(-x_1 - \cdots - x_m)) = 0$. Now recall that, by Lemma 6.9, the curve Γ is smoothable in $\mathbb{G}_E$. This means that there exists a flat family

$$\{X_t,\ t \in T\}$$

of curves $X_t \subset \mathbb{G}_E$ such that: (1) T is integral and smooth, (2) for a given $o \in T$ one has $X_o = \Gamma$, (3) X_t is smooth for $t \neq o$. Let

$$E_t := \mathcal{U}_r^* \otimes \mathcal{O}_{X_t}.$$

For t general we have $h^1(E_t) = h^1(F) = 0$, by semicontinuity, and hence $h^0(E_t) = rm$. For the same reason, the determinant map $d_t : \wedge^r H^0(E_t) \to H^0(\mathcal{O}_{X_t}(1))$ has rank bigger or equal to the rank of $d_o : \wedge^r H^0(F) \to H^0(\mathcal{O}_\Gamma(1))$. This is equivalent to say that

$$\dim\langle X_t\rangle \geq \dim\langle\Gamma\rangle \geq r(m-1) + g + 1.$$

Then, for t general, the pair (X_t, E_t) satisfies the statement of Theorem 6.3.

To complete the proof of the theorem, it remains to show that E_t *is semistable for a general* t. It is well known that E_t is semistable if it admits theta divisor, see [B3]. This is equivalent to say that

$$\Theta_t := \{N \in \mathrm{Pic}^m(X_t) \mid h^0(E_t \otimes N^{-1}) \geq 1\} \neq \mathrm{Pic}^m(X_t),$$

therefore E_t is semistable if

$$D_t := \{(z_1, \ldots, z_m) \in X_t^m \mid h^0(E_t(-z_1 - \cdots - z_m)) \geq 1\} \neq X_t^m.$$

To prove that $D_t \neq X_t^m$ for a general t, we fix in ${\mathbb{G}_E}^m \times T$ the family

$$A := \{(z_1,\dots,z_m;t) \in {\mathbb{G}_E}^m \times T \mid z_1,\dots,z_m \in X_t - \mathrm{Sing}(X_t)\},$$

which is integral and smooth over T. Then we consider its closed subset

$$D := \{(z_1,\dots,z_m;t) \in A \mid h^0(\mathcal{U}_r^* \otimes \mathcal{O}_{X_t}(-z_1-\cdots-z_m)) \geq 1\}.$$

It suffices to show that D is proper, so that $D_t \neq X_t^m$ for a general t. Since $E_o = F$, Lemma 6.11 implies that $D \cap X_o^m$ is proper. Indeed there exists a point $(x_1,\dots,x_m) \in C^m \subset X_o^m$ so that $h^0(F(-x_1-\cdots-x_m)) = 0$. Hence D is proper. □

Proof of Lemma 6.9. (i) We will put $\mathbb{G} := \mathbb{G}_E$. We recall that Γ is *smoothable* in $\mathbb{G}$ if there exists an integral variety $\mathcal{X} \subset \mathbb{G} \times T$ such that:

(a) *the projection $p: \mathcal{X} \to T$ is flat,*

(b) *for some $o \in T$ the fibre $\mathcal{X}_o$ is Γ,*

(c) *if $t \in T - \{o\}$, the fibre $\mathcal{X}_t$ is smooth of genus $g+1$.*

To prove that Γ is smoothable we use a well known argument, see [S] or [HH]. Consider the natural map $\phi: \mathcal{T}_{\mathbb{G}|\Gamma} \to \mathcal{N}_{\Gamma|\mathbb{G}}$, where $\mathcal{N}_{\Gamma|\mathbb{G}}$ is the normal bundle of Γ in $\mathbb{G}$. The Cokernel of ϕ is a sheaf T_S^1, supported on $S := \mathrm{Sing}(\Gamma)$. It is known as the T^1-sheaf of Lichtenbaum-Schlessinger. Finally, ϕ fits into the following exact sequence induced by the inclusion $\Gamma \subset \mathbb{G}$:

$$0 \longrightarrow \mathcal{T}_\Gamma \longrightarrow \mathcal{T}_{\mathbb{G}|\Gamma} \xrightarrow{\phi} \mathcal{N}_{\Gamma|\mathbb{G}} \longrightarrow T_S^1 \longrightarrow 0.$$

Let $\mathcal{N}'$ be the image of ϕ in $\mathcal{N}_{\Gamma|\mathbb{G}}$. The condition $h^1(\mathcal{N}') = 0$ implies that Γ is smoothable in $\mathbb{G}$, [S] prop. 1.6. To show that $h^1(\mathcal{N}') = 0$ it is enough to show that $h^1(\mathcal{T}_{\mathbb{G}|\Gamma}) = 0$, this is a standard argument following from the exact sequence

$$0 \longrightarrow \mathcal{T}_\Gamma \longrightarrow \mathcal{T}_{\mathbb{G}|\Gamma} \longrightarrow \mathcal{N}' \longrightarrow 0.$$

To prove that $h^1(\mathcal{T}_{\mathbb{G}|\Gamma}) = 0$ we use the Mayer-Vietoris exact sequence

$$0 \longrightarrow \mathcal{T}_{\mathbb{G}|\Gamma} \longrightarrow \mathcal{T}_{\mathbb{G}|C} \oplus \mathcal{T}_{\mathbb{G}|R} \longrightarrow \mathcal{T}_{\mathbb{G}|S} \longrightarrow 0.$$

The associated long exact yields the restriction map

$$\rho: H^0(\mathcal{T}_{\mathbb{G}|C}) \oplus H^0(\mathcal{T}_{\mathbb{G}|R}) \longrightarrow H^0(\mathcal{T}_{\mathbb{G}|S}).$$

