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VORTEX KNOTS

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In this paper we present new results concerning the evolution and stability of vortex knots in the context of the Euler equations. For the first time, since Lord Kelvin's original conjecture of 1875, we have direct numerical evidence of stability of vortex filaments in the shape of torus knots. The results are based on the analytical solutions of Ricca [1] for thin vortex filaments and numerical integration of the Biot-Savart induction law. Moreover, a comparative study of vortex knot evolution under the so-called Localized Induction Approximation (LIA), which is a low-order approximation to the Biot-Savart law, confirms the stability results predicted by the LIA analysis. In particular, we show that thin vortex knots which are unstable under LIA have a greatly extended lifetime when the Biot-Savart law is used, but thick vortex knots have the same stability behaviour for both equations of motion.

Applications of ideas from modern topology to fluid mechanics have been pioneered by Moffatt [2] and co-workers [3], whose results clearly demonstrate the importance of the new techniques in the study of knotted and linked structures in fluid flows. The use of geometric and topological methods in fluid mechanics has indeed proven to be very useful in the analysis of the entanglement of filamentary vortex structures as observed in direct numerical simulations of homogeneous turbulence (see, for example [4] and [5]). The most advanced visiometrics of streamlines and vorticity lines associated with the formation of coherent structures reveal that a high degree of braiding, re-connection and formation of new linkings of field lines is a generic feature of turbulent flows. Moreover, the study of complex flow patterns using topological techniques finds useful applications in the study of filament structures present in a wide spectrum of physical scales, from magnetic flux tubes in solar physics to quantized vortex lines in superfluidity [6]. Yet, from a theoretical viewpoint very little is known about

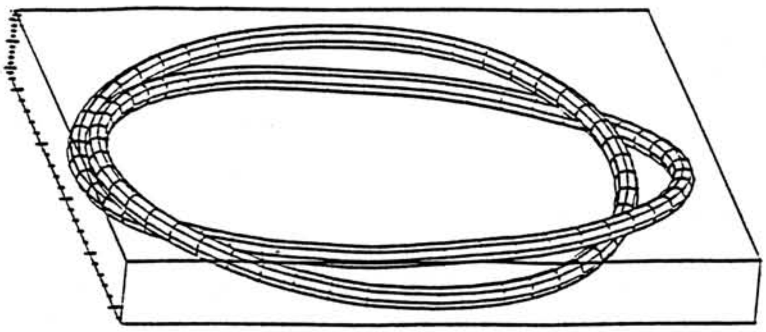

Figure 1. Evolution of torus knot $\mathcal{T}_{2,3}$ under LIA. The knot is found to be stable as predicted by the LIA analysis of Ricca. The knot is visualized by centering a thin tube on the knot axis, as shown in figure. Hence, the tube is a virtual object and its thickness is not measured by a_0.

the effects of topology on the evolution of complex structures, and there is therefore a call for more information about these processes and their mathematical modelling.

The aim of the present work is twofold: to investigate the relationship between geometry, topology and dynamics of topologically complex vortex structures; to model the topological entanglement of vortex structures using knotted vortex lines as elementary constituents. We have concentrated our attention to *torus knots* $\mathcal{T}_{p,q}$ which are thin vortex filaments wrapped around a mathematical torus, where p and q are relatively prime integers. A torus knot is characterized by its *winding number* $w = q/p$ which is a topological invariant and measures the number of wraps of the knot along the small circle of the torus per number of wraps along the large circle of the torus. In principle, vortex motion in the context of the Euler equations is governed by the Biot-Savart law which determines the self-induced velocity $\mathbf{u}(\mathbf{X})$ of a vortex line C of strength Γ in the following way:

$$\mathbf{u}(\mathbf{X}) = \frac{\Gamma}{4\pi} \int_C \frac{\hat{\mathbf{t}} \times (\mathbf{X} - \mathbf{R}(s))}{|\mathbf{X} - \mathbf{R}(s)|^3} \, ds \; . \tag{1}$$

