SHARP ASYMPTOTIC ESTIMATES FOR EIGENVALUES OF AHARONOV-BOHM OPERATORS WITH VARYING POLES

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Abstract. We investigate the behavior of eigenvalues for a magnetic Aharonov-Bohm operator with half-integer circulation and Dirichlet boundary conditions in a planar domain. We provide sharp asymptotics for eigenvalues as the pole is moving in the interior of the domain, approaching a zero of an eigenfunction of the limiting problem along a nodal line. As a consequence, we verify theoretically some conjectures arising from numerical evidences in preexisting literature. The proof relies on an Almgren-type monotonicity argument for magnetic operators together with a sharp blow-up analysis.

1. Introduction

The aim of this paper is to investigate the behavior of the eigenvalues of Aharonov-Bohm operators with moving poles. For $a = (a_1, a_2) \in \mathbb{R}^2$ and $\alpha \in \mathbb{R} \setminus \mathbb{Z}$, we consider the vector potential

$$A_a^\alpha(x) = \alpha \left(\frac{-(x_2 - a_2)}{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2} \right), \quad x = (x_1, x_2) \in \mathbb{R}^2 \setminus \{a\},$$

which generates the Aharonov-Bohm magnetic field in $\mathbb{R}^2$ with pole a and circulation α; such a field is produced by an infinitely long thin solenoid intersecting perpendicularly the plane (x_1, x_2) at the point a, as the radius of the solenoid goes to zero and the magnetic flux remains constantly equal to α (see e.g. [5, 6, 26]).

In this paper we will focus on the case of half-integer circulation, so we will assume $\alpha = 1/2$ and denote

$$A_a(x) = A_a^{1/2}(x) = A_0(x - a), \quad \text{where} \quad A_0(x_1, x_2) = \frac{1}{2}\left(-\frac{x_2}{x_1^2 + x_2^2}, \frac{x_1}{x_1^2 + x_2^2} \right).$$

In the spirit [8], [27] and [28], we are interested in studying the dependence on the pole a of the spectrum of Schrödinger operators with Aharonov-Bohm vector potentials, i.e. of operators $(i\nabla + A_a)^2$ acting on functions $u : \mathbb{R}^2 \to \mathbb{C}$ as

$$(i\nabla + A_a)^2 u = -\Delta u + 2iA_a \cdot \nabla u + |A_a|^2 u.$$

The interest in Aharonov-Bohm operators with half-integer circulation $\alpha = 1/2$ is motivated by the fact that nodal domains of eigenfunctions of such operators are strongly related to spectral minimal partitions of the Dirichlet laplacian with points of odd multiplicity, see [10, 28]. We refer to papers [9, 11, 15, 16, 17, 18, 19, 20, 21] for details on the deep relation between behavior of eigenfunctions, their nodal domains, and spectral minimal partitions. Furthermore, the investigation carried out in [8, 24, 27, 28] highlighted a strong connection between nodal properties of eigenfunctions and the critical points of the map which associates eigenvalues of the operator A_a to the position of pole a. Motivated by this, in the present paper we deepen the investigation started in [8, 27] about the dependence of eigenvalues of Aharonov-Bohm operators on the pole position, aiming at proving sharp asymptotic estimates for the convergence of eigenvalues associated to operators with a moving pole.

Let $\Omega \subset \mathbb{R}^2$ be a bounded, open and simply connected domain. For every $a \in \Omega$, we introduce the space $H^{1,a}(\Omega, \mathbb{C})$ as the completion of $\{u \in H^1(\Omega, \mathbb{C}) \cap C^\infty(\Omega, \mathbb{C}) : u \text{ vanishes in a neighborhood of } a\}$ with respect to the norm

$$\|u\|_{H^{1,a}(\Omega,\mathbb{C})} = \left(\|\nabla u\|^2_{L^2(\Omega,\mathbb{C}^2)} + \|u\|^2_{L^2(\Omega,\mathbb{C})} + \left\| \frac{u}{|x-a|} \right\|^2_{L^2(\Omega,\mathbb{C})} \right)^{1/2}.$$

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It is easy to verify that $H^{1,a}(\Omega,\mathbb{C}) = \{u \in H^1(\Omega,\mathbb{C}) : \frac{u}{|x-a|} \in L^2(\Omega,\mathbb{C})\}$. We also observe that, in view of the Hardy type inequality proved in [23] (see (17)), an equivalent norm in $H^{1,a}(\Omega,\mathbb{C})$ is given by

$$\left(\|(i\nabla + A_a)u\|^2_{L^2(\Omega,\mathbb{C}^2)} + \|u\|^2_{L^2(\Omega,\mathbb{C})}\right)^{1/2}. \tag{1}$$

We also consider the space $H^{1,a}_0(\Omega,\mathbb{C})$ as the completion of $C^\infty_c(\Omega \setminus \{a\},\mathbb{C})$ with respect to the norm $\|\cdot\|_{H^1_a(\Omega,\mathbb{C})}$, so that $H^{1,a}_0(\Omega,\mathbb{C}) = \{u \in H^1_0(\Omega,\mathbb{C}) : \frac{u}{|x-a|} \in L^2(\Omega,\mathbb{C})\}$.

For every $a \in \Omega$, we consider the eigenvalue problem

$$\begin{cases} (i\nabla + A_a)^2 u = \lambda u, & \text{in } \Omega, \\ u = 0, & \text{on } \partial\Omega, \end{cases} \tag{E_a}$$

in a weak sense, i.e. we say that $\lambda \in \mathbb{R}$ is an eigenvalue of problem (E_a) if there exists $u \in H^{1,a}_0(\Omega,\mathbb{C})\setminus\{0\}$ (called eigenfunction) such that

$$\int_\Omega (i\nabla u + A_a u)\cdot \overline{(i\nabla v + A_a v)}\,dx = \lambda \int_\Omega u\overline{v}\,dx \quad \text{for all } v \in H^{1,a}_0(\Omega,\mathbb{C}).$$

From classical spectral theory, the eigenvalue problem (E_a) admits a sequence of real diverging eigenvalues $\{\lambda_k^a\}_{k\geq 1}$ with finite multiplicity; in the enumeration $\lambda_1^a \leq \lambda_2^a \leq \cdots \leq \lambda_j^a \leq \ldots$, we repeat each eigenvalue as many times as its multiplicity. We are interested in the behavior of the function $a \mapsto \lambda_j^a$ in a neighborhood of a fixed point $b \in \Omega$. Up to a translation, it is not restrictive to consider $b = 0$. Thus, we assume that $0 \in \Omega$.

In [8, Theorem 1.1] and [24, Theorem 1.2] it is proved that, for all $j \geq 1$,

$$\text{the function } a \mapsto \lambda_j^a \text{ is continuous in } \Omega. \tag{2}$$

A strong improvement of the regularity (2) holds under simplicity of the eigenvalue. Indeed in [8, Theorem 1.3] it is proved that, if there exists $n_0 \geq 1$ such that

$$\lambda^0_{n_0} \quad \text{is simple,} \tag{3}$$

then the function $a \mapsto \lambda^a_{n_0}$ is of class C^∞ in a neighborhood of 0; this regularity result is improved in [24, Theorem 1.3], where, in the more general setting of Aharonov-Bohm operators with many singularities, it is shown that, under assumption (3) the function $a \mapsto \lambda^a_{n_0}$ is analytic in a neighborhood of 0. Then the question of what is the leading term in the asymptotic expansion of such a function (at least on a single straight path around the limit point 0) naturally arises. The main purpose of the present paper is to answer such a question. This may also shed some light on the nature of 0 as a critical point for the map $a \mapsto \lambda_a$ when the limit eigenfunction has in 0 a zero of order $k/2$ with $k \geq 3$ odd.

At a deep insight into the problem, papers [8] and [28] suggest a high reliability of the behavior of the eigenvalue $\lambda^a_{n_0}$ on the structure of the nodal lines of *the* eigenfunction relative to $\lambda^0_{n_0}$. In order to enter into the issue, let us establish the setting and some notation.

Let us assume that there exists $n_0 \geq 1$ such that (3) holds and denote $\lambda_0 = \lambda^0_{n_0}$ and, for any $a \in \Omega$, $\lambda_a = \lambda^a_{n_0}$. From (2) it follows that, if $a \to 0$, then $\lambda_a \to \lambda_0$. Let $\varphi_0 \in H^{1,0}_0(\Omega,\mathbb{C})\setminus\{0\}$ be an eigenfunction of problem (E_0) associated to the eigenvalue $\lambda_0 = \lambda^0_{n_0}$, i.e. solving

$$\begin{cases} (i\nabla + A_0)^2 \varphi_0 = \lambda_0 \varphi_0, & \text{in } \Omega, \\ \varphi_0 = 0, & \text{on } \partial\Omega, \end{cases} \tag{4}$$

such that

$$\int_\Omega |\varphi_0(x)|^2\,dx = 1. \tag{5}$$

In view of [13, Theorem 1.3] (see also Proposition 2.1 below) we have that

$$\varphi_0 \text{ has at 0 a zero of order } \frac{k}{2} \text{ for some odd } k \in \mathbb{N}, \tag{6}$$

see [8, Definition 1.4]. We recall from [13, Theorem 1.3] and [28, Theorem 1.5] that (6) implies that the eigenfunction φ_0 has got exactly k nodal lines meeting at 0 and dividing the whole angle into k equal parts.

A first result relating the rate of convergence of λ_a to λ_0 with the order of vanishing of φ_0 at 0 can be found in [8], where the following estimate is proved.

Theorem 1.1 ([8], Theorem 1.7)**.** *If assumptions* (3) *and* (6) *with* $k \geq 3$ *are satisfied, then*

$$|\lambda_a - \lambda_0| \leq C|a|^{\frac{k+1}{2}} \qquad \textit{as } a \to 0$$

for a constant $C > 0$ *independent of* a.

As already mentioned, the latter theorem pursue the idea that the asymptotic expansion of the function $a \mapsto \lambda_a$ has to do with the nodal properties of the related limit eigenfunction.

The main result of the present paper establishes the exact order of the asymptotic expansion of $\lambda_a - \lambda_0$ along a suitable direction as $|a|^k$, where k is the number of nodal lines of φ_0 at 0 which coincides with twice the order of vanishing of φ_0 in assumption (6). In addition, we detect the sharp coefficient of the asymptotics, which can be characterized in terms of the limit profile of a blow-up sequence obtained by a suitable scaling of approximating eigenfunctions.

In order to state our main result, we need to recall some known facts and to introduce some additional notation. By [13, Theorem 1.3] (see Proposition 2.1 below), if φ_0 is an eigenfunction of $(i\nabla + A_0)^2$ on Ω satisfying assumption (6), there exist $\beta_1, \beta_2 \in \mathbb{C}$ such that $(\beta_1, \beta_2) \neq (0, 0)$ and

$$r^{-k/2}\varphi_0(r(\cos t, \sin t)) \to \beta_1 e^{i\frac{t}{2}} \cos\left(\frac{k}{2}t\right) + \beta_2 e^{i\frac{t}{2}} \sin\left(\frac{k}{2}t\right) \quad \text{in } C^{1,\tau}([0, 2\pi], \mathbb{C}) \tag{7}$$

as $r \to 0^+$ for any $\tau \in (0, 1)$.

Let s_0 be the positive half-axis $s_0 = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0 \text{ and } x_1 \geq 0\}$. We observe that, for every odd natural number k, there exists a unique (up to a multiplicative constant) function ψ_k which is harmonic on $\mathbb{R}^2 \setminus s_0$, homogeneous of degree $k/2$ and vanishing on s_0. Such a function is given by

$$\psi_k(r\cos t, r\sin t) = r^{k/2} \sin\left(\frac{k}{2}t\right), \quad r \geq 0, \quad t \in [0, 2\pi]. \tag{8}$$

Let $s := \{(x_1, x_2) \in \mathbb{R}^2 : x_2 = 0 \text{ and } x_1 \geq 1\}$ and $\mathbb{R}^2_+ = \{(x_1, x_2) \in \mathbb{R}^2 : x_2 > 0)\}$. We denote as $\mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$ the completion of $C^\infty_c(\overline{\mathbb{R}^2_+} \setminus s)$ under the norm $(\int_{\mathbb{R}^2_+} |\nabla u|^2 \, dx)^{1/2}$. From the Hardy type inequality proved in [23] (see (17)) and a change of gauge, it follows that functions in $\mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$ satisfy the following Hardy type inequality:

$$\int_{\mathbb{R}^2} |\nabla\varphi(x)|^2 \, dx \geq \frac{1}{4} \int_{\mathbb{R}^2} \frac{|\varphi(x)|^2}{|x - \mathbf{e}|^2} \, dx, \quad \text{for all } \varphi \in \mathcal{D}^{1,2}_s(\mathbb{R}^2_+),$$

where $\mathbf{e} = (1, 0)$. Then $\mathcal{D}^{1,2}_s(\mathbb{R}^2_+) = \left\{u \in L^1_{\mathrm{loc}}(\overline{\mathbb{R}^2_+} \setminus s) : \nabla u \in L^2(\mathbb{R}^2_+), \ \frac{u}{|x - \mathbf{e}|} \in L^2(\mathbb{R}^2_+), \text{ and } u = 0 \text{ on } s\right\}$. The functional

$$J_k : \mathcal{D}^{1,2}_s(\mathbb{R}^2_+) \to \mathbb{R}, \quad J_k(u) = \frac{1}{2} \int_{\mathbb{R}^2_+} |\nabla u(x)|^2 \, dx - \int_{\partial\mathbb{R}^2_+ \setminus s} u(x_1, 0) \frac{\partial\psi_k}{\partial x_2}(x_1, 0) \, dx_1, \tag{9}$$

is well-defined on the space $\mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$; we notice that $\frac{\partial\psi_k}{\partial x_2}$ vanishes on $\partial\mathbb{R}^2_+ \setminus s_0$, so that

$$\int_{\partial\mathbb{R}^2_+ \setminus s} u(x_1, 0) \frac{\partial\psi_k}{\partial x_2}(x_1, 0) \, dx_1 = \int_0^1 u(x_1, 0) \frac{\partial\psi_k}{\partial x_2}(x_1, 0) \, dx_1.$$

By standard minimization methods, J_k achieves its minimum over the whole space $\mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$ at some function $w_k \in \mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$, i.e. there exists $w_k \in \mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$ such that

$$\mathfrak{m}_k = \min_{u \in \mathcal{D}^{1,2}_s(\mathbb{R}^2_+)} J_k(u) = J_k(w_k). \tag{10}$$

We note that

$$\mathfrak{m}_k = J_k(w_k) = -\frac{1}{2} \int_{\mathbb{R}^2_+} |\nabla w_k(x)|^2 \, dx = -\frac{1}{2} \int_0^1 \frac{\partial_+\psi_k}{\partial x_2}(x_1, 0) \, w_k(x_1, 0) \, dx_1 < 0, \tag{11}$$

where, for all $x_1 > 0$, $\frac{\partial_+\psi_k}{\partial x_2}(x_1, 0) = \lim_{t\to0^+} \frac{\psi_k(x_1,t)-\psi_k(x_1,0)}{t} = \frac{k}{2} x_1^{\frac{k}{2}-1}$.

We are now in a position to state our main theorem.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, open and simply connected domain such that $0 \in \Omega$ and let $n_0 \geq 1$ be such that the n_0-th eigenvalue $\lambda_0 = \lambda^0_{n_0}$ of $(i\nabla + A_0)^2$ on Ω is simple with associated eigenfunctions having in 0 a zero of order $k/2$ with $k \in \mathbb{N}$ odd. For $a \in \Omega$ let $\lambda_a = \lambda^a_{n_0}$ be the n_0-th eigenvalue of $(i\nabla + A_a)^2$ on Ω. Let $\mathfrak{r}$ be the half-line tangent to a nodal line of eigenfunctions associated to λ_0 ending at 0. Then, as $a \to 0$ with $a \in \mathfrak{r}$,*

$$\frac{\lambda_0 - \lambda_a}{|a|^k} \to -4\left(|\beta_1|^2 + |\beta_2|^2\right) \mathfrak{m}_k \tag{12}$$

with $(\beta_1, \beta_2) \neq (0, 0)$ being as in (7) and $\mathfrak{m}_k$ being as in (10)–(11).

Remark 1.3. Due to the analyticity of the function $a \mapsto \lambda_a$ established in [24, Theorem 1.3], from Theorem 1.2 it follows that

$$\frac{\lambda_0 - \lambda_a}{|a|^k} \to 4\left(|\beta_1|^2 + |\beta_2|^2\right) \mathfrak{m}_k$$

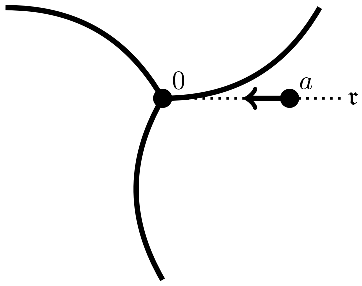

FIGURE 1. a approaches 0 along the tangent $\mathfrak{r}$ to a nodal line of φ_0.

as $a \to 0$ along the half-line opposite to the tangent to a nodal line of φ_0. In particular, we have that the restriction of the function $\lambda_0 - \lambda_a$ on the straight line tangent to a nodal line of φ_0 changes sign at 0 (is positive on the side of the nodal line of φ_0 and negative on the opposite side). Hence, if λ_0 is simple, then 0 cannot be an extremal point of the map $a \mapsto \lambda_a$.

We remark that Theorem 1.2 is significant not only from a pure "analytic" point of view (detecting of sharp asymptotics), but also from a quite theoretical point of view. Indeed Theorem 1.2 and the consequent Remark 1.3 allow completing some results of papers [8, 27, 28] concerning the investigation of critical and extremal points of the map $a \mapsto \lambda_a$. It is worth recalling from [8, Corollary 1.2] that the function $a \mapsto \lambda_a$ must have an extremal point in Ω. More precisely, in [8] the following result is proved.

Proposition 1.4 ([8], Corollary 1.8)**.** *Fix any $j \in \mathbb{N}$. If 0 is an extremal point of $a \mapsto \lambda_j^a$, then either λ_j^0 is not simple, or the eigenfunction of $(i\nabla + A_0)^2$ associated to λ_j^0 has at 0 a zero of order $k/2$ with $k \geq 3$ odd.*

In view of Theorem 1.2 and Remark 1.3, we can exclude the second alternative in Proposition 1.4, obtaining the following result.

Corollary 1.5. *Fix any $j \in \mathbb{N}$. If 0 is an extremal point of $a \mapsto \lambda_j^a$, then λ_j^0 is not simple.*

The simulations in [8] suggest that extremal points of the map $a \mapsto \lambda_a$ are generally attained at points where the function itself is not differentiable. Taking into account Corollary 1.5, we may conjecture that the missed differentiability is produced by the dropping of assumption (3).

Furthermore, several numerical simulations presented in [8] are validated and confirmed by Theorem 1.2. Indeed, Theorem 1.2 proves that the asymptotic expansion of $\lambda_0 - \lambda_a$ has a leading term of odd degree, hence, if $k \geq 3$, 0 is a stationary inflexion point along k directions (corresponding to the nodal lines of φ_0), as experimentally predicted by numerical simulations in [8, Section 7]. More precisely, as a consequence of Theorem 1.2 and Remark 1.3 we can state the following result.

Corollary 1.6. *Under assumptions (3) and (6) with $k \geq 3$, 0 is a saddle point for the map $a \mapsto \lambda_a$. In particular, 0 is a stationary and not extremal point.*

On the other hand, under assumptions (3) and (6) with $k = 1$, Theorem 1.2 implies that the gradient of the function $a \mapsto \lambda_a$ in 0 is different from zero, then 0 is not a stationary point, a fortiori not even an extremal point; we then recover a result stated in [28, Corollary 1.2].

The proof of Theorem 1.2 is based on the Courant-Fisher minimax characterization of eigenvalues. The asymptotics for eigenvalues is derived by combining estimates from above and below of the Rayleigh quotient. To obtain sharp estimates, we construct proper test functions for the Rayleigh quotient by suitable manipulation of eigenfunctions. In this way, we obtain upper and lower bounds whose limit as $a \to 0$ can be explicitly computed taking advantage of a fine blow-up analysis for scaled eigenfunctions. More precisely, we prove (see Theorem 8.2) that the blow-up sequence

$$\frac{\varphi_a(|a|x)}{|a|^{k/2}} \tag{13}$$

converges as $|a| \to 0^+$, $a \in \mathfrak{r}$, to a limit profile, which can be identified, up to a phase and a change of coordinates, with $w_k + \psi_k$, being w_k and ψ_k as in (10) and (8) respectively. The proof of the energy estimates for the blow-up sequence uses a monotonicity argument inspired by [7], based on the study of an Almgren-type frequency function given by the ratio of the local magnetic energy over mass near the origin; see [13, 22, 27] for Almgren-type monotonicity formulae for elliptic operators with magnetic potentials. We mention that a similar approach based on estimates of the Rayleigh quotient, blow-up analysis and monotonicity formula was used in [3] to prove a sharp control of the rate of convergence of the eigenvalues and eigenfunctions of the Dirichlet laplacian in a perturbed domain (obtained by attaching a shrinking handle to a smooth region) to the relative eigenvalue and eigenfunction in the limit domain (see also [4, 14] for blow-up analysis and monotonicity formula); however, in [4, 3, 14] only the particular case of limit eigenfunctions having at the singular point the lowest vanishing order (corresponding to the case $k = 1$ in our setting) was considered. In the present paper we do not prescribe any restriction on the order of the zero of the limit eigenfunction: this produces significant additional difficulties with respect to [3], the main of which relies in the identification of the limit profile of the blow-up sequence (13). Such a difficulty is overcome here by fine energy estimates of the difference between approximating and limit eigenfunctions, performed exploiting the invertibility of an operator associated to the limit eigenvalue problem.

From Theorem 1.2, it follows that, under the assumptions of Theorem 1.2, the Taylor polynomials of the function $a \mapsto \lambda_0 - \lambda_a$ with center 0 and degree strictly smaller than k vanish, since by Theorem 1.2 they vanish on the k independent directions corresponding to the nodal lines of φ_0 (see [2, Lemma 1.1] and [8, Lemma 6.6]). Then we obtain the following Taylor expansion at 0:

$$\lambda_0 - \lambda_a = P(a) + o(|a|^k), \quad \text{as } |a| \to 0^+,$$

for some

$$P \not\equiv 0, \quad P(a) = P(a_1, a_2) = \sum_{j=0}^{k} \alpha_j a_1^{k-j} a_2^j$$

homogeneous polynomial of degree k. The detection of the exact value of all coefficients of the polynomial (and hence the sharp asymptotics along any direction) is studied in the subsequent paper [2]. In the asymptotic analysis along any direction performed in [2], we will not be able to construct explicitly the limit profile of blown-up eigenfunctions as done in the present paper for directions of nodal lines; such difficulty is treated in [2] studying the dependence of the limit profile on the position of the pole and the symmetry/periodicity properties of the homogeneous polynomial P. The complete classification of homogeneous k-degree polynomials with such periodicity/symmetry invariances (which will allow us to determine explicitly the polynomial P in [2]) requires the result of Theorem 1.2 as a crucial ingredient; in particular the information that the limit in (12) is strictly positive is the starting point in [2], since it provides, besides the exact degree of the polynomial P, informations about locations of zeroes and factorization.

The paper is organized as follows. Sections 2 and 3 are devoted to set up the framework, recall some useful known facts, introduce notation and prove some basic inequalities. Section 4 contains the construction of a suitable limit profile which will be used to describe the limit of the blowed-up sequence. The study of the behavior of such a blow-up sequence can proceed thanks to the Almgren-type monotonicity argument which is presented in section 5. Via the energy estimates proved within section 5, in section 6 we present some preliminary upper and lower bounds for the difference $\lambda_0 - \lambda_a$, relying on the well-known minimax characterization for eigenvalues. Section 7 contains energy estimates of the difference between approximating and limit eigenfunctions which are used to identify the limit profile in the sharp blow-up analysis which is performed in section 8. Finally, section 9 concludes the proof of Theorem 1.2.

1.1. **Notation and review of known formulas.** We list below some notation used throughout the paper.

