Schrödinger equations with an external magnetic field: spectral problems & semiclassical states

Université Libre de Bruxelles
Faculté des Sciences
Département de Mathématiques

Università degli Studi di Milano-Bicocca
Facoltà di Scienze Matematiche, Fisiche e Naturali

Schrödinger equations with an external magnetic field: spectral problems & semiclassical states

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Manon Nys

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Directeurs de thèse :
Denis Bonheure et Susanna Terracini

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[1] V. Bonnaillie-Noël, B. Noris, M. Nys, and S. Terracini. "On the eigenvalues of Aharonov-Bohm operators with varying poles". *Anal. PDE* 7.6 (2014), pp. 1365–1395. ISSN: 2157-5045.

[2] B. Noris, M. Nys, and S. Terracini. "On the Aharonov–Bohm Operators with Varying Poles: The Boundary Behavior of Eigenvalues". *Comm. Math. Phys.* 339.3 (2015), pp. 1101–1146. ISSN: 0010-3616.

[3] D. Bonheure, S. Cingolani, and M. Nys. "Nonlinear Schrödinger equation: concentration on circles driven by an external magnetic field". *Submitted* (2015).

Notations

$\mathbb{N}_0$	set of naturals without 0.
$\mathbb{R}_0$	set of reals without 0.
$\mathbb{Z}_0$	set of integers without 0.
$B_r(b)$	ball of radius r centred at b.
$D_r^+(b)$	half-ball of radius r, centred at b, p 57.
$C_0^\infty(\Omega)$	space of C^∞ functions on Ω with compact support.
$\mathcal{D}^{1,2}_{A_a}(\Omega)$	completion of $C_0^\infty(\Omega\setminus\{a\},\mathbb{C})$ with respect to the norm $\\|(i\nabla+A_a)\cdot\\|_{L^2(\Omega)}$, p. 12.
$H^1_{A_a}(\Omega)$	completion of $\{u \in C^\infty(\Omega,\mathbb{C}) : u = 0$ in a neighbourhood of $a\}$ with respect to the norm $(\\|(i\nabla+A_a)\cdot\\|^2_{L^2(\Omega)}+\\|\cdot\\|^2_{L^2(\Omega)})^{1/2}$, p. 12.
$\mathcal{D}^{1,2}_{A,\varepsilon}(\mathbb{R}^N,\mathbb{C})$	completion of $C_0^\infty(\mathbb{R}^N,\mathbb{C})$ with respect to the norm $\\|(i\varepsilon\nabla+A)\cdot\\|_{L^2(\mathbb{R}^N)}$, p 129.
$H^1_{A,\varepsilon}(\mathbb{R}^N,\mathbb{C})$	$\{u\in L^2(\mathbb{R}^N,\mathbb{C}) : (i\varepsilon\nabla+A)u\in L^2(\mathbb{R}^N)\}$, p 130.
$H^1_{A,V,\varepsilon}(\mathbb{R}^N,\mathbb{C})$	$\{u\in \mathcal{D}^{1,2}_{A,\varepsilon}(\mathbb{R}^N,\mathbb{C}) : \int_{\mathbb{R}^N} V\|u\|^2<+\infty\}$, p 131.
$\mathcal{X}_\varepsilon$	$\{u\in H^1_{A,V,\varepsilon}(\mathbb{R}^N,\mathbb{C}) : u\circ g = u\}$, p 137.
$\\|\cdot\\|_\varepsilon$	$\\|\cdot\\|_{H^1_{A,V,\varepsilon}}$, p 131.
$\mathrm{d}_{cyl}(x,y)$	$\big((\rho_x-\rho_y)^2+(x_3-y_3)^2\big)^{1/2}$, p 126.
$B_{cyl}(x,r)$	$\{y\in\mathbb{R}^3 : \mathrm{d}_{cyl}(x,y)<r\}$, p 126.
φ_k	k-th eigenfunction of the Laplacian with Dirichlet boundary conditions.
λ_k	k-th eigenvalue of the Laplacian with Dirichlet boundary conditions.
φ_k^a	k-th eigenfunction of the magnetic operator $(i\nabla+A_a)^2$, p 12.

λ_k^a	k-th eigenvalue of the magnetic operator $(i\nabla + A_a)^2$, p 12.
$f(x) = O(g(x))$	$\limsup_{x\to x_0} \lvert f(x)/g(x)\rvert < +\infty$, p. 61.
$f(x) \sim g(x)$	$\lim_{x\to x_0} \lvert f(x)/g(x)\rvert \in (0,+\infty)$, p. 61.
$f(x) = o(g(x))$	$\lim_{x\to x_0} f(x)/g(x) = 0$, p. 61.
θ_a	polar angle around a, p 29.
interior zero of order $h/2$	critical point at which meet h arcs of nodal lines, p 17.
boundary zero of order $h/2$	boundary point at which meet $h/2-1$ arcs of nodal lines, p 21.

Contents

Chapter 1

Introduction

This thesis is based on three articles [1–3]. All the simulations were provided by Virginie Bonnaillie-Noël.

1.1. General setting

The main interest of those works is to study Schrödinger equations in the presence of a magnetic force, with magnetic potential $A : \Omega \to \mathbb{R}^N$, where $A(x) = (A_1(x), \ldots, A_N(x))$. The domain Ω is an open subset of $\mathbb{R}^N$ or $\mathbb{R}^N$ itself and $N \geq 2$. The magnetic potential A is linked to the magnetic field $B = \{B_{ij}\}_{i,j=1\cdots N}$ by the relations

$$B_{ij} = \frac{\partial A_j}{\partial x_i} - \frac{\partial A_i}{\partial x_j}, \qquad i, j = 1, \ldots, N.$$

More precisely, in the differential geometry language, the two-form $\omega_B = \sum_{i,j=1}^N B_{ij}\,\mathrm{d}x_i \wedge \mathrm{d}x_j$ is the exterior derivative of the one-form $\omega_A = \sum_{i=1}^N A_i\,\mathrm{d}x_i$ canonically associated to the vector field A. In the special case of dimension $N = 3$, B is the standard magnetic field, where

$$B = (B_{23}, B_{31}, B_{12}) = \nabla \times A.$$

In dimension $N = 2$, B is given by the unique component B_{12} but can also be seen as a field in dimension 3, orthogonal to the plane, that is

$$A = (A_1, A_2, 0) \qquad \text{and} \qquad B = (0, 0, B_{12}).$$

The magnetic momentum is given by $(p + A)$, where p was the usual momentum of a physical system without external magnetic field, and thus the magnetic operator, defined as

$$(i\varepsilon\nabla + A)^2 = -\varepsilon^2\,\Delta + 2i\varepsilon\,A \cdot \nabla + i\varepsilon\,\nabla \cdot A + |A|^2,$$

will replace the standard Laplacian. Physically, $\varepsilon = \hbar$ is the Planck constant. There are two physical relevant situations, the first for which ε is fixed, and then we can set $\varepsilon = 1$, and the other one for which $\varepsilon \to 0$. Those two different cases are respectively studied in Chapter 2 and Chapter 3. We remark that the magnetic field B remains unchanged under the transformation $A \mapsto A' = A + \nabla\chi$. This corresponds to a spectral invariance with respect to the gauge group. In this thesis, a gauge transformation is given by $\psi \mapsto \psi' = e^{i\chi/\varepsilon}\psi$, under which the operator changes as follows

$$e^{i\chi/\varepsilon}(i\varepsilon\nabla + A)^2 e^{-i\chi/\varepsilon} = (i\varepsilon\nabla + A')^2.$$

A large amount of magnetic potentials may be considered, which need not necessarily to be bounded or smooth. For example in dimension $N = 3$, the physically interesting case of the constant magnetic field $B = (0, 0, 1)$ is associated (up to a gauge transformation) to the magnetic potential $A = \frac{1}{2}(-x_2, x_1, 0)$ which growths linearly at infinity. For those two reasons, when working in the whole $\mathbb{R}^N$, we usually assume that $A \in L^2_{\text{loc}}(\mathbb{R}^N, \mathbb{R}^N)$, such that $(i\varepsilon\nabla + A)u$ makes sense as a distribution when acting on functions $u \in L^2(\mathbb{R}^N)$. For more details, see for example [4].

Here, we recall the diamagnetic inequality. The physical idea of such inequality is that the energy of the system is increased by the addition of an external magnetic field.

Theorem 1.1.1 (Diamagnetic inequality, [5, Theorem 2.1.1], [4, Theorem 7.21])**.** *Let $\varepsilon > 0$ and $A : \mathbb{R}^N \to \mathbb{R}^N$ be in $L^2_{loc}(\mathbb{R}^N)$ and assume that u is in $L^2(\mathbb{R}^N)$ with $(i\varepsilon\nabla + A)u \in L^2(\mathbb{R}^N)$. Then $|u| \in H^1(\mathbb{R}^N)$ and*

$$\varepsilon|\nabla|u|| \leq |(i\varepsilon\nabla + A)u|, \qquad \textit{a.e. in } \mathbb{R}^N.$$

By using the inequality above, we can obtain a magnetic Hardy inequality for functions $u \in L^2(\mathbb{R}^N)$ such that $(i\varepsilon\nabla + A)u \in L^2(\mathbb{R}^N)$

$$\varepsilon^2 \int_{\mathbb{R}^N} \frac{|u|^2}{|x|^2}\,\mathrm{d}x \leq \left(\frac{2}{N-2}\right)^2 \int_{\mathbb{R}^N} |(i\varepsilon\nabla + A)u|^2\,\mathrm{d}x.$$

We immediately notice that this inequality is not valid in dimension $N = 2$.

Nevertheless, in dimension $N = 2$, another interesting magnetic potential was consider by Bohm and Aharonov. For $a = (a_1, a_2) \in \mathbb{R}^2$ and $x = (x_1, x_2) \in \mathbb{R}^2 \backslash \{a\}$, this potential has the form (up to some gauge transformation)

$$A_a(x_1, x_2) = \alpha\left(-\frac{x_2 - a_2}{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2}\right). \quad (1.1)$$

This potential has the particularity to be singular at $a \in \mathbb{R}^2$, and to have a non zero circulation $\alpha \in \mathbb{R}_0$ around a (of course the case $\alpha = 0$ is trivial since we recover the usual Laplacian), given by

$$\alpha = \frac{1}{2\pi} \oint_\sigma A_a \cdot \mathrm{d}x,$$

for a closed path σ winding once around the singularity a. The associated magnetic field is a $2\pi\alpha$-multiple of the singular Dirac delta distribution at a, orthogonal to the plane.

Obviously A_a is not in $L^2_{loc}(\mathbb{R}^2)$ since it has a strong singularity at a. However, it is in $L^1_{loc}(\mathbb{R}^2) \cap C^\infty(\mathbb{R}^2 \setminus \{a\})$.

Physically, this situation arises when we consider an infinitesimally thin solenoid, generating a magnetic field orthogonal to the plane and null outside the solenoid. This potential A_a gives rise to the Aharonov-Bohm effect, see [6], which says that a charged particle constrained in a region where the magnetic field is zero will nonetheless be influenced by the presence of the Aharonov-Bohm potential. From a topological point of view, we can say that A_a, by means of its circulation α, influences the particle trajectories not homotopic to a point (see [5, Notes p47]).

In that case, a diamagnetic inequality still holds as long as we take care of the strong singularity at a.

Theorem 1.1.2 ([7, Lemma A.1], Diamagnetic inequality)**.** *Let $\varepsilon > 0$ and A_a be as in (1.1). Then, the following inequality holds for every $u \in C_0^\infty(\mathbb{R}^2 \backslash \{a\})$.*

$$\varepsilon |\nabla |u|| \leq |(i\varepsilon\nabla + A_a)u|, \quad a.e. \ in \ \mathbb{R}^2.$$

Moreover, even if we are in dimension $N = 2$, we notice that we recover an Hardy inequality by adding this Aharonov-Bohm potential, as long as its circulation is not an integer.

Theorem 1.1.3 ([8, Theorem 3])**.** *Let $\varepsilon > 0$, A_a be as in (1.1) and $\alpha \notin \mathbb{Z}$. Then,*

$$\varepsilon^2 \int_{\mathbb{R}^2} \frac{|u|^2}{|x-a|^2} \, \mathrm{d}x \leq C \int_{\mathbb{R}^2} |(i\varepsilon\nabla + A_a)u|^2 \, \mathrm{d}x, \quad u \in C_0^\infty(\mathbb{R}^2 \backslash \{a\}),$$

where $C = \left(\min_{k \in \mathbb{Z}} |\alpha - k|\right)^{-2}$. Moreover, this constant is sharp.

When the circulation is an integer, the magnetic Hardy inequality does not hold any more. This is due to the fact that $(i\varepsilon\nabla + A_a)^2$ is gauge equivalent to the Laplacian in that case, see for example [5, proposition 2.1.3] (more details will be given in Chapter 2).

For more details about Schrödinger equations with magnetic potentials, we refer to [5, 9–11].

1.2. The linear Schrödinger equation: an eigenvalue problem involving the Bohm-Aharonov potential

The non-relativistic linear Schrödinger equation in presence of a magnetic potential A has the form

$$i\frac{\partial}{\partial t}\Psi(x,t) = \frac{1}{2m}\left(i\hbar\nabla - qA\right)^2\Psi(x,t) + U(x)\Psi(x,t), \qquad x \in \mathbb{R}^N.$$

Here, $\hbar$ is the Plank constant, m the mass of the particle, q its charge and $U : \mathbb{R}^N \to \mathbb{R}$ some time-independent scalar potential. In the following, we will consider $m = 1/2$ and $q = -1$ for simplicity, $\hbar \equiv 1$ since it plays no role. This equation appears in non-relativistic quantum mechanics and describes the state of a particle of mass m and charge q constraint in the potential well U, in presence of an external magnetic field B, having source in A. The quantum interpretation of the wave function Ψ is probabilistic, in the sense that $|\Psi(x,t)|$ represents the probability amplitude of the particle to be in a position $x \in \mathbb{R}^N$ at a time t. It is the reason why the interesting physical states are those such that $\int_{\mathbb{R}^N} |\Psi(x,t)|^2 \,\mathrm{d}x$ is finite. For more informations, see for example [12].

The search of stationary states solutions given by

$$\Psi(x,t) = e^{-iEt}u(x), \tag{1.2}$$

leads to consider the time-independent linear Schrödinger equation

$$\left(i\nabla + A\right)^2 u + (U(x) - E)u = 0, \qquad x \in \mathbb{R}^N. \tag{1.3}$$

Those stationary states correspond to states of fixed energy E. Equation (1.3) may be seen as an eigenvalues equation, where u and E are respectively the eigenfunction and eigenvalue of the linear operator

$$\left(i\nabla + A\right)^2 + U,$$

the Hamiltonian which gives the energy of the particle.

Equation (1.3) can also be studied in an open bounded domain Ω of $\mathbb{R}^N$, where Dirichlet boundary conditions are imposed on $\partial\Omega$, that is $u = 0$ on $\partial\Omega$. This situation can be understood saying that the particle is trapped in an infinite potential well, its probability of presence being therefore zero outside Ω.

The Aharonov-Bohm effect mentioned above tells us that a charged particle confined in a region where the magnetic field is null will be influenced by the magnetic potential A_a. Therefore, the energy of such particles will be influenced by the presence of the potential, with respect to the case without magnetic potential. It is then interesting to study the spectrum of the Hamiltonian $(i\nabla + A_a)^2$, for A_a given by (1.1).

In Chapter 2, based on [1, 2], we work in a bounded and simply connected set $\Omega \subset \mathbb{R}^2$, and for any $a \in \Omega$, we consider the magnetic potential having the form (1.1). We look at the eigenvalue problem

$$\begin{cases} (i\nabla + A_a)^2 \varphi_k^a = \lambda_k^a \varphi_k^a & \text{in } \Omega \\ \varphi_k^a = 0 & \text{on } \partial\Omega. \end{cases}$$

As already said, this equation describes the motion of a charged particle, constrained in an infinite potential well, and influenced by the magnetic potential A_a. Standard spectral theory implies the existence of an increasing and diverging sequence of positive eigenvalues λ_k^a, counted with their multiplicity, and of associated eigenfunctions φ_k^a, normalized in $L^2(\Omega)$. Moreover, those eigenvalues verifies a min-max characterization.

More particularly, we are interested in studying the behaviour of the eigenvalues when the singular point a moves inside Ω and eventually approaches the boundary $\partial\Omega$. Our first result states that this map has a continuous extension up to the boundary $\partial\Omega$, and is regular inside Ω under some additional assumption.

Theorem 1.2.1. *Let $k \in \mathbb{N}_0$ and $b \in \Omega$.*

(i) The map $a \mapsto \lambda_k^a$ has a continuous extension up to the boundary $\partial\Omega$. More precisely,

$$\lambda_k^a \to \lambda_k, \qquad \text{as } a \to \partial\Omega,$$

where λ_k is the k-th eigenvalue of the Laplacian with Dirichlet boundary conditions on $\partial\Omega$.

(ii) If λ_k^b is simple, then the map $a \mapsto \lambda_k^a$ is C^∞ in a small neighbourhood of b.

The continuous extension up to the boundary may be understood from a topological point of view. When the singularity a is on $\partial\Omega$, the domain $\Omega \setminus \{a\}$ is simply connected, and equal to Ω since Ω is open. The particle will not experience any more the magnetic potential.

From this first result, we can deduce that the map $a \mapsto \lambda_k^a$ has at least one extremal point inside Ω, even if it is not necessarily differentiable. Then, we focus on the interior critical points of $a \mapsto \lambda_k^a$, and more particularly on the relation between the critical points of that map and the nodal set of the corresponding eigenfunctions. Those problems are related to some shape optimization problems. We refer to the discussion in the introduction of Chapter 2 for more details.

Next, we are interested in studying the behaviour of λ_k^a as $a \to \partial\Omega$. In that case, we need to impose some minimal regularity on Ω, and for simplicity we assume that Ω is of class C^∞. Some interesting questions are for instance to understand how fast λ_k^a converges to λ_k and what determines this rate of convergence; or does the convergence holds from above or from below. At first, we observe that $\lambda_1^a > \lambda_1$ as follows from the diamagnetic inequality. Nevertheless, this is not valid for the other eigenvalues. We can prove the two following theorems.

Theorem 1.2.2. *Let $\alpha = 1/2$, Ω be of class C^∞ and $b \in \partial\Omega$. Let φ_k, the k-th eigenfunction of the Laplacian, having $n \geq 1$ nodal lines ending at b and denote by Γ a piece of nodal line. Then, there exists $C > 0$ independent from a such that*

$$\lambda_k^a \leq \lambda_k - C|a-b|^{2n+2}, \qquad \textit{as } a \to b,\ a \in \Gamma.$$

Theorem 1.2.3. *Let $\alpha = 1/2$, Ω be of class C^∞ and $b \in \partial\Omega$. Assume that φ_k has no nodal lines ending at b. Then, there exists $C > 0$ independent from a such that*

$$\lambda_k \leq \lambda_k^a - C\left(\operatorname{dist}(a, \partial\Omega)\right)^2, \qquad \textit{as } a \to b.$$

This last theorem provides a kind of local diamagnetic inequality when a converges to a point where no nodal lines of the eigenfunction of the Laplacian ends.

1.3. Nonlinear Schrödinger equation: a semiclassical problem

The nonlinear Schrödinger equation in presence of an external magnetic field is given by

$$i\frac{\partial}{\partial t}\Psi(x,t) = (i\hbar\nabla + A)^2\,\Psi(x,t) + U(x)\Psi(x,t) - f(|\Psi|^2)\Psi(x,t), \qquad x \in \mathbb{R}^N.$$

We considered again $m = 1/2$ and $q = -1$. A typical example of non linearity is the following

$$f(|u|^2)u = \kappa|u|^{p-2}u,$$

for $p > 2$ and κ either positive or negative. When $A = 0$, this equation appears in many physical contexts. For example it provides a good description of Bose-Einstein condensates, where we can make the approximation that all the particles are in the same state, described by the wave function Ψ. The interactions between the particles of the system are modelled by some effective potential, generated by summing the contribution of all components of the system (mean field). This gives rise to the nonlinear term (in general with $p = 4$).

Another important context in which the nonlinear Schrödinger equation is used is the nonlinear optic, where it studies the propagation of a strong light in a nonlinear medium, for example in an optic fiber. The nonlinear term comes from the interaction with the medium in which the light is propagating, and is due to the fact that the refraction index depends on the intensity of the light.

Many other physical situations can be modelled or described by this equation, such as fluid mechanics where the nonlinear Schrödinger equation describes the motion of a wave envelop, plasma physics, supersolids. For more informations, we refer to [13–15].

We will concentrate on the case $\kappa = 1$ (which is equivalent to $\kappa > 0$ by a renormalization of the functions) and look for standing waves solutions (1.2). We are then reduced to study the time-independent nonlinear Schrödinger equation in presence of a magnetic field,

$$(i\hbar\nabla + A)^2 u + (U(x) - E)u = f(|u|^2)u, \qquad x \in \mathbb{R}^N.$$

We rewrite $V(x) = U(x) - E$ for simplicity.

An interesting situation to consider is the semiclassical limit of this equation. Formally, this arises when we let the parameter $\hbar \to 0$. This was first studied when $A = 0$. In that case, we can perform a rescale of the equation

$$v_\hbar(y) = u(y_0 + \hbar y)$$

around some $y_0 \in \mathbb{R}^N$. Then $v_\hbar$ solves

$$-\Delta v_\hbar + V(y_0 + \hbar y)v_\hbar = |v_\hbar|^{p-2}v_\hbar, \qquad y \in \mathbb{R}^N.$$

By making some assumptions on the potential V, let's say that V has a global and positive minimum (we stress that this hypothesis is not complete and neither the most general one. However, we do not enter in the details here since there is an extensive literature concerning semiclassical limit and the class of possible V is very large), we can deduce that the rescaled function $v_\hbar$ converges in some sense to a limit function $\bar{v}$ solving

$$-\Delta\bar{v} + V(y_0)\bar{v} = |\bar{v}|^{p-2}\bar{v}, \qquad y \in \mathbb{R}^N,$$

if y_0 is the minimal point of V. We then say that the solution u concentrates at the global minimum of V in the semiclassical limit. This can be understood since classically, a object will tend to place himself in a minimum of the potential. Later on, some semiclassical solutions concentrating around local minimum (or even other critical points) were found.

Next, we can also focus on solutions having a given symmetry. If we look for example at the radial symmetry, which is the first interesting case, with V radially symmetric, that is $V(x) = V(|x|)$, then we can exhibit solutions concentrating on a sphere. Moreover, the radius of this sphere has to be a critical point of the concentration function $\mathcal{M}(r) = r^{N-1}V^{\frac{p}{p-2}-\frac{1}{2}}$ (this was proved in [16]). Solutions concentrating around other higher dimensional sets were then studied. We remark that this discussion will be made in more details in the introduction of Chapter 3.

In presence of a magnetic field, a natural question is to ask if the location of the concentration of the semiclassical solutions will be affected by this field. In the case without any symmetries, it was showed that the semiclassical solutions concentrate around critical points of the potential V. In some sense, we can say that the semiclassical solutions do not see the magnetic potential.

However, some interesting effect arises when we look at semiclassical solutions carrying some symmetries and concentrating around higher dimensional sets. This is the situation discussed in Chapter 3, based on [3]. In that case, we show that the magnetic potential influences the location of the concentration. In Chapter 3, we consider the equation in a cylindrical setting in $\mathbb{R}^3$. Namely, we look for solutions being invariant under the rotations given by

$$g_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}, \qquad \alpha \in [0, 2\pi),$$

and concentrating around circles. We considered potentials V invariant under the same transformations, depending on $\rho \equiv (x_1^2+x_2^2)^{1/2}$ and $|x_3|$, and magnetic potentials A equivariant under those rotations, that is $g_\alpha A(g_\alpha^{-1}x) = A(x)$ for every $x \in \mathbb{R}^3$. A typical example of magnetic potential verifying that condition is $A = b\,(-x_2, x_1, 0) = b\,(-\rho\sin\theta, \rho\cos\theta, 0)$, corresponding to the uniform magnetic field $B = (0, 0, 2b)$. We prove that the radius of the circle around which the semiclassical solution is concentrating is determined by the concentration function

$$\mathcal{M}(\rho, x_3) = \rho\left(V + A_\tau^2\right)^{\frac{2}{p-2}},$$

where $A_\tau = A \cdot \mathbf{e}_\tau$ is the tangential component of A to the circle, $\mathbf{e}_\tau =$

$(-\sin\theta, \cos\theta, 0)$. If we restrict ourself to the particular case of cubic nonlinearity $p = 4$, we can for example prove the following.

Theorem 1.3.1. *Let $V(\rho) = 1/\rho^2$ and $A = b(-\rho\sin\theta, \rho\cos\theta, 0)$. Then, there exists a solution concentrating in the plane $\{x_3 = 0\}$, on a circle of radius $1/(3b^2)^{1/4}$, which corresponds to a minimum of the concentration function*

$$\mathcal{M}(\rho, x_3) = \rho\left(\frac{1}{\rho^2} + b^2\,\rho^2\right).$$

The presence of the magnetic field influences then the semiclassical limit to produce a solution concentrating on a sphere.

Chapter 2

Aharonov–Bohm operators with varying poles

2.1. Introduction

2.1.1. Aharonov-Bohm operators

Let $\Omega \subset \mathbb{R}^2$ be an open, simply connected and bounded domain. For $a = (a_1, a_2)$ varying in Ω and eventually approaching the boundary $\partial\Omega$, we consider the magnetic Schrödinger operator

$$(i\nabla + A_a)^2 = -\Delta + i\nabla \cdot A_a + 2iA_a \cdot \nabla + |A_a|^2 \tag{2.1}$$

acting on functions $u : \Omega \to \mathbb{C}$ with zero boundary conditions on $\partial\Omega$, A_a being a magnetic potential of Aharonov-Bohm type, singular at the point a. More specifically, the magnetic potential has the form

$$A_a(x) = \alpha \left(-\frac{x_2 - a_2}{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2} \right) + \nabla\chi, \tag{2.2}$$

where $x = (x_1, x_2) \in \Omega \setminus \{a\}$. The fixed constant $\alpha \in (0, 1)$ represents the circulation of A_a, while $\chi \in C^2(\Omega)$. Since the regular part χ does not play a significant role, throughout the paper we will suppose without loss of generality that $\chi \equiv 0$ (in other words, we fix the gauge, see Section 2.2 for more details).

Such magnetic vector potential is generated by an infinitesimally thin solenoid orthogonal to the plane, and the associated magnetic field $\nabla \times A_a$ is a $2\pi\alpha$-multiple of the Dirac delta distribution at a, orthogonal to the plane. The resulting magnetic Schrödinger equation describes the motion of a non relativistic, spinless and charged quantum particle constrained in an infinite potential well and interacting with the point magnetic field $\nabla \times A_a$. That particle moving in $\Omega \setminus \{a\}$ will be affected by the magnetic potential, although it remains in a region where the magnetic field is zero. This is the Aharonov-Bohm effect, see [6]. We can also think at the particle as being affected by the non-trivial topology of the set $\Omega \setminus \{a\}$.

Two natural spaces to study the operator (2.1) are given by $\mathcal{D}^{1,2}_{A_a}(\Omega)$ and $H^1_{A_a}(\Omega)$, respectively with and without vanishing boundary conditions on $\partial\Omega$. More precisely, $\mathcal{D}^{1,2}_{A_a}(\Omega)$ is defined as the completion of $C_0^\infty(\Omega\setminus\{a\},\mathbb{C})$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}_{A_a}(\Omega)} := \|(i\nabla + A_a)u\|_{L^2(\Omega)},$$

while $H^1_{A_a}(\Omega)$ is defined as the completion of the set

$$\{u \in C^\infty(\Omega,\mathbb{C})\,,\ u \text{ vanishes in a neighborhood of } a\}$$

with respect to the norm

$$\|u\|_{H^1_{A_a}(\Omega)} := \left(\|(i\nabla + A_a)u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)}\right)^{1/2}.$$

We recall several properties of those spaces in Section 2.2.

The domain of the operator (2.1) will be one of the two previously defined function spaces. The operator, with values into its dual space, is the Riesz one and is obviously self-adjoint: for any pair of functions $u, v \in \mathcal{D}^{1,2}_{A_a}(\Omega)$, $(i\nabla + A_a)^2 u \in \left(\mathcal{D}^{1,2}_{A_a}(\Omega)\right)'$ (or respectively for $H^1_{A_a}(\Omega)$) and

$$\left\langle (i\nabla + A_a)^2 u, v\right\rangle := \int_\Omega (i\nabla + A_a)u \cdot \overline{(i\nabla + A_a)v}.$$

We are interested in the following weighted eigenvalue problem

$$(i\nabla + A_a)^2 \varphi_k^a = \lambda_k^a\, p(x)\varphi_k^a, \quad \text{with} \quad \varphi_k^a \in \mathcal{D}^{1,2}_{A_a}(\Omega). \tag{2.3}$$

In Sections 2.3, 2.4, 2.5 and 2.6, we will consider

$$p(x) \equiv 1,$$

while in Section 2.7, in which we make the additional assumption that Ω is of class C^∞, p satisfies

$$p \in C^\infty(\overline{\Omega}), \quad p(x) > 0,\ x \in \Omega. \tag{2.4}$$

The weight $p(x)$ (when different from 1) has the role of incorporating the curvature of the domain when the boundary is locally flattened by a conformal change of variables, as we will see in Section 2.7. Of course, our main interest is the case $p(x) \equiv 1$. By standard spectral theory, and by the self-adjointness of the operator, the spectrum of the operator consists of an increasing positive sequence of eigenvalues, of finite multiplicity and counted with their multiplicity (for more details, see Section 2.2). We will denote those eigenvalues by λ_k^a, $k \in \mathbb{N}_0$, and the corresponding eigenfunctions, normalized in the $L^2(\Omega, p)$-norm, by φ_k^a. We will reserve the notation λ_k, φ_k for the eigenvalues and eigenfunctions of the Laplacian with weight $p(x)$ in Ω and with zero boundary conditions (again increasing and counted with their multiplicity).

At first, as a consequence of the diamagnetic inequality, the first eigenvalue is increased by the presence of the magnetic potential (see [8, 17, 18]). Indeed, since the first eigenvalue is defined by an infimum, the addition of a magnetic potential immediately affects its value by increasing it. This is not the case for higher eigenvalues, defined by a min-max (for more details, see Section 2.2), which show a more complex behaviour, which has been numerically detected and depicted in Figure 2.1. In that figure, the angular sector $\Sigma_{\pi/4}$ of aperture $\pi/4$ is represented with a dark thick line. Outside the angular sector are represented the eigenvalues λ_k of the Dirichlet-Laplacian on $\Sigma_{\pi/4}$ (which do not depend on a). As expected, the first eigenvalue presents a maximum, and its minimal value is attained on $\partial\Sigma_{\pi/4}$ by λ_1.

Such a rich structure calls for some theoretical explanation. In the following, we shall focus our attention on the extremal and critical points of the maps $a \mapsto \lambda_k^a$.

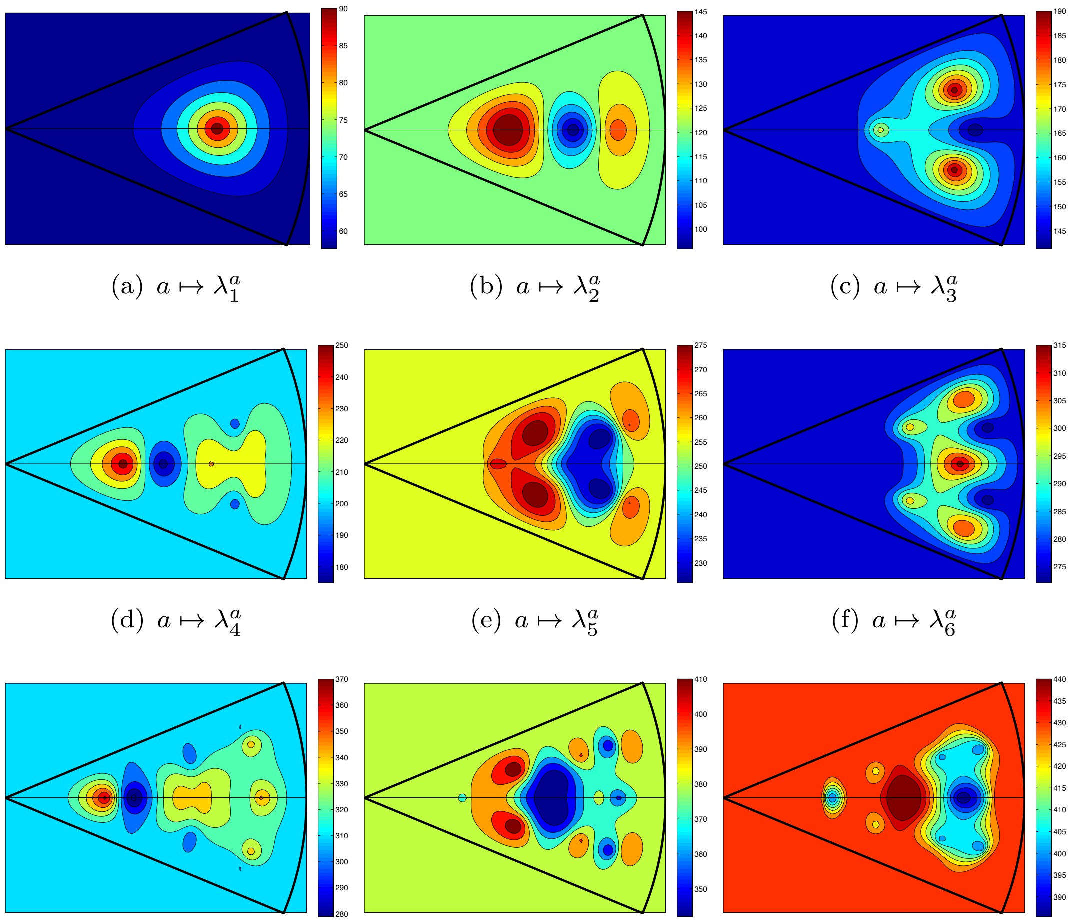
2.1.2. Motivations: spectral minimal partitions

A first motivation is, as said before, to understand how the presence of an Aharonov-Bohm potential can affect the spectrum of the operator (2.1).

Another important motivation is that the particular case of half-integer circulation, $\alpha = 1/2$, gives rise to a partition problem. In general, the nodal set of a complex regular function u, defined as

$$\mathcal{N}(u) := \overline{\{x \in \Omega \,:\, u(x) = 0\}},$$

is made of points since it corresponds to the intersection of the nodal sets of the real and the imaginary parts of u. However, in the particular case of half-integer circulation, the nodal set of the eigenfunctions φ_k^a of the magnetic operator (2.1) is composed of smooth curves and of clustering points (where the curves meet), determining then a partition of the domain Ω, in the same way that the nodal set of the eigenfunctions φ_k of the Laplacian determines a partition of the domain. Moreover, this set $\mathcal{N}(\varphi_k^a)$ can be characterized more precisely.



(a) $a \mapsto \lambda_1^a$ (b) $a \mapsto \lambda_2^a$ (c) $a \mapsto \lambda_3^a$

(d) $a \mapsto \lambda_4^a$ (e) $a \mapsto \lambda_5^a$ (f) $a \mapsto \lambda_6^a$

(g) $a \mapsto \lambda_7^a$ (h) $a \mapsto \lambda_8^a$ (i) $a \mapsto \lambda_9^a$

Figure 2.1.: First nine eigenvalues of $(i\nabla + A_a)^2$ for a varying in $\Sigma_{\pi/4}$.

Roughly speaking, we can say that, away from the singularity a, φ_k^a behaves in the same way as an eigenfunction of the Laplacian, while there is always an odd number of half-lines meeting at the point a. This last property is inherent to the magnetic operator, since in the case of the Laplacian, there is always an even number of nodal lines meeting at the clustering points. Therefore, critical positions of the moving pole can be related to optimal partition problems.

We first introduce briefly the notion of spectral minimal partition, for more details see for example [19–25]. For $k \in \mathbb{N}_0$, we define an open, connected k-partition of Ω as a family $D = (D_1, \ldots, D_k)$ such that the D_i are open and connected for $i = 1, \ldots, k$, $D_i \cap D_j = \emptyset$ for $i \neq j$ and $\cup_{i=1}^k D_i \subset \Omega$. We call $\mathcal{D}_k$ the set of all open, connected k-partitions. For a given partition $D \in \mathcal{D}_k$, we consider

$$\Lambda(D) := \max_{i=1\cdots k} \lambda_1(D_i),$$

where $\lambda_1(D_i)$ is the first eigenvalue of the Dirichlet-Laplacian on D_i. Then, we take the infimum over all the open, connected k-partitions

$$\mathcal{L}_k(\Omega) := \inf_{D\in\mathcal{D}_k} \Lambda(D).$$

The spectral minimal k-partition is the partition $D \in \mathcal{D}_k$ which achieves $\mathcal{L}_k(\Omega)$.

The existence of such partition has been proved in [26–28]. The relation between spectral minimal partitions and nodal domains of eigenfunctions has been investigated in full detail in [19–21, 25, 29–31]. By the results in [19], in two dimensions, the boundary of a spectral minimal partition is the union of finitely many regular arcs, meeting at some multiple intersection points and equally dividing the angle. Moreover, if the number of half-lines meeting at such intersection point is even, it was proved that the partition is nodal, i.e. it is the nodal set of an eigenfunction of the Dirichlet-Laplacian corresponding to the eigenvalue $\mathcal{L}_k(\Omega)$.

On the other hand, the results in [22–24, 30, 32] suggest that the minimal partitions featuring one, or more, clustering point of odd multiplicity should be related to the nodal domains of eigenfunctions of Aharonov-Bohm Hamiltonians which corresponds to a critical value of the eigenvalue with respect to the moving pole. In particular, in [30, Theorem 5.1], the authors proved the following. They considered a bounded, not necessarily simply connected domain $\Omega \subset \mathbb{R}^2$, with piecewise C^1 boundary and which presents m holes B_i, $i = 1, \dots, m$. Some singularities $a_1, \dots, a_l$ are disposed either in Ω or in the B_i, with at most one singularity in each hole. If it exists, they denoted by $L_k(\Omega)$ the lowest eigenvalue of the magnetic operator with l poles (for the precise definition, see their article) such that the corresponding eigenfunction exhibits k nodal domains. Otherwise, $L_k(\Omega) = +\infty$. They proved the following characterization for the spectral minimal k-partition

$$\mathcal{L}_k(\Omega) = \inf_{l\in\mathbb{N}} \inf_{a_1,\cdots,a_l} L_k(\Omega).$$

2.1.3. Main results

Our first result states the continuity of the magnetic eigenvalues with respect to the position of the singularity, up to the boundary.

Theorem 2.1.1. *For every $k \in \mathbb{N}_0$ and for every $\alpha \in (0,1)$, the function $a \in \Omega \mapsto \lambda_k^a \in \mathbb{R}$ admits a continuous extension on $\overline{\Omega}$. More precisely, as $a \to \partial\Omega$, we have that λ_k^a converges to λ_k, the k-th eigenvalue of $-\Delta$ in $H^1_0(\Omega)$.*

As an immediate consequence of this result, we have that the map $a \mapsto \lambda_k^a$, being continuous in $\overline{\Omega}$ and constant on $\partial\Omega$, always admits an interior extremal point.

Corollary 2.1.2. *For every $k \in \mathbb{N}_0$ and for every $\alpha \in (0,1)$, the function $a \in \Omega \mapsto \lambda_k^a \in \mathbb{R}$ has an extremal point in Ω.*

Heuristically, we can interpret the previous theorem thinking at a magnetic potential A_b, singular at $b \in \partial\Omega$. The domain $\Omega \setminus \{b\}$ coincides with Ω, so that it has a trivial topology. For this reason, the magnetic potential is not experienced by a particle moving in Ω and the operator acting on the particle is simply the Laplacian.

This result was first conjectured in the case $k = 1$ in [32], where it was applied to show that the function $a \mapsto \lambda_1^a$ has a global interior maximum, where it is not differentiable, corresponding to an eigenfunction of multiplicity exactly two. Numerical simulations in [23] supported the conjecture for every $k \in \mathbb{N}_0$, as well as the simulations presented in Figure 2.1. We note that the continuity of the eigenvalues with respect to multiple moving poles has been obtained independently in [33].

We remark that the continuous extension up to the boundary is a non-trivial issue because the nature of the operator changes as a approaches $\partial\Omega$. This fact can be seen in the more specific case $\alpha = 1/2$, which is equivalent to the standard Laplacian on the double covering defined as $\left\{(x,y) \in \mathbb{C}^2 \,|\, y^2 = x - a,\, x \in \Omega\right\}$, where we identify $\mathbb{R}^2$ and $\mathbb{C}$ (see [32, 34, 35] and Section 2.2). We can then consider either a problem on a fixed domain Ω, with a varying operator $(i\nabla + A_a)^2$ (which depends on the singularity a), or a problem with a fixed operator, the Laplacian and a varying domain, the double covering. The study of the continuity of the eigenvalues λ_k^a, when a moves in Ω, is equivalent to the study of the continuity of the eigenvalues of an elliptic operator when the domain is varying (for the convergence of the eigenvalues of elliptic operators on varying domains, we refer to [36, 37]). In this second case, the singularity is transferred from the operator into the domain. Indeed, when a approaches the boundary, the double covering develops a corner at the origin. In particular, [21, Theorem 7.1] cannot be applied in our case since there is no convergence in capacity of the domains.

In the light of the previous corollary it is natural to study additional properties of the interior extremal points of the map $a \mapsto \lambda_k^a$. Our aim is to establish a relation between the nodal properties of φ_k^b and the vanishing order of $|\lambda_k^a - \lambda_k^b|$ as $a \to b$. First of all we will need some additional regularity, which is guaranteed by the following theorem in case of simple eigenvalues. Before stating the theorem, we remark that by simple eigenvalue we mean that the eigenvalue is associated to an eigenspace of complex dimension 1. Then, if φ and ψ are two normalized eigenfunctions associated to λ, there exists a constant $\beta \in \mathbb{R}$ such that $\varphi = e^{i\beta}\psi$.

Theorem 2.1.3. *Let $b \in \Omega$ and $\alpha \in (0,1)$. If λ_k^b is simple, then, for every $k \in \mathbb{N}_0$, the map $a \in \Omega \mapsto \lambda_k^a$ is locally of class C^∞ in a neighbourhood of b.*

In order to examine the link with the nodal set of the eigenfunctions, we shall focus on the case $\alpha = 1/2$. In this case, it was proved in [32, 34, 35] (see also Proposition 2.2.10 below) that the eigenfunctions φ_k^a have an odd number of nodal lines ending at the pole a and an even number of nodal lines meeting at zeros different from a. We say that an eigenfunction has an interior zero of order $h/2$ at a point if it has h nodal lines meeting at such point. More precisely, we give the following definition.

Definition 2.1.4 (Interior zero of order $h/2$)**.** *Let $f : \Omega \to \mathbb{C}$, $b \in \Omega$ and $h \in \mathbb{N}_0$.*

(i) If h is even, we say that f has a zero of order $h/2$ at b if it is of class at least $C^{h/2}$ in a neighbourhood of b and $f(b) = \ldots = D^{h/2-1}f(b) = 0$, while $D^{h/2}f(b) \neq 0$.

(ii) If h is odd, we say that f has a zero of order $h/2$ at b if $f(x^2 + b)$ has a zero of order h at 0 (here x^2 is the complex square where we associated $\mathbb{R}^2$ and $\mathbb{C}$).

The following result is proved in [32].

Theorem 2.1.5 ([32, Theorem 1.1])**.** *Suppose that $\alpha = 1/2$. Fix any $k \in \mathbb{N}_0$. If φ_k^b has a zero of order $1/2$ at $b \in \Omega$ then either λ_k^b is not simple, or b is not a critical point of the map $a \mapsto \lambda_k^a$.*

Corollary 2.1.6. *By joining this with Corollary 2.1.2, we find that there is at least one interior point (critical with respect to the k-th eigenvalue) enjoying an alternative between degeneracy of the corresponding eigenvalue and the presence of a triple (or multiple) point nodal configuration.*

Under the assumption that λ_k^b is simple, we prove here that the converse of Theorem 2.1.5 also holds. In addition, we show that the number of nodal lines of φ_k^b at b determines the order of vanishing of $|\lambda_k^b - \lambda_k^a|$ as $a \to b$.

Theorem 2.1.7. *Suppose that $\alpha = 1/2$. Fix any $k \in \mathbb{N}_0$. If λ_k^b is simple and φ_k^b has a zero of order $h/2$ at $b \in \Omega$, with $h \geq 3$ odd, then*

$$|\lambda_k^a - \lambda_k^b| \leq C|a-b|^{(h+1)/2} \qquad \textit{as } a \to b, \tag{2.5}$$

for a constant $C > 0$ independent from a.

In conclusion, in case of half-integer circulation we have the following picture, which completes Corollary 2.1.2.

Corollary 2.1.8. *Suppose that $\alpha = 1/2$. Fix any $k \in \mathbb{N}_0$. If $b \in \Omega$ is a critical point of $a \mapsto \lambda_k^a$ then either λ_k^b is not simple, or φ_k^b has a zero of order $h/2$ at b, $h \geq 3$ odd. In this second case, the first $(h-1)/2$ terms of the Taylor expansion of λ_k^a at b cancel.*

When the order of the eigenfunction is at least $3/2$, the corresponding nodal set determines a regular partition of the domain, in the sense of [19], where such a notion has been introduced and linked with the properties of boundaries of spectral minimal partitions. It is interesting to connect the variational properties of the partition with the characterization of the pole a as a critical point of the map $a \mapsto \lambda_k^a$. To this aim, V. Bonnaillie-Noël performed a number of numerical computations. Rather surprisingly, the triple (or multiple) point configurations in the angular sector $\Sigma_{\pi/4}$ never appear in correspondence of the maximum or minimum values of the eigenvalues, which are always non-differentiability points, that is corresponding to degenerate eigenvalues. We observe in particular that any triple point configuration corresponds to a degenerate saddle point as illustrated in Figure 2.2, see also Figure 2.5. This fact could be interesting to understand and we will come back to it later.

We note that in a recent paper [38], the authors improved our result and obtained a sharp estimate for $\lambda_k^b - \lambda_k^a$, as $a \to b$, when a is moving on a half-line tangent to one of the nodal lines of φ_k^b ending at b.

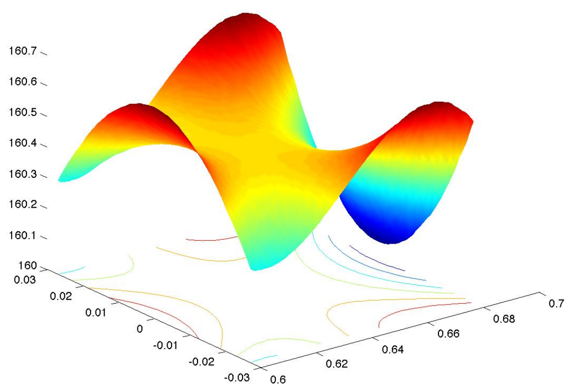
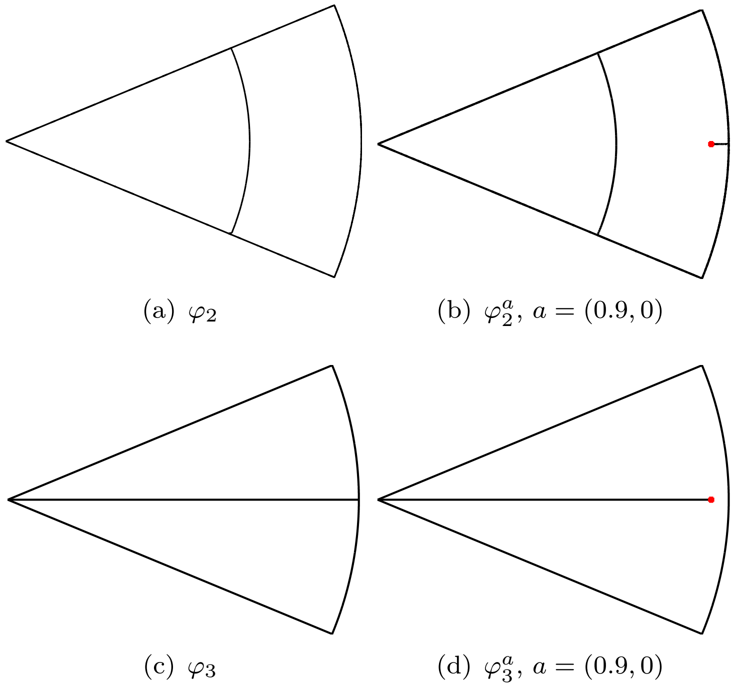


Figure 2.2.: $a \mapsto \lambda_3^a$, $a \in \left\{\left(\frac{m}{1000}, \frac{n}{1000}\right), 600 \leq m \leq 680, 0 \leq n \leq 30\right\}$.

We would like to mention that the relation between the presence of a magnetic field and the number of nodal lines of the eigenfunctions, as well as the consequences on the behaviour of the eigenvalues, have been recently studied in different contexts, giving rise to surprising conclusions. In [39, 40] the authors consider a magnetic Schrödinger operator on graphs and study the behaviour of its eigenvalues as the circulation of the magnetic field varies. In particular, they consider an arbitrary number of singular poles, having circulation close to 0. They prove that the simple eigenvalues of the Laplacian (zero circulation) are critical values of the function $\alpha \mapsto \lambda_k(\alpha)$, which associates to the circulation α the corresponding eigenvalue. In addition, they show that the number of nodal lines of the Laplacian eigenfunctions depends on the Morse index of $\lambda_k(0)$.

After the analysis of the interior critical points of the map $a \mapsto \lambda_k^a$, we are interested in studying the behaviour of the magnetic eigenvalues as the pole approaches the boundary of the domain. In that case, we have to assume more regularity on our domain. For simplicity, we choose Ω of class C^∞. Our first aim is to obtain an expression similar to (2.5) when $b \in \partial\Omega$. Moreover, some phenomenon enlightened by the numerical simulations attracted our attention. We see that the convergence $\lambda_k^a \to \lambda_k$ as $a \to \partial\Omega$ described in Theorem 2.1.1 can take place either from above or from below, depending on the value of k and on the position of the pole, see Figure 2.5. Of course, by the diamagnetic inequality, $\lambda_1^a > \lambda_1$ for every $a \in \Omega$. A more detailed analysis suggests that the different behaviours are related to the position of the pole a with respect to the nodal lines of φ_k. If the pole a moves from $\partial\Omega$ along a nodal line of φ_k, then the nodal line of φ_k^a should be shorter than the one of φ_k, locally for a close to the boundary, see Figures 2.3(c) and 2.3(d). Heuristically, this determines a decrease of the energy since φ_k is forced to cancel on a piece of line in which φ_k^a is not. Indeed, we can observe in Figure 2.4(b) and 2.5 that $\lambda_3^a < \lambda_3$ for a approaching the point $(1, 0) \in \partial\Omega$ along the symmetry axis of the circular sector. Conversely, if a moves from $\partial\Omega$ not on a nodal line of φ_k as in Figures 2.3(a)-2.3(b), this creates a new nodal line in the magnetic eigenfunction and consequently an increase of the energy, as we can see in Figures 2.4(a) and 2.5, in which $\lambda_2^a > \lambda_2$ for a approaching the point $(1, 0) \in \partial\Omega$ along the symmetry axis of the circular sector.

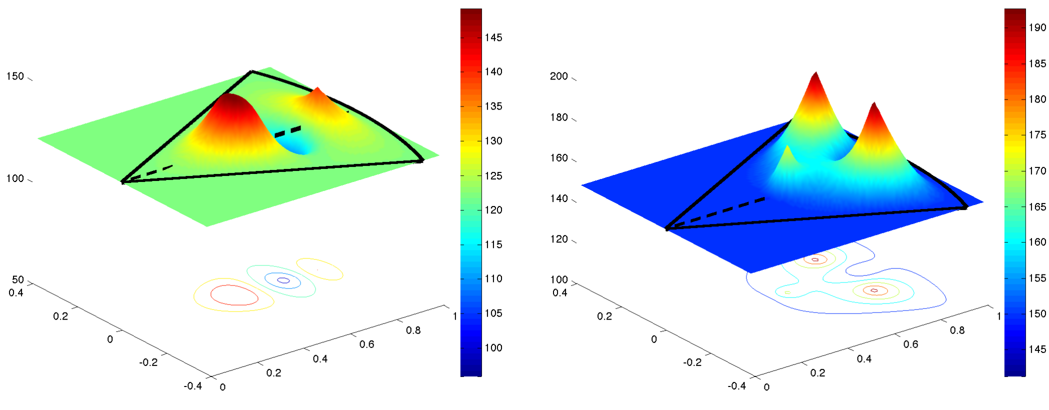
Remark 2.1.9. *We remark that this idea about the increase or the decrease of the energy depending on the position of the pole can be an attempt to explain why the triple point configurations in $\Sigma_{\pi/4}$ appear only for saddle points and never for minimum or maximum points, as observed before. Again, if a moves from b on a nodal line of φ_k^b, the nodal configuration of φ_k^a should present a piece of nodal line missing with respect to φ_k^b, and then produce a decrease of the energy. While if a moves from b on the bisector of two nodal lines of φ_k^b, we would expect a creation of a piece of nodal line and an increase of the energy.*



(a) φ_2 (b) φ_2^a, $a = (0.9, 0)$

(c) φ_3 (d) φ_3^a, $a = (0.9, 0)$

Figure 2.3.: Nodal lines of second and third eigenfunctions in $\Sigma_{\pi/4}$.



(a) $a \mapsto \lambda_2^a$ (b) $a \mapsto \lambda_3^a$

Figure 2.4.: 3-D Representation of the map $a \mapsto \lambda_k^a$ in $\Sigma_{\pi/4}$, for $k = 2, 3$.

When the zero is of order $3/2$, we obtain three directions where λ_k^a is increasing with respect to λ_k^b, and three opposite directions in which it decreases. This gives a saddle-point profile, as observed in the simulations.

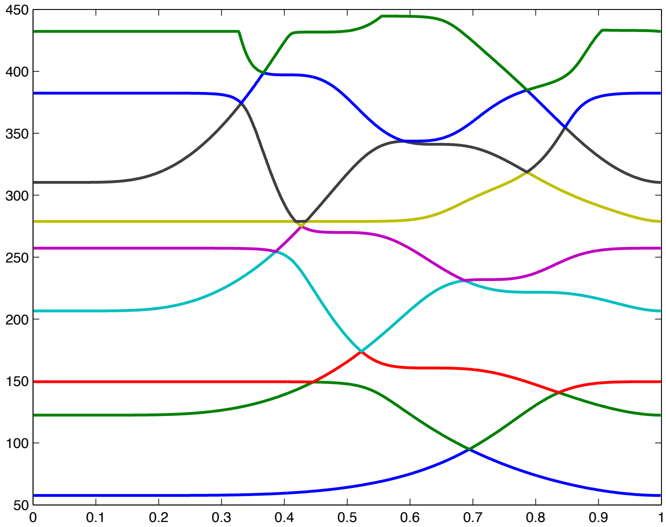


Figure 2.5.: $a \mapsto \lambda_k^a$, a belonging to the symmetry axis of $\Sigma_{\pi/4}$, for $k = 1, \dots, 9$.

The second aim is to provide a theoretical justification of these facts. Before stating our main results, let us give the definition of order of vanishing at a boundary point $b \in \partial\Omega$ of a function u, with $u = 0$ on $\partial\Omega$. As we will see, this definition makes sense only if $\partial\Omega$ is sufficiently regular (see the case of conical singularities in Appendix A.1).

Definition 2.1.10 (Boundary zero of order $h/2$)**.** *Let $\Omega \subset \mathbb{R}^2$ be open, bounded and of class C^∞. Assume that $u : \Omega \to \mathbb{C}$, $u = 0$ on $\partial\Omega$, $b \in \partial\Omega$ and h even.*

We say that u has a zero of order $h/2$ at b if there exists a neighbourhood $U(b)$ such that $u \in C^{h/2}(U(b) \cap \Omega)$ and $u(b) = \ldots = D^{h/2-1}u(b) = 0$ while $D^{h/2}u(b) \neq 0$ in $U(b) \cap \Omega$.

Note that, whereas when $b \in \Omega$ a zero of order $h/2$ corresponds to h arcs of nodal lines meeting at b, for $b \in \partial\Omega$ a zero of order $h/2$ corresponds to $h/2 - 1$ arcs of nodal lines meeting at b. This is due to the fact that we are considering zero boundary conditions. The following theorem explicit the behaviour of the eigenvalues when the singularity moves on a nodal line of the eigenfunction of the Laplacian.

Theorem 2.1.11. *Let $\Omega \subset \mathbb{R}^2$ be open, bounded, simply connected and of class C^∞ and $\alpha = 1/2$. Let p satisfy (2.4). Suppose that $\lambda_{k-1} < \lambda_k$ and that there exists an eigenfunction φ_k associated to λ_k having a zero of order $h/2 \geq 2$ at $b \in \partial\Omega$, i.e. at least one piece of nodal line ending at b. Denote by Γ any such* *piece of nodal line. Then there exists* $C > 0$*, not depending on* a*, such that*

$$\lambda_k^a \leq \lambda_k - C|a-b|^h \quad \text{for } a \in \Gamma,\ a \to b. \tag{2.6}$$

The study of the case when the pole approaches $\partial\Omega$ at a point where no nodal lines of φ_k end requires additional work. The difficulty is that, in order to prove the opposite inequality with respect to (2.6), we need some information about the behaviour of φ_k^a when a is close to the boundary. In this direction, we prove the uniqueness of the following limit profile.

Proposition 2.1.12. *Let* $e = (1,0)$ *and* $\alpha = 1/2$*. Let* $\psi \in H^1_{A_e,loc}(\mathbb{R}^2_+)$ *be a solution to*

$$\begin{cases} (i\nabla + A_e)^2\psi = 0 & \mathbb{R}^2_+ \\ \psi = 0 & \{x_1 = 0\}, \end{cases} \tag{2.7}$$

satisfying the normalization condition

$$\lim_{r\to+\infty} \frac{r\|(i\nabla + A_e)\psi\|^2_{L^2(D_r^+(0))}}{\|\psi\|^2_{L^2(\partial D_r^+(0))}} = 1, \tag{2.8}$$

where $\mathbb{R}^2_+ := \mathbb{R}^2 \cap \{x_1 > 0\}$ *and* $D_r^+(0) := D_r(0) \cap \{x_1 > 0\}$*. Then*

(i) ψ *is unique up to a complex multiplicative constant;*

(ii) for $r > 1$ *we have*

$$\psi(r,\theta) = Ce^{i\theta_e/2}\left(r\cos\theta - \frac{\beta}{\pi}\frac{\cos\theta}{r} + \sum_{n\geq 3, n\ odd} \frac{b_n}{r^n}\cos(n\theta)\right),$$

where $\beta > 0$ *is explicitly characterized in* (2.106) *and* $b_n \in \mathbb{R}$*.*

Here, r, θ are the polar coordinates around 0. Note that $\psi \in H^1_{A_e,\mathrm{loc}}(\mathbb{R}^2_+)$ if for every bounded, open set $B \subset \mathbb{R}^2_+$ such that $e \in B$, we have that $\psi \in H^1_{A_e}(B)$.

