APPROXIMATE FORWARD PROPAGATION

Motivation

Since $f_{q,\lambda} = -\frac{i}{4}\langle u_\lambda|_\Gamma\, \partial_N(E_{q,z}^{(\mathrm{inc})} + E_{q,z}^{(\mathrm{sc})})\rangle$, where $u_\lambda := \mathrm{Re}[v_\lambda]$, then

I) <u>project</u> $E_{q,z}^{(\mathrm{sc})}|_\Gamma$ <u>onto</u> $X_1^{(L)}$: $\quad E_{q,z}^{(\mathrm{sc},L)} := \sum_{\mu \in \Lambda(L)} c_{q,\mu}^{(L)} v_\mu$

II) <u>differentiate</u>: $\quad \partial_N E_{q,z}^{(\mathrm{sc},L)}|_\Gamma = \sum_{\mu \in \Lambda(L)} c_{q,\mu}^{(L)} \partial_N v_\mu$

III) <u>propagate</u> $\mathbf{c}_q^{(L)}$ <u>forward</u>: $\quad R\mathbf{L}_q{}^{(L)}.\mathbf{c}_q^{(L)} + \vec{\beta}_q^{(L)} = \mathbf{p}_q^{(L)}$.

This is the affine least squares (ALS) scheme in $X_1^{(L)}$.

Application

Prop B.5 *Given q, $E_{q,z}^{(\mathrm{inc})}$, $\vec{\psi}_{\mathrm{ref}}$ and L, an approximate (far zone) solution to the exterior Dirichlet BVP is represented by*

$$E_{q,z}^{(\mathrm{sc},L)} = \sum_{\lambda \in \Lambda(L)} p_{q,\lambda}^{(L)}[\vec{\psi}_{\mathrm{ref}}] v_\lambda,$$

where $\{p_{q,\lambda}^{(L)}\} = \mathbf{p}_q^{(L)} = -R\mathbf{L}_q{}^{(L)}.[\mathbf{P}_q{}^{(L)}]^{-1}.\mathbf{g}_q^{(L)} + \vec{\beta}_q^{(L)}$.

Consistency

Thm B.6

Fix q. <u>If</u> $\Gamma = S_R^{n-1}$ i.e., Ω is a disk ($n = 2$) or a sphere ($n = 3$) of radius R,

<u>then</u> the $X_1^{(L)}$ ALS scheme is consistent i.e.,

I) *$c_\lambda^{(L)}$ and $p_\lambda^{(L)}$ are independent of L*

II) $\forall L \geq 0 : c_\lambda^{(L)} = f_\lambda = p_\lambda^{(L)}, \ \forall \lambda \in \Lambda(L)$.

Proof: I) orthogonality of $\{v_\lambda\} \Rightarrow$ independence of L.
II) uniform convergence of $\sum_\lambda f_\lambda \partial_N v_\lambda$ on $S_R^{n-1} \Rightarrow$ direct verification. ■

THE $X_1^{(L)}$ ALS SCHEME AND SUCCESSIVE APPROXIMATIONS

THM B.7 (successive approximations in the infinite dimensional case). *Let $\Omega \in$ ROUND and fix q. Assume $\mathbf{f} \in \ell_2$ and RL: $\ell_2 \rightarrow \ell_2$ is bounded. If $r_\sigma[RL] < 1$, then, $\forall\ \mathbf{b} \in \ell_2$, there exists a unique solution, $\mathbf{f}$, to*

$$\mathbf{f} = \mathbf{b} + RL\,.\,\mathbf{f}\,,$$

obtained by successive approximations, where t is the iteration index i.e.,

$$\mathbf{p}^{[t+1]} = \mathbf{b} + RL\,.\,\mathbf{p}^{[t]}\,,\ t = 0, 1, 2, \ldots$$

started with an arbitrary $\mathbf{p}^{[0]} \in \ell_2$.

Proof: $\Omega \in$ ROUND $\Rightarrow f_\lambda = -\frac{i}{4}\langle u_\lambda|_\Gamma\ \partial_N E_z^{(\mathrm{inc})} + \sum_\mu f_\mu\, \partial_N v_\mu \rangle$; then

recall the contraction mapping THEOREM.

COR B.8 (to THM B.7; on the finite dimensional propagator)
Let $r_\sigma[RL^{(L)}] < 1$ for some (finite) L, then $\exists\,!$ fixed point $\bar{\mathbf{c}}^{(L)}$ such that

$$\bar{\mathbf{c}}^{(L)} = \mathbf{b}^{(L)} + RL^{(L)}.\bar{\mathbf{c}}^{(L)} \text{ i.e., } \bar{\mathbf{c}}^{(L)} = [\,\mathbf{1}^{(L)} - RL^{(L)}\,]^{-1}.\mathbf{b}^{(L)},$$

where $\mathbf{p}^{(L)[0]}$ arbitrary ($=\mathbf{c}^{(L)}$ in particular).