- For $r > 0$ and $a \in \mathbb{R}^2$, $D_r(a) = \{x \in \mathbb{R}^2 : |x - a| < r\}$ denotes the disk of center a and radius r.
- For all $r > 0$, $D_r = D_r(a)$ denotes the disk of center 0 and radius r.
- For every complex number $z \in \mathbb{C}$, $\overline{z}$ denotes its complex conjugate.
- For $z \in \mathbb{C}$, $\mathfrak{Re}\, z$ denotes its real part and $\mathfrak{Im}\, z$ its imaginary part.
- For $R > 0$, let $\eta_R : \mathbb{R}^2 \to \mathbb{R}$ be a smooth cut-off function such that

$$\eta_R \equiv 0 \text{ in } D_{R/2}, \quad \eta_R \equiv 1 \text{ on } \mathbb{R}^2 \setminus D_R, \quad 0 \le \eta_R \le 1 \quad \text{and} \quad |\nabla \eta_R| \le 4/R \text{ in } \mathbb{R}^2. \tag{14}$$

- For every $b = (b_1, b_2) \in \mathbb{R}^2$, we denote as θ_b the function $\theta_b : \mathbb{R}^2 \setminus \{b\} \to [0, 2\pi)$ defined as

$$\theta_b(x_1, x_2) = \begin{cases} \arctan \frac{x_2 - b_2}{x_1 - b_1}, & \text{if } x_1 > b_1,\ x_2 \ge b_2, \\ \frac{\pi}{2}, & \text{if } x_1 = b_1,\ x_2 > b_2, \\ \pi + \arctan \frac{x_2 - b_2}{x_1 - b_1}, & \text{if } x_1 < b_1, \\ \frac{3}{2}\pi, & \text{if } x_1 = b_1,\ x_2 < b_2, \\ 2\pi + \arctan \frac{x_2 - b_2}{x_1 - b_1}, & \text{if } x_1 > b_1,\ x_2 < b_2, \end{cases} \tag{15}$$

so that $\theta_b(b + r(\cos t, \sin t)) = t$ for all $r > 0$ and $t \in [0, 2\pi)$.

We also recall the Courant-Fisher *minimax characterization* of eigenvalues which will be used to estimate the eigenvalue variation in section 6. The Rayleigh quotient associated to the eigenvalue problem (E_a) is

$$\mathfrak{R}_a : H_0^{1,a}(\Omega, \mathbb{C}) \to \mathbb{R}, \quad \mathfrak{R}_a(u) = \frac{\int_\Omega |(i\nabla + A_a)u|^2 dx}{\int_\Omega |u|^2\, dx}.$$

It is well-known from classical spectral theory that the eigenvalues $\lambda_1^a \le \lambda_2^a \le \dots \le \lambda_j^a \le \dots$ of problem (E_a) admit the following variational characterization:

$$\lambda_j^a = \min\left\{ \max_{u \in F\setminus\{0\}} \mathfrak{R}_a(u) : F \text{ is a subspace of } H_0^{1,a}(\Omega,\mathbb{C}) \text{ with } \dim F = j \right\}. \tag{16}$$

2. Preliminaries

2.1. **Diamagnetic and Hardy inequalities.** We recall from [23] (see also [13, Lemma 3.1 and Remark 3.2]) the following Hardy type inequality

$$\int_{D_r(a)} |(i\nabla + A_a)u|^2\,dx \ge \frac{1}{4}\int_{D_r(a)} \frac{|u(x)|^2}{|x-a|^2}\,dx, \tag{17}$$

which holds for all $r > 0$, $a \in \mathbb{R}^2$ and $u \in H^{1,a}(D_r(a),\mathbb{C})$.

We also recall the well-known *diamagnetic inequality* (see e.g. [25] or [13, Lemma A.1] for a proof): if $a \in \Omega$ and $u \in H^{1,a}(\Omega,\mathbb{C})$, then

$$|\nabla|u|(x)| \le |i\nabla u(x) + A_a(x)u(x)| \quad \text{for a.e. } x \in \Omega. \tag{18}$$

2.2. **Approximating eigenfunctions.** For all $a \in \Omega$, let $\varphi_a \in H_0^{1,a}(\Omega,\mathbb{C}) \setminus \{0\}$ be an eigenfunction of problem (E_a) associated to the eigenvalue λ_a, i.e. solving

$$\begin{cases} (i\nabla + A_a)^2\varphi_a = \lambda_a\varphi_a, & \text{in } \Omega,\\ \varphi_a = 0, & \text{on } \partial\Omega, \end{cases} \tag{19}$$

such that

$$\int_\Omega |\varphi_a(x)|^2\,dx = 1 \quad \text{and} \quad \int_\Omega e^{\frac{i}{2}(\theta_0-\theta_a)(x)}\varphi_a(x)\overline{\varphi_0(x)}\,dx \text{ is a positive real number}, \tag{20}$$

where φ_0 is as in (4–5) and θ_a, θ_0 are defined in (15); we observe that, given an eigenfunction v of (E_a) associated to λ_a, to obtain an eigenfunction φ_a satisfying the normalization conditions (20) it is enough to consider $(\int_\Omega |v|^2\,dx)^{-1}e^{i\vartheta}v$ where $\vartheta = \arg\Big[\big(\int_\Omega |v|^2\,dx\big)\big(\int_\Omega e^{i(\theta_0-\theta_a)/2}v\overline{\varphi_0}\,dx\big)^{-1}\Big]$. Using (3), (4), (19), (20), and standard elliptic estimates, it is easy to prove that

$$\varphi_a \to \varphi_0 \quad \text{in } H^1(\Omega,\mathbb{C}) \text{ and in } C^2_{\rm loc}(\Omega\setminus\{0\},\mathbb{C}) \tag{21}$$

and

$$\int_\Omega \left|(i\nabla + A_a)\varphi_a(x)\right|^2\,dx \to \int_\Omega \left|(i\nabla + A_0)\varphi_0(x)\right|^2\,dx \tag{22}$$

as $a \to 0$. We notice that (21) and (22) imply that

$$(i\nabla + A_a)\varphi_a \to (i\nabla + A_0)\varphi_0 \quad \text{in } L^2(\Omega,\mathbb{C}). \tag{23}$$

2.3. **Local asymptotics of eigenfunctions.** We recall from [13] the description of the asymptotics at the singularity of solutions to elliptic equations with Aharonov-Bohm potentials. In the case of Aharonov-Bohm potentials with circulation $\frac{1}{2}$, such asymptotics is described in terms of eigenvalues and eigenfunctions of the following operator $\mathfrak{L}$ acting on 2π-periodic functions

$$\mathfrak{L}\psi = -\psi'' + i\psi' + \frac{1}{4}\psi. \tag{24}$$

It is easy to verify that the eigenvalues of $\mathfrak{L}$ are $\left\{\frac{j^2}{4} : j \in \mathbb{N},\ j \text{ is odd}\right\}$; moreover each eigenvalue $\frac{j^2}{4}$ has multiplicity 2 and an $L^2((0,2\pi),\mathbb{C})$-orthonormal basis of the eigenspace associated to the eigenvalue $\frac{j^2}{4}$ is formed by the functions

$$\psi_{j,1}(t) = \frac{e^{i\frac{t}{2}}}{\sqrt{\pi}}\cos\Big(\frac{j}{2}t\Big), \quad \psi_{j,2}(t) = \frac{e^{i\frac{t}{2}}}{\sqrt{\pi}}\sin\Big(\frac{j}{2}t\Big). \tag{25}$$

Proposition 2.1 ([13], Theorem 1.3)**.** *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set, $b \in \Omega$ and $h \in L^\infty_{\rm loc}(\Omega\setminus\{0\},\mathbb{C})$ such that $|h(x)| = O(|x|^{-2+\varepsilon})$ as $|x| \to 0$ for some $\varepsilon > 0$. Let $u \in H^{1,b}(\Omega,\mathbb{C})$ be a nontrivial weak solution to*

$$(i\nabla + A_b)^2 u = hu, \quad \text{in } \Omega, \tag{26}$$

i.e.

$$\int_\Omega (i\nabla u + A_b u)\cdot\overline{(i\nabla v + A_b v)}\,dx = \int_\Omega hu\overline{v}\,dx \quad \text{for all } v \in H_0^{1,b}(\Omega,\mathbb{C}). \tag{27}$$

Then there exists an odd $j \in \mathbb{N}$ such that

$$\lim_{r\to0^+} \frac{r\int_{D_r(b)} \left(|(i\nabla + A_b)u(x)|^2 - (\mathfrak{Re}\, h(x))\,|u(x)|^2\right) dx}{\int_{\partial D_r(b)} |u|^2\, ds} = \frac{j}{2}. \tag{28}$$

Furthermore,

$$r^{-j/2}u(b + r(\cos t, \sin t)) \to \sqrt{\pi}\beta_{j,1}(b, u, h)\psi_{j,1}(t) + \sqrt{\pi}\beta_{j,2}(b, u, h)\psi_{j,2}(t) \quad \text{in } C^{1,\alpha}([0, 2\pi], \mathbb{C}) \tag{29}$$

as $r \to 0^+$ for any $\alpha \in (0, 1)$, where, for $\ell = 1, 2$,

$$\beta_{j,\ell}(b, u, h) = \frac{1}{\sqrt{\pi}}\int_0^{2\pi} \bigg[(R^{-\frac{j}{2}}u(b + R(\cos t, \sin t)) + \int_0^R \frac{h(b + s(\cos t, \sin t))u(b + s(\cos t, \sin t))}{j}\left(s^{1-\frac{j}{2}} - \frac{s^{1+\frac{j}{2}}}{R^j}\right) ds\bigg]\overline{\psi_{j,\ell}(t)}\, dt \tag{30}$$

for all $R > 0$ such that $\{x \in \mathbb{R}^2 : |x - b| \le R\} \subset \Omega$ and $(\beta_{j,1}(b, u, h), \beta_{j,2}(b, u, h)) \neq (0, 0)$.

From Proposition 2.1 we have that, under assumption (6),

$$r^{-k/2}\varphi_0(r(\cos t, \sin t)) \to e^{i\frac{t}{2}}\left(\beta_{k,1}(0, \varphi_0, \lambda_0)\cos\left(\tfrac{k}{2}t\right) + \beta_{k,2}(0, \varphi_0, \lambda_0)\sin\left(\tfrac{k}{2}t\right)\right)$$

in $C^{1,\alpha}([0, 2\pi], \mathbb{C})$ as $r \to 0^+$ for any $\alpha \in (0, 1)$ with

$$(\beta_{k,1}(0, \varphi_0, \lambda_0), \beta_{k,2}(0, \varphi_0, \lambda_0)) \neq (0, 0), \tag{31}$$

where $\beta_{k,\ell}(0, \varphi_0, \lambda_0)$ are defined as in (30). We observe that, from [16] (see also [8, Lemma 2.3]), the function $e^{-i\frac{t}{2}}\varphi_0(r(\cos t, \sin t))$ is a multiple of a real-valued function and therefore

$$\text{either } \beta_{k,1}(0, \varphi_0, \lambda_0) = 0 \text{ or } \tfrac{\beta_{k,2}(0,\varphi_0,\lambda_0)}{\beta_{k,1}(0,\varphi_0,\lambda_0)} \text{ is real.} \tag{32}$$

Since (31) and (32) hold, the function

$$t \mapsto \beta_{k,1}(0, \varphi_0, \lambda_0)\cos\left(\tfrac{k}{2}t\right) + \beta_{k,2}(0, \varphi_0, \lambda_0)\sin\left(\tfrac{k}{2}t\right)$$

has exactly k zeroes $t_1, t_2, \ldots, t_k$ in $[0, 2\pi)$. Up to a change of coordinates in $\mathbb{R}^2$, it is not restrictive to assume that $0 \in \{t_1, t_2, \ldots, t_k\}$, i.e. to assume that

$$\beta_{k,1}(0, \varphi_0, \lambda_0) = 0. \tag{33}$$

Remark 2.2. Condition (33) can be interpreted as a suitable change of the cartesian coordinate system (x_1, x_2) in $\mathbb{R}^2$: we rotate the axes in such a way that the positive x_1-axis is tangent to one of the k nodal lines of φ_0 ending at 0 (see [28, Theorem 1.5] for the description of nodal lines of eigenfunctions near the pole). It is easy to verify that, besides the alignment of a nodal line of φ_0 along the x_1-axis, such a change of coordinates has also the effect of rotating the vector $(\beta_{k,1}(0, \varphi_0, \lambda_0), \beta_{k,2}(0, \varphi_0, \lambda_0))$; hence, since in the asymptotics stated in Theorem 1.2 only the norm of such a vector is involved, it is enough to prove the theorem for $\beta_{k,1}(0, \varphi_0, \lambda_0) = 0$.

By Proposition 2.1, under conditions (31) and (33), $\beta_{k,2}(0, \varphi_0, \lambda_0) \neq 0$ can be also characterized as

$$\beta_{k,2}(0, \varphi_0, \lambda_0) = \frac{1}{\pi}\lim_{r\to0^+} r^{-k/2}\int_0^{2\pi} \varphi_0(r(\cos t, \sin t))e^{-i\frac{t}{2}}\sin\left(\tfrac{k}{2}t\right) dt. \tag{34}$$

2.4. **Fourier coefficients of angular components of solutions.** Let $U \subseteq \mathbb{R}^2$ be an open set, $b \in U$ and $u \in H^{1,b}(U, \mathbb{C})$ be a weak solution (in the sense of (27)) to the problem

$$(i\nabla + A_b)^2u = \lambda u, \quad \text{in } U, \quad \text{for some } \lambda \in \mathbb{R}. \tag{35}$$

If $b \in \mathbb{R}^2$ is of the form $b = (|b|, 0)$, letting θ_b as in (15), we have that $\theta_b \in C^\infty(\mathbb{R}^2 \setminus ([|b|, +\infty) \times \{0\})$ and $\nabla\theta_b$ can be extended to be in $C^\infty(\mathbb{R}^2 \setminus \{b\})$ with $\nabla\left(\frac{\theta_b}{2}\right) = A_b$ in $\mathbb{R}^2 \setminus \{b\}$.

Let $b = (|b|, 0) \in U$ and let $u \in H^{1,b}(U, \mathbb{C})$ be a weak solution to (35). Let $R > 0$ be such that $R > |b|$ and $D_R \subset U$. For $\ell \in \{1, 2\}$ and j odd natural number we define, for all $r \in (|b|, R)$,

$$v_{j,\ell}(r) := \int_0^{2\pi} u(r(\cos t, \sin t))e^{-\frac{i}{2}\theta_b(r\cos t, r\sin t)}e^{i\frac{t}{2}}\overline{\psi_{j,\ell}(t)}\, dt. \tag{36}$$

We note that $\{v_{j,\ell}(r)\}_{j,\ell}$ are the Fourier coefficients of the function

$$t \mapsto u(r(\cos t, \sin t))e^{-\frac{i}{2}(\theta_b - \theta_0)(r\cos t, r\sin t)}$$

with respect to the orthonormal basis of the space of periodic-$L^2((0,2\pi),\mathbb{C})$ functions given in (25). Since the function $w = ue^{-\frac{i}{2}\theta_b}$ solves $-\Delta w = \lambda w$ in $\mathbb{R}^2 \setminus \{(x_1,0) : x_1 \geq |b|\}$ and jumps to its opposite across the crack $\{(x_1,0) : x_1 \geq |b|\}$ (as well as its derivative $\frac{\partial w}{\partial x_2}$), we have that $v_{j,\ell}$ is a solution to the equation

$$-\left(r^{1+j}\left(r^{-\frac{j}{2}}v_{j,\ell}\right)'\right)' = \lambda\, r^{1+\frac{j}{2}} v_{j,\ell}, \quad \text{in } (|b|, R). \tag{37}$$

3. Poincaré type inequalities

In this section we establish some Poincaré type inequalities uniformly with respect to varying poles.

Lemma 3.1 (Poincaré inequality)**.** *Let $r > 0$ and $a \in D_r$. For any $u \in H^{1,a}(\Omega,\mathbb{C})$ the following inequality holds true*

$$\frac{1}{r^2}\int_{D_r} |u|^2\, dx \leq \frac{1}{r}\int_{\partial D_r} |u|^2\, ds + \int_{D_r} |(i\nabla + A_a)u|^2\, dx. \tag{38}$$

Proof. From the Divergence Theorem, the Young inequality, and the diamagnetic inequality (18), it follows that

$$\begin{aligned}
\frac{2}{r^2}\int_{D_r} |u|^2\, dx &= \frac{1}{r^2}\int_{D_r}\Big(\mathrm{div}(|u|^2 x) - 2|u|\nabla|u|\cdot x\Big)\, dx = \frac{1}{r}\int_{\partial D_r}|u|^2\, ds - \frac{2}{r^2}\int_{D_r}|u|\nabla|u|\cdot x\, dx \\
&\leq \frac{1}{r}\int_{\partial D_r}|u|^2\, ds + \int_{D_r}|\nabla|u||^2\, dx + \frac{1}{r^2}\int_{D_r}|u|^2\, dx \\
&\leq \frac{1}{r}\int_{\partial D_r}|u|^2\, ds + \int_{D_r}|(i\nabla + A_a)u|^2\, dx + \frac{1}{r^2}\int_{D_r}|u|^2\, dx
\end{aligned}$$

which yields the conclusion. □

For every $b \in D_1$ we define

$$m_b := \inf_{\substack{v\in H^{1,b}(D_1,\mathbb{C})\\ v\not\equiv 0}} \frac{\int_{D_1}|(i\nabla + A_b)v|^2\, dx}{\int_{\partial D_1}|v|^2\, ds}. \tag{39}$$

Lemma 3.2. *For every $b \in D_1$, the infimum m_b defined in* (39) *is attained and $m_b > 0$.*

Proof. Let v_n be a minimizing sequence such that

$$\int_{\partial D_1}|v_n|^2\, dx = 1 \quad \text{and} \quad \int_{D_1}|(i\nabla + A_b)v_n|^2 = m_b + o(1) \quad \text{as } n\to\infty.$$

Then, by Lemma 3.1, we have that $\{v_n\}_{n\in\mathbb{N}}$ is bounded in $H^{1,b}(D_1,\mathbb{C})$. Hence there exists a subsequence v_{n_k} converging to some $v \in H^{1,b}(D_1,\mathbb{C})$ weakly in $H^{1,b}(D_1,\mathbb{C})$ and (by compactness of the trace embedding $H^{1,b}(D_1,\mathbb{C}) \hookrightarrow L^2(\partial D_1,\mathbb{C})$) strongly in $L^2(\partial D_1,\mathbb{C})$. Strong convergence in $L^2(\partial D_1,\mathbb{C})$ implies that $\int_{\partial D_1}|v|^2\, dx = 1$, so that $v \not\equiv 0$; moreover weak lower semicontinuity of the $H^{1,b}(D_1,\mathbb{C})$-norm implies that v attains m_b.

If, by contradiction, $m_b = 0$, then, via the diamagnetic inequality (18),

$$0 = \int_{D_1}|(i\nabla + A_b)v|^2\, dx \geq \int_{D_1}|\nabla|v||^2\, dx$$

which implies that $|v| \equiv C$, being $C \geq 0$ a real constant. Since $v \not\equiv 0$, we have that $C > 0$ and then $\int_{D_1}\frac{|v|^2}{|x-b|^2}\, dx = +\infty$, thus contradicting the fact that $v \in H^{1,b}(D_1,\mathbb{C})$. □

Lemma 3.3. *Let $r > 0$ and $a \in D_r$. Then*

$$\frac{m_{a/r}}{r}\int_{\partial D_r}|u|^2\, ds \leq \int_{D_r}|(i\nabla + A_a)u|^2\, dx \quad \textit{for all } u \in H^{1,a}(D_r,\mathbb{C}), \tag{40}$$

with $m_{a/r}$ as in (39) *with $b = \frac{a}{r}$.*

Proof. It follows from (39) and a standard dilation argument. □

Lemma 3.4. *The function $b \mapsto m_b$, with m_b defined in* (39)*, is continuous in D_1. Moreover $m_0 = \frac{1}{2}$.*

Proof. The proof that m_b is continuous follows by classical compactness arguments; we omit it for the sake of brevity and refer to [1] for details. To prove that $m_0 = \frac{1}{2}$, we observe that from Lemma 3.2, the infimum m_0 is attained by a function $v_0 \in H^{1,0}(D_1,\mathbb{C})\setminus\{0\}$, which weakly solves $(i\nabla + A_0)^2 v_0 = 0$ in D_1 in the sense of (27). From [13, Lemma 5.4], we have that

$$N(v_0,r) := \frac{r\int_{D_r}|(i\nabla + A_0)v_0|^2\, dx}{\int_{\partial D_r}|v_0|^2\, ds} \quad \text{is monotone nondecreasing w.r.t. } r;$$

furthermore (see Proposition 2.1) $\lim_{r\to 0^+} N(v_0, r) \geq \frac{1}{2}$. Hence $m_0 = N(v_0, 1) \geq \frac{1}{2}$. It is easy to verify that, letting $\tilde{v}(r\cos t, r\sin t) = r^{1/2} e^{i\frac{t}{2}} \sin(\frac{t}{2})$, we have that $\tilde{v} \in H^{1,0}(D_1, \mathbb{C})$ and

$$\frac{1}{2} = \frac{\int_{D_1} |(i\nabla + A_0)\tilde{v}|^2 \, dx}{\int_{\partial D_1} |\tilde{v}|^2 \, ds} \geq m_0,$$

thus implying $m_0 = \frac{1}{2}$. The proof is thereby complete. ☐

As a direct consequence of Lemma 3.4, we obtain the following result which provides a Poincaré type inequality with a control on the best constant which is uniform with respect to the variation of the pole.

Corollary 3.5. *For any $\delta \in (0, \frac{1}{2})$, there exists some sufficiently large $\mu_\delta > 1$ such that $m_b \geq \frac{1}{2} - \delta$ for every $b \in D_1$ with $|b| < \frac{1}{\mu_\delta}$.*

Proof. The proof is a straightforward consequence of Lemma 3.4. ☐

4. Limit profile

In the present section we construct the profile which will be used to describe the limit of blowed-up sequences of eigenfunctions with poles approaching 0 along the half-line tangent to a nodal line of φ_0.

Lemma 4.1. *For every odd natural number k there exists $\Phi_k \in \bigcup_{R>0} H^1(D_R^+)$ (where D_R^+ denotes the half-disk $\{(x_1, x_2) \in D_R(0) : x_2 > 0\}$) such that $\Phi_k - \psi_k \in \mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$,*

$$-\Delta \Phi_k = 0, \quad \textit{in } \mathbb{R}^2_+ \textit{ in a distributional sense,}$$

$\Phi_k = 0$ on s, and $\frac{\partial \Phi_k}{\partial \nu} = 0$ on $\partial\mathbb{R}^2_+ \setminus s$, where $\nu = (0, -1)$ is the outer normal unit vector on $\partial\mathbb{R}^2_+$.

Proof. The function $w_k \in \mathcal{D}^{1,2}_s(\mathbb{R}^2_+)$ minimizing the functional J_k defined in (9) weakly solves

$$\begin{cases} -\Delta w_k = 0, & \text{in } \mathbb{R}^2_+, \\ w_k = 0, & \text{on } s, \\ \frac{\partial w_k}{\partial \nu} = -\frac{\partial \psi_k}{\partial \nu}, & \text{on } \partial\mathbb{R}^2_+ \setminus s. \end{cases} \tag{41}$$

Taking

$$\Phi_k = \psi_k + w_k \tag{42}$$

we reach the conclusion. ☐

From now on, with a little abuse of notation, Φ_k will denote the even extension of the function Φ_k in the previous Lemma 4.1 on the whole $\mathbb{R}^2$, i.e. $\Phi_k(x_1, -x_2) = \Phi_k(x_1, x_2)$. Let us now set $\mathbf{e} = (1, 0)$ and define, for every odd natural number k,

$$\Psi_k = e^{i\frac{\theta_{\mathbf{e}}}{2}} \Phi_k, \tag{43}$$

where $\theta_{\mathbf{e}}$ is as in (15) (with $b = \mathbf{e}$) and Φ_k is the extension (even in x_2) of the function in Lemma 4.1.

We denote as $H^{1,\mathbf{e}}_{\rm loc}(\mathbb{R}^2, \mathbb{C})$ the space of functions belonging to $H^{1,\mathbf{e}}(D_r, \mathbb{C})$ for all $r > 0$, as $\mathcal{D}^{1,2}_s(\mathbb{R}^2)$ the completion of $C^\infty_c(\mathbb{R}^2 \setminus s)$ with respect to the norm $(\int_{\mathbb{R}^2} |\nabla u|^2 \, dx)^{1/2}$ and as $\mathcal{D}^{1,2}_{\mathbf{e}}(\mathbb{R}^2)$ the completion of $C^\infty_c(\mathbb{R}^2 \setminus \{\mathbf{e}\})$ with respect to the norm $(\int_{\mathbb{R}^2} |(i\nabla + A_{\mathbf{e}})u|^2 \, dx)^{1/2}$.