Once the uniqueness is established, we can prove that this limit profile provides a good description of the asymptotic behaviour of φ_k^a. In the following theorem, we locally flatten the boundary of Ω near the boundary point through a conformal transformation (see Section 2.7.1 for more details), which allows us to work in half-balls. Then we perform a normalized blow up, which converges to the previous profile on the half-space in the limit.

Theorem 2.1.13. *Let* $\alpha = 1/2$*. Let* $\Omega \subset \mathbb{R}^2$ *be open, bounded, simply connected and of class* C^∞*. Let* p *satisfy* (2.4)*. Suppose that* φ_k *has a zero of order* 1 *at* $b \in \partial\Omega$ *(where no nodal lines end).*

Let Φ be a conformal map such that $\Phi^{-1} \in C^\infty(\overline{\Omega})$, $\Phi^{-1}(b) = 0$ and, for some small $r > 0$,

$$\Phi^{-1}(\Omega) \cap D_r(0) = \{x \in D_r(0) : x_1 > 0\} =: D_r^+(0).$$

Then there exists $K > 1$ such that, denoting by $a' = (a'_1, a'_2) = \Phi^{-1}(a)$ and

$$\psi_k^a(y) = \frac{\sqrt{Ka'_1}}{\|\varphi_k^a \circ \Phi\|_{L^2(\partial D^+_{Ka'_1}(0,a'_2))}} \varphi_k^a(\Phi(a'_1 y_1, a'_1 y_2 + a'_2))$$

we have

$$\psi_k^a \to \psi \quad \text{in } H^1_{A_e,loc}(\mathbb{R}^2_+) \text{ as } a \to b,$$

where $e = (1, 0)$ and ψ is the unique solution to (2.7)-(2.8) with multiplicative constant C given explicitely in (2.137).

The previous two results are obtained by exploiting an Almgren-type frequency formula [41, 42] for magnetic eigenfunctions, see Definition 2.7.7. This tool has been introduced in the context of magnetic operators in [7] to obtain, among other results, sharp regularity results for Aharonov-Bohm eigenfunctions.

The asymptotic analysis above allows us to prove the following result.

Theorem 2.1.14. *Let $\Omega \subset \mathbb{R}^2$ be open, bounded, simply connected and of class C^∞. Let $\alpha = 1/2$ and p satisfy (2.4). Suppose that λ_k is simple and that φ_k has a zero of order 1 at $b \in \partial\Omega$ (where no nodal lines end). Then there exists $C > 0$, not depending on a, such that*

$$\lambda_k^a \geq \lambda_k + C\,(dist(a, \partial\Omega))^2 \quad \text{as } a \to b. \tag{2.9}$$

Finally, we succeed to prove an exact asymptotic in the case where there is no nodal lines ending at the boundary point.

Theorem 2.1.15. *Let $\Omega \subset \mathbb{R}^2$ be open, bounded, simply connected and of class C^∞. Let $\alpha = 1/2$ and p satisfy (2.4). Suppose that λ_k is simple and that φ_k has a zero of order 1 at $b \in \partial\Omega$ (where no nodal lines end). Then the following asymptotic holds*

$$\lim_{|a-b|\to 0} \frac{\lambda_k^a - \lambda_k}{(\mathrm{dist}(a, \partial\Omega))^2} = (\nabla\varphi_k(b) \cdot \nu)^2\,\beta,$$

where ν is the normal at $b \in \partial\Omega$ and β is defined in (2.104).

Remark 2.1.16. *(i) Theorem 2.1.14 establishes a local diamagnetic spectral inequality when the pole approaches the boundary away from the nodal lines of the eigenfunction of the Dirichlet-Laplacian.*
(ii) The exact behaviour as $a \to b \in \partial\Omega$, b being the endpoint of one or more nodal lines of φ_k, but a not belonging to any such nodal line, remains an open problem. However, when $b \in \partial\Omega$ is such point, we expect that λ_k^a would increase when a is moving from b on the bisector of two nodal lines of the Dirichlet-Laplacian, or on the bisector of one nodal line and the boundary line.
(iii) The regularity assumption on Ω can be weakened: it is enough to have $\partial\Omega \in C^{2,\gamma}$ for some $\gamma > 0$, see Remark 2.7.2.
(iv) If Ω presents a conical singularity, estimates (2.6) and (2.9) do not hold at the vertex of the cone. This can be observed in the numerical simulations: we see in Figure 2.4 that the curve $a \mapsto \lambda_k^a$ is flat as a approaches the acute angle of the circular sector. Concerning for example λ_2^a, we see that relation (2.9) does not hold, despite the absence of nodal lines of φ_2. We treat this topic in Appendix A.1. If we particularize Theorem A.1.1 to the case of the second eigenvalue in the cone of aperture $\pi/4$, we obtain for example

$$\lambda_2^a \geq \lambda_2 + C|a|^8,$$

as a moves along the angle bisector.

2.1.4. Organisation of the chapter

This chapter is organized as follows. In Section 2.2, we establish more properties of the two functional space $\mathcal{D}^{1,2}_{A_a}(\Omega)$ and $H^1_{A_a}(\Omega)$. We also recall an Hardy-type inequality and a theorem about the regularity of the eigenfunctions φ_k^a. Finally, in the case of half-integer circulation, we recall the equivalence between the problem involving an Aharonov-Bohm type potential (2.2) in Ω and the standard Laplacian problem in the double covering. This equivalence leads us to characterize the asymptotic behaviour of the eigenfunctions φ_k^a around zeros of order $h/2$. The first part of Theorem 2.1.1, concerning the interior continuity of the eigenvalues λ_k^a is proved in Section 2.3 and the second part concerning the extension to the boundary is studied in Section 2.4. In Section 2.5, we prove Theorem 2.1.3. Section 2.6 contains the proof of Theorem 2.1.7. The aim of Section 2.7 is to prove Theorems 2.1.11 and 2.1.14. For this we first use a Riemann mapping theorem in order to flatten locally the boundary $\partial\Omega$ around the point b, see Subsection 2.7.1. By doing this, we recover an equation similar to (2.3) as the new weight verify the same assumptions as the old one, see (2.4), thanks to the regularity hypothesis on $\partial\Omega$. Then, we prove some Poincaré-type inequalities in half balls. Theorem 2.1.11 is proved in Subsection 2.7.2. To prove Theorem 2.1.14, we need to do some more work as explained before. First, we introduce in Subsection 2.7.3 the Almgren's frequency formula for the eigenfunctions φ_k^a and we study some of its properties. Next, in Subsection 2.7.4 we prove Proposition 2.1.12. Subsection 2.7.5 is devoted to the proof of Theorem 2.1.14. Finally, Section 2.8 presents some numerical simulations in the case of the angular sector $\Sigma_{\pi/4}$ and the square which illustrate the previous results.

2.2. Preliminaries

2.2.1. Functional settings

First, we give an Hardy-type inequality involving Aharonov-Bohm potentials of the form (2.2).

Lemma 2.2.1 (Hardy-type inequality, [8, Theorem 2], [17, Lemma 7.4])**.** *Let $\Omega \subset \mathbb{R}^2$ be simply connected and of class C^∞ with $a \in \Omega$. Let A_a be as in (2.2). There exists a constant $C > 0$ such that, for every $u \in H^1_{A_a}(\Omega)$, the following inequality holds*

$$\left\| \frac{u}{|x-a|} \right\|_{L^2(\Omega)} \leq C \left(\min_{k \in \mathbb{Z}} |k - \alpha| \right)^{-2} \|(i\nabla + A_a)u\|_{L^2(\Omega)}. \tag{2.10}$$

We notice that the constant depends on the circulation of the magnetic potential A_a and remains finite whenever the circulation is not an integer. As a reference on Aharonov-Bohm operators we cite [43]. This inequality was first obtained by Laptev and Weidl [8, Theorem 2] for functions in $\mathcal{D}^{1,2}_{A_a}(\Omega)$ (and for any domain Ω) and has been extended to functions in $H^1_{A_a}(\Omega)$ in [17, Lemma 7.4] (see also [18]).

As already said in the introduction, if $\Omega \subset \mathbb{R}^2$ is bounded and simply connected, the space $\mathcal{D}^{1,2}_{A_a}(\Omega)$ is defined as the completion of $C_0^\infty(\Omega \backslash \{a\}, \mathbb{C})$ with respect to the norm

$$\|u\|_{\mathcal{D}^{1,2}_{A_a}(\Omega)} := \|(i\nabla + A_a)u\|_{L^2(\Omega)},$$

while $H^1_{A_a}(\Omega)$ is defined as the completion of the set

$$\{u \in C^\infty(\Omega, \mathbb{C}), \, u \text{ vanishes in a neighborhood of } a\}$$

with respect to the norm

$$\|u\|_{H^1_{A_a}(\Omega)} := \left(\|(i\nabla + A_a)u\|^2_{L^2(\Omega)} + \|u\|^2_{L^2(\Omega)} \right)^{1/2}.$$

We remark that those Hilbert spaces are considered as complex spaces.

It was proved for example in [32, Lemma 2.1] that those spaces admit the equivalent characterizations

$$\mathcal{D}^{1,2}_{A_a}(\Omega) = \left\{ u \in H^1_0(\Omega) \, : \, \frac{u}{|x-a|} \in L^2(\Omega) \right\}$$

and

$$H^1_{A_a}(\Omega) = \left\{ u \in H^1(\Omega) \, : \, \frac{u}{|x-a|} \in L^2(\Omega) \right\},$$

and moreover we have that $\mathcal{D}^{1,2}_{A_a}(\Omega)$ (respectively $H^1_{A_a}(\Omega)$) is continuously embedded in $H^1_0(\Omega)$ (respectively $H^1(\Omega)$) : there exists a constant $C > 0$ such that for every $u \in \mathcal{D}^{1,2}_{A_a}(\Omega)$ or $u \in H^1_{A_a}(\Omega)$ we have

$$\|u\|_{H^1_0(\Omega)} \leq C\|u\|_{\mathcal{D}^{1,2}_{A_a}(\Omega)} \quad \text{or} \quad \|u\|_{H^1(\Omega)} \leq C\|u\|_{H^1_{A_a}(\Omega)}. \tag{2.11}$$

This is proved by making use of the Hardy-type inequality (2.10).

As a consequence of the continuous embedding, we have the following.

Lemma 2.2.2. *Let Im be the compact immersion of $\mathcal{D}^{1,2}_{A_a}(\Omega)$ into its dual $(\mathcal{D}^{1,2}_{A_a}(\Omega))'$. Then, the operator $((i\nabla + A_a)^2)^{-1} \circ Im : \mathcal{D}^{1,2}_{A_a}(\Omega) \to \mathcal{D}^{1,2}_{A_a}(\Omega)$ is compact.*

As seen in the introduction, the operator (2.1) is the Riesz one and is self-adjoint and positive by definition. Then, $((i\nabla + A_a)^2)^{-1} \circ Im$ is also self-adjoint and positive in $\mathcal{D}^{1,2}_{A_a}(\Omega)$. We deduce that the spectrum of $(i\nabla + A_a)^2$ in $\mathcal{D}^{1,2}_{A_a}(\Omega)$ consists of a diverging sequences of real positive eigenvalues, having finite multiplicity, and counted with their multiplicity. They also admit the following variational characterization

$$\lambda^a_k = \inf_{\substack{W_k \subset \mathcal{D}^{1,2}_{A_a}(\Omega) \\ \dim W_k = k}} \sup_{\Phi \in W_k} \frac{\|\Phi\|^2_{\mathcal{D}^{1,2}_{A_a}(\Omega)}}{\|\Phi\|^2_{L^2(\Omega,p)}}.$$

The corresponding eigenvalues φ^a_k are taken normalized in the $L^2(\Omega, p)$-norm.

The value of the circulation strongly affects the behaviour of the eigenfunctions, starting from their regularity, as the following lemma shows.

Lemma 2.2.3 ([7, Section 7])**.** *Let $\alpha \in (0,1)$. If A_a has the form (2.2) then $\varphi^a_k \in C^{0,\gamma}(\Omega)$, where $\gamma = \operatorname{dist}(\alpha, \mathbb{Z})$, α being the circulation of A_a.*

2.2.2. Gauge invariance

In the introduction, we said that the magnetic potential has the form (2.2), where $\chi \in C^2$. Moreover, we assumed that we could take $\chi \equiv 0$ and $\alpha \in (0,1)$. Those choices are in fact not restrictive as we will explain below.

Physically, as explained in the introduction, we are interested in magnetic potentials A_a such that the corresponding magnetic field is zero in $\Omega\backslash\{a\}$ (corresponding to the case of an infinitesimally solenoid), that is

$$\nabla \times A_a = 0 \text{ in } \Omega\backslash\{a\}, \tag{2.12}$$

with circulation

$$\frac{1}{2\pi}\oint_\sigma A_a \cdot \mathrm{d}\sigma = \alpha \in \mathbb{R}, \tag{2.13}$$

where σ is a closed path in $\Omega\backslash\{a\}$ winding once around a. We also want to impose some regularity on our potential

$$A_a \in L^1(\Omega) \cap C^1(\Omega\backslash\{a\}). \tag{2.14}$$

We restrict ourself to consider a C^1-regularity outside of a since in the following we do not work with more than one derivative of A_a.

The following Lemma tells us that if the circulations of two magnetic potentials verifying (2.12) and (2.14) differ by an integer number, the corresponding operators are equivalent under a gauge transformation, so that they have the same spectrum (see for example [44, Lemma 2.7], [34, Theorem 1.1], [32, Lemma 3.2], [5, Proposition 2.1.3]).

Lemma 2.2.4 (Gauge equivalence)**.** *Assume that A_a and A'_a satisfy* (2.14) *and verify*

$$\nabla \times (A_a - A'_a) = 0 \ \textit{in} \ \Omega\backslash\{a\}, \qquad \frac{1}{2\pi}\oint_\sigma (A_a - A'_a)\cdot \mathrm{d}\sigma \in \mathbb{Z},$$

for every closed path σ in $\Omega\backslash\{a\}$ winding once around a. Then, there exists a multivalued function Θ and a smooth function χ such that,

$$A_a - A'_a = \nabla\Theta + \nabla\chi \ \textit{in} \ \Omega\backslash\{a\}, \qquad e^{i(\Theta+\chi)} \ \textit{is smooth and univalued in} \ \Omega,$$

and

$$(i\nabla + A_a)^2\left(e^{i(\Theta+\chi)}u\right) = e^{i(\Theta+\chi)}(i\nabla + A'_a)^2 u.$$

We say that A_a and A'_a are gauge equivalent.

Proof. First, we assume that $\oint_\sigma (A_a - A'_a) \cdot \mathrm{d}\sigma = 0$ for every closed path σ in $\Omega\backslash\{a\}$ winding once around a. Then, there exists a smooth function $\chi : \Omega \to \mathbb{R}$ such that $A_a - A'_a = \nabla\chi$. The phase $e^{i\chi}$ is univalued and regular and the last claim follows from it.

Now, when $\oint_\sigma (A_a - A'_a)\, \mathrm{d}\sigma \in \mathbb{Z}_0$, we consider the universal covering manifold $\tilde{\Omega}_a$ of $\Omega\backslash\{a\}$ and the associated projection $\Pi : \tilde{\Omega}_a \to \Omega\backslash\{a\}$. Since $\tilde{\Omega}_a$ is simply connected (differently from $\Omega\backslash\{a\}$), we have that $\oint_{\tilde{\sigma}} \left(\tilde{A}_a - \tilde{A}'_a\right) \cdot \mathrm{d}\tilde{\sigma} = 0$ for any closed path $\tilde{\sigma}$ in $\tilde{\Omega}_a$. Then, there exists a smooth real valued function $\tilde{\Theta} : \tilde{\Omega}_a \to \mathbb{R}$ such that $\tilde{A}_a - \tilde{A}'_a = \nabla\tilde{\Theta}$. Let's consider two different points p and p' in $\tilde{\Omega}_a$ such that $\Pi(p) = \Pi(p')$ in $\Omega\backslash\{a\}$. For every path $\tilde{\sigma}$ connecting p and p', we have

$$\begin{aligned} \tilde{\Theta}(p) - \tilde{\Theta}(p') &= \int_{\tilde{\sigma}} \nabla\tilde{\Theta} \cdot \mathrm{d}\tilde{\sigma} = \int_{\tilde{\sigma}} \left(\tilde{A}_a - \tilde{A}'_a\right) \cdot \mathrm{d}\tilde{\sigma} \\ &= \oint_\sigma (A_a - A'_a)\, \mathrm{d}\sigma = 2\pi z, \end{aligned}$$

for some $z \in \mathbb{Z}$, and where $\sigma = \Pi(\tilde{\sigma})$ is a closed path. Therefore, $\Theta = \tilde{\Theta} \circ \Pi^{-1}$ is multivalued in $\Omega\backslash\{a\}$ but the phase $e^{i\Theta}$ is well-defined and univalued in $\Omega\backslash\{a\}$. We can then proceed similarly as in the first part to conclude. ◻

Lemma 2.2.5. *A_a has the form given in (2.2) if and only if A_a verifies (2.12), (2.13) and (2.14).*

Proof. First, if we assume that A_a is as in (2.2), it is immediate to conclude that it verifies the three hypothesis (2.12)-(2.14).

Next, if we assume that A_a satisfies (2.12)-(2.14), we can look at

$$A'_a = \alpha \left(-\frac{x_2 - a_2}{(x_1 - a_1)^2 + (x_2 - a_2)^2}, \frac{x_1 - a_1}{(x_1 - a_1)^2 + (x_2 - a_2)^2} \right),$$

for which know that the same three assumptions are verified. Then, by using the gauge equivalence of Lemma 2.2.4 (more precisely the first part of the proof), we conclude that $A_a = A'_a + \nabla\chi$, for $\chi \in C^2$. ◻

We can deduce two other things from the gauge equivalence of Lemma 2.2.4. First, the regular part $\nabla\chi$ in (2.2) can be neglected since it has no influence. It is only the gradient of a regular function that can be added leaving the magnetic field $\nabla \times A_a$ unchanged. The second thing is that we can restrict ourself to circulations $\alpha \in (0, 1)$ in (2.2). Indeed, if we consider two magnetic potentials whose circulation differ by an integer, the associated magnetic operators are equivalent and then the spectrum remains the same. We can then always turn back to the case $\alpha \in [0, 1)$. We will neither consider the integer circulation since in that case the magnetic operator is equivalent to the Laplacian.

Here, we give the definition of the polar angle around a point $b \in \mathbb{R}^2$.

Definition 2.2.6. *Let $b = (b_1, b_2) \in \mathbb{R}^2$. For $\beta \in \mathbb{R}$, we define $\theta_b : \mathbb{R}^2 \setminus \{b\} \to [0, 2\pi)$ as*

$$\theta_b = \begin{cases} \arctan \dfrac{x_2 - b_2}{x_1 - b_1} & x_1 > b_1,\ x_2 \geq \beta x_1 + (b_2 - \beta b_1) \\ \pi/2 & x_1 = b_1,\ x_2 > b_2 \\ \pi + \arctan \dfrac{x_2 - b_2}{x_1 - b_1} & x_1 < b_1 \\ 3\pi/2 & x_1 = b_1,\ x_2 < b_2 \\ 2\pi + \arctan \dfrac{x_2 - b_2}{x_1 - b_1} & x_1 > b_1,\ x_2 < \beta x_1 + (b_2 - \beta b_1), \end{cases} \tag{2.15}$$

where we consider the following branch of the arctangent

$$\arctan : \mathbb{R} \to \left(-\frac{\pi}{2}, \frac{\pi}{2} \right).$$

We remark that this function θ_b is multivalued. Moreover, by definition, it is regular except on the half-line ending at the point b

$$\Gamma \equiv x_2 = \beta x_1 + (b_2 - \beta b_1), \quad x_1 > b_1.$$

Here, the choice of the half-line is totally arbitrary. Note that we could also consider Γ to be a regular curve ending at b and not only a line.

Lemma 2.2.7. *Let $a \in \Omega$ and A_a as in (2.2) (for $\chi \equiv 0$). If we consider a regular half-curve Γ and the multivalued function θ_a defined in (2.15) with $\theta_a \in C^\infty(\Omega \setminus \Gamma)$, then in its regularity set*

$$\alpha \nabla \theta_a = A_a.$$

Proof. By using (2.15) and an explicit calculation, we easily prove the claim. □

2.2.3. Half-integer circulation

Now, we focus on the particular case of half-integer circulation $\alpha = 1/2$, for which we can deduce several properties, despite the fact that $\alpha \notin \mathbb{Z}$. We recall a result proved in [34] concerning the magnetic operator (2.1) with $\alpha = 1/2$, which is fundamental for our analysis. The eigenfunctions of such operator coincide, up to the complex phase $e^{i\theta_a/2}$, with the antisymmetric eigenfunctions of the Laplace-Beltrami operator (real valued) on the twofold covering manifold of Ω. The twofold covering manifold, given by

$$\Sigma_a = \left\{(x, y) \in \mathbb{C}^2 \,|\, y^2 = x - a,\, x \in \Omega\right\},$$

is the Riemann surface associated to the complex square root, so that $e^{i\theta_a/2}$ is continuous therein. We prefer to state the result in the following form, taken from [32], where a projection is applied from the twofold covering manifold into some flat, bounded domain of $\mathbb{R}^2$.

Lemma 2.2.8 ([34, Lemma 3.3], [32, Lemma 3.14])**.** *Suppose that A_a has the form (2.2). Then the function*

$$e^{-i\theta(y)}\varphi_k^a(y^2 + a) \quad \textit{defined in } \{y \in \mathbb{C} : \; y^2 + a \in \Omega\},$$

(here θ is the angle of the polar coordinates around 0, we omit the index) is real valued and solves the following equation on its domain

$$-\Delta(e^{-i\theta(y)}\varphi_k^a(y^2 + a)) = \lambda_k^a p'(y)e^{-i\theta(y)}\varphi_k^a(y^2 + a), \qquad p'(y) = 4|y|^2 p(y^2 + a).$$

The projection allows us to recover the continuity of the phase thanks to the relation $e^{i\theta_a(y^2+a)/2} = e^{i\theta(y)}$. As a consequence of this result, the magnetic eigenfunctions behave, up to a complex phase, as the real eigenfunctions of the Laplacian far from the singular point a. The behaviour near the singularity is, up to a complex phase, that of the square root of an elliptic eigenfunction. Moreover, because of the equivalence with the antisymmetric eigenfunctions of the Laplacian, the magnetic eigenfunctions always possess an odd number of nodal lines ending at a, as stated below in Proposition 2.2.10.

Since we are interested in the shape of the nodal lines, let us first recall the known results concerning the elliptic eigenfunctions in the plane.

Proposition 2.2.9 ([45, equation (5")], [19, Theorem 2.1])**.** *Let $\Omega \subset \mathbb{R}^2$ be open, bounded, simply connected and of class C^∞. Let p satisfy (2.4). If φ_k has a zero of order $h/2$ at $0 \in \overline{\Omega}$, then h is even and we have*

$$\varphi_k(r, \theta) = r^{h/2}\left[c_h \cos\left(\frac{h}{2}\theta\right) + d_h \sin\left(\frac{h}{2}\theta\right)\right] + g(r, \theta),$$

with $x = re^{i\theta} \in \Omega$, $c_h^2 + d_h^2 \neq 0$ and

$$\lim_{r \to 0} \frac{\|g(r, \cdot)\|_{C^1(\partial D_r)}}{r^{h/2}} = 0. \tag{2.16}$$

In addition, there is a positive radius R such that *(i)* *if* $0 \in \Omega$ *then* $(\varphi_k)^{-1}(\{0\}) \cap D_R(0)$ *consists of* h *arcs of class* C^∞*, whose tangent lines divide the disk into* h *equal sectors;*

(ii) *if* $0 \in \partial\Omega$ *then* $(\varphi_k)^{-1}(\{0\}) \cap D_R(0) \cap \Omega$ *consists of* $h/2 - 1$ *arcs of class* C^∞*, whose tangent lines divide the half disk into* $h/2$ *equal sectors.*

The behaviour near the boundary at point (ii) above can be deduced from point (i). Indeed, since the boundary is regular, it can be locally rectified as described in Lemma 2.7.1 below. By performing an odd extension, we transform the boundary point into an interior point of type (i).

We summarize below the local properties of the magnetic eigenfunction near the pole. The proofs can be found in [7, Theorem 1.3], [34, Theorem 2.1] and [32, Theorem 1.5] (see also [45]).

Proposition 2.2.10. *There exists an odd integer* $h \geq 1$ *such that* φ_k^a *has a zero of order* $h/2$ *at* a*. Moreover, the following asymptotic expansion holds near* a

$$\varphi_k^a(r_a, \theta_a) = e^{i\frac{\theta_a}{2}} r_a^{h/2} \left[c_h(a) \cos\left(h\frac{\theta_a}{2}\right) + d_h(a) \sin\left(h\frac{\theta_a}{2}\right)\right] + g(r_a, \theta_a) \quad (2.17)$$

where $x - a = r_a e^{i\theta_a}$*,* $c_h(a)^2 + d_h(a)^2 \neq 0$ *and* g *satisfies* (2.16)*. In addition, there is a positive radius* R *such that* $(\varphi_k^a)^{-1}(\{0\}) \cap D_R(a)$ *consists of* h *arcs of class* C^∞*. If* $h \geq 3$ *then the tangent lines to the arcs at the point* a *divide the disk into* h *equal sectors.*

We remark that we can put in front of (2.17) an additional complex phase.

2.3. Continuity of the eigenvalues with respect to the pole in the interior of the domain

In this section we prove the first part of Theorem 2.1.1 for $p(x) \equiv 1$, that is the continuity of the function $a \mapsto \lambda_k^a$ when the pole a varies inside Ω.

Lemma 2.3.1. *Given* $a, b \in \Omega$*, there exists a radial cut-off function* $\eta_a : \mathbb{R}^2 \to \mathbb{R}$ *such that* $\eta_a(x) = 0$ *for* $|x-a| < 2|b-a|$*,* $\eta_a(x) = 1$ *for* $|x - a| \geq \sqrt{2|b-a|}$ *and moreover*

$$\int_{\mathbb{R}^2} \left(|\nabla \eta_a|^2 + (1 - \eta_a^2)\right) \, \mathrm{d}x \to 0 \qquad \text{as } a \to b.$$

Proof. Given any $0 < \varepsilon < 1$ we set

$$\eta(x) = \begin{cases} 0 & 0 \leq |x| \leq \varepsilon \\ \dfrac{\log \varepsilon - \log |x|}{\log \varepsilon - \log \sqrt{\varepsilon}} & \varepsilon \leq |x| \leq \sqrt{\varepsilon} \\ 1 & x \geq \sqrt{\varepsilon}. \end{cases}$$

Choosing $\varepsilon = 2|b-a|$ (provided the fact that a is sufficiently closed to b) and $\eta_a(x) = \eta(x-a)$, an explicit calculation shows that the properties are satisfied. □

Lemma 2.3.2. *Given $a, b \in \Omega$, there exist θ_a and θ_b such that $\theta_a - \theta_b \in C^\infty(\Omega \setminus \{ta + (1-t)b,\, t \in [0,1]\})$ and moreover in this set we have*

$$\alpha \nabla(\theta_a - \theta_b) = A_a - A_b.$$

Proof. Let $a = (a_1, a_2)$ and $b = (b_1, b_2)$. Suppose that $a_1 < b_1$, the other cases can be treated in a similar way. By Lemma 2.2.7, we know the existence of a multivalued function θ_a given by (2.15) (for $b = a$). We choose the discontinuity Γ to be the half-line starting from a and passing through b

$$\Gamma \equiv x_2 = \frac{b_2 - a_2}{b_1 - a_1} x_1 + \frac{a_2 b_1 - b_2 a_1}{b_1 - a_1}, \quad x_1 > a_1.$$

The definition of θ_b is analogous: we keep the same half-line Γ, whereas we replace (a_1, a_2) with (b_1, b_2) in the definition of the function. One can explicitely verify that $\theta_a - \theta_b$ is regular except on the segment from a to b. □

Recall that in the following φ_k^a is an eigenfunction associated to λ_k^a, normalized in the L^2-norm. Moreover, we can assume that the eigenfunctions are orthogonal.

Lemma 2.3.3. *Given $a, b \in \Omega$, let η_a be defined as in Lemma 2.3.1 and let θ_a, θ_b be defined as in Lemma 2.3.2. Fix an integer $k \geq 1$ and set, for $i = 1, \dots, k$,*

$$\tilde{\varphi}_i = e^{i\alpha(\theta_a - \theta_b)} \eta_a \varphi_i^b.$$

Then $\tilde{\varphi}_i \in \mathcal{D}^{1,2}_{A_a}(\Omega)$ and moreover for every $(\alpha_1, \dots, \alpha_k) \in \mathbb{C}^k$ it holds

$$(1 - \varepsilon_a) \| \sum_{i=1}^k \alpha_i \varphi_i^b \|^2_{L^2(\Omega)} \leq \| \sum_{i=1}^k \alpha_i \tilde{\varphi}_i \|^2_{L^2(\Omega)} \leq k \| \sum_{i=1}^k \alpha_i \varphi_i^b \|^2_{L^2(\Omega)},$$

where $\varepsilon_a \to 0$ as $a \to b$.

Proof. Let us prove first that $\tilde{\varphi}_i \in \mathcal{D}^{1,2}_{A_a}(\Omega)$. By Lemmas 2.3.1 and 2.3.2 we have that $\theta_a - \theta_b \in C^\infty(\mathrm{supp}\{\eta_a\})$, so that $\tilde{\varphi}_i \in H^1_0(\Omega)$. Moreover $\tilde{\varphi}_i(x) = 0$ if $|x-a| < 2|b-a|$, hence $\tilde{\varphi}_i/|x-a| \in L^2(\Omega)$. We conclude by using the other characterization of $\mathcal{D}^{1,2}_{A_a}(\Omega)$. Concerning the inequalities, we compute on one hand

$$\|\sum_{i=1}^k \alpha_i \tilde{\varphi}_i\|^2_{L^2(\Omega)} \leq k \sum_{i=1}^k |\alpha_i|^2 \|\eta_a \varphi_i^b\|^2_{L^2(\Omega)} \leq k \sum_{i=1}^k |\alpha_i|^2 = k \|\sum_{i=1}^k \alpha_i \varphi_i^b\|^2_{L^2(\Omega)},$$

where we used the inequality $\sum_{i,j=1}^k \alpha_i \overline{\alpha_j} \leq k \sum_{i=1}^k |\alpha_i|^2$ and the fact that the eigenfunctions are orthogonal and normalized in the $L^2(\Omega)$-norm. On the other hand we compute

$$\|\sum_{i=1}^k \alpha_i \varphi_i^b\|^2_{L^2(\Omega)} - \|\sum_{i=1}^k \alpha_i \tilde{\varphi}_i\|^2_{L^2(\Omega)} = \sum_{i,j=1}^k \alpha_i \overline{\alpha_j} \int_\Omega (1-\eta_a^2) \varphi_i^b \overline{\varphi_j^b} \,\mathrm{d}x.$$

Thanks to the regularity result proved by Felli, Ferrero and Terracini (see Lemma 2.2.3), we have that φ_i^b are bounded in $L^\infty(\Omega)$. Therefore the last quantity is bounded by

$$Ck \sum_{i=1}^k |\alpha_i|^2 \int_\Omega (1-\eta_a^2) \,\mathrm{d}x = Ck \|\sum_{i=1}^k \alpha_i \varphi_i^b\|^2_{L^2(\Omega)} \int_\Omega (1-\eta_a^2) \,\mathrm{d}x$$

and the conclusion follows from Lemma 2.3.1. □

We have all the tools to prove the first part of Theorem 2.1.1. We will use some ideas from [21, Theorem 7.1].

Theorem 2.3.4. *For every $k \in \mathbb{N}_0$, the function $a \in \Omega \mapsto \lambda_k^a \in \mathbb{R}$ is continuous.*

Proof. We divide the proof in two steps.

Step 1. First we prove that

$$\limsup_{a \to b} \lambda_k^a \leq \lambda_k^b.$$

To this aim it will be sufficient to exhibit a k-dimensional space $E_k \subset \mathcal{D}^{1,2}_{A_a}(\Omega)$ with the property that

$$\|\Phi\|^2_{\mathcal{D}^{1,2}_{A_a}(\Omega)} \leq \left(\lambda_k^b + \varepsilon'_a\right) \|\Phi\|^2_{L^2(\Omega)} \quad \text{for every } \Phi \in E_k, \tag{2.18}$$

with $\varepsilon'_a \to 0$ as $a \to b$. Let $\mathrm{span}\{\varphi_1^b, \dots, \varphi_k^b\}$ be any spectral space attached to $\lambda_1^b, \dots, \lambda_k^b$. Then we define

$$E_k := \mathrm{span}\{\tilde{\varphi}_1, \dots, \tilde{\varphi}_k\} \quad \text{with} \quad \tilde{\varphi}_i = e^{i\alpha(\theta_a - \theta_b)}\eta_a \varphi_i^b,\ i = 1, \dots, k.$$

We know from Lemma 2.3.3 that $E_k \subset \mathcal{D}^{1,2}_{A_a}(\Omega)$. Moreover, it is immediate to see that $\dim E_k = k$. Let us now verify (2.18) with $\Phi = \sum_{i=1}^k \alpha_i \tilde{\varphi}_i$, $\alpha_i \in \mathbb{C}$. We compute

$$\begin{aligned} \|\Phi\|^2_{\mathcal{D}^{1,2}_{A_a}(\Omega)} &= \int_\Omega \left| \sum_{i=1}^k \alpha_i (i\nabla + A_b)(\eta_a \varphi_i^b) \right|^2 \mathrm{d}x \\ &= \int_\Omega \sum_{i,j=1}^k \alpha_i \overline{\alpha_j} (i\nabla + A_b)^2 (\eta_a \varphi_i^b)(\eta_a \overline{\varphi_j^b})\, \mathrm{d}x, \end{aligned} \tag{2.19}$$

where we have used the equality

$$(i\nabla + A_a)\, \tilde{\varphi}_i = e^{i\alpha(\theta_a - \theta_b)}\, (i\nabla + A_b)\, (\eta_a \varphi_i^b)$$

and integration by parts. Next notice that

$$(i\nabla + A_b)(\eta_a \varphi_i^b) = \eta_a (i\nabla + A_b)\varphi_i^b + i\varphi_i^b \nabla \eta_a,$$

so that

$$(i\nabla + A_b)^2(\eta_a \varphi_i^b) = \eta_a (i\nabla + A_b)^2 \varphi_i^b + 2i(i\nabla + A_b)\varphi_i^b \cdot \nabla \eta_a - \varphi_i^b \Delta \eta_a.$$

By replacing in (2.19), we obtain

$$\begin{aligned} \|\Phi\|^2_{\mathcal{D}^{1,2}_{A_a}(\Omega)} &= \int_\Omega \sum_{i,j=1}^k \alpha_i \overline{\alpha_j} \left(\lambda_i^b \varphi_i^b \eta_a + 2i(i\nabla + A_b)\varphi_i^b \cdot \nabla \eta_a - \varphi_i^b \Delta \eta_a \right) \overline{\varphi_j^b} \eta_a \, \mathrm{d}x \\ &\le \lambda_k^b \| \sum_{i=1}^k \alpha_i \varphi_i^b \|^2_{L^2(\Omega)} + \beta_a \end{aligned} \tag{2.20}$$

where

$$\begin{aligned} \beta_a = \int_\Omega \sum_{i,j=1}^k \alpha_i \overline{\alpha_j} \Big\{ & \lambda_i^b (\eta_a^2 - 1)\varphi_i^b \overline{\varphi_j^b} + 2i\overline{\varphi_j^b} \eta_a (i\nabla + A_b)\varphi_i^b \cdot \nabla \eta_a \\ & - \varphi_i^b \overline{\varphi_j^b} \eta_a \Delta \eta_a \Big\} \, \mathrm{d}x. \end{aligned} \tag{2.21}$$

We need to estimate β_a. From Lemma 2.2.3 we deduce the existence of a constant $C > 0$ such that $\|\varphi_i^b\|_{L^\infty(\Omega)} \le C$ for every $i = 1, \dots, k$. Hence

$$\left|\int_\Omega \sum_{i,j=1}^k \alpha_i \overline{\alpha_j} \lambda_i^b (\eta_a^2 - 1)\varphi_i^b \overline{\varphi_j^b}\ \mathrm{d}x\right| \le C \sum_{i=1}^k |\alpha_i|^2 \int_\Omega (1-\eta_a^2)\ \mathrm{d}x.$$

Using the fact that $\|\varphi_i^b\|^2_{H_0^1(\Omega)} \le C\|\varphi_i^b\|^2_{\mathcal{D}^{1,2}_{A_b}(\Omega)} = C\lambda_i^b$ (see equation (2.11)), we have

$$\left|\int_\Omega \sum_{i,j=1}^k \alpha_i \overline{\alpha_j} \overline{\varphi_j^b} \eta_a \nabla\varphi_i^b \cdot \nabla\eta_a\ \mathrm{d}x\right| \le C \sum_{j=1}^k |\alpha_j|^2 \left(\int_\Omega |\nabla\eta_a|^2\ \mathrm{d}x\right)^{1/2}.$$

Next we apply the Hardy inequality (2.10) to obtain

$$\begin{aligned}
&\left|\int_\Omega \sum_{i,j=1}^k \alpha_i \overline{\alpha_j} \varphi_i^b \overline{\varphi_j^b} \eta_a A_b \cdot \nabla\eta_a\, \mathrm{d}x\right| \le C \sum_{i=1}^k |\alpha_i|^2 \int_\Omega |\varphi_i^b A_b \cdot \nabla\eta_a|\, \mathrm{d}x \\
&\le C \sum_{i=1}^k |\alpha_i|^2 \left\|\frac{\varphi_i^b}{x-b}\right\|_{L^2(\Omega)} \|(x-b)A_b\|_{L^\infty(\Omega)} \|\nabla\eta_a\|_{L^2(\Omega)} \\
&\le C \sum_{i=1}^k |\alpha_i|^2 \|\nabla\eta_a\|_{L^2(\Omega)}.
\end{aligned}$$

Concerning the last term in (2.21), similar estimates give

$$\begin{aligned}
&\left|\int_\Omega \sum_{i,j=1}^k \alpha_i \overline{\alpha_j} \varphi_i^b \overline{\varphi_j^b} \eta_a \Delta\eta_a\ \mathrm{d}x\right| \\
&= \left|\int_\Omega \sum_{i,j=1}^k \alpha_i \overline{\alpha_j} \left(|\nabla\eta_a|^2 \varphi_i^b \overline{\varphi_j^b} + \eta_a \nabla\eta_a \cdot \nabla(\varphi_i^b \overline{\varphi_j^b})\right)\ \mathrm{d}x\right| \\
&\le C \sum_{i=1}^k |\alpha_i|^2 \left(\int_\Omega |\nabla\eta_a|^2\ \mathrm{d}x\right)^{1/2}.
\end{aligned}$$

In conclusion we have obtained

$$\begin{aligned}
|\beta_a| &\le C \|\sum_{i=1}^k \alpha_i \varphi_i^b\|^2_{L^2(\Omega)} \left\{\int_\Omega (1-\eta_a^2)\, \mathrm{d}x + \left(\int_\Omega |\nabla\eta_a|^2\, \mathrm{d}x\right)^{1/2}\right\} \\
&= \|\sum_{i=1}^k \alpha_i \varphi_i^b\|^2_{L^2(\Omega)} \varepsilon_a'',
\end{aligned}$$

with $\varepsilon''_a \to 0$ as $a \to b$ by Lemma 2.3.1. By inserting the last estimate into (2.20) and then using Lemma 2.3.3 we obtain (2.18) with $\varepsilon'_a = (\varepsilon''_a + \lambda_k^b \varepsilon_a)/(1-\varepsilon_a)$.

Step 2. We want now to prove the second inequality

$$\liminf_{a\to b} \lambda_k^a \geq \lambda_k^b.$$

From relation (2.11) and Step 1, we deduce

$$\|\varphi_i^a\|^2_{H_0^1(\Omega)} \leq C\|\varphi_i^a\|^2_{\mathcal{D}^{1,2}_{A_a}(\Omega)} \leq C\lambda_i^b.$$

Hence there exists $\varphi_i^\star \in H_0^1(\Omega)$ such that (up to subsequences) $\varphi_i^a \rightharpoonup \varphi_i^\star$ weakly in $H_0^1(\Omega)$ and $\varphi_i^a \to \varphi_i^\star$ strongly in $L^2(\Omega)$, as $a \to b$. In particular we have

$$\int_\Omega |\varphi_i^\star|^2 \,\mathrm{d}x = 1 \quad \text{and} \quad \int_\Omega \varphi_i^\star \varphi_j^\star \,\mathrm{d}x = 0 \text{ if } i \neq j. \tag{2.22}$$

Moreover, Fatou's lemma, relation (2.10) and Step 1 provide

$$\begin{aligned}\|\varphi_i^\star/|x-b|\|_{L^2(\Omega)} \leq \liminf_{a\to b} \|\varphi_i^a/|x-a|\|_{L^2(\Omega)} &\leq C \liminf_{a\to b} \|\varphi_i^a\|_{\mathcal{D}^{1,2}_{A_a}(\Omega)} \\ &= C \liminf_{a\to b} \sqrt{\lambda_i^a} \leq C\sqrt{\lambda_i^b},\end{aligned}$$

so we deduce that $\varphi_i^\star \in \mathcal{D}^{1,2}_{A_b}(\Omega)$.

Given a test function $\phi \in C_0^\infty(\Omega\backslash\{b\})$, consider a sufficiently close to b so that $a \notin \operatorname{supp}\{\phi\}$. We have that

$$\begin{aligned}&\int_\Omega \lambda_i^a \varphi_i^a \bar{\phi}\,\mathrm{d}x = \int_\Omega \varphi_i^a \overline{(i\nabla + A_a)^2\phi}\,\mathrm{d}x \\ &= \int_\Omega \left\{-\Delta\varphi_i^a\bar{\phi} + \varphi_i^a\overline{[i\nabla\cdot A_a\phi + 2iA_a\cdot\nabla\phi + |A_a|^2\phi]}\right\}\,\mathrm{d}x \\ &= \int_\Omega \{(i\nabla + A_b)^2\varphi_i^a\bar{\phi} - i\nabla\cdot(A_a + A_b)\varphi_i^a\bar{\phi} - 2i(A_a\cdot\nabla\bar{\phi}\varphi_i^a + A_b\cdot\nabla\varphi_i^a\bar{\phi}) \\ &+(|A_a|^2 - |A_b|^2)\varphi_i^a\bar{\phi}\}\,\mathrm{d}x = \int_\Omega \{(i\nabla + A_b)^2\varphi_i^a\bar{\phi} - i\nabla\cdot(A_a - A_b)\varphi_i^a\bar{\phi} \\ &-2i\varphi_i^a(A_a - A_b)\cdot\nabla\bar{\phi} + (|A_a|^2 - |A_b|^2)\varphi_i^a\bar{\phi}\}\,\mathrm{d}x,\end{aligned}$$

where in the last step we used the identity

$$-2i\int_\Omega A_b\cdot\nabla\varphi_i^a\bar{\phi}\,\mathrm{d}x = 2i\int_\Omega (\nabla\cdot A_b\varphi_i^a\bar{\phi} + A_b\varphi_i^a\nabla\bar{\phi})\,\mathrm{d}x.$$

Since $a, b \notin \operatorname{supp}\{\phi\}$ then $A_a \to A_b$ in $C^\infty(\operatorname{supp}\{\phi\})$. Hence for a suitable subsequence we can pass to the limit in the previous expression obtaining

$$\int_\Omega (i\nabla + A_b)^2 \varphi_i^\star \bar{\phi} \, \mathrm{d}x = \int_\Omega \lambda_i^\star \varphi_i^\star \bar{\phi} \, \mathrm{d}x, \qquad \text{for every } \phi \in C_0^\infty(\Omega \backslash \{b\}),$$

where $\lambda_i^\star := \liminf_{a\to b} \lambda_i^a$. By density, the same is valid for $\phi \in \mathcal{D}^{1,2}_{A_b}(\Omega)$. As a consequence of the last equation and of (2.22), the functions $\varphi_i^\star$ are orthogonal in $\mathcal{D}^{1,2}_{A_b}(\Omega)$ and hence

$$\begin{aligned}
\lambda_k^b &= \inf_{\substack{W_k \subset \mathcal{D}^{1,2}_{A_b}(\Omega) \\ \dim W_k = k}} \sup_{\Phi \in W_k} \frac{\int_\Omega |(i\nabla + A_b)\Phi|^2 \, \mathrm{d}x}{\int_\Omega |\Phi|^2 \, \mathrm{d}x} \\
&\le \sup_{(\alpha_1,\dots,\alpha_k)\neq 0} \frac{\int_\Omega |(i\nabla + A_b)\left(\sum_{i=1}^k \alpha_i \varphi_i^\star\right)|^2 \, \mathrm{d}x}{\int_\Omega |\sum_{i=1}^k \alpha_i \varphi_i^\star|^2 \, \mathrm{d}x} \\
&= \sup_{(\alpha_1,\dots,\alpha_k)\neq 0} \frac{\sum_{i=1}^k |\alpha_i|^2 \lambda_i^\star}{\sum_{i=1}^k |\alpha_i|^2} \\
&\le \lambda_k^\star = \liminf_{a\to b} \lambda_k^a.
\end{aligned}$$

This concludes Step 2 and the proof of the theorem. □

2.4. Continuity of the eigenvalues with respect to the pole up to the boundary of the domain

In this section we prove the second part of Theorem 2.1.1 for $p(x) \equiv 1$, that is the continuous extension up to the boundary of the domain. Recall that we denote by φ_k an eigenfunction associated to λ_k, the k-th eigenvalue of the Laplacian in $H_0^1(\Omega)$. As usual, we suppose that the eigenfunctions are normalized in L^2 and orthogonal. The following two lemmas can be proved exactly as the corresponding ones in the previous section.

Lemma 2.4.1. *Given $a \in \Omega$ and $b \in \partial\Omega$, there exist θ_a and θ_b such that $\theta_a \in C^\infty(\Omega \setminus \{ta + (1-t)b, t \in [0,1]\})$, $\theta_b \in C^\infty(\Omega)$ and moreover in the respective sets of regularity the following holds*

$$\alpha \nabla \theta_a = A_a \qquad \alpha \nabla \theta_b = A_b.$$

Lemma 2.4.2. *Given $a \in \Omega$ and $b \in \partial\Omega$, let η_a be defined in Lemma 2.3.1 and let θ_a be defined in Lemma 2.2.7. Set, for $i = 1, \dots, k$,*

$$\tilde{\varphi}_i = e^{i\alpha\theta_a} \eta_a \varphi_i.$$

Then for every $(\alpha_1, \dots, \alpha_k) \in \mathbb{C}^k$ *it holds*

$$(1-\varepsilon_a)\|\sum_{i=1}^{k}\alpha_i\varphi_i\|^2_{L^2(\Omega)} \leq \|\sum_{i=1}^{k}\alpha_i\tilde{\varphi}_i\|^2_{L^2(\Omega)} \leq k\|\sum_{i=1}^{k}\alpha_i\varphi_i\|^2_{L^2(\Omega)},$$

where $\varepsilon_a \to 0$ *as* $a \to b$.

We can now prove the main theorem of this section.

Theorem 2.4.3. *Suppose that* $a \in \Omega$ *converges to* $b \in \partial\Omega$*. Then, for every* $k \in \mathbb{N}_0$*, we have that* λ_k^a *converges to* λ_k*.*

Proof. Following the scheme of the proof of Theorem 2.3.4 we proceed in two steps.

Step 1. First we show that

$$\limsup_{a\to b} \lambda_k^a \leq \lambda_k. \tag{2.23}$$

Since the proof is very similar to the one of Step 1 in Theorem 2.3.4 we will only point out the main differences. We define

$$E_k := \left\{\Phi = \sum_{i=1}^{k}\alpha_i\tilde{\varphi}_i,\ \alpha_i \in \mathbb{C}\right\} \quad \text{with} \quad \tilde{\varphi}_i = e^{i\alpha\theta_a}\eta_a\varphi_i$$

given in Lemma 2.4.2. We can verify the equality

$$(i\nabla + A_a)\left(e^{i\alpha\theta_a}\eta_a\varphi_i\right) = ie^{i\alpha\theta_a}\nabla(\eta_a\varphi_i),$$

so that we have

$$\|\Phi\|^2_{\mathcal{D}^{1,2}_{A_a}(\Omega)} = \int_\Omega \left|\sum_{i=1}^{k}\alpha_i\nabla(\eta_a\varphi_i)\right|^2 \,\mathrm{d}x \leq \lambda_k\|\sum_{i=1}^{k}\alpha_i\varphi_i\|^2_{L^2(\Omega)} + \beta_a,$$

with

$$\beta_a = \sum_{i,j=1}^{k}\alpha_i\overline{\alpha_j}\left(\int_\Omega |\nabla\eta_a|^2\varphi_i\varphi_j + 2\eta_a\nabla\eta_a\cdot\nabla\varphi_j\varphi_i + (\eta_a^2-1)\nabla\varphi_i\cdot\nabla\varphi_j\right)\,\mathrm{d}x.$$

Proceeding similarly to the proof of Theorem 2.3.4 we can estimate

$$|\beta_a| \leq \varepsilon''_a\|\sum_{i=1}^{k}\alpha_i\varphi_i\|^2_{L^2(\Omega)},$$

with $\varepsilon_a'' \to 0$ as $a \to b$. In conclusion, using Lemma 2.4.2, we have obtained

$$\|\Phi\|^2_{\mathcal{D}^{1,2}_{A_a}(\Omega)} \leq \left(\lambda_k + \frac{\varepsilon_a'' + \lambda_k \varepsilon_a}{1-\varepsilon_a}\right) \|\Phi\|^2_{L^2(\Omega)} \quad \text{for every } \Phi \in E_k,$$

with $\varepsilon_a, \varepsilon_a'' \to 0$ as $a \to b$. Therefore (2.23) is proved.

Step 2. We will now prove the second inequality

$$\liminf_{a\to b} \lambda_k^a \geq \lambda_k.$$

Given a test function $\phi \in C_0^\infty(\Omega)$, for a sufficiently close to b we have that

$$\{ta + (1-t)b,\ t \in [0,1]\} \subset \Omega \setminus \{\mathrm{supp}\phi\}.$$

Then $\phi \in \mathcal{D}^{1,2}_{A_a}(\Omega)$ and Lemma 2.4.1 implies that $e^{i\alpha\theta_a}\phi \in C_0^\infty(\Omega)$. For this reason we can compute the following

$$\int_\Omega \nabla(e^{-i\alpha\theta_b}\varphi_i^a) \cdot \nabla\bar{\phi}\,\mathrm{d}x = \int_\Omega e^{-i\alpha\theta_b}\varphi_i^a(\overline{-\Delta(e^{-i\alpha\theta_a}\phi e^{i\alpha\theta_a})})\,\mathrm{d}x. \tag{2.24}$$

Since

$$-\Delta(e^{-i\alpha\theta_a}\phi e^{i\alpha\theta_a}) = (i\nabla + A_a)^2\phi - 2iA_a\cdot\nabla\phi - i\nabla\cdot A_a\phi - |A_a|^2\phi,$$

the right hand side in (2.24) can be rewritten as

$$\int_\Omega \left((i\nabla + A_a)^2(e^{-i\alpha\theta_b}\varphi_i^a)\bar{\phi} + e^{-i\alpha\theta_b}\varphi_i^a(2iA_a\cdot\nabla\bar{\phi} + i\nabla\cdot A_a\bar{\phi} - |A_a|^2\bar{\phi})\right)\,\mathrm{d}x.$$

At this point notice that

$$\begin{aligned}(i\nabla + A_a)^2(e^{-i\alpha\theta_b}\varphi_i^a) = e^{-i\alpha\theta_b}\big(&(i\nabla + A_a)^2\varphi_i^a + i\nabla\cdot A_b\varphi_i^a + 2iA_b\cdot\nabla\varphi_i^a \\ &+|A_b|^2\varphi_i^a + 2A_a\cdot A_b\varphi_i^a\big).\end{aligned}$$

By inserting these information in (2.24) we obtain

$$\int_\Omega \nabla(e^{-i\alpha\theta_b}\varphi_j^a)\cdot\nabla\bar{\phi}\,\mathrm{d}x = \lambda_j^a\int_\Omega e^{-i\alpha\theta_b}\varphi_j^a\bar{\phi}\,\mathrm{d}x + \beta_a, \tag{2.25}$$

with

$$\begin{aligned}\beta_a = &\int_\Omega e^{-i\alpha\theta_b}\bar{\phi}\left(i\nabla\cdot A_b\varphi_i^a + 2iA_b\cdot\nabla\varphi_i^a + |A_b|^2\varphi_i^a + 2A_a\cdot A_b\varphi_i^a\right)\,\mathrm{d}x \\ &+ \int_\Omega e^{-i\alpha\theta_b}\varphi_i^a\left(2iA_a\cdot\nabla\bar{\phi} + i\nabla\cdot A_a\bar{\phi} - |A_a|^2\bar{\phi}\right)\,\mathrm{d}x.\end{aligned}$$

Integration by parts leads

$$\beta_a = \int_\Omega e^{-i\alpha\theta_b}\varphi_i^a \left(-\bar{\phi}|A_a - A_b|^2 + 2i\nabla\bar{\phi}\cdot(A_a - A_b) + i\bar{\phi}\nabla\cdot(A_a - A_b)\right)\,\mathrm{d}x,$$

so that $|\beta_a| \to 0$ as $a \to b$, since $A_a \to A_b$ in $C^\infty(\mathrm{supp}\{\phi\})$. Therefore we can pass to the limit in (2.25) to obtain

$$\int_\Omega \nabla\varphi_i^\star \cdot \nabla\bar{\phi}\,\mathrm{d}x = \lambda_i^\star \int_\Omega \varphi_i^\star\bar{\phi}\,\mathrm{d}x \quad \text{for every } \phi \in C_0^\infty(\Omega),$$

where $\varphi_i^\star$ is the weak limit of a suitable subsequence of $e^{-i\alpha\theta_b}\varphi_i^a$ (which exists by Step 1) and $\lambda_i^\star := \liminf_{a\to b}\lambda_i^a$. The conclusion of the proof is as in Theorem 2.3.4. □

Remark 2.4.4. *As a consequence of Theorem 2.4.3 we obtain that* $e^{-i\alpha\theta_a}\varphi_k^a \to \varphi_k$ *in* $H_0^1(\Omega)$ *as* $a \to b \in \partial\Omega$. *Indeed, an inspection of the previous proof provides the weak convergence* $e^{-i\alpha\theta_a}\varphi_k^a \rightharpoonup \varphi_k$ *in* $H_0^1(\Omega)$ *and the convergence of the norms*

$$\|e^{-i\alpha\theta_a}\varphi_k^a\|^2_{H_0^1(\Omega)} = \|\varphi_k^a\|^2_{\mathcal{D}^{1,2}_{A_a}(\Omega)} = \lambda_k^a \to \lambda_k = \|\varphi_k\|^2_{H_0^1(\Omega)},$$

as $a \to b \in \partial\Omega$, *for every* $k \in \mathbb{N}_0$.

2.5. Differentiability of the simple eigenvalues with respect to the pole

In this section we prove Theorem 2.1.3 for $p \equiv 1$. We omit the subscript in the notation of the eigenvalues and eigenfunctions; with this notation, λ^a is any eigenvalue of $(i\nabla + A_a)^2$ and φ^a is an associated eigenfunction. Moreover, for any $b \in \Omega$, we momentarily consider the space $\mathcal{D}^{1,2}_{A_b}(\Omega)$ as a real Hilbert space, such that, for any u and $v \in \mathcal{D}^{1,2}_{A_b}(\Omega)$, the real scalar product is defined as

$$(u,v)_{\mathcal{D}^{1,2}_{A_b}(\Omega),\mathbb{R}} = \mathrm{Re}\left(\int_\Omega (i\nabla + A_b)u \cdot \overline{(i\nabla + A_b)v}\,\mathrm{d}x\right).$$

The following is inspired by [38, Section 7].

Proof of Theorem 2.1.3. Let $b \in \Omega$ be such that λ^b is simple, as in the assumptions of the theorem. For R such that $B_{2R}(b) \subset \Omega$, let ξ be a cut-off function satisfying $\xi \in C^\infty(\Omega)$, $0 \le \xi \le 1$, $\xi(x) = 1$ for $x \in B_R(b)$ and $\xi(x) = 0$ for $x \in \Omega \setminus B_{2R}(b)$. For every $a \in B_R(b)$ we define the transformation

$$\Phi_a : \Omega \to \Omega, \qquad \Phi_a(x) = \xi(x)(x - b + a) + (1 - \xi(x))x.$$

Then $\varphi^a \circ \Phi_a \in \mathcal{D}^{1,2}_{A_b}(\Omega)$ and satisfies, for every $a \in B_R(b)$,

$$(i\nabla + A_b)^2(\varphi^a \circ \Phi_a) + \mathcal{L}(\varphi^a \circ \Phi_a) = \lambda^a \, (\varphi^a \circ \Phi_a) \tag{2.26}$$

and

$$\int_\Omega |\varphi^a \circ \Phi_a|^2 |\det(\Phi_a')| \, \mathrm{d}x = 1, \tag{2.27}$$

where $\mathcal{L}$ is a second-order operator of the form

$$\mathcal{L}v = -\sum_{i,j=1}^{2} a^{ij}(x) \frac{\partial^2 v}{\partial x_i \partial x_j} + \sum_{i=1}^{2} b^i(x) \frac{\partial v}{\partial x_i} + c(x)v,$$

with $a^{ij}, b^i, c \in C^\infty(\Omega, \mathbb{C})$ vanishing in $B_R(b)$ and outside of $B_{2R}(b)$. Notice that

$$\Phi_a'(x) = I + \nabla\xi(x) \otimes (a - b)$$

is a small perturbation of the identity whenever $|b - a|$ is sufficiently small, so that the operator in the left hand side of (2.26) is elliptic (see for example [46, Lemma 9.8]).

To prove the differentiability, we will use the implicit function theorem in Banach spaces. To this aim, we define the operator

$$\begin{aligned} &F : B_R(b) \times \mathcal{D}^{1,2}_{A_b}(\Omega) \times \mathbb{C} \to (\mathcal{D}^{1,2}_{A_b}(\Omega))' \times \mathbb{R} \times \mathbb{R} \\ &(a, v, \lambda) \mapsto ((i\nabla + A_b)^2 v + \mathcal{L}v - \lambda v, \int_\Omega |v|^2 |\det(\Phi_a')| \, \mathrm{d}x - 1, 2\,\mathrm{Im} \int_\Omega v\overline{\varphi^b}), \end{aligned} \tag{2.28}$$

where $(i\nabla + A_b)^2 v + \mathcal{L}v - \lambda v \in (\mathcal{D}^{1,2}_{A_b}(\Omega))'$ acts as

$$\begin{aligned} &\langle (i\nabla + A_b)^2 v + \mathcal{L}v - \lambda v, u \rangle_{(\mathcal{D}^{1,2}_{A_b}(\Omega), \mathbb{R})'} \\ &\qquad = \mathrm{Re}\left(\int_\Omega (i\nabla + A_b)v \cdot \overline{(i\nabla + A_b)u} + \mathcal{L}v\overline{u} - \lambda v \overline{u} \, \mathrm{d}x \right), \end{aligned} \tag{2.29}$$

for every $u, v \in \mathcal{D}^{1,2}_{A_b}(\Omega)$.

