$\Gamma \equiv \partial\Omega$ $\qquad$ $\underline{b}^{(L)} \equiv \vec{\beta}^{(L)}$

THE $X_1^{(L)}$ ALS SCHEME AND SUCCESSIVE APPROXIMATIONS

DEF: $\Omega \in$ RAYLEIGHObstacleUniformNormalDerivative class if $\sum_\lambda f_\lambda \, \partial_N v_\lambda[\mathbf{x}]$ converges <u>uniformly</u> to $\partial_N E_z^{(sc)}\big|_\Gamma$, $\forall \mathbf{x} \in \Gamma$.

THM J2.3 (successive approximations in the infinite dimensional case). *Let $\Omega \in$ ROUND. Assume $\mathbf{f} \in \ell_2$ and $R\mathbf{L}: \ell_2 \to \ell_2$ is bounded. If $r_\sigma[R\mathbf{L}] < 1$, then, $\forall\, \mathbf{b} \in \ell_2$, there exists a unique solution, $\mathbf{f}$, to*

$$\mathbf{f} = \mathbf{b} + R\mathbf{L}\,.\,\mathbf{f},$$

obtained by successive approximations, where t is the iteration index i.e.,

$$\mathbf{p}^{[t+1]} = \mathbf{b} + R\mathbf{L}\,.\,\mathbf{p}^{[t]}\,, t = 0, 1, 2, \ldots$$

started with an arbitrary $\mathbf{p}^{[0]} \in \ell_2$.

Proof: $\Omega \in$ ROUND $\Rightarrow f_\lambda = -\frac{i}{4}\langle u_\lambda|_\Gamma \, \partial_N E_z^{(inc)} + \sum_\mu f_\mu \, \partial_N v_\mu \rangle$; then

recall the contraction mapping THEOREM.

THM J2.4 (infinite *vs.* finite dimensional propagators)
<u>Let</u> $\Omega \in$ ROUND, $\mathbf{f}$, $\mathbf{b}$ and $R\mathbf{L}$ as in THM J2.3. <u>Assume</u>

$$r_\sigma[R\mathbf{L}^{(L)}] < 1\,, \forall\, L \text{ and } r_\sigma[R\mathbf{L}] < 1$$

<u>then</u> the following hold

I) $\quad \forall L\ \exists!$ *fixed point* $\bar{\mathbf{c}}^{(L)}$ *such that*

$$\bar{\mathbf{c}}^{(L)} = \mathbf{b}^{(L)} + R\mathbf{L}^{(L)}.\bar{\mathbf{c}}^{(L)} \text{ i.e., } \bar{\mathbf{c}}^{(L)} = [\, \mathbf{1}^{(L)} - R\mathbf{L}^{(L)}\,]^{-1}.\mathbf{b}^{(L)}$$

II) $\quad \lim_{L \to \infty} \bar{\mathbf{c}}^{(L)} = \mathbf{f}$.

III) $\quad \lim_{L \to \infty} \mathbf{p}^{(L)} = \mathbf{f}$.

Proof : Since $\mathbf{b}^{(L)} \to \mathbf{b}$, $\mathbf{1}^{(L)} - R\mathbf{L}^{(L)} \to \mathbf{1} - R\mathbf{L}$ and $[\mathbf{1}^{(L)} - R\mathbf{L}^{(L)}]^{-1} \to [\mathbf{1} - R\mathbf{L}]^{-1}$ then $\bar{\mathbf{c}}^{(L)} \to \mathbf{f}$ (convergence of the projection method).

THE NUMERICAL DETERMINATION OF SPECTRAL RADII

Scope: find a non trivial, non empty set such that, THM J2.3 holds.

PROP J2.5 [DÉMIDOVITCH & MARON, 1979, Ch. XII, § 11]

$$r_\sigma[R\mathrm{L}^{(L)}] \approx r_\sigma^{(M)}[R\mathrm{L}^{(L)}] := \left|\frac{\mathrm{Tr}[R\mathrm{L}^{(L)}]^{M+1}}{\mathrm{Tr}[R\mathrm{L}^{(L)}]^{M}}\right|, \; M \text{ large.}$$

(Stop @ M such that, $|r_\sigma^{(M+1)} - r_\sigma^{(M)}| < 10^{-7}$).

REMARKS

◆ If Γ is a disk, then $r_\sigma[R\mathrm{L}^{(L)}] = \max_{0 \le l \le L} |R\mathrm{L}^{(L)}{}_{plpl}|$, where

