THE ALGORITHM

The gradient of $F_{DV}^{(L)}$ with respect to $\vec{\psi}$ @ fixed q is

$$\nabla_{\vec{\psi}}\,\mathbf{p}^{(L)} = -\nabla_{\vec{\psi}}\,R\mathbf{L}^{(L)}.(-\mathbf{c}^{(L)}) - R\mathbf{L}^{(L)}.(-\nabla_{\vec{\psi}}\,\mathbf{c}^{(L)}) + \nabla_{\vec{\psi}}\,\vec{\beta}^{(L)}$$

where

$$\mathbf{P}^{(L)}.(-\nabla_{\vec{\psi}}\,\mathbf{c}^{(L)}) = \nabla_{\vec{\psi}}\,\mathbf{P}^{(L)}.(-\mathbf{c}^{(L)}) - \nabla_{\vec{\psi}}\,\mathbf{g}^{(L)}.$$

FEATURES

1) ◆ The scattering <u>data</u> are $\{A[\mathbf{k}_q\,;\phi_j\,;\gamma_q],\, j \in N_q\,,\, q \in Q\}$.

2) ◆ Superposition of limited aperture – multiple views allowed.

3) ◆ The <u>cost function</u> to be minimized is $F_{DV}^{(L)}[\vec{\psi}\,,\,\vec{\gamma}\,]$.

4) ◆ The <u>unknowns</u> are $\vec{\psi} \in \Psi_{\mathrm{adm}}$ and $\vec{\gamma} \in ([0,\, 2\pi\,])^{\mathrm{card}[Q]}$.

5) The <u>DIRICHLET BC</u> and the <u>boundary to far zone</u> relation are re–presented by approximate forward propagation.

6) $\mathbf{P}_q{}^{(L)}$, $R\mathbf{L}_q{}^{(L)}$ depend on $\vec{\psi}$ and on k_q (not on $\mathbf{k}_q$).

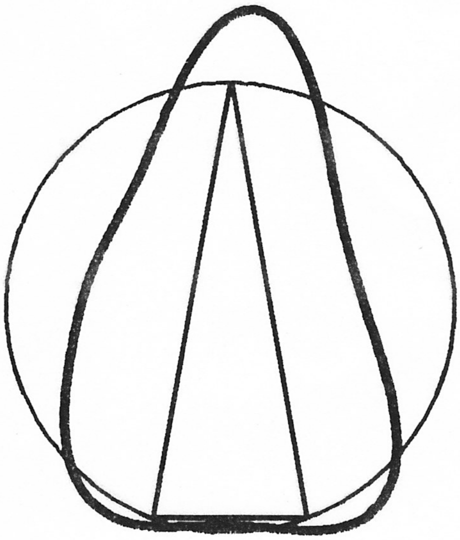
7) $\{\vec{\beta}_q^{(L)}\}$ and $\{\mathbf{g}_q^{(L)}\}$ depend on $\vec{\psi}$ and on $\mathbf{k}_q$.

8) ◆ Only $\mathbf{P}_q{}^{(L)}$ has to be inverted; $[\,\mathbf{P}_q^{(L)}\,]^{-1}$ $\exists$ unconditionally.

NUMERICAL RESULTS FROM $F_{DV}^{(L)}$ MINIMIZATION

IPSWICH files *ips009/ips009FV.**

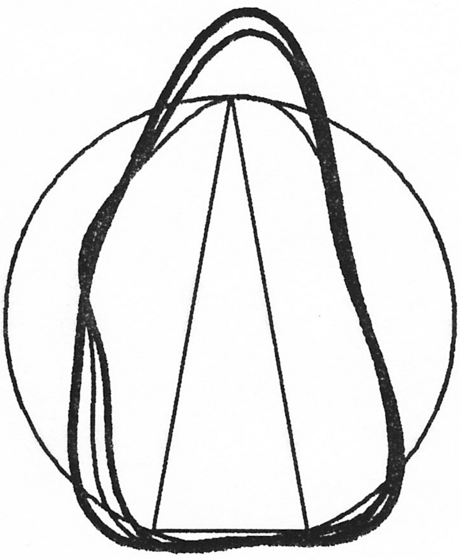
<u>Full aperture</u> data obtained via reciprocity of $A[\mathbf{k}_q\ ;\ \phi_j\ ;\ \gamma_q]$