Notice that F is of class C^∞ by the ellipticity of the operator, provided that R is suficiently small. Moreover, there exists $\omega_a \in \mathbb{R}$ such that $F(a, e^{i\omega_a}\varphi^a \circ \Phi_a, \lambda^a) = 0$ for every $a \in B_R(b)$, as we saw in (2.26), (2.27). Indeed, as long as $\int_\Omega (\varphi^a \circ \Phi_a)\,\overline{\varphi^b} \neq 0$, we can define

$$e^{i\omega_a} = \frac{\overline{\int_\Omega (\varphi^a \circ \Phi_a)\,\overline{\varphi^b}\,\mathrm{d}x}}{\left|\int_\Omega (\varphi^a \circ \Phi_a)\,\overline{\varphi^b}\,\mathrm{d}x\right|}. \tag{2.30}$$

In particular we have $F(b, \varphi^b, \lambda^b) = 0$, since Φ_b is the identity. We now have to verify that the differential of F with respect to the variables (v, λ), evaluated at the point $(b, \varphi^b, \lambda^b)$, that we denote by $\mathrm{d}_{(v,\lambda)}F(b, \varphi^b, \lambda^b)$, belongs to $Inv(\mathcal{D}^{1,2}_{A_b}(\Omega) \times \mathbb{C}, (\mathcal{D}^{1,2}_{A_b}(\Omega))' \times \mathbb{R} \times \mathbb{R})$. The differential is given by

$$\mathrm{d}_{(v,\lambda)}F(b, \varphi^b, \lambda^b) = \begin{pmatrix} (i\nabla + A_b)^2 - \lambda^b Im & -Im\,\varphi^b \\ 2\,\mathrm{Re}\int_\Omega \overline{\varphi^b}\,\mathrm{d}x & 0 \\ 2\,\mathrm{Im}\int_\Omega \overline{\varphi^b}\,\mathrm{d}x & 0 \end{pmatrix},$$

where Im is the compact immersion of $\mathcal{D}^{1,2}_{A_b}(\Omega)$ in $(\mathcal{D}^{1,2}_{A_b}(\Omega))'$, which was introduced in Lemma 2.2.2.

Let us first prove that it is injective. To this aim we have to show that, if $(w, s) \in \mathcal{D}^{1,2}_{A_b}(\Omega) \times \mathbb{C}$ is such that

$$(i\nabla + A_b)^2 w - \lambda^b w = s\varphi^b, \tag{2.31}$$

$$2\,\mathrm{Re}\int_\Omega w\overline{\varphi^b}\,\mathrm{d}x = 0, \tag{2.32}$$

$$2\,\mathrm{Im}\int_\Omega w\overline{\varphi^b}\,\mathrm{d}x = 0, \tag{2.33}$$

then $(w, s) = (0, 0)$. First, if we test (2.31) on φ^b, we have

$$\mathrm{Re}\left(\int_\Omega (i\nabla + A_b)w \cdot \overline{(i\nabla + A_b)\varphi^b} - \lambda^b w\overline{\varphi^b} - s|\varphi^b|^2\right) = 0,$$

thanks to the action of the operator given in (2.29). Inversely, by testing by w the equation satisfied by φ^b, we conclude that

$$\mathrm{Re}(s) = 0.$$

Next, we test (2.31) on $i\varphi^b$ and we obtain

$$\mathrm{Im}\left(\int_\Omega (i\nabla + A_b)w \cdot \overline{(i\nabla + A_b)\varphi^b} - \lambda^b w\overline{\varphi^b} - s|\varphi^b|^2\,\mathrm{d}x\right) = 0.$$

If we test by w the equation satisfied by $-i\varphi^b$, we immediately obtain that

$$\mathrm{Im}(s) = 0.$$

This allows us to conclude that $s = 0$, so that (2.31) becomes

$$(i\nabla + A_b)^2 w = \lambda^b w.$$

The assumption λ^b simple implies that there exists $\alpha \in \mathbb{C}$ such that $w = \alpha\varphi^b$. If we introduce this in (2.32) and (2.33), we obtain successively that $\mathrm{Re}(\alpha) = 0$ and $\mathrm{Im}(\alpha) = 0$, that is $w \equiv 0$.

For the surjectivity, we have to show that for all $(f, r, t) \in (\mathcal{D}^{1,2}_{A_b}(\Omega))' \times \mathbb{R} \times \mathbb{R}$ there exist $(w, s) \in \mathcal{D}^{1,2}_{A_b}(\Omega) \times \mathbb{C}$ which verifies the following equalities

$$(i\nabla + A_b)^2 w - \lambda^b w = f + s\varphi^b, \tag{2.34}$$

$$2\,\mathrm{Re}\int_\Omega w\overline{\varphi^b}\,\mathrm{d}x = r, \tag{2.35}$$

$$2\,\mathrm{Im}\int_\Omega w\overline{\varphi^b}\,\mathrm{d}x = t. \tag{2.36}$$

We recall that the operator $(i\nabla + A_b)^2 - \lambda^b Im : \mathcal{D}^{1,2}_{A_b}(\Omega) \to (\mathcal{D}^{1,2}_{A_b}(\Omega))'$ is Fredholm of index 0. This a standard fact, which can be proved for example noticing that this operator is isomorphic to $Id - \lambda^b((i\nabla + A_b)^2)^{-1} \circ Im$ through the Riesz isomorphism and because the operator $(i\nabla + A_b)^2$ is invertible. This is Fredholm of index 0 because it has the form identity minus compact, the compactness coming from Lemma 2.2.2. Therefore we have (through Riesz isomorphism)

$$\mathrm{Range}((i\nabla + A_b)^2 - \lambda^b Im) = (\mathrm{Ker}((i\nabla + A_b)^2 - \lambda^b Im))^\perp = (\mathrm{span}\{\varphi^b, i\varphi^b\})^\perp, \tag{2.37}$$

where we used the assumption λ^b simple in the last equality, where simple is meant in the complex sense (as explained in the Introduction), that is corresponding to an eigenspace of complex dimension one, or equivalently real dimension two. Here we intend span as a real span. As a consequence, if we test successively (2.34) by φ^b and $i\varphi^b$, we obtain an expression for s

$$s = -\int_\Omega f\overline{\varphi^b}\,\mathrm{d}x.$$

Next we can decompose w in $w_0 + w_1$ such that $w_0 \in \mathrm{Ker}((i\nabla + A_b)^2 - \lambda^b Im)$ and w_1 is in the orthogonal space. Condition (2.34) becomes

$$(i\nabla + A_b)^2 w_1 - \lambda^b w_1 = f - \varphi^b \int_\Omega f\overline{\varphi^b}\,\mathrm{d}x$$

and (2.37) ensures the existence of a solution w_1. Given such w_1, conditions (2.35) and (2.36) determine w_0 as follows

$$w_0 = \frac{1}{2}(r + it)\varphi^b,$$

so that the surjectivity is also proved.

We conclude that the implicit function theorem applies, so that the maps $a \in \Omega \mapsto \lambda^a \in \mathbb{R}$ and $a \in \Omega \mapsto \varphi^a \circ \Phi_a \in \mathcal{D}^{1,2}_{A_b}(\Omega)$ are of class C^∞ locally in a neighbourhood of b. $\square$

By combining the previous result with a standard lemma of local inversion we deduce the following fact, which we will need in the next section.

Corollary 2.5.1. *Let $b \in \Omega$. If λ^b is simple then the map $\Psi : \Omega \times \mathcal{D}^{1,2}_{A_b}(\Omega) \times \mathbb{C} \to \Omega \times (\mathcal{D}^{1,2}_{A_b}(\Omega))' \times \mathbb{R} \times \mathbb{R}$ given by*

$$\Psi(a, v, \lambda) = (a, F(a, v, \lambda)),$$

with F defined in (2.28), is locally invertible in a neighbourhood of $(b, \varphi^b, \lambda^b)$, with inverse Ψ^{-1} of class C^∞.

Proof. We saw in the proof of Theorem 2.1.3 that, if λ^b is simple, then $\mathrm{d}_{(v,\lambda)}F(b, \varphi^b, \lambda^b)$ is invertible. It is sufficient to apply Lemma 2.1 in Chapter 2 of the book of Ambrosetti and Prodi [47]. $\square$

2.6. Vanishing of the derivative at a multiple zero

In this section we prove Theorem 2.1.7 still for $p(x) \equiv 1$. But, we recall that here $\alpha = 1/2$ differently from the previous sections. We will need the following preliminary results.

Lemma 2.6.1. *Let $\lambda > 0$ and let $D_r = D_r(0) \subset \mathbb{R}^2$. Consider the following set of equations for $r > 0$ small*

$$\begin{cases} -\Delta u = \lambda u & \text{in } D_r \\ u = r^{k/2} f + g(r, \cdot) & \text{on } \partial D_r, \end{cases} \tag{2.38}$$

where $f, g(r, \cdot) \in H^1(\partial D_r)$ and g satisfies

$$\lim_{r \to 0} \frac{\|g(r, \cdot)\|_{H^1(\partial D_r)}}{r^{k/2}} = 0 \tag{2.39}$$

for some integer $k \geq 3$. Then for r sufficiently small there exists a unique solution to (2.38), which moreover satisfies

$$\|u\|_{L^2(D_r)} \leq C r^{(k+2)/2} \quad \text{and} \quad \left\| \frac{\partial u}{\partial \nu} \right\|_{L^2(\partial D_r)} \leq C r^{(k-1)/2},$$

where $C > 0$ is independent of r.

Proof. Let first z_1 solve

$$\begin{cases} -\Delta z_1 = 0 & \text{in } D_1 \\ z_1 = f + r^{-k/2} g(r, \cdot) & \text{on } \partial D_1. \end{cases} \tag{2.40}$$

Since the quadratic form

$$\int_{D_1} \left(|\nabla v|^2 - \lambda r^2 v^2 \right) \, \mathrm{d}x \tag{2.41}$$

is coercive for $v \in H^1_0(D_1)$ and for r sufficiently small, there exists a unique solution z_2 to the equation

$$\begin{cases} -\Delta z_2 - \lambda r^2 z_2 = \lambda r^2 z_1 & \text{in } D_1 \\ z_2 = 0 & \text{on } \partial D_1. \end{cases} \tag{2.42}$$

Then $u(x) = r^{k/2}(z_1(x/r) + z_2(x/r))$ is the unique solution to (2.38). In order to obtain the desired bounds on u we will estimate separately z_1 and z_2. Assumption (2.39) implies

$$\|z_1\|_{H^1(D_1)} = \|f + r^{-k/2} g(r, \cdot)\|_{H^{1/2}(\partial D_1)} \leq C \|f\|_{H^1(\partial D_1)}, \tag{2.43}$$

for r sufficiently small. We compare the function z_1 to its limit function when $r \to 0$, which is the harmonic extension of f in D_1, which we will denote w. Then we have

$$\begin{cases} -\Delta(z_1 - w) = 0 & \text{in } D_1 \\ z_1 - w = r^{-k/2} g(r, \cdot) & \text{on } \partial D_1, \end{cases}$$

and hence (2.39) implies

$$\left\| \frac{\partial}{\partial \nu}(z_1 - w) \right\|_{L^2(\partial D_1)} \leq C \|z_1 - w\|_{H^1(\partial D_1)} = C \frac{\|g(r, \cdot)\|_{H^1(\partial D_1)}}{r^{k/2}} \to 0.$$

Then we estimate z_2 as follows

$$\begin{aligned} \|z_2\|^2_{L^2(D_1)} &\leq C \int_{D_1} |\nabla z_2|^2 \, \mathrm{d}x \leq C \int_{D_1} (|\nabla z_2|^2 - \lambda r^2 z_2^2) \, \mathrm{d}x \\ &\leq C \|\lambda r^2 z_1\|_{L^2(D_1)} \|z_2\|_{L^2(D_1)}, \end{aligned}$$

where we used Poincaré inequality, the coercivity of the quadratic form (2.41) and the defintion of z_2 (2.42). Hence estimate (2.43) implies

$$\|z_2\|_{L^2(D_1)} \le Cr^2 \|f\|_{H^1(\partial D_1)} \to 0 \quad \text{as } r \to 0.$$

This and (2.43) provide, by a change of variables in the integral, the desired estimate on $\|u\|_{L^2(D_r)}$. Now, the standard bootstrap argument for elliptic equations applied to (2.42) provides

$$\|z_2\|_{H^2(D_1)} \le C(\|\lambda r^2 z_1\|_{L^2(D_1)} + \|z_2\|_{L^2(D_1)}) \to 0,$$

and hence the trace embedding implies

$$\left\| \frac{\partial z_2}{\partial \nu} \right\|_{L^2(\partial D_1)} \le C\|\nabla z_2\|_{H^1(D_1)} \le C\|z_2\|_{H^2(D_1)} \to 0.$$

So, we have obtained that there exists $C > 0$ independent of r such that

$$\left\| \frac{\partial}{\partial \nu}(z_1 + z_2) \right\|_{L^2(\partial D_1)} \le C.$$

Finally, going back to the function u, we have

$$\left\| \frac{\partial u}{\partial \nu} \right\|_{L^2(\partial D_r)} = r^{(k-1)/2} \left\| \frac{\partial}{\partial \nu}(z_1 + z_2) \right\|_{L^2(\partial D_1)} \le Cr^{(k-1)/2}$$

where we used the change of variable $x = ry$. □

Lemma 2.6.2. *Let $\phi \in \mathcal{D}^{1,2}_{A_a}(\Omega)$ ($a \in \Omega$ close to 0). Then*

$$\frac{1}{|a|^{1/2}} \|\phi\|_{L^2(\partial D_{|a|})} \le C\|\phi\|_{\mathcal{D}^{1,2}_{A_a}(\Omega)} \tag{2.44}$$

where C only depends on Ω.

Proof. Set $\tilde{\phi}(y) = \phi(|a|y)$ defined for $y \in \tilde{\Omega} = \{x/|a| : \ x \in \Omega\}$. We apply this change of variables to the left hand side in (2.44) and then use the trace embedding to obtain

$$\frac{1}{|a|^{1/2}} \|\phi\|_{L^2(\partial D_{|a|})} = \|\tilde{\phi}\|_{L^2(\partial D_1)} \le C\|\tilde{\phi}\|_{H^1(D_1)} \le C\|\tilde{\phi}\|_{H^1(D_2)}.$$

We have that $\tilde{\phi} \in \mathcal{D}^{1,2}_{A_e}(\tilde{\Omega})$, where $e = a/|a|$. Therefore we can apply relation (2.10) as follows

$$\|\tilde{\phi}\|_{L^2(D_2)} \le \|y - e\|_{L^\infty(D_2)} \left\| \frac{\tilde{\phi}}{|y - e|} \right\|_{L^2(D_2)} \le C\|(i\nabla + A_e)\tilde{\phi}\|_{L^2(D_2)},$$

$$
\begin{aligned}
\|\nabla\tilde{\phi}\|_{L^2(D_2)} &\leq \|(i\nabla + A_e)\tilde{\phi}\|_{L^2(D_2)} + \|A_e\tilde{\phi}\|_{L^2(D_2)} \\
&\leq \|(i\nabla + A_e)\tilde{\phi}\|_{L^2(D_2)} + \|(y-e)A_e\|_{L^\infty(D_2)} \left\| \frac{\tilde{\phi}}{|y-e|} \right\|_{L^2(D_2)} \\
&\leq C\|(i\nabla + A_e)\tilde{\phi}\|_{L^2(D_2)}.
\end{aligned}
$$

We combine the previous inequalities obtaining

$$
\frac{1}{|a|^{1/2}}\|\phi\|_{L^2(\partial D_{|a|})} \leq C\|(i\nabla + A_e)\tilde{\phi}\|_{L^2(D_2)} \leq C\|\phi\|_{\mathcal{D}^{1,2}_{A_a}(\Omega)},
$$

where in the last step we used the fact that the quadratic form is invariant under dilations. □

To simplify the notations we suppose without loss of generality that $0 \in \Omega$ and we take $b = 0$. Moreover, we omit the subscript in the notation of the eigenvalues as we did in the previous section. As a first step in the proof of Theorem 2.1.7, we shall estimate $|\lambda^a - \lambda^0|$ in the case when the pole a belongs to a nodal line of φ^0 ending at 0. We make this restriction because all the constructions in the following proposition require that φ^0 vanishes at a.

Proposition 2.6.3. *Suppose that λ^0 is simple and that φ^0 has a zero of order $h/2$ at the origin, with $h \geq 3$ odd. Denote by Γ a nodal line of φ^0 with endpoint at 0 (which exists by Proposition 2.2.10) and take $a \in \Gamma$. Then there exists a constant $C > 0$ independent of a such that*

$$
|\lambda^a - \lambda^0| \leq C|a|^{h/2} \qquad \text{as } a \to 0,\ a \in \Gamma.
$$

Proof. The idea of the proof is to construct a function $u_a \in \mathcal{D}^{1,2}_{A_a}(\Omega)$ satisfying

$$
(i\nabla + A_a)^2 u_a - \lambda^0 u_a = g_a, \qquad \|u_a\|_{L^2(\Omega)} = 1 - \epsilon_a \tag{2.45}
$$

with

$$
\|g_a\|_{(\mathcal{D}^{1,2}_{A_a}(\Omega))'} \leq |a|^{h/2} \qquad \text{and} \qquad |\epsilon_a| \leq |a|^{(h+2)/2} \tag{2.46}
$$

and then to apply the Corollary 2.5.1. For the construction of the function u_a we will heavily rely on the assumption $a \in \Gamma$.

Step 1. Construction of u_a. We define it separately in $D_{|a|} = D_{|a|}(0)$ and in its complement $\Omega \setminus D_{|a|}$, using the following notation

$$
u_a = \begin{cases} u_a^{ext} & \Omega \setminus D_{|a|} \\ u_a^{int} & D_{|a|}. \end{cases} \tag{2.47}
$$

Concerning the exterior function we set

$$u_a^{ext} = e^{i\frac{(\theta_a - \theta_0)}{2}}\varphi^0, \tag{2.48}$$

where θ_a, θ_0 are defined as in Lemma 2.2.7 and Lemma 2.3.2 in such a way that $\theta_a - \theta_0$ is regular in $\Omega \setminus D_{|a|}$ (here $\theta_0 = \theta$ is the the angle in the usual polar coordinates, but we emphasize the position of the singularity in the notation). Therefore u_a^{ext} solves the following magnetic equation

$$\begin{cases} (i\nabla + A_a)^2 u_a^{ext} = \lambda^0 u_a^{ext} & \Omega \backslash D_{|a|} \\ u_a^{ext} = e^{i\frac{(\theta_a - \theta_0)}{2}}\varphi^0 & \partial D_{|a|} \\ u_a^{ext} = 0 & \partial\Omega. \end{cases} \tag{2.49}$$

For the definition of u_a^{int}, we will first consider a related elliptic problem. Notice that, by our choice $a \in \Gamma$, we have that $e^{-i\frac{\theta_0}{2}}\varphi^0$ is continuous on $\partial D_{|a|}$. Indeed, $e^{-i\frac{\theta_0}{2}}$ restricted to $\partial D_{|a|}$ is discontinuous only at the point a, where φ^0 vanishes. Moreover, note that this boundary trace is at least $H^1(\partial D_{|a|})$. Indeed, the eigenfunction φ^0 is C^∞ far from the singularity 0 and $e^{-i\frac{\theta_0}{2}}$ is also regular except on the point a. Then, the boundary trace is differentiable almost everywhere.

This allows to apply Lemma 2.6.1, thus providing the existence of a unique function ψ_a^{int}, solution of the following equation

$$\begin{cases} -\Delta\psi_a^{int} = \lambda^0 \psi_a^{int} & D_{|a|} \\ \psi_a^{int} = e^{-i\frac{\theta_0}{2}}\varphi^0 & \partial D_{|a|}. \end{cases} \tag{2.50}$$

Then we complete our construction of u_a by setting

$$u_a^{int} = e^{i\frac{\theta_a}{2}}\psi_a^{int}, \tag{2.51}$$

which is well defined since θ_a is regular in $D_{|a|}$. Note that u_a^{int} solves the following elliptic equation

$$\begin{cases} (i\nabla + A_a)^2 u_a^{int} = \lambda^0 u_a^{int} & D_{|a|} \\ u_a^{int} = u_a^{ext} & \partial D_{|a|}. \end{cases} \tag{2.52}$$

Step 2. Estimate of the normal derivative of u_a^{int} along $\partial D_{|a|}$. By assumption, φ^0 has a zero of order $h/2$ at the origin, with $h \geq 3$ odd. Hence, by Proposition 2.2.10, the following asymptotic expansion holds on $\partial D_{|a|}$ as $a \to 0$

$$\varphi^0(|a|, \theta_0) = e^{i\frac{\theta_0}{2}}|a|^{h/2}[c_h \cos(\frac{h}{2}\theta_0) + d_h \sin(\frac{h}{2}\theta_0)] + g(|a|, \theta_0), \tag{2.53}$$

with

$$\lim_{|a|\to 0} \frac{\|g(|a|,\cdot)\|_{C^1(\partial D_{|a|})}}{|a|^{h/2}} = 0. \tag{2.54}$$

Hence Lemma 2.6.1 applies to ψ_a^{int} given in (2.50), providing the existence of a constant C independent of a such that

$$\|\psi_a^{int}\|_{L^2(D_{|a|})} \le C|a|^{(h+2)/2} \quad \text{and} \quad \left\| \frac{\partial \psi_a^{int}}{\partial \nu} \right\|_{L^2(\partial D_{|a|})} \le C|a|^{(h-1)/2}. \tag{2.55}$$

Finally, differentiating (2.51) we see that

$$(i\nabla + A_a)u_a^{int} = ie^{i\frac{\theta_a}{2}} \nabla \psi_a^{int},$$

so that, integrating, we obtain the following L^2-estimate for the magnetic normal derivative of u_a^{int} along $\partial D_{|a|}$

$$\|(i\nabla + A_a)u_a^{int} \cdot \nu\|_{L^2(\partial D_{|a|})} \le C|a|^{(h-1)/2}. \tag{2.56}$$

Step 3. Estimate of the normal derivative of u_a^{ext} along $\partial D_{|a|}$. We differentiate (2.48) to obtain

$$(i\nabla + A_a)u_a^{ext} = A_0 u_a^{ext} + ie^{i\frac{(\theta_a-\theta_0)}{2}} \nabla \varphi^0. \tag{2.57}$$

On the other hand, the following holds a.e.

$$\nabla \varphi^0 = iA_0 \varphi^0 + e^{i\frac{\theta_0}{2}} \nabla (e^{-i\frac{\theta_0}{2}} \varphi^0),$$

so that

$$ie^{i\frac{(\theta_a-\theta_0)}{2}} \nabla \varphi^0 = -A_0 u_a^{ext} + ie^{i\frac{\theta_a}{2}} \nabla (e^{-i\frac{\theta_0}{2}} \varphi^0).$$

Combining the last equality with (2.57) we obtain a.e.

$$(i\nabla + A_a)u_a^{ext} = ie^{i\frac{\theta_a}{2}} \nabla (e^{-i\frac{\theta_0}{2}} \varphi^0)$$

and hence $|(i\nabla + A_a)u_a^{ext}| \le C|a|^{h/2-1}$ on $\partial D_{|a|}$ a.e., for some C not depending on a, by (2.53) and (2.54). Integrating on $\partial D_{|a|}$ we arrive at the same estimate as for u_a^{int}, that is

$$\|(i\nabla + A_a)u_a^{ext} \cdot \nu\|_{L^2(\partial D_{|a|})} \le C|a|^{(h-1)/2}. \tag{2.58}$$

Step 4. Proof of (2.46). We test equation (2.49) with a test function $\phi \in \mathcal{D}^{1,2}_{A_a}(\Omega)$ and apply the formula of integration by parts to obtain

$$\int_{\Omega \setminus D_{|a|}} \left\{ (i\nabla + A_a)u_a^{ext} \cdot \overline{(i\nabla + A_a)\phi} - \lambda^0 u_a^{ext} \bar{\phi} \right\} \, dx$$
$$= i \int_{\partial D_{|a|}} (i\nabla + A_a)u_a^{ext} \cdot \nu \bar{\phi} \, d\sigma.$$

Similarly, equation (2.52) provides

$$\int_{D_{|a|}} \left\{(i\nabla + A_a)u_a^{int} \cdot \overline{(i\nabla + A_a)\phi} - \lambda^0 u_a^{int}\bar{\phi}\right\} \,\mathrm{d}x = \\ -i\int_{\partial D_{|a|}} (i\nabla + A_a)u_a^{int} \cdot \nu\bar{\phi}\,\mathrm{d}\sigma.$$

Then, we test the equation in (2.45) with ϕ, we integrate by parts and we replace the previous equalities to get

$$\int_{\Omega} g_a\bar{\phi}\,\mathrm{d}x = i\int_{\partial D_{|a|}} (i\nabla + A_a)(u_a^{ext} - u_a^{int}) \cdot \nu\bar{\phi}\,\mathrm{d}\sigma.$$

To the previous expression we apply first the Hölder inequality and then the estimates obtained in the previous steps (2.56) and (2.58) to obtain

$$\left|\int_{\Omega} g_a\bar{\phi}\,\mathrm{d}x\right| \leq \|(i\nabla + A_a)u_a^{int} \cdot \nu\|_{L^2(\partial D_{|a|})}\|\phi\|_{L^2(\partial D_{|a|})} \\ +\|(i\nabla + A_a)u_a^{ext} \cdot \nu\|_{L^2(\partial D_{|a|})}\|\phi\|_{L^2(\partial D_{|a|})} \leq C|a|^{(h-1)/2}\|\phi\|_{L^2(\partial D_{|a|})}.$$

Finally, Lemma 2.6.2 provides the desired estimate on g_a. Then we estimate ϵ_a as follows. Since $\|u_a^{ext}\|_{L^2(\Omega\setminus D_{|a|})} = \|\varphi^0\|_{L^2(\Omega\setminus D_{|a|})}$ we have

$$\left|\|u_a\|^2_{L^2(\Omega)} - 1\right| = \left|\|u_a^{int}\|^2_{L^2(D_{|a|})} - \|\varphi^0\|^2_{L^2(D_{|a|})}\right| \leq C|a|^{h+2}, \tag{2.59}$$

where in the last inequality we used the fact that $\|\varphi^0\|^2_{L^2(D_{|a|})} \leq C|a|^{h+2}$ by (2.53) and (2.54), and that $\|u_a^{int}\|^2_{L^2(D_{|a|})} = \|\psi_a^{int}\|^2_{L^2(D_{|a|})} \leq C|a|^{h+2}$, by (2.55).

Step 5. Local inversion theorem. To conclude the proof we apply the Corollary 2.5.1.

First, we consider $e^{i\omega_a}$ as in (2.30) (with $b = 0$). Since we proved in Theorem 2.3.4 (see also Remark 2.4.4) that

$$\|\varphi^a \circ \Phi_a - \varphi^0\|_{\mathcal{D}^{1,2}_{A_0}(\Omega)} \to 0, \quad \text{as } a \to 0,$$

we conclude that $e^{i\omega_a} \to 1$. Then, we also have that

$$\|e^{i\omega_a}\varphi^a \circ \Phi_a - \varphi^0\|_{\mathcal{D}^{1,2}_{A_0}(\Omega)} \to 0, \quad \text{as } a \to 0,$$

such that $e^{i\omega_a}\varphi^a \circ \Phi_a$ is also in a small neighbourhood of φ^0.

In the same way, if $\int_\Omega (u_a \circ \Phi_a)\,\overline{\varphi^0}\,\mathrm{d}x \neq 0$, we define

$$e^{i\beta_a} = \frac{\overline{\int_\Omega (u_a \circ \Phi_a)\,\overline{\varphi^0}\,\mathrm{d}x}}{\left|\int_\Omega (u_a \circ \Phi_a)\,\overline{\varphi^0}\,\mathrm{d}x\right|}.$$

This clearly implies that

$$\mathrm{Im} \int_\Omega (u_a \circ \Phi_a)\,\overline{\varphi^0}\,\mathrm{d}x = 0.$$

Since it is not difficult to see that

$$\|u_a \circ \Phi_a - \varphi^0\|_{\mathcal{D}^{1,2}_{A_0}(\Omega)} \to 0, \qquad \text{as } a \to 0,$$

we conclude that $e^{i\beta_a} \to 1$. Then, $e^{i\beta_a} u_a \circ \Phi_a$ is also in a small neighbourhood of φ^0.

Let Ψ be the function defined in Corollary 2.5.1 (recall that here $b = 0$). The construction that we did in the previous steps ensures that

$$\begin{aligned} \Psi(a, e^{i\omega_a}\varphi^a \circ \Phi_a, \lambda^a) &= (a, 0, 0, 0) \\ \Psi(a, e^{i\beta_a} u_a \circ \Phi_a, \lambda^0) &= (a, e^{i\beta_a} g_a \circ \Phi_a, \epsilon_a, 0), \end{aligned}$$

with g_a, ϵ_a satisfying (2.46). Since, we proved in Theorem 2.3.4 that $|\lambda^a - \lambda^0| \to 0$ as $a \to 0$, and by the considerations above, the points $(a, e^{i\omega_a}\varphi^a \circ \Phi_a, \lambda^a)$ and $(a, e^{i\beta_a} u_a \circ \Phi_a, \lambda^0)$ are approaching $(0, \varphi^0, \lambda^0)$ in the space $\Omega \times \mathcal{D}^{1,2}_{A_0}(\Omega) \times \mathbb{C}$ as $a \to 0$. Since Ψ admits an inverse of class C^∞ in a neighbourhood of $(0, \varphi^0, \lambda^0)$ (recall that λ^0 is simple), we deduce that

$$\|(\varphi^a - u_a) \circ \Phi_a\|_{\mathcal{D}^{1,2}_{A_0}(\Omega)} + |\lambda^a - \lambda^0| \leq C(\|g_a\|_{(\mathcal{D}^{1,2}_{A_a}(\Omega))'} + |\epsilon_a|) \leq C|a|^{h/2},$$

for some constant C independent of a, which concludes the proof of the proposition. □

At this point we have proved the desired property only for a pole a belonging to the nodal lines of φ^0. We would like to extend this result to all a sufficiently closed to 0. We will proceed in the following way. Thanks to Theorem 2.1.3, we can consider the Taylor expansion of the function $a \mapsto \lambda^a$ in a neighbourhood of 0. Then Proposition 2.6.3 provides h vanishing conditions, corresponding to the h nodal lines of φ^0. Finally, we will use these conditions to show that in fact the first terms of the polynome are identically zero. Let us begin with a lemma on the existence and the form of the Taylor expansion.

Lemma 2.6.4. *If λ^0 is simple then the following expansion is valid for $a \in \Omega$ sufficiently close to 0 and for all $M \in \mathbb{N}$*

$$\lambda^a - \lambda^0 = \sum_{m=1}^{M} |a|^h P_m(\vartheta(a)) + o(|a|^M),$$

where $a = |a|(\cos\vartheta(a), \sin\vartheta(a))$ and

$$P_m(\vartheta) = \sum_{j=0}^{m} \beta_{j,m} \cos^j \vartheta \sin^{m-j} \vartheta \tag{2.60}$$

for some $\beta_{j,m} \in \mathbb{R}$ not depending on a.

Proof. Since λ^0 is simple, λ^a is also simple for a sufficiently closed to 0. Then we proved in Theorem 2.1.3 that λ^a is C^∞ in the variable a. As a consequence, we can consider the first terms of the Taylor expansion, with Peano rest, of λ^a

$$\lambda^a - \lambda^0 = \sum_{m=1}^{M} \sum_{j=0}^{m} \frac{1}{j!(m-j)!} \frac{\partial^m \lambda^a}{\partial^j a_1 \partial^{m-j} a_2}\Big|_{a=0} a_1^j a_2^{m-j} + o(|a|^M),$$

where $a = (a_1, a_2)$. Setting

$$\beta_{j,m} = \frac{1}{j!(m-j)!} \frac{\partial^m \lambda^a}{\partial^j a_1 \partial^{m-j} a_2}\Big|_{a=0}$$

and $a_1 = |a| \cos\vartheta(a)$, $a_2 = |a| \sin\vartheta(a)$ the thesis follows. □

The following lemma tells us that on the h nodal lines of φ^0, the first low-order polynomials cancel.

Lemma 2.6.5. *Suppose that λ^0 is simple and that φ^0 has a zero of order $h/2$ at 0, with $h \geq 3$ odd. Then there exist an angle $\tilde{\vartheta} \in [0, 2\pi)$ and non-negative quantities $\varepsilon_0, \dots, \varepsilon_{h-1}$ arbitrarily small such that*

$$P_m\left(\tilde{\vartheta} + \frac{2\pi l}{h} + \varepsilon_l\right) = 0 \quad \textit{for every integers } l \in [0, h-1], m \in \left[1, \frac{h-1}{2}\right]$$

where P_m is defined in (2.60).

Proof. We know from Proposition 2.2.10 that φ^0 has h nodal lines with endpoint at 0, which we denote Γ_l, $l = 0, \dots, h-1$. Take points $a_l \in \Gamma_l$, $l = 0, \dots, h-1$, satisfying $|a_0| = \ldots = |a_{h-1}|$ and denote

$$a_l = |a_l|(\cos\vartheta(a_l), \sin\vartheta(a_l)).$$

First we claim that

$$P_m(\vartheta(a_l)) = 0 \quad \text{for every integers } l \in [0, h-1], m \in \left[1, \frac{h-1}{2}\right].$$

Indeed, suppose by contradiction that this is not the case for some l, m belonging to the intervals defined above. Then for such l, m the following holds by Lemma 2.6.4

$$\lambda^{a_l} - \lambda^0 = C|a_l|^m + o(|a_l|^m) \quad \text{for some } C \neq 0.$$

On the other hand we proved in Proposition 2.6.3 that there exists $C > 0$ independent of a such that, for every $l = 0, \dots, h-1$, we have

$$|\lambda^{a_l} - \lambda^0| \leq C|a_l|^{h/2} \quad \text{as } a_l \to 0.$$

This contradicts the last estimate because $m \leq (h-1)/2$, so that the claim is proved.

Finally setting $\tilde{\vartheta} := \vartheta(a_0)$, Proposition 2.2.10 implies

$$\vartheta(a_l) = \tilde{\vartheta} + \frac{2\pi l}{h} + \varepsilon_l \qquad l = 1, \dots, h-1$$

with $\varepsilon_l \to 0$ as $a_l \to 0$. □

The next lemma extends the previous property to all a close to 0.

Lemma 2.6.6. *Fix $h \geq 3$ odd. For any integer $m \in [1, (h-1)/2]$ consider any polynomial of the form*

$$P_m(\vartheta) = \sum_{j=0}^{m} \beta_{j,m} \cos^j \vartheta \sin^{m-j} \vartheta, \tag{2.61}$$

with $\beta_{j,m} \in \mathbb{R}$. Suppose that there exist $\tilde{\vartheta} \in [0, 2\pi)$ and $\varepsilon_0, \dots, \varepsilon_{h-1}$ satisfying $0 \leq \varepsilon_l \leq \pi/(4h)$ such that

$$P_m\left(\tilde{\vartheta} + \frac{2\pi l}{h} + \varepsilon_l\right) = 0 \quad \textit{for every integer } l \in [0, h-1].$$

Then $P_m \equiv 0$.

Proof. We prove the result by induction on m.

Step 1. Let $m = 1$, then

$$P_1(\vartheta) = \beta_{0,1}\sin\vartheta + \beta_{1,1}\cos\vartheta$$

and the following conditions hold for $l = 0, \dots, h-1$

$$\beta_{0,1}\sin\left(\tilde{\vartheta} + \frac{2\pi l}{h} + \varepsilon_l\right) + \beta_{1,1}\cos\left(\tilde{\vartheta} + \frac{2\pi l}{h} + \varepsilon_l\right) = 0. \tag{2.62}$$

In case for every $l = 0, \dots, h-1$ we have

$$\sin\left(\tilde{\vartheta} + \frac{2\pi l}{h} + \varepsilon_l\right) \neq 0 \quad \text{and} \quad \cos\left(\tilde{\vartheta} + \frac{2\pi l}{h} + \varepsilon_l\right) \neq 0,$$

system (2.62) has two unknowns $\beta_{0,1}, \beta_{1,1}$ and $h \geq 3$ linearly independent equations. Hence in this case $\beta_{0,1} = \beta_{1,1} = 0$ and $P_1 \equiv 0$. In case there exists l such that

$$\sin\left(\tilde{\vartheta} + \frac{2\pi l}{h} + \varepsilon_l\right) = 0$$

then of course $\cos\left(\tilde{\vartheta} + 2\pi l/h + \varepsilon_l\right) \neq 0$, which implies $\beta_{1,1} = 0$. We claim that in this case

$$\sin\left(\tilde{\vartheta} + \frac{2\pi l'}{h} + \varepsilon_{l'}\right) \neq 0 \tag{2.63}$$

for every integer $l' \in [0, h-1]$ different from l. To prove the claim we proceed by contradiction. We can suppose without loss of generality that

$$\tilde{\vartheta} + \frac{2\pi l}{h} + \varepsilon_l = 0 \quad \text{and} \quad \tilde{\vartheta} + \frac{2\pi l'}{h} + \varepsilon_{l'} = \pi.$$

Then

$$l = -\frac{h}{2\pi}(\tilde{\vartheta} + \varepsilon_l) \quad \text{and} \quad l' = \frac{h}{2\pi}(\pi - \tilde{\vartheta} - \varepsilon_{l'})$$

so that

$$l' - l = \frac{h}{2} + h\frac{\varepsilon_l - \varepsilon_{l'}}{2\pi}.$$

The assumption $0 \leq \varepsilon_l \leq \pi/(4h)$ implies

$$\frac{h}{2} - \frac{1}{8} \leq l' - l \leq \frac{h}{2} + \frac{1}{8}.$$

Being $h \geq 3$ an odd integer, the last estimate provides $l' - l \notin \mathbb{N}$, which is a contradiction. Therefore we have proved (2.63). Now consider any of the equations in (2.62) for $l' \neq l$. Inserting the information $\beta_{1,1} = 0$ and (2.63) we get $\beta_{0,1} = 0$ and hence $P_1 \equiv 0$. In case one of the cosine vanishes one can proceed in the same way, so we have proved the basis of the induction.

Step 2. Suppose that the statement is true for some $m \leq (h-3)/2$ and let us prove it for $m+1$. The following conditions hold for $l = 0, \dots, h-1$

$$\sum_{j=0}^{m+1} \beta_{j,m+1} \cos^j \left(\tilde{\vartheta} + \frac{2\pi l}{h} + \varepsilon_l \right) \sin^{m+1-j} \left(\tilde{\vartheta} + \frac{2\pi l}{h} + \varepsilon_l \right) = 0. \tag{2.64}$$

We can proceed similarly to Step 1. If none of the sinus, cosinus vanishes then we have a system with $m+2 \leq (h+1)/2$ unknowns and h linearly independent equations, hence $P_{m+1} \equiv 0$. Otherwise suppose that there exists l such that

$$\sin \left(\tilde{\vartheta} + \frac{2\pi l}{h} + \varepsilon_l \right) = 0.$$

Then we saw in Step 1 that

$$\cos \left(\tilde{\vartheta} + \frac{2\pi l}{h} + \varepsilon_l \right) \neq 0 \quad \text{and} \quad \sin \left(\tilde{\vartheta} + \frac{2\pi l'}{h} + \varepsilon_{l'} \right) \neq 0$$

for every integer $l' \in [0, h-1]$ different from l. By rewriting P_{m+1} in the following form

$$P_{m+1}(\vartheta) = \sin \vartheta P_m(\vartheta) + \beta_{m+1,m+1} \cos^{m+1} \vartheta,$$

with P_m as in (2.61), we deduce both that $\beta_{m+1,m+1} = 0$ and that

$$P_m \left(\tilde{\vartheta} + \frac{2\pi l'}{h} + \varepsilon_{l'} \right) = 0$$

for every $l' \in [0, h-1]$ different from l. These are $h-1$ conditions for a polynomial of order $m \leq (h-3)/2$, so the induction hypothesis implies $P_m \equiv 0$ and in turn $P_{m+1} \equiv 0$. □

End of the proof of Theorem 2.1.7. Take any $a \in \Omega$ sufficiently close to 0, then by Lemma 2.6.4

$$\lambda^a - \lambda^0 = \sum_{m=1}^{M} |a|^m P_m(\vartheta(a)) + o(|a|^M).$$

By combining Lemmas 2.6.5 and 2.6.6 we obtain that $P_m \equiv 0$ for every $m \in [1, (h-1)/2]$, therefore $|\lambda^a - \lambda^0| \leq C|a|^{(h+1)/2}$ for some constant C independent of a. □

2.7. Asymptotic behaviour of simple eigenvalues when the pole is close to the boundary

The aim of this section is to prove Theorems 2.1.11 and 2.1.14. For simplicity, we will assume without loss of generality that $0 \in \partial\Omega$ and that $a \to 0$. Remember that in this section, we work with a circulation $\alpha = 1/2$, with a domain Ω of class C^∞ and with a weight $p(x)$ satisfying (2.4).

First of all, we need to rectify locally the boundary of Ω to be able to work in half-balls. This is the aim of the following section.

2.7.1. Equation on a domain with locally rectified boundary

The local analysis near $0 \in \partial\Omega$ is easier if the boundary is locally flat at 0, i.e. if there exists $r > 0$ such that

$$\Omega \cap D_r(0) = \{x \in D_r(0) \,|\, x_1 > 0\} =: D_r^+(0). \tag{2.65}$$

If $\partial\Omega$ is sufficiently regular, we can locally rectify it without altering the nature of the problem, as we show in the following lemma. This is not the case when $\partial\Omega$ presents an angle, as for the angular sector presented in the Introduction, see Appendix A.1.

Lemma 2.7.1. *Let $\Omega, \Omega' \subset \mathbb{C}$ be open, bounded and simply connected domains with C^∞ boundary. There exist a conformal transformation $\Phi : \Omega' \to \Omega$, $\Phi \in C^\infty(\overline{\Omega'})$ and a function $\chi \in C^\infty(\overline{\Omega'})$ such that, letting $w_k^{a'} = e^{-i\chi}(\varphi_k^a \circ \Phi)$, we have*

$$(i\nabla + A_{a'})^2 w_k^{a'} = \lambda_k^a p'(x) w_k^{a'}, \quad w_k^{a'} \in \mathcal{D}^{1,2}_{A_{a'}}(\Omega'),$$

where $\Phi(a') = a$ and p' satisfies (2.4).

Proof. The existence of a regular conformal map is ensured by the Riemann Mapping Theorem. We note that Φ is regular up to the boundary thanks to the assumption that Ω, Ω' have C^∞ boundary (see for example [48, Theorem 5.2.4]).

First define $v_k^{a'} = \varphi_k^a \circ \Phi$. This function solves

$$(i\nabla + B_{a'})^2 v_k^{a'} = \lambda_k^a p'(x) v_k^{a'},$$

where the weight is given by $p'(x) = |\Phi'|^2 p \circ \Phi(x)$, $B_{a'} = (\Phi'^{\,t} \cdot A_a) \circ \Phi$ and where Φ' is the matrix of the derivatives of Φ. Indeed, remember that the magnetic potential A_a is defined as $A_a = \nabla\theta_a/2$ almost everywhere in Ω. Under the conformal transformation, the magnetic potential will transform as

$$B_{a'} = \frac{\nabla(\theta_a \circ \Phi)}{2} = (\Phi'^{\,t} \cdot A_a) \circ \Phi.$$

Next, we can verify explicitly that this new magnetic potential $B_{a'}$ has the same circulation than A_a. Indeed, if we consider a closed path γ, winding once around the point a' in Ω', we obtain

$$\begin{aligned}\frac{1}{2\pi}\oint_{x\equiv\gamma} B_{a'}^t(x)|_{x=\gamma(t)} \cdot \gamma'(t)\,\mathrm{d}t &= \frac{1}{2\pi}\oint_{x\equiv\gamma} (A_a^t \cdot \Phi')(\Phi(x))|_{x=\gamma(t)} \cdot \gamma'(t)\,\mathrm{d}t \\ = \frac{1}{2\pi}\oint_{y\equiv\sigma} (A_a^t \cdot \Phi')(y)|_{y=\sigma(t)} \cdot \Phi'^{\,-1}\sigma'(t)\,\mathrm{d}t &= \frac{1}{2\pi}\oint_{y\equiv\sigma} A_a^t(y)|_{y=\sigma(t)} \cdot \sigma'(t)\,\mathrm{d}t,\end{aligned}$$

where we used the change of variable $y = \Phi(x)$. The new path $\sigma = \Phi \circ \gamma$ is a closed path winding once around a in Ω and its derivative is $\gamma' = \Phi'^{\,-1} \cdot \sigma'$. Moreover, we can explicitly verify that $\nabla \times B_{a'} = 0$ everywhere in $\Omega' \backslash \{a'\}$ by using the fact that Φ is conformal and since A_a satisfies (2.12). Then, by Lemma 2.2.5 and the regularity of Ω', there exists $\chi \in C^\infty(\overline{\Omega'})$ such that $B_{a'} = A_{a'} + \nabla\chi$, where $A_{a'}$ has the form given in (2.2) (the gradient term being 0).

Finally, we can gauge away the term $\nabla\chi$ by letting $w_k^{a'} = e^{-i\chi} v_k^{a'}$. □

Remark 2.7.2. *In fact, in what follows, it will be enough to have a weight p of class $C^1(\overline{\Omega'})$. Thanks to [48, Theorem 5.2.4], to this aim it is sufficient to assume $\partial\Omega \in C^{2,\gamma}$ for some positive γ. This ensures that Φ in the previous lemma is $C^2(\overline{\Omega'})$, so that the transformed weight $p'(x) = |\Phi'|^2 p \circ \Phi(x)$ is $C^1(\overline{\Omega'})$.*

In the following, we want to work in the rectified domain Ω', and with the singular point a'. Indeed, in that case, we can prove some nice inequalities (see below). In Sections 2.7.3 and 2.7.5, we will study the behaviour of the eigenfunctions $\varphi_j^{a'}$ as $a' = (a_1', a_2')$ approaches $0 \in \partial\Omega'$. As we will see later, the significant parameter will be the distance from a' to the boundary $\partial\Omega'$. Such distance is a_1' if $\partial\Omega'$ is locally flat around 0 and $|a'|$ is sufficiently small. We will perform the analysis in half balls centred at $(0, a_2')$; all the future estimates will be independent from a_2'. To simplify the notations, we write

$$\pi(a') = (0, a_2'),$$

π corresponding then to the projection onto the x_2-axis, so that

$$D_r^+(\pi(a')) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 - a_2')^2 < r^2,\ x_1 > 0\}.$$

However, our main interest is the general domain Ω. When we have proved our theorems in the locally flattened domain Ω', we would like to come back to the initial domain. Since Φ is a conformal transformation, it has the nice property of preserving locally the angles. Then, heuristically we can say that locally it consists of a rotation and a dilatation. Therefore, we can find a relation between the distance from a' to the boundary $\partial\Omega'$ (which is as said before given by a'_1) and the distance from a to the boundary $\partial\Omega$, in the case where a' and a are respectively closed to $0 \in \partial\Omega'$ and $b = \Phi(0) \in \partial\Omega$. Indeed, $\operatorname{dist}(a, \partial\Omega) = \inf_{x\in\partial\Omega} |a - x| = \inf_{x'\in\partial\Omega'} |\Phi(a') - \Phi(x')|$. By developing in series locally around 0, we get $\inf_{x'\in\partial\Omega'} |\Phi'(0)|\, |a' - x'| + o(|a' - x'|)$. This infimum is reached for some $x' = \pi(a') + o(a'_1)$, in such a way that

$$\operatorname{dist}(a, \partial\Omega) = |\Phi'(0)|\, a'_1 + o(a'_1). \tag{2.66}$$

Then, for a' close to 0 and a close to b, both distances differ by the conformal factor $|\Phi'(0)|$, at the limit.

In the following, by simply renaming the weight $p(x)$ in equation (2.3), we will always work in a new domain satisfying (2.65). Moreover, for simplicity we call this rectified domain Ω (and not Ω') and the singular point a (and not a'). Below, we obtain some general inequalities for functions $u \in H^1_{A_a}(D_r^+(\pi(a)))$ with $u = 0$ on $\{x_1 = 0\}$, $a \in D_r^+(\pi(a))$.

Lemma 2.7.3 (Poincaré inequality)**.** *Let $a \in D_r^+(\pi(a))$. For every $u \in H^1_{A_a}(D_r^+(\pi(a)))$, with $u = 0$ on $\{x_1 = 0\}$, the following inequality is verified*

$$\frac{1}{r^2}\int_{D_r^+(\pi(a))} |u|^2\,\mathrm{d}x \le \frac{1}{r}\int_{\partial D_r^+(\pi(a))} |u|^2\,\mathrm{d}\sigma + \int_{D_r^+(\pi(a))} |(i\nabla + A_a)u|^2\,\mathrm{d}x. \tag{2.67}$$

Proof. By explicit calculation we see that, for every $u \in H^1_{A_a}(D_r^+(\pi(a)))$, we have for almost every $x \in D_r^+(\pi(a))$

$$\nabla\cdot(|u|^2(x - \pi(a))) = 2\operatorname{Re}\left(iu\,\overline{(i\nabla + A_b)u\cdot(x - \pi(a))}\right) + 2|u|^2. \tag{2.68}$$

Then

$$\begin{aligned}
&\int_{D_r^+(\pi(a))} |u|^2\mathrm{d}x \\
&= -\operatorname{Re}\left(i\int_{D_r^+(\pi(a))} u\,\overline{(i\nabla + A_a)u\cdot(x - \pi(a))}\mathrm{d}x\right) + \frac{r}{2}\int_{\partial D_r^+(\pi(a))} |u|^2\,\mathrm{d}\sigma \\
&\le \frac{1}{2}\int_{D_r^+(\pi(a))} |u|^2\mathrm{d}x + \frac{r^2}{2}\int_{D_r^+(\pi(a))} |(i\nabla + A_a)u|^2\mathrm{d}x + \frac{r}{2}\int_{\partial D_r^+(\pi(a))} |u|^2\,\mathrm{d}\sigma,
\end{aligned}$$

where we used the Young inequalities and the fact that $(x - \pi(a)) = r\nu$ on $\partial D_r^+(\pi(a))$, ν being the unit normal vector. This proves the statement. $\square$

Similarly, we can prove the corresponding Poincaré inequality for all functions v in $H^1(D_r^+(0), \mathbb{R})$ with zero boundary conditions on $\{x_1 = 0\}$,

$$\frac{1}{r^2} \int_{D_r^+(0)} |v|^2 \, \mathrm{d}x \leq \frac{1}{r} \int_{\partial D_r^+(0)} |v|^2 \, \mathrm{d}\sigma + \int_{D_r^+(0)} |\nabla v|^2 \, \mathrm{d}x. \tag{2.69}$$

Lemma 2.7.4. *Let $a \in D_r^+(\pi(a))$. For $u \in H^1_{A_a}(D_r^+(\pi(a)))$, with $u = 0$ on $\{x_1 = 0\}$, the following holds*

$$\frac{1}{r} \int_{\partial D_r^+(\pi(a))} |u|^2 \, \mathrm{d}\sigma \leq \int_{D_r^+(\pi(a))} |(i\nabla + A_a)u|^2 \, \mathrm{d}x. \tag{2.70}$$

Proof. We will prove the following statement: for all $v \in H^1(D_r^+(\pi(a)), \mathbb{R})$, with $v = 0$ on $\{x_1 = 0\}$, we have

$$\frac{1}{r} \int_{\partial D_r^+(\pi(a))} |v|^2 \, \mathrm{d}\sigma \leq \int_{D_r^+(\pi(a))} |\nabla v|^2 \, \mathrm{d}x. \tag{2.71}$$

The lemma follows from it by taking $v = |u|$ and by applying the diamagnetic inequality. It is sufficient to prove (2.71) in the ball $D_1^+(0)$ since the general case can be recovered by performing a translation and a dilation. Let

$$\beta = \inf \Big\{ \int_{D_1^+(0)} |\nabla w|^2 \mathrm{d}x \ : \\ \int_{\partial D_1^+(0)} |w|^2 \, \mathrm{d}\sigma = 1 \,, \, w \in H^1(D_1^+(0), \mathbb{R}) \,, \, w = 0 \text{ on } \{x_1 = 0\} \Big\}.$$

Let w_k be a minimizing sequence. Then $\sup_k \int_{D_1^+(0)} |\nabla w_k|^2 \mathrm{d}x \leq C$ and by the Poincaré inequality (2.69) we have $\sup_k \|w_k\|_{H^1(D_1^+(0), \mathbb{R})} \leq C'$. Hence there exists a function $\bar{w} \in H^1(D_1^+(0), \mathbb{R})$ such that, up to a subsequence,

$$w_k \rightharpoonup \bar{w} \ \text{ in } \ H^1(D_1^+(0), \mathbb{R}) \quad \text{ and } \quad w_k \to \bar{w} \ \text{ in } \ L^2(D_1^+(0), \mathbb{R})$$

by the compact injection. Using the lower semi-continuity of the H^1-norm we obtain

$$\int_{D_1^+(0)} |\nabla \bar{w}|^2 \mathrm{d}x \leq \liminf_{k \to +\infty} \int_{D_1^+(0)} |\nabla w_k|^2 \mathrm{d}x = \beta.$$

By the compact embedding $H^1(\Omega) \hookrightarrow L^2(\partial\Omega)$ (see e.g. [49, Theorem 5.36]) we also have $\int_{\partial D_1^+(0)} |\bar{w}|^2 \, d\sigma = 1$. Therefore $\bar{w}$ is a minimizer and solves the associated Euler-Lagrange equation

$$\begin{cases} -\Delta \bar{w} = 0 & D_1^+(0) \\ \dfrac{\partial \bar{w}}{\partial \nu} = \beta \bar{w} & \partial D_1^+(0) \\ \bar{w} = 0 & x_1 = 0. \end{cases}$$

If we decompose the boundary trace in Fourier series in the following way $\bar{w}(1,\theta) = \sum_{k \text{ odd}} a_k \cos(k\theta) + \sum_{k \text{ even}} b_k \sin(k\theta)$, $\theta \in [-\frac{\pi}{2}, \frac{\pi}{2}]$, then $\bar{w}(r,\theta) = \sum_{k \text{ odd}} a_k r^k \cos(k\theta) + \sum_{k \text{ even}} b_k r^k \sin(k\theta)$. The boundary conditions imply $k\, a_k = \beta\, a_k$ and $k\, b_k = \beta\, b_k$ for every $k \geq 1$. We deduce that $\bar{w}(r,\theta) = a_\beta r^\beta \cos(\beta\theta)$ or $\bar{w}(r,\theta) = b_\beta r^\beta \sin(\beta\theta)$ for some integer $\beta \geq 0$. Since $\bar{w}$ can not vanish because of the constraint, the infimum is assumed by $\beta = 1$ and $\bar{w} = \sqrt{\frac{2}{\pi}} x_1$. □

2.7.2. Pole approaching the boundary on a nodal line of φ_k

In this section we prove Theorem 2.1.11. We use some of the ideas introduced in Section 2.6. As we already mentioned, we modify the argument therein both in order to avoid the use of local inversion methods and in order to prove that the convergence $\lambda_k^a \to \lambda_k$ takes place from below.

We adopt the usual notation $f(x) = o(g(x))$ as $x \to x_0$ if $\lim_{x\to x_0} f(x)/g(x)$ is zero, $f(x) = O(g(x))$ as $x \to x_0$ if $\limsup_{x\to x_0} |f(x)/g(x)|$ is finite, $f(x) \sim g(x)$ as $x \to x_0$ if $\lim_{x\to x_0} f(x)/g(x)$ is finite and different from zero.

Lemma 2.7.5. *Let $\lambda > 0$ and p satisfy (2.4). Let $a \to 0$ so that $\frac{a}{|a|} \to e \notin \{x_1 = 0\}$. Consider the following set of equations in the parameter a*

$$\begin{cases} (i\nabla + A_a)^2 v = \lambda p(x) v & \text{in } D_{2|a|}^+(0) \\ v = 0 & \text{on } \{x_1 = 0\} \\ v = e^{i\frac{\theta_a}{2}} \left\{ |a|^{n+1} f + g(2|a|, \cdot) \right\} & \text{on } \partial D_{2|a|}^+(0) \cap \mathbb{R}^2_+, \end{cases} \tag{2.72}$$

where $f, g(2|a|,\cdot) \in H^1(\partial D_{2|a|}^+(0) \cap \mathbb{R}^2_+)$ are real valued, vanish at $-\pi/2$ and at $\pi/2$, $f \not\equiv 0$ and, for some $n \in \mathbb{N}_0$,

$$\lim_{|a|\to 0} \frac{\|g(2|a|,\cdot)\|_{H^1(\partial D_{2|a|}^+(0) \cap \mathbb{R}^2_+)}}{|a|^{n+1}} = 0.$$

Then for $|a|$ sufficiently small, there exists a unique solution of (2.72) *in* $H^1_{A_a}(D^+_{2|a|}(0) \cap \mathbb{R}^2_+)$*, which moreover satisfies*

$$\|v\|_{L^2(\partial D^+_{2|a|}(0))} \sim |a|^{n+\frac{3}{2}}, \; \|(i\nabla + A_a)v \cdot \nu\|_{L^2(\partial D^+_{2|a|}(0) \cap \mathbb{R}^2_+)} \sim |a|^{n+\frac{1}{2}},$$
$$\|v\|_{L^2(D^+_{2|a|}(0))} \sim |a|^{n+2}, \; \left| \int_{\partial D^+_{2|a|}(0)} (i\nabla + A_a)v \cdot \nu v \, \mathrm{d}\sigma \right| \sim |a|^{2n+2}.$$

Proof. Notice that the boundary trace is continuous on $\partial D^+_{2|a|}(0)$. Indeed, by Lemma 2.2.7, we can choose θ_a to be discontinuous on the segment joining a with the origin, so that $e^{i\theta_a/2}$ restricted to the boundary is discontinuous only at the origin, where the boundary trace vanishes. The existence and uniqueness of the minimizer follow plainly by the fact that the quadratic form

$$\int_{D^+_{2|a|}(0)} \left[|(i\nabla + A_a)v|^2 - \lambda p(x)|v|^2 \right] \mathrm{d}x$$

is coercive for $|a|$ sufficiently small.

The estimate on the $L^2(\partial D^+_{2|a|}(0))$-norm is immediate. In order to prove the remaining estimates, we will make use of some computations contained in the proof of Lemma 2.6.1.