THM B.9 (infinite *vs.* finite dimensional propagators)
<u>Let</u> $\Omega \in$ ROUND, $\mathbf{f}$, $\mathbf{b}$ and RL as in THM B.7. <u>Assume</u>

$$r_\sigma[RL^{(L)}] < 1\,,\ \forall\, L \text{ and } r_\sigma[RL] < 1$$

<u>then</u> the following hold

I) $\forall L\ \exists\,!$ fixed point $\bar{\mathbf{c}}^{(L)}$ as in COR B.8.

II) $\lim_{L \rightarrow \infty} \bar{\mathbf{c}}^{(L)} = \mathbf{f}$.

III) $\lim_{L \rightarrow \infty} \mathbf{p}^{(L)} = \mathbf{f}$.

Proof: Since

$$\mathbf{b}^{(L)} \rightarrow \mathbf{b}\,,\ \mathbf{1}^{(L)} - RL^{(L)} \rightarrow \mathbf{1} - RL \text{ and } [\mathbf{1}^{(L)} - RL^{(L)}]^{-1} \rightarrow [\mathbf{1} - RL]^{-1},$$

then

$\bar{\mathbf{c}}^{(L)} \rightarrow \mathbf{f}$ (convergence of the projection method).

CONVERGENCE, TRUST AND EFFECTIVENESS

Def The $X_1^{(L)}$ ALS scheme is <u>effective</u> whenever, in the far zone (i.e., ϱ sufficiently large)

$$\Big\| \sum_{\lambda \in \Lambda(L)} p_{q,\lambda}^{(L)} \tilde{v}_\lambda - E_{q,z}^{(sc)} \Big\|_{L^2(S_\varrho^{n-1})}^2 < \Big\| \sum_{\lambda \in \Lambda(L)} c_{q,\lambda}^{(L)} \tilde{v}_\lambda - E_{q,z}^{(sc)} \Big\|_{L^2(S_\varrho^{n-1})}^2 .$$

Recall: $\mathbf{c}^{(L)}$ = initial value ; $\mathbf{p}^{(L)}$ = first iterate.

Def A method is <u>trusted</u> with respect to **f** if it returns $\bar{\mathbf{c}}^{(L)}$ such that, given $\epsilon > 0$,

$$\left\| \mathbf{f}^{(L)} - \bar{\mathbf{c}}^{(L)} \right\|_{C^M} < \epsilon \left\| \bar{\mathbf{c}}^{(L)} \right\|_{C^M}, \; M := \mathrm{card}[\Lambda(L)].$$

Thm B.10 ({convergence; trust} $\Rightarrow$ effectiveness) *Let $\epsilon > 0$, small and .* *<u>If</u> $r_\sigma[RL^{(L)}] < 1$ such that,*

(a) $\quad \left\| \mathbf{p}^{(L)} - \bar{\mathbf{c}}^{(L)} \right\|_{C^M} + 2\epsilon \left\| \bar{\mathbf{c}}^{(L)} \right\|_{C^M} < \left\| \mathbf{c}^{(L)} - \bar{\mathbf{c}}^{(L)} \right\|_{C^M}$

(b) $\quad \left\| \mathbf{f}^{(L)} - \bar{\mathbf{c}}^{(L)} \right\|_{C^M} < \epsilon \left\| \bar{\mathbf{c}}^{(L)} \right\|_{C^M}$ *and*

(c) $\quad k^2\varrho^2 >> \dfrac{4L^2 - 1}{8}$ *(if $n = 2$) or* $k^2\varrho^2 >> \dfrac{L+1}{2} L$ *(if $n = 3$)*

<u>then</u>

$$\Big\| \sum_{\lambda \in \Lambda(L)} p_{q,\lambda}^{(L)} \tilde{v}_\lambda - E_{q,z}^{(sc)} \Big\|_{L^2(S_\varrho^{n-1})}^2 \overset{(d)}{<} \Big\| \sum_{\lambda \in \Lambda(L)} c_{q,\lambda}^{(L)} \tilde{v}_\lambda - E_{q,z}^{(sc)} \Big\|_{L^2(S_\varrho^{n-1})}^2 .$$

Proof. $\{ (a) \,\&\, (b) \} \Rightarrow \left\| \mathbf{p}^{(L)} - \mathbf{f}^{(L)} \right\|_{C^M} \overset{(e)}{<} \left\| \mathbf{c}^{(L)} - \mathbf{f}^{(L)} \right\|_{C^M}$.