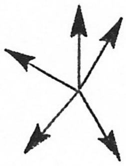


$\{\mathbf{k}_q\ ,\ 1 \leq q \leq 4\}$
$L = 12, I = 10$

$$\frac{\psi_4^{(0)}}{\psi_1^{(0)}} = -0.4$$

MF9x4CA r_0 = .495868D + 01 It. = 15
$B^{[12]}$ = .425680D + 03 ; $\|\nabla B\|$ = .258242D + 05

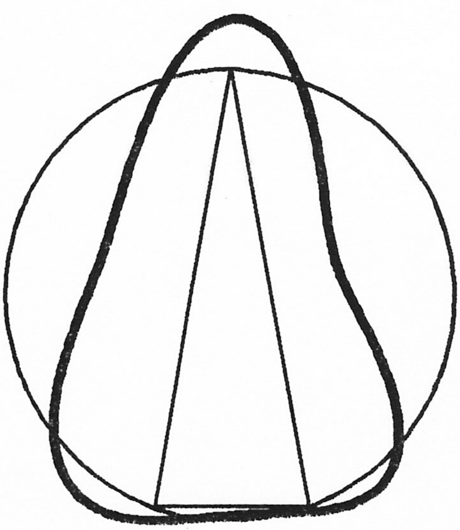


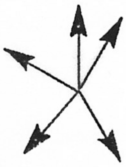


$\{\mathbf{k}_q\ ,\ 1 \leq q \leq 5\}$
$L = 12, I = 10$

$$\frac{\psi_4^{(0)}}{\psi_1^{(0)}} = -0.4$$

MF9x5CA r_0 = .495868D + 01 It. = 15
$B^{[12]}$ = .485822D + 03 ; $\|\nabla B\|$ = .195682D + 05
MF9x5CA r_0 = .409808D + 01 It. = 15
$B^{[12]}$ = .494881D + 03 ; $\|\nabla B\|$ = .177667D + 05
MF9x5CA r_0 = .450789D + 01 It. = 15
$B^{[12]}$ = .514020D + 03 ; $\|\nabla B\|$ = .195457D + 05