Proposition 4.2. *The functions Ψ_k defined in (43) satisfies the following properties:*

$$\Psi_k \in H^{1,\mathbf{e}}_{\rm loc}(\mathbb{R}^2, \mathbb{C}); \tag{44}$$

$$(i\nabla + A_{\mathbf{e}})^2 \Psi_k = 0 \quad \textit{in } \mathbb{R}^2 \textit{ in a weak } H^{1,\mathbf{e}}\textit{-sense}; \tag{45}$$

$$\int_{\mathbb{R}^2} \left|(i\nabla + A_{\mathbf{e}})(\Psi_k - e^{i\theta_{\mathbf{e}}/2}\psi_k)\right|^2 dx < +\infty; \tag{46}$$

$$e^{i\frac{\theta_{\mathbf{e}}(x)}{2}} w_k = \Psi_k(x) - e^{i\frac{\theta_{\mathbf{e}}(x)}{2}} \psi_k(x) = O(|x|^{-1/2}), \quad \textit{as } |x| \to +\infty. \tag{47}$$

Proof. Statements (44–45) follow by direct calculations together with the asymptotic expansion of solutions to elliptic problems with cracks which is proved in [12] and which yields that $\Phi_k(\mathbf{e} + r(\cos t, \sin t)) = O(r^{1/2})$ as $r \to 0^+$. (46) follows from Lemma 4.1 and direct calculations.

To prove (47), we write

$$\Psi_k = e^{i\frac{\theta_{\mathbf{e}}}{2}} \psi_k + v$$

where $v = e^{i\frac{\theta_{\mathbf{e}}}{2}}(\Phi_k - \psi_k)$. We note that $w_k = \Phi_k - \psi_k \in \mathcal{D}^{1,2}_s(\mathbb{R}^2)$ and hence $v \in \mathcal{D}^{1,2}_{\mathbf{e}}(\mathbb{R}^2)$. Since w_k weakly solves $-\Delta w_k = 0$ in $\mathbb{R}^2 \setminus s_0$, its Kelvin transform $\tilde{w}_k(x) = w_k(\frac{x}{|x|^2})$ weakly solves $-\Delta \tilde{w}_k = 0$ in $D_1 \setminus \{(x_1, 0) : 0 \leq x_1 < 1\}$ and vanishes on $\{(x_1, 0) : 0 \leq x_1 < 1\}$. From the asymptotics of solutions to elliptic problems with cracks proved in [12] it follows that $|\tilde w_k(x)| = O(|x|^{1/2})$ as $|x| \to 0^+$, which yields $|w_k(x)| = O(|x|^{-1/2})$ as $|x| \to +\infty$. Therefore we have that

$$|v(x)| = O(|x|^{-1/2}) \quad \text{as } |x| \to +\infty, \tag{48}$$

thus proving (47). □

The following result establishes that Ψ_k is the unique function satisfying (44), (45), and (46).

Proposition 4.3. *If $\Phi \in H^{1,\mathbf{e}}_{\rm loc}(\mathbb{R}^2)$ weakly satisfies*

$$(i\nabla + A_{\mathbf{e}})^2\Phi = 0, \quad \text{in } \mathbb{R}^2, \tag{49}$$

and

$$\int_{\mathbb{R}^2} |(i\nabla + A_{\mathbf{e}})(\Phi - e^{\frac{i}{2}\theta_{\mathbf{e}}}\psi_k)|^2 < +\infty, \tag{50}$$

then $\Phi = \Psi_k$, with Ψ_k being the function defined in (43).

Proof. Suppose that $\Phi \in H^{1,\mathbf{e}}_{\rm loc}(\mathbb{R}^2)$ satisfies (49) and (50). Then, in view of (45), the difference $\Psi = \Phi - \Psi_k$ weakly solves $(i\nabla + A_{\mathbf{e}})^2\Psi = 0$ in $\mathbb{R}^2$. Moreover from (46) and (50) it follows that

$$\int_{\mathbb{R}^2} |(i\nabla + A_{\mathbf{e}})\Psi(x)|^2 dx < +\infty, \tag{51}$$

which, in view of (17), implies that

$$\int_{\mathbb{R}^2} \frac{|\Psi(x)|^2}{|x-\mathbf{e}|^2}\, dx < +\infty. \tag{52}$$

For $R > 1$, let $\eta_R : \mathbb{R}^2 \to \mathbb{R}$ as in (14). Testing the equation for Ψ by $(1-\eta_R)^2\Psi$ we obtain that

$$\begin{aligned}\int_{\mathbb{R}^2} (1-\eta_R)^2 |(i\nabla + A_{\mathbf{e}})\Psi|^2\, dx &= -2i \int_{\mathbb{R}^2} (1-\eta_R)\overline{\Psi}(i\nabla + A_{\mathbf{e}})\Psi \cdot \nabla\eta_R\, dx \\ &\leq \frac12 \int_{\mathbb{R}^2} (1-\eta_R)^2 |(i\nabla + A_{\mathbf{e}})\Psi|^2\, dx + 2\int_{\mathbb{R}^2} |\Psi|^2 |\nabla\eta_R|^2\, dx\end{aligned}$$

which implies that

$$\begin{aligned}\int_{D_{R/2}} |(i\nabla + A_{\mathbf{e}})\Psi|^2\, dx &\leq \int_{\mathbb{R}^2} (1-\eta_R)^2 |(i\nabla + A_{\mathbf{e}})\Psi|^2\, dx \leq 4\int_{\mathbb{R}^2} |\Psi|^2 |\nabla\eta_R|^2\, dx \\ &\leq \frac{64}{R^2}\int_{D_R\setminus D_{R/2}} |\Psi|^2 dx \leq 64\frac{(R+1)^2}{R^2}\int_{D_R\setminus D_{R/2}} \frac{|\Psi|^2}{|x-\mathbf{e}|^2}\, dx \to 0\end{aligned}$$

as $R \to +\infty$ thanks to (52). It follows that $\int_{\mathbb{R}^2} |(i\nabla + A_{\mathbf{e}})\Psi|^2\, dx = 0$ and then $\int_{\mathbb{R}^2} |x-\mathbf{e}|^{-2}|\Psi(x)|^2\, dx = 0$ in view of (17). Hence $\Psi \equiv 0$ in $\mathbb{R}^2$ and $\Phi = \Psi_k$. □

The following lemma establishes a deep relation between the function Ψ_k and the constant $\mathfrak{m}_k$ in (10).

Lemma 4.4. *Let Ψ_k be the function defined in* (43). *Then*

$$\pi - \int_0^{2\pi} \Psi_k(\cos t, \sin t) e^{-\frac{i}{2}\theta_{\mathbf{e}}(\cos t, \sin t)} \sin\left(\tfrac{k}{2}t\right) dt = \frac{4}{k}\mathfrak{m}_k \tag{53}$$

with $\mathfrak{m}_k$ as in (10).

Proof. Let w_k be the function introduced in (10) and (41), extended evenly in x_2 to the whole $\mathbb{R}^2$ (i.e. $w_k(x_1,-x_2) = w_k(x_1,x_2)$); from (41) we have that w_k is harmonic on $\mathbb{R}^2 \setminus s_0$. Taking into account (25), (42), and (8), we have that

$$-\frac{1}{\sqrt{\pi}}\left(\pi - \int_0^{2\pi} \Psi_k(\cos t, \sin t) e^{-\frac{i}{2}\theta_{\mathbf{e}}(\cos t, \sin t)} \sin\left(\tfrac{k}{2}t\right) dt\right) = \int_0^{2\pi} w_k(\cos t, \sin t) e^{i\frac{t}{2}}\overline{\psi_{k,2}(t)}\, dt = \omega(1)$$

where $\omega(r) := \int_0^{2\pi} w_k(r\cos t, r\sin t) e^{i\frac{t}{2}}\overline{\psi_{k,2}(t)}\, dt$. As observed in §2.4, $\omega(r)$ satisfies, for some $C_\omega \in \mathbb{C}$, $\left(r^{-k/2}\omega(r)\right)' = C_\omega r^{-(1+k)}$, for $r > 1$. Integrating the previous equation over $(1,r)$ we obtain that

$$\frac{\omega(r)}{r^{k/2}} - \omega(1) = \frac{C_\omega}{k}\left(1 - \frac{1}{r^k}\right), \quad \text{for all } r \geq 1.$$

From (47) it follows that $\omega(r) = O(r^{-1/2})$ as $r \to +\infty$, hence, letting $r \to +\infty$ in the previous identity, we obtain that necessarily $C_\omega = -k\omega(1)$ and then

$$\omega(r) = \omega(1) r^{-k/2}, \quad \omega'(r) = -\frac{k}{2}\omega(1) r^{-\frac{k}{2}-1}, \quad \text{for all } r \geq 1. \tag{54}$$

On the other hand,

$$\omega'(r) = \frac{r^{-\frac{k}{2}-1}}{\sqrt{\pi}} \int_{\partial D_r} \frac{\partial w_k}{\partial \nu} \psi_k \, ds. \tag{55}$$

Combining (54) and (55) we obtain that

$$\omega(1) = -\frac{2}{k\sqrt{\pi}} \int_{\partial D_1} \frac{\partial w_k}{\partial \nu} \psi_k \, ds. \tag{56}$$

Multiplying the equation $-\Delta w_k = 0$ (which is weakly satisfied in $\mathbb{R}^2 \setminus s_0$) by ψ_k and integrating by parts on $D_1 \setminus s_0$, we obtain that

$$\int_{\partial D_1} \frac{\partial w_k}{\partial \nu} \psi_k \, ds = \int_{D_1} \nabla w_k \cdot \nabla \psi_k \, dx, \tag{57}$$

whereas multiplying $-\Delta \psi_k = 0$ (which is weakly satisfied in $\mathbb{R}^2 \setminus s_0$) by w_k and integrating by parts on $D_1 \setminus s_0$ we obtain that

$$\int_{\partial D_1} \frac{\partial \psi_k}{\partial \nu} w_k \, ds - 2\int_0^1 \frac{\partial_+ \psi_k}{\partial x_2}(x_1, 0) \, w_k(x_1, 0) \, dx_1 = \int_{D_1} \nabla w_k \cdot \nabla \psi_k \, dx. \tag{58}$$

Collecting (57) and (58) we have that

$$\int_{\partial D_1} \frac{\partial w_k}{\partial \nu} \psi_k \, ds = \int_{\partial D_1} \frac{\partial \psi_k}{\partial \nu} w_k \, ds - 2\int_0^1 \frac{\partial_+ \psi_k}{\partial x_2}(x_1, 0) \, w_k(x_1, 0) \, dx_1.$$

Since $\int_{\partial D_1} \frac{\partial \psi_k}{\partial \nu} w_k \, ds = \frac{k\sqrt{\pi}}{2}\omega(1)$, (56) now reads $\omega(1) = -\omega(1) + \frac{4}{k\sqrt{\pi}} \int_0^1 \frac{\partial_+ \psi_k}{\partial x_2}(x_1, 0) \, w_k(x_1, 0) \, dx_1$ and thus

$$\omega(1) = \frac{2}{k\sqrt{\pi}} \int_0^1 \frac{\partial_+ \psi_k}{\partial x_2}(x_1, 0) \, w_k(x_1, 0) \, dx_1.$$

Letting $\mathfrak{m}_k$ as in (10), in view of (11) we conclude that $\omega(1) = -\frac{4}{k\sqrt{\pi}}\mathfrak{m}_k$, thus proving (53). □

5. Monotonicity formula and energy estimates for blow-up sequences

In this section we prove some energy estimates for eigenfunctions using an adaption of the Almgren monotonicity argument inspired by [27, Section 5] and [13].

Definition 5.1. Let $\lambda \in \mathbb{R}$, $b \in \mathbb{R}^2$, and $u \in H^{1,b}(D_r, \mathbb{C})$. For any $r > |b|$, we define the Almgren-type frequency function as

$$N(u, r, \lambda, A_b) = \frac{E(u, r, \lambda, A_b)}{H(u, r)},$$

where

$$E(u, r, \lambda, A_b) = \int_{D_r} |(i\nabla + A_b)u|^2 \, dx - \lambda \int_{D_r} |u|^2 \, dx, \quad H(u, r) = \frac{1}{r} \int_{\partial D_r} |u|^2 \, ds.$$

When we study the quotient $N = E/H$ for any magnetic eigenfunction, we find several specific relations to hold true. We are interested in the derivative of such a quotient, since it provides some information about the possible vanishing behavior of eigenfunctions near the pole of the magnetic potential.

For all $1 \le j \le n_0$ and $a \in \Omega$, let $\varphi_j^a \in H_0^{1,a}(\Omega, \mathbb{C}) \setminus \{0\}$ be an eigenfunction of problem (E_a) associated to the eigenvalue λ_j^a, i.e. solving

$$\begin{cases} (i\nabla + A_a)^2 \varphi_j^a = \lambda_j^a \varphi_j^a, & \text{in } \Omega, \\ \varphi_j^a = 0, & \text{on } \partial\Omega, \end{cases} \tag{59}$$

such that

$$\int_\Omega |\varphi_j^a(x)|^2 \, dx = 1 \quad \text{and} \quad \int_\Omega \varphi_j^a(x) \overline{\varphi_\ell^a(x)} \, dx = 0 \text{ if } j \neq \ell. \tag{60}$$

For $j = n_0$, we choose

$$\varphi_{n_0}^a = \varphi_a, \tag{61}$$

with φ_a as in (19)–(20). We observe that, since $a \in \Omega \mapsto \lambda_j^a$ admits a continuous extension on $\overline{\Omega}$ as proved in [8, Theorem 1.1], we have that

$$\Lambda := \sup_{\substack{a \in \Omega \\ 1 \le j \le n_0}} \lambda_j^a \in (0, +\infty). \tag{62}$$

Lemma 5.2.

(i) *There exists $R_0 \in (0, (5\Lambda)^{-1/2})$ such that $D_{R_0} \subset \Omega$ and, if $|a| < R_0$, $H(\varphi_j^a, r) > 0$ for all $r \in (|a|, R_0)$ and $1 \le j \le n_0$.*

(ii) *There exist* $C_0 > 0$ *and* $\alpha_0 \in (0, R_0)$ *such that* $H(\varphi_j^a, R_0) \geq C_0$ *for all* a *with* $|a| < \alpha_0$ *and* $1 \leq j \leq n_0$.

Proof. To prove (i) we argue by contradiction and assume that, for all n sufficiently large, there exist $a_n \in \Omega$ with $|a_n| < \frac{1}{n}$, $r_n \in \big(|a_n|, \frac{1}{n}\big)$, and $j_n \in \{1, \dots, n_0\}$ such that $H(\varphi_{j_n}^{a_n}, r_n) = 0$, i.e. $\varphi_{j_n}^{a_n} \equiv 0$ on ∂D_{r_n}. Testing (59) with $\varphi_{j_n}^{a_n}$ and integrating on D_{r_n}, in view of Lemma 3.1 we obtain

$$0 = \int_{D_{r_n}} \Big(|(i\nabla + A_{a_n})\varphi_{j_n}^{a_n}|^2 - \lambda_{j_n}^{a_n} |\varphi_{j_n}^{a_n}|^2 \Big)\, dx \geq (1 - \Lambda r_n^2) \int_{D_{r_n}} |(i\nabla + A_{a_n})\varphi_{j_n}^{a_n}|^2\, dx.$$

Since $r_n \to 0$, for n large $1 - \Lambda r_n^2 > 0$ and hence the above inequality yields $\int_{D_{r_n}} |(i\nabla + A_{a_n})\varphi_{j_n}^{a_n}|^2\, dx = 0$. Lemma 3.1 then implies that $\|\varphi_{j_n}^{a_n}\|_{H^{1,a_n}(D_{r_n},\mathbb{C})} = 0$ and hence $\varphi_{j_n}^{a_n} \equiv 0$ in D_{r_n}. From the unique continuation principle (see [13, Corollary 1.4]) we conclude that $\varphi_{j_n}^{a_n} \equiv 0$ in Ω, a contradiction.

To prove (ii), we argue by contradiction and assume that, for all n sufficiently large, there exist $a_n \in \Omega$ with $a_n \to 0$ and $j_n \in \{1, \dots, n_0\}$ such that $\lim_{n\to\infty} H(\varphi_{j_n}^{a_n}, R_0) = 0$. Letting $\varphi_n := \varphi_{j_n}^{a_n}$ and $\lambda_n := \lambda_{j_n}^{a_n}$, using (59) and (60) it is easy to prove that, along a subsequence, $\lambda_{n_k} \to \lambda_{j_0}^0$ for some $j_0 \in \{1, \dots, n_0\}$ and $\varphi_{n_k} \to \varphi$ weakly in $H^1(\Omega, \mathbb{C})$ for some $\varphi \in H_0^{1,0}(\Omega, \mathbb{C})$ satisfying $(i\nabla + A_0)^2\varphi = \lambda_{j_0}^0 \varphi$ in a weak sense in Ω and $\int_\Omega |\varphi(x)|^2\, dx = 1$. In particular $\varphi \not\equiv 0$. Furthermore, by compactness of the trace embedding $H^1(D_{R_0}, \mathbb{C}) \hookrightarrow L^2(\partial D_{R_0}, \mathbb{C})$, we have that

$$0 = \lim_{k\to\infty} \frac{1}{R_0} \int_{\partial D_{R_0}} |\varphi_{n_k}|^2\, ds = \frac{1}{R_0} \int_{\partial D_{R_0}} |\varphi|^2\, ds,$$

which implies that $\varphi = 0$ on ∂D_{R_0}. Testing $(i\nabla + A_0)^2\varphi = \lambda_{j_0}^0 \varphi$ with φ and integrating on D_{R_0}, in view of Lemma 3.1 we obtain

$$0 = \int_{D_{R_0}} \Big(|(i\nabla + A_0)\varphi|^2 - \lambda_{j_0}^0 |\varphi|^2 \Big)\, dx \geq (1 - \Lambda R_0^2) \int_{D_{R_0}} |(i\nabla + A_0)\varphi|^2\, dx.$$

Since $1 - \Lambda R_0^2 > 0$, we deduce that $\int_{D_{R_0}} |(i\nabla + A_0)\varphi|^2\, dx = 0$. Lemma 3.1 then implies that $\varphi \equiv 0$ in D_{R_0}. From the unique continuation principle (see [13, Corollary 1.4]) we conclude that $\varphi \equiv 0$ in Ω, thus giving rise to a contradiction. ☐

We notice that, thanks to Lemma 5.2, the function $r \mapsto N(\varphi_j^a, r, \lambda_j^a, A_a)$ is well defined in $(|a|, R_0)$.

Lemma 5.3. *Let* $1 \leq j \leq n_0$, $a \in \Omega$, *and* $\varphi_j^a \in H_0^{1,a}(\Omega, \mathbb{C})$ *be a solution to* (59)–(60). *Then* $r \mapsto H(\varphi_j^a, r)$ *is smooth in* $(|a|, R_0)$ *and*

$$\frac{d}{dr} H(\varphi_j^a, r) = \frac{2}{r} E(\varphi_j^a, r, \lambda_j^a, A_a).$$

Proof. Since the proof is similar to that of [27, Lemma 5.2], we omit it. ☐

Lemma 5.4. *For* $\delta \in (0, 1/4)$, *let* μ_δ *be as in Corollary 3.5. Let* $r_0 \leq R_0$ *and* $j \in \{1, \dots, n_0\}$. *If* $\mu_\delta |a| \leq r_1 < r_2 \leq r_0$ *and* φ_j^a *is a solution to* (59)–(60), *then*

$$\frac{H(\varphi_j^a, r_2)}{H(\varphi_j^a, r_1)} \geq e^{-\frac{5}{2}\Lambda r_0^2} \left(\frac{r_2}{r_1}\right)^{1-2\delta}.$$

Proof. Combining Lemma 3.1 with Lemma 3.3 and Corollary 3.5 we obtain that, for every $\mu_\delta|a| < r < R_0$,

$$\frac{1}{r^2} \int_{D_r} \left|\varphi_j^a\right|^2 dx \leq \left(1 + \frac{2}{1 - 2\delta}\right) \int_{D_r} \left|(i\nabla + A_a)\varphi_j^a\right|^2 dx < 5 \int_{D_r} \left|(i\nabla + A_a)\varphi_j^a\right|^2 dx.$$

From above, Lemma 5.3, Lemma 3.3, recalling that $R_0 < (5\Lambda)^{-1/2}$, for every $\mu_\delta|a| < r < r_0$ we have that

$$\begin{aligned} \frac{d}{dr} H(\varphi_j^a, r) &= \frac{2}{r} \int_{D_r} \left(\left|(i\nabla + A_a)\varphi_j^a\right|^2 - \lambda_j^a \left|\varphi_j^a\right|^2 \right) dx \geq \frac{2}{r} \left(1 - 5\Lambda r^2\right) \int_{D_r} \left|(i\nabla + A_a)\varphi_j^a\right|^2 dx \\ &\geq \frac{2}{r} \left(1 - 5\Lambda r^2\right) m_{a/r} H(\varphi_j^a, r) \geq \frac{2}{r} \left(1 - 5\Lambda r^2\right) \left(\frac{1}{2} - \delta\right) H(\varphi_j^a, r), \end{aligned}$$

so that, in view of Lemma 5.2,

$$\frac{d}{dr} \log H(\varphi_j^a, r) \geq \frac{1 - 2\delta}{r} - \Lambda(5 - 10\delta) r \geq \frac{1 - 2\delta}{r} - 5\Lambda r.$$

Integrating between r_1 and r_2 we obtain the desired inequality. ☐

Lemma 5.5. *For $1 \le j \le n_0$ and $a \in \Omega$, let $\varphi_j^a \in H_0^{1,a}(\Omega,\mathbb{C})$ be a solution to (59)–(60). Then, for all $|a| < r < R_0$, we have that*

$$\frac{d}{dr}E(\varphi_j^a, r, \lambda_j^a, A_a) = 2\int_{\partial D_r} \left|(i\nabla + A_a)\varphi_j^a \cdot \nu\right|^2 ds - \frac{2}{r}\left(M_j^a + \lambda_j^a \int_{D_r} \left|\varphi_j^a\right|^2 dx\right) \tag{63}$$

where $\nu(x) = \frac{x}{|x|}$ denotes the unit normal vector to ∂D_r and

$$M_j^a = \frac{1}{4}\Big(a_1(c_{a,j}^2 - d_{a,j}^2) + 2a_2 c_{a,j} d_{a,j}\Big), \tag{64}$$

with $a = (a_1, a_2)$, $c_{a,j} = \sqrt{\pi}\beta_{1,1}(a,\varphi_j^a,\lambda_j^a)$, and $d_{a,j} = \sqrt{\pi}\beta_{1,2}(a,\varphi_j^a,\lambda_j^a)$, being $\beta_{1,1}(a,\varphi_j^a,\lambda_j^a)$ and $\beta_{1,2}(a,\varphi_j^a,\lambda_j^a)$ the coefficients defined in (30). Furthermore, letting μ_δ as in Corollary 3.5,

$$\frac{|M_j^a|}{H(\varphi_j^a, \mu_\delta|a|)} \le C_\delta,$$

for some $C_\delta > 0$ independent of a.