Let Φ_a be a family of conformal transformations in the parameter a satisfying the properties

$$\Phi_a^{-1} \in C(\overline{D^+_2(0)}), \quad \Phi_a^{-1}(D^+_2(0)) = D^+_2(0), \quad \text{and} \quad \Phi_a^{-1}\left(\frac{a}{|a|}\right) = e.$$

Since $a/|a| \to e$, we have that Φ_a is a small perturbation of the identity. Moreover, thanks to the fact that $e \notin \{x_1 = 0\}$, we obtain that

$$|\Phi_a'| \leq C \text{ uniformly in } a. \tag{2.73}$$

Making use of such maps Φ_a, we decompose v into the sum of two functions

$$v(x) = |a|^{1+n} z_1 \left(\Phi_a^{-1}\left(\frac{x}{|a|}\right)\right) + |a|^{1+n} z_2 \left(\Phi_a^{-1}\left(\frac{x}{|a|}\right)\right),$$

where

$$(i\nabla + A_e)^2 z_1 = 0 \text{ in } D^+_2(0), \qquad z_1 = |a|^{-(1+n)} v(|a|\Phi_a) \text{ on } \partial D^+_2(0),$$
$$(i\nabla + A_e)^2 z_2 = \lambda \, |a|^2 \, p(|a|\Phi_a) \, |\Phi_a'|^2 \, (z_1 + z_2) \text{ in } D^+_2(0), \qquad z_2 = 0 \text{ on } \partial D^+_2(0).$$

These transformed equations have the advantage that both the domain of validity, $D_2^+(0)$, and the position of the singularity, e, are fixed. The dependence on the parameter a has been transferred to the coefficients of the equation (and is still visible on the boundary trace).

Notice that $e^{i\theta_a(|a|\Phi_a)/2} = e^{i\chi}e^{i\theta_e/2}$ for some regular function χ and let

$$\tilde{\Omega} = \left\{ y \in \mathbb{C} \,:\, y^2 + e \in D_2^+(0) \right\}$$

be the double covering, which does not depend on a. By applying Lemma 2.2.8 we see that the new functions

$$\tilde{z}_i(y) = e^{-i\frac{\theta_e}{2} - i\chi} z_i(y^2 + e)$$

are real valued and solve the following elliptic equations in $\tilde{\Omega}$

$$-\Delta \tilde{z}_1 = 0 \text{ in } \tilde{\Omega}, \qquad \tilde{z}_1 = e^{-i\frac{\theta_e}{2} - i\chi}|a|^{-(1+n)}v(|a|\Phi_a(y^2+e)) \text{ on } \partial\tilde{\Omega},$$

which is the same as the relation (2.40), and

$$-\Delta \tilde{z}_2 = \lambda\, |a|^2\, p(|a|\Phi_a(y^2+e))\, |\Phi_a'(y^2+e)|^2\, (\tilde{z}_1 + \tilde{z}_2) \text{ in } \tilde{\Omega}, \qquad \tilde{z}_2 = 0 \text{ on } \partial\tilde{\Omega},$$

which is the same as (2.42). Thanks to the bound (2.73), one can proceed exactly as in the proof of Lemma 2.6.1 and show that $\tilde{z}_2$ provides a negligible contribution in the sense that

$$\|\tilde{z}_2\|_{H^1(\tilde{\Omega})} + \|\nabla \tilde{z}_2 \cdot \nu\|_{L^2(\partial\tilde{\Omega})} \leq C|a|^2,$$

while the contribution of $\tilde{z}_1$ is the following

$$c_1 \leq \|\tilde{z}_1\|_{H^1(\tilde{\Omega})} \leq c_2, \qquad c_1 \leq \|\nabla \tilde{z}_1 \cdot \nu\|_{L^2(\partial\tilde{\Omega})} \leq c_2,$$

for positive constants c_1, c_2 not depending on a. This also provides

$$c_1 \leq \left| \int_{\partial\tilde{\Omega}} \nabla(\tilde{z}_1 + \tilde{z}_2) \cdot \nu (\tilde{z}_1 + \tilde{z}_2)\, \mathrm{d}\sigma \right| \leq c_2.$$

By performing all the inverse changes of variables and going back to the original domain $D_{2|a|}^+(0)$, we obtain the statement. □

Lemma 2.7.6. *Let $k \geq 2$ and let $M = M(|a|) = (m_{ij})$ be a $k \times k$ hermitian matrix depending in a smooth way on the parameter $|a|$. Suppose that there exist $n \in \mathbb{N}_0$ and $C_k > 0$ such that, as $a \to 0$, we have*

$$m_{ii} = \lambda_i + O(|a|^2),\ i = 1, \ldots, k-1, \quad m_{ij} = O(|a|^2),\ i \neq j,\ i, j = 1, \ldots, k-1,$$
$$m_{kk} = \lambda_k - C_k|a|^{2n+2} + o(|a|^{2n+2}), \quad m_{ik} = O(|a|^{n+2}),\ i \neq k.$$

If $\lambda_{k-1} < \lambda_k$, then the greatest eigenvalue of M satisfies

$$\lambda_{max}(M) = \lambda_k - C_k|a|^{2n+2} + o(|a|^{2n+2}) \quad \text{as } a \to 0.$$

Proof. In order to estimate the eigenvalues of M, we compute the determinant of the matrix $B = M - tId = (b_{ij})$. We have

$$\det(B) = \sum_{\sigma \in P_k} \operatorname{sgn}(\sigma) \prod_{i=1}^{k} b_{i\sigma(i)}, \tag{2.74}$$

where σ is a permutation of the set $\{1, \dots, k\}$, P_k is the set of all such permutations, $\operatorname{sgn}(\sigma)$ is the sign of σ and $\sigma(i)$ is the image of the element $i \in \{1, \dots, k\}$ under the action of σ. We recall that a fixed point of σ is an element i such that $\sigma(i) = i$. We define, for $r = 0, \dots, k$,

$$P_{k,r} = \{\sigma \in P_k : \ \sigma \text{ has exactly } r \text{ fixed points}\}.$$

Notice that $P_{k,k} = \{Id\}$ and $P_{k,k-1} = \emptyset$. We split the sum in (2.74) in the following way

$$\det(B) = \prod_{i=1}^{k} b_{ii} + \sum_{r=1}^{k-2} \sum_{\substack{\sigma \in P_{k,r} \\ \sigma(k)=k}} \operatorname{sgn}(\sigma) \prod_{i=1}^{k} b_{i\sigma(i)} + \sum_{r=0}^{k-2} \sum_{\substack{\sigma \in P_{k,r} \\ \sigma(k)\neq k}} \operatorname{sgn}(\sigma) \prod_{i=1}^{k} b_{i\sigma(i)}.$$

Due to the specific form of M we can estimate each piece as follows

$$\prod_{i=1}^{k} b_{ii} = (\lambda_k - C_k|a|^{2n+2} + o(|a|^{2n+2}) - t) \prod_{i=1}^{k-1} (\lambda_i + O(|a|^2) - t),$$

for $r = 1, \dots, k-2$

$$\sum_{\substack{\sigma \in P_{k,r} \\ \sigma(k)=k}} \operatorname{sgn}(\sigma) \prod_{i=1}^{k} b_{i\sigma(i)} = (\lambda_k - C_k|a|^{2n+2} - t) O\left(|a|^{2(k-r)}\right) Q_{r-1}(t, |a|^2),$$

and for $r = 0, \dots, k-2$

$$\sum_{\substack{\sigma \in P_{k,r} \\ \sigma(k)\neq k}} \operatorname{sgn}(\sigma) \prod_{i=1}^{k} b_{i\sigma(i)} = O\left(|a|^{2(n+2)}\right) O\left(|a|^{2(k-r-2)}\right) Q_r(t, |a|^2),$$

where $Q_r(t, |a|^2)$ denotes a polynomial of degree r in the variable t, which depends on $|a|$ with terms of order $O(|a|^2)$. More explicitly, $Q_r(t, |a|^2)$ is given by the sum over any possible choice of r numbers in the set $\{1, \dots, k-1\}$ of a product of r terms of the form $(\lambda_i + O(|a|^2) - t)$, for some $i \neq k$. We can also define $Q_{k-1}(t, |a|^2) = \prod_{i=1}^{k-1}(\lambda_i + O(|a|^2) - t)$ and remark that

$$Q_{k-1}(\lambda_k, 0) = \prod_{i=1}^{k-1}(\lambda_i - \lambda_k) \neq 0, \tag{2.75}$$

where we used the assumption that $\lambda_{k-1} < \lambda_k$.

Let $\varepsilon = |a|^2$. We rewrite the determinant in terms of ε, obtaining

$$\det(B) = (\lambda_k - C_k\varepsilon^{n+1} + o(\varepsilon^{n+1}) - t)\left\{Q_{k-1}(t,\varepsilon) + \sum_{r=1}^{k-2} O\left(\varepsilon^{k-r}\right) Q_{r-1}(t,\varepsilon)\right\} + \sum_{r=0}^{k-2} O\left(\varepsilon^{n+k-r}\right) Q_r(t,\varepsilon) =: f(t,\varepsilon).$$

The assumptions of the implicit function theorem hold for $f(t,\varepsilon)$ at the point $(\lambda_k, 0)$. Indeed, $f(\lambda_k, 0) = 0$, f is at least C^{n+1} in a neighbourhood of $(\lambda_k, 0)$, and $\frac{\partial f}{\partial t}(\lambda_k, 0) = -Q_{k-1}(\lambda_k, 0) \neq 0$ thanks to (2.75). Then there exists a function $\lambda(\varepsilon) \in C^{n+1}$, defined in a neighbourhood of $\varepsilon = 0$, such that $f(\lambda(\varepsilon), \varepsilon) = 0$.

Let us first differentiate this relation with respect to ε and estimate it in $(\lambda_k, 0)$

$$\frac{\partial f}{\partial t}(\lambda_k, 0)\, \lambda'(0) + \frac{\partial f}{\partial \varepsilon}(\lambda_k, 0) = 0.$$

Since $n \geq 1$, $\frac{\partial f}{\partial \varepsilon}(\lambda_k, 0) = 0$ and we conclude that $\lambda'(0) = 0$. We can differentiate $n+1$ times the identity $f(\lambda(\varepsilon), \varepsilon) = 0$ and each time use the relations obtained in the previous step. We have

$$\frac{\partial f}{\partial t}(\lambda_k, 0)\, \lambda^{(j)}(0) + \frac{\partial^j f}{\partial \varepsilon^j}(\lambda_k, 0) = 0,\ j = 1, \dots, n+1.$$

Thanks to the fact that $n + k - r > n + 1$ for all $r = 0, \dots, k-2$, and using (2.75), we deduce

$$\frac{\partial^j f}{\partial \varepsilon^j}(\lambda_k, 0) = 0,\ j = 1, \dots, n, \qquad \frac{\partial^{n+1} f}{\partial \varepsilon^{n+1}}(\lambda_k, 0) = -C_k(n+1)!Q_{k-1}(\lambda_k, 0) \neq 0.$$

Then

$$\lambda^{(j)}(0) = 0,\ j = 1, \dots, n, \qquad \lambda^{(n+1)}(0) = -C_k(n+1)!$$

and we conclude that $\lambda(\varepsilon) = \lambda_k - C_k\varepsilon^{n+1} + o(\varepsilon^{n+1})$ as $\varepsilon \to 0$. □

Proof of Theorem 2.1.11. We can assume without loss of generality that $b = 0$ and moreover, by Lemma 2.7.1, that Ω satisfies (2.65). Let φ_k have a zero of order $h/2 \geq 2$ at 0, corresponding to

$$n = \frac{h}{2} - 1 \geq 1$$

arcs of nodal line ending at 0. Denote by Γ any such piece of nodal line and let $a \in \Gamma$. We shall take advantage of the min-max characterization of the eigenvalues, which we exploit by constructing suitable finite dimensional spaces of competitor functions.

Step 1. Construction of the space of competitors. As shown in Lemma 2.2.7, we can choose the discontinuity of θ_a on the piece of Γ connecting a with the origin, so that

$$\frac{\nabla\theta_a}{2} = A_a \quad \text{globally in } \Omega \setminus D^+_{|a|}(0).$$

For $i = 1, \dots, k$ we define

$$v_i^{ext}(x) = e^{i\frac{\theta_a}{2}(x)}\varphi_i(x), \qquad x \in \Omega \setminus D^+_{2|a|}(0).$$

Since $e^{i\frac{\theta_a}{2}}$ is univalued and regular in $\Omega \backslash D^+_{|a|}(0)$, the gauge invariance implies

$$(i\nabla + A_a)^2 v_i^{ext} = \lambda_i p(x) v_i^{ext} \quad \text{in } \Omega\backslash D^+_{2|a|}(0). \tag{2.76}$$

In the interior of the small disk we take the solution of the magnetic equation in $H^1_{A_a}(D^+_{2|a|}(0))$, having the same boundary trace, that is, for $i = 1, \dots, k$,

$$(i\nabla + A_a)^2 v_i^{int} = \lambda_i p(x) v_i^{int} \quad \text{in } D^+_{2|a|}(0), \qquad v_i^{int} = e^{i\frac{\theta_a}{2}}\varphi_i \quad \text{on } \partial D^+_{2|a|}(0). \tag{2.77}$$

By uniqueness, v_i^{int} can also be characterized as the function which achieves

$$\inf\Big\{ \int_{D^+_{2|a|}(0)} \left[|(i\nabla + A_a)v|^2 - \lambda_i p(x)|v|^2\right] dx : \\ v \in H^1_{A_a}(D^+_{2|a|}(0)),\ v = e^{i\frac{\theta_a}{2}}\varphi_i \text{ on } \partial D^+_{2|a|}(0)\Big\}. \tag{2.78}$$

Though v_i^{int} and v_i^{ext} solve the same equation on the respective domains, the competitor functions defined as

$$v_i = \begin{cases} v_i^{int} & D^+_{2|a|}(0) \\ v_i^{ext} & \Omega\backslash D^+_{2|a|}(0) \end{cases} \tag{2.79}$$

do not solve the equation in Ω. Indeed, we have, for all $\phi \in \mathcal{D}^{1,2}_{A_a}(\Omega)$,

$$\int_\Omega \left[(i\nabla + A_a)v_i \cdot \overline{(i\nabla + A_a)\phi} - \lambda_i p(x) v_i \overline{\phi}\right] \mathrm{d}x \\ = i \int_{\partial D^+_{2|a|}(0)} (i\nabla + A_a)(v_i^{ext} - v_i^{int}) \cdot \nu \overline{\phi} \, \mathrm{d}\sigma, \tag{2.80}$$

where we used the formula of integration by parts, (2.76) and (2.77).

Step 2. Estimates on the single competitor functions. By Proposition 2.2.9, φ_i has the following behaviour on $\partial D^+_{2|a|}(0)$, for $i < k$, as $a \to 0$

$$\varphi_i|_{\partial D^+_{2|a|}(0)} = 2|a| c_1 \cos\theta + o(|a|),$$

with c_1 eventually 0, whereas for φ_k we have as $a \to 0$

$$\varphi_k|_{\partial D^+_{2|a|}(0)} = (2|a|)^{1+n} \left(c_{1+n} \cos[(1+n)\theta] + d_{1+n} \sin[(1+n)\theta]\right) + o(|a|^{1+n}),$$

with $c_{1+n} \neq 0, d_{1+n} = 0$ if n is even and $c_{1+n} = 0, d_{1+n} \neq 0$ if n is odd. Since a belongs to one of the nodal lines of φ_k, Γ, and the tangents to the nodal lines divide π into equal angles, we have that $a/|a| \to e \notin \{x_1 = 0\}$ and we recover the property (2.73). Hence Lemma 2.7.5 applies providing the following estimates

$$\|v_i^{int}\|_{L^2(\partial D^+_{2|a|}(0))} = O(|a|^{\frac{3}{2}}), \ \|v_i^{int}\|_{L^2(D^+_{2|a|}(0))} = O(|a|^2), \\ \|(i\nabla + A_a)v_i^{int} \cdot \nu\|_{L^2(\partial D^+_{2|a|}(0) \cap \Omega)} = O(|a|^{\frac{1}{2}}), \tag{2.81}$$

for $i = 1, \dots, k-1$, and

$$\|v_k^{int}\|_{L^2(\partial D^+_{2|a|}(0))} \sim |a|^{\frac{3}{2}+n}, \ \|(i\nabla + A_a)v_k^{int} \cdot \nu\|_{L^2(\partial D^+_{2|a|}(0) \cap \Omega)} \sim |a|^{\frac{1}{2}+n}, \tag{2.82}$$

$$\|v_k^{int}\|_{L^2(D^+_{2|a|}(0))} \sim |a|^{2+n}, \ \int_{\partial D^+_{2|a|}(0)} (i\nabla + A_a)v_k^{int} \cdot \nu v_k^{int} \, \mathrm{d}\sigma \sim |a|^{2n+2}. \tag{2.83}$$

Step 3. We claim that there exists a constant $C_k > 0$ such that

$$i \int_{\partial D^+_{2|a|}(0)} (i\nabla + A_a)(v_k^{ext} - v_k^{int}) \cdot \nu \overline{v_k} \, \mathrm{d}\sigma = -C_k |a|^{2n+2} + o(|a|^{2n+2}). \tag{2.84}$$

The asymptotic behaviour is consequence of (2.83). Let us prove that the quantity we want to estimate is negative. To this aim, we extend the function $v_k^{ext} = e^{i\frac{\theta_a}{2}}\varphi_k$ to all Ω. Such extension is continuous in $D^+_{2|a|}(0)$, since φ_k vanishes on Γ and θ_a is regular outside Γ, and solves

$$\begin{cases} (i\nabla + A_a)^2 v_k^{ext} = \lambda_k p(x) v_k^{ext} & \Omega\backslash\Gamma \\ v_k^{ext} = 0 & \partial\Omega. \end{cases}$$

Since $v_k^{ext} = 0$ on Γ, we can test this equation by v_k^{ext} itself in $D^+_{2|a|}(0)$ and apply the formula of integration by parts to obtain

$$i\int_{\partial D^+_{2|a|}(0)} (i\nabla + A_a) v_k^{ext}\cdot\nu\overline{v_k}\,\mathrm{d}\sigma \\ = -\int_{D^+_{2|a|}(0)} \left[|(i\nabla + A_a)v_k^{ext}|^2 - \lambda_k p(x)|v_k^{ext}|^2\right]\,\mathrm{d}x.$$

On the other hand, by testing (2.77) by v_k^{int} we obtain the same expression

$$i\int_{\partial D^+_{2|a|}(0)} (i\nabla + A_a) v_k^{int}\cdot\nu\overline{v_k}\,\mathrm{d}\sigma \\ = -\int_{D^+_{2|a|}(0)} \left[|(i\nabla + A_a)v_k^{int}|^2 - \lambda_k p(x)|v_k^{int}|^2\right]\,\mathrm{d}x.$$

By subtracting the two equalities, and recalling the characterization of v_k^{int} in (2.78), we obtain that the boundary integral in (2.84) is negative.

Step 4. Estimate of the eigenvalue. Let

$$F_k = \left\{\Phi = \sum_{i=1}^k \alpha_i v_i:\ \alpha = (\alpha_1,\dots,\alpha_k)\in\mathbb{C}^k\right\} \subset \mathcal{D}^{1,2}_{A_a}(\Omega),$$

where v_i are the competitor functions defined in (2.79). By (2.81), (2.82) we have, for $i\neq j$,

$$\left|\int_\Omega p(x) v_i\overline{v_j}\mathrm{d}x\right| = \left|\int_\Omega p(x)\varphi_i\overline{\varphi_j}\mathrm{d}x + \int_{D^+_{2|a|}(0)} p(x)(v_i^{int}\overline{v_j^{int}} - \varphi_i\overline{\varphi_j})\,\mathrm{d}x\right| \\ \leq C|a|^4$$

(the last estimate improves to $|a|^{4+n}$ in case $i = k$ or $j = k$). Therefore F_k is a k-dimensional subspace of $\mathcal{D}^{1,2}_{A_a}(\Omega)$ for $|a|$ sufficiently small and we have

$$\lambda_k^a \leq \sup_{\Phi\in F_k} \frac{\|\Phi\|^2_{D^{1,2}_{A_a}(\Omega)}}{\int_\Omega p(x)|\Phi|^2\,\mathrm{d}x}.$$

Relation (2.80) provides

$$\|\Phi\|^2_{\mathcal{D}^{1,2}_{A_a}(\Omega)} = \sum_{i,j=1}^{k} \alpha_i \overline{\alpha_j} \Bigg\{ \lambda_i \int_\Omega p(x)\, v_i \overline{v_j}\, \mathrm{d}x \\ + i \int_{\partial D^+_{2|a|}(0)} (i\nabla + A_a)(v_i^{ext} - v_i^{int}) \cdot \nu \overline{v_j}\, \mathrm{d}\sigma \Bigg\}.$$

Thus we can write

$$\lambda_k^a \le \sup_{\alpha \in \mathbb{C}^k} \frac{\overline{\alpha}^T M \alpha}{\overline{\alpha}^T N \alpha} = \lambda_{max}(N^{-1}M),$$

where $\overline{\alpha}^T$ denotes the transposed and the complex conjugation of the vector α, $\lambda_{max}(\cdot)$ is the largest eigenvalue of a matrix and M, N are by construction $k \times k$ hermitian matrices with entries

$$m_{ij} = \lambda_i \int_\Omega p(x)\, v_i \overline{v_j}\, \mathrm{d}x + i \int_{\partial D^+_{2|a|}(0)} (i\nabla + A_a)(v_i^{ext} - v_i^{int}) \cdot \nu \overline{v_j}\, \mathrm{d}\sigma$$

$$n_{ij} = \int_\Omega p(x) v_i \overline{v_j}\, \mathrm{d}x.$$

By exploiting (2.81)-(2.83), we see that M has the form in Lemma 2.7.6 and

$$N = \begin{pmatrix} 1 + O(|a|^4) & & O(|a|^4) & O(|a|^{n+4}) \\ O(|a|^4) & \ddots & & \vdots \\ & & 1 + O(|a|^4) & O(|a|^{n+4}) \\ O(|a|^{n+4}) & \dots & O(|a|^{n+4}) & 1 + O(|a|^{2n+4}) \end{pmatrix}.$$

By writing $N = Id + \mathcal{E}(|a|)$ we have $N^{-1} = \sum_{j=0}^{\infty} (-1)^j \mathcal{E}(|a|)^j \sim Id - \mathcal{E}(|a|)$ as $a \to 0$, so that N^{-1} has the same form as N. Therefore $N^{-1}M$ has the same form as M and we can apply Lemma 2.7.6 obtaining

$$\lambda_k^a \le \lambda_k - C_k |a|^{2n+2} + o(|a|^{2n+2}).$$

The result follows recalling that $2n + 2 = h$. □

2.7.3. Frequency formula for magnetic eigenfunctions at boundary points

Throughout this section we assume that Ω is regular and that $p(x)$ satisfies (2.4). Given a pole $b = (b_1, b_2) \in \mathbb{R}^2$, we recall the following notation from

Section 2.7.1

$$\pi(b) = (0, b_2), \quad D_r^+(\pi(b)) = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + (x_2 - b_2)^2 < r^2, \, x_1 > 0\}.$$

We define a Almgren-type frequency function in $D_r^+(\pi(b))$ as follows.

Definition 2.7.7. *Let $b \in \mathbb{C}$, $r > 0$ such that $b \in D_r^+(\pi(b))$ and $u \in H^1_{A_b}(D_r^+(\pi(b)))$ with $u = 0$ on $\{x_1 = 0\}$. Let $\lambda \in \mathbb{R}$, and $p(x)$ satisfy (2.4) in $D_r^+(\pi(b))$. We define*

$$E(u, r, \pi(b), \lambda, A_b) = \int_{D_r^+(\pi(b))} \left(|(i\nabla + A_b)u|^2 - \lambda p(x)|u|^2 \right) \mathrm{d}x,$$

$$H(u, r, \pi(b)) = \frac{1}{r} \int_{\partial D_r^+(\pi(b))} |u|^2 \, \mathrm{d}\sigma,$$

and the frequency function

$$N(u, r, \pi(b), \lambda, A_b) = \frac{E(u, r, \pi(b), \lambda, A_b)}{H(u, r, \pi(b))}.$$

In the notation above we keep track of all the parameters involved, apart from the weight p, since we will need to let them change from section to section. The weight is not explicitly mentioned because it does not play a significant role, as long as it satisfies (2.4).

In particular, in this section we will estimate the frequency function for $u = \varphi_k^a$ and $\lambda = \lambda_k^a$. We shall omit the index k since we will work with a fixed k from now up to Subsection 2.7.5. By Lemma 2.7.1, we can assume that $\partial\Omega$ is locally flat near the origin, so that we consider the following equation

$$\begin{cases} (i\nabla + A_a)^2 \varphi^a = \lambda^a p(x) \varphi^a & D_{2r_0}^+(0) \\ \varphi^a = 0 & \{x_1 = 0\} \\ \varphi^a \in H^1_{A_a}(D_{2r_0}^+(0)). & \end{cases} \tag{2.85}$$

Here r_0 is chosen such that

$$r_0 < (2\lambda^a \|p\|_{L^\infty})^{-1/2} \text{ for } |a| \text{ sufficiently small,} \tag{2.86}$$

which is possible due to the fact that p is bounded and that $\lambda^a \to \lambda$ as $a \to 0$, as recalled in Theorem 2.1.1. For $r < r_0$ and $|a| < r$ we have that $D_r^+(\pi(a)) \subset D_{2r_0}^+(0)$ so that, for such r and a, the frequency function for solutions of (2.85) is well defined.

Estimates on $H(\varphi^a, r, \pi(a))$

We can compute the derivative of H with respect to r similarly to the standard frequency function for non-magnetic eigenfunctions. In the following $a = (a_1, a_2)$.

Lemma 2.7.8. *If φ^a is a solution of (2.85), for $a_1 < r < r_0$ we have*

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r} H(\varphi^a, r, \pi(a)) &= -\frac{2i}{r} \int_{\partial D_r^+(\pi(a))} (i\nabla + A_a)\varphi^a \cdot \nu \, \overline{\varphi^a} \, \mathrm{d}\sigma \\ &= \frac{2}{r} E(\varphi^a, r, \pi(a), \lambda^a, A_a). \end{aligned} \tag{2.87}$$

Proof. By the change of variables $y = (x - \pi(a))/r$ we have

$$H(\varphi^a, r, \pi(a)) = \frac{1}{r} \int_{\partial D_r^+(\pi(a))} |\varphi^a|^2(x) \, \mathrm{d}\sigma(x) = \int_{\partial D_1^+(0)} |\varphi^a|^2(ry + \pi(a)) \, \mathrm{d}\sigma(y).$$

By taking the derivative with respect to r we obtain

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r} H(\varphi^a, r, \pi(a)) &= 2 \operatorname{Re} \int_{\partial D_1^+(0)} \nabla\varphi^a(ry + \pi(a)) \cdot y \, \overline{\varphi^a(ry + \pi(a))} \, \mathrm{d}\sigma(y) \\ &= \frac{2}{r} \operatorname{Re} \int_{\partial D_r^+(\pi(a))} \nabla\varphi^a \cdot \nu \, \overline{\varphi^a} \, \mathrm{d}\sigma(x) \\ &= \frac{2}{r} \operatorname{Re} \left\{ -i \int_{\partial D_r^+(\pi(a))} (i\nabla + A_a)\varphi^a \cdot \nu \, \overline{\varphi^a} \, \mathrm{d}\sigma \right\}, \end{aligned}$$

where we used the fact that $\operatorname{Re}(-i|\varphi^a|^2 A_a \cdot \nu) = 0$. On the other hand, by testing equation (2.85) by φ^a and integrating by parts, we see that

$$E(\varphi^a, r, \pi(a), \lambda^a, A_a) = -i \int_{\partial D_r^+(\pi(a))} (i\nabla + A_a)\varphi^a \cdot \nu \, \overline{\varphi^a} \, \mathrm{d}\sigma \quad \in \mathbb{R},$$

which concludes the proof. $\square$

We can prove the following estimate.

Lemma 2.7.9. *Let φ^a be a solution of (2.85) and r_0 be as in (2.86). If $a_1 < r_1 < r_2 < r_0$ then*

$$\frac{H(\varphi^a, r_2, \pi(a))}{H(\varphi^a, r_1, \pi(a))} \geq e^{-Cr_0^2} \left(\frac{r_2}{r_1} \right)^2.$$

If $|a|$ is sufficiently small, we can choose $C = 4\lambda \|p\|_{L^\infty}$, where λ is the limit of λ^a as $a \to 0$.

Proof. By combining Lemmas 2.7.3 and 2.7.4 we see that

$$\frac{1}{r^2}\int_{D_r^+(\pi(a))} |\varphi^a|^2 \,\mathrm{d}x \le 2\int_{D_r^+(\pi(a))} |(i\nabla + A_a)\varphi^a|^2 \,\mathrm{d}x. \tag{2.88}$$

Next we apply Lemma 2.7.8 and, in order, the inequalities (2.88) and (2.70) in the following way

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}r}H(\varphi^a, r, \pi(a)) &= \frac{2}{r}\int_{D_r^+(\pi(a))} (|(i\nabla + A_a)\varphi^a|^2 - \lambda^a p(x)|\varphi^a|^2)\,\mathrm{d}x \\ &\ge \frac{2}{r}(1 - 2\lambda^a \|p\|_{L^\infty} r^2)\int_{D_r^+(\pi(a))} |(i\nabla + A_a)\varphi^a|^2 \,\mathrm{d}x \\ &\ge \frac{2}{r}(1 - 2\lambda^a \|p\|_{L^\infty} r^2) H(\varphi^a, r, \pi(a))).\end{aligned}$$

Integrating the last inequality between r_1 and r_2 we obtain

$$\begin{aligned}\log\left(\frac{H(\varphi^a, r_2, \pi(a))}{H(\varphi^a, r_1, \pi(a))}\right) &\ge \log\left(\frac{r_2}{r_1}\right)^2 - 2\lambda^a \|p\|_{L^\infty}(r_2^2 - r_1^2) \\ &\ge \log\left(\frac{r_2}{r_1}\right)^2 - 2\lambda^a \|p\|_{L^\infty} r_0^2.\end{aligned}$$

Taking the exponential of both sides and recalling that $\lambda^a \to \lambda$ we obtain the statement. □

Remark 2.7.10. *Relation* (2.88) *shows that* $E(\varphi^a, r, \pi(a), \lambda^a, A_a) \ge 0$ *for* $r < r_0$.

Estimates on $E(\varphi^a, r, \pi(a), \lambda^a, A_a)$

We will need the following Pohozaev-type identity for the solution φ^a of (2.85). Also compare with [7, Section 4].

Lemma 2.7.11 (Pohozaev-type identity)**.** *If* φ^a *is the solution of* (2.85) *and* r_0 *given in* (2.86), *the following identity is valid for* $a_1 < r < r_0$

$$\begin{aligned}&\frac{r}{2}\int_{\partial D_r^+(\pi(a))} \left\{|(i\nabla + A_a)\varphi^a|^2 - 2|(i\nabla + A_a)\varphi^a \cdot \nu|^2 - \lambda^a p|\varphi^a|^2\right\} \mathrm{d}\sigma \\ &+ \lambda^a \int_{D_r^+(\pi(a))} |\varphi^a|^2 \left(p + \frac{\nabla p \cdot (x - \pi(a))}{2}\right) \mathrm{d}x + M_a = 0,\end{aligned} \tag{2.89}$$

where

$$M_a = \lim_{\varepsilon\to 0}\int_{\partial D_\varepsilon(a)} \Big\{ \mathrm{Re}\Big[(i\nabla + A_a)\varphi^a\cdot\nu\,\overline{(i\nabla + A_a)\varphi^a\cdot(x-\pi(a))}\Big] \\ -\frac{1}{2}|(i\nabla + A_a)\varphi^a|^2(x-\pi(a))\cdot\nu\Big\}\,\mathrm{d}\sigma. \tag{2.90}$$

Proof. We test the equation (2.85) with the vector field $\xi = (i\nabla + A_a)\varphi^a\cdot(x-\pi(a))$ in $D_r^+(\pi(a))\setminus D_\varepsilon(a)$. We need to remove a small ball around the singularity because $\nabla\varphi^a$ may be singular at a (it is singular in the case that φ^a has a zero of order $1/2$ at a). Multiplying by i and taking the real part we obtain

$$\mathrm{Re}\left(i\int_{D_r^+(\pi(a))\setminus D_\varepsilon(a)}(i\nabla + A_a)^2\varphi^a\,\overline{\xi}\,\mathrm{d}x\right) = \mathrm{Re}\left(i\lambda^a\int_{D_r^+(\pi(a))\setminus D_\varepsilon(a)}\varphi^a p\,\overline{\xi}\,\mathrm{d}x\right). \tag{2.91}$$

Similarly to (2.68), the following identity with the weight holds almost everywhere in $D_r^+(\pi(a))\setminus D_\varepsilon(a)$

$$\frac{1}{2}\nabla\cdot\left(p|\varphi^a|^2(x-\pi(a))\right) = |\varphi^a|^2\left(p + \frac{\nabla p\cdot(x-\pi(a))}{2}\right) + \mathrm{Re}\left(ip\varphi^a\overline{\xi}\right).$$

It allows to rewrite the right hand side of (2.91) as

$$\frac{\lambda^a}{2}\int_{\partial(D_r^+(\pi(a))\setminus D_\varepsilon(a))} p|\varphi^a|^2\,(x-\pi(a))\cdot\nu\,\mathrm{d}\sigma \\ -\lambda^a\int_{D_r^+(\pi(a))\setminus D_\varepsilon(a)}|\varphi^a|^2\left(p + \frac{\nabla p\cdot(x-\pi(a))}{2}\right)\mathrm{d}x.$$

Taking the limit as $\varepsilon\to 0$, we obtain

$$\frac{\lambda^a r}{2}\int_{\partial D_r^+(\pi(a))} p|\varphi^a|^2\,\mathrm{d}\sigma - \lambda^a\int_{D_r^+(\pi(a))}|\varphi^a|^2\left(p + \frac{\nabla p\cdot(x-\pi(a))}{2}\right)\mathrm{d}x.$$

The integral on $\partial D_\varepsilon(a)$ vanishes as $\varepsilon\to 0$ because $|\varphi^a|$ behaves at least like $\varepsilon^{1/2}$ on $\partial D_\varepsilon(a)$ by (2.17). Next we look at the left-hand side of (2.91). Integrating by parts and using the following identity valid for a.e. $x\in D_r^+(\pi(a))\setminus D_\varepsilon(a)$

$$\mathrm{Re}\left(i(i\nabla + A_a)\varphi^a\cdot\overline{(i\nabla + A_a)\xi}\right) = \frac{1}{2}\nabla\cdot(|(i\nabla + A_a)\varphi^a|^2\,(x-\pi(a))),$$

we rewrite it as

$$
\begin{aligned}
&\mathrm{Re}\left\{ i\int_{D_r^+(\pi(a))\setminus D_\varepsilon(a)} (i\nabla + A_a)\varphi^a \cdot \overline{(i\nabla + A_a)\xi}\,\mathrm{d}x \right. \\
&\left. - \int_{\partial(D_r^+(\pi(a))\setminus D_\varepsilon(a))} (i\nabla + A_a)\varphi^a \cdot \nu\,\overline{\xi}\,\mathrm{d}\sigma \right\} \\
&= \int_{\partial(D_r^+(\pi(a))\setminus D_\varepsilon(a))} \frac{1}{2}|(i\nabla + A_a)\varphi^a|^2 (x-\pi(a))\cdot\nu\,\mathrm{d}\sigma \\
&- \mathrm{Re}\int_{\partial(D_r^+(\pi(a))\setminus D_\varepsilon(a))} (i\nabla + A_a)\varphi^a \cdot \nu\,\overline{\xi}\ \mathrm{d}\sigma \\
&= r\int_{\partial D_r^+(\pi(a))} \left\{ \frac{1}{2}|(i\nabla + A_a)\varphi^a|^2 - |(i\nabla + A_a)\varphi^a\cdot\nu|^2 \right\}\,\mathrm{d}\sigma \\
&+ \int_{\partial D_\varepsilon(a)} \left\{ \mathrm{Re}[(i\nabla + A_a)\varphi^a\cdot\nu\overline{\xi}] - \frac{1}{2}|(i\nabla + A_a)\varphi^a|^2(x-\pi(a))\cdot\nu \right\}\,\mathrm{d}\sigma.
\end{aligned}
$$

By taking the limit as $\varepsilon \to 0$ and by combining the two contributions in (2.91) we obtain the result. $\square$

This identity allows to compute the derivative of $E(\varphi^a, r, \pi(a), \lambda^a, A_a)$ with respect to r.

Lemma 2.7.12. *If φ^a is a solution of (2.85) and r_0 is given in (2.86), then for $a_1 < r < r_0$ we have*

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}r}E(\varphi^a, r, \pi(a), \lambda^a, A_a) = 2\int_{\partial D_r^+(\pi(a))} &|(i\nabla + A_a)\varphi^a\cdot\nu|^2\,\mathrm{d}\sigma \\
- \frac{\lambda^a}{r}\int_{D_r^+(\pi(a))} &|\varphi^a|^2(2p + \nabla p\cdot(x-\pi(a)))\,\mathrm{d}x - \frac{2}{r}M_a,
\end{aligned}
\tag{2.92}
$$

where M_a is defined in (2.90).

Proof. We have

$$
\frac{\mathrm{d}}{\mathrm{d}r}E(\varphi^a, r, \pi(a), \lambda^a, A_a) = \int_{\partial D_r^+(\pi(a))} \left(|(i\nabla + A_a)\varphi^a|^2 - \lambda^a p|\varphi^a|^2\right)\,\mathrm{d}\sigma.
$$

Then we use the Pohozaev identity (2.89) to conclude. $\square$

In what follows we will estimate the remainder M_a which appears in the derivative of E in equation (2.92).

Lemma 2.7.13. *Let $v(y) = e^{-i\theta(y)}\varphi^a(a_1 y^2 + a)$, defined in the set $\{y : \ a_1 y^2 + a \in D^+_{2r_0}(0)\}$. Then*

$$M_a = \pi \operatorname{Re}\left\{\left(\frac{\partial v(0)}{\partial y}\right)^2\right\}. \tag{2.93}$$

Proof. First we shall prove that

$$\begin{aligned} M_a &= \lim_{\varepsilon\to 0}\int_{\partial D_\varepsilon(a)} \operatorname{Re}\left((i\nabla + A_a)\varphi^a \cdot \nu\, \overline{(i\nabla + A_a)\varphi^a \cdot (x - \pi(a))}\right) \mathrm{d}\sigma \\ &= \frac{a_1\pi}{4}(c_1^2 - d_1^2), \end{aligned} \tag{2.94}$$

where $c_1 = c_1(a)$, $d_1 = d_1(a)$ are the coefficients appearing in the asymptotic expansion (2.17) of φ^a, and $a = (a_1, a_2)$. Indeed, by differentiating (2.17) we obtain almost everywhere

$$(i\nabla + A_a)\varphi^a = \frac{ie^{i\theta_a/2}}{2\sqrt{r_a}}\left(c_1 \cos\frac{\theta_a}{2} - d_1 \sin\frac{\theta_a}{2}, c_1 \sin\frac{\theta_a}{2} + d_1 \cos\frac{\theta_a}{2}\right) + o(r_a^{-1/2}),$$

as $r_a \to 0$ and hence

$$|(i\nabla + A_a)\varphi^a|^2 = \frac{1}{4r_a}(c_1^2 + d_1^2) + o(r_a^{-1}) \quad \text{as } r_a \to 0.$$

Moreover notice that $x - \pi(a) = (a_1, 0) + \varepsilon(\cos\theta_a, \sin\theta_a)$ and $\nu = (\cos\theta_a, \sin\theta_a)$ on $\partial D_\varepsilon(a)$. Therefore

$$\lim_{\varepsilon\to 0}\int_{\partial D_\varepsilon(a)} |(i\nabla + A_a)\varphi^a|^2 (x - \pi(a)) \cdot \nu \, \mathrm{d}\sigma =$$

$$\lim_{\varepsilon\to 0} \varepsilon \int_0^{2\pi} \left[\frac{1}{4\varepsilon}(c_1^2 + d_1^2) + o(\varepsilon^{-1})\right][a_1 \cos\theta_a + \varepsilon]\, \mathrm{d}\theta_a = 0,$$

and we have estimated the second term in (2.90). By a direct calculation one estimates the first term in (2.90) and obtains (2.94).

Now, by changing variables in (2.17), we obtain the following expansion for v

$$v(r, \theta) = \sqrt{a_1} r\,(c_1 \cos\theta + d_1 \sin\theta) + o(r) \quad \text{as } r \to 0.$$

Hence we have $\frac{\partial v(0)}{\partial y} = \frac{\sqrt{a_1}}{2}(c_1 - id_1)$ and (2.93) follows by combining with (2.94). ∎

Remark 2.7.14. *From (2.94), we remark that the constant M_a is identically zero if the eigenfunction φ^a has a zero of order stricly greater than $1/2$ at a.*

Moreover, we can give a geometrical meaning to this constant M_a. In case φ^a has a zero of order $1/2$ at a, for which $M_a \neq 0$, we can rewrite (2.17) in the following way

$$\varphi^a = e^{i\frac{\theta_a}{2}} \sqrt{c_1^2 + d_1^2}\, \sqrt{r_a} \cos\left(\frac{\theta_a - \alpha}{2}\right) + o(r_a^{1/2}), \qquad \textit{as } r_a \to 0,$$

where $\cos(\alpha/2) = c_1/\sqrt{c_1^2 + d_1^2}$ *and* $\sin(\alpha/2) = d_1/\sqrt{c_1^2 + d_1^2}$*. The unique nodal line of φ^a ending at a leaves then the point a with an angle given by $\theta_a = \alpha + \pi$ $(\mathrm{mod}\, 2\pi)$. We immediately see that $M_a = 0$ if and only if $\cos \alpha = 0$, that is $\alpha = \pi/2$ or $3\pi/2$. This corresponds to a nodal line leaving a in parallel with the boundary of the domain.*

Lemma 2.7.15. *There exists $C > 0$ not depending on a_1 such that*

$$\frac{|M_a|}{H(\varphi^a, 2a_1, \pi(a))} \leq C.$$

Proof. The quantity M_a is expressed in terms of v in Lemma 2.7.13. We also rewrite

$$H(\varphi^a, 2a_1, \pi(a)) = \int_\gamma v^2 |y| \,\mathrm{d}\sigma, \tag{2.95}$$

where, letting $\Omega = \{y : \ a_1 y^2 + a \in D^+_{2a_1}(\pi(a))\}$, we have $\gamma = \partial\Omega$. By Lemma 2.2.8, v solves $-\Delta v = 4a_1^2 |y|^2 \tilde{p} \lambda^a v$ in Ω, where $\tilde{p}(y) = p(a_1 y^2 + a)$ has the same properties as $p(x)$. Since γ does not depend on a, Lemma A.2.1 applies, providing for a_1 sufficiently small the representation formula

$$v(x) = -\int_\gamma v(y)\, \partial_\nu G(x, y) \,\mathrm{d}\sigma(y),$$

for $x \in \Omega$ and moreover,

$$\|\partial_{x_i} G(x, \cdot) - \partial_{x_i} \Phi(x, \cdot)\|_{W^{2,q}(\Omega)} \leq C a_1^2 \quad \text{ for } 1 \leq q < 2.$$

Therefore we have, by the Hölder and traces inequalities (for the trace embedding, see for example [49, Theorem 5.36]) and the estimate above, taking for example $q = 4/3$, we have

$$\begin{aligned} |\partial_{x_i} v(0)|^2 &= \left(\int_\gamma v \partial_\nu \partial_{x_i} G(0, y) \mathrm{d}\sigma(y) \right)^2 \\ &\leq \int_\gamma v^2 \,\mathrm{d}\sigma \int_\gamma |\partial_\nu \partial_{x_i} G(0, y)|^2 \,\mathrm{d}y \leq C \int_\gamma v^2 |y| \,\mathrm{d}\sigma, \end{aligned}$$

where we divided and multiplicated by $|y|$, recording that $|y| > 0$ on γ. Hence, by Lemma 2.7.13 and by (2.95), we have

$$\frac{|M_a|}{H(\varphi^a, 2a_1, \pi(a))} \leq C \frac{|\nabla v(0)|^2}{H(\varphi^a, 2a_1, \pi(a))} \leq C. \qquad \square$$

Lemma 2.7.16. *There exists $C > 0$ independent of a_1 such that*

$$\frac{|M_a|}{H(\varphi^a, ka_1, \pi(a))} \leq \frac{C}{k^2} \text{ for every } k > 2.$$

Proof. It is a straightforward consequence of Lemmas 2.7.9 and 2.7.15

$$\frac{|M_a|}{H(\varphi^a, ka_1, \pi(a))} = \frac{|M_a|}{H(\varphi^a, 2a_1, \pi(a))} \cdot \frac{H(\varphi^a, 2a_1, \pi(a))}{H(\varphi^a, ka_1, \pi(a))} \leq \frac{C}{k^2}. \qquad \square$$

Estimates on $N(\varphi^a, r, \pi(a), \lambda^a, A_a)$

The function N may not be increasing, because of the remainder M_a which appears in the derivative of E in (2.92). Nonetheless, we can use the estimates proved in the previous paragraph to obtain a bound from below on the derivative of N.

Lemma 2.7.17. *Let φ^a be a solution of (2.85) and r_0 be as in (2.86). For $a_1 < r < r_0$ we have*

$$\frac{1}{r^2} \int_{D_r^+(\pi(a))} |\varphi^a|^2 \, dx \leq \frac{E(\varphi^a, r, \pi(a), \lambda^a, A_a) + H(\varphi^a, r, \pi(a))}{1 - Cr_0^2}.$$

If $|a|$ is sufficiently small, we can choose $C = 2\lambda \|p\|_{L^\infty}$, where λ is the limit of λ^a as $a \to 0$.

Proof. On one hand, the Poincaré inequality (2.67) provides

$$\frac{1}{r^2} \int_{D_r^+(\pi(a))} |\varphi^a|^2 \, dx - \lambda^a \int_{D_r^+(\pi(a))} p(x) |\varphi^a|^2 \, dx \\ \leq E(\varphi^a, r, \pi(a), \lambda^a, A_a) + H(\varphi^a, r, \pi(a)).$$

On the other hand, if we take $r < r_0$, we obtain that

$$\frac{1 - r_0^2 \lambda^a \|p\|_{L^\infty}}{r^2} \int_{D_r^+(\pi(a))} |\varphi^a|^2 \, dx \\ \leq \frac{1}{r^2} \int_{D_r^+(\pi(a))} |\varphi^a|^2 \, dx - \lambda^a \int_{D_r^+(\pi(a))} p(x) |\varphi^a|^2 \, dx.$$

The result follows by combining the previous two inequalities. $\square$

Lemma 2.7.18. *Let φ^a be a solution of (2.85) and r_0 be as in (2.86). For every $k > 1$, $a_1 < r_0/k$ and $ka_1 < r < r_0$ we have*

$$\begin{aligned} &N(\varphi^a, r, \pi(a), \lambda^a, A_a) \\ &\leq (N(\varphi^a, r_0, \pi(a), \lambda^a, A_a) + 1) \exp\left(\frac{Ce^{Cr_0^2}}{k^2} + \frac{Cr_0^2}{1 - Cr_0^2}\right) - 1, \end{aligned}$$

with $C > 0$ independent from a_1, k, r, r_0.

Proof. Let for the moment $N = N(\varphi^a, r, \pi(a), \lambda^a, A_a)$ and analogously for H and E. We use Lemmas 2.7.8 and 2.7.12 to obtain, for $r > a_1$,

$$\begin{aligned} \frac{\mathrm{d}N}{\mathrm{d}r} = \frac{1}{H^2} \Bigg\{ &\frac{2}{r} \int_{\partial D_r^+(\pi(a))} |(i\nabla + A_a)\varphi^a \cdot \nu|^2 \,\mathrm{d}\sigma \int_{\partial D_r^+(\pi(a))} |\varphi^a|^2 \,\mathrm{d}\sigma \\ &- \frac{2}{r} \left(i \int_{\partial D_r^+(\pi(a))} (i\nabla + A_a)\varphi^a \cdot \nu \, \overline{\varphi^a} \,\mathrm{d}\sigma \right)^2 \Bigg\} \\ &- \frac{1}{rH} \left\{ 2M_a + \lambda^a \int_{D_r^+(\pi(a))} |\varphi^a|^2 (2p + \nabla p \cdot (x - \pi(a))) \,\mathrm{d}x \right\} \\ \geq -\frac{1}{rH} &\left\{ 2|M_a| + \lambda^a \|2p + \nabla p \cdot (x - \pi(a))\|_{L^\infty} \int_{D_r^+(\pi(a))} |\varphi^a|^2 \,\mathrm{d}x \right\}. \end{aligned} \tag{2.96}$$

In the last step we used the Schwarz inequality and the regularity assumption on p (2.4). Therefore we have

$$\begin{aligned} &\frac{\mathrm{d}}{\mathrm{d}r} \log(N + 1) \\ &\geq -\frac{1}{r(E + H)} \left\{ 2|M_a| + \lambda^a \|2p + \nabla p \cdot (x - \pi(a))\|_{L^\infty} \int_{D_r^+(\pi(a))} |\varphi^a|^2 \,\mathrm{d}x \right\}. \end{aligned} \tag{2.97}$$

We look at the first term in the right hand side of (2.97). By Lemma 2.7.9 we have

$$\frac{H(\varphi^a, r, \pi(a))}{H(\varphi^a, ka_1, \pi(a))} \geq e^{-Cr_0^2} \frac{r^2}{(ka_1)^2}, \qquad ka_1 < r < r_0.$$

This together with Remark 2.7.10 and Lemma 2.7.16 provides

$$-\frac{|M_a|}{r(E + H)} \geq -\frac{|M_a|}{rH} \geq -\frac{e^{Cr_0^2}}{r^3} \frac{|M_a|(ka_1)^2}{H(\varphi^a, ka_1, \pi(a))} \geq -Ce^{Cr_0^2} \frac{a_1^2}{r^3}$$

for $ka_1 < r < r_0$. Concerning the second term in the right hand side of (2.97), we apply Lemma 2.7.17 to obtain

$$-\frac{1}{r(E+H)}\int_{D_r^+(\pi(a))}|\varphi^a|^2\,\mathrm{d}x \geq -\frac{r}{1-Cr_0^2}, \qquad ka_1 < r < r_0.$$

Thus we have obtained

$$\frac{\mathrm{d}}{\mathrm{d}r}\log(N+1) \geq -\frac{Ce^{Cr_0^2}a_1^2}{r^3} - \frac{Cr}{1-Cr_0^2}, \qquad ka_1 < r < r_0.$$

By integrating between r and r_0 we arrive at

$$\begin{aligned}\log\frac{N(\varphi^a, r_0, \pi(a), \lambda^a, A_a)+1}{N(\varphi^a, r, \pi(a), \lambda^a, A_a)+1} &\geq Ca_1^2\left(\frac{e^{Cr_0^2}}{r_0^2} - \frac{e^{Cr_0^2}}{r^2}\right) - \frac{C}{1-Cr_0^2}(r_0^2-r^2)\\ &\geq -Ce^{Cr_0^2}\frac{a_1^2}{r^2} - \frac{Cr_0^2}{1-Cr_0^2}\\ &\geq -\frac{Ce^{Cr_0^2}}{k^2} - \frac{Cr_0^2}{1-Cr_0^2},\end{aligned}$$

for $ka_1 < r < r_0$. The statement follows by exponentiating both terms. $\square$

2.7.4. Proof of Proposition 2.1.12

In this section, we prove the existence and uniqueness of the limit profile of Proposition 2.1.12.

Proof of Proposition 2.1.12. **Step 1.** Suppose by contradiction that there are two solutions ψ and $\tilde{\psi}$ to (2.7) in $H^1_{A_e,\mathrm{loc}}(\mathbb{R}^2_+)$, which do not differ by a multiplicative constant, that is $\frac{\tilde{c}_1}{c_1} \neq \frac{\tilde{d}_1}{d_1}$ if $c_1, d_1 \neq 0$, $\tilde{c}_1 \neq 0$ if $c_1 = 0$, $\tilde{d}_1 \neq 0$ if $d_1 = 0$. By Proposition 2.2.10 we have

$$\psi(r_e,\theta_e) = e^{i\frac{\theta_e}{2}}\sqrt{r_e}\left(c_1\cos\frac{\theta_e}{2} + d_1\sin\frac{\theta_e}{2}\right) + o(\sqrt{r_e}),$$

$$\tilde{\psi}(r_e,\theta_e) = e^{i\frac{\theta_e}{2}}\sqrt{r_e}\left(\tilde{c}_1\cos\frac{\theta_e}{2} + \tilde{d}_1\sin\frac{\theta_e}{2}\right) + o(\sqrt{r_e}),$$

as $r_e = |x-e| \to 0$, $e = (1,0)$. Suppose first that $c_1^2 + d_1^2 \neq 0$. We consider the linear combination $t\psi + \tilde{\psi}$ that we can write, thanks to the expressions above,

$$(t\psi+\tilde{\psi})(r_e,\theta_e) = e^{i\frac{\theta_e}{2}}\sqrt{r_e}\left[(tc_1+\tilde{c}_1)\cos\frac{\theta_e}{2} + (td_1+\tilde{d}_1)\sin\frac{\theta_e}{2}\right] + o(\sqrt{r_e}). \tag{2.98}$$

The parameter t is chosen in such a way that

$$M := \pi \frac{(tc_1 + \tilde{c}_1)^2 - (td_1 + \tilde{d}_1)^2}{4} = 0,$$

where M is the constant associated to $t\psi + \tilde{\psi}$, see (2.94). More explicitly we have $t = (\tilde{d}_1 - \tilde{c}_1)/(c_1 - d_1)$ if $c_1 \neq d_1$ or $t = -(\tilde{c}_1 + \tilde{d}_1)/(c_1 + d_1)$ if $c_1 \neq -d_1$. Exactly as in (2.96) we have for $r > 1$

$$\frac{\mathrm{d}}{\mathrm{d}r} N(t\psi + \tilde{\psi}, r, 0, 0, A_e) \geq -\frac{2M}{rH(t\psi + \tilde{\psi}, r, 0)} = 0, \tag{2.99}$$

thanks to our choice of t, so that $N(t\psi + \tilde{\psi}, \cdot, 0, 0, A_e)$ is increasing.

We claim that

$$\lim_{r\to\infty} N(t\psi + \tilde{\psi}, r, 0, 0, A_e) \leq 1. \tag{2.100}$$

Assume by contradiction that there exist $\delta > 0$ and $R_\delta > 0$ such that, for every $r > R_\delta$, $N(t\psi + \tilde{\psi}, r, 0, 0, A_e) \geq 1 + \delta$. Then, since $t\psi + \tilde{\psi}$ solves the equation, proceeding as in (2.87), we find

$$\frac{\mathrm{d}}{\mathrm{d}r} \log H(t\psi + \tilde{\psi}, r, 0) = \frac{2}{r} N(t\psi + \tilde{\psi}, r, 0, 0, A_e) \geq \frac{2}{r}(1+\delta), \qquad r > R_\delta. \tag{2.101}$$

Integrating between R_δ and r we obtain

$$H(t\psi + \tilde{\psi}, r, 0) \geq Cr^{2(1+\delta)}, \qquad r > R_\delta. \tag{2.102}$$

On the other hand, by assumption (2.8), we have (by eventually taking a larger R_δ)

$$N(\psi, r, 0, 0, A_e) < 1 + \frac{\delta}{2}, \quad N(\tilde{\psi}, r, 0, 0, A_e) < 1 + \frac{\delta}{2}, \qquad r > R_\delta.$$

Proceeding as above, this implies $H(\psi, r, 0) + H(\tilde{\psi}, r, 0) \leq Cr^{2(1+\delta/2)}$ for $r > R_\delta$. Hence, by the Young inequality, we obtain

$$H(t\psi + \tilde{\psi}, r, 0) \leq 2\left[H(t\psi, r, 0) + H(\tilde{\psi}, r, 0)\right] \leq Cr^{2(1+\delta/2)}, \qquad r > R_\delta,$$

which contradicts (2.102). Hence (2.100) is proved.

On the other hand, it is not difficult to see that, since $t\psi + \tilde{\psi}$ vanishes on $\{x_1 = 0\}$ but is not identically zero, we must have

$$N(t\psi + \tilde{\psi}, 0, 0, 0, A_e) \geq 1. \tag{2.103}$$

Indeed, suppose by contradiction that there exist $\varepsilon > 0$, $r_\varepsilon < 1$ such that

$$N(t\psi + \tilde{\psi}, r, 0, 0, A_e) < 1 - \varepsilon, \qquad r < r_\varepsilon.$$

Using this inequality as we did in (2.101), and then integrating between r and r_ε, we arrive at

$$\frac{H(t\psi + \tilde{\psi}, r_\varepsilon, 0)}{H(t\psi + \tilde{\psi}, r, 0)} \leq \left(\frac{r_\varepsilon}{r}\right)^{2-2\varepsilon}.$$

Conversely, Lemmas 2.7.8 and 2.7.4 provide

$$\frac{\mathrm{d}}{\mathrm{d}r} H(t\psi + \tilde{\psi}, r, 0) = \frac{2}{r} \int_{D_r^+(0)} |(i\nabla + A_e)(t\psi + \tilde{\psi})|^2 \,\mathrm{d}x \geq \frac{2}{r} H(t\psi + \tilde{\psi}, r, 0),$$

and hence

$$\frac{H(t\psi + \tilde{\psi}, r_\varepsilon, 0)}{H(t\psi + \tilde{\psi}, r, 0)} \geq \left(\frac{r_\varepsilon}{r}\right)^{2},$$

which contradicts the previous inequality for $r < r_\varepsilon$.

We conclude from (2.99), (2.100) and (2.103) that $N(t\psi + \tilde{\psi}, r, 0, 0, A_e) \equiv 1$ and, in turn, that $t\psi + \tilde{\psi} = e^{i\theta_e/2} r g(\theta)$, for some function g depending only on the angle. This contradicts the asymptotic behaviour (2.98) of $t\psi + \tilde{\psi}$ at e. We have obtained uniqueness up to a multiplicative constant in case $c_1^2 + d_1^2 \neq 0$. If $c_1 = d_1 = 0$ then all the previous considerations apply with $\tilde{\psi}$ in place of $t\psi + \tilde{\psi}$ and we still obtain a contradiction.

We remark that a more precise analysis of the results above and (2.99) tells us that M must be positive, otherwise we obtain again a contradiction. As explained in Remark 2.7.14, $M > 0$ is equivalent to have the nodal line of ψ leaving e with $\theta_e \in (\pi/2, 3\pi/2)$. Moreover, we cannot have neither a zero of order $h/2$ at e with $h \geq 3$ since in that case $M \equiv 0$.