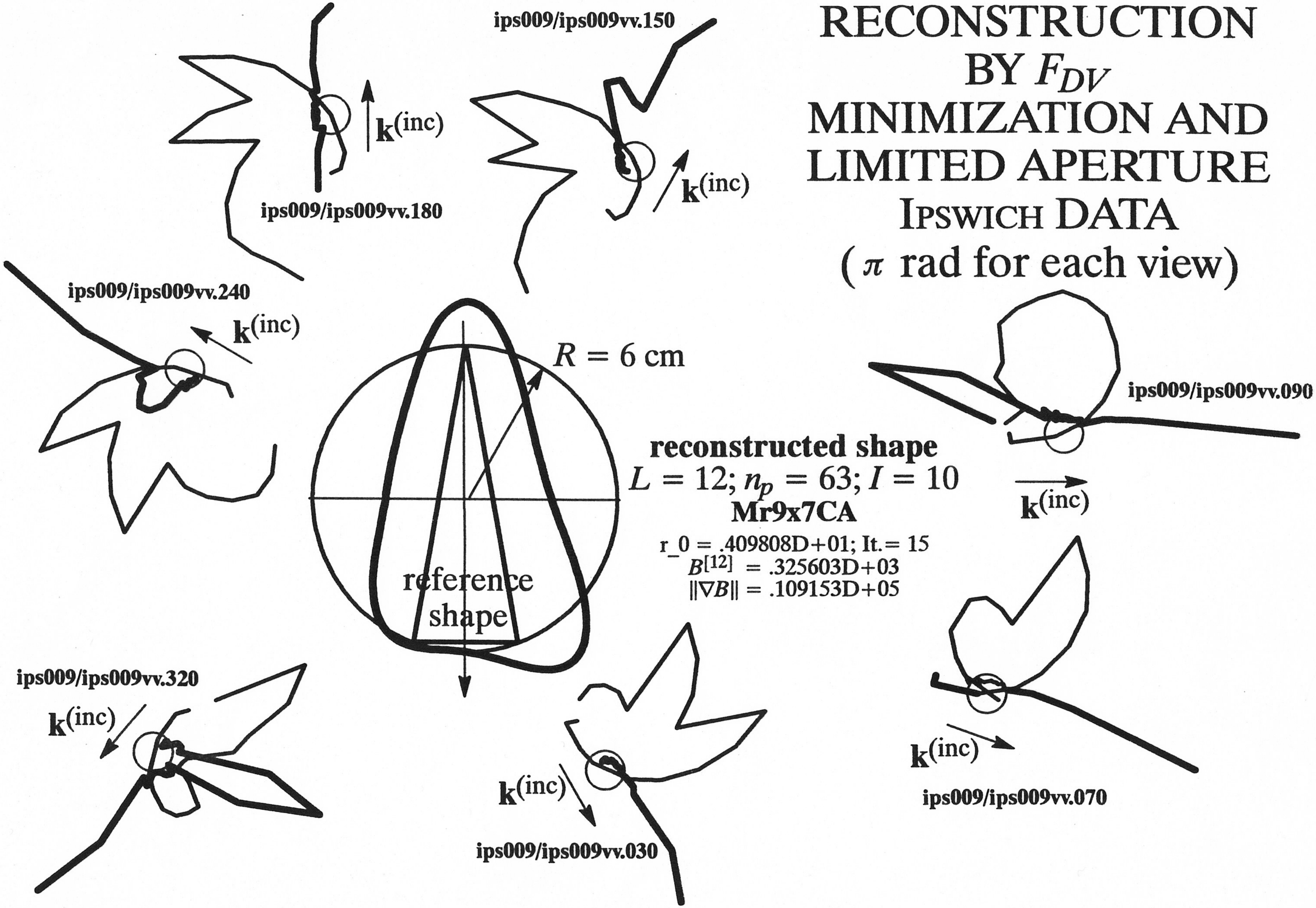
$\{\mathbf{k}_q\ ,\ 1 \leq q \leq 5\}$
$L = 14, I = 10$

$$\frac{\psi_4^{(0)}}{\psi_1^{(0)}} = -0.4$$

MF9x5EA r_0 = .495868D + 01 It. = 15
$B^{[14]}$ = .464231D + 03 ; $\|\nabla B\|$ = .537194D + 05