Proof. Since the proof is similar to that of [27, Lemmas 5.6, 5.7, 5.8, 5.9], we omit it. □

Lemma 5.6. *For $1 \le j \le n_0$ and $a \in \Omega$, let $\varphi_j^a \in H_0^{1,a}(\Omega,\mathbb{C})$ be a solution to (59)–(60). Let $\delta \in (0, 1/4)$, μ_δ as in Corollary 3.5 and $r_0 \le R_0$. Then there exists $c_{\delta,r_0} > 0$ such that, for all $\mu > \mu_\delta$, $|a| < \frac{r_0}{\mu}$, $\mu|a| \le r < r_0$, and $1 \le j \le n_0$,*

$$e^{\frac{\Lambda r^2}{1-\Lambda {r_0}^2}}\big(N(\varphi_j^a, r, \lambda_j^a, A_a) + 1\big) \le e^{\frac{\Lambda {r_0}^2}{1-\Lambda {r_0}^2}}\big(N(\varphi_j^a, r_0, \lambda_j^a, A_a) + 1\big) + \frac{c_{\delta,r_0}}{\mu^{1-2\delta}} \tag{65}$$

and

$$N(\varphi_j^a, r, \lambda_j^a, A_a) + 1 > 0. \tag{66}$$

Proof. By direct computations and Schwarz inequality (see [27, Lemma 5.11]), we obtain that, for all $|a| < r < R_0$,

$$\begin{aligned}&\frac{dN(\varphi_j^a, r, \lambda_j^a, A_a)}{dr}\\ &= \frac{\frac{2}{r}\Big(\big(\int_{\partial D_r}|(i\nabla + A_a)\varphi_j^a\cdot\nu|^2\,ds\big)\big(\int_{\partial D_r}|\varphi_j^a|^2\,ds\big) - \big(i\int_{\partial D_r}(i\nabla + A_a)\varphi_j^a\cdot\nu\,\overline{\varphi_j^a}\,ds\big)^2\Big)}{H^2(\varphi_j^a, r)}\\ &\quad - \frac{2}{rH(\varphi_j^a, r)}\left(M_j^a + \lambda_j^a\int_{D_r}\left|\varphi_j^a\right|^2 dx\right) \ge -\frac{2}{rH(\varphi_j^a, r)}\left(|M_j^a| + \lambda_j^a\int_{D_r}\left|\varphi_j^a\right|^2 dx\right).\end{aligned}$$

Via Lemmas 5.4 and 5.5 we estimate, for all $\mu_\delta|a| \le r < r_0$,

$$\frac{2|M_j^a|}{H(\varphi_j^a, r)} = 2\frac{|M_j^a|}{H(\varphi_j^a, \mu_\delta|a|)}\frac{H(\varphi_j^a, \mu_\delta|a|)}{H(\varphi_j^a, r)} \le \mathrm{const}_\delta\left(\frac{|a|}{r}\right)^{1-2\delta},$$

where $\mathrm{const}_\delta > 0$ is independent of a (but depends on δ). On the other hand, by Lemma 3.1 we have that, for all $\mu_\delta|a| \le r < r_0$,

$$\frac{1-\Lambda r^2}{r^2}\int_{D_r}\left|\varphi_j^a\right|^2 \le H(\varphi_j^a, r) + E(\varphi_j^a, r, \lambda_j^a, A_a)$$

which implies

$$\frac{2\lambda_j^a}{rH(\varphi_j^a, r)}\int_{D_r}|\varphi_j^a|^2\,dx \le \frac{2\Lambda r}{1-\Lambda {r_0}^2}\Big(N(\varphi_j^a, r, \lambda_j^a, A_a) + 1\Big).$$

Therefore (66) follows. Moreover, for all $\mu_\delta|a| \le r < r_0$,

$$\frac{dN(\varphi_j^a, r, \lambda_j^a, A_a)}{dr} \ge -\mathrm{const}_\delta\frac{|a|^{1-2\delta}}{r^{2-2\delta}} - \frac{2\Lambda r}{1-\Lambda {r_0}^2}\Big(N(\varphi_j^a, r, \lambda_j^a, A_a) + 1\Big)$$

which is read as

$$\left(e^{\frac{\Lambda r^2}{1-\Lambda {r_0}^2}}\big(N(\varphi_j^a, r, \lambda_j^a, A_a) + 1\big)\right)' e^{-\frac{\Lambda r^2}{1-\Lambda {r_0}^2}} \ge -\mathrm{const}_\delta\frac{|a|^{1-2\delta}}{r^{2-2\delta}}.$$

Letting $r \in [\mu_\delta|a|, r_0)$ and integrating from r to r_0 we obtain

$$e^{\frac{\Lambda r^2}{1-\Lambda {r_0}^2}}\big(N(\varphi_j^a, r, \lambda_j^a, A_a) + 1\big) \le e^{\frac{\Lambda {r_0}^2}{1-\Lambda {r_0}^2}}\big(N(\varphi_j^a, r_0, \lambda_j^a, A_a) + 1\big) + e^{\frac{\Lambda {r_0}^2}{1-\Lambda {r_0}^2}}\frac{\mathrm{const}_\delta}{1-2\delta}\left(\frac{|a|}{r}\right)^{1-2\delta}.$$

Letting $\mu > \mu_\delta$, $|a| < \frac{r_0}{\mu}$, $\mu|a| \le r < r_0$, and taking $c_{\delta,r_0} = e^{\frac{\Lambda {r_0}^2}{1-\Lambda {r_0}^2}}\frac{\mathrm{const}_\delta}{1-2\delta}$, the above estimates yields (65). □

A first consequence of Lemma 5.6 is the following estimate of the Almgren quotient of φ_a at radii of size $|a|$ in terms of the order of vanishing of φ_0 at the pole.

Lemma 5.7. *For $a \in \Omega$, let $\varphi_a \in H_0^{1,a}(\Omega, \mathbb{C})$ be a solution to (19-20). For every $\delta \in (0, 1/4)$ there exist $r_\delta > 0$ and $K_\delta > \mu_\delta > 0$ such that, if $\mu \geq K_\delta$, $|a| < \frac{r_\delta}{\mu}$, and $\mu|a| \leq r < r_\delta$, then $N(\varphi_a, r, \lambda_a, A_a) \leq \frac{k}{2} + \delta$.*

Proof. Let $p > 0$ be sufficiently small so that $p(2 + \frac{k}{2} + \frac{p}{2}) < \frac{1}{2}$. Let $\delta \in (0, \frac{1}{4})$. Since, in view of Proposition 2.1,

$$\lim_{r \to 0^+} N(\varphi_0, r, \lambda_0, A_0) = \frac{k}{2},$$

we can choose $r_\delta > 0$ sufficiently small so that $r_\delta < \min\{R_0, (5\Lambda)^{-1/2}\}$, $e^{\frac{\Lambda {r_\delta}^2}{1-\Lambda {r_\delta}^2}} \leq 1 + \delta p$, and $N(\varphi_0, r_\delta, \lambda_0, A_0) < \frac{k}{2} + \delta p$.

Since, in view of (21) and (23), $N(\varphi_a, r_\delta, \lambda_a, A_a) \to N(\varphi_0, r_\delta, \lambda_0, A_a)$ as $|a| \to 0$, there exists some $\alpha_\delta > 0$ such that if $|a| < \alpha_\delta$ then $N(\varphi_a, r_\delta, \lambda_a, A_a) < \frac{k}{2} + \delta p$. From Lemma 5.6 it follows that, if $\mu > \mu_\delta$, $|a| < \min\{\frac{r_\delta}{\mu}, \alpha_\delta\}$, and $\mu|a| \leq r < r_\delta$, then

$$\begin{aligned} N(\varphi_a, r, \lambda_a, A_a) + 1 &\leq (1 + \delta p)\big(\tfrac{k}{2} + \delta p + 1\big) + \frac{c_{\delta, r_\delta}}{\mu^{1-2\delta}} \\ &= 1 + \frac{k}{2} + \delta\Big(2p + p\frac{k}{2} + \delta p^2\Big) + \frac{c_{\delta, r_\delta}}{\mu^{1-2\delta}} < 1 + \frac{k}{2} + \frac{1}{2}\delta + \frac{c_{\delta, r_\delta}}{\mu^{1-2\delta}}. \end{aligned}$$

If $K_\delta > \max\{\mu_\delta, \big(\frac{2c_{\delta, r_\delta}}{\delta}\big)^{\frac{1}{1-2\delta}}, r_\delta/\alpha_\delta\}$, we conclude that, if $\mu \geq K_\delta$, $|a| < \frac{r_\delta}{\mu}$, and $\mu|a| \leq r < r_\delta$, then $N(\varphi_a, r, \lambda_a, A_a) < \frac{k}{2} + \delta$, thus concluding the proof. □

A second consequence of Lemma 5.6 is the following estimate of the energy of eigenfunctions φ_j^a in disks of radius of order $|a|$.

Lemma 5.8. *For $1 \leq j \leq n_0$ and $a \in \Omega$, let $\varphi_j^a \in H_0^{1,a}(\Omega, \mathbb{C})$ be a solution to (59)–(60). Let R_0 be as in Lemma 5.2. For every $\delta \in (0, 1/4)$, there exist $\tilde{K}_\delta > 1$ and $\tilde{C}_\delta > 0$ such that, for all $\mu \geq \tilde{K}_\delta$, $a \in \Omega$ with $|a| < \frac{R_0}{\mu}$, and $1 \leq j \leq n_0$,*

$$\int_{\partial D_{\mu|a|}} |\varphi_j^a|^2 \, ds \leq \tilde{C}_\delta (\mu|a|)^{2-2\delta}, \tag{67}$$

$$\int_{D_{\mu|a|}} |(i\nabla + A_a)\varphi_j^a|^2 \, dx \leq \tilde{C}_\delta (\mu|a|)^{1-2\delta}, \tag{68}$$

$$\int_{D_{\mu|a|}} |\varphi_j^a|^2 \, dx \leq \tilde{C}_\delta (\mu|a|)^{3-2\delta}. \tag{69}$$

Proof. Let us fix $\delta \in (0, 1/4)$ and let μ_δ be as in Corollary 3.5. From Lemma 5.6 it follows that, if $\mu > \mu_\delta$ and $|a| < \frac{R_0}{\mu}$ then, for all $1 \leq j \leq n_0$,

$$N(\varphi_j^a, \mu|a|, \lambda_j^a, A_a) \leq e^{\frac{\Lambda {R_0}^2}{1-\Lambda {R_0}^2}} \big(N(\varphi_j^a, R_0, \lambda_j^a, A_a) + 1\big) + \frac{c_{\delta, R_0}}{\mu_\delta^{1-2\delta}} - 1. \tag{70}$$

From (59), (60), and (62) we deduce that

$$\int_{D_{R_0}} \big|(i\nabla + A_a)\varphi_j^a\big|^2 \, dx \leq \int_\Omega \big|(i\nabla + A_a)\varphi_j^a\big|^2 \, dx = \lambda_j^a \leq \Lambda, \tag{71}$$

therefore, in view of Lemma 5.2, if $|a| < \alpha_0$,

$$N(\varphi_j^a, R_0, \lambda_j^a, A_a) = \frac{\int_{D_{R_0}} |(i\nabla + A_a)\varphi_j^a|^2 \, dx - \lambda_j^a \int_{D_{R_0}} |\varphi_j^a|^2 \, dx}{H(\varphi_j^a, R_0)} \leq \frac{\Lambda}{C_0}. \tag{72}$$

Combining (70) and (72) we obtain that, if $\mu \geq \tilde{K}_\delta$ with $\tilde{K}_\delta > \max\{\mu_\delta, R_0/\alpha_0\}$ and $|a| < \frac{R_0}{\mu}$, then

$$\int_{D_{\mu|a|}} |(i\nabla + A_a)\varphi_j^a|^2 \, dx - \lambda_j^a \int_{D_{\mu|a|}} |\varphi_j^a|^2 \, dx \leq \text{const}_\delta H(\varphi_j^a, \mu|a|)$$

for some positive $\text{const}_\delta > 0$ depending on δ. Hence, from Lemma 3.1,

$$(1 - \Lambda\mu^2|a|^2) \int_{D_{\mu|a|}} |(i\nabla + A_a)\varphi_j^a|^2 \, dx - \Lambda(\mu|a|)^2 H(\varphi_j^a, \mu|a|) \leq \text{const}_\delta H(\varphi_j^a, \mu|a|)$$

which implies

$$\int_{D_{\mu|a|}} |(i\nabla + A_a)\varphi_j^a|^2 \, dx \leq \frac{\Lambda R_0^2 + \text{const}_\delta}{1 - \Lambda R_0^2} H(\varphi_j^a, \mu|a|). \tag{73}$$

From Lemma 5.4 it follows that, if $\mu \geq \tilde{K}_\delta$ and $|a| < \frac{R_0}{\mu}$,

$$H(\varphi_j^a, \mu|a|) \leq e^{\frac{5}{2}\Lambda R_0^2} \left(\frac{\mu|a|}{R_0}\right)^{1-2\delta} H(\varphi_j^a, R_0). \tag{74}$$

On the other hand, Lemma 3.3, Corollary 3.5, and (71) yield

$$H(\varphi_j^a, R_0) \leq \frac{1}{m_{a/R_0}} \int_{D_{R_0}} \left|(i\nabla + A_a)\varphi_j^a\right|^2 dx \leq \frac{2\Lambda}{1-2\delta}. \tag{75}$$

Estimate (67) follows combining (74), and (75), whereas estimate (68) follows from (73), (74), and (75). Finally, (69) can be deduced from (67), (68) and Lemma 3.1. □

We blow-up the family of eigenfunctions $\{\varphi_a\}$ with $a = (|a|, 0)$ as $|a| \to 0$, i.e. we introduce the family of functions

$$\tilde{\varphi}_a(x) := \frac{\varphi_a(|a|x)}{\sqrt{H(\varphi_a, K_\delta|a|)}}, \quad a = (|a|, 0) = |a|\mathbf{e}, \tag{76}$$

with K_δ being as in Lemma 5.7 for some fixed $\delta \in (0, 1/4)$. We observe that $\tilde{\varphi}_a$ weakly solves

$$(i\nabla + A_{\mathbf{e}})^2 \tilde{\varphi}_a = |a|^2 \lambda_a \tilde{\varphi}_a, \quad \text{in } \tfrac{1}{|a|}\Omega = \{x \in \mathbb{R}^2 : |a|x \in \Omega\}, \tag{77}$$

and

$$\frac{1}{K_\delta} \int_{\partial D_{K_\delta}} |\tilde{\varphi}_a|^2 \, ds = 1. \tag{78}$$

In section 8 we will prove that $\tilde{\varphi}_a$ converges to a limit profile which is a multiple of the function Ψ_k introduced in (43). To this aim, the energy estimates below will play a crucial role.

Theorem 5.9. *For all $R \geq K_\delta$,*

$$\text{the family of functions } \left\{\tilde{\varphi}_a : a = |a|\mathbf{e}, |a| < \tfrac{r_\delta}{R}\right\} \text{ is bounded in } H^{1,\mathbf{e}}(D_R, \mathbb{C}). \tag{79}$$

In particular, for all $R \geq K_\delta$,

$$\int_{D_{R|a|}} |(i\nabla + A_a)\varphi_a|^2 \, dx = O(H(\varphi_a, K_\delta|a|)), \quad \text{as } |a| \to 0^+, \tag{80}$$

$$\int_{\partial D_{R|a|}} |\varphi_a|^2 dx = O(|a| H(\varphi_a, K_\delta|a|)), \quad \text{as } |a| \to 0^+, \tag{81}$$

$$\int_{D_{R|a|}} |\varphi_a|^2 dx = O(|a|^2 H(\varphi_a, K_\delta|a|)), \quad \text{as } |a| \to 0^+. \tag{82}$$

Proof. For $\delta \in (0, 1/4)$ fixed, let $r_\delta > 0$ and $K_\delta > \mu_\delta$ be as in Lemma 5.7, so that Lemma 5.7 yields

$$N(\varphi_a, R|a|, \lambda_a, A_a) \leq \frac{k}{2} + \delta, \quad \text{for all } R \geq K_\delta \text{ and } |a| < \frac{r_\delta}{R}. \tag{83}$$

Let us observe that, by a standard change of variables in the integrals and (83),

$$\begin{aligned} N(\varphi_a, R|a|, \lambda_a, A_a) &= \frac{R|a| \left(\int_{D_{R|a|}} |(i\nabla + A_a)\varphi_a|^2 \, dx - \lambda_a \int_{D_{R|a|}} |\varphi_a|^2 \, dx\right)}{\int_{\partial D_{R|a|}} |\varphi_a|^2 \, ds} \\ &= \frac{R \left(\int_{D_R} |(i\nabla + A_{\mathbf{e}})\tilde{\varphi}_a|^2 \, dx - |a|^2 \lambda_a \int_{D_R} |\tilde{\varphi}_a|^2 \, dx\right)}{\int_{\partial D_R} |\tilde{\varphi}_a|^2 \, ds} \leq \frac{k}{2} + \delta. \end{aligned} \tag{84}$$

Thus, via Corollary 3.5, Lemma 3.1 and (84), for all $R \geq K_\delta$ and $|a| < \frac{r_\delta}{R}$ there holds

$$\begin{aligned} (1 - 5\Lambda r_\delta^2) \int_{D_R} |(i\nabla + A_{\mathbf{e}})\tilde{\varphi}_a|^2 \, dx &\leq \left(1 - \lambda_a |a|^2 R^2 (1 + m_{\mathbf{e}/R}^{-1})\right) \int_{D_R} |(i\nabla + A_{\mathbf{e}})\tilde{\varphi}_a|^2 \, dx \\ \leq \int_{D_R} |(i\nabla + A_{\mathbf{e}})\tilde{\varphi}_a|^2 \, dx - |a|^2 \lambda_a \int_{D_R} |\tilde{\varphi}_a|^2 \, dx &\leq H(\tilde{\varphi}_a, R)\left(\frac{k}{2} + \delta\right) = \frac{H(\varphi_a, R|a|)}{H(\varphi_a, K_\delta|a|)}\left(\frac{k}{2} + \delta\right). \end{aligned} \tag{85}$$

From Lemmas 5.3 and 5.7, there holds that, if $R \geq K_\delta$ and $|a| < \frac{r_\delta}{R}$,

$$\frac{1}{H(\varphi_a, r)} \frac{d}{dr} H(\varphi_a, r) = \frac{2}{r} N(\varphi_a, r, \lambda_a, A_a) \leq \frac{2}{r}\left(\frac{k}{2} + \delta\right) \quad \text{for all } K_\delta|a| \leq r \leq r_\delta, \tag{86}$$

hence integration between $K_\delta|a|$ and $R|a|$ yields

$$H(\tilde{\varphi}_a, R) = \frac{H(\varphi_a, R|a|)}{H(\varphi_a, K_\delta|a|)} \leq \left(\frac{R}{K_\delta}\right)^{k+2\delta}. \tag{87}$$

From (85) and (87) we obtain that, if $R \geq K_\delta$ and $|a| < \frac{r_\delta}{R}$,

$$\int_{D_R} |(i\nabla + A_{\mathbf{e}})\tilde{\varphi}_a|^2 \, dx \leq \frac{1}{1 - 5\Lambda r_\delta^2}\Big(\frac{k}{2} + \delta\Big)\Big(\frac{R}{K_\delta}\Big)^{k+2\delta}. \tag{88}$$

Moreover (87) yields

$$\int_{\partial D_R} |\tilde{\varphi}_a|^2 \, ds \leq R\Big(\frac{R}{K_\delta}\Big)^{k+2\delta}. \tag{89}$$

Estimates (88) and (89) together with Lemma 3.1 imply (79). To conclude, we observe that (88) yields (80), (87) imply (81), while (82) follows from (80) and (81) in view of Lemma 3.1. □

6. Preliminary estimates for the difference of eigenvalues

To obtain both upper and lower estimates for the eigenvalue variation $\lambda_0 - \lambda_a$, we will use the following technical lemma.

Lemma 6.1. *For every $a = (|a|, 0) \in \Omega$ let us consider a quadratic form*

$$Q_a : \mathbb{C}^{n_0} \to \mathbb{R}, \quad Q_a(z_1, z_2, \dots, z_{n_0}) = \sum_{j,n=1}^{n_0} M_{j,n}(a) z_j \overline{z_n},$$

with $M_{j,n}(a) \in \mathbb{C}$ such that $M_{j,n}(a) = \overline{M_{n,j}(a)}$. Let us assume that there exist $\alpha \in (0, +\infty)$, $a \mapsto \sigma(a) \in \mathbb{R}$ with $\sigma(a) \geq 0$ and $\sigma(a) = O(|a|^{2\alpha})$ as $|a| \to 0^+$, and $a \mapsto \mu(a) \in \mathbb{R}$ with $\mu(a) = O(1)$ as $|a| \to 0^+$, such that the coefficients $M_{j,n}(a)$ satisfy the following conditions:

$$M_{n_0,n_0}(a) = \sigma(a)\mu(a), \tag{90}$$

$$\text{for all } j < n_0 \ M_{j,j}(a) \to M_j \text{ as } |a| \to 0^+ \text{ for some } M_j \in \mathbb{R},\ M_j < 0, \tag{91}$$

$$\text{for all } j < n_0 \ M_{j,n_0}(a) = \overline{M_{n_0,j}(a)} = O(|a|^\alpha \sqrt{\sigma(a)}) \text{ as } |a| \to 0^+, \tag{92}$$

$$\text{for all } j, n < n_0 \text{ with } j \neq n \ M_{j,n}(a) = O(|a|^{2\alpha}) \text{ as } |a| \to 0^+, \tag{93}$$

$$\text{there exists } M \in \mathbb{N} \text{ such that } |a|^{(2+M)\alpha} = o(\sigma(a)) \text{ as } |a| \to 0^+. \tag{94}$$

Then

$$\max_{\substack{z \in \mathbb{C}^{n_0} \\ \|z\|=1}} Q_a(z) = \sigma(a)\big(\mu(a) + o(1)\big) \quad \text{as } |a| \to 0^+,$$

where $\|z\| = \|(z_1, z_2, \dots, z_{n_0})\| = \big(\sum_{j=1}^{n_0} |z_j|^2\big)^{1/2}$.

Proof. For every a let $z(a) = (z_1(a), \dots, z_{n_0}(a)) \in \mathbb{C}^{n_0}$ be such that

$$\|z(a)\| = 1 \quad \text{and} \quad Q_a(z(a)) = \max_{\substack{z \in \mathbb{C}^{n_0} \\ \|z\|=1}} Q_a(z). \tag{95}$$

From

$$M_{n_0,n_0}(a) \leq \sum_{j,n=1}^{n_0} M_{j,n}(a) z_j(a) \overline{z_n(a)} \tag{96}$$

it follows that

$$\big(1 - |z_{n_0}(a)|^2\big)\Big(M_{n_0,n_0}(a) - \max_{j<n_0} M_{j,j}(a)\Big) \leq \sum_{\substack{j,n=1 \\ j\neq n}}^{n_0} M_{j,n}(a) z_j(a) \overline{z_n(a)}$$

and hence, by (90) and (91),

$$\big(1 - |z_{n_0}(a)|^2\big)\Big(-\max_{j<n_0} M_j + o(1)\Big) \leq \sum_{\substack{j,n=1 \\ j\neq n}}^{n_0} M_{j,n}(a) z_j(a) \overline{z_n(a)}, \tag{97}$$

as $|a| \to 0^+$. Due to (92), (93) and the assumption $\sigma(a) = O(|a|^{2\alpha})$ we then have

$$1 - |z_{n_0}(a)|^2 = O(|a|^{2\alpha}) \quad \text{as } |a| \to 0^+. \tag{98}$$

Since $1 - |z_{n_0}(a)|^2 = \sum_{j<n_0} |z_j(a)|^2$, we also have that

$$|z_j(a)|^2 = O(|a|^{2\alpha}), \quad \text{for all } j < n_0, \tag{99}$$

as $|a| \to 0^+$. We claim that

$$\sum_{\substack{j,n=1\\ j\neq n}}^{n_0} M_{j,n}(a)z_j(a)\overline{z_n(a)} = o\big(\sigma(a)\big) \tag{100}$$

as $|a| \to 0^+$. To prove (100) it is enough to show that

$$\begin{cases} \text{for every sequence } a_l = |a_l|\mathbf{e} \to 0 \text{ there exists a subsequence } a_{l_p} \text{ such that} \\ \sum_{j\neq n} M_{j,n}(a_{l_p})z_j(a_{l_p})\overline{z_n(a_{l_p})} = o\big(\sigma(a_{l_p})\big) \text{ as } p \to +\infty. \end{cases} \tag{101}$$

Let $a_l = |a_l|\mathbf{e} \to 0$. From (92), (93), (99), and the assumption $\sigma(a) = O(|a|^{2\alpha})$ we deduce that

$$\begin{aligned} &\sum_{\substack{j,n=1\\ j\neq n}}^{n_0} M_{j,n}(a)z_j(a)\overline{z_n(a)} \\ &\quad= \sum_{\substack{j,n=1\\ j\neq n}}^{n_0-1} M_{j,n}(a)z_j(a)\overline{z_n(a)} + \sum_{j=1}^{n_0-1} M_{j,n_0}(a)z_j(a)\overline{z_{n_0}(a)} + \sum_{j=1}^{n_0-1} M_{n_0,j}(a)z_{n_0}(a)\overline{z_j(a)} \\ &\quad= O(|a|^{4\alpha}) + O(|a|^{2\alpha}\sqrt{\sigma(a)}) = O(|a|^{3\alpha}) \end{aligned}$$

as $a = a_l$, $l \to \infty$. If $|a_l|^{3\alpha} = o(\sigma(a_l))$, we have proved claim (101); if not, there holds

$$\sigma(a) = O(|a|^{3\alpha}) \tag{102}$$

along a subsequence of a_l (still denoted as a_l). Hence estimate (99) is improved as

$$|z_j(a)|^2 = O(|a|^{3\alpha}), \quad \text{for all } j < n_0, \tag{103}$$

along the subsequence. We now perform a recursive argument, improving the previous estimates step by step. Proceeding as above and exploiting the improved estimates (102) and (103), together with (92) and (93), along the subsequence we have

$$\begin{aligned} &\sum_{\substack{j,n=1\\ j\neq n}}^{n_0} M_{j,n}(a)z_j(a)\overline{z_n(a)} \\ &\quad= \sum_{\substack{j,n=1\\ j\neq n}}^{n_0-1} M_{j,n}(a)z_j(a)\overline{z_n(a)} + \sum_{j=1}^{n_0-1} M_{j,n_0}(a)z_j(a)\overline{z_{n_0}(a)} + \sum_{j=1}^{n_0-1} M_{n_0,j}(a)z_{n_0}(a)\overline{z_j(a)} \\ &\quad= O(|a|^{5\alpha}) + O\big(|a|^{\frac{5}{2}\alpha}\sqrt{\sigma(a)}\big) = O(|a|^{4\alpha}). \end{aligned}$$

If $|a|^{4\alpha} = o(\sigma(a))$ along the subsequence, we have proved claim (101); if not, up to passing to a subsequence again, there holds

$$\sigma(a) = O(|a|^{4\alpha}). \tag{104}$$

Hence we improve estimate (103) as

$$|z_j(a)|^2 = O(|a|^{4\alpha}), \quad \text{for all } j < n_0, \tag{105}$$

along the subsequence. Repeating the above argument M times with M as in (94), we obtain that, along a subsequence,

$$\sum_{\substack{j,n=1\\ j\neq n}}^{n_0} M_{j,n}(a)z_j(a)\overline{z_n(a)} = O\big(|a|^{\alpha(2+M)}\big) = o(\sigma(a)),$$

thus proving (101) and then (100).