Here $\mathbf{X}$ is the position vector, $\mathbf{X} = \mathbf{R}(s)$ the vector equation for $\mathcal{C}$, s the arc-length and $\hat{\mathbf{t}} = d\mathbf{R}/ds$ the unit tangent along $\mathcal{C}$. Note that the integral of eq. (1) is a global geometric functional and retains all the induction effects associated with the geometry of $\mathcal{C}$, while preserving topology. Unfortunately, explicit analytic solutions to the Biot-Savart law are only known for very simple geometries and in general are very difficult to obtain. Moreover, numerical simulations based on (1) are rather expensive to run, because the motion of each single vortex point depends on the motion of *all* the other points in which the vortex line is discretized. A standard way to overcome these difficulties is to use a cut-off technique based on the Localized Induction Approximation to (1). Under LIA the filament motion is governed essentially by local curvature effects and in the limit of very thin vortex filaments (1) is replaced by

$$\mathbf{u}(\mathbf{X}) = \frac{\Gamma}{4\pi} \ln\left(\frac{R_{\text{eff}}}{a_0}\right) \mathbf{R}' \times \mathbf{R}'' , \qquad (2)$$

where R_{eff} is some length-scale, which we choose equal to $8c$ (c local radius of curvature) in order to reproduce the correct velocity for vortex rings; a_0 represents a very small cut-off parameter, and typically $a_0 \approx 10^{-8}$.

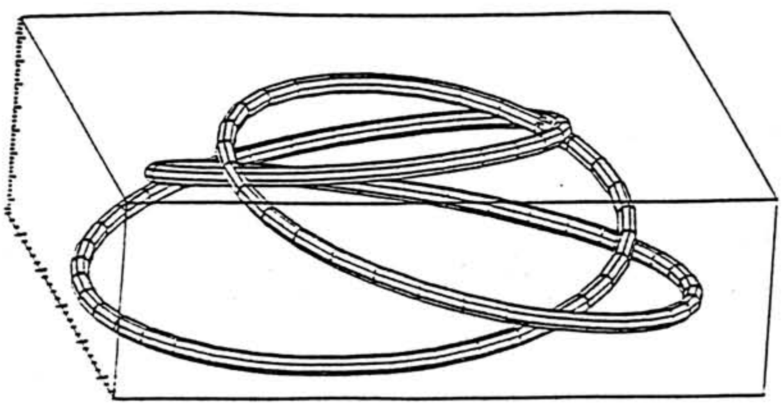

Figure 2. Evolution of torus knot $\mathcal{T}_{3,2}$ under LIA. The knot is found to be unstable as predicted by the LIA analysis of Ricca. The knot is however stabilized when its evolution is governed by the Biot-Savart law.

The numerical calculations which we have performed [7] confirm the validity of Ricca's [1] stability criterion under LIA evolution, i.e. that torus knots are stable if $w > 1$. Figure 1 shows the stable knot $\mathcal{T}_{2,3}$ and Figure 2 shows the knot $\mathcal{T}_{3,2}$ as it becomes unstable and unfolds. These results provide useful information for studying more sophisticate models of vortex structures under LIA. Another interesting result that we have found is the discovery of a strong stabilizing effect due to the Biot-Savart law. Take for example the knot $\mathcal{T}_{3,2}$: this knot becomes immediately unstable under LIA, whereas it remains stable under Biot-Savart, travelling a considerable distance. Although we find that these knots eventually de-stabilize (remember that numerical noise is always present), the time which elapses and the distance over which the knots travel before breaking-up is very large and has physical significance. Moreover, there are cases (for relatively thin knots) in which the evolution under Biot-Savart is almost identical to that given by LIA, an unexpected result worth investigating.

Finally, let us point out that unstable vortex knots evolve under Biot-Savart towards a reconnection event. This is another interesting feature of vortex knot evolution. In view of the great interest in the formation of singularities in the Euler equations, an issue that represents an outstanding problem in the mathematics of ideal fluid mechanics, unstable vortex torus knots prove to be a simple and effective means of investigation. No doubt that these results will stimulate more numerical work and will certainly give new impetus to the mathematical search for the existence of steady and stable vortex knot solutions to the Euler equations.

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