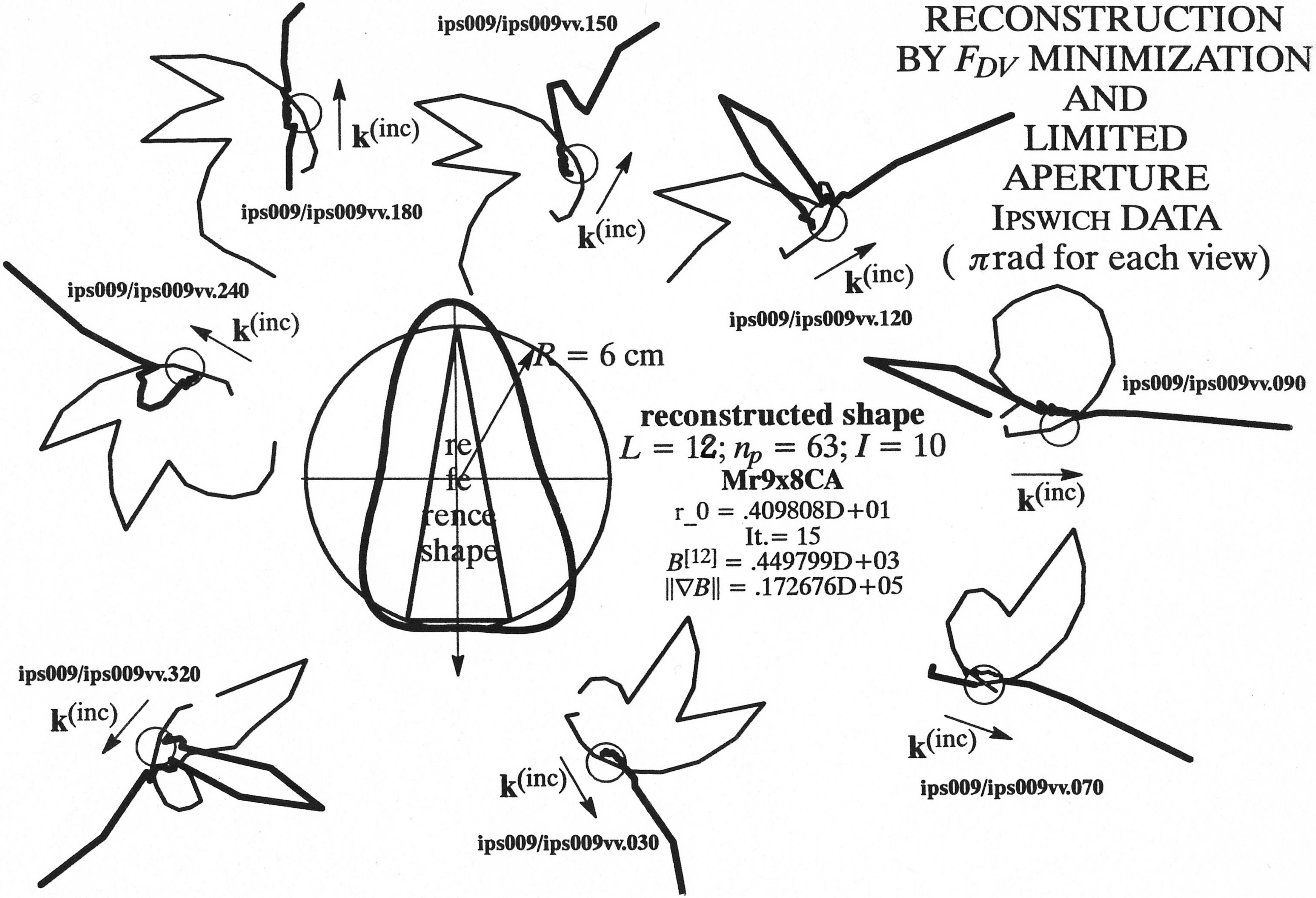
RECONSTRUCTION BY F_{DV} MINIMIZATION AND LIMITED APERTURE IPSWICH DATA

(π rad for each view)