From (97) and (100), it follows that

$$|z_{n_0}(a)|^2 = 1 + o(\sigma(a)) \quad \text{and} \quad |z_j(a)|^2 = o(\sigma(a)) \quad \text{for all } j < n_0, \tag{106}$$

as $|a| \to 0^+$. From (90), (91), (95), (100), and (106), we obtain the conclusion. □

6.1. **Upper bound for $\lambda_0 - \lambda_a$: the Rayleigh quotient for λ_0.** We are now going to estimate the Rayleigh quotient for λ_0. Let $R > 2$. Being R_0 as in Lemma 5.2, for every $a = (|a|, 0)$ with $|a| < R_0/R$ we define the functions $v_{j,R,a}$ as follows:

$$v_{j,R,a} = \begin{cases} v_{j,R,a}^{ext}, & \text{in } \Omega \setminus D_{R|a|}, \\ v_{j,R,a}^{int}, & \text{in } D_{R|a|}, \end{cases} \qquad j = 1, \dots, n_0,$$

where

$$v_{j,R,a}^{ext} := e^{\frac{i}{2}(\theta_0 - \theta_a)} \varphi_j^a \quad \text{in } \Omega \setminus D_{R|a|}, \tag{107}$$

with φ_j^a as in (59)–(61) and θ_a, θ_0 as in (15) (notice that $e^{\frac{i}{2}(\theta_0-\theta_a)}$ is smooth in $\Omega \setminus D_{R|a|}$), so that it solves

$$\begin{cases} (i\nabla + A_0)^2 v_{j,R,a}^{ext} = \lambda_j^a v_{j,R,a}^{ext}, & \text{in } \Omega \setminus D_{R|a|}, \\ v_{j,R,a}^{ext} = e^{\frac{i}{2}(\theta_0-\theta_a)} \varphi_j^a & \text{on } \partial(\Omega \setminus D_{R|a|}), \end{cases}$$

whereas $v_{j,R,a}^{int}$ is the unique solution to the minimization problem

$$\begin{aligned} &\int_{D_{R|a|}} |(i\nabla + A_0) v_{j,R,a}^{int}(x)|^2 \, dx \\ &\quad = \min \left\{ \int_{D_{R|a|}} |(i\nabla + A_0) u(x)|^2 \, dx : \, u \in H^{1,0}(D_{R|a|}, \mathbb{C}), \ u = e^{\frac{i}{2}(\theta_0-\theta_a)} \varphi_j^a \text{ on } \partial D_{R|a|} \right\}, \end{aligned} \tag{108}$$

so that it solves

$$\begin{cases} (i\nabla + A_0)^2 v_{j,R,a}^{int} = 0, & \text{in } D_{R|a|}, \\ v_{j,R,a}^{int} = e^{\frac{i}{2}(\theta_0-\theta_a)} \varphi_j^a, & \text{on } \partial D_{R|a|}. \end{cases} \tag{109}$$

It is easy to verify that

$$\dim\big(\operatorname{span}\{v_{1,R,a}, \dots, v_{n_0,R,a}\}\big) = n_0. \tag{110}$$

Lemma 6.2. *For $\delta \in (0, 1/4)$, let $\tilde K_\delta > 1$ be as in Lemma 5.8 and let R_0 be as in Lemma 5.2. For all $R > \max\{2, \tilde K_\delta\}$, $a = (|a|, 0) \in \Omega$ with $|a| < \frac{R_0}{R}$, and $1 \le j \le n_0$, let $v_{j,R,a}^{int}$ be defined in* (108)-(109). *Then there exists $\hat C_\delta > 0$ (depending only on δ) such that*

$$\int_{D_{R|a|}} |(i\nabla + A_0) v_{j,R,a}^{int}|^2 \, dx \le \hat C_\delta (R|a|)^{1-2\delta}, \tag{111}$$

$$\int_{\partial D_{R|a|}} |v_{j,R,a}^{int}|^2 \, ds \le \hat C_\delta (R|a|)^{2-2\delta}, \tag{112}$$

$$\int_{D_{R|a|}} |v_{j,R,a}^{int}|^2 \, dx \le \hat C_\delta (R|a|)^{3-2\delta}. \tag{113}$$

Proof. Let $\eta_{|a|R}$ be as in (14). From (108) it follows that

$$\begin{aligned} \int_{D_{R|a|}} |(i\nabla + A_0) v_{j,R,a}^{int}(x)|^2 \, dx &\le \int_{D_{R|a|}} \Big| (i\nabla + A_0)\big(e^{\frac{i}{2}(\theta_0-\theta_a)} \varphi_j^a \eta_{|a|R}\big)(x) \Big|^2 dx \\ &\le 2 \int_{D_{R|a|} \setminus D_{\frac{R|a|}{2}}} \Big| (i\nabla + A_0)\big(e^{\frac{i}{2}(\theta_0-\theta_a)} \varphi_j^a\big)(x) \Big|^2 dx + 2 \int_{D_{R|a|}} |\varphi_j^a(x)|^2 |\nabla \eta_{|a|R}(x)|^2 \, dx \\ &\le 2 \int_{D_{R|a|} \setminus D_{\frac{R|a|}{2}}} \big| (i\nabla + A_a) \varphi_j^a(x) \big|^2 dx + \frac{32}{R^2|a|^2} \int_{D_{R|a|}} |\varphi_j^a(x)|^2 \, dx, \end{aligned} \tag{114}$$

which yields (111) in view of estimates (68) and (69). Estimate (112) follows directly from (109) and (67). We finally conclude by observing that (113) follows from Lemma 3.1 and estimates (111) and (112). □

For all $R > 2$ and $a = (|a|, 0) \in \Omega$ with $|a| < \frac{R_0}{R}$, we define

$$Z_a^R(x) := \frac{v_{n_0,R,a}^{int}(|a|x)}{\sqrt{H(\varphi_a, K_\delta |a|)}}. \tag{115}$$

Lemma 6.3. *For all $R > 2$,*

$$\text{the family of functions } \big\{ Z_a^R : a = |a|\mathbf{e}, |a| < \tfrac{r_\delta}{R} \big\} \text{ is bounded in } H^{1,0}(D_R, \mathbb{C}). \tag{116}$$

In particular, for all $R > 2$,

$$\int_{D_{R|a|}} \left|(i\nabla + A_0)v^{int}_{n_0,R,a}\right|^2 dx = O(H(\varphi_a, K_\delta|a|)), \quad \text{as } |a| \to 0^+, \tag{117}$$

$$\int_{\partial D_{R|a|}} |v^{int}_{n_0,R,a}|^2 dx = O(|a|H(\varphi_a, K_\delta|a|)), \quad \text{as } |a| \to 0^+, \tag{118}$$

$$\int_{D_{R|a|}} |v^{int}_{n_0,R,a}|^2 dx = O(|a|^2 H(\varphi_a, K_\delta|a|)), \quad \text{as } |a| \to 0^+. \tag{119}$$

Proof. We notice that Z_a^R solves

$$\begin{cases} (i\nabla + A_0)^2 Z_a^R = 0, & \text{in } D_R \\ Z_a^R = e^{\frac{i}{2}(\theta_0 - \theta_{\mathbf{e}})}\tilde{\varphi}_a, & \text{on } \partial D_R, \end{cases}$$

and, by the Dirichlet principle and Theorem 5.9,

$$\begin{aligned} \int_{D_R} |(i\nabla + A_0)Z_a^R|^2 dx &\le \int_{D_R} \left|(i\nabla + A_0)\big(\eta_R\, e^{\frac{i}{2}(\theta_0-\theta_{\mathbf{e}})}\tilde{\varphi}_a\big)\right|^2 dx \\ &\le 2\int_{D_R} |\nabla\eta_R|^2|\tilde{\varphi}_a|^2 dx + 2\int_{D_R\setminus D_{R/2}} \eta_R^2\left|(i\nabla + A_{\mathbf{e}})\tilde{\varphi}_a\right|^2 dx \le C_R, \end{aligned} \tag{120}$$

for some $C_R > 0$ and η_R being as in (14). Then, taking into account (17), we obtain (116). Estimate (117) follows directly from (120) and (115) while (118) is a direct consequence of the definition of $v^{int}_{n_0,R,a}$ (see (109)) and (81). (119) follows from (117) and (118) in view of Lemma 3.1. ☐

Lemma 6.4. *There exists $\tilde{R} > 2$ such that for all $R > \tilde{R}$ and $a = (|a|, 0) \in \Omega$ with $|a| < \frac{R_0}{R}$,*

$$\frac{\lambda_0 - \lambda_a}{H(\varphi_a, K_\delta|a|)} \le f_R(a)$$

where

$$\begin{aligned} f_R(a) &= \int_{D_R} |(i\nabla + A_0)Z_a^R|^2\, dx - \int_{D_R} |(i\nabla + A_{\mathbf{e}})\tilde{\varphi}_a|^2\, dx + o(1), \quad \text{as } |a| \to 0^+, \\ f_R(a) &= O(1), \quad \text{as } |a| \to 0^+, \end{aligned} \tag{121}$$

with $\tilde{\varphi}_a$ and Z_a^R defined in (76) *and* (115) *respectively.*

Proof. Let $\tilde{K}_\delta > 1$ be as in Lemma 5.8 and fix $R > \max\{2, \tilde{K}_\delta\}$.

In (16) with $j = n_0$ and $a = 0$, we choose F as the space of functions $\{\tilde{v}_{j,R,a}\}$ which result from $\{v_{j,R,a}\}$ by a Gram–Schmidt process, that is

$$\tilde{v}_{j,R,a} := \frac{\hat{v}_{j,R,a}}{\|\hat{v}_{j,R,a}\|_{L^2(\Omega,\mathbb{C})}}, \quad j = 1, \dots, n_0,$$

where $\hat{v}_{n_0,R,a} := v_{n_0,R,a}$ and

$$\hat{v}_{j,R,a} := v_{j,R,a} - \sum_{\ell=j+1}^{n_0} \frac{\int_\Omega v_{j,R,a}\overline{\hat{v}_{\ell,R,a}}\, dx}{\|\hat{v}_{\ell,R,a}\|^2_{L^2(\Omega,\mathbb{C})}}\hat{v}_{\ell,R,a} \quad \text{for } j = 1, \dots, n_0 - 1.$$

For notation convenience we also set

$$d_{\ell,j}^{R,a} := \frac{\int_\Omega v_{j,R,a}\overline{\hat{v}_{\ell,R,a}}\, dx}{\|\hat{v}_{\ell,R,a}\|^2_{L^2(\Omega,\mathbb{C})}}.$$

From (60), Lemmas 5.8 and 6.2, and an induction argument, il follows that

$$\|\hat{v}_{j,R,a}\|^2_{L^2(\Omega,\mathbb{C})} = 1 + O(|a|^{3-2\delta}) \quad \text{and} \quad d_{\ell,j}^{R,a} = O(|a|^{3-2\delta}) \text{ for } \ell \ne j \tag{122}$$

as $|a| \to 0^+$. Furthermore, from (60), (82), and (119) we deduce that

$$\|\hat{v}_{n_0,R,a}\|^2_{L^2(\Omega,\mathbb{C})} = \|v_{n_0,R,a}\|^2_{L^2(\Omega,\mathbb{C})} = 1 + O\big(|a|^2 H(\varphi_a, K_\delta|a|)\big) \quad \text{as } |a| \to 0^+, \tag{123}$$

and

$$d_{n_0,j}^{R,a} = O\big(|a|^{\frac{5}{2}-\delta}\sqrt{H(\varphi_a, K_\delta|a|)}\big) \quad \text{as } |a| \to 0^+, \quad \text{for all } j < n_0. \tag{124}$$

From (16) and (110) it follows that

$$\lambda_0 \le \max_{\substack{(\alpha_1,\dots,\alpha_{n_0}) \in \mathbb{C}^{n_0} \\ \sum_{j=1}^{n_0}|\alpha_j|^2 = 1}} \int_\Omega \left|(i\nabla + A_0)\Big(\sum_{j=1}^{n_0}\alpha_j\tilde{v}_{j,R,a}\Big)\right|^2 dx.$$

Hence

$$\lambda_0 - \lambda_a \le \max_{\substack{(\alpha_1,\dots,\alpha_{n_0})\in\mathbb{C}^{n_0}\\ \sum_{j=1}^{n_0}|\alpha_j|^2=1}} \sum_{j,n=1}^{n_0} m_{j,n}^{a,R}\alpha_j\overline{\alpha_n}, \tag{125}$$

where

$$m_{j,n}^{a,R} = \int_\Omega (i\nabla + A_0)\tilde v_{j,R,a}\cdot\overline{(i\nabla + A_0)\tilde v_{n,R,a}}\,dx - \lambda_a\delta_{jn},$$

with $\delta_{jn}=1$ if $j=n$ and $\delta_{jn}=0$ if $j\neq n$. We will show that the quadratic form with coefficients $m_{j,n}^{a,R}$ satisfies the assumptions of Lemma 6.1 with $\sigma(a)=H(\varphi_a,K_\delta|a|)$, $\mu(a)=f_R(a)$ and $\alpha=\frac12-\delta$.

To this aim, we first observe that integration of (86) over the interval $(K_\delta|a|,r_\delta)$ yields

$$H(\varphi_a,K_\delta|a|)\ge C_\delta|a|^{k+2\delta},\quad \text{if } |a|<\frac{r_\delta}{K_\delta}, \tag{126}$$

for some $C_\delta>0$ independent of a, thus yielding (94) if M is such that $1+\frac{M}{2}-(2+M)\delta>k+2\delta$. Estimate (67) implies that

$$H(\varphi_a,K_\delta|a|)=O(|a|^{1-2\delta})\quad \text{as } |a|\to 0. \tag{127}$$

From (123), (115), (76), Theorem 5.9, and Lemma 6.3 we deduce that

$$\begin{aligned} m_{n_0,n_0}^{a,R} &= \frac{\lambda_a(1-\|v_{n_0,R,a}\|^2_{L^2(\Omega,\mathbb{C})})}{\|v_{n_0,R,a}\|^2_{L^2(\Omega,\mathbb{C})}} + \frac{\Big(\int_{D_{R|a|}}\big|(i\nabla+A_0)v^{int}_{n_0,R,a}\big|^2dx - \int_{D_{R|a|}}\big|(i\nabla+A_a)\varphi_a\big|^2dx\Big)}{\|v_{n_0,R,a}\|^2_{L^2(\Omega,\mathbb{C})}} \\ &= H(\varphi_a,K_\delta|a|)\bigg(\int_{D_R}|(i\nabla+A_0)Z_a^R|^2\,dx - \int_{D_R}|(i\nabla+A_{\mathbf e})\tilde\varphi_a|^2\,dx + o(1)\bigg), \end{aligned} \tag{128}$$

as $|a|\to 0^+$, thus yielding (90). From [8, Theorem 1.1] (which ensures that $\lambda_j^a\to\lambda_j^0$ as $|a|\to 0$), (122), (59), (60), and Lemmas 5.8 and 6.2, we obtain that, if $j<n_0$,

$$\begin{aligned} m_{j,j}^{a,R} &= -\lambda_a + \frac{1}{\|\hat v_{j,R,a}\|^2_{L^2(\Omega,\mathbb{C})}}\bigg(\lambda_j^a - \int_{D_{R|a|}}\big|(i\nabla+A_a)\varphi_j^a\big|^2\,dx + \int_{D_{R|a|}}\big|(i\nabla+A_0)v^{int}_{j,R,a}\big|^2\,dx\bigg) \\ &\quad + \frac{1}{\|\hat v_{j,R,a}\|^2_{L^2(\Omega,\mathbb{C})}}\int_\Omega\bigg|(i\nabla+A_0)\Big(\sum_{\ell>j}d_{\ell,j}^{R,a}\hat v_{\ell,R,a}\Big)\bigg|^2dx \\ &\quad - \frac{2}{\|\hat v_{j,R,a}\|^2_{L^2(\Omega,\mathbb{C})}}\mathfrak{Re}\bigg(\int_\Omega(i\nabla+A_0)v_{j,R,a}\cdot\overline{(i\nabla+A_0)\Big(\sum_{\ell>j}d_{\ell,j}^{R,a}\hat v_{\ell,R,a}\Big)}\,dx\bigg) \\ &= (\lambda_j^0-\lambda_0)+o(1)\quad \text{as } |a|\to 0. \end{aligned}$$

so that (91) is satisfied. From (122), (124), (59), (60), (80), Lemmas 5.8 and 6.2, and (117), it follows that, for all $j<n_0$,

$$\begin{aligned} &\|\hat v_{j,R,a}\|_{L^2(\Omega,\mathbb{C})}\|\hat v_{n_0,R,a}\|_{L^2(\Omega,\mathbb{C})}m_{j,n_0}^{a,R} \\ &\quad = \int_{D_{R|a|}}\Big((i\nabla+A_0)v^{int}_{j,R,a}\cdot\overline{(i\nabla+A_0)v^{int}_{n_0,R,a}} - (i\nabla+A_a)\varphi_j^a\cdot\overline{(i\nabla+A_a)\varphi_a}\Big)\,dx \\ &\qquad - \int_\Omega(i\nabla+A_0)\Big(\sum_{\ell>j}d_{\ell,j}^{R,a}\hat v_{\ell,R,a}\Big)\cdot\overline{(i\nabla+A_0)v_{n_0,R,a}}\,dx = O\Big(|a|^{\frac12-\delta}\sqrt{H(\varphi_a,K_\delta|a|)}\Big). \end{aligned}$$

Hence, by (122) and (123), we have that

$$m_{j,n_0}^{a,R}=O\Big(|a|^{\frac12-\delta}\sqrt{H(\varphi_a,K_\delta|a|)}\Big)\quad \text{and}\quad m_{n_0,j}^{a,R}=\overline{m_{j,n_0}^{a,R}}=O\Big(|a|^{\frac12-\delta}\sqrt{H(\varphi_a,K_\delta|a|)}\Big)$$

as $|a| \to 0^+$, thus yielding (92). From (122), (59), (60), and Lemmas 5.8 and 6.2, we deduce that, for all $j, n < n_0$ with $j \neq n$,

$$\begin{aligned}
&\|\hat v_{j,R,a}\|_{L^2(\Omega,\mathbb{C})}\|\hat v_{n,R,a}\|_{L^2(\Omega,\mathbb{C})} m^{a,R}_{j,n} \\
&\quad= \int_{D_{R|a|}} \Big((i\nabla + A_0) v^{int}_{j,R,a} \cdot \overline{(i\nabla + A_0) v^{int}_{n,R,a}} - (i\nabla + A_a)\varphi^a_j \cdot \overline{(i\nabla + A_a)\varphi^a_n}\Big)\,dx \\
&\qquad + \int_\Omega (i\nabla + A_0)\big(\sum_{\ell>j} d^{R,a}_{\ell,j} \hat v_{\ell,R,a}\big) \cdot \overline{(i\nabla + A_0)\big(\sum_{h>n} d^{R,a}_{h,n} \hat v_{h,R,a}\big)}\,dx \\
&\qquad - \int_\Omega (i\nabla + A_0)\big(\sum_{\ell>j} d^{R,a}_{\ell,j} \hat v_{\ell,R,a}\big) \cdot \overline{(i\nabla + A_0) v_{n,R,a}}\,dx \\
&\qquad - \int_\Omega (i\nabla + A_0) v_{j,R,a} \cdot \overline{(i\nabla + A_0)\big(\sum_{h>n} d^{R,a}_{h,n} \hat v_{h,R,a}\big)}\,dx = O(|a|^{1-2\delta}) \quad \text{as } |a| \to 0.
\end{aligned}$$

Hence, in view of (122),

$$m^{a,R}_{j,n} = O(|a|^{1-2\delta}) \quad \text{as } |a| \to 0,$$

so that also (93) is verified. Then we can apply Lemma 6.1 to deduce that

$$\max_{\substack{(\alpha_1,\dots,\alpha_{n_0})\in\mathbb{C}^{n_0}\\ \sum_{j=1}^{n_0}|\alpha_j|^2=1}} \sum_{j,n=1}^{n_0} m^{a,R}_{j,n}\alpha_j\overline{\alpha_n} = H(\varphi_a, K_\delta|a|)\bigg(\int_{D_R} |(i\nabla + A_0)Z^R_a|^2\,dx - \int_{D_R} |(i\nabla + A_{\mathbf{e}})\tilde\varphi_a|^2\,dx + o(1)\bigg)$$

as $|a| \to 0^+$, which, in view of (125), yields $\frac{\lambda_0-\lambda_a}{H(\varphi_a,K_\delta|a|)} \le f_R(a)$ with f_R as in (121). We notice that, from Theorem 5.9 and Lemma 6.3, for all $R > \tilde R$, $f_R(a) = O(1)$ as $|a| \to 0^+$. The proof is now complete. □

As a direct consequence of Lemma 6.4 the following corollary holds.