Step 2. We will use some ideas in [50], in particular Lemmas 2.4 and 2.9 (see also [51]). Let $Q_1 = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$ and let $\Gamma_1 = \{(x_1, 0) \in \mathbb{R}^2 : 0 < x_1 < 1\}$. We consider the following minimization problem

$$\frac{\beta}{2} = \inf \left\{ \int_{Q_1} |\nabla u|^2 \,\mathrm{d}x : u \in \mathcal{D}^{1,2}(Q_1),\, u = 0 \text{ on } \{x_1 = 0\},\, u = -x_1 \text{ on } \Gamma_1 \right\}, \tag{2.104}$$

where we denote by $\mathcal{D}^{1,2}(Q_1)$ the closure of $C_0^\infty(Q_1)$ with respect to $\|\nabla u\|_{L^2(Q_1)}$. By standard variational methods, the infimum is achieved by a unique function $w \in \mathcal{D}^{1,2}(Q_1)$ (see for example [52, Theorem 8.4]). Due to the symmetries of the problem, we can extend w to $\mathbb{R}^2_+$ in such a way that $w(x_1, -x_2) = w(x_1, x_2)$ and moreover w satisfies the following properties

$$-\Delta w = 0 \text{ in } \mathbb{R}^2_+ \setminus \Gamma_1, \quad w = 0 \text{ on } \{x_1 = 0\}, \quad w = -x_1 \text{ on } \Gamma_1$$

and

$$\int_{\mathbb{R}^2_+} |\nabla w|^2 \, \mathrm{d}x = \beta < \infty.$$

By the maximum principle we have $w < 0$ in $\mathbb{R}^2_+$. If the discontinuity segment of θ_e is chosen on Γ_1, one can check that

$$\tilde{\psi} = e^{i\theta_e/2}(x_1 + w)$$

solves (2.7), by passing to the double covering as in Lemma 2.2.8. We aim at showing that $\tilde{\psi} = \psi$; thanks to step 1, it will be sufficient to prove that $\tilde{\psi}$ satisfies (2.8). Let $\tilde{w}$ be the Kelvin transform of w, that is $\tilde{w}(y) = w(y/|y|^2)$ for $|y| < 1$. Because w is harmonic outside of $D_1^+(0)$, $\tilde{w}$ is harmonic in $D_1^+(0)$ with zero boundary conditions on $\{y_1 = 0\}$. Moreover, $\int_{D_1^+(0)} |\nabla \tilde{w}|^2 \, \mathrm{d}x = \int_{\mathbb{R}^2_+ \setminus D_1^+(0)} |\nabla w|^2 \, \mathrm{d}x < \beta$, then $\tilde{w}$ has finite energy. Therefore $\tilde{w}$ is analytic in $D_1^+(0)$ and admits the following expansion in $D_1^+(0)$

$$\tilde{w}(y) = \sum_{n=1}^{\infty} \mathrm{Re}(\tilde{b}_n y^n), \ \tilde{b}_n \in \mathbb{C},$$

so that

$$w(x) = \sum_{n=1}^{\infty} \mathrm{Re}\left(\tilde{b}_n \frac{x^n}{|x|^{2n}}\right), \qquad \text{for } |x| > 1.$$

By passing to polar coordinates and taking into account the symmetries of w, we find

$$w(r, \theta) = \sum_{n \text{ odd}} \frac{b_n}{r^n} \cos(n\theta), \quad r > 1, \quad b_n \in \mathbb{R}, \text{ with } b_1 < 0. \tag{2.105}$$

Hence $w(r, \theta) = b_1 \cos\theta / r + O(r^{-3})$ as $r \to \infty$, and an explicit calculation provides

$$\lim_{r \to \infty} N(\tilde{\psi}, r, 0, 0, A_e) = 1 \quad \text{ and hence } \quad \psi = \tilde{\psi} = e^{i\theta_e/2}(x_1 + w),$$

by the uniqueness proved in Step 1. To conclude the proof of point (ii) it remains to show that $b_1 = -\beta/\pi$. By testing the equation $-\Delta w = 0$ in $\mathbb{R}^2_+ \setminus \Gamma_1$ by w we deduce

$$\beta = \int_{\mathbb{R}^2_+} |\nabla w|^2 \,\mathrm{d}x = -2\int_{\Gamma_1} x_1 \nabla w \cdot \nu \,\mathrm{d}\sigma. \tag{2.106}$$

On the other hand, by testing the equation of w by x_1 in $D_R^+(0)$, $R > 1$, and the equation $-\Delta x_1 = 0$ by w in $D_R^+(0)$, and subtracting them, we obtain

$$\int_{\partial D_R^+(0)} (w\nabla x_1 - x_1 \nabla w)\cdot \nu \,\mathrm{d}\sigma - 2\int_{\Gamma_1} x_1 \nabla w \cdot \nu \,\mathrm{d}\sigma = 0. \tag{2.107}$$

By combining (2.105)-(2.107) we obtain

$$\begin{aligned}
\beta &= \lim_{R\to\infty} \int_{\partial D_R^+(0)} (x_1 \nabla w - w \nabla x_1)\cdot \nu \,\mathrm{d}\sigma \\
&= \lim_{R\to\infty} \left\{ -\sum_{n \text{ odd}} \frac{(n+1)b_n}{R^{n-1}} \int_{-\pi/2}^{\pi/2} \cos(n\theta)\cos\theta \,\mathrm{d}\theta \right\} \\
&= -2b_1 \int_{-\pi/2}^{\pi/2} \cos^2\theta \,\mathrm{d}\theta = -\pi b_1
\end{aligned}$$

which concludes the proof. □

We can interpret β as the cost, in terms of energy, needed to impose that w vanishes on Γ_1, or equivalently as the energy cost of the nodal line of ψ.

2.7.5. Pole approaching the boundary not on a nodal line of φ_k: estimate from below

Without loss of generality, we assume that $b = 0 \in \partial\Omega$ and that condition (2.65) is satisfied, meaning that the boundary is flat around 0. Let φ^a be a solution of (2.85). In this section we treat the case when $a \to 0$ and φ has a zero of order 1 at 0 (no nodal lines). In this case, if $\pi(a) = (0, a_2)$ is sufficiently close to 0, then φ has a zero of order 1 also at $\pi(a)$: there exists $\bar{a}_2 > 0$ such that, for $|\pi(a)| < \bar{a}_2$, we have

$$\varphi(r_{\pi(a)}, \theta_{\pi(a)}) = r_{\pi(a)} c_1(\pi(a)) \cos\theta_{\pi(a)} + O(r^2_{\pi(a)}), \tag{2.108}$$

as $r_{\pi(a)} = |x - \pi(a)| \to 0$, where $x - \pi(a) = r_{\pi(a)}(\cos\theta_{\pi(a)}, \sin\theta_{\pi(a)})$ and $c_1(\pi(a)) \neq 0$. We remark that only the $\cos\theta_{\pi(a)}$-term remains since φ vanishes on the boundary $\{x_1 = 0\}$. In the following, we keep the notation used in Section 2.7.3.

Estimates on the frequency function

Lemma 2.7.19. *Let φ^a be a solution of (2.85) and suppose that φ has a zero of order 1 at 0. Let $|\pi(a)| = |a_2| < \bar{a}_2$ so that (2.108) holds. For every $\varepsilon > 0$, there exists $\tilde{r}_\varepsilon > 0$ such that for all $r_\varepsilon \leq \tilde{r}_\varepsilon$ there exists $\bar{a}_{1,\varepsilon,r_\varepsilon} > 0$ such that*

$$1 \leq N(\varphi^a, r_\varepsilon, \pi(a), \lambda^a, A_a) \leq 1 + \frac{\varepsilon}{2} \quad \text{for all } a_1 < \bar{a}_{1,\varepsilon,r_\varepsilon}.$$

Proof. The bound from below can be proved as in (2.103). Let us concentrate on the bound from above. Relation (2.108) implies that $1 \leq N(\varphi, r, \pi(a), \lambda, 0) \leq 1 + O(r)$ as $r \to 0$ (see for example [53, Corollary 2.2.4]). Let $\tilde{r}_\varepsilon$ be such that $N(\varphi, \tilde{r}_\varepsilon, \pi(a), \lambda, 0) \leq 1 + \varepsilon/8$. By the monotonicity property of the Almgren function for the eigenfunctions of the Laplacian (see for example [53, Corollary 3.1.2]), we have that $N(\varphi, r, \pi(a), \lambda, 0) \leq 1 + \varepsilon/4$ for every $r \leq \tilde{r}_\varepsilon$. Fix $0 < r_\varepsilon < \tilde{r}_\varepsilon$. Since we know from Remark 2.4.4 that

$$\lambda^a \to \lambda \quad \text{and} \quad \|e^{-i\theta_a/2}\varphi^a - \varphi\|_{H^1(\Omega)} \to 0, \quad \text{as } a \to 0,$$

we deduce $|N(\varphi^a, r_\varepsilon, \pi(a), \lambda^a, A_a) - N(\varphi, r_\varepsilon, \pi(a), \lambda, 0)| \leq \varepsilon/4$ for $|\pi(a)| < \bar{a}_2$ and $a_1 < \bar{a}_{1,\varepsilon,r_\varepsilon}$. ☐

So far we have obtained an estimate on N for a fixed radius r_ε. Since N is not increasing, this is not sufficient to obtain the estimate for $r \to 0$. Nonetheless, we can provide a bound on N for r sufficiently far from the singularity. This is done by exploiting the estimates proved in the Section 2.7.3.

Lemma 2.7.20. *Let φ^a be a solution of (2.85) and suppose that φ has a zero of order 1 at 0. Let $|\pi(a)| < \bar{a}_2$ so that (2.108) holds. For every $\varepsilon > 0$, there exist $r_\varepsilon, \bar{a}_{1,\varepsilon} > 0$ and $k_\varepsilon > 1$ such that*

$$N(\varphi^a, r, \pi(a), \lambda^a, A_a) \leq 1 + \varepsilon \tag{2.109}$$

for every $a_1 < \bar{a}_{1,\varepsilon}$ and for every $k_\varepsilon a_1 < r < r_\varepsilon$, and

$$\frac{H(\varphi^a, r_2, \pi(a))}{H(\varphi^a, r_1, \pi(a))} \leq \left(\frac{r_2}{r_1}\right)^{2(1+\varepsilon)}. \tag{2.110}$$

for every $a_1 < \bar{a}_{1,\varepsilon}$ and $k_\varepsilon a_1 < r_1 < r_2 < r_\varepsilon$.

Proof. To prove the first inequality we combine the previous lemma with Lemma 2.7.18. In Lemma 2.7.18 we choose $r_0 = r_\varepsilon < \tilde{r}_\varepsilon$. For every $k > 1$, $a_1 < \min\{r_\varepsilon/k, \bar{a}_{1,\varepsilon,r_\varepsilon}\}$ and $ka_1 < r < r_\varepsilon$ we have

$$N(\varphi^a, r, \pi(a), \lambda^a, A_a) \leq \left(2 + \frac{\varepsilon}{2}\right) \exp\left(\frac{Ce^{Cr_\varepsilon^2}}{k^2} + \frac{2\lambda^a r_\varepsilon^2}{1 - \lambda^a r_\varepsilon^2}\right) - 1.$$

We can impose that the right hand side above is less than $1+\varepsilon$ by choosing r_ε sufficiently small and $k = k_\varepsilon$ sufficiently large. Then we let $\bar{a}_{1,\varepsilon} < \min\{r_\varepsilon/k_\varepsilon, \bar{a}_{1,\varepsilon,r_\varepsilon}\}$.

Let us look at the second inequality. We deduce from Lemma 2.7.8 and from (2.109) that

$$\frac{\mathrm{d}}{\mathrm{d}r}\log H(\varphi^a, r, \pi(a)) = \frac{2}{r}N(\varphi^a, r, \pi(a), \lambda^a, A_a) \le \frac{2(1+\varepsilon)}{r}$$

for $a_1 < \bar{a}_{1,\varepsilon}$ and $k_\varepsilon a_1 < r < r_\varepsilon$. Integrating between r_1 and r_2 we obtain the result. □

In Lemma 2.7.9, we obtained a superior bound on the function $H(\varphi^a, r, \pi(a))$. In the case where r is sufficiently far from the singularity, we can obtain an inferior bound on $H(\varphi^a, r, \pi(a))$, which can be improved with respect to (2.110). This is the object of the following lemma.

Lemma 2.7.21. *Let φ^a be a solution of (2.85) and suppose that φ has a zero of order 1 at 0. Let $|\pi(a)| < \bar{a}_2$ so that (2.108) holds. Let $k_\varepsilon > \max\{1, \sqrt{\beta/\pi}\}$, $\bar{a}_{1,\varepsilon}$, r_ε be as in Lemma 2.7.20. Then, there exists $C > 0$ independent from a_1 such that*

$$H(\varphi^a, k_\varepsilon a_1, \pi(a)) \ge C(k_\varepsilon a_1)^2 \quad \textit{for every } a_1 < \bar{a}_{1,\varepsilon}.$$

Moreover,

$$\liminf_{k_\varepsilon\to+\infty}\ \liminf_{a_1\to 0} \frac{\sqrt{H(\varphi^a, k_\varepsilon a_1, \pi(a))}}{k_\varepsilon a_1} \ge \sqrt{\frac{\pi}{2}}c_1,$$

where $c_1 \in \mathbb{R}$ is defined in (2.108).

Proof. We consider the function ψ which has been introduced in the statement of Proposition 2.1.12 (with abuse of notation we divide by the multiplicative constant C which is not relevant in this context). The rescaled function

$$\Phi(x) = a_1\psi\left(\frac{x-\pi(a)}{a_1}\right)$$

satisfies

$$\begin{cases} (i\nabla + A_a)^2\Phi = 0 & \mathbb{R}^2_+ \\ \Phi = 0 & \{x_1 = 0\}, \end{cases}$$

and the following expansion, where $x - \pi(a) = \rho\,(\cos\theta_{\pi(a)}, \sin\theta_{\pi(a)})$

$$\Phi(\rho, \theta_{\pi(a)}) = e^{i\frac{\theta_a}{2}}\left(\rho\cos\theta_{\pi(a)} - \frac{\beta}{\pi}a_1^2\frac{\cos\theta_{\pi(a)}}{\rho} + \sum_{\substack{n\ge 3\\ n \text{ odd}}} b_n a_1^{n+1}\frac{\cos(n\theta_{\pi(a)})}{\rho^n}\right), \tag{2.111}$$

for $\rho > a_1$. By testing the equation satisfied by φ^a by Φ in $D_r^+(\pi(a))$ $(r > a_1)$, we obtain

$$\lambda^a \int_{D_r^+(\pi(a))} p(x)\varphi^a \overline{\Phi}\, \mathrm{d}x$$
$$= i \int_{\partial D_r^+(\pi(a))} \Big\{ (i\nabla + A_a)\varphi^a \cdot \nu\, \overline{\Phi} + \varphi^a\, \overline{(i\nabla + A_a)\Phi \cdot \nu} \Big\}\, \mathrm{d}\sigma. \tag{2.112}$$

Fix $k_\varepsilon > \sqrt{\beta/\pi}$. For $\rho > a_1$ we also define

$$\Gamma(\rho, \theta_{\pi(a)}) = e^{i\frac{\theta_a}{2}} \left\{ \left(k_\varepsilon^2 - \frac{\beta}{\pi}\right) a_1^2 \frac{\cos\theta_{\pi(a)}}{\rho} + \sum_{\substack{n\geq 3\\ n \text{ odd}}} b_n a_1^{n+1} \frac{\cos(n\theta_{\pi(a)})}{\rho^n} \right\}, \tag{2.113}$$

so that

$$\begin{cases} (i\nabla + A_a)^2\Gamma = 0 & \mathbb{R}^2_+ \setminus D_{a_1}^+(\pi(a)) \\ \Gamma = 0 & \{x_1 = 0\} \\ \Gamma = \Phi & \partial D_{k_\varepsilon a_1}^+(\pi(a)). \end{cases}$$

By testing the equation satisfied by φ^a by Γ in an annulus $(D_{r_\varepsilon}^+ \setminus D_r^+)(\pi(a))$ $(r_\varepsilon > r > k_\varepsilon a_1)$, we obtain

$$\lambda^a \int_{(D_{r_\varepsilon}^+ \setminus D_r^+)(\pi(a))} p(x)\varphi^a \overline{\Gamma}\, \mathrm{d}x$$
$$= i \int_{\partial(D_{r_\varepsilon}^+ \setminus D_r^+)(\pi(a))} \Big\{ (i\nabla + A_a)\varphi^a \cdot \nu\, \overline{\Gamma} + \varphi^a\, \overline{(i\nabla + A_a)\Gamma \cdot \nu} \Big\}\, \mathrm{d}\sigma. \tag{2.114}$$

In equations (2.112) and (2.114) we choose $r = k_\varepsilon a_1$. Adding the two equations we obtain

$$i \int_{\partial D_{k_\varepsilon a_1}^+(\pi(a))} \varphi^a \Big\{ \overline{(i\nabla + A_a)\Phi \cdot \nu} - \overline{(i\nabla + A_a)\Gamma \cdot \nu} \Big\}\, \mathrm{d}\sigma$$
$$= \lambda^a \int_{D_{k_\varepsilon a_1}^+(\pi(a))} p(x)\varphi^a \overline{\Phi}\, \mathrm{d}x + \lambda^a \int_{(D_{r_\varepsilon}^+ \setminus D_{k_\varepsilon a_1}^+)(\pi(a))} p(x)\varphi^a \overline{\Gamma}\, \mathrm{d}x$$
$$- i \int_{\partial D_{r_\varepsilon}^+(\pi(a))} \Big\{ (i\nabla + A_a)\varphi^a \cdot \nu \overline{\Gamma} + \varphi^a \overline{(i\nabla + A_a)\Gamma \cdot \nu} \Big\}\, \mathrm{d}\sigma. \tag{2.115}$$

Noticing that

$$(i\nabla + A_a)\Phi \cdot \nu \,|_{\partial D_{k_\varepsilon a_1(\pi(a))}^+}$$
$$= i e^{i\frac{\theta_a}{2}} \Big\{ \Big(1 + \frac{\beta}{\pi k_\varepsilon^2}\Big) \cos\theta_{\pi(a)} - \sum_{\substack{n\geq 3\\ n \text{ odd}}} \frac{n b_n}{k_\varepsilon^{n+1}} \cos(n\theta_{\pi(a)}) \Big\},$$

and

$$(i\nabla + A_a)\Gamma \cdot \nu \,|_{\partial D^+_{k_\varepsilon a_1}(\pi(a))} = -ie^{i\frac{\theta_a}{2}} \Big\{ \Big(1 - \frac{\beta}{\pi k_\varepsilon^2}\Big) \cos\theta_{\pi(a)} + \sum_{\substack{n\geq 3 \\ n \text{ odd}}} \frac{nb_n}{k_\varepsilon^{n+1}} \cos(n\theta_{\pi(a)}) \Big\},$$

we can estimate the left hand side of (2.115) from above as follows

$$\begin{aligned} &\left| i \int_{\partial D^+_{k_\varepsilon a_1}(\pi(a))} \varphi^a \ \Big\{ \overline{(i\nabla + A_a)\Phi \cdot \nu} - \overline{(i\nabla + A_a)\Gamma \cdot \nu} \Big\} \, \mathrm{d}\sigma \right| \\ &\qquad = \left| \int_{\partial D^+_{k_\varepsilon a_1}(\pi(a))} \varphi^a e^{-i\frac{\theta_a}{2}} 2\cos\theta_{\pi(a)} \, \mathrm{d}\sigma \right| \\ &\qquad \leq 2 \|\varphi^a\|_{L^2(\partial D^+_{k_\varepsilon a_1}(\pi(a)))} \|\cos\theta_{\pi(a)}\|_{L^2(\partial D^+_{k_\varepsilon a_1}(\pi(a)))} \\ &\qquad = k_\varepsilon a_1 \sqrt{2\pi H(\varphi^a, k_\varepsilon a_1, \pi(a))}. \end{aligned} \tag{2.116}$$

In what follows we will estimate the right hand side of (2.115). To this aim, recall that for every $r > 0$ it holds

$$\|e^{-i\frac{\theta_a}{2}} \varphi^a - \varphi\|_{C^\infty(\Omega \setminus D^+_r(0))} \to 0 \quad \text{as } a \to 0.$$

Moreover, φ satisfies (2.108). Hence we have

$$\varphi^a \,|_{\partial D^+_\rho(\pi(a))} = e^{i\frac{\theta_a}{2}} c_1 \rho \cos\theta_{\pi(a)} + h(\rho, \theta_{\pi(a)}) + o_{a_1}(1), \qquad \text{for every } \rho > a_1, \tag{2.117}$$

where $c_1 \in \mathbb{R}$ was defined in (2.108) and h satisfies (see (2.16))

$$\lim_{\rho \to 0} \frac{\|h(\rho, \cdot)\|_{C^1(\partial D^+_\rho(\pi(a)))}}{\rho} = 0. \tag{2.118}$$

Let's first look at the boundary term in the right hand side of (2.115). Taking into account that r_ε is fixed and $a_1 \to 0$, we have

$$(i\nabla + A_a)\varphi^a \cdot \nu \overline{\Gamma} \,|_{\partial D^+_{r_\varepsilon}(\pi(a))} = i \left(k_\varepsilon^2 - \frac{\beta}{\pi} \right) \frac{a_1^2}{r_\varepsilon} \left\{ c_1 \cos^2\theta_{\pi(a)} + \frac{\partial h}{\partial \rho}(r_\varepsilon, \theta_{\pi(a)}) \cos\theta_{\pi(a)} \right\} + o(a_1^2),$$

and

$$\varphi^a \, \overline{(i\nabla + A_a)\Gamma \cdot \nu} \,|_{\partial D^+_{r_\varepsilon}(\pi(a))} = i \left(k_\varepsilon^2 - \frac{\beta}{\pi} \right) \frac{a_1^2}{r_\varepsilon} \left\{ c_1 \cos^2\theta_{\pi(a)} + \frac{h(r_\varepsilon, \theta_{\pi(a)})}{r_\varepsilon} \cos\theta_{\pi(a)} \right\} + o(a_1^2),$$

so that

$$\begin{aligned}
&- i \int_{\partial D^+_{r_\varepsilon}(\pi(a))} \left\{ (i\nabla + A_a)\varphi^a \cdot \nu \overline{\Gamma} + \varphi^a \overline{(i\nabla + A_a)\Gamma \cdot \nu} \right\} \mathrm{d}\sigma = c_1(\pi k_\varepsilon^2 - \beta) a_1^2 \\
&+ \left(k_\varepsilon^2 - \frac{\beta}{\pi} \right) \frac{a_1^2}{r_\varepsilon} \int_{\partial D^+_{r_\varepsilon}(\pi(a))} \left(\frac{h(r_\varepsilon, \theta_{\pi(a)})}{r_\varepsilon} + \frac{\partial h}{\partial \rho}(r_\varepsilon, \theta_{\pi(a)}) \right) \cos\theta_{\pi(a)} \, \mathrm{d}\sigma + o(a_1^2) \\
&\geq c_1(\pi k_\varepsilon^2 - \beta) a_1^2 - C a_1^2 \left\| \frac{h(r_\varepsilon, \cdot)}{r_\varepsilon} + \frac{\partial h}{\partial \rho}(r_\varepsilon, \cdot) \right\|_{L^\infty(\partial D^+_{r_\varepsilon}(\pi(a)))} + o(a_1^2) \\
&\geq C' k_\varepsilon^2 a_1^2, \qquad (2.119)
\end{aligned}$$

for suitable $C' > 0$ and a_1 sufficiently small, thanks to (2.118).

Concerning the integral in the annulus in (2.115), we replace (2.113) and (2.117) to obtain

$$\begin{aligned}
&\left| \int_{(D^+_{r_\varepsilon} \setminus D^+_{k_\varepsilon a_1})(\pi(a))} p(x) \varphi^a \overline{\Gamma} \, \mathrm{d}x \right| \leq \|p\|_{L^\infty} \left| c_1 \frac{\pi}{4} \left(k_\varepsilon^2 - \frac{\beta}{\pi} \right) a_1^2 {r_\varepsilon}^2 \right| \\
&+ \|p\|_{L^\infty} \|h\|_{L^\infty} \int_{k_\varepsilon a_1}^{r_\varepsilon} \int_{\partial D^+_\rho(\pi(a))} \left| \sum_{\substack{n \geq 3 \\ n \text{ odd}}} b_n a_1^{n+1} \frac{\cos(n\theta_{\pi(a)})}{\rho^n} \right| \mathrm{d}\sigma \, \mathrm{d}\rho + o(a_1^2) \\
&\leq C \left\{ a_1^2 k_\varepsilon^2 r_\varepsilon^2 + \sum_{\substack{n \geq 3 \\ n \text{ odd}}} |b_n| a_1^{n+1} \left| \frac{1}{r_\varepsilon^{n-2}} - \frac{1}{(k_\varepsilon a_1)^{n-2}} \right| \right\} + o(a_1^2) \\
&\leq C a_1^2 k_\varepsilon^2 r_\varepsilon^2 + o(a_1^2), \qquad (2.120)
\end{aligned}$$

since $r_\varepsilon, k_\varepsilon$ are fixed while $a_1 \to 0$.

In order to estimate the last term, we apply Lemma 2.7.3, the equation satisfied by Φ and the expansion (2.111) with $\rho = k_\varepsilon a_1 > a_1$, as follows

$$\begin{aligned}
&\|\Phi\|^2_{L^2(D^+_{k_\varepsilon a_1}(\pi(a)))} \\
&\leq k_\varepsilon a_1 \int_{\partial D^+_{k_\varepsilon a_1}(\pi(a))} |\Phi|^2 \, \mathrm{d}\sigma + (k_\varepsilon a_1)^2 \int_{D^+_{k_\varepsilon a_1}(\pi(a))} |(i\nabla + A_a)\Phi|^2 \, \mathrm{d}x \\
&= k_\varepsilon a_1 \int_{\partial D^+_{k_\varepsilon a_1}(\pi(a))} |\Phi|^2 \, \mathrm{d}\sigma - i(k_\varepsilon a_1)^2 \int_{\partial D^+_{k_\varepsilon a_1}(\pi(a))} (i\nabla + A_a)\Phi \cdot \nu \overline{\Phi} \, \mathrm{d}\sigma \\
&= O(a_1^4).
\end{aligned}$$

In a similar way

$$\begin{aligned}\|\varphi^a\|^2_{L^2(D^+_{k_\varepsilon a_1}(\pi(a)))} \leq k_\varepsilon a_1 &\int_{\partial D^+_{k_\varepsilon a_1}(\pi(a))} |\varphi^a|^2 \, \mathrm{d}\sigma \\ &+ (k_\varepsilon a_1)^2 \lambda^a \int_{D^+_{k_\varepsilon a_1}(\pi(a))} p(x)|\varphi^a|^2 \, \mathrm{d}x \\ &- (k_\varepsilon a_1)^2 i \int_{\partial D^+_{k_\varepsilon a_1}(\pi(a))} (i\nabla + A_a)\varphi^a \cdot \nu \overline{\varphi^a} \, \mathrm{d}\sigma\end{aligned}$$

so that, using (2.117),

$$(1 - \lambda^a \|p\|_{L^\infty}(k_\varepsilon a_1)^2)\|\varphi^a\|^2_{L^2(D^+_{k_\varepsilon a_1}(\pi(a)))} = O(a_1^4).$$

The Hölder inequality provides

$$\left| \int_{D^+_{k_\varepsilon a_1}(\pi(a))} \varphi^a \overline{\Phi} \, \mathrm{d}x \right| \leq \|\varphi^a\|_{L^2(D^+_{k_\varepsilon a_1}(\pi(a)))} \|\Phi\|_{L^2(D^+_{k_\varepsilon a_1}(\pi(a)))} = O(a_1^4). \tag{2.121}$$

By combining (2.115), (2.116), (2.119), (2.120) and (2.121), we obtain

$$k_\varepsilon a_1 \sqrt{2\pi H(\varphi^a, k_\varepsilon a_1, \pi(a))} \geq C(k_\varepsilon a_1)^2 - C'(k_\varepsilon a_1 r_\varepsilon)^2 + o(a_1^2) \geq C''(k_\varepsilon a_1)^2,$$

for a_1 sufficiently small, and hence the first thesis.

To obtain the more precise estimate, we remark that, combining the same expressions (2.116), (2.119), (2.120) and (2.121), we get

$$\begin{aligned}&\sqrt{H(\varphi^a, k_\varepsilon a_1, \pi(a))} \\ &\qquad \geq \frac{1}{\sqrt{2\pi} k_\varepsilon a_1} \left[c_1(\pi k_\varepsilon^2 - \beta) a_1^2 + a_1^2 o_{r_\varepsilon}(1) + k_\varepsilon^2 a_1^2 o_{r_\varepsilon}(1) + o(a_1^2) \right]\end{aligned}$$

Then, dividing by $k_\varepsilon a_1$ and remembering that the estimate is valid for every $a_1 < \bar{a}_{1,\varepsilon}$, we can consider $\liminf_{a_1 \to 0}$

$$\liminf_{a_1 \to 0} \frac{\sqrt{H(\varphi^a, k_\varepsilon a_1, \pi(a))}}{k_\varepsilon a_1} \geq \frac{1}{\sqrt{2\pi} k_\varepsilon^2} \left[c_1(\pi k_\varepsilon^2 - \beta) + o_{r_\varepsilon}(1) + k_\varepsilon^2 o_{r_\varepsilon}(1) \right].$$

Next, k_ε can be taken as large as we want

$$\liminf_{k_\varepsilon \to +\infty} \liminf_{a_1 \to 0} \frac{\sqrt{H(\varphi^a, k_\varepsilon a_1, \pi(a))}}{k_\varepsilon a_1} \geq \left(c_1 \sqrt{\frac{\pi}{2}} + o_{r_\varepsilon}(1) \right).$$

Finally, since r_ε can be taken arbitrary small (see Lemma 2.7.19), we obtain the thesis. □

Lemma 2.7.9 and 2.7.21 allow us to say that $H(\varphi^a, k_\varepsilon a_1, \pi(a)) = O\left((k_\varepsilon a_1)^2\right)$ for $k_\varepsilon > \max\{\beta/\pi, 1\}$ and $a_1 < \bar{a}_{1,\varepsilon}$.

Normalized blow-up at the pole

In order to analyse the behaviour of φ^a near a (for $|a|$ close to 0), we perform a normalized blow-up of the function near the pole. For a fixed $\varepsilon > 0$, let

$$r_\varepsilon, \bar{a}_{1,\varepsilon}, k_\varepsilon \text{ be as in Lemma 2.7.20.}$$

We define

$$\psi^a(y) = \frac{\varphi^a(a_1 y + \pi(a))}{\sqrt{H(\varphi^a, k_\varepsilon a_1, \pi(a))}}, \quad y \in D^+_{R_\varepsilon}(0), \; R_\varepsilon = \frac{r_\varepsilon}{a_1}. \tag{2.122}$$

Note that these are the functions which appear in the statement of Theorem 2.1.13 (with $K = k_\varepsilon$) and that they are singular at $e = (0, 1)$, independently of a. We also remark that ψ^a solves the problem

$$\begin{cases} (i\nabla + A_e)^2 \psi^a = \lambda^a a_1^2 \hat{p}(y) \psi^a & D^+_{R_\varepsilon}(0) \\ \psi^a = 0 & \{y_1 = 0\}, \end{cases}$$

where $\hat{p}(y) = p(a_1 y + \pi(a))$.

A direct calculation provides the following relations between the frequency function for φ^a and that for ψ^a

$$E(\psi^a, R, 0, \lambda^a a_1^2, A_e) = \frac{E(\varphi^a, R a_1, \pi(a), \lambda^a, A_a)}{H(\varphi^a, k_\varepsilon a_1, \pi(a))}, \tag{2.123}$$

$$H(\psi^a, R, 0) = \frac{H(\varphi^a, R a_1, \pi(a))}{H(\varphi^a, k_\varepsilon a_1, \pi(a))}, \tag{2.124}$$

$$N(\psi^a, R, 0, \lambda^a a_1^2, A_e) = N(\varphi^a, R a_1, \pi(a), \lambda^a, A_a), \tag{2.125}$$

for $R > 1$. Here, with an abuse of notation, the frequency function for φ^a contains the weight $p(x)$, while in the frequency function for ψ^a appears $\hat{p}(y)$ due to the change of variables in the integral. This has no influence in the calculations, since both p and $\hat{p}$ satisfy (2.4).

We will show that the boundedness of the Almgren's function implies the convergence of the blow-up sequence as $a_1 \to 0$. To this aim, notice that Lemma 2.7.20 and relations (2.123)-(2.125) provide the following bounds.

Lemma 2.7.22. *Given $\varepsilon > 0$, take the same assumptions and notations of Lemma 2.7.20. Let ψ^a be as in (2.122). Then*

$$N(\psi^a, R, 0, \lambda^a a_1^2, A_e) \leq 1 + \varepsilon \tag{2.126}$$

for every $a_1 < \bar{a}_{1,\varepsilon}$ *and* $k_\varepsilon < R < r_\varepsilon/a_1$*, and*

$$\frac{H(\psi^a, R_2, 0)}{H(\psi^a, R_1, 0)} \leq \left(\frac{R_2}{R_1}\right)^{2(1+\varepsilon)}. \tag{2.127}$$

for every $a_1 < \bar{a}_{1,\varepsilon}$ *and* $k_\varepsilon < R_1 < R_2 < r_\varepsilon/a_1$*.*

Lemma 2.7.23. *Given* $\varepsilon > 0$*, take the same assumptions and notations of Lemma 2.7.20. Let* ψ^a *be as in* (2.122)*. For every* $R > k_\varepsilon$*, there exists a constant* $C(\varepsilon, R) > 0$ *such that*

$$\|\psi^a\|_{H^1_{A_e}(D^+_R(0))} \leq C(\varepsilon, R) \qquad \textit{for every } a_1 < \min\left\{\frac{r_\varepsilon}{R}, \bar{a}_{1,\varepsilon}\right\}. \tag{2.128}$$

Proof. Relation (2.127) and our choice of the normalization provide

$$H(\psi^a, R, 0) = \frac{H(\psi^a, R, 0)}{H(\psi^a, k_\varepsilon, 0)} \leq \left(\frac{R}{k_\varepsilon}\right)^{2(1+\varepsilon)} \leq C(\varepsilon) R^{2(1+\varepsilon)}. \tag{2.129}$$

This, together with the definition of N and (2.126), implies

$$E(\psi^a, R, 0, \lambda^a a_1^2, A_e) = N(\psi^a, R, 0, \lambda^a a_1^2, A_e) H(\psi^a, R, 0) \leq C(\varepsilon) R^{2(1+\varepsilon)}.$$

Both relations hold for $R > k_\varepsilon$ and $a_1 < \min\{r_\varepsilon/R, \bar{a}_{1,\varepsilon}\}$. Then

$$\begin{aligned}
&\int_{D^+_R(0)} |(i\nabla + A_e)\psi^a|^2 \, dy \leq C(\varepsilon) R^{2(1+\varepsilon)} + \lambda^a a_1^2 \int_{D^+_R(0)} \hat{p}(y) |\psi^a|^2 \, dy \\
&\leq C(\varepsilon) R^{2(1+\varepsilon)} + \lambda^a a_1^2 \|p\|_{L^\infty} R^2 \left(H(\psi^a, R, 0) + \int_{D^+_R(0)} |(i\nabla + A_e)\psi^a|^2 \, dy \right) \\
&\leq C(\varepsilon) R^{2(1+\varepsilon)} + \lambda^a \|p\|_{L^\infty} r_\varepsilon^2 C(\varepsilon) R^{2(1+\varepsilon)} \\
&+ \lambda^a \|p\|_{L^\infty} r_\varepsilon^2 \int_{D^+_R(0)} |(i\nabla + A_e)\psi^a|^2 \, dy.
\end{aligned}$$

At the second line we used the Poincaré inequality (2.67), at the third line we used (2.129) and the fact that $R \leq r_\varepsilon/a_1$. Then, thanks to (2.86), we have

$$\int_{D^+_R(0)} |(i\nabla + A_e)\psi^a|^2 \, dy \leq C(\varepsilon) R^{2(1+\varepsilon)} \frac{1 + \lambda^a \|p\|_{L^\infty} r_\varepsilon^2}{1 - \lambda^a \|p\|_{L^\infty} r_\varepsilon^2}.$$

We look then at the second part of the norm. Using Poincaré inequality (2.67), we obtain

$$\int_{D^+_R(0)} |\psi^a|^2 \, dy \leq R^2 H(\psi^a, R, 0) + R^2 \int_{D^+_R(0)} |(i\nabla + A_e)\psi^a|^2 \, dy \leq C(\varepsilon, R),$$

where we used the previous inequality and (2.129). Finally, we combine the two contributions and obtain a constant depending only on R and ε. ☐

Lemma 2.7.24. *Given $\varepsilon > 0$, take the same assumptions and notations of Lemma 2.7.20. Let ψ^a be as in (2.122). There exists $\psi \in H^1_{A_e,loc}(\mathbb{R}^2_+)$, $\psi \not\equiv 0$, such that for every $R > k_\varepsilon$ we have, up to a subsequence, $\psi^a \to \psi$ in $H^1_{A_e}(D^+_R(0))$ as $a_1 \to 0$. Moreover, ψ solves*

$$\begin{cases} (i\nabla + A_e)^2\psi = 0 & \mathbb{R}^2_+ \\ \psi = 0 & \{y_1 = 0\}. \end{cases} \tag{2.130}$$

Proof. By Lemma 2.7.23, there exists ψ such that, up to a subsequence, $\psi^a \rightharpoonup \psi$ in $H^1_{A_e}(D^+_R(0))$ and $\psi^a \to \psi$ in $L^2(D^+_R(0))$ as $a_1 \to 0$. Due to the compactness of the trace embedding, we have $\int_{\partial D^+_{k_\varepsilon}(0)} |\psi|^2 \,\mathrm{d}\sigma = k_\varepsilon$, so that $\psi \not\equiv 0$. [1] For every $R > k_\varepsilon$ and for every test function $\phi \in C_0^\infty(D^+_R(0)\backslash\{e\})$, we have

$$\int_{D^+_R(0)} (i\nabla + A_e)\psi^a \cdot \overline{(i\nabla + A_e)\phi} \,\mathrm{d}y = \lambda^a a_1^2 \int_{D^+_R(0)} \hat{p}(y)\psi^a \bar{\phi} \,\mathrm{d}y.$$

By the weak convergence in $H^1_{A_e}(D^+_R(0))$, the first term converges

$$\int_{D^+_R(0)} (i\nabla + A_e)\psi^a \cdot \overline{(i\nabla + A_e)\phi} \,\mathrm{d}y \to \int_{D^+_R(0)} (i\nabla + A_e)\psi \cdot \overline{(i\nabla + A_e)\phi} \,\mathrm{d}y.$$

We estimate the second term as follows by means of Lemma 2.7.23

$$\begin{aligned} \left| \lambda^a a_1^2 \int_{D^+_R(0)} \hat{p}(y)\psi^a \bar{\phi} \,\mathrm{d}y \right| &\le \lambda^a a_1^2 \|p\|_{L^\infty} \|\phi\|_{L^2(D^+_R(0))} \|\psi^a\|_{L^2(D^+_R(0))} \\ &\le C a_1^2 \|\psi^a\|_{H^1_{A_e}(D^+_R(0))} \le C(\varepsilon, R) a_1^2 \to 0, \end{aligned}$$

so that ψ solves the limit equation (2.130). In order to prove the strong convergence, we consider the equation satisfied by $\psi^a - \psi$. We have

$$(i\nabla + A_e)^2(\psi^a - \psi) = \lambda^a a_1^2 \hat{p}(y)\psi^a \quad \text{in } D^+_R(0).$$

[1] We remark that to use the embeddings, we first need to use the equivalent characterization of $H^1_{A_e}(D^+_R(0))$ given in Section 2.2 and then [49, Theorem 5.36]. This equivalent characterization makes use of the Hardy inequality of Lemma 2.2.1. However, this Lemma requires that the domain is smooth, which is not the case of B^+_R. Nevertheless, since the functions we work with are identically zero on the boundary $\{y_1 = 0\}$, we extend them in the entire ball $B_R(0)$ by zero and we obtained functions in $H^1_{A_e}(B_R(0))$. This argument repeat every time we use the compact embedding of the traces.

By Lemma 2.7.23 and the Sobolev embeddings, the right hand side above converges to 0 in $L^p(D_R^+(0))$ for every $p < \infty$ as $a_1 \to 0$. The Kato inequality

$$-\Delta|\psi^a - \psi| \leq |(i\nabla + A_e)^2(\psi^a - \psi)|$$

(see for example [9]) and the standard regularity theory for elliptic equations, imply that $|\psi^a - \psi| \to 0$ in $W^{2,p}(D_R^+(0))$ for every $p < \infty$ as $a_1 \to 0$. This in turn implies that the convergence is $C^{1,\tau}_{\rm loc}(D_R^+(0) \setminus \{e\})$ for every $\tau \in (0,1)$ and $H^1(D_r^+(0))$. $\square$

As a consequence of the strong convergence and of Lemma 2.7.22, we deduce the following.

Lemma 2.7.25. *Let ψ be defined in Lemma 2.7.24. We have*

$$N(\psi, R, 0, 0, A_e) \leq 1 + \varepsilon \quad \text{for every } R > k_\varepsilon, \tag{2.131}$$

$$\frac{H(\psi, R_2, 0)}{H(\psi, R_1, 0)} \leq \left(\frac{R_2}{R_1}\right)^{2(1+\varepsilon)} \qquad \text{for every } k_\varepsilon < R_1 < R_2.$$

Lemma 2.7.26. *Let ψ be defined in Lemma 2.7.24. There exists $d \in [0, +\infty]$ such that*

$$\lim_{R \to +\infty} N(\psi, R, 0, 0, A_e) = d.$$

Proof. Reasoning as in (2.96) we find

$$\frac{\mathrm{d}}{\mathrm{d}R} N(\psi, R, 0, 0, A_e) \geq -\frac{2M}{RH(\psi, R, 0)},$$

where M is now the constant

$$M = \lim_{\varepsilon \to 0} \mathrm{Re} \int_{\partial D_\varepsilon(e)} (i\nabla + A_e)\psi \cdot \nu \overline{(i\nabla + A_e)\psi \cdot y} \, \mathrm{d}\sigma > 0.$$

We can prove as in Lemma 2.7.9 that

$$\frac{H(\psi, R, 0)}{H(\psi, k_\varepsilon, 0)} \geq \left(\frac{R}{k_\varepsilon}\right)^2,$$

for $R > k_\varepsilon$. Recalling that $H(\psi, k_\varepsilon, 0) = 1$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}R} N(\psi, R, 0, 0, A_e) \geq -\frac{Ck_\varepsilon^2}{R^3},$$

for a positive constant C. Let us show that this implies the existence of the limit. Let for the moment $N(R) = N(\psi, R, 0, 0, A_e)$. Integrating the last inequality in (R_1, R_2), with $k_\varepsilon < R_1 < R_2$, we obtain

$$N(R_2) - N(R_1) \geq Ck_\varepsilon^2 \left(\frac{1}{R_2^2} - \frac{1}{R_1^2} \right). \tag{2.132}$$

If $d = +\infty$ there is nothing to prove. Otherwise, we claim that $N(R)$ is bounded. Indeed, $d \neq \infty$ implies the existence of $K > 0$ and of a sequence $R_n \to \infty$ such that $N(R_n) < K$ for every n, so that for R sufficiently large and $R_n > R$ we have, by (2.132)

$$N(R) \leq N(R_n) - Ck_\varepsilon^2 \left(\frac{1}{R_n^2} - \frac{1}{R^2} \right) \leq K + o(1) \quad \text{as } R \to \infty,$$

so that N is bounded. Suppose by contradiction that $N(R)$ does not admit limit $d \in [0, \infty)$. Then for every $\delta > 0$ there exists a sequence $R_n \to \infty$ such that $|N(R_n) - N(R_{n+1})| \geq \delta$. The case $N(R_n) \geq N(R_{n+1}) + \delta$ contradicts (2.132) if R_n is great enough, the case $N(R_{n+1}) \geq N(R_n) + \delta$ contradicts the fact that N is bounded. □

In the next subsection we will prove that $d = 1$.

Proof of Theorem 2.1.13

In order to study the behaviour of the limit function ψ at infinity, we perform a rescaling (blow-down) on the independent variable by a factor R. As before, using the boundedness of the Almgren's frequence of ψ, we prove the convergence of the blow-down function as $R \to \infty$. Moreover, we will prove that the limit function is an homogeneous harmonic function of degree 1. Then, this aims us to conclude that sufficiently far from the singularity ψ behaves like an harmonic function of degree 1, up to a complex phase. More specifically, we prove that this function ψ verifies the conditions of Proposition 2.1.12.

Lemma 2.7.27. *Let ψ be the function introduced in Lemma 2.7.24. We define*

$$w_R(x) = \frac{\psi(Rx)}{\sqrt{H(\psi, R, 0)}}. \tag{2.133}$$

For every $r > 0$, there exists a constant $C(\varepsilon, r)$ such that $\|w_R\|_{H^1_{A_{e/R}}(D_r^+(0))} \leq C(\varepsilon, r)$ for every $R > k_\varepsilon$.

Proof. For $r > 1$ and $R > k_\varepsilon$ we have

$$N(w_R, r, 0, 0, A_{e/R}) = N(\psi, rR, 0, 0, A_e) \le 1 + \varepsilon$$

and

$$H(w_R, r, 0) = \frac{H(\psi, rR, 0)}{H(\psi, R, 0)} \le r^{2(1+\varepsilon)},$$

by Lemma 2.7.25. By combining the two, we obtain

$$E(w_R, r, 0, 0, A_{e/R}) \le (1 + \varepsilon) r^{2(1+\varepsilon)}$$

for every $r > 1$ and $R > k_\varepsilon$. As a consequence, using Lemma 2.7.3 we estimate

$$\|w_R\|_{H^1_{A_{e/R}}(D_r^+(0))} \le (1 + r^2) E(w_R, r, 0, 0, A_{e/R}) + r^2 H(w_R, r, 0) \le C(\varepsilon, r)$$

for $R > k_\varepsilon$. □

Lemma 2.7.28. *Let w_R be defined in (2.133). There exists $w \in H^1_{loc}(\mathbb{R}^2_+)$, $w \not\equiv 0$, such that $e^{-i\theta_{e/R}/2} w_R \rightharpoonup w$ in $H^1_{loc}(\mathbb{R}^2_+)$. In addition, w is harmonic in $\mathbb{R}^2_+$ with zero boundary conditions and, for almost every $r > 0$, we have*

$$\lim_{R \to \infty} E(w_R, r, 0, 0, A_{e/R}) = E(w, r, 0, 0, 0). \tag{2.134}$$

Proof. Fix $r > 0$. By (2.11) and Lemma 2.7.27, there exists a constant $C(\varepsilon, r) > 0$ (not depending on R) such that

$$\|w_R\|_{H^1(D_r^+(0))} \le C \|w_R\|_{H^1_{A_{e/R}}(D_r^+(0))} \le C(\varepsilon, r).$$

In order to check that the constant C in the previous inequality does not depend on the position of the singularity e/R, one can extend functions in $H^1_{A_{e/R}}(D_r^+(0))$ which vanish on $\{x_1 = 0\}$ trivially to functions belonging to $H^1_{A_{e/R}}(D_r(0))$, and then proceed as in the proof of [17, Lemma 7.4]. Hence there exists $\tilde{w} \in H^1(D_r^+(0))$ such that $w_R \rightharpoonup \tilde{w}$ in $H^1(D_r^+(0))$ and $w_R \to \tilde{w}$ in $L^2(D_r^+(0))$, as $R \to +\infty$. Since $H(w_R, 1, 0) = 1$ for every R, the trace embeddings provide $\tilde{w} \not\equiv 0$.

Let $w = e^{-i\theta_0/2}\tilde{w}$. In order to prove that w is harmonic, notice first that $(i\nabla + A_{e/R})^2 w_R = 0$ in $\mathbb{R}^2_+$ for every R. Given a test function $\phi \in C_0^\infty(D_r^+(0))$, let R be so large that $e/R \notin \operatorname{supp}\{\phi\}$. Consequently we have

$$-\Delta(e^{-i\theta_{e/R}/2} w_R) = 0 \tag{2.135}$$

in supp$\{\phi\}$. This implies, using the weak convergence,

$$0 = \int_{D_r^+(0)} \nabla(e^{-i\theta_{e/R}/2} w_R) \cdot \nabla\phi \,\mathrm{d}x \to \int_{D_r^+(0)} \nabla w \cdot \nabla\phi \,\mathrm{d}x \quad \text{as } R \to \infty,$$

so that w is harmonic in $D_r^+(0)$.

To prove the last part of the statement, fix two concentric semi-annuli $\mathcal{A}_1 \subset \mathcal{A}_2$, centred at the origin and having positive distance from it. Let η be a cut-off function which is 1 in $\mathcal{A}_1$ and vanishes outside $\mathcal{A}_2$. For R sufficiently large, we have that (2.135) holds in $\mathcal{A}_2$. By testing the equation satisfied by $e^{-i\theta_{e/R}/2} w_R - w$ by $(e^{-i\theta_{e/R}/2} w_R - w)\eta$ in $\mathcal{A}_2$, we obtain

$$\begin{aligned}
&\int_{\mathcal{A}_1} |\nabla(e^{-i\theta_{e/R}/2} w_R - w)|^2 \,\mathrm{d}x \\
&\leq \left| \int_{\mathcal{A}_2} \nabla(e^{-i\theta_{e/R}/2} w_R - w) \nabla\eta (e^{-i\theta_{e/R}/2} w_R - w) \,\mathrm{d}x \right| \to 0, \text{ as } R \to +\infty,
\end{aligned}$$

by the weak convergence. This implies that

$$\int_{\partial D_\rho^+(0)} \left(|\nabla(e^{-i\theta_{e/R}/2} w_R - w)|^2 + |e^{-i\theta_{e/R}/2} w_R - w|^2 \right) \mathrm{d}\sigma \to 0, \tag{2.136}$$

for almost every ρ such that $\partial D_\rho^+(0) \subset \mathcal{A}_1$, as $R \to +\infty$.

Finally, we use integration by parts as follows (the second equality is well defined provided for $R > 1/r$)

$$\begin{aligned}
&\left| E(w_R, r, 0, 0, A_{e/R}) - E(w, r, 0, 0, 0) \right| \\
&\leq \int_{\partial D_r^+(0)} \left| -i(i\nabla + A_{e/R}) w_R \cdot \nu \overline{w_R} - \nabla w \cdot \nu w \right| \,\mathrm{d}\sigma \\
&= \int_{\partial D_r^+(0)} \left| \nabla(e^{-i\theta_{e/R}/2} w_R) \cdot \nu \overline{e^{-i\theta_{e/R}/2} w_R} - \nabla w \cdot \nu w \right| \,\mathrm{d}\sigma \to 0,
\end{aligned}$$

where the convergence to 0 comes from (2.136) for almost every $r > 0$. □

End of the proof of theorem 2.1.13. By combining (2.134) and Lemma 2.7.26 we obtain, for almost every $r > 0$,

$$N(w, r, 0, 0, 0) = \lim_{R \to \infty} N(w_R, r, 0, 0, A_{e/R}) = \lim_{R \to \infty} N(\psi, rR, 0, 0, A_e) = d$$

(recall that ψ was introduced in Lemma 2.7.24). Since $N(w, \cdot, 0, 0, 0)$ is continuous, it is constant. Since we proved in the previous lemma that w is harmonic with zero boundary conditions on $\{x_1 = 0\}$, we deduce from standard arguments (see for example [54, Proposition 3.9]) that $w(r,\theta) = Cr^d \cos(d\theta)$, for some $d \in \mathbb{N}_0$ odd or $w(r,\theta) = Cr^d \sin\theta$, for some $d \in \mathbb{N}_0$ even. Comparing with (2.131), taking for example $\varepsilon = 1/2$, we conclude that $d = 1$. In conclusion, by Proposition 2.1.12, ψ solves (2.7)-(2.8). Then,

$$\psi = Ce^{i\theta_e/2}\left(r\cos\theta - \frac{\beta}{\pi}\frac{\cos\theta}{r} + O(r^{-3})\right), \qquad \text{for } r > 1,$$

(the exact asymptotic is given in (2.105)). Moreover, since $H(\psi, k_\varepsilon, 0) = 1$, the constant C is given by

$$|C|^2 = \frac{1}{\frac{k_\varepsilon^2\pi}{2} - \beta + O(\frac{1}{k_\varepsilon^2})}. \tag{2.137}$$

This constant will be useful in the exact estimates computed later.

□

Proof of Theorem 2.1.14

We can assume without loss of generality that $b = 0$ and moreover, by Lemma 2.7.1, that Ω satisfies (2.65). Let φ_k have a zero of order 1 at $0 \in \partial\Omega$, meaning that there are no nodal lines of φ_k ending at 0. Let $k_\varepsilon > \sqrt{\beta/\pi}$ large be such that the statement of Theorem 2.1.13 holds, $\bar{a}_{1,\varepsilon}$, r_ε be as in Lemma 2.7.20. We proceed similarly to the proof of Theorem 2.1.11.

For $i = 1, \dots, k$ let

$$v_i^{ext} = e^{-i\frac{\theta_a}{2}}\varphi_i^a \quad \text{ in } \Omega \setminus D_{k_\varepsilon a_1}(\pi(a)).$$

For a_1 sufficiently small, v_i^{int} is defined as the unique function which achieves

$$\inf\Big\{\int_{D^+_{k_\varepsilon a_1}(\pi(a))}\left(|\nabla v|^2 - \lambda_i^a p(x)|v|^2\right)\,dx : \\ v \in H^1(D^+_{k_\varepsilon a_1}(\pi(a))),\ v = v_i^{ext} \text{ on } \partial D^+_{k_\varepsilon a_1}(\pi(a))\Big\}.$$

We let $v_i = v_i^{int}$ in $D^+_{k_\varepsilon a_1}(\pi(a))$, $v_i = v_i^{ext}$ in $\Omega \setminus D_{k_\varepsilon a_1}(\pi(a))$. Notice that estimate (2.81) holds in this case for every $1 \le i \le k$ since φ_k has no nodal line at 0. We take

$$F_k = \left\{\Phi = \sum_{i=1}^k \alpha_i v_i : \alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{R}^k\right\} \subset H^1_0(\Omega),$$

so that

$$\lambda_k \leq \sup_{\Phi \in F_k} \frac{\|\nabla \Phi\|^2_{L^2(\Omega)}}{\int_\Omega p(x)|\Phi|^2 \mathrm{d}x} = \sup_{\alpha \in \mathbb{R}^k} \frac{\alpha^T M \alpha}{\alpha^T N \alpha} = \lambda_{max}(N^{-1}M). \tag{2.138}$$

Here $\lambda_{max}(\cdot)$ is the largest eigenvalue of a matrix and M, N are $k \times k$ matrices with entries

$$\begin{aligned} m_{ij} &= \int_\Omega \nabla v_i \cdot \nabla v_j \, \mathrm{d}x \\ &= \lambda_i^a \int_\Omega p(x) v_i v_j \, \mathrm{d}x + \int_{\partial D^+_{k_\varepsilon a_1}(\pi(a))} \nabla(v_i^{int} - v_i^{ext}) \cdot \nu v_j \, \mathrm{d}\sigma, \end{aligned}$$

and

$$n_{ij} = \int_\Omega p(x) v_i v_j \, \mathrm{d}x.$$

As in the proof of Theorem 2.1.11, the significant part is m_{kk}, and more precisely the boundary term in m_{kk}. Let us then estimate it. We perform the following change of variables in order to work with the function ψ_k^a defined in (2.122)

$$\begin{aligned} f_k^{a,ext}(y) &= e^{-i\theta_e/2}\psi_k^a(y) = \frac{v_k^{ext}(a_1 y + \pi(a))}{\sqrt{H(\varphi_k^a, k_\varepsilon a_1, \pi(a))}}, \\ f_k^{a,int}(y) &= \frac{v_k^{int}(a_1 y + \pi(a))}{\sqrt{H(\varphi_k^a, k_\varepsilon a_1, \pi(a))}}. \end{aligned}$$

By Theorem 2.1.13, we have that $f_k^{a,ext} \to e^{-i\theta_e/2}\psi_k$ in $H^1(D^+_{k_\varepsilon}(0))$ as $a_1 \to 0$. Moreover, we have $f_k^{a,int} \to f_k^{int}$ in $H^1(D^+_{k_\varepsilon}(0))$ as $a_1 \to 0$ and $-\Delta f_k^{int} = 0$ in $D^+_{k_\varepsilon}(0)$, $f_{k_\varepsilon}^{int} = e^{-i\theta_e/2}\psi_k$ on $\partial D^+_{k_\varepsilon}(0)$ (with constant C given by (2.137)). From Proposition 2.1.12 (ii), we deduce the following behaviour of the harmonic extension f_k^{int}

$$f_k^{int}(r,\theta) = C\left(1 - \frac{\beta}{\pi k_\varepsilon^2}\right) r\cos\theta + C \sum_{\substack{n \geq 3 \\ n \text{ odd}}} \frac{b_n}{k_\varepsilon^{2n}} \cos(n\theta)\, r^n, \quad r < k_\varepsilon,$$

for b_n as in Proposition 2.1.12, (ii), and C given by (2.137). Therefore we have

$$\begin{aligned}
&\int_{\partial D^+_{k_\varepsilon a_1}(\pi(a))} \nabla(v_k^{int} - v_k^{ext}) \cdot \nu v_k \, \mathrm{d}\sigma \\
&\qquad = H(\varphi_k^a, k_\varepsilon a_1, \pi(a)) \int_{\partial D^+_{k_\varepsilon}(0)} \nabla(f_k^{a,int} - f_k^{a,ext}) \cdot \nu f_k^{ext} \, \mathrm{d}\sigma \\
&\qquad = H(\varphi_k^a, k_\varepsilon a_1, \pi(a))\, C^2 \left\{ -\beta + O(k_\varepsilon^{-2}) + o_{a_1}(1) \right\} \qquad (2.139) \\
&\qquad = -C_k a_1^2 + o(a_1^2)
\end{aligned}$$

for some $C_k > 0$ as soon as k_ε is sufficiently large and a_1 is sufficiently small, where in the last step we used Lemma 2.7.21.

We deduce that the matrices M and N appearing in (2.138) have the following form

$$M = \begin{pmatrix} \lambda_1^a + O(a_1^2) & & O(a_1^2) & \\ & \ddots & & O(a_1^2) \\ O(a_1^2) & & \lambda_{k-1}^a + O(a_1^2) & \\ & O(a_1^2) & & \lambda_k^a - C_k a_1^2 + o(a_1^2) \end{pmatrix}$$

$$N = \begin{pmatrix} 1 + O(a_1^4) & & O(a_1^4) \\ & \ddots & \\ O(a_1^4) & & \\ & & 1 + O(a_1^4) \end{pmatrix}.$$

Since λ_k is simple, proceeding similarly to Lemma 2.7.6, we obtain

$$\lambda_k \leq \lambda_k^a - C_k a_1^2 + o(a_1^2), \tag{2.140}$$

which concludes the proof. Indeed, $N^{-1}M$ has the same form as M. When looking for the eigenvalues of this matrix we search the t such that

$$(\lambda_k^a - C_k a_1^2 - t) Q_{k-1}(t, a_1^2) + a_1^4 Q_{k-2}(t, a_1^2) = 0,$$

where

$$Q_{k-1}(t, a_1^2) = \prod_{i=1}^{k-1} (\lambda_i^a + O(a_1^2) - t)$$

and $Q_{k-2}(t, a_1^2)$ is a polynomial of degree $k-2$ in the variable t, which depends on a_1 with terms of order $O(a_1^2)$. We set $\varepsilon = a_1^2$ and we apply the implicit function theorem to

$$f(\varepsilon, x, t) = (x - t) Q_{k-1}(t, \varepsilon) + \varepsilon^2 Q_{k-2}(t, \varepsilon)$$

at the point $(0, \lambda_k^a - C_k a_1^2, \lambda_k^a - C_k a_1^2)$. We see that at any point $(0, \bar{x}, \bar{x})$ we have

$$\frac{\partial f}{\partial t}(0, \bar{x}, \bar{x}) = -Q_{k-1}(\bar{x}, 0) = -\prod_{i=1}^{k-1}(\lambda_i - \bar{x}),$$

so that the implicit function theorem applies at any point $(0, \bar{x}, \bar{x})$ such that $\bar{x} \neq \lambda_i$ for every $i = 1, \dots, k-1$, and we have $t(\varepsilon, \bar{x}) = \bar{x} + o(\varepsilon)$ in a neighborhood of $(\varepsilon, x) = (0, \bar{x})$. Taking $\bar{x} = \lambda_k^a - C_k a_1^2$ we obtain (2.140).