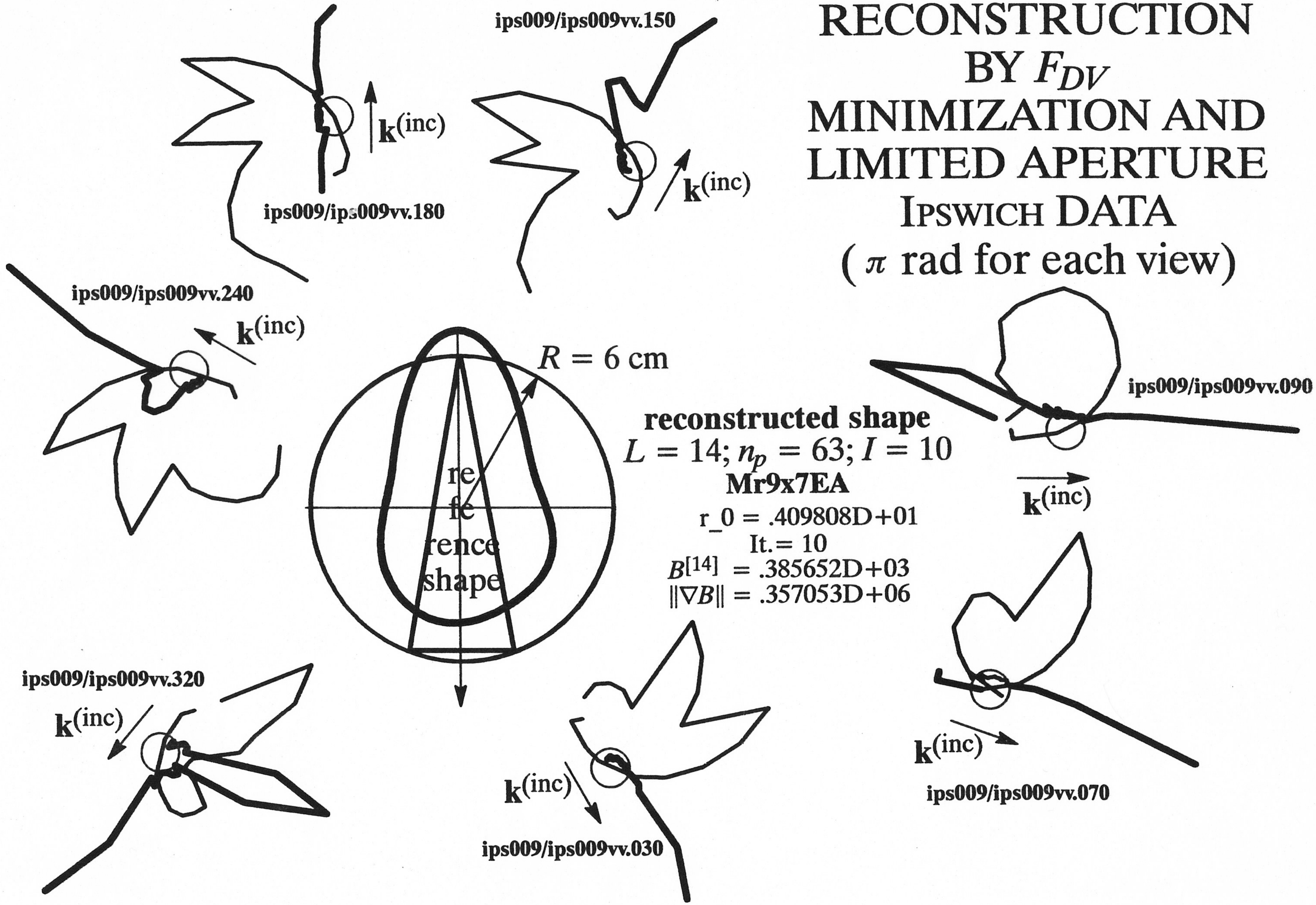
RECONSTRUCTION BY F_{DV} MINIMIZATION AND LIMITED APERTURE IPSWICH DATA

(π rad for each view)



RECONSTRUCTION BY F_{DV} MINIMIZATION AND LIMITED APERTURE IPSWICH DATA

(π rad for each view)



THE HEURISTICS OF SHAPE RECONSTRUCTION BY $F_{DV}^{(L)}$ MINIMIZATION

Prop A2.1 *Given* q, $E_{q,z}^{(\mathrm{inc})}$, $\vec{\psi}_{\mathrm{ref}}$ *and* L, *an approximate (far zone) solution to the exterior Dirichlet BVP is represented by*

$$E_{q,z}^{(\mathrm{sc},L)} = \sum_{\lambda \in \Lambda(L)} p_{q,\lambda}^{(L)}[\vec{\psi}_{\mathrm{ref}}]\nu_\lambda,$$

where $\{p_{q,\lambda}^{(L)}\} = \mathbf{p}_q^{(L)} = -R\mathbf{L}^{(L)}.[\mathbf{P}^{(L)}]^{-1}.\mathbf{g}_q^{(L)} + \vec{\beta}_q^{(L)}$.