Corollary 6.5. *There exists positive constants $C^*, r^* > 0$ such that, for all $a = (|a|, 0) \in \Omega$ with $|a| < r^*$,*

$$\lambda_0 - \lambda_a \le C^* H(\varphi_a, K_\delta|a|).$$

6.2. **Lower bound for $\lambda_0 - \lambda_a$: the Rayleigh quotient for λ_a.** Being R_0 as in Lemma 5.2, for every $R > 2$ and $a = (|a|, 0) \in \Omega$ with $|a| < R_0/R$ we define the functions $w_{j,R,a}$ as

$$w_{j,R,a} = \begin{cases} w^{ext}_{j,R,a}, & \text{in } \Omega \setminus D_{R|a|}, \\ w^{int}_{j,R,a}, & \text{in } D_{R|a|}, \end{cases} \qquad j = 1, \dots, n_0,$$

where $w^{ext}_{j,R,a} := e^{\frac{i}{2}(\theta_a-\theta_0)}\varphi^0_j$ in $\Omega \setminus D_{R|a|}$, with φ^0_j as in (59)–(61) with $a = 0$, so that it solves

$$\begin{cases} (i\nabla + A_a)^2 w^{ext}_{j,R,a} = \lambda^0_j w^{ext}_{j,R,a}, & \text{in } \Omega \setminus D_{R|a|}, \\ w^{ext}_{j,R,a} = e^{\frac{i}{2}(\theta_a-\theta_0)}\varphi^0_j & \text{on } \partial(\Omega \setminus D_{R|a|}), \end{cases}$$

whereas $w^{int}_{j,R,a}$ is the unique solution to the minimization problem

$$\int_{D_{R|a|}} |(i\nabla + A_a) w^{int}_{j,R,a}(x)|^2\,dx = \min_{\substack{u\in H^{1,a}(D_{R|a|},\mathbb{C})\\ u=e^{\frac{i}{2}(\theta_a-\theta_0)}\varphi^0_j \text{ on } \partial D_{R|a|}}} \int_{D_{R|a|}} |(i\nabla + A_a)u(x)|^2\,dx,$$

thus solving $(i\nabla + A_a)^2 w^{int}_{j,R,a} = 0$ in $D_{R|a|}$ with $w^{int}_{j,R,a} = e^{\frac{i}{2}(\theta_a-\theta_0)}\varphi^0_j$ on $\partial D_{R|a|}$. It is easy to verify that

$$\dim\big(\operatorname{span}\{w_{1,R,a}, \dots, w_{n_0,R,a}\}\big) = n_0. \tag{129}$$

As a direct consequence of [13, Theorem 1.3] (see also Proposition 2.1), there exists some $\tilde K > 0$ such that, for every $R > 2$, $a = (|a|, 0) \in \Omega$ with $|a| < \frac{R_0}{R}$, and $1 \le j \le n_0$,

$$\int_{\partial D_{R|a|}} |\varphi^0_j|^2\,ds \le \tilde K(R|a|)^2, \quad \int_{D_{R|a|}} |(i\nabla + A_0)\varphi^0_j|^2\,dx \le \tilde K(R|a|), \quad \int_{D_{R|a|}} |\varphi^0_j|^2\,dx \le \tilde K(R|a|)^3. \tag{130}$$

Arguing as in the proof of Lemma 6.2 (using estimates (130) instead of (67)–(69)) we obtain (up to enlarging the constant $\tilde K$) that, for every $R > 2$, $a = (|a|, 0) \in \Omega$ with $|a| < \frac{R_0}{R}$, and $1 \le j \le n_0$,

$$\int_{D_{R|a|}} |(i\nabla + A_a) w^{int}_{j,R,a}|^2\,dx \le \tilde K(R|a|), \tag{131}$$

$$\int_{\partial D_{R|a|}} |w^{int}_{j,R,a}|^2\,ds \le \tilde K(R|a|)^2, \quad \int_{D_{R|a|}} |w^{int}_{j,R,a}|^2\,dx \le \tilde K(R|a|)^3. \tag{132}$$

For all $R > 2$ and $a = (|a|, 0) \in \Omega$ with $|a| < \frac{R_0}{R}$, we define

$$U_a^R(x) := \frac{w^{int}_{n_0,R,a}(|a|x)}{|a|^{k/2}}, \quad W_a(x) := \frac{\varphi_0(|a|x)}{|a|^{k/2}}. \tag{133}$$

Under assumptions (6) and (33), from [13, Theorem 1.3 and Lemma 6.1] we have that

$$W_a \to \beta e^{\frac{i}{2}\theta_0}\psi_k \quad \text{as } |a| \to 0 \tag{134}$$

in $H^{1,0}(D_R, \mathbb{C})$ for every $R > 1$, where ψ_k is defined in (8) and

$$\beta := \beta_{k,2}(0, \varphi_0, \lambda_0) \tag{135}$$

with $\beta_{k,2}(0, \varphi_0, \lambda_0)$ as in (30) and (34).

We also denote as w_R the unique solution to the minimization problem

$$\int_{D_R} |(i\nabla + A_{\mathbf{e}})w_R(x)|^2\,dx = \min\left\{\int_{D_R} |(i\nabla + A_{\mathbf{e}})u(x)|^2\,dx : u \in H^{1,\mathbf{e}}(D_R,\mathbb{C}),\ u = e^{\frac{i}{2}\theta_{\mathbf{e}}}\psi_k \text{ on } \partial D_R\right\},$$

which then solves

$$\begin{cases} (i\nabla + A_{\mathbf{e}})^2 w_R = 0, & \text{in } D_R, \\ w_R = e^{\frac{i}{2}\theta_{\mathbf{e}}}\psi_k, & \text{on } \partial D_R. \end{cases} \tag{136}$$

By the Dirichlet principle and (134), we have that

$$\begin{aligned} &\int_{D_R} \left|(i\nabla + A_{\mathbf{e}})(U_a^R - \beta w_R)\right|^2 dx \le \int_{D_R} \left|(i\nabla + A_{\mathbf{e}})\big(\eta_R\, e^{\frac{i}{2}(\theta_{\mathbf{e}}-\theta_0)}(W_a - \beta e^{\frac{i}{2}\theta_0}\psi_k)\right|^2 dx \\ &\quad \le 2\int_{D_R} |\nabla\eta_R|^2 \left|W_a - \beta e^{\frac{i}{2}\theta_0}\psi_k\right|^2 dx + 2\int_{D_R\setminus D_{R/2}} \eta_R^2 \left|(i\nabla + A_0)(W_a - \beta e^{\frac{i}{2}\theta_0}\psi_k)\right|^2 dx = o(1) \end{aligned}$$

as $|a| \to 0^+$, where $\eta_R : \mathbb{R}^2 \to \mathbb{R}$ is a smooth cut-off function as in (14). Hence, for all $R > 2$,

$$U_a^R \to \beta w_R, \quad \text{in } H^{1,\mathbf{e}}(D_R, \mathbb{C}), \tag{137}$$

as $|a| \to 0$, where β is defined in (135).

Lemma 6.6. *For every $r > 1$, $w_R \to \Psi_k$ in $H^{1,\mathbf{e}}(D_r, \mathbb{C})$ as $R \to +\infty$.*

Proof. Let $r > 2$. For every $R > r$, by the Dirichlet Principle, (46), and (47) we have that, letting η_R as in (14),

$$\begin{aligned} &\int_{D_r} |(i\nabla + A_{\mathbf{e}})(w_R - \Psi_k)(x)|^2\,dx \le \int_{D_R} |(i\nabla + A_{\mathbf{e}})(\eta_R(e^{\frac{i}{2}\theta_{\mathbf{e}}}\psi_k - \Psi_k))(x)|^2\,dx \\ &\quad \le 2\int_{\mathbb{R}^2\setminus D_{R/2}} |(i\nabla + A_{\mathbf{e}})(e^{\frac{i}{2}\theta_{\mathbf{e}}}\psi_k - \Psi_k)|^2\,dx + \frac{32}{R^2}\int_{D_R\setminus D_{R/2}} |e^{\frac{i}{2}\theta_{\mathbf{e}}}\psi_k - \Psi_k|^2\,dx = o(1) \end{aligned}$$

as $R \to +\infty$. □

Lemma 6.7. *For $a = (|a|, 0) \in \Omega$, let $\varphi_a \in H^{1,a}_0(\Omega, \mathbb{C})$ solve (19-20) and $\varphi_0 \in H^{1,0}_0(\Omega, \mathbb{C})$ be a solution to (4–5). If* (3) *and* (6) *hold and* (33) *is satisfied, then, for all $R > \tilde{R}$ and $a = (|a|, 0) \in \Omega$, $\frac{\lambda_0 - \lambda_a}{|a|^k} \ge g_R(a)$ where $\lim_{|a|\to 0} g_R(a) = i|\beta|^2\tilde{\kappa}_R$, with β as in* (135) *and*

$$\tilde{\kappa}_R = \int_{\partial D_R} \Big(e^{-\frac{i}{2}\theta_{\mathbf{e}}}(i\nabla + A_{\mathbf{e}})w_R \cdot \nu - (i\nabla)\psi_k \cdot \nu\Big)\psi_k\,ds \tag{138}$$

being ψ_k as in (8).

Proof. In (16) with $j = n_0$ we choose F as the space of functions $\{\tilde{w}_{j,R,a}\}$ which result from $\{w_{j,R,a}\}$ by a Gram–Schmidt process, that is

$$\tilde{w}_{j,R,a} := \frac{\hat{w}_{j,R,a}}{\|\hat{w}_{j,R,a}\|_{L^2(\Omega,\mathbb{C})}}, \quad j = 1, \dots, n_0,$$

where $\hat{w}_{n_0,R,a} := w_{n_0,R,a}$,

$$\hat{w}_{j,R,a} := w_{j,R,a} - \sum_{\ell=j+1}^{n_0} c^{R,a}_{\ell,j}\hat{w}_{\ell,R,a} \quad \text{for } j = 1, \dots, n_0 - 1, \quad c^{R,a}_{\ell,j} := \frac{\int_\Omega w_{j,R,a}\overline{\hat{w}_{\ell,R,a}}\,dx}{\|\hat{w}_{\ell,R,a}\|^2_{L^2(\Omega,\mathbb{C})}}.$$

From (60), (130), and (132) and an induction argument, it follows that

$$\|\hat{w}_{j,R,a}\|^2_{L^2(\Omega,\mathbb{C})} = 1 + O(|a|^3) \quad \text{and} \quad c^{R,a}_{\ell,j} = O(|a|^3) \text{ for } \ell \neq j \tag{139}$$

as $|a| \to 0^+$. Furthermore, from (60), (134), and (137) we deduce that

$$\|\hat{w}_{n_0,R,a}\|^2_{L^2(\Omega,\mathbb{C})} = \|w_{n_0,R,a}\|^2_{L^2(\Omega,\mathbb{C})} = 1 + O\big(|a|^{2+k}\big) \quad \text{as } |a| \to 0^+, \tag{140}$$

and

$$c^{R,a}_{n_0,j} = O\big(|a|^{\frac{5}{2}+\frac{k}{2}}\big) \quad \text{as } |a| \to 0^+, \quad \text{for all } j < n_0. \tag{141}$$

From (16) and (129) it follows that

$$\lambda_a \le \max_{\substack{(\alpha_1,\dots,\alpha_{n_0})\in\mathbb{C}^{n_0} \\ \sum_{j=1}^{n_0} |\alpha_j|^2=1}} \int_\Omega \left|(i\nabla + A_a)\Big(\sum_{j=1}^{n_0} \alpha_j \tilde w_{j,R,a}\Big)\right|^2 dx.$$

Hence

$$\lambda_a - \lambda_0 \le \max_{\substack{(\alpha_1,\dots,\alpha_{n_0})\in\mathbb{C}^{n_0} \\ \sum_{j=1}^{n_0} |\alpha_j|^2=1}} \sum_{j,n=1}^{n_0} p^{a,R}_{j,n} \alpha_j \overline{\alpha_n}, \tag{142}$$

where $p^{a,R}_{j,n} = \int_\Omega (i\nabla + A_a)\tilde w_{j,R,a} \cdot \overline{(i\nabla + A_a)\tilde w_{n,R,a}}\, dx - \lambda_0 \delta_{jn}$. By (133), (134), (137), (140), and integration by parts we obtain that

$$\begin{aligned} p^{a,R}_{n_0,n_0} &= \frac{\lambda_0(1 - \|w_{n_0,R,a}\|^2_{L^2(\Omega,\mathbb{C})})}{\|w_{n_0,R,a}\|^2_{L^2(\Omega,\mathbb{C})}} + \frac{|a|^k\big(\int_{D_R} |(i\nabla + A_{\mathbf{e}})U^R_a|^2\, dx - \int_{D_R} |(i\nabla + A_0)W_a|^2\, dx\big)}{\|w_{n_0,R,a}\|^2_{L^2(\Omega,\mathbb{C})}} \\ &= |a|^k|\beta|^2\bigg(\int_{D_R} |(i\nabla + A_{\mathbf{e}})w_R|^2\, dx - \int_{D_R} |\nabla\psi_k|^2\, dx + o(1)\bigg) = -i|a|^k|\beta|^2\big(\tilde\kappa_R + o(1)\big), \end{aligned}$$

as $|a| \to 0$, with $\tilde\kappa_R$ as in (138). From (130), (131), and (139), we have that, for all $j < n_0$,

$$\begin{aligned} p^{a,R}_{j,j} &= -\lambda_0 + \frac{1}{\|\hat w_{j,R,a}\|^2_{L^2(\Omega,\mathbb{C})}} \left(\lambda^0_j - \int_{D_{R|a|}} \big|(i\nabla + A_0)\varphi^0_j\big|^2\, dx + \int_{D_{R|a|}} \big|(i\nabla + A_a)w^{int}_{j,R,a}\big|^2\, dx\right) \\ &\quad + \frac{1}{\|\hat w_{j,R,a}\|^2_{L^2(\Omega,\mathbb{C})}} \int_\Omega \left|(i\nabla + A_a)\Big(\sum_{\ell>j} c^{R,a}_{\ell,j} \hat w_{\ell,R,a}\Big)\right|^2 dx \\ &\quad - \frac{2}{\|\hat w_{j,R,a}\|^2_{L^2(\Omega,\mathbb{C})}} \mathfrak{Re}\left(\int_\Omega (i\nabla + A_a)w_{j,R,a} \cdot \overline{(i\nabla + A_a)\Big(\sum_{\ell>j} c^{R,a}_{\ell,j} \hat w_{\ell,R,a}\Big)}\, dx\right) \\ &= (\lambda^0_j - \lambda_0) + o(1) \quad \text{as } |a| \to 0. \end{aligned}$$

From (130), (131), (134), (137), and (141) it follows that, for all $j < n_0$,

$$\begin{aligned} &\|\hat w_{j,R,a}\|_{L^2(\Omega,\mathbb{C})} \|\hat w_{n_0,R,a}\|_{L^2(\Omega,\mathbb{C})} p^{a,R}_{j,n_0} \\ &\quad = \int_{D_{R|a|}} \Big((i\nabla + A_a)w^{int}_{j,R,a} \cdot \overline{(i\nabla + A_a)w^{int}_{n_0,R,a}} - (i\nabla + A_0)\varphi^0_j \cdot \overline{(i\nabla + A_0)\varphi_0}\Big)\, dx \\ &\qquad - \int_\Omega (i\nabla + A_a)\Big(\sum_{\ell>j} c^{R,a}_{\ell,j} \hat w_{\ell,R,a}\Big) \cdot \overline{(i\nabla + A_a)w_{n_0,R,a}}\, dx = O\Big(|a|^{\frac{k+1}{2}}\Big) \quad \text{as } |a| \to 0, \end{aligned}$$

and hence, in view of (139) and (140),

$$p^{a,R}_{j,n_0} = \overline{p^{a,R}_{n_0,j}} = O\Big(|a|^{\frac{k+1}{2}}\Big) \quad \text{as } |a| \to 0.$$

In a similar way, we have that, for all $j, n < n_0$ with $j \ne n$, $p^{a,R}_{j,n} = O(|a|)$ as $|a| \to 0$.

Then the quadratic form with coefficients $p^{a,R}_{j,n}$ satisfies the assumptions of Lemma 6.1 with $\sigma(a) = |a|^k$ and $\alpha = \frac{1}{2}$. Then Lemma 6.1 implies that

$$\max_{\substack{(\alpha_1,\dots,\alpha_{n_0})\in\mathbb{C}^{n_0} \\ \sum_{j=1}^{n_0} |\alpha_j|^2=1}} \sum_{j,n=1}^{n_0} p^{a,R}_{j,n} \alpha_j \overline{\alpha_n} = |a|^k\Big(-i|\beta|^2\tilde\kappa_R + o(1)\Big), \quad \text{as } |a| \to 0,$$

which, in view of (142), yields $\frac{\lambda_0 - \lambda_a}{|a|^k} \ge g_R(a)$ where $\lim_{|a|\to 0} g_R(a) = i|\beta|^2\tilde\kappa_R$. The proof is thereby complete. □

Lemma 6.8. *Let $\tilde\kappa_R$ be as in* (138). *Then $\lim_{R\to+\infty} \tilde\kappa_R = 4i\mathfrak{m}_k$, with $\mathfrak{m}_k$ as in* (10).

Proof. We claim that

$$\tilde\kappa_R = ik\sqrt{\pi}(\sqrt{\pi} - \xi(1)) + o(1), \quad \text{as } R \to +\infty, \tag{143}$$

where

$$\xi(r) := \int_0^{2\pi} e^{\frac{i}{2}(\theta_0 - \theta_{\mathbf{e}})(r\cos t, r\sin t)} \Psi_k(r\cos t, r\sin t)\overline{\psi_{k,2}(t)}\, dt, \quad r \ge 1. \tag{144}$$

To prove claim (143), we note that, according to (36) and (37), the function υ_R defined as

$$\upsilon_R(r) := \int_0^{2\pi} w_R(r(\cos t, \sin t)) e^{-\frac{i}{2}\theta_{\mathbf{e}}(r\cos t, r\sin t)} e^{i\frac{t}{2}} \overline{\psi_{k,2}(t)}\, dt, \quad r \in [1, R],$$

satisfies, for some $c_R \in \mathbb{C}$, $\big(r^{-k/2}\upsilon_R(r)\big)' = \frac{c_R}{r^{1+k}}$ in $(1, R)$. Integrating the previous equation over $(1, r)$ we obtain

$$r^{-k/2}\upsilon_R(r) - \upsilon_R(1) = \frac{c_R}{k}\left(1 - \frac{1}{r^k}\right), \quad \text{for all } r \in (1, R]. \tag{145}$$

We notice that, in view of (25) and (8),

$$\psi_k(r\cos t, r\sin t) = \sqrt{\pi} r^{k/2} e^{-\frac{i}{2}t}\psi_{k,2}(t), \quad \text{for all } t \in (0, 2\pi) \text{ and } r > 0. \tag{146}$$

Since (136) and (146) imply that $\upsilon_R(R) = \sqrt{\pi}R^{k/2}$, from (145) we deduce that $c_R = k\,\frac{R^k}{R^k-1}(\sqrt{\pi} - \upsilon_R(1))$ and then

$$\upsilon_R(r) = r^{k/2}\upsilon_R(1) + r^{k/2}\frac{R^k(\sqrt{\pi} - \upsilon_R(1))}{R^k - 1}\left(1 - \frac{1}{r^k}\right) = r^{k/2}\frac{R^k\sqrt{\pi} - \upsilon_R(1)}{R^k - 1} - r^{-k/2}\frac{R^k(\sqrt{\pi} - \upsilon_R(1))}{R^k - 1},$$

for all $r \in (1, R]$. By differentiation of the previous identity, we obtain that

$$\upsilon_R'(R) = \frac{k}{2}\,\frac{R^{\frac{k}{2}-1}}{R^k - 1}\Big((R^k + 1)\sqrt{\pi} - 2\upsilon_R(1)\Big). \tag{147}$$

On the other hand, writing υ_R as $\upsilon_R(r) = \frac{1}{r}\int_{\partial D_r} w_R(x) e^{-\frac{i}{2}(\theta_{\mathbf{e}} - \theta_0)(x)}\,\overline{\psi_{k,2}(\theta_0(x))}\, ds(x)$, differentiating and using (146), we obtain that

$$\upsilon_R'(r) = -\frac{i}{\sqrt{\pi}} r^{-1-\frac{k}{2}} \int_{\partial D_r} e^{-\frac{i}{2}\theta_{\mathbf{e}}}(i\nabla + A_{\mathbf{e}})w_R \cdot \nu\, \psi_k\, ds. \tag{148}$$

Combination of (147) and (148) yields

$$\int_{\partial D_R} e^{-\frac{i}{2}\theta_{\mathbf{e}}}(i\nabla + A_{\mathbf{e}})w_R \cdot \nu\, \psi_k\, ds = \frac{ik\sqrt{\pi}}{2}\,\frac{R^k}{R^k - 1}\Big((R^k + 1)\sqrt{\pi} - 2\upsilon_R(1)\Big). \tag{149}$$

Moreover, (8) directly gives

$$\int_{\partial D_R} (i\nabla)\psi_k \cdot \nu \psi_k\, ds = \frac{k}{2}\, iR^k\pi. \tag{150}$$

From (149), (150), and (138), it follows that

$$\begin{aligned}
\tilde{\kappa}_R &= \frac{ik\sqrt{\pi}}{2}\,\frac{R^k}{R^k - 1}\Big((R^k + 1)\sqrt{\pi} - 2\upsilon_R(1)\Big) - \frac{k}{2}\, iR^k\pi \\
&= \frac{ik\sqrt{\pi}}{2}\,\frac{R^k}{R^k - 1}\Big(\sqrt{\pi}R^k + \sqrt{\pi} - 2\upsilon_R(1) - \sqrt{\pi}(R^k - 1)\Big) \\
&= \frac{ik\sqrt{\pi}R^k}{2(R^k - 1)}\big(2\sqrt{\pi} - 2\upsilon_R(1)\big) = \frac{ik\sqrt{\pi}R^k}{R^k - 1}\big(\sqrt{\pi} - \upsilon_R(1)\big).
\end{aligned}$$

Since Lemma 6.6 and (144) imply that $\lim_{R\to+\infty}\upsilon_R(1) = \xi(1)$, we obtain claim (143). The conclusion follows by combining (143) and the identity

$$\sqrt{\pi} - \xi(1) = \frac{4}{k\sqrt{\pi}}\mathfrak{m}_k, \tag{151}$$

which results from Lemma 4.4. □

Combining Lemma 6.7 and Lemma 6.8 we deduce the following result.

Proposition 6.9. *For $a = (|a|, 0) \in \Omega$, let $\varphi_a \in H^{1,a}_0(\Omega, \mathbb{C})$ solve (19-20) and $\varphi_0 \in H^{1,0}_0(\Omega, \mathbb{C})$ be a solution to (4–5). If* (3) *and* (6) *hold and* (33) *is satisfied, then*

$$\liminf_{|a|\to 0} \frac{\lambda_0 - \lambda_a}{|a|^k} \geq -4|\beta|^2\mathfrak{m}_k > 0$$

with β as in (135) *and $\mathfrak{m}_k$ as in (10-11).*

Remark 6.10. As a consequence of Proposition 6.9, we have that, if $a \in \Omega$ approaches 0 along the half-line tangent to a nodal line of eigenfunctions associated to the simple eigenvalue λ_0, then $\lambda_a < \lambda_0$.

Combining Corollary 6.5 with Proposition 6.9 we obtain the following upper/lower estimates for $\lambda_0 - \lambda_a$.

Proposition 6.11. *For $a = (|a|, 0) = |a|\mathbf{e} \in \Omega$, let $\varphi_a \in H^{1,a}_0(\Omega, \mathbb{C})$ solve (19-20) and $\varphi_0 \in H^{1,0}_0(\Omega, \mathbb{C})$ be a solution to (4–5). Let* (3)*,* (6)*, and* (33) *hold. Then there exists a positive constant $C^* > 0$ such that*

$$-4|\beta|^2 \mathfrak{m}_k\, |a|^k (1 + o(1)) \leq \lambda_0 - \lambda_a \leq C^*\, H(\varphi_a, K_\delta |a|), \quad \text{as } |a| \to 0,$$

with β as in (135) *and $\mathfrak{m}_k < 0$ as in (10-11).*

7. Energy estimates

To obtain our main result, we aim at proving that the difference of the eigenvalues $\lambda_0 - \lambda_a$ is estimated even from above by the rate $|a|^k$, i.e. we have to determine the exact asymptotic behavior of the normalization term in (76), i.e. of $\sqrt{H(\varphi_a, K_\delta|a|)}$. To this purpose, in this section we obtain some preliminary energy estimates of the difference between approximating and limit eigenfunctions after blow-up, based on the invertibility of the differential of the function F defined below.

Throughout this section, we will treat the space $H^{1,0}_0(\Omega, \mathbb{C})$ defined in §1 as a real Hilbert space endowed with the scalar product

$$(u, v)_{H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C})} = \mathfrak{Re}\left(\int_\Omega (i\nabla + A_0)u \cdot \overline{(i\nabla + A_0)v}\, dx \right),$$

which induces on $H^{1,0}_0(\Omega, \mathbb{C})$ the norm

$$\|u\|_{H^{1,0}_0(\Omega,\mathbb{C})} = \left(\int_\Omega \left|(i\nabla + A_0)u\right|^2 dx \right)^{1/2}$$

which is equivalent to the norm (1) (see Lemma 3.1). To emphasize the fact that here $H^{1,0}_0(\Omega, \mathbb{C})$ is meant as a vector space over $\mathbb{R}$ we denote it as $H^{1,0}_{0,\mathbb{R}}(\Omega, \mathbb{C})$. We will denote as $(H^{1,0}_{0,\mathbb{R}}(\Omega, \mathbb{C}))^\star$ the real dual space of $H^{1,0}_{0,\mathbb{R}}(\Omega, \mathbb{C})$.