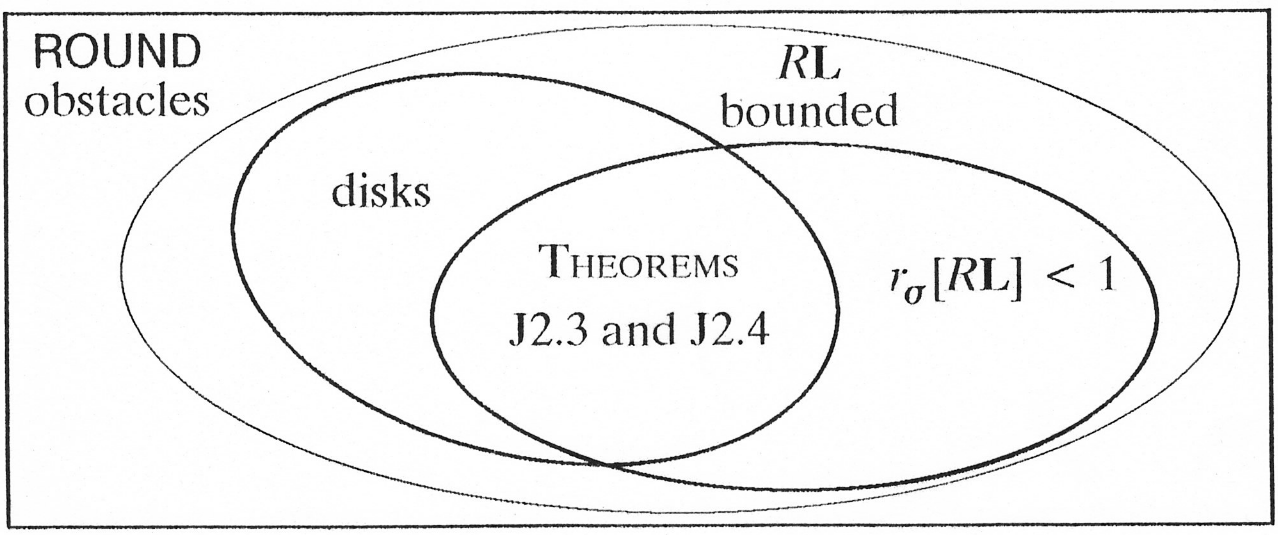
$$R\mathrm{L}^{(L)}{}_{plpl} = -\frac{\mathrm{i}}{2}\pi kR\, J_l[kR]\frac{\mathrm{d}}{\mathrm{d}z}H_l^{(1)}[kR], p = 0, 1.$$

◆ $r_\sigma[R\mathrm{L}^{(L)}]$ depends on $\{ L, k, \vec{\psi} \}$, **not** on $\hat{\mathbf{k}}$.

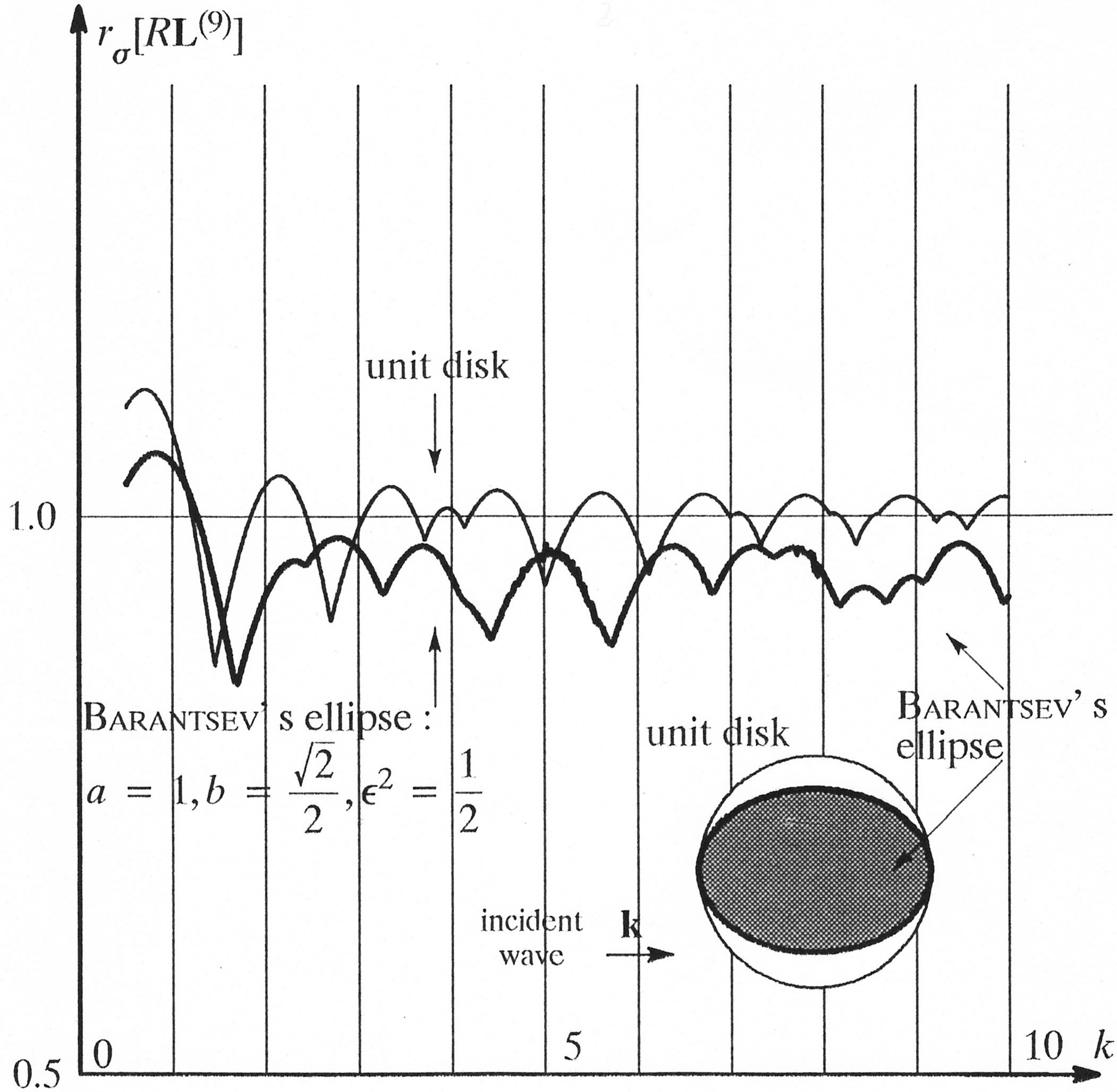
◆ When $\Gamma = S_R^{n-1}$, there is no contradiction between the consistency property (THM J2.1) and the result $r_\sigma[R\mathrm{L}^{(L)}] > 1$ for some $\{L, kR\}$. Namely, consistency is verified directly and does not involve any iterative scheme.