Pbm A2.2 (reconstruction from card$[Q]$ sets of Ipswich data)

Given $\{A[\mathbf{k}_q ; \phi_j ; \gamma_q], j \in N_q, q \in Q\}$, *find* $\vec{\psi}^\star \in \Psi_{\mathrm{adm}}$ *and* $\vec{\gamma}^\star$ *such that*

$$F_{DV}^{(L)}[\vec{\psi}^\star ; \vec{\gamma}^\star] :=$$

$$= \frac{1}{2}\sum_{q \in Q}\sum_{j \in N_q}\Big|\sum_{\lambda \in \Lambda(L)} p_{q,\lambda}^{(L)}[\vec{\psi}^\star]\nu_\lambda[\phi_j] - A[\mathbf{k}_q ; \phi_j ; \vec{\gamma}^\star]\Big|^2 = \min$$

Rem. @ fixed q

$$\nabla_{\vec{\psi}}\,\mathbf{p}^{(L)} = -\nabla_{\vec{\psi}}\,R\mathbf{L}^{(L)}.(-\mathbf{c}^{(L)}) - R\mathbf{L}^{(L)}.(-\nabla_{\vec{\psi}}\,\mathbf{c}^{(L)}) + \nabla_{\vec{\psi}}\,\vec{\beta}^{(L)}$$

where

$$\mathbf{P}^{(L)}.(-\nabla_{\vec{\psi}}\,\mathbf{c}^{(L)}) = \nabla_{\vec{\psi}}\,\mathbf{P}^{(L)}.(-\mathbf{c}^{(L)}) - \nabla_{\vec{\psi}}\,\mathbf{g}^{(L)}.$$

Features

1) ◆ The scattering data are $\{A[\mathbf{k}_q ; \phi_j ; \gamma_q], j \in N_q, q \in Q\}$.

2) ◆ Superposition of limited aperture – multiple views allowed.

3) ◆ The cost function to be minimized is $F_{DV}^{(L)}[\vec{\psi}, \vec{\gamma}]$.

4) ◆ The unknowns are $\vec{\psi} \in \Psi_{\mathrm{adm}}$ and $\vec{\gamma} \in ([0, 2\pi])^{\mathrm{card}[Q]}$.

5) The Dirichlet BC and the boundary to far zone relation are represented by **AFP**.

6) $\mathbf{P}^{(L)}$, $R\mathbf{L}^{(L)}$ depend on $\vec{\psi}$ and on k_q (not on $\mathbf{k}_q$).

7) $\{\vec{\beta}_q^{(L)}\}$ and $\{\mathbf{g}_q^{(L)}\}$ depend on $\vec{\psi}$ and on $\mathbf{k}_q$.

8) ◆ Only $\mathbf{P}^{(L)}$ has to be inverted; $[\mathbf{P}^{(L)}]^{-1}$ $\exists$ unconditionally.

BOTTOM – UP JUSTIFICATION:
THE $X_1^{(L)}$ – LEAST SQUARES BOUNDARY COEFFICIENTS

Def $X_1^{(L)} := \mathrm{span}\{ v_\lambda \mid \lambda \in \Lambda[L] \}$.

Pbm *Let q be fixed. Given $\vec{\psi}$, L, $E_{q,z}^{(\mathrm{inc})}$, find $\{ c_{q,\lambda}^{(L)} \}$ such that,*

$$B_{DV}^{(L)} = \frac{1}{2} \Big\| \sum_{\lambda \in \Lambda(L)} c_{q,\lambda}^{(L)} v_\lambda + E_{q,z}^{(\mathrm{inc})} \Big\|_{L^2(\Gamma)}^2 = \min.$$

Solution

$$\mathbf{P}^{(L)}.\mathbf{c}^{(L)} = -\mathbf{g}^{(L)}$$

Thm B.1

I) *$\forall\ \epsilon > 0$, $\exists$ at least an approximation order $L[\epsilon]$ and a vector of coefficients $\mathbf{c}^{(L[\epsilon])}$ such that $B_{DV}^{(L[\epsilon])} < \epsilon^2$.*