Proof of the exact estimate from below

Finally, to obtain the exact estimate, we use what was done before and rewrite (2.139) and (2.140) as

$$\frac{\lambda_k^a - \lambda_k}{a_1^2} \geq \frac{H(\varphi_k^a, k_\varepsilon a_1, \pi(a))}{k_\varepsilon^2 a_1^2} \left(k_\varepsilon^2 C^2\right) \left(\beta - O(k_\varepsilon^{-2}) + o_{a_1}(1)\right).$$

We can first take the $\liminf_{a_1 \to 0}$. Then, since k_ε can be taken as large as we want, we use the expression of the normalization constant C given by (2.137) to see that $\lim_{k_\varepsilon \to +\infty} k_\varepsilon^2 C^2 = \frac{2}{\pi}$. By the exact estimate of Lemma 2.7.21, we get

$$\liminf_{a_1 \to 0} \frac{\lambda_k^a - \lambda_k}{a_1^2} \geq c_1^2 \beta, \tag{2.141}$$

where c_1 was given in (2.108).

2.7.6. Pole approaching the boundary not on a nodal line of φ_k: estimate from above

Some ideas and proofs of this section were first developed in [38]. Let's consider $K > 2$ sufficiently big, and $\bar{a}_1 > 0$ (depending on K) small enough. As in the previous section, we assume that $b = 0 \in \partial\Omega$ and that (2.65) is satisfied around 0, meaning that $\partial\Omega$ is locally flat around 0. As already said at the begin of Section 2.7.5, if φ has a zero of order 1 at 0, then if $\pi(a) = (0, a_2)$ is sufficiently close to 0, φ has a zero of order 1 also at $\pi(a)$: there exists $\bar{a}_2 > 0$ such that, for $|\pi(a)| < \bar{a}_2$, we have

$$\varphi(r_{\pi(a)}, \theta_{\pi(a)}) = r_{\pi(a)} c_1(\pi(a)) \cos\theta_{\pi(a)} + O(r_{\pi(a)}^2),$$

as $r_{\pi(a)} = |x - \pi(a)| \to 0$, where $x - \pi(a) = r_{\pi(a)}(\cos\theta_{\pi(a)}, \sin\theta_{\pi(a)})$ and $c_1(\pi(a)) \neq 0$ is the same as in (2.108).

In the same way as in Sections 2.7.2 and 2.7.5, the idea is to use the variational characterization of the eigenvalues and to construct suitable functions to obtain the right estimates. We first begin by the construction of those functions.

Construction of the test functions

First, for $i = 1, \dots, k$ we define

$$v_i^{ext} = e^{i\frac{\theta_a}{2}}\varphi_i, \qquad \text{in } \Omega \backslash D_{Ka_1}(\pi(a)).$$

Here, we choose the discontinuity segment of θ_a in such a way that $e^{i\theta_a/2}$ is regular in $\Omega \backslash D_{Ka_1}(\pi(a))$. In its domain, we have that

$$\begin{cases} (i\nabla + A_a)^2 v_i^{ext} = \lambda_i p(x) v_i^{ext} & \Omega \backslash D_{Ka_1}(\pi(a)) \\ v_i^{ext} = e^{i\frac{\theta_a}{2}}\varphi_i & \partial\left(\Omega \backslash D_{Ka_1}(\pi(a))\right). \end{cases}$$

Next, in $D^+_{Ka_1}(\pi(a))$, we consider the unique function v_i^{int}, $i = 1, \dots, k$, which achieves the infimum

$$\inf\Big\{ \int_{D^+_{Ka_1}(\pi(a))} |(i\nabla + A_a)u|^2 \ : \\ u \in H^1_{A_a}(D^+_{Ka_1}(\pi(a))),\, u = e^{i\frac{\theta_a}{2}}\varphi_i \text{ on } \partial D^+_{Ka_1}(\pi(a)) \Big\}.$$

In its domain, this function solves the equation

$$\begin{cases} (i\nabla + A_a)^2 v_i^{int} = 0 & D^+_{Ka_1}(\pi(a)) \\ v_i^{int} = e^{i\frac{\theta_a}{2}}\varphi_i & \partial D^+_{Ka_1}(\pi(a)). \end{cases}$$

Finally, for $i = 1, \dots, k$, the function v_i is equal to v_i^{int} in $D^+_{Ka_1}(\pi(a))$ and to v_i^{ext} in $\Omega \backslash D_{Ka_1}(\pi(a))$.

We consider the k-dimensional subspace of $\mathcal{D}^{1,2}_{A_a}(\Omega)$

$$F_k = \left\{ \Phi = \sum_{i=1}^k \alpha_i v_i \,|\, \alpha = (\alpha_1, \dots, \alpha_k) \in \mathbb{C}^k \right\} \subset \mathcal{D}^{1,2}_{A_a}(\Omega)$$

so that

$$\lambda_k^a \le \sup_{\Phi \in F_k} \frac{\|(i\nabla + A_a)\Phi\|^2_{L^2(\Omega)}}{\int_\Omega p(x)|\Phi|^2 \mathrm{d}x} = \sup_{\alpha \in \mathbb{C}^k} \frac{\overline{\alpha}^T M \alpha}{\overline{\alpha}^T N \alpha} = \lambda_{max}(N^{-1}M).$$

Here $\lambda_{max}(\cdot)$ is the largest eigenvalue of a matrix and M, N are $k \times k$ hermitian matrices with entries

$$\begin{aligned} m_{ij} &= \int_\Omega (i\nabla + A_a)v_i \cdot \overline{(i\nabla + A_a)v_j}\,\mathrm{d}x \\ &= \lambda_i \int_\Omega p(x) v_i \overline{v_j}\,\mathrm{d}x - \lambda_i \int_{D^+_{Ka_1}(\pi(a))} p(x)\, v_i^{int} \overline{v_j^{int}}\,\mathrm{d}x \\ &\quad + i \int_{\partial D^+_{Ka_1}(\pi(a))} (i\nabla + A_a)(v_i^{ext} - v_i^{int}) \cdot \nu\, \overline{v_j}\,\mathrm{d}\sigma, \end{aligned}$$

and

$$n_{ij} = \int_\Omega p(x) v_i \overline{v_j}\,\mathrm{d}x.$$

We notice that for all $i = 1, \dots, k$, the estimates of the m_{ij} and n_{ij} are provided by (2.81) and (2.82), (2.83) (with $n = 0$ for the last ones) since φ_k does not have any nodal lines ending at 0. Then, we deduce that the matrices M and N have the following form

$$M = \begin{pmatrix} \lambda_1 + O(a_1^2) & & O(a_1^2) & \\ & \ddots & & O(a_1^2) \\ O(a_1^2) & & \lambda_{k-1} + O(a_1^2) & \\ & O(a_1^2) & & \lambda_k + C_k a_1^2 + o(a_1^2) \end{pmatrix}$$

$$N = \begin{pmatrix} 1 + O(a_1^4) & & O(a_1^4) \\ & \ddots & \\ O(a_1^4) & & 1 + O(a_1^4) \end{pmatrix}.$$

Since λ_k is simple, proceeding similarly to Lemma 2.7.6, we obtain

$$\lambda_k^a \le \lambda_k + C_k a_1^2 + o(a_1^2).$$

Finally, we want to estimate more precisely the constant C_k above. To this aim, in the same way as before, the significant part to look at is the boundary term in m_{kk}, that is

$$i \int_{\partial D^+_{Ka_1}(\pi(a))} (i\nabla + A_a)(v_k^{ext} - v_k^{int}) \cdot \nu \overline{v_k}\,\mathrm{d}\sigma. \tag{2.142}$$

To estimate (2.142), we will perform as before a convenient blow-up on v_i^{int} and v_i^{ext}. Those blow-up functions will converge to their own limit profile (in the same way as $f_k^{a,int}$ and $f_k^{a,ext}$ were converging in Section 2.7.5). The next section studies more precisely the limit profile associated to v_k^{int}, as we will see later.

Limit profile

Let W_K be the unique function which achieves the infimum

$$\inf\Big\{\int_{D_K^+(0)}|(i\nabla+A_e)u|^2\,\mathrm{d}y\ :\\ u\in H^1_{A_e}(D_K^+(0)),\ u=e^{i\frac{\theta_e}{2}}c_1y_1\ \text{ on } \partial D_K^+(0)\Big\}. \tag{2.143}$$

Again the discontinuity segment of θ_e is chosen in such a way that the boundary trace is continuous. We notice that the boundary trace can be rewritten by making use of the polar coordinates as $e^{i\frac{\theta_e}{2}}c_1K\cos\theta$. That function W_K verifies the equation

$$\begin{cases}(i\nabla+A_e)^2W_K=0 & D_K^+(0)\\ W_K=e^{i\frac{\theta_e}{2}}c_1y_1 & \partial D_K^+(0).\end{cases}$$

The following lemma proves that, locally, W_K converges to the limit profile ψ defined in (2.7) (by adapting suitably the constant in front of ψ).

Lemma 2.7.29. *For every $r>1$, $W_K\to\psi$ in $H^1_{A_e}(D_r^+(0))$ as $K\to+\infty$, where ψ is the limit profile defined in (2.7), with constant $C=c_1$, c_1 being as in (2.108).*

Proof. First, since we remember that ψ has the form $\psi=e^{i\frac{\theta_e}{2}}c_1\,(y_1+w)$, w being defined in (2.105), we conclude that

$$\begin{cases}(i\nabla+A_e)^2(W_K-\psi)=0 & D_K^+(0)\\ W_K-\psi=-e^{i\frac{\theta_e}{2}}c_1w & \partial D_K^+(0).\end{cases}$$

In the same way as in [38], we introduce a smooth cut-off function η_K such that $\eta_K=0$ in $D_{K/2}^+(0)$, $\eta_K=1$ in $\mathbb{R}^2_+\setminus D_K^+(0)$ and $|\nabla\eta_K|\le C/K$. We remark that the function $\eta_K\left(e^{i\frac{\theta_e}{2}}c_1y_1-\psi\right)\in H^1_{A_e}(D_K^+(0))$ thanks to the fact that the singularity e lies in a region where $\eta_K=0$, $K\ge 2$. Moreover, it is equal to $W_K-\psi$ on $\partial D_K^+(0)$. Then, since $W_K-\psi$ realises the infimum of the energy, we have for K large enough

$$\int_{D_r^+(0)} |(i\nabla + A_e)(W_K - \psi)|^2 \leq \int_{D_K^+(0)} |(i\nabla + A_e)(W_K - \psi)|^2$$
$$\leq \int_{D_K^+(0)} \left|(i\nabla + A_e)\left(\eta_K(e^{i\frac{\theta_e}{2}}c_1 y_1 - \psi)\right)\right|^2$$
$$\leq \int_{D_K^+(0)\setminus D_{K/2}^+(0)} \left(2\eta_K^2 \left|(i\nabla + A_e)\left(e^{i\frac{\theta_e}{2}}c_1 y_1 - \psi\right)\right|^2 + \frac{2C^2}{K^2}\left|e^{i\frac{\theta_e}{2}}c_1 y_1 - \psi\right|^2\right)$$
$$= 2\int_{D_K^+(0)\setminus D_{K/2}^+(0)} \left(\eta_K^2 c_1^2 |\nabla w|^2 + \frac{C^2 c_1^2}{K^2}|w|^2\right).$$

Since $\int_{\mathbb{R}^2_+} |\nabla w|^2 = \beta < +\infty$, the first term becomes

$$\int_{D_K^+(0)\setminus D_{K/2}^+(0)} \eta_K^2 |\nabla w|^2 \leq \int_{\mathbb{R}^2_+\setminus D_{K/2}^+(0)} |\nabla w|^2 \to 0 \quad \text{as } K \to +\infty.$$

If we remember the asymptotic expansion $w = -\frac{\beta}{\pi r} + O(r^{-3})$, for $r > 1$, see (2.105), the second term gives

$$\int_{D_K^+(0)\setminus D_{K/2}^+(0)} \frac{C^2}{K^2}|w|^2 \to 0 \quad \text{as } K \to +\infty.$$

Finally, we use the Poincaré-type inequalities (2.67), (2.71) to obtain

$$\int_{D_r^+(0)} |W_K - \psi|^2 \leq 2r^2 \int_{D_r^+(0)} |(i\nabla + A_e)(W_K - \psi)|^2 \to 0 \quad \text{as } K \to +\infty,$$

by the previous step. This concludes the proof. □

In the following, we would like to estimate the boundary integral

$$-i\int_{\partial D_K^+(0)} (i\nabla + A_e)W_K \cdot \nu\, \overline{W_K}\, \mathrm{d}\sigma.$$

Thanks to (2.143), we already know W_k on $\partial D_K^+(0)$ but it still misses the gradient part. The function $Z_K = e^{-i\frac{\theta_e}{2}} W_K$ solves in the annulus $D_K^+(0)\setminus D_1^+(0)$

$$\begin{cases} -\Delta Z_K = 0 & D_K^+(0)\setminus D_1^+(0) \\ Z_K = c_1 y_1 & \partial D_K^+(0) \\ Z_K = 0 & \{y_1 = 0\}. \end{cases}$$

In this annulus, we can develop Z_K in Fourier series, in the basis $\{\cos m\theta, \sin n\theta\}$, for $\theta \in (-\pi/2, \pi/2)$, m odd and n even, corresponding to the functions being zero on the boundary $\{y_1 = 0\}$

$$Z_K(r,\theta) = \sum_{m \text{ odd}} a_{K,m}(r) \cos m\theta + \sum_{n \text{ even}} b_{K,n}(r) \sin n\theta, \quad 1 < r < K,$$

where

$$a''_{K,m}(r) + \frac{1}{r} a'_{K,m}(r) - \frac{m^2}{r^2} a_{K,m}(r) = 0 \tag{2.144}$$

and

$$b''_{K,n}(r) + \frac{1}{r} b'_{K,n}(r) - \frac{n^2}{r^2} b_{K,n}(r) = 0,$$

for $1 < r < K$.

Lemma 2.7.30. *We have that*

$$-i \int_{\partial D_K^+(0)} (i\nabla + A_e) W_K \cdot \nu \, \overline{W_K} \, \mathrm{d}\sigma = c_1 \frac{\pi}{2} K^2 \left(\frac{K^2+4}{K^2-4} c_1 - \frac{4 a_{K,1}(2)}{K^2-4} \right),$$

where c_1 is defined in (2.108) and $a_{K,1}(2)$ is the Fourier coefficient in $r = 2$.

Moreover, we have that

$$a_{K,1}(2) \to 2c_1 \left(1 - \frac{\beta}{4\pi}\right), \qquad \text{as } K \to +\infty,$$

β being defined in (2.104).

Proof. In the following, we use in order the definition of Z_K, its expression in Fourier series, the fact that $Z_K = c_1 y_1$ on $\partial D_K^+(0)$, and the orthogonality of the element of the basis

$$\begin{aligned} -i \int_{\partial D_K^+(0)} (i\nabla + A_e) W_K \cdot \nu \, \overline{W_K} \, \mathrm{d}\sigma &= \int_{\partial D_K^+(0)} \nabla Z_K \cdot \nu \, \overline{Z_K} \, \mathrm{d}\sigma \\ &= c_1 \frac{\pi}{2} K^2 \, a'_{K,1}(K). \end{aligned}$$

It remains us to estimate $a'_{K,1}(K)$. To this aim, we remember that $a_{K,1}(r)$ solves (2.144). This equation may be rewritten as

$$\left(r^3 \left(\frac{a_{K,1}(r)}{r} \right)' \right)' = 0.$$

By integrating between 2 and r, we obtain

$$a_{K,1}(r) = -\frac{A}{2r} + \frac{A}{8}r + \frac{a_{K,1}(2)}{2}r, \qquad 2 < r < K,$$

where A is an integration constant. Next, we use the expression of Z_K in Fourier series and its value on $\partial D_K^+(0)$ to obtain

$$K\frac{\pi}{2}\, a_{K,1}(K) = \frac{1}{K}\int_{\partial D_K^+(0)} Z_K\, y_1 \,\mathrm{d}\sigma = c_1 K^2 \frac{\pi}{2},$$

that is

$$a_{K,1}(K) = c_1 K.$$

By simple calculations, we can deduce the value of the integration constant A and obtain

$$a_{K,1}(r) = \frac{c_1 K^2}{K^2-4}\left(r - \frac{4}{r}\right) + \frac{2a_{K,1}(2)}{K^2-4}\left(\frac{K^2}{r} - r\right).$$

By computing the derivative and estimating it in $r = K$, we obtain the first claim.

To prove the second claim, we use the same tools as before to obtain

$$a_{K,1}(2) = \frac{1}{2\pi}\int_{\partial D_2^+(0)} Z_K\, y_1 \,\mathrm{d}\sigma.$$

Since we proved in Lemma 2.7.29 the strong convergence of W_K to ψ in $H^1_{A_e}(D_r^+(0))$ for every $r > 1$, this is in particular valid for $r = 2$. By using a trace embedding (see for example [49, Theorem 5.36]), the L^2-convergence of the traces holds. Then, by remembering the expression of $\psi = c_1 e^{i\frac{\theta_e}{2}}\left(r - \frac{\beta}{\pi r}\right)\cos\theta + O(r^{-3})$, given exactly in (2.105), we obtain

$$\frac{1}{2\pi}\int_{\partial D_2^+(0)} Z_K\, y_1 \,\mathrm{d}\sigma \to 2c_1\left(1 - \frac{\beta}{4\pi}\right), \qquad \text{as } K \to +\infty.$$

□

Normalized Blow-up sequences

We define first the blow-up function related to v_k^{ext}.

$$U_k^a(y) = \frac{\varphi_k(a_1 y + \pi(a))}{a_1}. \tag{2.145}$$

Since we know that φ_k has a zero of order 1 and satisfies the asymptotic expansion (2.108), we can rewrite U_k^a as

$$U_k^a(r,\theta) = c_1 r\cos\theta + \frac{O((ra_1)^2)}{a_1}, \tag{2.146}$$

where $y = r(\cos\theta, \sin\theta)$.

Lemma 2.7.31. *For every $R>0$, $U_k^a \to c_1 y_1$ in $H^1(D_R^+(0))$, as $a_1 \to 0$.*

Proof. The proof is immediate by the shape of U_k^a. □

Next, we define the blow-up sequence

$$W_k^a(y) = \frac{v_k^{int}(a_1 y + \pi(a))}{a_1}, \tag{2.147}$$

which solves

$$\begin{cases} (i\nabla + A_e)^2 W_k^a = 0 & D_K^+(0) \\ W_k^a = e^{i\frac{\theta_e}{2}} U_k^a & \partial D_K^+(0). \end{cases}$$

The following Lemma tells us that the blow-up function W_k^a converges in fact to the limit profile W_K introduced above.

Lemma 2.7.32. *The function $W_k^a \to W_K$ in $H^1_{A_e}(D_K^+(0))$, as $a_1 \to 0$, where W_K was defined in* (2.143).

Proof. We use the same ideas and the cut-off function η_K as in the proof of Lemma 2.7.29. The function $e^{i\frac{\theta_e}{2}}\eta_K\,(U_k^a - c_1 y_1)$ is in $H^1_{A_e}(D_K^+(0))$ and moreover $e^{i\frac{\theta_e}{2}}\eta_K\,(U_k^a - c_1 y_1) = W_k^a - W_K$ on $\partial D_K^+(0)$. Then, since $W_k^a - W_K$ realizes the infimum of the energy

$$\int_{D_K^+(0)} |(i\nabla + A_e)(W_k^a - W_K)|^2 \le \int_{D_K^+(0)} |(i\nabla + A_e)\left(e^{i\frac{\theta_e}{2}}\eta_K(U_k^a - c_1 y_1)\right)|^2$$
$$\le \int_{D_K^+(0)\setminus D_{K/2}^+(0)} \left(\eta_K^2 |\nabla(U_k^a - c_1 y_1)|^2 + \frac{C^2}{K^2}|U_k^a - c_1 y_1|^2\right) \to 0, \quad \text{as } a_1 \to 0,$$

because of Lemma 2.7.31. □

Proof of the exact estimate from above

First, we look at the first term in (2.142). By using (2.145), (2.146) and the convergence of Lemma 2.7.31, we obtain that

$$
\begin{aligned}
i \int_{\partial D^+_{Ka_1}(\pi(a))} (i\nabla + A_a) v_k^{ext} \cdot \nu \, \overline{v_k} \, \mathrm{d}\sigma &= - \int_{\partial D^+_{Ka_1}(\pi(a))} \nabla \varphi_k \cdot \nu \, \varphi_k \, \mathrm{d}\sigma \\
= -K a_1^2 \int_{-\pi/2}^{\pi/2} \nabla U_k^a \cdot \nu \, U_k^a \, \mathrm{d}\theta &= -K^2 a_1^2 c_1^2 \frac{\pi}{2} + o((Ka_1)^2).
\end{aligned} \tag{2.148}
$$

Next, we look at the second part in (2.142). For this, we use the blow-up function defined in (2.147) and the strong convergence of Lemma 2.7.32

$$
\begin{aligned}
-i \int_{\partial D^+_{Ka_1}(\pi(a))} & (i\nabla + A_a) v_k^{int} \cdot \nu \, \overline{v_k} \, \mathrm{d}\sigma \\
&= -i\, a_1^2 \int_{\partial D^+_K(0)} (i\nabla + A_e) W_k^a \cdot \nu \, \overline{W_k^a} \, \mathrm{d}\sigma \\
&= -i\, a_1^2 \int_{\partial D^+_K(0)} (i\nabla + A_e) W_K \cdot \nu \, \overline{W_K} \, \mathrm{d}\sigma + o(a_1^2) \\
&= K^2 a_1^2 c_1 \frac{\pi}{2} \left(\frac{K^2+4}{K^2-4} c_1 - \frac{4 a_{K,1}(2)}{K^2-4} \right) + o(a_1^2),
\end{aligned} \tag{2.149}
$$

where in the last step, we used Lemma 2.7.30. Finally, if we add the two contributions (2.148) and (2.149), we obtain

$$
\begin{aligned}
i \int_{\partial D^+_{Ka_1}(\pi(a))} & (i\nabla + A_a)(v_k^{ext} - v_k^{int}) \cdot \nu \, \overline{v_k} \, \mathrm{d}\sigma \\
&= a_1^2 c_1^2 \frac{\pi}{2} \frac{8K^2}{K^2-4} - a_1^2 c_1 \frac{\pi}{2} \frac{4K^2}{K^2-4} a_{K,1}(2) + o(a_1^2) \\
&= a_1^2 g_K + o(a_1^2).
\end{aligned}
$$

This gives us a more precise estimate from above on the eigenvalues

$$
\limsup_{a_1 \to 0} \frac{\lambda_k^a - \lambda_k}{a_1^2} \le g_K.
$$

Finally, since this is valid for any large $K > 2$, we can consider $\lim_{K \to +\infty}$. We use the second claim of Lemma 2.7.30 to obtain

$$
g_K \to c_1^2 \beta, \qquad \text{as } K \to +\infty,
$$

that is

$$\limsup_{a_1 \to 0} \frac{\lambda_k^a - \lambda_k}{a_1^2} \le c_1^2 \beta. \tag{2.150}$$

Proof of Theorem 2.1.15

By putting together (2.141) and (2.150), we obtain that

$$\lim_{a_1 \to 0} \frac{\lambda_k^a - \lambda_k}{a_1^2} = c_1^2 \beta.$$

Here, we remember that $c_1 = c_1(\pi(a))$ depends in fact on $\pi(a)$. Since we are interested in the limit $a \to 0$, $\pi(a)$ can be taken as small as possible and by the continuity of the eigenvalues of the Laplacian, we have $c_1(\pi(a)) \to \frac{\partial \varphi_k}{\partial x_1}(0)$. Then,

$$\lim_{a \to 0} \frac{\lambda_k^a - \lambda_k}{a_1^2} = \beta \left(\frac{\partial \varphi_k}{\partial x_1}(0) \right)^2 = \beta \left(\frac{\partial \varphi_k}{\partial \nu}(0) \right)^2.$$

Finally, to obtain the result in the general domain (and not the rectified one), we recall that

$$\varphi_k^r(x) = \varphi_k^g(\Phi(x)),$$

where φ_k^r and φ_k^g are respectively the eigenfunctions of the Laplacian in the rectified domain and in the general domain (with different weights). Then,

$$\frac{\partial \varphi_k^r}{\partial x_1}(0) = |\Phi'(0)| \left(\frac{\partial \varphi_k^g}{\partial y_1}(b)\, \nu_1 + \frac{\partial \varphi_k^g}{\partial y_2}(b)\, \nu_2 \right) = |\Phi'(0)|\, \nabla \varphi_k^g(b) \cdot \nu,$$

where

$$\nu = (\nu_1, \nu_2) = \frac{1}{|\Phi'(0)|} \left(\frac{\partial \Phi_1}{\partial x_1}(0), \frac{\partial \Phi_2}{\partial x_1}(0) \right)$$

is the interior normal to the general domain at $b = \Phi(0)$. This combined with the discussion of Section 2.7.1, and more particularly with (2.66), gives us the result in the general domain.

2.8. Numerical illustration

Let us now illustrate the previous results with some numerical simulations provided by Virginie Bonnaillie-Noël. Those simulations were made for circulation $\alpha = 1/2$.

Those simulations were made for the angular sector of aperture $\pi/4$

$$\Sigma_{\pi/4} = \left\{ (x_1, x_2) \in \mathbb{R}^2,\ x_1 > 0,\ |x_2| < x_1 \tan \frac{\pi}{8},\ x_1^2 + x_2^2 < 1 \right\}.$$

An analysis of the spectral minimal partitions of angular sectors can be found in [31] and more details about the methods used to perform the simulations can be found in [1]. She considered a discretization grid of step $1/N$ with $N = 100$ or $N = 1000$

$$a \in \Pi_N := \left\{ \left(\frac{m}{N}, \frac{n}{N} \right) \ :\ 0 < m < N,\ 0 < \frac{|n|}{m} < \tan \frac{\pi}{8},\ \frac{m^2 + n^2}{N^2} < 1 \right\}.$$

Figure 2.6 gives the first nine eigenvalues λ_k^a for $a \in \Sigma_{\pi/4}$. In these figures, the angular sector is represented by a dark thick line. Outside the angular sector are represented the eigenvalues λ_k of the Dirichlet Laplacian on $\Sigma_{\pi/4}$ (which do not depend on a). We observe the convergence proved in Theorem 2.1.1

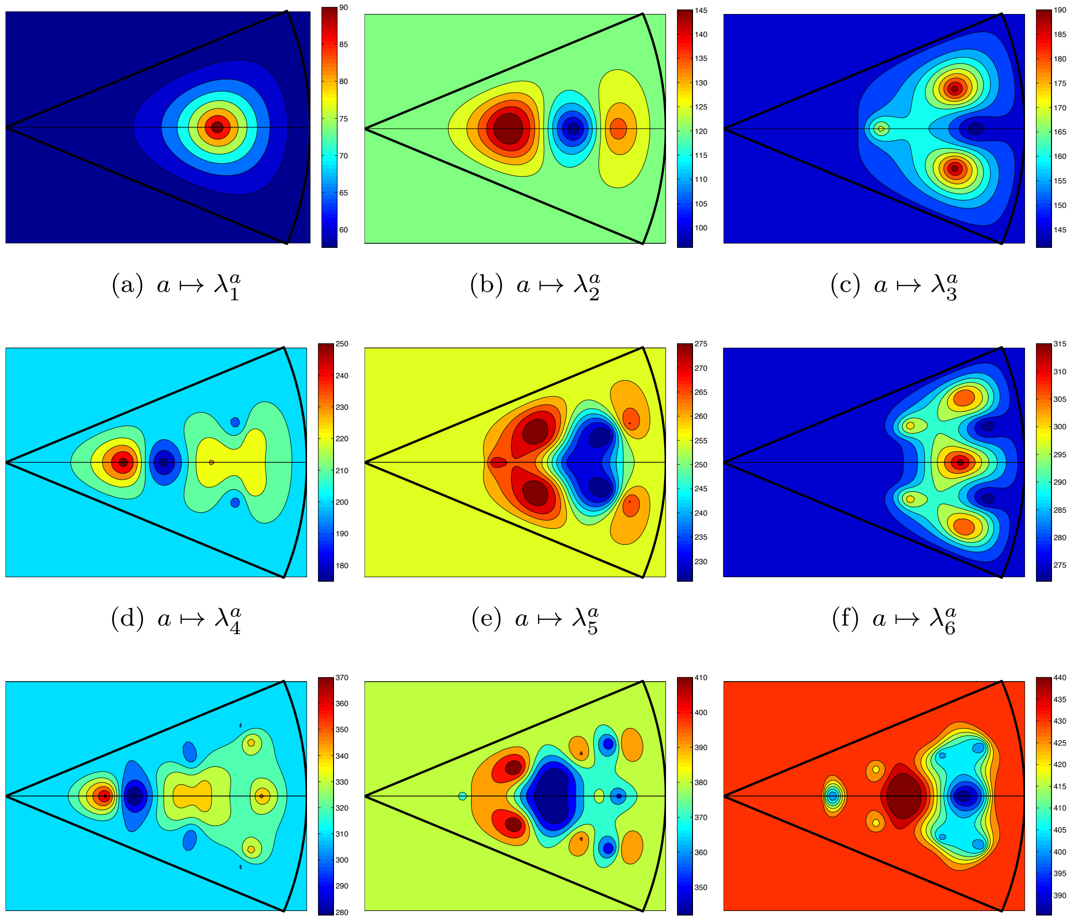
$$\forall k \geq 1, \quad \lambda_k^a \to \lambda_k \quad \text{ as } a \to \partial \Sigma_{\pi/4}.$$

Moreover, Figure 2.6(a) corroborate the fact that $\lambda_1^a > \lambda_1$, given by the diamagnetic inequality. This fact is obviously not true for the other eigenvalues, as we can see in the others eight figures.

Figure 2.7 provides the 3-D representation of Figures 2.6(a) and 2.6(b). Here, we remark that the maximum and minimum points of both λ_1^a and λ_2^a correspond to non differentiable points.

Let us now deal more accurately with the singular points on the symmetry axis $\{y = 0\}$ of $\Sigma_{\pi/4}$. Numerically, the discretization step is equal to $1/1000$ and $a \in \{(m/1000, 0), 1 \leq m \leq 1000\}$. Figure 2.8 gives the first nine eigenvalues when a lies on the symmetry axis. Here we can identify the points a such that λ_k^a is not simple. If we look for example at the first and second eigenvalues, we see that they are not simple respectively for one and three values of a on the symmetry axis. At such values, the function $a \mapsto \lambda_k^a$, $k = 1, 2$, is not differentiable, as can be seen in Figure 2.7. Figure 2.7 and Figure 2.8 illustrate Theorem 2.1.3 for a domain with a piecewise C^∞ boundary: we see that the function $a \mapsto \lambda_k^a$, $k = 1, 2$, is regular except at the points where the eigenvalue λ_k^a is not simple.

By looking at Figure 2.8, we see that the only critical points of λ_k^a which correspond to simple eigenvalues are inflexion points. As an example, the inflexion points for λ_3^a, λ_4^a, λ_5^a when $a = (a_1, 0)$ with $a_1 \in (0.6, 0.7)$, $a_1 \in (0.75, 0.85)$ and $a_1 \in (0.45, 0.55)$ respectively were analysed. Those inflexion



(a) $a \mapsto \lambda_1^a$ (b) $a \mapsto \lambda_2^a$ (c) $a \mapsto \lambda_3^a$

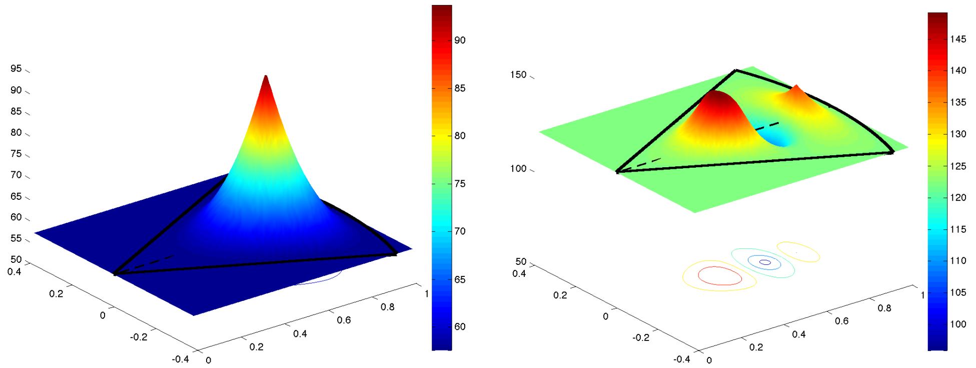
(d) $a \mapsto \lambda_4^a$ (e) $a \mapsto \lambda_5^a$ (f) $a \mapsto \lambda_6^a$

(g) $a \mapsto \lambda_7^a$ (h) $a \mapsto \lambda_8^a$ (i) $a \mapsto \lambda_9^a$

Figure 2.6.: First nine eigenvalues of $(i\nabla + A_a)^2$ for a varying in $\Sigma_{\pi/4}$.

points are denoted by $a_{(k)}$, $k = 3, 4, 5$. Figure 2.9 gives the nodal lines of λ_3^a for three different points $a = (a_1, 0)$ on the symmetry axis with $a_1 = 0.6$, 0.63 and 0.65, that is a closed to $a_{(k)}$. This figure illustrates the emergence of a triple point when the pole is moved along the symmetry axis. Figure 2.10 represents the nodal lines of the eigenfunctions $\varphi_k^{a_{(k)}}$ associated with $\lambda_k^{a_{(k)}}$, $k = 3, 4, 5$. We observe that each $\varphi_k^{a_{(k)}}$ has a zero of order $3/2$ at $a_{(k)}$. Correspondingly, the derivative of λ_k^a at $a_{(k)}$ vanishes in Figure 2.8, thus illustrating Theorem 2.1.7. In the three examples proposed here, also the second derivative of λ_k^a vanishes at $a_{(k)}$. We can then think that our estimate (2.5) in Theorem 2.1.7 is not optimal, and can be improved as seen in [38].

Let us now move a little the singular point around $a_{(k)}$. We use here a



(a) $a \mapsto \lambda_1^a$, $a \in \Pi_{100}$ (b) $a \mapsto \lambda_2^a$, $a \in \Pi_{100}$

Figure 2.7.: 3-D representation of Figures 2.6(a) and 2.6(b).

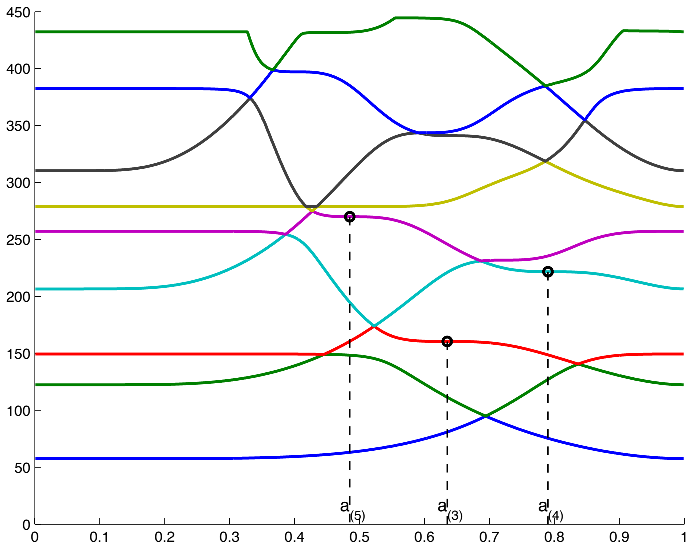
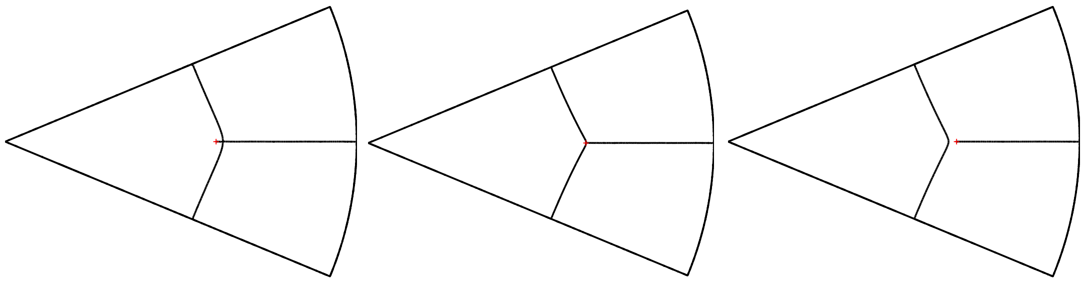


Figure 2.8.: $a \mapsto \lambda_j^a$, $a \in \left\{(\frac{m}{1000}, 0), 0 < m < 1000\right\}$, $1 \leq j \leq 9$.

discretization step of 1/1000.

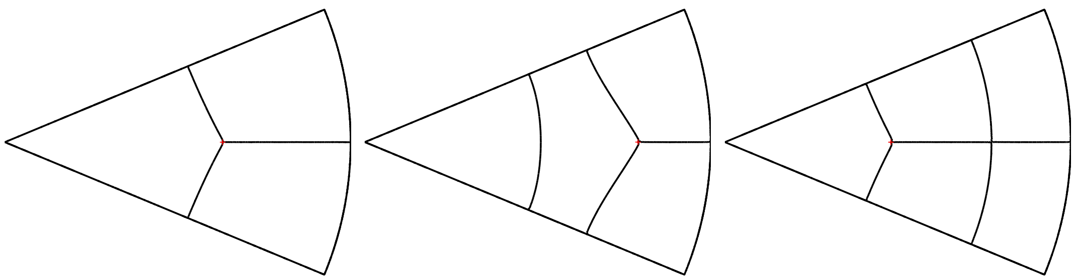
Figure 2.11 represents a zoom of the behaviour of λ_k^a for a close to $a_{(k)}$, $k = 3, 4, 5$. It indicates that these points are degenerated saddle points. The behaviour of the function $a \mapsto \lambda_k^a$, $k = 3, 4, 5$, around $a_{(k)}$ is quite similar to that of the function $(t, x) \mapsto t(t^2 - x^2)$ around the origin $(0, 0)$.

We remark that computing the first twelve eigenvalues of $(i\nabla + A_a)^2$ on $\Sigma_{\pi/4}$, we have never found an eigenfunction for which five or more nodal lines end at



(a) λ_3^a, $a = (0.60, 0) \simeq a_{(3)}$ (b) λ_3^a, $a = (0.63, 0) \simeq a_{(3)}$ (c) λ_3^a, $a = (0.65, 0) \simeq a_{(3)}$

Figure 2.9.: Nodal lines of an eigenfunction associated with λ_3^a, $a = (a_1, 0)$, $a_1 = 0.6, 0.63, 0.65$.

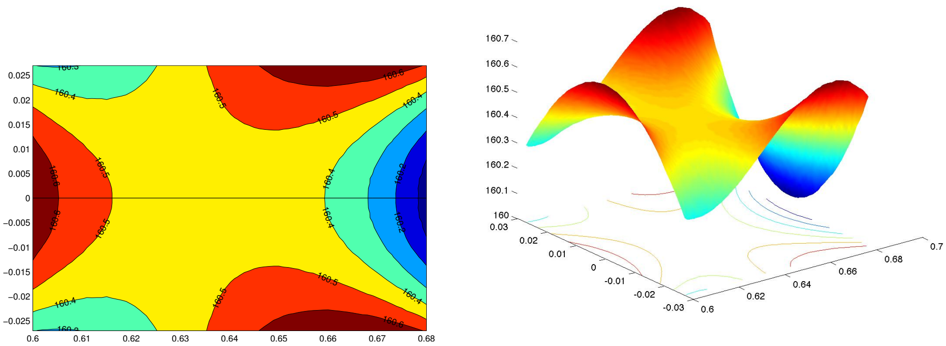


(a) λ_3^a, $a = (0.63, 0) \simeq a_{(3)}$ (b) λ_4^a, $a = (0.79, 0) \simeq a_{(4)}$ (c) λ_5^a, $a = (0.48, 0) \simeq a_{(5)}$

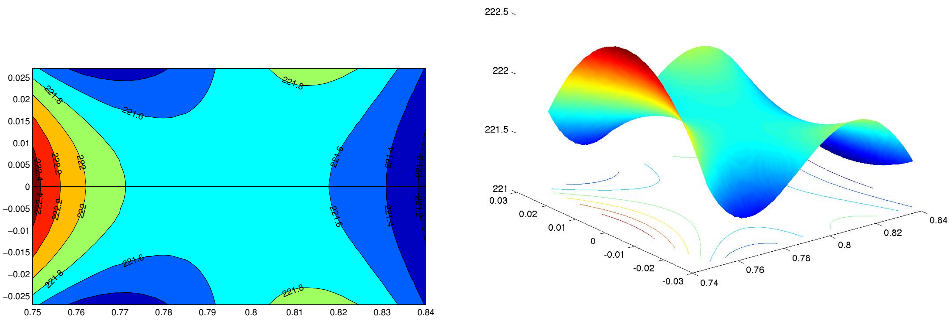
Figure 2.10.: Nodal lines of an eigenfunction associated with $\lambda_j^{a_{(j)}}$, $j = 3, 4, 5$.

a singular point a.

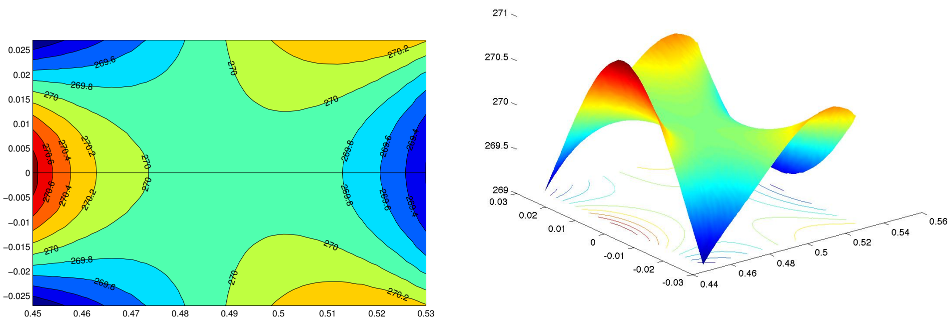
As we have already remarked, all the local maxima and minima of λ_k^a in Figure 2.8 correspond to non-simple eigenvalues. The plots of the nodal lines of the corresponding eigenfunctions show that they all have a zero of order $1/2$ at a, i.e. one nodal line ending at a. Nonetheless, this is not a general fact: in performing the same analysis in the case where Ω is a square $[0, 1] \times [0, 1]$, we have found that the third and fourth eigenfunctions have a zero of order $3/2$ at the center $a = (\frac{1}{2}, \frac{1}{2})$, see Figure 2.12, which is in this case a maximum of $a \mapsto \lambda_3^a$ and a minimum of $a \mapsto \lambda_4^a$, see Figures 2.13, 2.14. We observe in Figure 2.13 that the first and second derivatives of λ_3^a and of λ_4^a seem to vanish at the center $a = (\frac{1}{2}, \frac{1}{2})$. Therefore, in that case we do not encounter the saddle point profile that we would except, as explained in Remark 2.1.9. However, λ_3^a and λ_4^a have the particular property to be non simple at $a = (\frac{1}{2}, \frac{1}{2})$ but still differentiable. This situation seems to be related to the strong symmetries of



(a) $a \mapsto \lambda_3^a$, $a \in \left\{ \left(\frac{m}{1000}, \frac{n}{1000}\right), 600 \leq m \leq 680, 0 \leq n \leq 30 \right\}$.



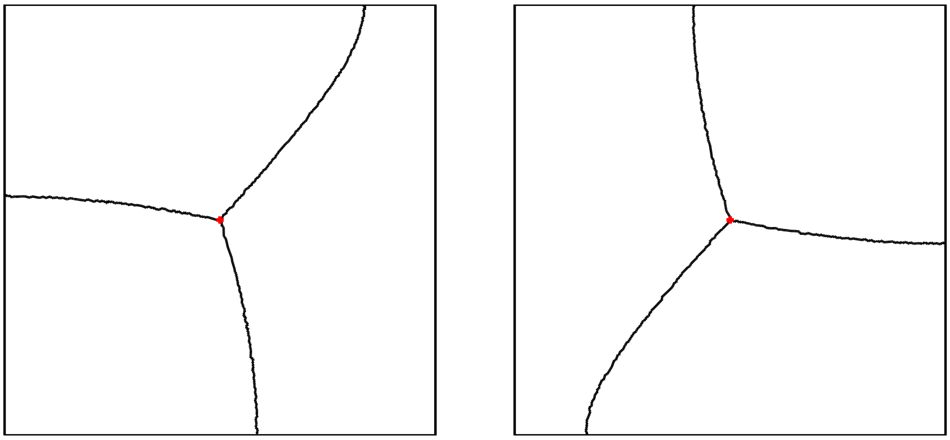
(b) $a \mapsto \lambda_4^a$, $a \in \left\{ \left(\frac{m}{1000}, \frac{n}{1000}\right), 750 \leq m \leq 840, 0 \leq n \leq 30 \right\}$.



(c) $a \mapsto \lambda_5^a$, $a \in \left\{ \left(\frac{m}{1000}, \frac{n}{1000}\right), 450 \leq m \leq 530, 0 \leq n \leq 30 \right\}$.

Figure 2.11.: λ_j^a vs. a for a around the inflexion point $a_{(j)}$, $j = 3, 4, 5$.

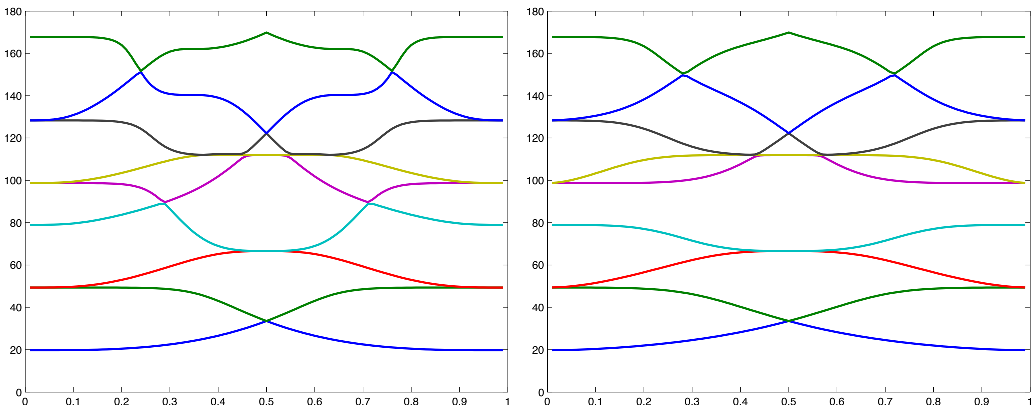
the square.



(a) λ_3^a, $a = (\frac{1}{2}, \frac{1}{2})$

(b) λ_4^a, $a = (\frac{1}{2}, \frac{1}{2})$

Figure 2.12.: Nodal lines of an eigenfunction associated with λ_j^a, $j = 3, 4$, $a = (\frac{1}{2}, \frac{1}{2})$.

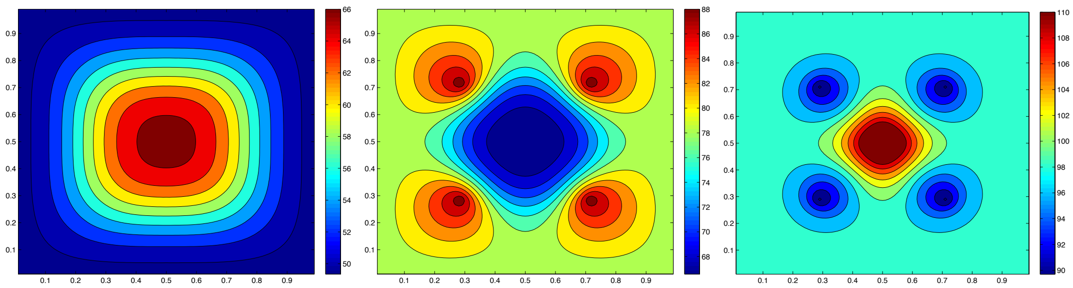


(a) $a \mapsto \lambda_j^a$, $a \in \left\{(\frac{n}{100}, \frac{n}{100}), 0 < n < 200\right\}$, $1 \leq j \leq 9$ (square).

(b) $a \mapsto \lambda_j^a$, $a \in \left\{(\frac{n}{100}, \frac{1}{2}), 0 < n < 200\right\}$, $1 \leq j \leq 9$ (square).

Figure 2.13.: $a \mapsto \lambda_j^a$ for a along the perpendicular bisector or the diagonal of a square

Finally, as already explained in the Introduction, Figure 2.8 illustrates Theorems 2.1.11 and 2.1.14. Indeed, focus for example on λ_1^a. Since we know that φ_1, the eigenfunction of the Laplacian associated to λ_1 does not have any nodal



(a) λ_3^a vs. a (b) λ_4^a vs. a (c) λ_5^a vs. a

Figure 2.14.: Eigenvalues of $(i\nabla + A_a)^2$ in $[0,1]\times[0,1]$, $a \in \Pi_{50}$.

lines, by Theorem 2.1.14, the convergence of λ_1^a to λ_1 should take place from above, as observed in the Figure. The same discussion, already made in the introduction, can be made on λ_2^a and λ_3^a.

Chapter 3

Equivariant potentials under cylindrical symmetries in $\mathbb{R}^3$

3.1. Introduction

As already said in the main Introduction, the nonlinear Schrödinger equation (NLS) with a magnetic field B, having source in the magnetic vector potential A, and a scalar (electric) potential U has the form

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{1}{2m}\left(i\hbar\nabla - qA(x)\right)^2\Psi + U(x)\Psi = f(|\Psi|^2)\Psi, \qquad x \in \mathbb{R}^N,$$

where $N \geq 3$, $i^2 = -1$, $\hbar$ is the Planck constant, m the mass of the particle and q its charge. For simplicity, we will take $m = 1/2$ and $q = -1$. The magnetic Laplacian is defined by

$$\left(i\hbar\nabla + A(x)\right)^2 = -\hbar^2\Delta + 2\hbar i A(x)\cdot\nabla + i\hbar \operatorname{div} A(x) + |A(x)|^2,$$

and $f(|\psi|^2)\psi$ is a nonlinear term. In dimension $N = 3$, it is known that the magnetic potential A is related to the magnetic field B by the relation $B = \nabla \times A$. Such evolution equation arises in various physical contexts, such as nonlinear optics or plasma physics, where one simulates the interaction effect among many particles by introducing a nonlinear term.

The search of standing waves solutions $\Psi(x,t) = e^{-i\frac{E}{\hbar}t}u(x)$ leads to study the stationary nonlinear magnetic Schrödinger equation

$$\left(i\hbar\nabla + A(x)\right)^2 u + (U(x) - E)u = f(|u|^2)u, \qquad x \in \mathbb{R}^N. \tag{3.1}$$

In the following, we write $V(x) = U(x) - E$ and for simplicity we fix $f(t) = t^{(p-2)/2}$, even if a more general nonlinearity could be considered, as long as it verifies some growth properties at zero and at infinity and some monotonicity, see e.g. [55].

For $\hbar > 0$ fixed, the existence of a solution of (3.1) whose modulus $|u_\hbar|$ vanishes at infinity was first proved by Esteban and Lions in [56] using a constrained minimization approach. They applied concentration-compactness arguments to solve such minimization problem for special classes of magnetic fields, typically the uniformly constant magnetic field. Successively in [57], Arioli and Szulkin studied the existence of infinitely many solutions of (3.1) assuming that V and B are periodic.

Here, we are interested in the semiclassical analysis of the magnetic nonlinear Schrödinger equation (3.1). From a mathematical point of view, the transition from quantum to classical mechanics can be formally performed by letting $\hbar \to 0$. For small values of $\hbar > 0$, solutions $u_\hbar : \mathbb{R}^N \to \mathbb{C}$ of (3.1) are usually referred to as semiclassical bound states.

When $A = 0$, the study of the nonlinear Schrödinger equation

$$-\hbar^2 \Delta u + V(x)u = f(|u|^2)u, \qquad x \in \mathbb{R}^N, \tag{3.2}$$

in the semiclassical regime was first performed by Floer and Weinstein in [58], in dimension $N = 1$ and for the cubic nonlinearity. In that paper, they used a Lyapunov-Schmidt reduction method to prove that if V is bounded and has a positive global and non degenerate minimum, then there exists a solution of (3.2) concentrating at that minimum point, for $\hbar > 0$ small. By using the same methods, this work was extended by Oh in [59] and [60], where in the second article the author proved the existence of a multi-peaks solution, that is a solution concentrating simultaneously at several critical points of V. A bit later, variational methods were used by Rabinowitz in [55] to prove the existence of a positive ground state solution to (3.2) for small $\hbar > 0$. To use a mountain pass argument, he required some global condition on the potential V, more precisely $0 < \inf_{\mathbb{R}^N} V < \liminf_{|x| \to +\infty} V(x)$. Successively in [61], by using variational methods with the same assumptions on V, Wang studied the concentration of the ground state solution around a global minimum of the potential. Later on, in [62], del Pino and Felmer exhibited positive solutions concentrating at an arbitrary, not necessarily non degenerate, minimum of V, and not any more at the global minimum. As global assumption on V, they only assumed $0 < \inf_{\mathbb{R}^N} V$, without any condition at infinity on V as in [55]. Moreover, they also required the existence of a smooth bounded set Λ in which V possesses a local minimum, that is

$$\inf_{x \in \Lambda} V(x) < \inf_{x \in \partial\Lambda} V(x).$$

To prove their result, they performed a penalization on the energy functional associated to (3.2) outside this set Λ, in order to recover enough compactness and use a mountain pass theorem. Furthermore, this methods allows them to catch a solution having a maximum on this set, for $\hbar > 0$ small, and then concentrating at the local minimum in Λ as $\hbar \to 0$. Finally, they showed that this solution is in fact a solution to the original problem, if $\hbar > 0$ is small enough.

We remark that in all the above papers, the arguments used rely strongly on the positivity condition on V, that is $\inf_{\mathbb{R}^N} V > 0$. The study of the case $\inf_{\mathbb{R}^N} V = 0$, including the cases where V vanishes at some points in $\mathbb{R}^N$, or V vanishes at infinity, was then investigated. When $\inf_{\mathbb{R}^N} V$ is achieved, Byeon and Wang studied in [63, 64] the concentration of solutions around that minimum point of V. This case is often referred to as critical frequency case, since the behaviour of the semiclassical solutions differs strongly from the non critical frequency case (for more details, see Section 3.7.1). On the other hand, the case $V > 0$ but $\inf_{\mathbb{R}^N} V = 0$ has been initiated by Ambrosetti, Felli and Malchiodi in [65]. They considered (3.2) with a slow decaying potential V and an additional function $K(x)$ in the right-hand side, in front of the nonlinearity, and they proved existence of solutions assuming that K decreases sufficiently fast at infinity. By slow decay, we mean that V must decay slower that the Hardy potential $1/|x|^2$. We also quote the paper of Ambrosetti, Malchiodi and Ruiz [66] which relies on a Lyapunov-Schmidt reduction argument. Later on in several papers [67–70], by using the penalization argument introduced by del Pino and Felmer, the authors extended the concentration results to fast decaying potentials, i.e. decaying faster than the Hardy potential $1/|x|^2$. This Hardy potential represents a critical threshold, in the sense that solutions to (3.2) with slow decaying potentials present an exponential decay at infinity, while the decay becomes polynomial in case of fast decaying potentials. This last situation requires additional tools which are explained more precisely in Section 3.4 (see also [69, 70]).

In addition to those papers, we quote several others dealing with the semiclassical regime [71–84], in which is proved the existence of one-peak or multiple-peaks solutions. Those solutions concentrates at some critical points of the electric potential V (and then, not always minimum points), for $\hbar > 0$ small.

Successively, the question of existence of semiclassical solutions to NLS equations concentrating on higher dimensional sets has been investigated. In [16], Ambrosetti, Malchiodi and Ni considered the case of a radial potential $V(|x|)$ and constructed radial solutions exhibiting concentration on a sphere, whose radius is a non degenerate critical point of the auxiliary function $\mathcal{M}(r) = r^{N-1}V^{\sigma}(r)$, where $\sigma = p/(p-2) - 1/2$. Moreover they conjectured that this phenomenon takes place, at least along a sequence $\hbar_n \to 0$, whenever the sphere is replaced by a closed hypersurface Γ, stationary and non degenerate for the weighted area functional $\int_\Gamma V^\sigma$. We also cite the following papers [85–89], that focus on concentration around spheres in the radial case. In [90], Molle and Passaseo studied concentration on spheres of co-dimension 2. In [91], the above conjecture of Ambrosetti and al. was completely solved in the planar case. We also quote [92], where the authors considered the NLS equation on a smooth bounded domain Ω in $\mathbb{R}^2$ with Neumann boundary conditions and proved, for a suitable sequence $\hbar_n \to 0$, the existence of positive solutions $u_{\hbar_n}$ concentrating at the whole boundary of Ω or at some components. In [93, 94], this boundary concentration on a geodesic of the boundary has been treated in the three dimensional case. More recently in [95], Bonheure, Di Cosmo and Van Schaftingen proved the existence of semiclassical solutions to (3.2) concentrating around a k-dimensional sphere, where $1 \leq k \leq N-1$, for a large class of symmetric potentials V including fast decaying potentials at infinity. They argued with the penalization techniques from [62] adapted with the symmetries, and arguments from [69, 70] to deal with the possible fast decay of V. We also quote the book of Ambrosetti and Malchiodi, in which they study concentration of solutions on k-dimensional spheres, see in particular [96, Theorem 10.11].