Moreover $(c) \Rightarrow \| \tilde{v}_\lambda \|_{L^2(S_\varrho^{n-1})}^2 \approx$ independent of λ, hence $(e) \Rightarrow (d)$ ■

ASIDE:

Prop B.11 (How small shall ϵ be for a method to be trusted ?)
*<u>If</u> **f** unknown, <u>then</u> a method, which returns $\bar{\mathbf{c}}^{(L)}$, is a candidate for trust provided it is ϵ–accurate, where*

$$\epsilon < \frac{\| \bar{\mathbf{c}}^{(L)} - \mathbf{c}^{(L)} \|_{C^M} - \| \bar{\mathbf{c}}^{(L)} - \mathbf{p}^{(L)} \|_{C^M}}{2 \, \| \bar{\mathbf{c}}^{(L)} \|_{C^M}} .$$

THE NUMERICAL DETERMINATION OF SPECTRAL RADII

Scope: find a non trivial, non empty set such that, THM B.7 holds.

PROP B.12 [DÉMIDOVITCH & MARON, 1979, Ch. XII, § 11]

$$r_\sigma[RL^{(L)}] \approx r_\sigma{}^{(M)}[RL^{(L)}] := \left|\frac{\mathrm{Tr}[RL^{(L)}]^{M+1}}{\mathrm{Tr}[RL^{(L)}]^{M}}\right|, \ M \text{ large.}$$

(Stop @ M such that, $|r_\sigma{}^{(M+1)} - r_\sigma{}^{(M)}| < 10^{-7}$).

REMARKS

◈ If Γ is a disk, then $r_\sigma[RL^{(L)}] = \max_{0 \le l \le L} |RL^{(L)}{}_{plpl}|$, where

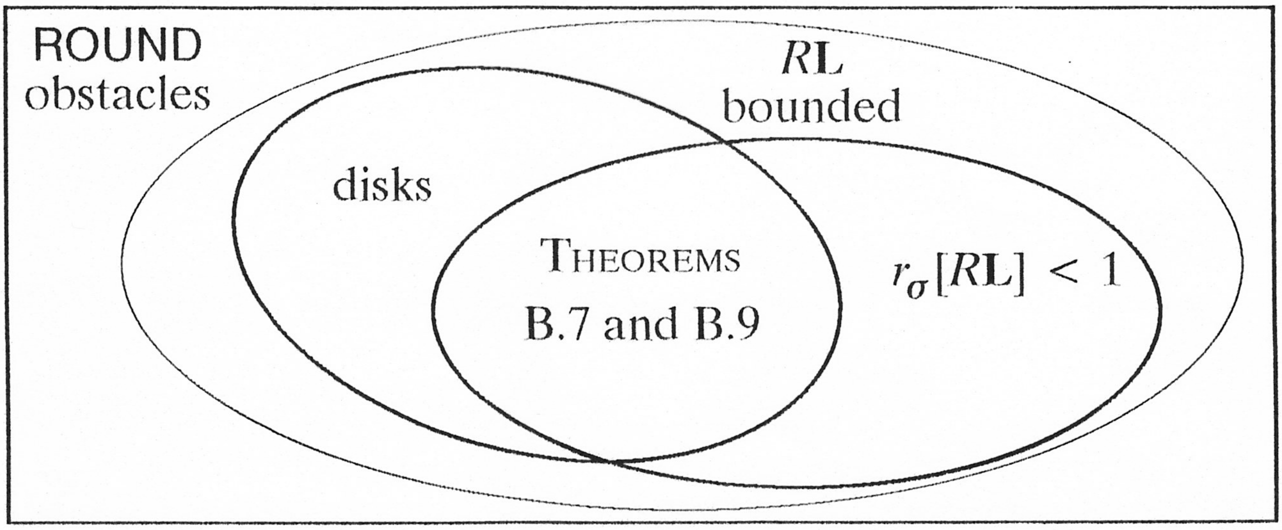
$$RL^{(L)}{}_{plpl} = -\frac{i}{2}\pi kR\, J_l[kR] \frac{d}{dz} H_l^{(1)}[kR]\,, p = 0, 1.$$

◈ $r_\sigma[RL^{(L)}]$ depends on $\{ L, k, \vec{\psi} \}$, **not** on $\hat{\mathbf{k}}$.