Let us consider the function

$$\begin{aligned} F : \mathbb{C} \times H^{1,0}_{0,\mathbb{R}}(\Omega, \mathbb{C}) &\longrightarrow \mathbb{R} \times \mathbb{R} \times (H^{1,0}_{0,\mathbb{R}}(\Omega, \mathbb{C}))^\star \\ (\lambda, \varphi) &\longmapsto \Big(\|\varphi\|^2_{H^{1,0}_0(\Omega,\mathbb{C})} - \lambda_0,\ \mathfrak{Im}\big(\textstyle\int_\Omega \varphi \overline{\varphi_0}\, dx \big),\ (i\nabla + A_0)^2 \varphi - \lambda\varphi \Big), \end{aligned} \tag{152}$$

where $(i\nabla + A_0)^2\varphi - \lambda\varphi \in (H^{1,0}_{0,\mathbb{R}}(\Omega, \mathbb{C}))^\star$ acts as

$${}_{(H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star}\Big\langle (i\nabla + A_0)^2\varphi - \lambda\varphi, u \Big\rangle_{H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C})} = \mathfrak{Re}\left(\textstyle\int_\Omega (i\nabla + A_0)\varphi \cdot \overline{(i\nabla + A_0)u}\, dx - \lambda \int_\Omega \varphi \overline{u}\, dx \right)$$

for all $\varphi \in H^{1,0}_{0,\mathbb{R}}(\Omega, \mathbb{C})$. In (152) $\mathbb{C}$ is also meant as a vector space over $\mathbb{R}$. From (4) and (5) it follows that $F(\lambda_0, \varphi_0) = (0, 0, 0)$.

Lemma 7.1. *Under assumptions* (3)*,* (4) *and* (5)*, the function F defined in* (152) *is Fréchet-differentiable at (λ_0, φ_0) and its Fréchet-differential $dF(\lambda_0, \varphi_0) \in \mathcal{L}\big(\mathbb{C} \times H^{1,0}_{0,\mathbb{R}}(\Omega, \mathbb{C}), \mathbb{R} \times \mathbb{R} \times (H^{1,0}_{0,\mathbb{R}}(\Omega, \mathbb{C}))^\star\big)$ is invertible.*

Proof. By direct calculations it is easy to verify that F is Fréchet-differentiable at (λ_0, φ_0) and

$$dF(\lambda_0, \varphi_0)(\lambda, \varphi) = \left(2\,\mathfrak{Re}\left(\textstyle\int_\Omega (i\nabla + A_0)\varphi_0 \cdot \overline{(i\nabla + A_0)\varphi}\, dx \right), \mathfrak{Im}\left(\int_\Omega \varphi\overline{\varphi_0}\, dx \right), (i\nabla + A_0)^2\varphi - \lambda_0\varphi - \lambda\varphi_0 \right)$$

for every $(\lambda, \varphi) \in \mathbb{C} \times H^{1,0}_{0,\mathbb{R}}(\Omega, \mathbb{C})$.

It remains to prove that $dF(\lambda_0, \varphi_0) : \mathbb{C} \times H^{1,0}_{0,\mathbb{R}}(\Omega, \mathbb{C}) \to \mathbb{R} \times \mathbb{R} \times (H^{1,0}_{0,\mathbb{R}}(\Omega), \mathbb{C})^\star$ is invertible. To this aim, by exploiting the compactness of the map $T : H^{1,0}_{0,\mathbb{R}}(\Omega, \mathbb{C}) \to (H^{1,0}_{0,\mathbb{R}}(\Omega, \mathbb{C}))^\star$, $u \mapsto \lambda_0 u$, it is easy to prove that, if $\mathcal{R} : (H^{1,0}_{0,\mathbb{R}}(\Omega, \mathbb{C}))^\star \to H^{1,0}_{0,\mathbb{R}}(\Omega, \mathbb{C})$ is the Riesz isomorphism and $\mathcal{I}$ denotes the standard identification of $\mathbb{R} \times \mathbb{R}$ onto $\mathbb{C}$, then the operator $(\mathcal{I} \times \mathcal{R}) \circ dF(\lambda_0, \varphi_0) \in \mathcal{L}(\mathbb{C} \times H^{1,0}_{0,\mathbb{R}}(\Omega))$ is a compact perturbation of the identity. Indeed, since by definition

$${}_{(H^{1,0}_{0,\mathbb{R}}(\Omega))^\star}\big\langle (i\nabla + A_0)^2\varphi, u \big\rangle_{H^{1,0}_{0,\mathbb{R}}(\Omega)} = \mathfrak{Re}\left(\int_\Omega (i\nabla + A_0)\varphi \cdot \overline{(i\nabla + A_0)u}\, dx \right) = (\varphi, u)_{H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C})},$$

we have that $\mathcal{R}\big((i\nabla + A_0)^2\varphi - \lambda_0\varphi - \lambda\varphi_0\big) = \varphi - \mathcal{R}(\lambda_0\varphi) - \mathcal{R}(\lambda\varphi_0)$, being $\mathcal{R}(\lambda_0\varphi)$ the image of φ by a compact operator (composition of the Riesz isomorphism and the compact operator T), as well as $\mathcal{R}(\lambda\varphi_0)$. Therefore, from the Fredholm alternative, $dF(\lambda_0, \varphi_0)$ is invertible if and only if it is injective. So, to conclude the proof, it is enough to prove that $\ker(dF(\lambda_0, \varphi_0)) = \{0, 0\}$. Let $(\lambda, \varphi) \in \mathbb{C} \times H^{1,0}_{0,\mathbb{R}}(\Omega, \mathbb{C})$ be such that

$$\mathfrak{Re}\left(\textstyle\int_\Omega (i\nabla + A_0)\varphi_0 \cdot \overline{(i\nabla + A_0)\varphi}\, dx \right) = 0, \quad \mathfrak{Im}\left(\int_\Omega \varphi\overline{\varphi_0}\, dx \right) = 0, \quad (i\nabla + A_0)^2\varphi - \lambda_0\varphi - \lambda\varphi_0 = 0. \tag{153}$$

The last equation in (153) means that $\mathfrak{Re}\left(\int_\Omega \left((i\nabla + A_0)\varphi \cdot \overline{(i\nabla + A_0)u} - \lambda_0\varphi\overline{u} - \lambda\varphi_0\overline{u}\right) dx\right) = 0$, for all $u \in H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C})$. Plugging $u = \varphi_0$ and $u = i\varphi_0$ into the previous identity and recalling (4) and (5), we obtain $\mathfrak{Re}\lambda = 0$ and $\mathfrak{Im}\lambda = 0$, respectively. Then the last equation in (153) becomes $(i\nabla+A_0)^2\varphi-\lambda_0\varphi = 0$ in $(H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star$, which, by assumption (3), implies that $\varphi = (\alpha + i\beta)\varphi_0$ for some $\alpha, \beta \in \mathbb{R}$. The first and the second equation in (153) imply $\alpha = 0$ and $\beta = 0$, respectively, so that $\varphi = 0$. Then we conclude that the only element in the kernel of $dF(\lambda_0, \varphi_0)$ is $(0, 0) \in \mathbb{C} \times H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C})$. □

Theorem 7.2. *For every $R > 2$ and $a = (|a|, 0) \in \Omega$ with $|a| < R_0/R$ and R_0 being as in Lemma 5.2, let $\varphi_a \in H^{1,a}_0(\Omega,\mathbb{C})$ solve (19-20), $\varphi_0 \in H^{1,0}_0(\Omega,\mathbb{C})$ be a solution to (4–5) satisfying* (3), (6), *and* (33), *and $v_{n_0,R,a}$ be as in §6.1 (see also* (107) *and* (108)*). Then $\|v_{n_0,R,a} - \varphi_0\|_{H^{1,0}_0(\Omega,\mathbb{C})} = O\Big(\sqrt{H(\varphi_a, K_\delta|a|)}\Big)$ as $|a| \to 0^+$ for every $R > 2$, with K_δ as in Lemma 5.7 for some fixed $\delta \in (0, 1/4)$.*

Proof. Let $R > 2$. We first notice that $v_{n_0,R,a} \to \varphi_0$ in $H^{1,0}_0(\Omega,\mathbb{C})$ as $|a| \to 0^+$. Indeed, recalling definitions (76), (115) and (133), we have that

$$\begin{aligned}\int_\Omega \left|(i\nabla + A_0)(v_{n_0,R,a} - \varphi_0)\right|^2 dx &= \int_\Omega |e^{\frac{i}{2}(\theta_0-\theta_a)}(i\nabla + A_a)\varphi_a - (i\nabla + A_0)\varphi_0|^2\, dx\\ &+ H(\varphi_a, K_\delta|a|)\int_{D_R} \Big|(i\nabla + A_0)\Big(Z^R_a - \tfrac{|a|^{k/2}}{\sqrt{H(\varphi_a,K_\delta|a|)}}W_a\Big)\Big|^2 dx\\ &- H(\varphi_a, K_\delta|a|)\int_{D_R} \Big|e^{\frac{i}{2}(\theta_0-\theta_{\mathbf{e}})}(i\nabla + A_{\mathbf{e}})\tilde\varphi_a - \tfrac{|a|^{k/2}}{\sqrt{H(\varphi_a,K_\delta|a|)}}(i\nabla + A_0)W_a\Big)\Big|^2 dx.\end{aligned}$$

Estimate (127) implies that $H(\varphi_a, K_\delta|a|) = o(1)$ whereas Proposition 6.11 yields $\frac{|a|^{k/2}}{\sqrt{H(\varphi_a,K_\delta|a|)}} = O(1)$ as $|a| \to 0^+$. Then Theorem 5.9, Lemma 6.3, (134), and (23) imply that $v_{n_0,R,a} \to \varphi_0$ in $H^{1,0}_0(\Omega,\mathbb{C})$ as $|a| \to 0^+$. Therefore, from Lemma 7.1, we have that

$$F(\lambda_a, v_{n_0,R,a}) = dF(\lambda_0, \varphi_0)(\lambda_a - \lambda_0, v_{n_0,R,a} - \varphi_0) + o\big(|\lambda_a - \lambda_0| + \|v_{n_0,R,a} - \varphi_0\|_{H^{1,0}_0(\Omega,\mathbb{C})}\big) \tag{154}$$

as $|a| \to 0^+$. In view of Lemma 7.1, the operator $dF(\lambda_0, \varphi_0)$ is invertible (and its inverse is continuous by the Open Mapping Theorem), then (154) implies that

$$\begin{aligned}&|\lambda_a - \lambda_0| + \|v_{n_0,R,a} - \varphi_0\|_{H^{1,0}_0(\Omega,\mathbb{C})}\\ &\quad\le \|(dF(\lambda_0, \varphi_0))^{-1}\|_{\mathcal{L}(\mathbb{R}\times\mathbb{R}\times(H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star,\mathbb{C}\times H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))}\|F(\lambda_a, v_{n_0,R,a})\|_{\mathbb{R}\times\mathbb{R}\times(H^{1,0}_{0,\mathbb{R}}(\Omega))^\star}(1 + o(1))\end{aligned}$$

as $|a| \to 0^+$. In order to prove the theorem, it remains to estimate the norm of

$$\begin{aligned}F(\lambda_a, v_{n_0,R,a}) &= (\alpha_a, \beta_a, w_a)\\ &= \Big(\|v_{n_0,R,a}\|^2_{H^{1,0}_0(\Omega,\mathbb{C})} - \lambda_0, \mathfrak{Im}\left(\textstyle\int_\Omega v_{n_0,R,a}\overline{\varphi_0}\, dx\right), (i\nabla + A_0)^2 v_{n_0,R,a} - \lambda_a v_{n_0,R,a}\Big)\end{aligned} \tag{155}$$

in $\mathbb{R} \times \mathbb{R} \times (H^{1,0}_{0,\mathbb{R}}(\Omega))^\star$. As far as α_a is concerned, arguing as in (128), we have that, in view of (76), (115), Theorem 5.9, Lemma 6.3, and Proposition 6.11,

$$\begin{aligned}\alpha_a &= \left(\int_{D_{R|a|}} |(i\nabla + A_0)v^{int}_{n_0,R,a}|^2\, dx - \int_{D_{R|a|}} |(i\nabla + A_a)\varphi_a|^2\, dx\right) + (\lambda_a - \lambda_0)\\ &= H(\varphi_a, K_\delta|a|)\left(\int_{D_R} |(i\nabla + A_0)Z^R_a|^2\, dx - \int_{D_R} |(i\nabla + A_{\mathbf{e}})\tilde\varphi_a|^2\, dx\right) + (\lambda_a - \lambda_0) = O(H(\varphi_a, K_\delta|a|))\end{aligned}$$

as $|a| \to 0^+$. As far as β_a is concerned, by Theorem 5.9, Lemma 6.3, (134), and the normalization condition (20) required on φ_a, we have that

$$\begin{aligned}\beta_a &= \mathfrak{Im}\left(\int_{D_{R|a|}} v^{int}_{n_0,R,a}\overline{\varphi_0}\, dx - \int_{D_{R|a|}} e^{\frac{i}{2}(\theta_0-\theta_a)}\varphi_a\overline{\varphi_0}\, dx + \int_\Omega e^{\frac{i}{2}(\theta_0-\theta_a)}\varphi_a\overline{\varphi_0}\, dx\right)\\ &= \sqrt{H(\varphi_a, K_\delta|a|)}|a|^{\frac{k}{2}+2}\,\mathfrak{Im}\left(\int_{D_R} Z^R_a\overline{W_a}\, dx - \int_{D_R} e^{\frac{i}{2}(\theta_0-\theta_{\mathbf{e}})}\tilde\varphi_a\overline{W_a}\, dx\right) = o\Big(\sqrt{H(\varphi_a, K_\delta|a|)}\Big)\end{aligned}$$

as $|a| \to 0^+$. Let $\eta_{|a|R}$ be a cut-off function as in (14). Then, for every $\varphi \in H_0^{1,0}(\Omega,\mathbb{C})$ we have that $\eta_{|a|R} e^{\frac{i}{2}(\theta_a-\theta_0)}\varphi \in H_0^{1,a}(\Omega,\mathbb{C})$. Hence testing (19) with $\eta_{|a|R} e^{\frac{i}{2}(\theta_a-\theta_0)}\varphi$ we obtain that

$$\int_{\Omega\setminus D_{R|a|}} e^{\frac{i}{2}(\theta_0-\theta_a)}(i\nabla + A_a)\varphi_a \cdot \overline{(i\nabla + A_0)\varphi}\, dx - \lambda_a \int_{\Omega\setminus D_{R|a|}} e^{\frac{i}{2}(\theta_0-\theta_a)}\varphi_a\overline{\varphi}\, dx$$
$$= -\int_{D_{R|a|}} (i\nabla + A_a)\varphi_a \cdot \overline{(i\nabla + A_0)\varphi}\,\eta_{|a|R} e^{\frac{i}{2}(\theta_0-\theta_a)}\, dx - i\int_{D_{R|a|}} (i\nabla + A_a)\varphi_a \cdot \nabla\eta_{|a|R}\overline{\varphi} e^{\frac{i}{2}(\theta_0-\theta_a)}\, dx$$
$$+ \lambda_a \int_{D_{R|a|}} \varphi_a \eta_{|a|R} e^{\frac{i}{2}(\theta_0-\theta_a)}\overline{\varphi}\, dx$$

and hence, by Hölder inequality and (17),

$$\left| \int_{\Omega\setminus D_{R|a|}} e^{\frac{i}{2}(\theta_0-\theta_a)}(i\nabla + A_a)\varphi_a \cdot \overline{(i\nabla + A_0)\varphi}\, dx - \lambda_a \int_{\Omega\setminus D_{R|a|}} e^{\frac{i}{2}(\theta_0-\theta_a)}\varphi_a\overline{\varphi}\, dx \right|$$
$$\leq \left(9\left(\int_{D_{R|a|}} |(i\nabla + A_a)\varphi_a|^2\, dx \right)^{1/2} + 2\lambda_a|a|R\left(\int_{D_{R|a|}} |\varphi_a|^2\, dx\right)^{1/2}\right) \|\varphi\|_{H_0^{1,0}(\Omega,\mathbb{C})}. \tag{156}$$

By Hölder inequality and (17), we also have that

$$\left| \int_{D_{R|a|}} (i\nabla + A_0)v^{int}_{n_0,R,a} \cdot \overline{(i\nabla + A_0)\varphi}\, dx - \lambda_a \int_{D_{R|a|}} v^{int}_{n_0,R,a}\overline{\varphi}\, dx\right|$$
$$\leq \left(\int_{D_{R|a|}} |(i\nabla + A_0)v^{int}_{n_0,R,a}|^2\, dx\right)^{1/2} (1 + 4\lambda_a|a|^2R^2)\|\varphi\|_{H_0^{1,0}(\Omega,\mathbb{C})}. \tag{157}$$

From (156), (157), (80), (82), and (117) it follows that

$$\|w_a\|_{(H^{1,0}_{0,\mathbb{R}}(\Omega,\mathbb{C}))^\star} = \sup_{\substack{\varphi\in H_0^{1,0}(\Omega,\mathbb{C})\\ \|\varphi\|_{H_0^{1,0}(\Omega,\mathbb{C})}=1}} \left| \mathfrak{Re}\left(\int_\Omega (i\nabla + A_0)v_{n_0,R,a} \cdot \overline{(i\nabla + A_0)\varphi}\, dx - \lambda_a\int_\Omega v_{n_0,R,a}\overline{\varphi}\, dx\right)\right|$$
$$= O\big(\sqrt{H(\varphi_a, K_\delta|a|)}\big), \quad \text{as } |a| \to 0^+.$$

The proof is thereby complete. □

As a consequence of Theorem 7.2, we obtain the following uniform energy estimate.

Theorem 7.3. *For $a = (|a|, 0) \in \Omega$, let $\varphi_a \in H_0^{1,a}(\Omega,\mathbb{C})$ solve (19-20), $\varphi_0 \in H_0^{1,0}(\Omega,\mathbb{C})$ be a solution to (4–5) satisfying* (3), (6), *and* (33), *$\tilde\varphi_a$ be as in* (76) *and W_a as in* (133). *Then, for every $R > 2$,*

$$\int_{\left(\frac{1}{|a|}\Omega\right)\setminus D_R} \left| (i\nabla + A_{\mathbf{e}})\left(\tilde\varphi_a(x) - e^{\frac{i}{2}(\theta_{\mathbf{e}}-\theta_0)} \frac{|a|^{k/2}}{\sqrt{H(\varphi_a,K_\delta|a|)}} W_a\right)\right|^2 dx = O(1), \quad \text{as } |a| \to 0^+. \tag{158}$$

Proof. The proof follows directly from scaling and Theorem 7.2. □

8. Blow-up analysis

In this section we study the limit of the blow-up sequence introduced in (76).

Theorem 8.1. *For $a = (|a|, 0) \in \Omega$, let $\varphi_a \in H_0^{1,a}(\Omega,\mathbb{C})$ solve (19-20) and $\varphi_0 \in H_0^{1,0}(\Omega,\mathbb{C})$ be a solution to (4–5) satisfying* (3), (6), *and* (33). *Let $\tilde\varphi_a$ and K_δ be as in* (76), *$\beta_{k,2}(0,\varphi_0,\lambda_0)$ as in* (34), *and Ψ_k be the function defined in* (43). *Then*

$$\lim_{|a|\to 0^+} \frac{|a|^{k/2}}{\sqrt{H(\varphi_a,K_\delta|a|)}} = \frac{1}{|\beta_{k,2}(0,\varphi_0,\lambda_0)|}\sqrt{\frac{K_\delta}{\int_{\partial D_{K_\delta}} |\Psi_k|^2 ds}} \tag{159}$$

and

$$\tilde\varphi_a \to \frac{\beta_{k,2}(0,\varphi_0,\lambda_0)}{|\beta_{k,2}(0,\varphi_0,\lambda_0)|}\sqrt{\frac{K_\delta}{\int_{\partial D_{K_\delta}} |\Psi_k|^2 ds}}\ \Psi_k \quad \text{as } |a| \to 0^+, \tag{160}$$

in $H^{1,\mathbf{e}}(D_R,\mathbb{C})$ for every $R > 1$, almost everywhere and in $C^2_{\rm loc}(\mathbb{R}^2\setminus\{\mathbf{e}\},\mathbb{C})$.

Proof. From Theorem 5.9 we know that the family of functions $\left\{\tilde\varphi_a : a = |a|\mathbf{e}, |a| < \frac{r_\delta}{R}\right\}$ is bounded in $H^{1,\mathbf{e}}(D_R,\mathbb{C})$ for all $R \geq K_\delta$. Furthermore, from Proposition 6.11, $\frac{|a|^{k/2}}{\sqrt{H(\varphi_a,K_\delta|a|)}} = O(1)$ as $|a| \to 0^+$. It follows that, for every sequence $a_n = (|a_n|, 0) = |a_n|\mathbf{e}$ with $|a_n| \to 0$, by a diagonal process there exist $c \in [0, +\infty)$, $\tilde{\Phi} \in H^{1,\mathbf{e}}_{\rm loc}(\mathbb{R}^2, \mathbb{C})$, and a subsequence a_{n_ℓ} such that

$$\lim_{\ell\to+\infty} \frac{|a_{n_\ell}|^{k/2}}{\sqrt{H(\varphi_{a_{n_\ell}}, K_\delta|a_{n_\ell}|)}} = c \quad \text{and} \quad \tilde{\varphi}_{a_{n_\ell}} \rightharpoonup \tilde{\Phi} \quad \text{weakly in } H^{1,\mathbf{e}}(D_R, \mathbb{C})$$

for every $R > 1$ and almost everywhere. We notice that $\tilde{\Phi} \not\equiv 0$ since

$$\frac{1}{K_\delta} \int_{\partial D_{K_\delta}} |\tilde{\Phi}|^2\, ds = 1 \tag{161}$$

thanks to (78) and the compactness of the trace embedding $H^{1,\mathbf{e}}(D_{K_\delta}, \mathbb{C}) \hookrightarrow L^2(\partial D_{K_\delta}, \mathbb{C})$.

Multiplying (77) by $\eta \in C^\infty_{\rm c}(\mathbb{R}^2 \setminus \{\mathbf{e}\}, \mathbb{C})$ and integrating by parts, we have that, if $|a|$ is sufficiently small so that $\operatorname{supp}\eta \subset \frac{1}{|a|}\Omega$,

$$\int_{\mathbb{R}^2} (i\nabla + A_{\mathbf{e}})\tilde{\varphi}_a \cdot \overline{(i\nabla + A_{\mathbf{e}})\eta}\, dx = \lambda_a |a|^2 \int_{\mathbb{R}^2} \tilde{\varphi}_a \overline{\eta}\, dx.$$

Along $a = a_{n_\ell}$ with $\ell \to \infty$, the left hand side converges to $\int_{\mathbb{R}^2} (i\nabla + A_{\mathbf{e}})\tilde{\Phi} \cdot \overline{(i\nabla + A_{\mathbf{e}})\eta}\, dx$ via the weak $H^{1,\mathbf{e}}(D_R, \mathbb{C})$-convergence, where $R > 1$ is such that $\operatorname{supp}\eta \subset D_R$, whereas, in view of (79), the right hand side can be estimated as

$$\left| \lambda_{a_{n_\ell}} |a_{n_\ell}|^2 \int_{\mathbb{R}^2} \tilde{\varphi}_{a_{n_\ell}} \overline{\eta}\, dx \right| \leq \lambda_{a_{n_\ell}} |a_{n_\ell}|^2 \|\tilde{\varphi}_{a_{n_\ell}}\|_{H^{1,\mathbf{e}}(D_R,\mathbb{C})} \|\eta\|_{L^2(\mathbb{R}^2,\mathbb{C})} = O(|a_{n_\ell}|^2) \quad \text{as } \ell \to \infty,$$

thus proving that $\tilde{\Phi}$ weakly solves

$$(i\nabla + A_{\mathbf{e}})^2 \tilde{\Phi} = 0, \quad \text{in } \mathbb{R}^2. \tag{162}$$

We now claim that the convergence of the subsequence $\tilde{\varphi}_{a_{n_\ell}}$ to $\tilde{\Phi}$ is actually strong in $H^{1,\mathbf{e}}(D_R, \mathbb{C})$ for every $R > 1$. By classical elliptic estimates, we can easily prove that $\tilde{\varphi}_{a_{n_l}} \to \tilde{\Phi}$ in $C^{2,\alpha}(D_{R_2} \setminus D_{R_1}, \mathbb{C})$ for every $1 < R_1 < R_2$. Therefore, multiplying by $\tilde{\Phi}$ equation (162) and integrating by parts in D_R for $R > 1$, we obtain

$$-i \int_{\partial D_R} \big((i\nabla + A_{\mathbf{e}})\tilde{\varphi}_{a_{n_\ell}} \cdot \nu\big) \overline{\tilde{\varphi}_{a_{n_\ell}}}\, ds \to -i \int_{\partial D_R} \big((i\nabla + A_{\mathbf{e}})\tilde{\Phi} \cdot \nu\big) \overline{\tilde{\Phi}}\, ds = \int_{D_R} |(i\nabla + A_{\mathbf{e}})\tilde{\Phi}|^2 dx \tag{163}$$

as $\ell \to \infty$. On the other hand, multiplying equation (77) by $\tilde{\varphi}_{a_{n_\ell}}$ with ℓ large and integrating by parts in D_R for $R > 1$, we obtain

$$\int_{D_R} |(i\nabla + A_{\mathbf{e}})\tilde{\varphi}_{a_{n_\ell}}|^2 dx = \lambda_{a_{n_\ell}} |a_{n_\ell}|^2 \int_{D_R} |\tilde{\varphi}_{a_{n_\ell}}|^2 dx - i \int_{\partial D_R} \big((i\nabla + A_{\mathbf{e}})\tilde{\varphi}_{a_{n_\ell}} \cdot \nu\big) \overline{\tilde{\varphi}_{a_{n_\ell}}}\, ds. \tag{164}$$

From (163) and (164), we obtain that $\int_{D_R} |(i\nabla + A_{\mathbf{e}})\tilde{\varphi}_{a_{n_\ell}}|^2 dx \to \int_{D_R} |(i\nabla + A_{\mathbf{e}})\tilde{\Phi}|^2 dx$ as $\ell \to \infty$, whereas the compactness of the trace embedding $H^{1,\mathbf{e}}(D_R, \mathbb{C}) \hookrightarrow L^2(\partial D_R, \mathbb{C})$ yields $\int_{\partial D_R} |\tilde{\varphi}_{a_{n_\ell}}|^2 ds \to \int_{\partial D_R} |\tilde{\Phi}|^2 ds$ as $\ell \to \infty$, so that, in view of Lemma 3.1, we can conclude that $\|\tilde{\varphi}_{a_{n_\ell}}\|_{H^{1,\mathbf{e}}(D_R,\mathbb{C})} \to \|\tilde{\Phi}\|_{H^{1,\mathbf{e}}(D_R,\mathbb{C})}$ as $\ell \to \infty$, and hence $\tilde{\varphi}_{a_{n_\ell}} \to \tilde{\Phi}$ strongly in $H^{1,\mathbf{e}}(D_R, \mathbb{C})$ for every $R > 1$ as desired.