In presence of a magnetic field ($A \neq 0$), a challenging question is to establish how the magnetic field influences the existence and the concentration of the modula of the complex-valued solutions of (3.1), for $\hbar > 0$ small. A first result dealing with the concentration of least-energy solutions for the magnetic NLS equation was obtained in [97]. In this paper, Kurata proved that if $(u_\hbar)_\hbar$ is a sequence of least-energy solution to (3.1) with $f(t) = t^{(p-2)/2}$, then the sequence $(|u_\hbar|)_\hbar$ of their modula must concentrate at a global minimum x_0 of V, for $\hbar > 0$ small. More precisely, there exist a sequence of points $(x_n)_n \subset \mathbb{R}^N$ and a subsequence still denoted by $(\hbar_n)_n$, with $x_n \to x_0$ and $\hbar_n \to 0$ as $n \to +\infty$, such that the rescaled function $v_{\hbar_n}(y) = u_{\hbar_n}(x_n + \hbar_n y)$ converges to some $v \in C^2_{\mathrm{loc}}$ and also converges weakly in L^p. Moreover, v satisfies weakly the limit equation

$$(i\nabla + A(x_0))^2\, v + V(x_0)v = |v|^{p-2}v, \qquad x \in \mathbb{R}^3.$$

If we let $w(x) = e^{-iA(x_0)\cdot x}\, v(x)$, it follows that w satisfies weakly the equation

$$-\Delta w + V(x_0)w = |w|^{p-2}w, \qquad x \in \mathbb{R}^3.$$

Hence the concentration of the least-energy solutions is driven by the electric potential while the magnetic potential influences the phase factor of the solutions, but does not affect the location of the peaks of their modula.

The existence of such semiclassical least-energy solutions for magnetic NLS equations was established in [98]. Successively in [99], Cingolani and Secchi proved the existence of semiclassical bound state solutions to (3.1), concentrating at local minima of V, in case of bounded potentials. Then in [100], by using a penalization argument, the same authors extended the result to a large class of magnetic potentials, covering the case of potentials having polynomial growth, as for instance those corresponding to a constant magnetic field in a given direction. We also refer to [101] for existence results of multi-spikes solutions to (3.1), whose modula have multiple concentration points around local minima of V, dealing with a large class of nonlinear terms (possibly not monotone) (see also [102]); and to [103] for semiclassical solutions having specific symmetries concentrating around orbits of critical points of V. In [104], the authors have established necessary conditions for a sequence of standing wave solutions of (3.1) to concentrate, in different senses, around a given point. More precisely, they showed that if $f(t) = t^{(p-2)/2}$, then the modula of the peaks have to locate at critical points of V, independently of A, confirming what was conjectured in [99].

We point out the fact that in all the above cited papers, the concentration of the modula of the complex valued solutions occurs at one or a finite set of critical points of the electric potential V, while the magnetic field only influences the phase factor of the standing waves, for $\hbar > 0$ small.

In the present work, we are interested in studying concentration phenomena on higher dimensional sets in the presence of a magnetic field and for a large class of electric potentials. More particularly, we aim to understand how and in which situations the magnetic field influences such concentration. This is totally new with respect to the results available in the literature, in which the constructed solutions do not see the magnetic field in the semiclassical limit. In the following, we restrict ourself to consider (3.1) in $\mathbb{R}^3$, for which we can already detect some interesting phenomena.

More specifically, we consider the class of scalar potentials V invariant under a group G of orthogonal transformations, and the class of magnetic potentials A equivariant under the same group, that is

$$g\,A(g^{-1}x) = A(x), \tag{3.3}$$

for every $g \in G$.

In dimension 3, the simplest group is $G = O(3)$ which corresponds to a radially symmetric setting. The potential V then depends only on $|x|$, while A satisfies the equivariance condition (3.3) for every $g \in O(3)$. However, this last constraint on A is too strong, in the sense that the only possible vector potential satisfying this condition is a multiple of the normal vector to the sphere depending only on the radius of the sphere. Indeed, if x is a point on a sphere of radius r, there always exist rotations $g_x \in O(3)$ that leave the axis going through the center of the sphere and x invariant, that is $g_x x = x$ for those particular g_x. Then, at that point x, the equivariance condition (3.3) rewrites

$$g_x\, A(g_x^{-1}x) = A(x) \qquad \Rightarrow \qquad g_x\, A(x) = A(x).$$

This means that at that point x, $A(x) = f(x)x$, where $f(x)$ is any arbitrary real function of x. Finally, if we consider any $g \in O(3)$ with A having the above expression, we obtain

$$g\, f(g^{-1}x)\left(g^{-1}x\right) = f(g^{-1}x)x = f(x)x, \qquad \text{for all } g \in O(3).$$

Then $A(x) = f(r)x$ is a normal vector to the sphere, depending only on the radius r. Furthermore, we immediately notice that $\nabla \times A = B = 0$ and therefore A is a conservative field. We remark that this result was already obtained in [105, Theorem 1.3]. Then physically, for that class of magnetic potentials, (3.1) is equivalent to (3.2). In particular, the concentration on spheres of the solutions of (3.1) is only driven by the scalar potential V and we are exactly in the case studied by Ambrosetti, Malchiodi and Ni in [16].

A physically relevant case occurs in $\mathbb{R}^3$ in presence of magnetic and electric potentials having cylindrical symmetries. In that setting, we obtain a new surprising result for (3.1). We prove that the existence and the concentration of semiclassical bound states is influenced by the magnetic field when the concentration occurs on a circle. We conjecture that this result should also occur in more general situations. More specifically, we consider the class of invariant scalar potentials and equivariant magnetic potentials under the action of the group

$$G := \{g_\alpha \in O(3),\ \alpha \in [0, 2\pi[\,\}, \tag{3.4}$$

where

$$g_\alpha = \begin{pmatrix} \cos\alpha & -\sin\alpha & 0 \\ \sin\alpha & \cos\alpha & 0 \\ 0 & 0 & \pm 1 \end{pmatrix}.$$

Namely, we assume that $A = (A_1, A_2, A_3) \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ satisfies (3.3) for every $g \in G$ given by (3.4). If we use the cylindrical coordinates $(x_1, x_2, x_3) = (\rho\cos\theta, \rho\sin\theta, x_3)$, the condition (3.3) can be rewritten as

$$\begin{aligned} A_1(\rho, \theta - \alpha, \pm x_3) &= \cos\alpha\, A_1(\rho, \theta, x_3) + \sin\alpha\, A_2(\rho, \theta, x_3) \\ A_2(\rho, \theta - \alpha, \pm x_3) &= -\sin\alpha\, A_1(\rho, \theta, x_3) + \cos\alpha\, A_2(\rho, \theta, x_3) \\ A_3(\rho, \theta - \alpha, \pm x_3) &= \pm A_3(\rho, \theta, x_3). \end{aligned}$$

If we denote by

$$\mathbf{e}_\tau = (-\sin\theta, \cos\theta, 0)\,, \qquad \mathbf{e}_n = (\cos\theta, \sin\theta, 0)\,, \qquad \mathbf{e}_3 = (0,0,1)$$

an orthonormal basis of $\mathbb{R}^3$ adapted to the cylindrical setting, we therefore observe that A has the form

$$A(\rho,\theta,x_3) = \phi(\rho,|x_3|)\,\mathbf{e}_n \;+\; c(\rho,|x_3|)\,\mathbf{e}_\tau \;+\; A_3(\rho,x_3)\,\mathbf{e}_3,$$

for some functions $\phi, c \in C^1(\mathbb{R}^+ \times \mathbb{R}^+)$ and some $A_3 \in C^1(\mathbb{R}^+ \times \mathbb{R})$ which is odd in x_3. The typical example $\phi \equiv 0 \equiv A_3$ and $c = b\rho/2$, $b \in \mathbb{R}_0$ corresponds to the constant magnetic field $B = b$ in the direction x_3 which is the simplest but also one of the most relevant case.

Next, we consider nonnegative potentials $V \in C(\mathbb{R}^3 \setminus \{0\})$ being cylindrically invariant, that is $V(gx) = V(x)$ for every $g \in G$. This is equivalent to assume that V depends only on ρ and $|x_3|$. Moreover, we impose a growth condition at infinity when $p \in (2,4]$:

$$(V^\infty) \qquad \text{there exists } \alpha \leq 2 \text{ such that } \liminf_{|x|\to+\infty} V(x)|x|^\alpha > 0.$$

This hypothesis (V^∞) states that V must have a slower decay, in comparison with the Hardy potential. An explanation to this assumption for $p \in (2,4]$ is that the equation (3.2), without magnetic potential and with fast decaying scalar potential, does not have any solution outside a large ball for $p \leq N/(N-2)+1$, see e.g. [106, 69]. When $N = 3$, which is the case we are interested in, we exactly recover that p should be greater than 4 to allow fast decaying potentials. When $p > 4$, we do not impose any restriction, so that V could verify (V^∞) as well as it could be fast decaying or even compactly supported. Nonetheless, as we will see later, we cannot consider $V \equiv 0$. Note that the additional assumption (V^∞) induces a better decrease at infinity on the solutions (see Lemma 3.6.5).

At the origin, we do not require any specific assumptions on V. For instance, V can behave singularly at the origin, or be locally bounded at the origin. However, if V has a singularity, one can single out the Hardy potential as a threshold behaviour as in [95]. If we assume in addition that

$$(V^0) \qquad \text{there exists } \alpha \geq 2 \text{ such that } \liminf_{|x|\to 0} V(x)|x|^\alpha > 0,$$

then one can deduce a strong flatness of the solutions at the origin which depends on the order of the singularity.

We denote by $\mathcal{H} \subset \mathbb{R}^3$ the 1-dimensional vectorial subspace spanned by $\mathbf{e}_3$, and by $\mathcal{H}^\perp$ its orthogonal complement. We also introduce some notations and tools adapted to the cylindrical symmetry of the problem. For $y, z \in \mathbb{R}^3$, we define the pseudo-metric

$$\mathrm{d}_{\mathrm{cyl}}(y,z) = \left((\rho_y - \rho_z)^2 + (y_3 - z_3)^2\right)^{1/2},$$

where $\rho_y = (y_1^2 + y_2^2)^{1/2}$ and $\rho_z = (z_1^2 + z_2^2)^{1/2}$. This function accounts for the distance between two circles. Then, for $r > 0$ and $x \in \mathbb{R}^3$, we denote by $B_{\mathrm{cyl}}(x,r)$ the ball (which is torus shaped)

$$B_{\mathrm{cyl}}(x,r) = \left\{y \in \mathbb{R}^3 \,|\, \mathrm{d}_{\mathrm{cyl}}(x,y) < r\right\}.$$

We are interested in solutions $u : \mathbb{R}^3 \to \mathbb{C}$ to the problem

$$\begin{cases} (i\hbar\nabla + A)^2\, u + Vu = |u|^{p-2}\, u, & x \in \mathbb{R}^3, \\ u \in L^2(\mathbb{R}^3, \mathbb{C}), \\ (i\hbar\nabla + A)\, u \in L^2(\mathbb{R}^3, \mathbb{C}^3), \end{cases} \tag{3.5}$$

which satisfy the symmetry condition

$$u(gx) = u(x), \qquad \forall g \in G, \quad \forall x \in \mathbb{R}^3$$

and, due to the symmetry of the problem, concentrate around a circle in $\mathcal{H}^\perp$ for $\hbar > 0$ small. To this aim, we introduce the concentration function $\mathcal{M} : \mathbb{R}^3 \to \mathbb{R}^+$

$$\mathcal{M}(x) = \mathcal{M}(\rho, |x_3|) = 2\pi\rho \left[c^2(\rho, |x_3|) + V(\rho, |x_3|)\right]^{\frac{2}{p-2}} \mathcal{E}(0,1), \tag{3.6}$$

where $\mathcal{E}(0,1)$ is a positive irrelevant constant (see Section 3.3 for the details about the definition of the concentration function).

We assume the existence of a smooth bounded and open set $\Lambda \subset \mathbb{R}^3$ such that

$$\bar{\Lambda} \cap \mathcal{H} = \emptyset, \quad \Lambda \cap \mathcal{H}^\perp \neq \emptyset, \quad \text{and for every } g \in G \quad g(\Lambda) = \Lambda. \tag{3.7}$$

Furthermore we assume that

$$\inf_{\Lambda \cap \mathcal{H}^\perp} \mathcal{M} < \inf_{\partial\Lambda \cap \mathcal{H}^\perp} \mathcal{M} \quad \text{and} \quad \inf_{\Lambda \cap \mathcal{H}^\perp} \mathcal{M} < 2 \inf_{\Lambda} \mathcal{M}, \tag{3.8}$$

and that

$$\inf_{\bar{\Lambda}} V > 0. \tag{3.9}$$

We remark that the second assumption in (3.8) is just a technical condition and is in fact not restrictive if we take Λ sufficiently small, since Λ is smooth.

Using the facts that A and V have cylindrical symmetry, the equation (3.5) can be reduced to a problem in $\mathbb{R}^2$. Let $\rho_0 > 0$ be a fixed radius and consider the constant magnetic potential $A_0 : \mathbb{R}^2 \to \mathbb{R}^2$, and the constant potential $a_0 > 0$, respectively defined by

$$A_0 = (\phi(\rho_0, 0), 0) \quad \text{and} \quad a_0 = c^2(\rho_0, 0) + V(\rho_0, 0).$$

The following two-dimensional problem

$$(i\nabla + A_0)^2 u + a_0 u = |u|^{p-2}u, \qquad y \in \mathbb{R}^2, \tag{3.10}$$

can be regarded as a limiting problem for (3.5). In this way, following the approach in [95], we will obtain the existence of cylindrically symmetric solutions of (3.1) concentrating around a circle in $\Lambda \cap \mathcal{H}^\perp$, for $\hbar > 0$ small.

We stress that the two-dimensional limiting problem (3.10), as well as the concentration function $\mathcal{M}$, take into account the magnetic field which will therefore influence the location of the concentration set of the semiclassical solutions of (3.1). This feature is new and, up to our knowledge, different from all the previous results in literature when dealing with an exterior magnetic field.

Moreover, if we let $A_\tau(\rho) = A(\rho, \theta, 0) \cdot \mathbf{e}_\tau = c(\rho, 0)$ be the tangential component of $A(\rho, \theta, 0)$ and $A_n(\rho) = A(\rho, \theta, 0) \cdot \mathbf{e}_n = \phi(\rho, 0)$ be its normal component, the solution of the two-dimensional limit problem (3.10) is given by $e^{i(A_n(\rho_0),0)\cdot y} w$, where w is the ground state solution of

$$-\Delta w + a_0 w = |w|^{p-2} w,$$

and $y \in \mathbb{R}^2$. In this equation, $a_0 = c^2(\rho_0, 0) + V(\rho_0, 0) = A_\tau^2(\rho_0) + V(\rho_0, 0)$. Therefore, our result below shows that the location of the concentration of the semiclassical bound states is influenced by the tangential component of A and by the scalar potential V, while the phase factor of the semiclassical wave depends on the normal component of A. We conjecture that this is a general fact and that it is not just a consequence of the symmetry assumptions.

Our main theorem states, for $\hbar > 0$ sufficiently small, the existence of solutions of (3.5) that concentrate around a circle in the plane $\mathcal{H}^\perp$, centred at the origin and of radius $\rho_\hbar$, where $\rho_\hbar$ converges to a minimizer of $\mathcal{M}$ in $\Lambda \cap \mathcal{H}^\perp$.

Theorem 3.1.1. *Let $p > 2$. Let $V \in C(\mathbb{R}^3 \setminus \{0\})$ and $A \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ be such that $V(gx) = V(x)$ and $g\,A(g^{-1}x) = A(x)$, for every $g \in G$ defined in (3.4). Moreover, if $p \in (2, 4]$, we suppose that V satisfies (V^∞). Assume that there exists a smooth bounded set $\Lambda \subset \mathbb{R}^3$ such that (3.7), (3.8) and (3.9) are satisfied. Then there exists $\hbar_0 > 0$ such that for every $0 < \hbar < \hbar_0$, and for $0 < \alpha < 1$,* *(i)* *the problem* (3.5) *has at least one solution* $u_\hbar \in C^{1,\alpha}_{loc}(\mathbb{R}^3 \setminus \{0\})$ *such that* $u_\hbar(gx) = u_\hbar(x)$ *for every* $g \in G$.

Moreover, for every $0 < \hbar < \hbar_0$, $|u_\hbar|$ *attains its maximum at some* $x_\hbar = (\rho_\hbar \cos\theta, \rho_\hbar \sin\theta, x_{3,\hbar}) \subset \Lambda$, $\theta \in [0, 2\pi[$, *such that*

(ii) $\liminf\limits_{\hbar\to 0} |u_\hbar(x_\hbar)| > 0$;

(iii) $\lim\limits_{\hbar\to 0} \mathcal{M}(x_\hbar) = \inf\limits_{\Lambda\cap\mathcal{H}^\perp} \mathcal{M}$;

(iv) $\limsup\limits_{\hbar\to 0} \dfrac{\mathrm{d}_{cyl}(x_\hbar, \mathcal{H}^\perp)}{\hbar} < +\infty$, *that is* $x_{3,\hbar} \to 0$;

(v) $\liminf\limits_{\hbar\to 0} \mathrm{d}_{cyl}(x_\hbar, \partial\Lambda) > 0$.

Finally, for every $0 < \hbar < \hbar_0$, *there exist* $C > 0$ *and* $\lambda > 0$ *such that the following asymptotic holds*

$$(vi) \quad 0 < |u_\hbar(x)| \leq C \exp\left(-\frac{\lambda}{\hbar} \frac{\mathrm{d}_{cyl}(x, x_\hbar)}{1 + \mathrm{d}_{cyl}(x, x_\hbar)} \right) (1 + |x|)^{-1} \quad \forall x \in \mathbb{R}^3 \setminus \{0\}.$$

We remark that the last assertion (vi) in Theorem 3.1.1 combines a concentration estimate with a decay as $|x| \to \infty$. This decay at infinity is not enough to guarantee that our solutions are L^2 (since the ambient space is $\mathbb{R}^3$). However, this is only a rough estimate valid without further assumption on V and it can be improved when assuming a slower decay of V at infinity. Namely if we assume that (V^∞) holds, then the solutions decay fast enough to be square integrable and thus they are true bounded state solutions. We mention also that when (V^0) holds, we can estimate the flatness of the solution at the origin. We refer to Lemma 3.6.5 for more details.

Example 3.1.2. *As a striking example, which is the one presented in the main Introduction, we observe that the presence of a constant magnetic field can produce a concentration phenomenon when coupled with a decaying electric potential. If we consider for instance the cubic nonlinearity, i.e.* $p = 4$, *and the cylindrical Hardy potential* $V(\rho) = 1/\rho^2$, *where* $\rho^2 = x_1^2 + x_2^2$, *there is no concentrated bound state (probably no bound state at all) of the equation without magnetic field. The presence of a constant magnetic field* $B = b$ *in the direction* x_3 *produces a solution that concentrates on the circle of radius* $2^{1/2}/(3b^2)^{1/4}$.

As a particular case of Theorem 3.1.1, we also deduce another somewhat surprising result that if the scalar potential V is constant, the existence and the location of such semiclassical state is only driven by the magnetic field.

Corollary 3.1.3. *Assume $V \equiv \omega$, where ω is a positive constant. Then, under the assumptions of Theorem 3.1.1, the concentration of the solutions $u_\hbar$ holds at*

$$\inf_{\Lambda\cap\mathcal{H}^\perp} \mathcal{M} = \inf_{\Lambda\cap\mathcal{H}^\perp} 2\pi\rho\left(c^2+\omega\right)^{\frac{2}{p-2}} \mathcal{E}(0,1).$$

Finally, we remark that in the present work our existence results is established for any value of $p > 2$. So we can treat critical and supercritical exponent problems looking for cylindrical symmetric solutions to (3.1) which concentrate outside the origin.

The chapter is organized as follows. In Section 3.2, we give the variational framework and some related properties. Section 3.3 is devoted to the study of the two dimensional limit problem (3.10). A penalization scheme similar to the ones in [62, 95] is introduced in Section 3.4 and the existence of least-energy solutions of this penalized problem is proved. The asymptotics of those solutions is studied in Section 3.5, while their concentration behaviour is established in Section 3.6, showing that the solutions of the penalized problem solve the original one. Theorem 3.1.1 is our main result. Finally, Section 3.7 is devoted to another class of symmetric solutions in the special case of a Lorentz magnetic potential. In particular, in that Section, we treat the critical frequency case already mentioned above.

3.2. The variational framework

In this section, we will fix our variational framework. In particular, we will define the Hilbert spaces adapted to the presence of a magnetic potential. We emphasize that in all Hilbert spaces we use, the scalar product will always be taken as the real scalar product, i.e. for every $z, w \in \mathbb{C}$, the scalar product will be defined by $(w|z) = \mathrm{Re}(w\bar{z})$. Moreover, in the following we will always use the parameter $\varepsilon \equiv \hbar$.

The spaces and inequalities introduced below are written making use of magnetic potentials $A \in L^2_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$ (as usually found in the literature). Since we consider $A \in C^1$, in particular A is also in L^2_{loc}.

3.2.1. The magnetic spaces

Let $N \geq 2$. For $A \in L^2_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$, we define the space $\mathcal{D}^{1,2}_{A,\varepsilon}(\mathbb{R}^N, \mathbb{C})$ as the closure of $C_0^\infty(\mathbb{R}^N, \mathbb{C})$ with respect to the norm defined through

$$\|u\|^2_{\mathcal{D}^{1,2}_{A,\varepsilon}} := \int_{\mathbb{R}^N} |(i\varepsilon\nabla + A)u|^2.$$

Similarly, $\mathcal{D}^{1,2}(\mathbb{R}^N, \mathbb{C})$ (resp. $\mathcal{D}^{1,2}(\mathbb{R}^N, \mathbb{R})$) is the closure of $C_0^\infty(\mathbb{R}^N, \mathbb{C})$ (resp. $C_0^\infty(\mathbb{R}^N, \mathbb{R})$) with respect to the norm defined through

$$\|u\|^2_{\mathcal{D}^{1,2}} := \int_{\mathbb{R}^N} |\nabla u|^2.$$

Remember that the Sobolev inequality implies that $\mathcal{D}^{1,2}(\mathbb{R}^N, \mathbb{C})$ (and resp. $\mathcal{D}^{1,2}(\mathbb{R}^N, \mathbb{R})$) is embedded in $L^{2^\star}(\mathbb{R}^N, \mathbb{C})$ (resp. $L^{2^\star}(\mathbb{R}^N, \mathbb{R})$). We also consider the space

$$H^1_{A,\varepsilon}(\mathbb{R}^N, \mathbb{C}) = \left\{ u \in L^2(\mathbb{R}^N, \mathbb{C}) \,|\, (i\varepsilon\nabla + A)u \in L^2(\mathbb{R}^N, \mathbb{C}^N) \right\},$$

endowed with the norm

$$\|u\|^2_{H^1_{A,\varepsilon}} = \int_{\mathbb{R}^N} |(i\varepsilon\nabla + A)u|^2 + |u|^2.$$

We remark that, in general, this space is not embedded in $H^1(\mathbb{R}^N, \mathbb{C})$ (and inversely). However, if $u \in H^1_{A,\varepsilon}(\mathbb{R}^N, \mathbb{C})$, then $|u| \in H^1(\mathbb{R}^N, \mathbb{R})$. This is the diamagnetic inequality that we recall here.

Lemma 3.2.1 (Diamagnetic inequality)**.** *Let $A : \mathbb{R}^N \to \mathbb{R}^N$ be in $L^2_{loc}(\mathbb{R}^N, \mathbb{R}^N)$ and let $u \in \mathcal{D}^{1,2}_{A,\varepsilon}(\mathbb{R}^N, \mathbb{C})$. Then, $|u| \in \mathcal{D}^{1,2}(\mathbb{R}^N, \mathbb{R})$ and the diamagnetic inequality*

$$\varepsilon\, |\nabla |u|(x)| \leq |(i\varepsilon\nabla + A)u(x)| \tag{3.11}$$

holds for almost every $x \in \mathbb{R}^N$ and for every $\varepsilon > 0$.

Proof. We compute

$$\varepsilon\nabla|u| = \operatorname{Im}\left(i\varepsilon\nabla u \frac{\bar{u}}{|u|}\right) = \operatorname{Im}\left((i\varepsilon\nabla + A)u\, \frac{\bar{u}}{|u|}\right) \qquad \text{a.e.}$$

because A is real-valued. We conclude using the fact that $|\operatorname{Im}(z)| \leq |z|$ for any complex number z. ☐

Using (3.11), we can verify that, for every $u \in \mathcal{D}^{1,2}_{A,\varepsilon}(\mathbb{R}^N, \mathbb{C})$,

$$\varepsilon^2 \int_{\mathbb{R}^N} |\nabla |u||^2 \,\mathrm{d}x \leq \int_{\mathbb{R}^N} |(i\varepsilon\nabla + A)u|^2 \,\mathrm{d}x, \tag{3.12}$$

for any $\varepsilon > 0$.

In some particular cases, the spaces $H^1_{A,\varepsilon}(\mathbb{R}^N,\mathbb{C})$ and $H^1(\mathbb{R}^N,\mathbb{C})$ are equivalent, as for instance if A is bounded. The following lemma is proved for example in [99, Lemma 3.1].

Lemma 3.2.2. *Let $A:\mathbb{R}^N\to\mathbb{R}^N$ be such that $\|A\|_{L^\infty}\le C$ where $C\ge 0$. Then, for every $\varepsilon>0$, the spaces $H^1_{A,\varepsilon}(\mathbb{R}^N,\mathbb{C})$ and $H^1(\mathbb{R}^N,\mathbb{C})$ are equivalent.*

Finally, we introduce the following Hilbert space

$$H^1_{A,V,\varepsilon}(\mathbb{R}^N,\mathbb{C})=\left\{u\in\mathcal{D}^{1,2}_{A,\varepsilon}(\mathbb{R}^N,\mathbb{C})\mid\int_{\mathbb{R}^N}V(x)|u|^2<\infty\right\},$$

endowed with the norm

$$\|u\|^2_{H^1_{A,V,\varepsilon}}=\int_{\mathbb{R}^N}|(i\varepsilon\nabla+A)u|^2+V(x)|u|^2.$$

In what follows, for simplicity, we write $\|u\|_\varepsilon$ instead of $\|u\|_{H^1_{A,V,\varepsilon}}$.

We remark that the Hardy potential $1/|x|^2$ represents also in that case a critical potential, both at zero and at infinity. Indeed, if V behaves like $1/|x|^\alpha$ at infinity, for $\alpha\ge 2$, and as $1/|x|^\alpha$ for $\alpha\le 2$ at zero, then we have an equivalence between $H^1_{A,V,\varepsilon}(\mathbb{R}^N,\mathbb{C})$ and $\mathcal{D}^{1,2}_{A,\varepsilon}(\mathbb{R}^N,\mathbb{C})$. It means that the condition $\int_{\mathbb{R}^N}V(x)|u|^2<\infty$ is unnecessary in that case.

3.2.2. Hardy and Kato inequalities

In dimensions $N\ge 3$, the Hardy inequality for functions $u\in\mathcal{D}^{1,2}(\mathbb{R}^N,\mathbb{C})$ writes

$$\left(\frac{N-2}{2}\right)^2\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^2}\,\mathrm{d}x\le\int_{\mathbb{R}^N}|\nabla u|^2,\tag{3.13}$$

and for functions $u\in\mathcal{D}^{1,2}_{A,\varepsilon}(\mathbb{R}^N,\mathbb{C})$

$$\varepsilon^2\left(\frac{N-2}{2}\right)^2\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^2}\,\mathrm{d}x\le\varepsilon^2\int_{\mathbb{R}^N}|\nabla|u||^2\le\int_{\mathbb{R}^N}|(i\varepsilon\nabla+A)u|^2,\tag{3.14}$$

for any $\varepsilon>0$. Furthermore, we recall the following Kato's inequalities. First, for functions $u\in L^1_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{C})$ with $\nabla u\in L^1_{\mathrm{loc}}(\mathbb{R}^N,\mathbb{C}^N)$, we define

$$\mathrm{sign}(u)(x)=\begin{cases}\dfrac{\bar u(x)}{|u(x)|} & u(x)\neq 0\\ 0 & u(x)=0.\end{cases}$$

We have

$$\Delta|u| \geq \operatorname{Re}\left(\operatorname{sign}(u)\Delta u\right). \tag{3.15}$$

We also have a similar inequality in presence of a magnetic potential $A \in L^2_{\mathrm{loc}}(\mathbb{R}^N, \mathbb{R}^N)$,

$$\varepsilon^2 \Delta|u| \geq -\operatorname{Re}\left(\operatorname{sign}(u)(i\varepsilon\nabla + A)^2 u\right). \tag{3.16}$$

Throughout the text, we will use an auxiliary Hardy-type potential. This potential was first introduced in [69, 70] to extend the penalization method of Del Pino and Felmer to compactly supported potentials V. For $N \geq 3$, we define the function $H : \mathbb{R}^N \to \mathbb{R}$ by

$$H(x) = \frac{\kappa}{|x|^2\left((\log|x|)^2 + 1\right)^{\frac{1+\beta}{2}}},$$

for $\beta > 0$ and $0 < \kappa < \left(\frac{N-2}{2}\right)^2$. Notice that, for all $x \in \mathbb{R}^N$, we have

$$H(x) \leq \frac{\kappa}{|x|^2}, \quad \text{or} \quad H(x) \leq \frac{\kappa}{|x|^2\left|\log|x|\right|^{1+\beta}}. \tag{3.17}$$

The interest of this auxiliary potential comes mainly from the following comparison principle for $-\Delta - H$ which was proved in [95, Proposition 3.1].

Lemma 3.2.3. *Let $N \geq 3$ and $\Omega \subset \mathbb{R}^N$ be a smooth domain. Let $v, w \in H^1_{loc}(\Omega, \mathbb{R})$ be such that $\nabla(w - v)_- \in L^2(\Omega)$, $(w - v)_-/|x| \in L^2(\Omega)$ and*

$$-\Delta w - H(x)w \geq -\Delta v - H(x)v, \quad \forall x \in \Omega.$$

Moreover, if $\partial\Omega \neq \emptyset$, assume that $w \geq v$ on $\partial\Omega$. Then, $w \geq v$ in Ω.

3.2.3. Notations adapted to the cylindrical symmetry

From now on, we deal with dimension $N = 3$. As we will work with functions having cylindrical symmetry, that is functions such that $u \circ g = u$ for $g \in G$, where G is defined in (3.4), the significant variables are $\rho \in \mathbb{R}^+$ and $x_3 \in \mathbb{R}$, where $\rho = (x_1^2 + x_2^2)^{1/2}$, $x = (x_1, x_2, x_3)$. The angular variable $\theta \in [0, 2\pi)$ plays no role. However, even if those functions only depend on ρ and $|x_3|$, there are still functions defined in $\mathbb{R}^3$ and with some abuse of notations we will write either $u(\rho, x_3)$ or $u(x_1, x_2, x_3)$ depending on the situation.

We also recall the distance adapted to cylindrical symmetry already mentioned in the introduction: for $y, z \in \mathbb{R}^3$,

$$\mathrm{d}_{\mathrm{cyl}}(y, z) = \left((\rho_y - \rho_z)^2 + (y_3 - z_3)^2\right)^{1/2},$$

for $\rho_y = (y_1^2 + y_2^2)^{1/2}$ and $\rho_z = (z_1^2 + z_2^2)^{1/2}$, as well as the cylindrical ball

$$B_{\mathrm{cyl}}(x, r) = \left\{ y \in \mathbb{R}^3 \,|\, \mathrm{d}_{\mathrm{cyl}}(x, y) < r \right\},$$

for $r > 0$ and $x \in \mathbb{R}^3$.

The following lemma gives us some compact embedding of the magnetic Sobolev spaces with cylindrical symmetry.

Lemma 3.2.4. *Assume that* $\Omega \subset \mathbb{R}^3$ *is an open bounded set such that*

$$g(\Omega) = \Omega \text{ for every } g \in G$$

and

$$0 < \rho_0 < \rho < \rho_1 \quad \text{for every } (\rho\cos\theta, \rho\sin\theta, x_3) \in \Omega.$$

Then, the space $\left\{ u \in H^1_{A,\varepsilon}(\Omega, \mathbb{C}) \,|\, u \circ g = u \text{ for every } g \in G \right\}$ *is compactly embedded in* $L^q(\Omega)$*, for* $2 \le q < +\infty$.

Proof. First, since Ω is bounded and $A \in C^1(\mathbb{R}^3, \mathbb{R}^3)$, we have seen in Lemma 3.2.2 that this space is equivalent to $\left\{ u \in H^1(\Omega, \mathbb{C}) \,|\, u \circ g = u \text{ for every } g \in G \right\}$. Since u depends only on ρ and x_3, we can write the square of the H^1-norm of u as

$$\int_\Omega \left(|\nabla u|^2 + |u|^2\right) \mathrm{d}x_1 \mathrm{d}x_2 \mathrm{d}x_3 = 2\pi \int_{\Omega_0} \left(|\partial_\rho u|^2 + |\partial_{x_3} u|^2 + |u|^2\right) \rho \,\mathrm{d}\rho \mathrm{d}x_3,$$

where Ω_0 is the parametrization of Ω in the ρ, x_3 variables. Now, take a bounded sequence $(u_n)_n \subset H^1_{A,\varepsilon}(\Omega, \mathbb{C})$. Considering each u_n as a function of the two variables ρ, x_3 in $\mathbb{R}^2$, we infer that the sequence is bounded as a sequence $(u_n)_n \subset H^1(\Omega_0)$. We can then use the compact embedding in dimension 2 to conclude. $\square$

3.3. The limit problem

Because of the symmetry, our solutions will concentrate on circles and the limit problem will hold in $\mathbb{R}^2$. The aim of this section is to describe such a limit problem. Consider a constant potential $A_0 : \mathbb{R}^2 \to \mathbb{R}^2$ and a positive constant $a_0 > 0$. The equation

$$(i\nabla + A_0)^2 u + a_0 u = |u|^{p-2} u, \qquad y = (y_1, y_2) \in \mathbb{R}^2 \tag{3.18}$$

will be referred to as the limit equation associated to the problem (3.1). For solutions concentrating around a circle of radius $\rho_0 > 0$, we will have

$$A_0 = (\phi(\rho_0, 0), 0) \quad \text{and} \quad a_0 = c(\rho_0, 0)^2 + V(\rho_0, 0).$$

By lemma 3.2.2, the weak solutions of (3.18) are critical points of the functional $\mathcal{J}_{a_0}^{A_0} : H^1(\mathbb{R}^2, \mathbb{C}) \to \mathbb{R}$ defined by

$$\mathcal{J}_{a_0}^{A_0}(u) = \frac{1}{2}\int_{\mathbb{R}^2} \left[|(i\nabla + A_0)u|^2 + a_0|u|^2\right] \mathrm{d}y - \frac{1}{p}\int_{\mathbb{R}^2} |u|^p \,\mathrm{d}y. \tag{3.19}$$

Any non trivial critical point $u \in H^1(\mathbb{R}^2, \mathbb{C})$ of $J_{a_0}^{A_0}$ belongs to the Nehari manifold

$$\mathcal{N}_{a_0}^{A_0} = \left\{u \in H^1(\mathbb{R}^2, \mathbb{C}) \,|\, u \not\equiv 0 \text{ and } \langle \mathcal{J}_{a_0}^{A_0}{}'(u), u\rangle = 0\right\}.$$

A solution $u \in H^1(\mathbb{R}^2, \mathbb{C})$ is called a least energy solution, or ground state, of (3.18) if

$$\mathcal{J}_{a_0}^{A_0}(u) = \inf_{v \in \mathcal{N}_{a_0}^{A_0}} \mathcal{J}_{a_0}^{A_0}(v).$$

The following lemma states that any least energy solution of the limit problem (3.18) is real up to a change of gauge and a complex phase.

Lemma 3.3.1. *Suppose v is a least energy solution of equation (3.18). Then*

$$v(y) = w(y - y_0)e^{i\alpha}e^{iA_0 \cdot y},$$

for some $\alpha \in \mathbb{R}$, $y_0 \in \mathbb{R}^2$ and where w is the unique radially symmetric real positive solution of the scalar equation

$$-\Delta w + a_0 w = |w|^{p-2}w, \qquad y \in \mathbb{R}^2. \tag{3.20}$$

Proof. First, we consider the functional $\mathcal{J}_{a_0}^0 : H^1(\mathbb{R}^2, \mathbb{C}) \to \mathbb{R}$ associated to equation (3.20)

$$\mathcal{J}_{a_0}^0(u) = \frac{1}{2}\int_{\mathbb{R}^2} |\nabla u|^2 + a_0|u|^2 \,\mathrm{d}y - \frac{1}{p}\int_{\mathbb{R}^2} |u|^p \,\mathrm{d}y.$$

Again, any nontrivial critical point $u \in H^1(\mathbb{R}^2, \mathbb{C})$ of $\mathcal{J}_{a_0}^0$ belongs to the Nehari manifold $\mathcal{N}_{a_0}^0$. By performing the change of gauge

$$v(y) = e^{iA_0 \cdot y}u(y) \tag{3.21}$$

on functions $v \in \mathcal{N}_{a_0}^{A_0}$ and $u \in \mathcal{N}_{a_0}^0$, we observe that there is an isomorphism between the two Nehari manifolds. Indeed, any least energy solution v of $\mathcal{J}_{a_0}^{A_0}$ provides a least energy solution u of $\mathcal{J}_{a_0}^0$ by (3.21) and vice-versa.

Since it is well-known, see for example [97, Lemma 7], that the set of complex valued least energy solutions u of $\mathcal{J}_{a_0}^0$ can be written as

$$\{u(y) = e^{i\alpha}w(y - y_0),\ \alpha \in \mathbb{R},\ y_0 \in \mathbb{R}^2\},$$

the proof is completed. $\square$

We may now define the ground energy function $\mathcal{E} : \mathbb{R}^2 \times \mathbb{R}_0^+ \to \mathbb{R}^+$ by

$$\mathcal{E}(A_0, a_0) = \inf_{v \in \mathcal{N}_{a_0}^{A_0}} J_{a_0}^{A_0}(v).$$

The following lemma gives some properties of this ground energy function. We refer to [95] or [55] for more details.

Lemma 3.3.2. *For every $(A_0, a_0) \in \mathbb{R}^2 \times \mathbb{R}_0^+$, $\mathcal{E}(A_0, a_0)$ is a critical value of $\mathcal{J}_{a_0}^{A_0}$ and we have the following variational characterization*

$$\mathcal{E}(0, a_0) = \mathcal{E}(A_0, a_0) = \inf_{v \in H^1(\mathbb{R}^2, \mathbb{C}) \setminus \{0\}} \max_{t \geq 0} \mathcal{J}_{a_0}^{A_0}(v).$$

Moreover,

(i) for every $A_0 \in \mathbb{R}^2$, $a_0 \in \mathbb{R}_0^+ \mapsto \mathcal{E}(A_0, a_0)$ is continuous;

(ii) for every $A_0 \in \mathbb{R}^2$, $a_0 \in \mathbb{R}_0^+ \mapsto \mathcal{E}(A_0, a_0)$ is strictly increasing.

In fact, for our nonlinearity

$$\mathcal{E}(0, a_0) = \mathcal{E}(A_0, a_0) = \mathcal{E}(A_0 a_0^{-\frac{1}{p-2}}, 1) a_0^{\frac{2}{p-2}} = \mathcal{E}(0, 1) a_0^{\frac{2}{p-2}}. \tag{3.22}$$

Finally, the concentration function $\mathcal{M} : \mathbb{R}^3 \to \mathbb{R}^+$, already introduced in (3.6), is defined more precisely by

$$\begin{aligned}\mathcal{M}(x) = \mathcal{M}(\rho, |x_3|) &= 2\pi\rho\, \mathcal{E}(0, c^2(\rho, |x_3|) + V(\rho, |x_3|)) \\ &= 2\pi\rho \left[c^2(\rho, |x_3|) + V(\rho, |x_3|)\right]^{\frac{2}{p-2}} \mathcal{E}(0, 1).\end{aligned}$$

We will look for solutions concentrating around local minima of $\mathcal{M}$ in $\Lambda \cap \mathcal{H}^{\perp}$.

3.4. The penalization scheme

The functional associated to equation (3.1) is given by

$$\int_{\mathbb{R}^3} \left(|(i\varepsilon\nabla + A)u|^2 + V|u|^2\right) \mathrm{d}x - \frac{1}{p}\int_{\mathbb{R}^3} |u|^p \,\mathrm{d}x.$$

It is natural to consider this functional in the Sobolev space $H^1_{A,V,\varepsilon}(\mathbb{R}^3, \mathbb{C})$. However, the mere assumptions on V, and more particularly the fact that V can decay to zero at infinity, do not ensure that $H^1_{A,V,\varepsilon}(\mathbb{R}^3, \mathbb{C})$ is embedded in the $L^p(\mathbb{R}^3, \mathbb{C})$. Then, the last term of the functional is not necessarily finite.

Moreover, even if we assume that V is bounded away from zero, the functional would have a mountain-pass geometry in $H^1_{A,V,\varepsilon}(\mathbb{R}^3, \mathbb{C})$, but the Palais-Smale condition could fail without further specific assumptions on V (as for example in [55]).

For those reasons, following del Pino and Felmer [62], we truncate the nonlinear term through a penalization outside the set where the concentration is expected. Basically, the penalization approach in [62] consists in modifying the nonlinearity outside the bounded set Λ, where Λ verifies (3.7), (3.8) and (3.9), in the following way

$$\tilde{f}(x,s) = \min\{\mu V(x)s, f(s)\},$$

where $0 < \mu < 1$. The penalized functional, given by

$$\int_{\mathbb{R}^3} \left(|(i\varepsilon\nabla + A)u|^2 + V|u|^2\right) \,\mathrm{d}x - \int_{\mathbb{R}^3} \tilde{F}(|u|) \,\mathrm{d}x,$$

where $\tilde{F}(\tau) = \int_0^\tau \tilde{f}(s)\,ds$, has the mountain-pass geometry and we recover the Palais-Smale condition thanks to the penalization, so that we can easily deduce the existence of a mountain pass critical point u. Then, if we succeed to show that $f(u) \leq \mu V(x)u$ outside the set Λ, we recover a solution of the initial problem.

We will argue slightly differently for two reasons. The first one is that this approach works fine when V stays bounded away from zero, or at least when V does not converge to fast to zero at infinity, that is slower than the Hardy potential $1/|x|^2$. Indeed, as said in the introduction, in that case solutions decay exponentially to zero at infinity, so that it is not difficult to prove $f(u) \leq \mu V(x)u$ and recover a solution of the initial problem. However, we do not want to restrict our assumptions on the potentials V to this class. To solve this issue, we will add the term $\varepsilon^2 H(x)$ to μV in the modified nonlinearity. This penalization approach was first introduced in [69, 70] and subsequently used in [95]. The second reason, as already said before, is that our functions are complex-valued. We will then perform the penalization on the modulus of the unknown.

3.4.1. The penalized functional

We fix $\mu \in (0,1)$. We define the penalized nonlinearity $g_\varepsilon : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}$ by

$$g_\varepsilon(x,s) = \chi_\Lambda(x)f(s) + (1 - \chi_\Lambda(x)) \min\left\{(\varepsilon^2 H(x) + \mu V(x)), f(s)\right\} \qquad (3.23)$$

for $f(s) = s^{\frac{p-2}{2}}$. Let $G_\varepsilon(x,s) = \frac{1}{2}\int_0^s g_\varepsilon(x,\sigma)\,\mathrm{d}\sigma$. There exists $2 < \theta \le p$ such that

$$0 < \theta\, G_\varepsilon(x,s) \le g_\varepsilon(x,s)s \qquad \forall x \in \Lambda,\ \forall s > 0, \tag{3.24}$$

$$0 < 2G_\varepsilon(x,s) \le g_\varepsilon(x,s)s \le (\varepsilon^2 H(x) + \mu V(x))s \qquad \forall x \notin \Lambda,\ \forall s > 0. \tag{3.25}$$

Moreover, we have that

$$g_\varepsilon(x,s^2) \quad \text{is nondecreasing} \quad \forall x \in \mathbb{R}^3, \tag{3.26}$$

which is a useful property, see for example [55].

In the following we look for cylindrically symmetric solutions of the penalized equation

$$(i\varepsilon\nabla + A)^2 u + V(x)u = g_\varepsilon(x,|u|^2)u, \qquad x \in \mathbb{R}^3. \tag{3.27}$$

Let us define the penalized functional $\mathcal{J}_\varepsilon : H^1_{A,V,\varepsilon}(\mathbb{R}^3,\mathbb{C}) \to \mathbb{R}$

$$\mathcal{J}_\varepsilon(u) = \frac{1}{2}\int_{\mathbb{R}^3} |(i\varepsilon\nabla + A)u|^2 + V(x)|u|^2 - \int_{\mathbb{R}^3} G_\varepsilon(x,|u|^2),$$

and the space

$$\mathcal{X}_\varepsilon = \left\{u \in H^1_{A,V,\varepsilon}(\mathbb{R}^3,\mathbb{C}) \,|\, u \circ g = u,\ \forall g \in G\right\}.$$

By the principle of symmetric criticality [107], the critical points of $\mathcal{J}_\varepsilon$ in $\mathcal{X}_\varepsilon$ are weak solutions of the penalized problem (3.27), having cylindrical symmetry.

Thanks to the properties (3.24) and (3.25), the functional has a mountain pass geometry.

Lemma 3.4.1. *Let $g_\varepsilon : \mathbb{R}^3 \times \mathbb{R}^+ \to \mathbb{R}$ be defined in (3.23) and satisfy (3.24), (3.25) and (3.26). Then, for every $\varepsilon > 0$, the functional $\mathcal{J}_\varepsilon : \mathcal{X}_\varepsilon \to \mathbb{R}$ has a mountain pass geometry.*

Proof. We first prove that the functional displays a local minimum at zero. To prove this, we use in order (3.24), (3.25), (3.17), the Hardy inequality (3.14) and the fact that $\{u \in H^1_{A,V,\varepsilon}(\Lambda,\mathbb{C}) \,|\, u \circ g = u \text{ for } g \in G\}$ is compactly embedded in $L^q(\Lambda)$, $2 \le q < +\infty$, thanks to Lemma 3.2.4, (3.7) and (3.9). Then, for any

$u \in \mathcal{X}_\varepsilon$, we consider

$$\begin{aligned}
\mathcal{J}_\varepsilon(u) &= \frac{1}{2}\int_{\mathbb{R}^3} |(i\varepsilon\nabla + A)u|^2 + V(x)|u|^2 - \int_{\mathbb{R}^3} G_\varepsilon(x,|u|^2) \\
&\geq \frac{1}{2}\int_{\mathbb{R}^3} |(i\varepsilon\nabla + A)u|^2 + V(x)|u|^2 - \frac{1}{p}\int_\Lambda |u|^p \\
&- \frac{1}{2}\int_{\mathbb{R}^3\setminus\Lambda} \left(\varepsilon^2 H(x) + \mu V(x)\right)|u|^2 \\
&\geq \frac{1}{2}\left(1 - \max\{4\kappa,\mu\}\right)\|u\|_\varepsilon^2 - \frac{1}{p}\int_\Lambda |u|^p \\
&\geq \frac{1}{2}\left(1 - \max\{4\kappa,\mu\}\right)\|u\|_\varepsilon^2 - \frac{C}{p}\|u\|_\varepsilon^p \\
&\geq C\|u\|_\varepsilon^2,
\end{aligned}$$

if $\|u\|_\varepsilon \leq \eta$, $\eta > 0$ small enough. and thanks to the fact that $(1 - \max\{4\kappa,\mu\}) > 0$. This clearly shows that 0 is a strict local minimum.

Now, if we consider any $u \not\equiv 0$ in $\mathcal{X}_\varepsilon$, thanks to the shape of the functional, we remark immediately that $\mathcal{J}_\varepsilon(tu) \to -\infty$, as $t \to +\infty$. □

Standard arguments imply then the existence of a Palais-Smale sequence $(u_n)_n \subset \mathcal{X}_\varepsilon$ for $\mathcal{J}_\varepsilon$, that is a sequence such that

$$\mathcal{J}_\varepsilon(u_n) \leq C \quad \text{and} \quad \mathcal{J}'_\varepsilon(u_n) \to 0 \text{ as } n \to \infty.$$

To secure the existence of a weak solution of (3.27) for every $\varepsilon > 0$, it only remains to prove that $\mathcal{J}_\varepsilon$ satisfies the Palais-Smale condition, i.e. each Palais-Smale sequence possesses a convergent subsequence. This is our next aim.

3.4.2. The Palais-Smale condition

Lemma 3.4.2. *For every $\varepsilon > 0$, every Palais-Smale sequence for $\mathcal{J}_\varepsilon$ in $\mathcal{X}_\varepsilon$ contains a convergent subsequence.*

Proof. We proceed in several steps.

Step 1. As usual, the first step of the proof consists in proving that the Palais-Smale sequence $(u_n)_n$ is bounded in $\mathcal{X}_\varepsilon$. By proceeding identically to Lemma 3.4.1 and using successively the properties of the Palais-Smale sequence $(u_n)_n$, (3.24), (3.25) and finally the magnetic Hardy inequality (3.14), we infer that

$$\begin{aligned}
\frac{1}{2}\|u_n\|_\varepsilon^2 &= \mathcal{J}_\varepsilon(u_n) + \int_{\mathbb{R}^3} G_\varepsilon(x,|u_n|^2) \\
&\le C + \int_\Lambda G_\varepsilon(x,|u_n|^2) + \int_{\mathbb{R}^3\setminus\Lambda} G_\varepsilon(x,|u_n|^2) \\
&\le C + \frac{1}{\theta}\int_{\mathbb{R}^3} g_\varepsilon(x,|u_n|^2)|u_n|^2 + \left(\frac{1}{2}-\frac{1}{\theta}\right)\int_{\mathbb{R}^3\setminus\Lambda} g_\varepsilon(x,|u_n|^2)|u_n|^2 \\
&\le C + \frac{1}{\theta}\|u_n\|_\varepsilon^2 - \frac{1}{\theta}\langle \mathcal{J}_\varepsilon'(u_n),u_n\rangle \\
&+ \left(\frac{1}{2}-\frac{1}{\theta}\right)\int_{\mathbb{R}^3\setminus\Lambda} \left(\varepsilon^2 H(x) + \mu V(x)\right)|u_n|^2 \\
&\le C + \frac{1}{\theta}\|u_n\|_\varepsilon^2 + o(1)\|u_n\|_\varepsilon + \left(\frac{1}{2}-\frac{1}{\theta}\right)\mu\int_{\mathbb{R}^3} V(x)|u_n|^2 \\
&+ \left(\frac{1}{2}-\frac{1}{\theta}\right)4\kappa\int_{\mathbb{R}^3} |(i\varepsilon\nabla + A)u_n|^2 \\
&\le C + \frac{1}{\theta}\|u_n\|_\varepsilon^2 + o(1)\|u_n\|_\varepsilon + \left(\frac{1}{2}-\frac{1}{\theta}\right)\max\{\mu,4\kappa\}\|u_n\|_\varepsilon^2.
\end{aligned}$$

Since $\theta > 2$ and $\max\{\mu,4\kappa\} < 1$, the inequality

$$\left(\frac{1}{2}-\frac{1}{\theta}\right)\left(1-\max\{\mu,4\kappa\}\right)\|u_n\|_\varepsilon^2 \le C + o(1)\|u_n\|_\varepsilon$$

leads to the conclusion.

From Step 1, we deduce the existence of a function $u \in \mathcal{X}_\varepsilon$ such that, up to a subsequence still denoted in the same way, u_n weakly converges to u.

Step 2. In this step, we prove two useful claims aiming to deduce the strong convergence. We define the closed set $A_\lambda = \overline{B(0,e^\lambda)} \setminus B(0,e^{-\lambda})$, where $\lambda \ge 0$.

Claim 1 - for every $\delta > 0$, there exists $\lambda_\delta \ge 0$ such that

$$\limsup_{n\to\infty} \varepsilon^2 \int_{\mathbb{R}^3\setminus A_{\lambda_\delta}} H(x)|u_n|^2 < \delta. \tag{3.28}$$

The inequality (3.17) together with Hardy inequality (3.14) yields

$$\int_{\mathbb{R}^3\setminus A_\lambda} H(x)|u_n|^2 \le \frac{4\kappa}{\lambda^{1+\beta}}\int_{\mathbb{R}^3} |(i\varepsilon\nabla + A)u_n|^2.$$

Since $(u_n)_n$ is bounded, we now infer that for every $\delta > 0$, there exists $\lambda_\delta \ge 0$ such that (3.28) holds.

Claim 2 - for every $\delta > 0$*, there exists* $\lambda_\delta \geq 0$ *(eventually bigger than the previous one) such that*

$$\limsup_{n\to\infty} \int_{\mathbb{R}^3\setminus A_{\lambda_\delta}} V(x)|u_n|^2 < \delta. \tag{3.29}$$

We first define $\xi \in C^\infty(\mathbb{R})$ such that $0 \leq \xi \leq 1$ and

$$\xi(s) = \begin{cases} 0 & \text{if } |s| \leq \frac{1}{2} \\ 1 & \text{if } |s| \geq 1 \end{cases}$$

to build the cut-off function $\eta_\lambda \in C^\infty(\mathbb{R}^3, \mathbb{R})$ as

$$\eta_\lambda(x) = \xi\left(\frac{\log|x|}{\lambda}\right).$$

Since $(u_n)_n$ is a bounded Palais-Smale sequence and $\eta_\lambda \leq 1$, we deduce that $\langle \mathcal{J}'_\varepsilon(u_n), \eta_\lambda u_n\rangle = o(1)$. We then infer that

$$\begin{aligned} &\int_{\mathbb{R}^3} \left(|(i\varepsilon\nabla + A)u_n|^2 + V(x)|u_n|^2\right)\eta_\lambda \\ &\quad = \int_{\mathbb{R}^3} g_\varepsilon(x,|u_n|^2)|u_n|^2\eta_\lambda + \operatorname{Re}\int_{\mathbb{R}^3} i\varepsilon(i\varepsilon\nabla + A)u_n\cdot\nabla\eta_\lambda\overline{u_n} + o(1). \end{aligned} \tag{3.30}$$

Since $\bar{\Lambda} \subset \mathbb{R}^3 \setminus \{0\}$, there exists $\lambda_0 \geq 0$ such that $\bar{\Lambda} \subset A_{\lambda_0}$. Then, if we take $\lambda \geq 2\lambda_0$, we have $\eta_\lambda = 0$ on Λ. Now, using (3.25) and the above remark, we get

$$\begin{aligned} \int_{\mathbb{R}^3} g_\varepsilon(x,|u_n|^2)|u_n|^2\eta_\lambda &= \int_{\mathbb{R}^3\setminus\Lambda} g_\varepsilon(x,|u_n|^2)|u_n|^2\eta_\lambda \\ &\leq \int_{\mathbb{R}^3\setminus\Lambda} \left(\varepsilon^2 H(x) + \mu V(x)\right)|u_n|^2\eta_\lambda \\ &\leq \int_{\mathbb{R}^3} \left(\varepsilon^2 H(x) + \mu V(x)\right)|u_n|^2\eta_\lambda. \end{aligned} \tag{3.31}$$

Next, using the properties of η_λ and the magnetic Hardy inequality (3.14), we deduce that

$$\begin{aligned} \operatorname{Re}\int_{\mathbb{R}^3} i\varepsilon(i\varepsilon\nabla + A)u_n\cdot\nabla\eta_\lambda\overline{u_n} &\leq \varepsilon\left|\int_{\mathbb{R}^3}(i\varepsilon\nabla + A)u_n)\cdot\nabla\eta_\lambda\overline{u_n}\right| \\ &\leq \frac{C\varepsilon}{\lambda}\left(\int_{\mathbb{R}^3}|(i\varepsilon\nabla + A)u_n|^2\right)^{1/2}\left(\int_{\mathbb{R}^3}\frac{|u_n|^2}{|x|^2}\right)^{1/2} \\ &\leq \frac{4C}{\lambda}\int_{\mathbb{R}^3}|(i\varepsilon\nabla + A)u_n|^2. \end{aligned} \tag{3.32}$$

Combining (3.30), (3.31) and (3.32), we get the estimate

$$\begin{aligned}\int_{\mathbb{R}^3\setminus A_\lambda} \left(|(i\varepsilon\nabla + A)u_n|^2 \right. &\left. +(1-\mu)V(x)|u_n|^2\right) \\ &\leq \int_{\mathbb{R}^3} \left(|(i\varepsilon\nabla + A)u_n|^2 + (1-\mu)V(x)|u_n|^2\right)\eta_\lambda \\ &\leq \frac{4C}{\lambda}\|u_n\|_\varepsilon^2 + \varepsilon^2\int_{\mathbb{R}^3} H(x)|u_n|^2\eta_\lambda + o(1),\end{aligned}$$

for $\lambda \geq 2\lambda_0$. Finally, thanks to (3.28), if we take $\lambda > 2\lambda_\delta$, the second term in the right hand side is smaller than δ. It follows that, for every $\delta > 0$, we can choose (a new) $\lambda_\delta \geq 2\max\{\lambda_0, \lambda_\delta\}$ such that (3.29) holds.

Step 3. We are now in a position to deduce the strong convergence. We compute

$$\begin{aligned}\|u_n - u\|_\varepsilon^2 = \langle J_\varepsilon'(u_n), u_n - u\rangle - \langle J_\varepsilon'(u), u_n - u\rangle \\ + \operatorname{Re}\int_{\mathbb{R}^3} \left[g_\varepsilon(x,|u_n|^2)u_n - g_\varepsilon(x,|u|^2)u\right](\overline{u_n - u}).\end{aligned}$$

From Step 1, we know that $(u_n)_n$ is bounded so that in the right hand side, the first two terms converge to zero. For the last term in the right hand side, we divide the integral in three pieces. We treat separately the integrals on Λ, $A_{\lambda_\delta} \setminus \Lambda$ and $\mathbb{R}^3 \setminus A_{\lambda_\delta}$ and we next prove that they converge to zero.

For the integral on Λ, we can use the fact that $u_n \in H^1_{A,\varepsilon}(\Lambda,\mathbb{C})$ because $\inf_\Lambda V > 0$ and $u_n \in \mathcal{X}_\varepsilon$. Then, we conclude by using the compact embedding of $H^1_{A,\varepsilon}(\Lambda,\mathbb{C})$ in $L^q(\Lambda,\mathbb{C})$ for $2 \leq q < +\infty$. Indeed, Lemma 3.2.4 applies since Λ is bounded away from $\mathcal{H}$.

For the integral in $A_{\lambda_\delta} \setminus \Lambda$, we use the fact that $u_n \in H^1(A_{\lambda_\delta} \setminus \Lambda, \mathbb{C})$. Indeed, $A_{\lambda_\delta} \setminus \Lambda$ is bounded. In dimension 3, this space is compactly embedded in $L^q(A_{\lambda_\delta} \setminus \Lambda, \mathbb{C})$ for $1 \leq q < 6$. Moreover, the penalization g_ε is bounded on this bounded set. This and the strong convergence in L^2 allows to conclude.

The claims in Step 2 were intended to treat the remaining integral. Indeed, using (3.28) and (3.29), we infer that

$$\begin{aligned}&\limsup_{n\to\infty}\left|\operatorname{Re}\int_{\mathbb{R}^3\setminus A_{\lambda_\delta}} \left[g_\varepsilon(x,|u_n|^2)u_n - g_\varepsilon(x,|u|^2)u\right](\overline{u_n - u})\right| \\ &\leq 2\limsup_{n\to\infty}\int_{\mathbb{R}^3\setminus A_{\lambda_\delta}} \left(\varepsilon^2 H(x) + \mu V(x)\right)\left(|u_n|^2 + |u|^2\right) \\ &\leq 4C(1+\mu)\delta.\end{aligned}$$

Then, since $\delta > 0$ is arbitrary, we are done. $\square$

As a direct consequence, we deduce the existence of a least energy solution of the penalized problem (3.27).