At first we show its surjectivity: it suffices to show that

$$\rho: 0 \oplus H^0(\mathcal{T}_{\mathbb{G}|R}) \longrightarrow H^0(\mathcal{T}_{\mathbb{G}|S})$$

is surjective. Recall that S consists of two points x, y and that $T_S^1 = \mathcal{O}_S$. Then, tensoring by $\mathcal{T}_{\mathbb{G}|R}$ the exact sequence

$$0 \longrightarrow \mathcal{O}_R(-x-y) \longrightarrow \mathcal{O}_R \longrightarrow \mathcal{O}_S \longrightarrow 0,$$

the surjectivity of ρ follows if $h^1(\mathcal{T}_{\mathbb{G}|R}(-x-y))=0$. To prove this consider the standard Euler sequence defining the tangent bundle to $\mathbb{G}$:

$$0 \longrightarrow \mathcal{U}_r \otimes \mathcal{U}_r^* \longrightarrow O_{\mathbb{G}}{}^{\oplus rm} \otimes \mathcal{U}_r^* \longrightarrow \mathcal{T}_{\mathbb{G}} \longrightarrow 0.$$

Then restrict it to R and tensor by $\mathcal{O}_R(-x-y)$. The term in the middle of such a sequence is $M := \mathcal{O}_{\mathbb{P}^1}^{\oplus rm} \otimes \mathcal{O}_{\mathbb{P}^1}(-1)^{\oplus r}$. This just follows because $\mathcal{U}_r^* \otimes \mathcal{O}_R \cong \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$. Since $h^1(M)=0$, it follows that $h^1(\mathcal{T}_{\mathbb{G}|R}(-x-y))=0$. Hence ρ is surjective. The surjectivity of ρ and the vanishing of $h^1(\mathcal{T}_{\mathbb{G}|R})$ and $h^1(\mathcal{T}_{\mathbb{G}|C})$ clearly imply that $h^1(\mathcal{T}_{\mathbb{G}|\Gamma})=0$. Hence we are left to show that $h^1(\mathcal{T}_{\mathbb{G}|R})=h^1(\mathcal{T}_{\mathbb{G}|C})=0$. Since $\mathcal{T}_{\mathbb{G}|R} \cong \mathcal{O}_{\mathbb{P}^1}^{\oplus rm} \otimes \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}$, the former vanishing is immediate. To prove that $h^1(\mathcal{T}_{\mathbb{G}|C})=0$ the argument is similar. Restricting the above Euler sequence to C we obtain the exact sequence

$$0 \longrightarrow E^* \otimes E \longrightarrow E^{\oplus rm} \longrightarrow \mathcal{T}_{\mathbb{G}|C} \longrightarrow 0,$$

since $\mathcal{U}^*_{r\,|C} \simeq E$. Then $h^1(E)=0$ implies $h^1(\mathcal{T}_{\mathbb{G}|C})=0$.

(ii) To prove $h^1(\mathcal{O}_\Gamma(1))=0$ it suffices to consider the long exact sequence associated to the Mayer-Vietoris exact sequence

$$0 \longrightarrow \mathcal{O}_\Gamma(1) \longrightarrow \mathcal{O}_C(1) \oplus \mathcal{O}_R(1) \longrightarrow \mathcal{O}_{x,y}(1) \longrightarrow 0.$$

For degree reasons we have $h^1(\mathcal{O}_C(1))=h^1(\mathcal{O}_R(1))=0$. Hence it suffices to show that the restriction $H^0(\mathcal{O}_C(1)) \oplus H^0(\mathcal{O}_R(1)) \to \mathcal{O}_{x,y}$ is surjective. This follows from the surjectivity of the restriction $H^0(\mathcal{O}_R(1)) \to \mathcal{O}_{x,y}$. □

Proof of Lemma 6.10. Tensoring by F the standard Mayer-Vietoris exact sequence

$$0 \longrightarrow \mathcal{O}_\Gamma \longrightarrow \mathcal{O}_C \oplus \mathcal{O}_R \longrightarrow \mathcal{O}_{x,y} \longrightarrow 0$$

we have the exact sequence

$$0 \longrightarrow F \longrightarrow E \oplus \mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r} \longrightarrow F \otimes \mathcal{O}_{x,y} \longrightarrow 0.$$

Passing to the associated long exact sequence we obtain

$$0 \longrightarrow H^0(F) \xrightarrow{u} H^0(E) \oplus H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}\big) \xrightarrow{\rho} H^0\big(F \otimes \mathcal{O}_{x,y}\big) \longrightarrow H^1(F) \cdots .$$

Restricting ρ to $H^0(E) \oplus 0$ or $0 \oplus H^0(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r})$ we have the following maps

$$\rho_C : H^0(E) \longrightarrow E_x \oplus E_y,$$

and

$$\rho_R : H^0\big(\mathcal{O}_{\mathbb{P}^1}(1)^{\oplus r}\big) \longrightarrow \mathcal{O}_{\mathbb{P}^1,x}(1)^{\oplus r} \oplus \mathcal{O}_{\mathbb{P}^1,y}(1)^{\oplus r}.$$

These are the usual evaluation maps and we know they are surjective. It follows from the surjectivity of ρ and the above long exact sequence that $h^0(F) = rm = h^0(\mathcal{U}_r^*)$ and $h^1(F) = 0$. Thus, to complete the proof, it suffices to show that $H^0(\mathcal{U}_r^*) \to H^0(F)$ is injective. This is clear because the composition of maps $H^0(\mathcal{U}_r^*) \to H^0(F) \to H^0(E)$ is injective. □

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