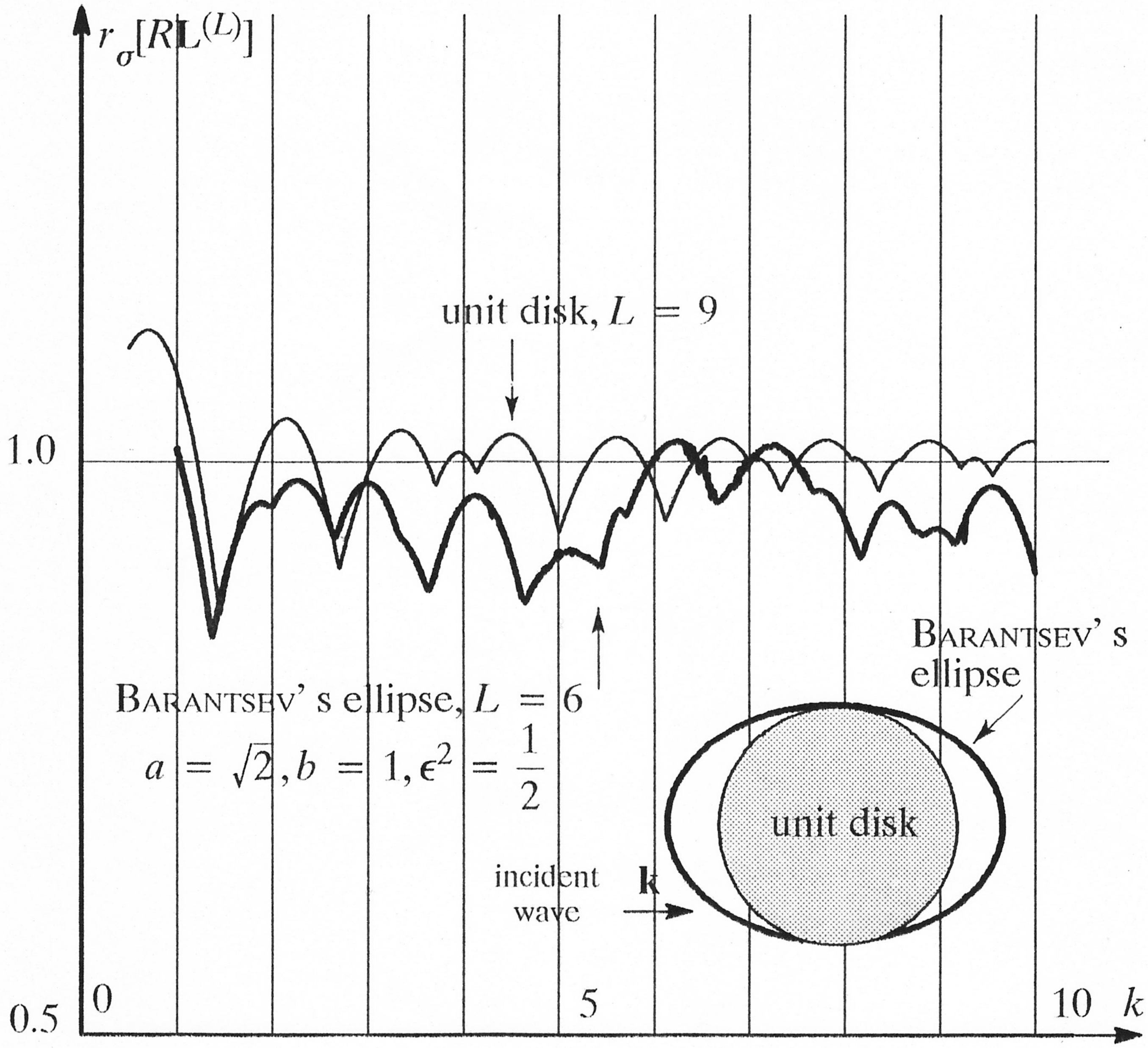
◈ When $\Gamma = S_R^{n-1}$, there is no contradiction between the consistency property (THM B.6) and the result $r_\sigma[RL^{(L)}] > 1$ for some $\{L, kR\}$. Namely, consistency is verified directly and does not involve any iterative scheme.

♠ THMS B.7 and B.9 only provide a sufficient condition for the convergence of forward propagation.

The relationships known to date among the conditions on RL.



COMPUTED SPECTRAL RADII



Unit disk: direct calculation.

BARANTSEV' s ellipse: by PROP B.12.
$10 < M[k] < 2000$, strongly affected by k.

MORE QUESTIONS AND OPEN PROBLEMS

Q1 How many IPSWICH data sets are needed to imply the uniqueness (if any) of the unknown shape ?

Q2 (approximation) In view of THM B.2, are cylindrical wave functions the appropriate basis ?

Q3 (approximation) What is the role of approximation and parameterization orders, $\{ I, L \}$, in uniqueness and stability ?

Q4 Is $\vec{\gamma}$ the proper unknown to represent phase corrections of the IPSWICH data ?

Q5 Are conjugate directions the best descent rule ?

Q6 How will other backpropagation methods e.g., the $\mathbf{W}^{(L)}$ method [1994, 1995] perform ?

Q7 How will penalty methods (ANGELL–KLEINMAN–ROACH [1987], RAMM [1994]) compare ?