Passing to the limit along a_{n_ℓ} in (158) and recalling (134), we obtain that

$$\int_{\mathbb{R}^2} \left| (i\nabla + A_{\mathbf{e}}) \Big(\tilde{\Phi}(x) - c\beta e^{\frac{i}{2}\theta_{\mathbf{e}}} \psi_k \Big) \right|^2 dx < +\infty. \tag{165}$$

Estimate (165) implies that $c > 0$. Indeed, $c = 0$ would imply that $\int_{\mathbb{R}^2} |(i\nabla + A_{\mathbf{e}})\tilde{\Phi}|^2 dx < +\infty$ and then, arguing as in the proof of Proposition 4.3, we could prove that $\tilde{\Phi} \equiv 0$, thus contradicting (161).

Then, from (162), (165), and Proposition 4.3 we deduce that necessarily $\tilde{\Phi} = c\beta\Psi_k$ with Ψ_k being the function defined in (43). From (161) and the fact that c is a positive real number, it follows that $c = \frac{1}{|\beta|}\big(\frac{K_\delta}{\int_{\partial D_{K_\delta}} |\Psi_k|^2 ds}\big)^{1/2}$. Hence we have that $\tilde{\varphi}_{a_{n_\ell}} \to \frac{\beta}{|\beta|}\big(\frac{K_\delta}{\int_{\partial D_{K_\delta}} |\Psi_k|^2 ds}\big)^{1/2} \Psi_k$ in $H^{1,\mathbf{e}}(D_R, \mathbb{C})$ for every $R > 1$ and a. e., and $\frac{|a_{n_\ell}|^{k/2}}{\sqrt{H(\varphi_{a_{n_\ell}}, K_\delta|a_{n_\ell}|)}} \to \frac{1}{|\beta|}\big(\frac{K_\delta}{\int_{\partial D_{K_\delta}} |\Psi_k|^2 ds}\big)^{1/2}$. Since the above limits depend neither on the sequence $\{a_n\}_n$ nor on the subsequence $\{a_{n_\ell}\}_\ell$, we conclude that the above convergences hold as $|a| \to 0^+$, thus concluding the proof of the theorem (the convergence in $C^2_{\rm loc}(\mathbb{R}^2 \setminus \{\mathbf{e}\}, \mathbb{C})$ follows easily from classical elliptic estimates). □

Theorem 8.2. *For $a = (|a|, 0) \in \Omega$, let $\varphi_a \in H^{1,a}_0(\Omega, \mathbb{C})$ solve (19-20) and $\varphi_0 \in H^{1,0}_0(\Omega, \mathbb{C})$ be a solution to (4–5) satisfying* (3), (6), *and* (33). *Then*

$$\frac{\varphi_a(|a|x)}{|a|^{k/2}} \to \beta_{k,2}(0, \varphi_0, \lambda_0)\Psi_k \quad \textit{as } |a| \to 0^+,$$

in $H^{1,\mathbf{e}}(D_R,\mathbb{C})$ *for every* $R>1$*, a.e. and in* $C^2_{\rm loc}(\mathbb{R}^2\setminus\{\mathbf{e}\},\mathbb{C})$*, with* $\beta_{k,2}(0,\varphi_0,\lambda_0)\neq 0$ *as in* (34) *and* Ψ_k *being the function defined in* (43).

Proof. It follows directly from convergences (159) and (160) established in Theorem 8.1. □

As a consequence of Theorem 8.1, we can prove convergence of the blow-up family of functions introduced in (115). Let z_R be the unique solution to the minimization problem

$$\int_{D_R}|(i\nabla+A_0)z_R(x)|^2\,dx=\min\left\{\int_{D_R}|(i\nabla+A_0)u(x)|^2\,dx:\ u\in H^{1,0}(D_R,\mathbb{C}),\ u=e^{\frac{i}{2}(\theta_0-\theta_{\mathbf{e}})}\Psi_k\text{ on }\partial D_R\right\},$$

which then solves

$$\begin{cases}(i\nabla+A_0)^2z_R=0, & \text{in } D_R,\\ z_R=e^{\frac{i}{2}(\theta_0-\theta_{\mathbf{e}})}\Psi_k, & \text{on }\partial D_R.\end{cases}\tag{166}$$

Lemma 8.3. *Under the same assumptions as in Theorem 8.1, let* Z_a^R *be as in* (115)*. Then, for all* $R>2$*,* $Z_a^R\to\gamma_\delta z_R$ *in* $H^{1,0}(D_R,\mathbb{C})$ *as* $|a|\to 0^+$*, where*

$$\gamma_\delta=\frac{\beta_{k,2}(0,\varphi_0,\lambda_0)}{|\beta_{k,2}(0,\varphi_0,\lambda_0)|}\sqrt{\frac{K_\delta}{\int_{\partial D_{K_\delta}}|\Psi_k|^2ds}}.$$

Proof. We notice that $Z_a^R-\gamma_\delta z_R$ solves $(i\nabla+A_0)^2\big(Z_a^R-\gamma_\delta z_R\big)=0$ in D_R with boundary condition $Z_a^R-\gamma_\delta z_R=e^{\frac{i}{2}(\theta_0-\theta_{\mathbf{e}})}\,(\tilde\varphi_a-\gamma_\delta\Psi_k)$ on ∂D_R. Then, by the Dirichlet principle and Theorem 8.1,

$$\begin{aligned}\int_{D_R}\big|(i\nabla+A_0)(Z_a^R-\gamma_\delta z_R)\big|^2\,dx&\le\int_{D_R}\Big|(i\nabla+A_0)\big(\eta_R\,e^{\frac{i}{2}(\theta_0-\theta_{\mathbf{e}})}(\tilde\varphi_a-\gamma_\delta\Psi_k)\big)\Big|^2\,dx\\&\le 2\int_{D_R}|\nabla\eta_R|^2\big|\tilde\varphi_a-\gamma_\delta\Psi_k\big|^2dx+2\int_{D_R\setminus D_{R/2}}\eta_R^2\big|(i\nabla+A_{\mathbf{e}})(\tilde\varphi_a-\gamma_\delta\Psi_k)\big|^2dx=o(1)\end{aligned}$$

as $|a|\to0^+$, where $\eta_R:\mathbb{R}^2\to\mathbb{R}$ is a smooth cut-off function as in (14). Then, taking into account (17), we conclude. □

9. Sharp asymptotics for convergence of eigenvalues

In view of the exact asymptotics of the term $H(\varphi_a,K_\delta|a|)$ established in (159), Proposition 6.11 yields a control of $\lambda_0-\lambda_a$ with $|a|^k$ both from above and below. To compute explicitly the limit of $\frac{\lambda_0-\lambda_a}{|a|^k}$ it remains to determine the limit of the function $f_R(a)$ in Lemma 6.4 as $|a|\to0$ and $R\to+\infty$.

Lemma 9.1. *For all* $R>\tilde R$ *and* $a=(|a|,0)\in\Omega$ *with* $|a|<\frac{R_0}{R}$*, let* $f_R(a)$ *be as in Lemma 6.4. Then*

$$\lim_{|a|\to0^+}f_R(a)=-i\frac{K_\delta}{\int_{\partial D_{K_\delta}}|\Psi_k|^2ds}\kappa_R\tag{167}$$

where

$$\kappa_R=\int_{\partial D_R}\Big(e^{-\frac{i}{2}\theta_0}(i\nabla+A_0)z_R\cdot\nu-e^{-\frac{i}{2}\theta_{\mathbf{e}}}(i\nabla+A_{\mathbf{e}})\Psi_k\cdot\nu\Big)e^{\frac{i}{2}\theta_{\mathbf{e}}}\overline{\Psi_k}\,ds.\tag{168}$$

Furthermore $\lim_{R\to+\infty}\kappa_R=-4i\mathfrak{m}_k$*, where* $\mathfrak{m}_k$ *is defined in* (10).

Proof. We first observe that, by Theorem 8.1, Lemma 8.3, (45), and (166),

$$\begin{aligned}&\lim_{|a|\to0^+}\int_{D_R}|(i\nabla+A_0)Z_a^R|^2\,dx-\int_{D_R}|(i\nabla+A_{\mathbf{e}})\tilde\varphi_a|^2\,dx\\&=\frac{K_\delta}{\int_{\partial D_{K_\delta}}|\Psi_k|^2ds}\left(\int_{D_R}|(i\nabla+A_0)z_R|^2\,dx-\int_{D_R}|(i\nabla+A_{\mathbf{e}})\tilde\Psi_k|^2\,dx\right)=\frac{-iK_\delta}{\int_{\partial D_{K_\delta}}|\Psi_k|^2ds}\kappa_R\end{aligned}$$

with κ_R as in (168). Hence (167) follows from (121). The computation of $\lim_{R\to+\infty}\kappa_R$ is divided into two steps.

Step 1. We claim that

$$\kappa_R=\int_{\partial D_R}\Big(e^{-\frac{i}{2}\theta_0}(i\nabla+A_0)z_R\cdot\nu-e^{-\frac{i}{2}\theta_{\mathbf{e}}}(i\nabla+A_{\mathbf{e}})\Psi_k\cdot\nu\Big)\psi_k\,ds+o(1)\tag{169}$$

as $R\to\infty$. To prove the claim, we observe that

$$\kappa_R=\int_{\partial D_R}\Big(e^{-\frac{i}{2}\theta_0}(i\nabla+A_0)z_R\cdot\nu-e^{-\frac{i}{2}\theta_{\mathbf{e}}}(i\nabla+A_{\mathbf{e}})\Psi_k\cdot\nu\Big)\psi_k\,ds+I_1(R)+I_2(R)\tag{170}$$

where

$$I_1(R) = \int_{\partial D_R} \Big(\overline{\Psi_k} - e^{-\frac{i}{2}\theta_{\mathbf{e}}}\psi_k\Big)(i\nabla + A_{\mathbf{e}})\Big(e^{\frac{i}{2}\theta_{\mathbf{e}}}\psi_k - \Psi_k\Big) \cdot \nu \, ds$$

$$I_2(R) = \int_{\partial D_R} \Big(e^{-\frac{i}{2}(\theta_0-\theta_{\mathbf{e}})}\overline{\Psi_k} - e^{-\frac{i}{2}\theta_0}\psi_k\Big)(i\nabla + A_0)\Big(z_R - e^{\frac{i}{2}\theta_0}\psi_k\Big) \cdot \nu \, ds.$$

Testing the equation $(i\nabla + A_{\mathbf{e}})^2\big(e^{\frac{i}{2}\theta_{\mathbf{e}}}\psi_k - \Psi_k\big) = 0$, which is satisfied in $\mathbb{R}^2 \setminus D_R$, with the function $\big(e^{\frac{i}{2}\theta_{\mathbf{e}}}\psi_k - \Psi_k\big)(1-\eta_{2R})^2$ (being η_{2R} as in (14)), we obtain that

$$\begin{aligned} I_1(R) &= i \int_{\mathbb{R}^2\setminus D_R} (i\nabla + A_{\mathbf{e}})\big(e^{\frac{i}{2}\theta_{\mathbf{e}}}\psi_k - \Psi_k\big) \cdot \overline{(i\nabla + A_{\mathbf{e}})\big((1-\eta_{2R})^2(e^{\frac{i}{2}\theta_{\mathbf{e}}}\psi_k - \Psi_k)\big)} \, dx \\ &= i \int_{\mathbb{R}^2\setminus D_R} |(i\nabla + A_{\mathbf{e}})\big(e^{\frac{i}{2}\theta_{\mathbf{e}}}\psi_k - \Psi_k\big)|^2 (1-\eta_{2R})^2 \, dx \\ &\quad + 2\int_{\mathbb{R}^2\setminus D_R} (1-\eta_{2R})(e^{-\frac{i}{2}\theta_{\mathbf{e}}}\psi_k - \overline{\Psi_k})(i\nabla + A_{\mathbf{e}})\big(e^{\frac{i}{2}\theta_{\mathbf{e}}}\psi_k - \Psi_k\big) \cdot \nabla\eta_{2R} \, dx, \end{aligned}$$

and hence, thanks to (46) and (47),

$$|I_1(R)| \leq 2\int_{\mathbb{R}^2\setminus D_R} |(i\nabla + A_{\mathbf{e}})\big(e^{\frac{i}{2}\theta_{\mathbf{e}}}\psi_k - \Psi_k\big)|^2 \, dx + \frac{4}{R^2}\int_{D_{2R}\setminus D_R} |e^{\frac{i}{2}\theta_{\mathbf{e}}}\psi_k - \Psi_k|^2 \, dx \to 0 \tag{171}$$

as $R \to +\infty$.

On the other hand, testing the equation $(i\nabla + A_0)^2(e^{\frac{i}{2}\theta_0}\psi_k - z_R) = 0$ in D_R with the function $\eta_R\big(e^{\frac{i}{2}(\theta_0-\theta_{\mathbf{e}})}\Psi_k - e^{\frac{i}{2}\theta_0}\psi_k\big)$ (with η_R as in (14)) and using the Dirichlet Principle, we have that

$$\begin{aligned} |I_2(R)| &= \left| -i\int_{D_R} (i\nabla + A_0)(e^{\frac{i}{2}\theta_0}\psi_k - z_R) \cdot \overline{(i\nabla + A_0)\Big(\eta_R\big(e^{\frac{i}{2}(\theta_0-\theta_{\mathbf{e}})}\Psi_k - e^{\frac{i}{2}\theta_0}\psi_k\big)\Big)} \, dx \right| \\ &\leq \int_{D_R} \left|(i\nabla + A_0)\Big(\eta_R\big(e^{\frac{i}{2}(\theta_0-\theta_{\mathbf{e}})}\Psi_k - e^{\frac{i}{2}\theta_0}\psi_k\big)\Big)\right|^2 dx \\ &\leq 2\int_{D_R\setminus D_{\frac{R}{2}}} \left|(i\nabla + A_{\mathbf{e}})\Big(\Psi_k - e^{\frac{i}{2}\theta_{\mathbf{e}}}\psi_k\Big)\right|^2 dx + \frac{32}{R^2}\int_{D_R\setminus D_{\frac{R}{2}}} \left|\Psi_k - e^{\frac{i}{2}\theta_{\mathbf{e}}}\psi_k\right|^2 dx \end{aligned}$$

which, in view of (46) and estimate (47), yields that $I_2(R) \to 0$ as $R \to +\infty$. Claim (169) then follows recalling (170) and (171).

Step 2. We claim that

$$\int_{\partial D_R} \Big(e^{-\frac{i}{2}\theta_0}(i\nabla + A_0)z_R \cdot \nu - e^{-\frac{i}{2}\theta_{\mathbf{e}}}(i\nabla + A_{\mathbf{e}})\Psi_k \cdot \nu\Big)\psi_k \, ds = ik\sqrt{\pi}\big(\xi(1) - \sqrt{\pi}\big), \tag{172}$$

where the function ξ is defined in (144). From (36) and (37), the function ξ satisfies $\big(r^{-k/2}\xi(r)\big)' = \frac{C_\xi}{r^{1+k}}$ in $(1, +\infty)$, for some $C_\xi \in \mathbb{C}$. Integrating the previous equation over $(1, r)$ we obtain that

$$r^{-k/2}\xi(r) - \xi(1) = \frac{C_\xi}{k}\left(1 - \frac{1}{r^k}\right). \tag{173}$$

From (8) and estimate (47) it follows that

$$\begin{aligned} \xi(r) &= \frac{1}{\sqrt{\pi}}\int_0^{2\pi} \psi_k(r\cos t, r\sin t)\sin\big(\tfrac{k}{2}t\big) \, dt \\ &\quad + \int_0^{2\pi} e^{\frac{i}{2}(\theta_0-\theta_{\mathbf{e}})(r\cos t, r\sin t)}\Big(\Psi_k(r\cos t, r\sin t) - e^{\frac{i}{2}\theta_{\mathbf{e}}(r\cos t, r\sin t)}\psi_k(r\cos t, r\sin t)\Big)\overline{\psi_{k,2}(t)} \, dt \\ &= \sqrt{\pi}\, r^{k/2} + O(r^{-1/2}), \quad \text{as } r \to +\infty, \end{aligned}$$

and hence $r^{-k/2}\xi(r) \to \sqrt{\pi}$ as $r \to +\infty$. Letting $r \to +\infty$ in (173), this implies that $\frac{C_\xi}{k} = \sqrt{\pi} - \xi(1)$, so that

$$\xi(r) = \sqrt{\pi}\, r^{k/2} + r^{-k/2}\big(\xi(1) - \sqrt{\pi}\big), \quad \xi'(r) = \frac{k}{2}\sqrt{\pi}r^{k/2-1} + (\sqrt{\pi} - \xi(1))\frac{k}{2}r^{-k/2-1}, \quad r > 1. \tag{174}$$

In particular, from (174) we have that

$$\sqrt{\pi} - \xi(1) = \sqrt{\pi}r^k - r^{k/2}\xi(r), \quad \text{for all } r > 1, \tag{175}$$

whose substitution into (174) yields

$$\xi'(r) = k\sqrt{\pi}r^{k/2-1} - \frac{k}{2}\frac{\xi(r)}{r}, \quad \text{for all } r > 1. \tag{176}$$

On the other hand, writing ξ as $\xi(r) = \frac{1}{r}\int_{\partial D_r} e^{\frac{i}{2}(\theta_0-\theta_{\mathbf{e}})}\Psi_k(x)\,\overline{\psi_{k,2}(\theta_0(x))}\,ds(x)$, differentiating and taking into account (25), (8) and the fact that $A_0\cdot\nu = 0$ on ∂D_r, we obtain also that

$$\xi'(r) = \frac{1}{r}\int_{\partial D_r}\nabla\Big(e^{\frac{i}{2}(\theta_0-\theta_{\mathbf{e}})}\Psi_k\Big)\cdot\nu\,\overline{\psi_{k,2}(\theta_0(x))}\,ds = -\frac{i}{\sqrt{\pi}}r^{-\frac{k}{2}-1}\int_{\partial D_r} e^{-\frac{i}{2}\theta_{\mathbf{e}}}(i\nabla + A_{\mathbf{e}})\Psi_k\cdot\nu\,\psi_k\,ds. \tag{177}$$

Combination of (176) and (177) yields that

$$\int_{\partial D_r} e^{-\frac{i}{2}\theta_{\mathbf{e}}}(i\nabla + A_{\mathbf{e}})\Psi_k\cdot\nu\,\psi_k\,ds = i\sqrt{\pi}r^{k/2+1}\xi'(r) = i\sqrt{\pi}r^{k/2+1}\bigg(k\sqrt{\pi}r^{k/2-1} - \frac{k}{2}\frac{\xi(r)}{r}\bigg), \tag{178}$$

for all $r > 1$. According to (36) and (37), the function ζ_R defined as

$$\zeta_R(r) := \int_0^{2\pi} z_R(r\cos t, r\sin t)\overline{\psi_{k,2}(t)}\,dt$$

satisfies, for some $C_R \in \mathbb{C}$, $\big(r^{-k/2}\zeta_R(r)\big)' = \frac{C_R}{r^{1+k}}$ in $(0,R)$. Integrating the previous equation over (r,R) we obtain $R^{-k/2}\zeta_R(R) - r^{-k/2}\zeta_R(r) = \frac{C_R}{k}\big(\frac{1}{r^k} - \frac{1}{R^k}\big)$, for all $r \in (0,R]$. Since by Proposition 2.1 $\zeta_R(r) = O(r^{1/2})$ as $r \to 0^+$, we necessarily have $C_R = 0$. Hence

$$\zeta_R(r) = \frac{\zeta_R(R)}{R^{k/2}}r^{k/2}, \quad \text{for all } r \in (0,R], \quad \zeta_R'(r) = \frac{k}{2}\frac{\zeta_R(R)}{R^{k/2}}r^{k/2-1}, \quad \text{for all } r \in (0,R]. \tag{179}$$

On the other hand, writing ζ_R as $\zeta_R(r) = \frac{1}{r}\int_{\partial D_r} z_R(x)\,\overline{\psi_{k,2}(\theta_0(x))}\,ds(x)$, differentiating and using (25), (8) and $A_0\cdot\nu = 0$ on ∂D_r, we obtain that

$$\zeta_R'(r) = \frac{1}{r}\int_{\partial D_r}\nabla z_R\cdot\nu\,\overline{\psi_{k,2}(\theta_0(x))}\,ds = -\frac{i}{\sqrt{\pi}}r^{-\frac{k}{2}-1}\int_{\partial D_r} e^{-\frac{i}{2}\theta_0}(i\nabla + A_0)z_R\cdot\nu\,\psi_k\,ds. \tag{180}$$

Combination of (179) and (180) yields that

$$\int_{\partial D_r} e^{-\frac{i}{2}\theta_0}(i\nabla + A_0)z_R\cdot\nu\,\psi_k\,ds = \frac{ik}{2}\sqrt{\pi}\frac{\zeta_R(R)}{R^{k/2}}r^k \tag{181}$$

for all $r \in (0,R]$. From the boundary condition in (166) it follows that $\xi(R) = \zeta_R(R)$. Hence, collecting (178), (181), and (175) we obtain that

$$\int_{\partial D_R}\Big(e^{-\frac{i}{2}\theta_0}(i\nabla + A_0)z_R\cdot\nu - e^{-\frac{i}{2}\theta_{\mathbf{e}}}(i\nabla + A_{\mathbf{e}})\Psi_k\cdot\nu\Big)\psi_k\,ds = ik\sqrt{\pi}\big(\xi(R)R^{\frac{k}{2}} - \sqrt{\pi}R^k\big) = ik\sqrt{\pi}\big(\xi(1) - \sqrt{\pi}\big),$$

thus proving claim (172).

Combining (169) with (172) we obtain that $\kappa_R = ik\sqrt{\pi}(\xi(1) - \sqrt{\pi}) + o(1)$ as $R \to +\infty$. The conclusion then follows recalling Lemma 4.4 (see also (151)). □

We are now in position to prove our main result.

Proof of Theorem 1.2. From Proposition 6.11, Lemma 6.4, Lemma 9.1, and (159) it follows that, for every $R > \tilde{R}$,

$$-4|\beta_{k,2}(0,\varphi_0,\lambda_0)|^2\mathfrak{m}_k \le \liminf_{|a|\to 0^+}\tfrac{\lambda_0-\lambda_a}{|a|^k} \le \limsup_{|a|\to 0^+}\tfrac{\lambda_0-\lambda_a}{|a|^k} \le -i\kappa_R|\beta_{k,2}(0,\varphi_0,\lambda_0)|^2.$$

Letting $R \to +\infty$, Lemma 9.1 yields the conclusion (see Remark 2.2). □

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