II) *Select one such $L[\epsilon]$. Let $\varrho >> r_{\max}$ and $C[\Gamma, \varrho, k]$ be a quantity, which depends on Γ, ϱ and k. Then the corresponding $\mathbf{c}^{(L[\epsilon])}$ is related to the far zone coefficients, $\{ f_\lambda \mid \lambda \in \Lambda[L[\epsilon]] \}$ by the error bound*

$$| f_\lambda - c_\lambda^{(L[\epsilon])} |^2 < \frac{k^2 \epsilon^2}{\pi} C[\Gamma, \varrho, k], \ \forall \lambda \in \Lambda(L[\epsilon]).$$

Remarks.

♠ In general, the determination of $\mathbf{c}^{(L)}$ is asymptotically ill posed, because $\mathbf{P}^{(L)}$ is asymptotically ill conditioned [Ramm, 1986].

♠ The dependence of $L[\epsilon]$ on ϵ has not yet been investigated.

ABOUT RAYLEIGH OBSTACLES

DEF An obstacle is of RAYLEIGH class if $\sum_{\lambda} f_\lambda \, v_\lambda[\mathbf{x}]$ converges, $\forall \mathbf{x} \in \Gamma$.

THM B.2 [BARANTSEV *et al.*, 1971].
Ellipses, $r[\phi] = \dfrac{1}{\sqrt{1-\varepsilon^2\cos^2\phi}}$, *of eccentricity* $\varepsilon < \dfrac{1}{\sqrt{2}}$ *are RAYLEIGH obstacles and*

$$f_\lambda \approx \frac{C\, l^{-\gamma}}{\sqrt{2\pi l}} \left[\frac{\mathrm{e}k}{2l}\right]^l \left[\frac{\varepsilon}{\sqrt{1-\varepsilon^2}}\right]^l \text{ as } l \to \infty \text{, where } \gamma > 0.$$

Proof : by asymptotics of $H_l^{(1)}$ and J_l, singular integral equations, and saddle point method.

THM B.3 [MILLAR, 1973]. *On the boundary of a RAYLEIGH obstacle*

$$\lim_{L \to \infty} \left| f_\lambda - c_\lambda^{(L)} \right| = 0, \ \forall \lambda \in \Lambda(L).$$

DEF (ROUND obstacle). $\Omega \in$ RAYLEIGHObstacleUniformNormalDerivative class if $\sum_{\lambda} f_\lambda \, \partial_N v_\lambda[\mathbf{x}]$ converges <u>uniformly</u> to $\partial_N E_z^{(sc)}\big|_\Gamma$, $\forall \mathbf{x} \in \Gamma$.

THM B.4. *Ellipses, such that* $\varepsilon < \dfrac{1}{\sqrt{2}}$ *are a class of ROUND obstacles.*

Proof : Denote $\zeta := kr[\phi]$. Let $r[\phi] \le 1$.

Since $\dfrac{\mathrm{d}H_l^{(1)}(\zeta)}{\mathrm{d}\zeta} f_\lambda \approx \mathrm{i}\dfrac{C\, l^{-\gamma}}{kr\pi} \left[\dfrac{\varepsilon}{r\sqrt{1-\varepsilon^2}}\right]^l$ as $l \to \infty$, then $\sum_{\lambda} f_\lambda \, \partial_N v_\lambda[\mathbf{x}]$ converges uniformly on Γ.
Moreover [KLEINMAN, ROACH, STRØM, 1984] the infinite dimensional matrix $\mathbf{L} := -\frac{\mathrm{i}}{4}\left[\langle v_\lambda|_\Gamma \, \partial_N v_\mu \rangle\right]$ is invertible and

$$\sum_{\lambda} f_\lambda \, \partial_N v_\lambda\big|_\Gamma = \partial_N E_z^{(sc)}\big|_\Gamma. \qquad \blacksquare$$

(the series converges to the normal derivative of the scattered field on the boundary !)