♠ THMS J2.3 and J2.4 only provide a sufficient condition for the convergence of forward propagation.

The relationships known to date among the conditions on RL.



COMPUTED SPECTRAL RADII



Unit disk: direct calculation.

BARANTSEV' s ellipse: by PROP J2.5.
$10 < M[k] < 2000$, strongly affected by k.

THE TRUSTED METHOD INEQUALITY

Recall: $\mathbf{c}^{(L)}$ = initial value ; $\mathbf{p}^{(L)}$ = first iterate.

COR J2.6 (to THM J2.4)

Fix L , then $\left\|\mathbf{c}^{(L)}-\bar{\mathbf{c}}^{(L)}\right\| < \left\|\mathbf{p}^{(L)}-\bar{\mathbf{c}}^{(L)}\right\|$. (Norms in $\mathbb{C}^{\,\mathrm{card}[\Lambda(L)]}$).

DEF A method is <u>trusted</u> with respect to **f** if it returns $\bar{\mathbf{c}}^{(L)}$ such that, given $\epsilon > 0$,

$$|f_\lambda - \bar{c}_\lambda^{(L)}| < \epsilon\, |\bar{c}_\lambda^{(L)}|,\ \forall \lambda \in \Lambda(L).$$

PROP J2.7 (a sufficient condition for the effectiveness of forward propagation)

<u>If</u> $\exists\, \epsilon > 0$, small such that, $\forall \lambda \in \Lambda(L)$

a) $|f_\lambda - \bar{c}_\lambda^{(L)}| < \epsilon\, |\bar{c}_\lambda^{(L)}|$ *<u>and</u>*

b) $|p_\lambda^{(L)} - \bar{c}_\lambda^{(L)}| + 2\epsilon\, |\bar{c}_\lambda^{(L)}| < |c_\lambda^{(L)} - \bar{c}_\lambda^{(L)}|$

<u>then</u> ALS forward propagation is effective i.e.,

$$|p_\lambda^{(L)} - f_\lambda| < |c_\lambda^{(L)} - f_\lambda|.$$

REMARKS

♠ COR J2.6 $\not\Rightarrow |c_\lambda^{(L)} - \bar{c}_\lambda^{(L)}| < |p_\lambda^{(L)} - \bar{c}_\lambda^{(L)}|$, $\lambda \in \Lambda(L)$.

♠ The effectiveness of **ALS** remains an open problem.

PROP J2.8 (How small shall ϵ be for a method to be trusted ?)

*<u>If</u> **f** unknown, <u>then</u> a method, which returns $\bar{\mathbf{c}}^{(L)}$, is a candidate for trust provided it is ϵ–accurate, where*

$$\epsilon < \max_{\lambda \in \Lambda(L)} \frac{|\bar{c}_\lambda^{(L)} - c_\lambda^{(L)}| - |\bar{c}_\lambda^{(L)} - p_\lambda^{(L)}|}{2|\bar{c}_\lambda^{(L)}|}\,.$$

MORE QUESTIONS AND OPEN PROBLEMS

Q1 How many IPSWICH data sets are needed to imply the uniqueness (if any) of the unknown shape ?

Q2 (approximation) In view of THM J1.2, are cylindrical wave functions the appropriate basis ?

Q3 (approximation) What is the role of approximation and parameterization orders, $\{ I, L \}$, in uniqueness and stability ?

Q4 Is $\vec{\gamma}$ the proper unknown to represent phase corrections of the IPSWICH data ?

Q5 Are conjugate directions the best descent rule ?

Q6 How will other backpropagation methods e.g., the $\mathbf{W}^{(L)}$ method [1994, 1995] perform ?

Q7 How will penalty methods (ANGELL–KLEINMAN–ROACH [1987], RAMM [1994]) compare ?