Theorem 3.4.3. *Let $g_\varepsilon : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ defined in (3.23) satisfying (3.24), (3.25), (3.26) and $V \in C(\mathbb{R}^3 \setminus \{0\})$ verifying the hypothesis of Section 2.2. Then, for every $\varepsilon > 0$, the functional $\mathcal{J}_\varepsilon$ has a non trivial critical point $u_\varepsilon \in \mathcal{X}_\varepsilon$, which is also a weak solution of (3.27), characterized by*

$$c_\varepsilon = \mathcal{J}_\varepsilon(u_\varepsilon) = \inf_{u \in \mathcal{X}_\varepsilon \setminus \{0\}} \max_{t>0} \mathcal{J}_\varepsilon(tu). \tag{3.33}$$

This solution u_ε belongs to $W^{2,q}_{\text{loc}}(\mathbb{R}^3 \setminus \{0\})$ for $2 \leq q < +\infty$ and therefore to $C^{1,\alpha}_{\text{loc}}(\mathbb{R}^3 \setminus \{0\})$. We cannot hope a better regularity since the penalization g_ε is not even continuous.

In the next section, we estimate the critical value c_ε from above. In the study of the asymptotics of the solutions u_ε, this upper estimate will be useful to determine that the concentration occurs exactly in Λ.

3.4.3. Upper estimate of the mountain pass level

Proposition 3.4.4 (Upper estimate of the critical value c_ε)**.** *Suppose that the assumptions of Theorem 3.4.3 are satisfied. For every $\varepsilon > 0$ small enough, the critical value c_ε defined in (3.33) satisfies*

$$\liminf_{\varepsilon \to 0} \varepsilon^{-2} c_\varepsilon \leq \inf_{\Lambda \cap \mathcal{H}^\perp} \mathcal{M}. \tag{3.34}$$

Moreover, there exists $C > 0$ such that the solution u_ε found in Theorem 3.4.3 satisfies

$$\|u_\varepsilon\|_\varepsilon^2 \leq C\varepsilon^2. \tag{3.35}$$

Proof. Let $x_0 = (\rho_0 \cos\theta, \rho_0 \sin\theta, 0) \in \Lambda \cap \mathcal{H}^\perp$, with $\rho_0 > 0$ and $\theta \in [0, 2\pi)$, be such that $\mathcal{M}(x_0) = \inf_{\Lambda \cap \mathcal{H}^\perp} \mathcal{M}$. The existence of x_0 is ensured by the continuity of $\mathcal{M}$ on Λ and (3.8). Consider the functional $\mathcal{J}_{a_0}^{A_0}$ defined by (3.19), with $a_0 = \big[c(x_0)^2 + V(x_0)\big]$ and $A_0 = (\phi(x_0), 0)$. Next, we define the cut-off function $\eta \in C_0^\infty(\mathbb{R}^+ \times \mathbb{R})$ such that $0 \leq \eta \leq 1$, $\eta = 1$ in a small neighbourhood of $(\rho_0, 0)$, and is compactly supported in a small neighbourhood of $(\rho_0, 0)$, and $\|\nabla \eta\|_{L^\infty} \leq C$. We define the cylindrically symmetric function

$$u(x_1, x_2, x_3) = u(\rho, x_3) = \eta(\rho, x_3) v\left(\frac{\rho - \rho_0}{\varepsilon}, \frac{x_3}{\varepsilon}\right),$$

where v is the least-energy solution of $\mathcal{J}_{a_0}^{A_0}$. If we perform the change of variables

$$y_1 = \frac{\rho - \rho_0}{\varepsilon} \quad \text{and} \quad y_2 = \frac{x_3}{\varepsilon}$$

in the computation of $\mathcal{J}_\varepsilon(tu)$, we get

$$\begin{aligned}
&\mathcal{J}_\varepsilon(tu)\\
&= \frac{t^2}{2}\int_{\mathbb{R}^3} (|(i\varepsilon\nabla + A)u|^2 + V(x)|u|^2)\,\mathrm{d}x - \int_{\mathbb{R}^3} G_\varepsilon(x, t^2|u|^2)\,\mathrm{d}x\\
&= 2\pi\varepsilon^2\frac{t^2}{2}\int_{-\frac{\rho_0}{\varepsilon}}^{\infty}\int_{\mathbb{R}} \Big\{\eta^2(\rho_0+\varepsilon y_1, \varepsilon y_2)\,|(i\nabla + A_{0,\varepsilon}(y))v|^2\\
&+ \eta^2(\rho_0+\varepsilon y_1,\varepsilon y_2)\big[c(\rho_0+\varepsilon y_1,\varepsilon y_2)^2 + V(\rho_0+\varepsilon y_1,\varepsilon y_2)\big]|v|^2\Big\}\,(\rho_0+\varepsilon y_1)\,\mathrm{d}y\\
&- 2\pi\varepsilon^2\int_{-\frac{\rho_0}{\varepsilon}}^{\infty}\int_{\mathbb{R}} G_\varepsilon(\rho_0+\varepsilon y_1, \varepsilon y_2, t^2|v|^2\eta^2)\,(\rho_0+\varepsilon y_1)\,\mathrm{d}y + o(\varepsilon^2),
\end{aligned}$$

where $A_{0,\varepsilon}(y) = (\phi(\rho_0+\varepsilon y_1,\varepsilon y_2), A_3(\rho_0+\varepsilon y_1,\varepsilon y_2))$. The $o(\varepsilon^2)$ includes the terms where the derivatives were applied to η instead of v. This term is controlled thanks the the compactness of the support of η and the control $\|\nabla\eta\|_{L^\infty} \le C$. Finally, as η is compactly supported around $(\rho_0, 0)$, $G_\varepsilon(x, t^2|v|^2\eta^2)$ will coincide with $F(t^2|v|^2\eta^2)$ for ε small enough. We then deduce that

$$\liminf_{\varepsilon\to 0} \varepsilon^{-2}\mathcal{J}_\varepsilon(tu) \le 2\pi\rho_0\,\mathcal{J}_{a_0}^{A_0}(tv).$$

Now we exploit the fact that c_ε is the least-energy level for $\mathcal{J}_\varepsilon$ and v is the least-energy function for $\mathcal{J}_{a_0}^{A_0}$, as well as Lemma 3.3.1 where w is the least-energy solution to $\mathcal{J}_{a_0}^0$, to obtain

$$\begin{aligned}
\liminf_{\varepsilon\to 0}\varepsilon^{-2}c_\varepsilon \le \liminf_{\varepsilon\to 0}\varepsilon^{-2}\max_{t>0}\mathcal{J}_\varepsilon(tu) &\le 2\pi\rho_0\,\max_{t>0}\mathcal{J}_{a_0}^{A_0}(tv)\\
&= 2\pi\rho_0\,\mathcal{J}_{a_0}^{A_0}(v) = 2\pi\rho_0\,\mathcal{J}_{a_0}^0(w).
\end{aligned}$$

The last equality follows from (3.22).

To deduce the second statement of the proposition, we argue as in Step 1 of the proof of Lemma 3.4.2, with the extra properties that $\mathcal{J}_\varepsilon'(u_\varepsilon) = 0$, because we have the additional information that u_ε is a critical point, and $\mathcal{J}_\varepsilon(u_\varepsilon) = c_\varepsilon \le C\varepsilon^2$. We then infer that

$$\left(\frac{1}{2} - \frac{1}{\theta}\right)(1 - \max\{4\kappa, \mu\})\,\|u_\varepsilon\|_\varepsilon^2 \le C\varepsilon^2.$$

□

3.5. Asymptotic estimates

In this section, we study the behaviour of solutions when $\varepsilon \to 0$. With those estimates at hand, we will be able to prove that the solutions of the penalized problem solve the original equation for ε small enough.

3.5.1. No uniform convergence to 0 on Λ

We start by proving that the solution u_ε does not converge uniformly to 0 in Λ as $\varepsilon \to 0$.

Proposition 3.5.1. *Suppose the assumptions of Theorem 3.4.3 are satisfied and let $(u_\varepsilon)_\varepsilon \subset \mathcal{X}_\varepsilon$ be the solutions found in Theorem 3.4.3. Then,*

$$\liminf_{\varepsilon\to 0} \|u_\varepsilon\|_{L^\infty(\Lambda)} > 0.$$

Proof. By contradiction, assume that there exists a sequence $(\varepsilon_n)_n \subset \mathbb{R}^+$ such that $\varepsilon_n \to 0$ and $\|u_{\varepsilon_n}\|_{L^\infty(\Lambda)} \to 0$ as $n \to +\infty$. Using Kato inequality (3.16) and the equation (3.27), we obtain

$$\begin{aligned}-{\varepsilon_n}^2 (\Delta + H) |u_{\varepsilon_n}| &+ (1-\mu)V|u_{\varepsilon_n}| \\ &\le -{\varepsilon_n}^2 H|u_{\varepsilon_n}| - \mu V|u_{\varepsilon_n}| + g_{\varepsilon_n}(x, |u_{\varepsilon_n}|^2)|u_{\varepsilon_n}|.\end{aligned}$$

By (3.25), we infer that the right hand side of the last inequality is non positive in $\mathbb{R}^3 \setminus \Lambda$. On the other hand, since we assume that $\|u_{\varepsilon_n}\|_{L^\infty(\Lambda)} \to 0$, the facts that $p > 2$ and $\inf_\Lambda V > 0$ imply $|u_{\varepsilon_n}|^{p-1} \le \mu V(x)|u_{\varepsilon_n}|$ in Λ for n large. We thus conclude that

$$-{\varepsilon_n}^2 (\Delta + H(x)|) |u_{\varepsilon_n}| + (1-\mu)V(x)|u_{\varepsilon_n}| \le 0, \qquad \text{in } \mathbb{R}^3.$$

We then reach a contradiction because the comparison principle (Lemma 3.2.3) implies that $|u_{\varepsilon_n}| = 0$ for large n. □

3.5.2. Estimates on the rescaled solutions

As we have seen in Proposition 3.4.4, the norm of the solution u_ε is of the order ε. It is then natural to rescale u_ε around some family of points $(x_\varepsilon)_\varepsilon = (\rho_\varepsilon \cos\theta, \rho_\varepsilon \sin\theta, x_{3,\varepsilon})_\varepsilon \subset \bar{\Lambda}$, $\theta \in [0, 2\pi[$ as

$$v_\varepsilon(y) = u_\varepsilon(\rho_\varepsilon + \varepsilon y_1, x_{3,\varepsilon} + \varepsilon y_2). \tag{3.36}$$

The rescaled solution is defined for $y = (y_1, y_2) \in (-\rho_\varepsilon/\varepsilon, +\infty) \times \mathbb{R}$. The following lemma shows the convergence of those rescaled sequences of solutions.

Lemma 3.5.2 (Convergence of the rescaled solutions)**.** *Suppose the assumptions of Theorem 3.4.3 are satisfied. Let $(x_n)_n = (\rho_n \cos\theta, \rho_n \sin\theta, x_{3,n})_n \subset \overline{\Lambda}$ and $(\varepsilon_n)_n \subset \mathbb{R}^+$ be such that $\varepsilon_n \to 0$ and $x_n \to \bar{x} = (\bar{\rho}\cos\theta, \bar{\rho}\sin\theta, \bar{x}_3) \in \overline{\Lambda}$, as $n \to +\infty$, $\theta \in [0, 2\pi)$. Set*

$$\bar{A} = (\phi(\bar{\rho}, \bar{x}_3), A_3(\bar{\rho}, \bar{x}_3)), \qquad \bar{a} = c^2(\bar{\rho}, \bar{x}_3) + V(\bar{\rho}, \bar{x}_3).$$

Consider the sequence of solutions $(u_{\varepsilon_n})_n \subset \mathcal{X}_{\varepsilon_n}$ found in Theorem 3.4.3. There exists $v \in H^1(\mathbb{R}^2, \mathbb{C})$ such that, up to a subsequence,

$$v_{\varepsilon_n} \to v \quad \text{in} \quad C^{1,\alpha}_{loc}(\mathbb{R}^2, \mathbb{C}) \quad \text{for } \alpha \in (0,1),$$

where $(v_{\varepsilon_n})_n$ is the sequence defined by (3.36), and v solves the equation

$$(i\nabla + \bar{A})^2 v + \bar{a} v = \bar{g}(y, |v|^2) v \quad \text{in } \mathbb{R}^2, \tag{3.37}$$

with

$$\bar{g}(y, |v|^2) = \chi(y) f(|v|^2) + (1 - \chi(y)) \min\{\mu V(\bar{\rho}, \bar{x}_3), f(|v|^2)\}, \tag{3.38}$$

χ being the limit a.e. of $\chi_n(y) = \chi_\Lambda(\rho_n + \varepsilon_n y_1, x_{3,n} + \varepsilon_n y_2)$. Moreover, we have

$$2\pi\bar{\rho} \int_{\mathbb{R}^2} \left(|(i\nabla + \bar{A}) v|^2 + \bar{a}|v|^2 \right) \mathrm{d}y = \lim_{R\to+\infty} \liminf_{n\to+\infty} \varepsilon_n^{-2} \int_{B_{cyl}(x_n, \varepsilon_n R)} \left(|(i\varepsilon_n \nabla + A) u_{\varepsilon_n}|^2 + V(x)|u_{\varepsilon_n}|^2 \right) \mathrm{d}x. \tag{3.39}$$

Proof. We proceed again in several steps.

Step 1. Convergence of the sequence $(v_{\varepsilon_n})_n$. First, the equation solved by v_{ε_n} in $(-\rho_{\varepsilon_n}/\varepsilon_n, +\infty) \times \mathbb{R}$ is the following

$$(i\nabla + A_n)^2 v_{\varepsilon_n} - \frac{\varepsilon_n}{\rho_n + \varepsilon_n y_1} \frac{\partial v_{\varepsilon_n}}{\partial y_1} + i \frac{\varepsilon_n}{\rho_n + \varepsilon_n y_1} \phi_n v_{\varepsilon_n} + \left[V_n + c_n^2 \right] v_{\varepsilon_n} = g_{\varepsilon_n, n}(y, |v_{\varepsilon_n}|^2) v_{\varepsilon_n}. \tag{3.40}$$

The two-dimensional magnetic potential is given by $A_n(y) = (\phi_n(y), A_{3,n}(y))$ and the other functions are defined by

$$\phi_n, A_{3,n}, c_n, V_n, g_{\varepsilon_n,n}(y) := \phi, A_3, c, V, g_{\varepsilon_n}(\rho_n + \varepsilon_n y_1, x_{3,n} + \varepsilon_n y_2).$$

By using the definition of v_{ε_n} and (3.35), we obtain the following inequality

$$\int_{-\frac{\rho_n}{\varepsilon_n}}^{+\infty} \int_{\mathbb{R}} \left[|(i\nabla + A_n) v_{\varepsilon_n}|^2 + (V_n + c_n^2)|v_{\varepsilon_n}|^2 \right] (\rho_n + \varepsilon_n y_1) \, \mathrm{d}y = \frac{1}{2\pi\varepsilon_n^2} \|u_{\varepsilon_n}\|^2 \le C, \tag{3.41}$$

for $C > 0$ independent from n.

Next, we choose a sequence R_n such that $R_n \to +\infty$ and $\varepsilon_n R_n \to 0$ as $n \to +\infty$, and we define the cut-off function $\eta_{R_n} \in C_0^\infty(\mathbb{R})$ such that $0 \le \eta_{R_n} \le 1$,

$$\eta_{R_n}(y) = \begin{cases} 0 & \text{if } |y| \ge R_n \\ 1 & \text{if } |y| \le R_n/2 \end{cases}$$

and $\|\nabla \eta_{R_n}\|_{L^\infty} \le C/R_n$ for some $C > 0$. Since $\overline{\Lambda} \cap \mathcal{H} = \emptyset$, we have that $\rho_n \to \bar{\rho} > 0$, and then, for n sufficiently large, $\varepsilon_n R_n < \bar{\rho}/2 < \rho_n$. Set $w_n(y) = \eta_{R_n}(y) v_{\varepsilon_n}(y)$, where v_{ε_n} is extended by 0 where it is not defined (anyway $\eta_{R_n} = 0$ therein).

We now estimate the L^2-norm of $|w_n|$. Observe that if $y_1^2 + y_2^2 \le R_n^2$, then $(\rho - \rho_n)^2 + (x_3 - x_{3,n})^2 \le \varepsilon_n^2 R_n^2$, so that for n large enough, $\rho_n + \varepsilon_n y_1 \in \overline{\Lambda}$. Hence, since by hypothesis $\inf_{\overline{\Lambda}}(c^2 + V) > 0$ and $(\rho_n - \varepsilon_n R_n) > \bar{\rho}/2$ for n large enough, we infer that

$$\begin{aligned} \int_{\mathbb{R}^2} |w_n|^2 \, \mathrm{d}y &\le \int_{B(0,R_n)} |v_{\varepsilon_n}|^2 \, \mathrm{d}y \\ &\le \frac{2}{\bar{\rho}} \sup_{\bar{\Lambda}} \frac{1}{c^2 + V} \int_{B(0,R_n)} |v_{\varepsilon_n}|^2 (c_n^2 + V_n)(\rho_n + \varepsilon_n y_1) \, \mathrm{d}y. \end{aligned}$$

Using the fact that $B(0,R_n) \subset \left(-\frac{\rho_n}{\varepsilon_n}, +\infty\right) \times \mathbb{R}$, for n large enough and (3.41), we deduce the estimate

$$\begin{aligned} \int_{\mathbb{R}^2} |w_n|^2 \, \mathrm{d}y &\le \frac{2}{\bar{\rho}} \sup_{\bar{\Lambda}} \frac{1}{c^2 + V} \int_{-\frac{\rho_n}{\varepsilon_n}}^{+\infty} \int_{\mathbb{R}} |v_{\varepsilon_n}|^2 (c_n^2 + V_n)(\rho_n + \varepsilon_n y_1) \, \mathrm{d}y \\ &\le C. \end{aligned}$$

Next, we study the L^2-norm of $\nabla |w_n|$. By using the diamagnetic inequality (3.12) and arguing as before, we get

$$\begin{aligned} &\int_{\mathbb{R}^2} |\nabla |w_n||^2 \, \mathrm{d}y \\ &\le \int_{\mathbb{R}^2} |(i\nabla + A_n)(\eta_{R_n} v_{\varepsilon_n})|^2 \, \mathrm{d}y \\ &\le 2 \int_{\mathbb{R}^2} |(i\nabla + A_n) v_{\varepsilon_n}|^2 \eta_{R_n}^2 \, \mathrm{d}y + 2 \int_{\mathbb{R}^2} |\nabla \eta_{R_n}|^2 |v_{\varepsilon_n}|^2 \, \mathrm{d}y \\ &\le \sup_{\bar{\Lambda}} \frac{4}{\bar{\rho}(c^2 + V)} \int_{B(0,R_n)} (|(i\nabla + A_n) v_{\varepsilon_n}|^2 + (c_n^2 + V_n)|v_{\varepsilon_n}|^2)(\rho_n + \varepsilon_n y_1) \, \mathrm{d}y \\ &\le C. \end{aligned}$$

We have just shown that $(|w_n|)_n \subset H^1(\mathbb{R}^2, \mathbb{R})$ is a bounded sequence. Hence, there exists a function $|v| \in H^1(\mathbb{R}^2, \mathbb{R})$ such that, up to a subsequence, $|w_n|$ converges weakly to $|v|$. Moreover, we deduce from Sobolev embeddings that the convergence is strong in $L^p_{\text{loc}}(\mathbb{R}^2, \mathbb{C})$ for $2 \leq p < +\infty$.

To prove the convergence in $C^{1,\alpha}_{\text{loc}}$, we consider any compact set $K \subset \mathbb{R}^2$. For n sufficiently large, we have $K \subset B(0, \frac{R_n}{2})$ which implies $w_n = v_n$ in K. In that compact set K, w_n solves the equation (3.40) and using a standard bootstrap argument (see for example [108, Theorem 9.1]) and the fact that $w_n \in L^p(K, \mathbb{C})$ for $2 \leq p < +\infty$, we conclude that

$$\sup_n \|w_n\|_{W^{2,p}(K)} \leq C.$$

Finally, since this estimate holds for all $2 \leq p < +\infty$, Sobolev embeddings imply that $w_n = v_n$ converges in $C^{1,\alpha}(K)$ to v. The claim then follows from a diagonal procedure.

Step 2. Limit equation satisfied by v. Since Λ is smooth, the characteristic functions converge almost everywhere to a measurable function $0 \leq \chi(y) \leq 1$. We therefore obtain equation (3.37) from (3.40) by using the $C^{1,\alpha}_{\text{loc}}$-convergence. Moreover, if $\bar{x} \in \Lambda$, we remark that $\bar{g}(y, |v|^2) = f(|v|^2)$, that is $\chi \equiv 1$.

Step 3. Proof of the estimate (3.39). Using the preceding arguments and the $C^{1,\alpha}_{\text{loc}}$-convergence, we have

$$\begin{aligned}
&\liminf_{n\to+\infty} \varepsilon_n^{-2} \int_{B_{\text{cyl}}(x_n, \varepsilon_n R)} \left(|(i\varepsilon_n \nabla + A)u_{\varepsilon_n}|^2 + V(x)|u_{\varepsilon_n}|^2\right) \mathrm{d}x \\
&= 2\pi \liminf_{n\to+\infty} \int_{B(0,R)} \left[|(i\nabla + A_n)v_{\varepsilon_n}|^2 + (c_n^2 + V_n)|v_{\varepsilon_n}|^2\right] (\rho_n + \varepsilon_n y_1)\,\mathrm{d}y \\
&= 2\pi\bar{\rho} \int_{B(0,R)} \left[|(i\nabla + \bar{A})v|^2 + \bar{a}|v|^2\right] \mathrm{d}y.
\end{aligned}$$

Finally, we let R go to $+\infty$ to complete the proof. □

Next, we examine the contribution of u_ε to the action functional in a neighbourhood of a circle. In particular, we derive a lower estimate on the action of u_ε which accounts for the number of circles around which u_ε is non negligible. By combining the next lemmas with the upper estimate on the critical level c_ε, we reach the conclusion that u_ε concentrates around exactly one circle.

Lemma 3.5.3 (lower bound in a small ball)**.** *Suppose that the assumptions of Theorem 3.4.3 are satisfied. Let $(x_n)_n = (\rho_n \cos\theta, \rho_n \sin\theta, x_{3,n})_n \subset \bar{\Lambda}$ and $(\varepsilon_n)_n \subset \mathbb{R}^+$ be such that $\varepsilon_n \to 0$ and $x_n \to \bar{x} = (\bar{\rho}\cos\theta, \bar{\rho}\sin\theta, \bar{x}_3) \in \bar{\Lambda}$, as*

$n \to +\infty$, $\theta \in [0, 2\pi)$. *Let* $(u_{\varepsilon_n})_n \subset \mathcal{X}_{\varepsilon_n}$ *be the solutions found in Theorem 3.4.3. If*

$$\liminf_{n\to+\infty} |u_{\varepsilon_n}(x_n)| > 0, \tag{3.42}$$

then, up to a subsequence, we have

$$\begin{aligned}\liminf_{R\to+\infty} \liminf_{n\to+\infty} \varepsilon_n^{-2} \Bigg\{ \int_{B_{cyl}(x_n,\varepsilon_n R)} & \frac{1}{2}(|(i\varepsilon_n\nabla + A)u_{\varepsilon_n}|^2 + V(x)|u_{\varepsilon_n}|^2) \\ & - \int_{B_{cyl}(x_n,\varepsilon_n R)} G_{\varepsilon_n}(x, |u_{\varepsilon_n}|^2) \Bigg\} \\ & \geq \mathcal{M}(\bar{\rho}, \bar{x}_3).\end{aligned}$$

Proof. We set again v_{ε_n} as in (3.36). First, by (3.42), we have that $|v(0)| = \lim_{n\to+\infty} |v_{\varepsilon_n}(0)| > 0$, then v is not identically zero. Moreover, we know from Lemma 3.36 that v satisfies the equation (3.37). This implies that v is a critical point of the functional $\mathcal{G}_{\bar{a}}^{\bar{A}} : H^1(\mathbb{R}^2, \mathbb{C}) \to \mathbb{R}$ defined by

$$\mathcal{G}_{\bar{a}}^{\bar{A}}(u) = \frac{1}{2}\int_{\mathbb{R}^2} |(i\nabla + \bar{A})u|^2 + \bar{a}|u|^2 \,\mathrm{d}y - \int_{\mathbb{R}^2} \bar{G}(y, |u|^2)\,\mathrm{d}y,$$

$\bar{a}$ and $\bar{A}$ being defined in Lemma 3.5.2, and where

$$\bar{G}(y, s) = \frac{1}{2}\int_0^s \bar{g}(y, \sigma)\,\mathrm{d}\sigma.$$

Since $\bar{g}(y, |u|^2) \leq f(|u|^2)$, it follows immediately that

$$\mathcal{G}_{\bar{a}}^{\bar{A}}(u) \geq \mathcal{J}_{\bar{a}}^{\bar{A}}(u).$$

Since v is a critical point of $\mathcal{G}_{\bar{a}}^{\bar{A}}$ and $\bar{g}$ satisfies the property (3.26), we have that

$$\begin{aligned}\mathcal{G}_{\bar{a}}^{\bar{A}}(v) = \sup_{t>0} \mathcal{G}_{\bar{a}}^{\bar{A}}(tv) & \geq \inf_{u\in H^1(\mathbb{R}^2,\mathbb{C})} \sup_{t>0} \mathcal{G}_{\bar{a}}^{\bar{A}}(tu) \\ & \geq \inf_{u\in H^1(\mathbb{R}^2,\mathbb{C})} \sup_{t>0} \mathcal{J}_{\bar{a}}^{\bar{A}}(tu) = \mathcal{E}(\bar{A}, \bar{a}) = \mathcal{E}(0, 1)\, \bar{a}^{\frac{2}{p-2}}.\end{aligned}$$

By using the $C^{1,\alpha}_{\mathrm{loc}}$-convergence of the sequence $(v_{\varepsilon_n})_n$, we obtain that

$$\liminf_{n\to+\infty} \varepsilon_n^{-2} \int_{B_{\mathrm{cyl}}(x_n,\varepsilon_n R)} \frac{1}{2}\left(|(i\varepsilon_n\nabla + A)u_{\varepsilon_n}|^2 + V(x)|u_{\varepsilon_n}|^2\right) - G_{\varepsilon_n}(x,|u_{\varepsilon_n}|^2)\,\mathrm{d}x$$

$$= 2\pi \liminf_{n\to+\infty} \left\{ \int_{B(0,R)} \left[\frac{1}{2}\left(|(i\nabla + A_n)v_{\varepsilon_n}|^2 + (V_n + c_n^2)|v_{\varepsilon_n}|^2\right)\right](\rho_n + \varepsilon_n y_1)\,\mathrm{d}y \right.$$

$$\left. - \int_{B(0,R)} G_{\varepsilon_n,n}(y,|v_{\varepsilon_n}|^2)(\rho_n + \varepsilon_n y_1)\,\mathrm{d}y \right\}$$

$$= 2\pi\bar{\rho} \int_{B(0,R)} \left[\frac{1}{2}\left(|(i\nabla + \bar{A})v|^2 + \bar{a}|v|^2\right) - \bar{G}(y,|v|^2)\right]\mathrm{d}y.$$

Finally, we let $R \to +\infty$ to conclude.

□

The following lemma estimates what happens outside those small balls where u_ε concentrates. In particular we show that the contribution to the action of u_ε is nonnegative so that the lower estimate from the preceding lemma is meaningful.

Lemma 3.5.4 (Inferior bound outside small balls)**.** *Assume that the assumptions of Theorem 3.4.3 are satisfied. Let $(\varepsilon_n)_n \subset \mathbb{R}^+$ and $(x_n^i)_n = (\rho_n^i\cos\theta, \rho_n^i\sin\theta, x_{3,n}^i)_n \subset \bar{\Lambda}$ be such that $x_n^i \to \bar{x}^i = (\bar{\rho}^i\cos\theta, \bar{\rho}^i\sin\theta, \bar{x}_3^i) \in \bar{\Lambda}$, for $1 \le i \le M$, and $\varepsilon_n \to 0$, as $n \to +\infty$, $\theta \in [0,2\pi)$. Let $(u_{\varepsilon_n})_n \subset \mathcal{X}_{\varepsilon_n}$ be the solutions found in Theorem 3.4.3. Then, up to a subsequence, we have*

$$\liminf_{R\to+\infty}\liminf_{n\to+\infty} \varepsilon_n^{-2} \left\{ \int_{\mathbb{R}^3\setminus\mathcal{B}_n(R)} \frac{1}{2}\left(|(i\varepsilon_n\nabla + A)u_{\varepsilon_n}|^2 + V(x)|u_{\varepsilon_n}|^2\right) - \int_{\mathbb{R}^3\setminus\mathcal{B}_n(R)} G_{\varepsilon_n}(x,|u_{\varepsilon_n}|^2) \right\} \ge 0, \tag{3.43}$$

where

$$\mathcal{B}_n(R) = \cup_{i=1}^M B_{cyl}(x_n^i, \varepsilon_n R). \tag{3.44}$$

Proof. We consider yet another smooth test function η_{R,ε_n} such that $\eta_{R,\varepsilon_n} = 0$ on $\mathcal{B}_n(R/2)$, $\eta_{R,\varepsilon_n} = 1$ on $\mathbb{R}^2\setminus\mathcal{B}_n(R)$ and $\|\nabla\eta_{R,\varepsilon_n}\|_{L^\infty} \le C/(\varepsilon_n R)$. From (3.24) and (3.25), we infer that

$$\int_{\mathbb{R}^3\setminus\mathcal{B}_n(R)} \left[\frac{1}{2}\left(|(i\varepsilon_n\nabla + A)u_{\varepsilon_n}|^2 + V(x)|u_{\varepsilon_n}|^2\right) - G_{\varepsilon_n}(x,|u_{\varepsilon_n}|^2)\right]\mathrm{d}x$$

$$\ge \int_{\mathbb{R}^3\setminus\mathcal{B}_n(R)} \left[\frac{1}{2}\left(|(i\varepsilon_n\nabla + A)u_{\varepsilon_n}|^2 + V(x)|u_{\varepsilon_n}|^2\right) - g_{\varepsilon_n}(x,|u_{\varepsilon_n}|^2)|u_{\varepsilon_n}|^2\right]\mathrm{d}x.$$

If we test the equation (3.27) on $(u_{\varepsilon_n}\eta_{R,\varepsilon_n})$, we obtain

$$\begin{aligned}
0 = &\int_{\mathbb{R}^3\setminus\mathcal{B}_n(R)} \left[|(i\varepsilon_n\nabla + A)u_{\varepsilon_n}|^2 + V(x)|u_{\varepsilon_n}|^2 - g_{\varepsilon_n}(x,|u_{\varepsilon_n}|^2)|u_{\varepsilon_n}|^2\right] \mathrm{d}x \\
&+ \int_{\mathcal{A}_n(R)} \left[|(i\varepsilon_n\nabla + A)u_{\varepsilon_n}|^2 + V(x)|u_{\varepsilon_n}|^2 - g_{\varepsilon_n}(x,|u_{\varepsilon_n}|^2)|u_{\varepsilon_n}|^2\right] \eta^2_{R,\varepsilon_n}\,\mathrm{d}x \\
&- i\varepsilon_n \int_{\mathcal{A}_n(R)} \nabla\eta_{R,\varepsilon_n}\cdot(i\varepsilon_n\nabla + A)u_{\varepsilon_n}\overline{u_{\varepsilon_n}}\,\mathrm{d}x,
\end{aligned}$$

where we defined $\mathcal{A}_n(R) = \mathcal{B}_n(R)\setminus\mathcal{B}_n(R/2)$. Then, to deduce the estimate (3.43), it is enough to estimate the last two integrals in $\mathcal{A}_n(R)$.

We start with the first of these two terms. First, we have that $\mathcal{A}_n(R)$ is a bounded set having the cylindrical symmetry and with $\overline{\mathcal{A}_n(R)}\cap\mathcal{H} = \emptyset$. Moreover, for every $R > 0$, we can consider n sufficiently large (and then ε_n sufficiently small) such that $\inf_{\mathcal{A}_n(R)} V > 0$. This allows us to use the compact embeddings from Lemma 3.2.4. Then, we conclude that

$$\begin{aligned}
\liminf_{n\to+\infty}\varepsilon_n^{-2}\Bigg|&\int_{\mathcal{A}_n(R)} \left[|(i\varepsilon_n\nabla + A)u_{\varepsilon_n}|^2 + V(x)|u_{\varepsilon_n}|^2\right]\eta^2_{R,\varepsilon_n}\,\mathrm{d}x \\
&- \int_{\mathcal{A}_n(R)} g_{\varepsilon_n}(x,|u_{\varepsilon_n}|^2)|u_{\varepsilon_n}|^2\eta^2_{R,\varepsilon_n}\,\mathrm{d}x\Bigg| \\
&\le \liminf_{n\to+\infty}\varepsilon_n^{-2} C\left(I^2_{n,R} + I^q_{n,R}\right),
\end{aligned}$$

where we denoted

$$I_{n,R} = \left[\int_{\mathcal{A}_n(R)} \left[|(i\varepsilon_n\nabla + A)u_{\varepsilon_n}|^2 + V(x)|u_{\varepsilon_n}|^2\right]\,\mathrm{d}x\right]^{\frac{1}{2}}.$$

Next, we estimate the second term

$$\begin{aligned}
\liminf_{n\to+\infty}\varepsilon_n^{-2}&\left|\varepsilon_n\int_{\mathcal{A}_n(R)}(i\varepsilon_n\nabla + A)u_{\varepsilon_n}\cdot\nabla\eta_{R,\varepsilon_n}\overline{u_{\varepsilon_n}}\,\mathrm{d}x\right| \\
&\le \liminf_{n\to+\infty} C\varepsilon_n^{-2}R^{-1}\int_{\mathcal{A}_n(R)} |(i\varepsilon_n\nabla + A)u_{\varepsilon_n}||u_{\varepsilon_n}|\,\mathrm{d}x \\
&\le \liminf_{n\to+\infty} 2C\varepsilon_n^{-2}R^{-1}\int_{\mathcal{A}_n(R)} \left(|(i\varepsilon_n\nabla + A)u_{\varepsilon_n}|^2 + |u_{\varepsilon_n}|^2\right)\,\mathrm{d}x \\
&\le \liminf_{n\to+\infty} 2C\varepsilon_n^{-2}R^{-1}I^2_{n,R}.
\end{aligned}$$

Finally, by taking the $\liminf_{R\to+\infty}$ and using relation (3.39), we obtain that both integrals converge to zero, which concludes the result. $\square$

The next lemma combines the informations from the two preceding ones and yields a lower bound on the action of u_ε as a function of the points in $\bar{\Lambda}$ where the solution concentrates.

Lemma 3.5.5 (lower bound on the critical level)**.** *Suppose that the assumptions of Theorem 3.4.3 are satisfied. Let* $(x_n^i)_n = (\rho_n^i \cos\theta, \rho_n^i \sin\theta, x_{3,n}^i)_n \subset \bar{\Lambda}$ *and* $(\varepsilon_n)_n \subset \mathbb{R}^+$ *be such that* $\varepsilon_n \to 0$ *and* $x_n^i \to \bar{x}^i = (\bar{\rho}^i \cos\theta, \bar{\rho}^i \sin\theta, x_3^i) \in \bar{\Lambda}$*, for* $1 \le i \le M$*, as* $n \to +\infty$*,* $\theta \in [0, 2\pi)$*. Let* $(u_{\varepsilon_n})_n \subset \mathcal{X}_{\varepsilon_n}$ *be the solutions found in Theorem 3.4.3. Assume that for every* $1 \le i < j \le M$*, we have*

$$\limsup_{n\to+\infty} \frac{\mathrm{d}_{cyl}(x_n^i, x_n^j)}{\varepsilon_n} = +\infty, \tag{3.45}$$

and

$$\liminf_{n\to+\infty} |u_{\varepsilon_n}(x_n^i)| > 0.$$

Then it holds

$$\liminf_{n\to+\infty} \varepsilon_n^{-2} c_{\varepsilon_n} \ge \sum_{i=1}^{M} \mathcal{M}(\bar{\rho}^i, \bar{x}_3^i).$$

Proof. We infer from the previous lemmas that for every $\delta > 0$, there exists $R_\delta > 0$ large enough, such that for all $R > R_\delta$

$$\liminf_{n\to+\infty} \varepsilon_n^{-2} \int_{\mathbb{R}^3\setminus\mathcal{B}_n(R)} \frac{1}{2}(|(i\varepsilon_n\nabla + A)u_{\varepsilon_n}|^2 + V(x)|u_{\varepsilon_n}|^2) - G_{\varepsilon_n}(x, |u_{\varepsilon_n}|^2)\,\mathrm{d}x$$
$$\ge -\delta$$
$$\liminf_{n\to+\infty} \varepsilon_n^{-2} \int_{B_{\mathrm{cyl}}(x_n^i,\varepsilon_n R)} \frac{1}{2}(|(i\varepsilon_n\nabla + A)u_{\varepsilon_n}|^2 + V(x)|u_{\varepsilon_n}|^2) - G_{\varepsilon_n}(x, |u_{\varepsilon_n}|^2)\,\mathrm{d}x$$
$$\ge \mathcal{M}(\bar{\rho}^i, \bar{x}_3^i) - \delta,$$

where $\mathcal{B}_n(R)$ is defined in (3.44). Then, thanks to the hypothesis (3.45), the balls are disjoint. We then decompose $\varepsilon_n^{-2}\mathcal{J}(u_{\varepsilon_n})$ as the sum of the M integrals on each ball $B_{\mathrm{cyl}}(x_n^i, \varepsilon_n R)$ and one integral in $\mathbb{R}^3 \setminus \mathcal{B}_n(R)$. We then have

$$\liminf_{n\to+\infty} \varepsilon_n^{-2}\mathcal{J}(u_{\varepsilon_n}) \ge \sum_{i=1}^{M} \mathcal{M}(\bar{\rho}^i, \bar{x}_3^i) - (M+1)\delta.$$

Since $\delta > 0$ is arbitrary, the conclusion follows. $\square$

The following proposition is a key result of the proof. It concludes to the existence of a sequence of maximum points for u_ε in $\bar{\Lambda}$ and tells us that sequence of maximum points will in fact converge to the point of infimum of our concentration function $\mathcal{M}$ at the interior of Λ.

Proposition 3.5.6. *Suppose that the assumptions of Theorem 3.4.3 are satisfied. Let $(u_\varepsilon)_\varepsilon \subset \mathcal{X}_\varepsilon$ be the solutions found in Theorem 3.4.3 for $\varepsilon > 0$. Then, there exist $(x_\varepsilon)_\varepsilon = (\rho_\varepsilon \cos\theta, \rho_\varepsilon \sin\theta, x_{3,\varepsilon})_\varepsilon \subset \bar{\Lambda}$ such that*

$$\liminf_{\varepsilon\to 0} |u_\varepsilon(x_\varepsilon)| > 0. \tag{3.46}$$

Moreover, we have

(i) $\displaystyle\limsup_{\varepsilon\to 0} \frac{\mathrm{d}_{cyl}(x_\varepsilon, \mathcal{H}^\perp)}{\varepsilon} < +\infty$, *that is* $x_{3,\varepsilon} \to 0$;

(ii) $\displaystyle\liminf_{\varepsilon\to 0} \mathrm{d}_{cyl}(x_\varepsilon, \partial\Lambda) > 0$;

(iii) $\displaystyle\lim_{\varepsilon\to 0} \mathcal{M}(x_\varepsilon) = \inf_{\Lambda\cap\mathcal{H}^\perp} \mathcal{M}$;

(iv) for every $\delta > 0$, there exists $R_\delta > 0$, such that for every $R > R_\delta$ there exist $\varepsilon_R > 0$ such that, for every $\varepsilon < \varepsilon_R$, $|u_\varepsilon| < \delta$ in $\Lambda \setminus B_{cyl}(x_\varepsilon, \varepsilon R)$.

Proof. First, observe that the existence of a sequence $(x_\varepsilon)_\varepsilon \subset \bar{\Lambda}$ of local maximum points of $|u_\varepsilon|$ in $\bar{\Lambda}$ follows from the continuity of u_ε in $\bar{\Lambda}$. The estimate (3.46) holds because we know from Proposition 3.5.1 that u_ε does not converge uniformly to zero in $\bar{\Lambda}$.

Proof of assertion (i). By contradiction, assume that there exist sequences $(\varepsilon_n)_n \subset \mathbb{R}^+$ and $(x_n)_n \subset \bar{\Lambda}$ such that $\varepsilon_n \to 0$ and $x_n \to \bar{x} = (\bar{\rho}\cos\theta, \bar{\rho}\sin\theta, \bar{x}_3) \in \bar{\Lambda}$, as $n \to +\infty$, $\theta \in [0, 2\pi)$ (this is always possible because of the compactness of $\bar{\Lambda}$),

$$\liminf_{n\to+\infty} |u_{\varepsilon_n}(x_n)| > 0,$$

and

$$\limsup_{n\to+\infty} \frac{\mathrm{d}(x_n, \mathcal{H}^\perp)}{\varepsilon_n} = +\infty.$$

Let $g_{ref} \in G$ be the reflection with respect to $\mathcal{H}^\perp$. We know that $u_{\varepsilon_n} \circ g_{ref} = u_{\varepsilon_n}$, so that

$$\liminf_{n\to+\infty} |u_{\varepsilon_n}(g_{ref}(x_n))| > 0.$$

Moreover, by our assumption

$$\lim_{n\to+\infty} \frac{\mathrm{d}_{\mathrm{cyl}}(g_{ref}(x_n), \mathcal{H}^{\perp})}{\varepsilon_n} = +\infty.$$

Therefore, we infer that

$$\limsup_{n\to+\infty} \frac{\mathrm{d}_{\mathrm{cyl}}(x_n, g_{ref}(x_n))}{\varepsilon_n} = +\infty,$$

We can now use Lemma 3.5.5 to deduce that

$$\liminf_{n\to+\infty} \varepsilon_n^{-2} c_{\varepsilon_n} \geq (\mathcal{M}(\bar{x}) + \mathcal{M}(g_{ref}(\bar{x}))) \geq 2 \inf_{\Lambda} \mathcal{M},$$

whereas we know from (3.34) in Proposition 3.4.4 that

$$\liminf_{n\to+\infty} \varepsilon_n^{-2} c_{\varepsilon_n} \leq \inf_{\Lambda\cap\mathcal{H}^{\perp}} \mathcal{M}.$$

This yields the inequality

$$2 \inf_{\Lambda} \mathcal{M} \leq \inf_{\Lambda\cap\mathcal{H}^{\perp}} \mathcal{M},$$

which is impossible because of the property (3.8) of the set Λ.

Proof of the assertion (ii). Arguing again by contradiction, assume that there exist sequences $(\varepsilon_n)_n \subset \mathbb{R}^+$ and $(x_n)_n \subset \bar{\Lambda}$ such that $\varepsilon_n \to 0$,

$$\liminf_{n\to+\infty} |u_{\varepsilon_n}(x_n)| > 0,$$

and

$$\lim_{n\to+\infty} \mathrm{d}_{\mathrm{cyl}}(x_n, \partial\Lambda) = 0,$$

that is $x_n \to \bar{x} = (\bar{\rho}\cos\theta, \bar{\rho}\sin\theta, \bar{x}_3) \in \partial\Lambda$, as $n \to +\infty$, $\theta \in [0, 2\pi)$. By assertion (i), we also know that $\bar{x} \in \mathcal{H}^{\perp}$. From Lemma 3.5.5, we have

$$\liminf_{n\to+\infty} \varepsilon_n^{-2} c_{\varepsilon_n} \geq \mathcal{M}(\bar{\rho}, \bar{x}_3) \geq \inf_{\partial\Lambda\cap\mathcal{H}^{\perp}} \mathcal{M},$$

so that (3.34) in Proposition 3.4.4 implies

$$\inf_{\partial\Lambda\cap\mathcal{H}^{\perp}} \mathcal{M} \leq \inf_{\Lambda\cap\mathcal{H}^{\perp}} \mathcal{M},$$

which is again a contradiction to (3.8).

Proof of assertion (iii). This is also an easy consequence of Proposition 3.4.4 and Lemma 3.5.5. Indeed, using (i) and (ii), we can assume by contradiction the existence of a sequence $(x_n)_n$ such that x_n converges to some $\bar{x} = (\bar{\rho}\cos\theta, \bar{\rho}\sin\theta, 0) \in \Lambda \cap \mathcal{H}^\perp$ and $\mathcal{M}(\bar{x}) > \inf_{\Lambda\cap\mathcal{H}^\perp} \mathcal{M}$. Then combining Lemma 3.5.5 and Proposition 3.4.4, we deduce that

$$\inf_{\Lambda\cap\mathcal{H}^\perp} \mathcal{M} < \mathcal{M}(\bar{x}) \le \liminf_{n\to+\infty} \varepsilon_n^{-2} c_{\varepsilon_n} \le \inf_{\Lambda\cap\mathcal{H}^\perp} \mathcal{M},$$

which is a contradiction.

Proof of assertion (iv). Assume by contradiction the existence of $\delta > 0$ and a sequence $y_n \in \bar{\Lambda}$ such that

$$|u_{\varepsilon_n}(y_n)| > \delta,$$

and

$$\lim_{n\to+\infty} \frac{\mathrm{d}_{\mathrm{cyl}}(x_n, y_n)}{\varepsilon_n} = +\infty.$$

Up to a subsequence, we know that $y_n \to \bar{y} \in \bar{\Lambda} \cap \mathcal{H}^\perp$, Then, using again Lemma 3.5.5, Proposition 3.4.4 and (3.8), we obtain

$$\inf_{\Lambda\cap\mathcal{H}^\perp} \mathcal{M} \ge \liminf_{n\to+\infty} \varepsilon_n^{-2} c_{\varepsilon_n} \ge (\mathcal{M}(\bar{x}) + \mathcal{M}(\bar{y})) \ge 2 \inf_{\Lambda\cap\mathcal{H}^\perp} \mathcal{M},$$

which is impossible. □

3.6. Solutions of the initial problem

All this section is inspired by [95], where they study concentration of solutions around k-spheres for Laplacian problems.

3.6.1. Linear inequation outside small balls

Lemma 3.6.1. *Suppose that the assumptions of Theorem 3.4.3 are satisfied. Let $(u_\varepsilon)_\varepsilon \subset \mathcal{X}_\varepsilon$ be the solutions found in Theorem 3.4.3. Let $(x_\varepsilon)_\varepsilon \subset \bar{\Lambda}$, found in Proposition 3.5.6, be such that*

$$\liminf_{\varepsilon\to 0} |u_\varepsilon(x_\varepsilon)| > 0.$$

Then, there exists $r_0 > 0$ such that for every $r > r_0$, there exists $\varepsilon_r > 0$ such that for every $\varepsilon < \varepsilon_r$,

$$-\varepsilon^2 (\Delta + H) |u_\varepsilon| + (1-\mu) V |u_\varepsilon| \le 0 \qquad \text{in } \mathbb{R}^3 \setminus B_{cyl}(x_\varepsilon, \varepsilon r).$$

Proof. First, we have that

$$\mu V(x) \geq \delta > 0,$$

for $x \in \Lambda$. By Proposition 3.5.6 (iv), there exists $r_0 > 0$ sufficiently large, such that, for every $r > r_0$ there exist $\varepsilon_r > 0$ such that for every $\varepsilon < \varepsilon_r$,

$$|u_\varepsilon(x)|^{p-2} < \delta \quad \text{in } \Lambda \setminus B_{\text{cyl}}(x_\varepsilon, \varepsilon r).$$

Then, we use the Kato inequality (3.15) to obtain in $\Lambda \setminus B_{\text{cyl}}(x_\varepsilon, \varepsilon r)$ that

$$-\varepsilon^2 (\Delta + H) |u_\varepsilon| + (1-\mu)V|u_\varepsilon| \leq |u_\varepsilon|^{p-1} - \mu V|u_\varepsilon| - \varepsilon^2 H|u_\varepsilon| < 0.$$

Now, in $\mathbb{R}^3 \setminus \Lambda$, we use again the Kato inequality to obtain

$$-\varepsilon^2 (\Delta + H) |u_\varepsilon| + (1-\mu)V|u_\varepsilon| \leq 0 \quad \text{in } \mathbb{R}^3 \setminus \Lambda,$$

by the definition of the nonlinearity g_ε in $\mathbb{R}^3 \setminus \Lambda$. This concludes the proof. $\square$

3.6.2. Barrier functions

Once we can construct functions w_ε verifying the opposite inequation

$$-\varepsilon^2 (\Delta + H) w_\varepsilon + (1-\mu)V w_\varepsilon \geq 0, \qquad \text{in } \mathbb{R}^3 \setminus B_{\text{cyl}}(x_\varepsilon, \varepsilon r),$$

with some convenient boundary conditions on $\partial B_{\text{cyl}}(x_\varepsilon, \varepsilon r)$, Lemma 3.6.1 suggests that we can use the comparison principle to obtain an upper bound on $|u_\varepsilon|$. Those functions w_ε will be chosen in such a good way that the bound $|u_\varepsilon| \leq C w_\varepsilon$ imply that $|u_\varepsilon|^{p-2} \leq \mu V(x) + \varepsilon^2 H(x)$ for all $x \in \mathbb{R}^3 \setminus \Lambda$, so that we recover solutions of the initial problem (3.1).

We now define more precisely the notion of barrier functions.

Definition 3.6.2. *Let* $(x_\varepsilon)_\varepsilon \subset \mathbb{R}^3$ *and* $r > 0$. *We say that* $(w_\varepsilon)_\varepsilon \subset C^{1,\alpha}(\mathbb{R}^3 \setminus (B_{cyl}(x_\varepsilon, \varepsilon r) \cup \{0\}))$ *is a family of barrier functions if there exists* $\varepsilon_0 > 0$ *such that, for every* $\varepsilon < \varepsilon_0$, *we have that*

(i) w_ε *satisfies the inequation*

$$-\varepsilon^2 (\Delta + H) w_\varepsilon + (1-\mu)V w_\varepsilon \geq 0 \quad \textit{in } \mathbb{R}^3 \setminus B_{cyl}(x_\varepsilon, \varepsilon r);$$

(ii) $\nabla w_\varepsilon \in L^2(\mathbb{R}^3 \setminus B_{cyl}(x_\varepsilon, \varepsilon r))$;

(iii) $w_\varepsilon \geq 1$ *on* $\partial B_{cyl}(x_\varepsilon, \varepsilon r)$.

We remark that the boundary condition on $\partial B_{\text{cyl}}(x_\varepsilon, \varepsilon r)$ is arbitrary as long as it is positive since we can always multiply w_ε by a positive constant.

Construction of the comparison functions

In this section, we recall how to construct some comparison functions in Λ and in $\mathbb{R}^3 \setminus \Lambda$. Those comparison functions will be used to construct the barrier functions. We first begin by the construction in $\mathbb{R}^3 \setminus \Lambda$.

Lemma 3.6.3. *For every $\varepsilon > 0$, there exists $\Psi_\varepsilon \in C^{1,\alpha}_{loc}(\mathbb{R}^3 \setminus (\Lambda \cup \{0\}))$ such that*

$$\begin{cases} -\varepsilon^2(\Delta + H)\Psi_\varepsilon + (1-\mu)V\Psi_\varepsilon = 0 & \text{in } \mathbb{R}^3 \setminus \Lambda, \\ \Psi_\varepsilon = 1 & \text{on } \partial\Lambda, \end{cases}$$

and

$$\int_{\mathbb{R}^3\setminus\Lambda} |\nabla \Psi_\varepsilon|^2 + \frac{|\Psi_\varepsilon|^2}{|x|^2} < +\infty. \tag{3.47}$$

We also have the following estimate for every $x \in \mathbb{R}^3 \setminus (\Lambda \cup \{0\})$ and $C > 0$

$$0 < \Psi_\varepsilon(x) \leq \frac{C}{1+|x|}. \tag{3.48}$$

Moreover, if we assume that

(V^∞) holds with $\alpha = 2$, then, for every $\nu > 1$ and for every large $R > 1$, with $\bar{\Lambda} \subset B(0,R)$, there exist $C > 0$ and $\varepsilon_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$ and for every $x \in \mathbb{R}^3 \setminus B(0,R)$,

$$0 < \Psi_\varepsilon(x) \leq \frac{C}{|x|^\nu}; \tag{3.49}$$

(V^∞) holds with $\alpha < 2$, then, for every $\nu > 0$ and for every large $R > 1$, with $\bar{\Lambda} \subset B(0,R)$, there exist $C > 0$ and $\varepsilon_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$ and for every $x \in \mathbb{R}^3 \setminus B(0,R)$,

$$0 < \Psi_\varepsilon(x) \leq C \exp\left(-\nu |x|^{\frac{2-\alpha}{2}}\right); \tag{3.50}$$

(V^0) holds with $\alpha = 2$, then, for every $\nu > 0$ and for every $0 < r < 1$, with $B(0,r) \cap \bar{\Lambda} = \emptyset$, there exist $C > 0$ and $\varepsilon_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$ and for every $x \in B(0,r) \setminus \{0\}$,

$$0 < \Psi_\varepsilon(x) \leq C|x|^\nu; \tag{3.51}$$

(V^0) *holds with* $\alpha > 2$*, then, for every* $\nu > 0$ *and for every* $0 < r < 1$*, with* $B(0,r) \cap \bar{\Lambda} = \emptyset$*, there exist* $C > 0$ *and* $\varepsilon_0 > 0$ *such that, for every* $\varepsilon < \varepsilon_0$ *and for every* $x \in B(0,r) \setminus \{0\}$*,*

$$0 < \Psi_\varepsilon(x) \leq C \exp\left(-\nu |x|^{\frac{2-\alpha}{2}}\right). \tag{3.52}$$

Proof. **Step 1.** Proof of the existence and regularity. To prove the existence of the function Ψ_ε, we minimize the functional

$$\int_{\mathbb{R}^3 \setminus \Lambda} \varepsilon^2 \left(|\nabla u|^2 - H u^2\right) + (1-\mu) V u^2$$

on the set of functions

$$\left\{ u \in \mathcal{D}^{1,2}(\mathbb{R}^3, \mathbb{R}) \mid \int_{\mathbb{R}^3} V u^2 < +\infty \text{ and } u = 1 \text{ on } \partial\Lambda \right\}.$$

Estimate (3.47) follows from the Hardy inequality (3.13). By standard regularity theory, $\Psi_\varepsilon \in W^{2,q}_{\mathrm{loc}}(\mathbb{R}^3 \setminus (\Lambda \cup \{0\}))$, for $2 \leq q < +\infty$. Then, by Sobolev embeddings, $\Psi_\varepsilon \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^3 \setminus (\Lambda \cup \{0\}))$, for $\alpha \in [0,1)$.

Step 2. Proof of the estimate (3.48). Now, we want to obtain the estimate (3.48) corresponding to the case in which we make no assumptions on the potential V. Using the comparison principle with the function 0 in $\mathbb{R}^3 \setminus (\Lambda \cup \{0\})$, we obtain immediately that $\Psi_\varepsilon > 0$ in $\mathbb{R}^3 \setminus (\Lambda \cup \{0\})$. It remains us to prove the upper estimate. To this aim, let us first introduce the function

$$f(x) = \frac{1}{|x|}\left(\beta - \kappa \left(\log |x|\right)^{-\beta}\right), \qquad \text{for } x \in \mathbb{R}^3 \setminus B(0,R),$$

where $R > 1$ is chosen in such a way that $\bar{\Lambda} \subset B(0,R)$, $f(x) > 0$ in $\mathbb{R}^3 \setminus B(0,R)$ and $f(x) > 0$ on $\partial B(0,R)$. Using (3.17), we compute that

$$-\varepsilon^2 \left(\Delta + H\right) f \geq \frac{\varepsilon^2 \kappa}{|x|^3 \left(\log |x|\right)^{1+\beta}} \left(\beta(\beta+1) + \frac{\kappa}{\left(\log |x|\right)^{\beta}}\right) \geq 0.$$

Then, for every $\varepsilon > 0$, the inequality

$$-\varepsilon^2 \left(\Delta + H\right) f + (1-\mu) V f \geq 0, \qquad \text{in } \mathbb{R}^3 \setminus B(0,R),$$

follows by the positivity of V and f. If we use the comparison principle of Lemma 3.2.3 with Ψ_ε and f in $\mathbb{R}^3 \setminus B(0,R)$, there exists a positive constant $C > 0$ such that

$$\Psi_\varepsilon(x) \leq C f(x) \leq \frac{C}{|x|}, \qquad \text{in } \mathbb{R}^3 \setminus B(0,R). \tag{3.53}$$

Next, we consider the function

$$g(x) = \beta - \kappa \left(\log \frac{1}{|x|}\right)^{-\beta}, \qquad \text{for } x \in B(0,r) \setminus \{0\},$$

where $r < 1$ is chosen in such a way that $B(0,r) \cap \Lambda = \emptyset$, $g(x) > 0$ in $B(0,r)$ and $g(x) > 0$ on $\partial B(0,r)$. Using again (3.17), we obtain that

$$-\varepsilon^2 \left(\Delta + H\right) g \geq \frac{\varepsilon^2 \kappa}{|x|^2} \left(\beta(\beta+1)\left(\log \frac{1}{|x|}\right)^{-\beta-2} + \kappa \left(\log \frac{1}{|x|}\right)^{-2\beta-1}\right) \geq 0.$$

Then, for every $\varepsilon > 0$, the inequation

$$-\varepsilon^2 \left(\Delta + H\right) g + (1-\mu) g \geq 0, \qquad \text{in } B(0,r) \setminus \{0\},$$

follows by the positivity of V and g. Using the comparison principle for Ψ_ε and g in $B(0,r) \setminus \{0\}$, there exists a positive constant C such that

$$\Psi_\varepsilon(x) \leq C g(x) \leq C \quad \text{in } B(0,r) \setminus \{0\}. \tag{3.54}$$

Finally, we use the continuity of Ψ_ε and the fact that $\Psi_\varepsilon = 1$ on $\partial\Lambda$ to say that Ψ_ε is bounded from above in $B(0,R) \setminus (\Lambda \cup B(0,r))$. Combining this, (3.53) and (3.54), we obtain (3.48) for every $x \in \mathbb{R}^3 \setminus (\Lambda \cup \{0\})$, for a well chosen constant.

Step 3. Proof of the other estimates (3.49)-(3.52). When we add some assumptions on the potential V, either at zero or at infinity, we would like to obtain more precise bounds on Ψ_ε. Indeed, if we do not assume anything on V, meaning that V can decreasing as fast as possible at infinity and can be as low as possible at zero, the better estimate we can obtain is (3.48). What we expect is that slower is the decrease of V at infinity, faster is the convergence of Ψ_ε to zero at infinity (and inversely at zero). As already mentioned in the introduction, the Hardy potential $1/|x|^2$ is critical in the sense that there is a transition from polynomial onto exponential decrease in the profile of Ψ_ε. The strategy to get the estimates is always the same. We find convenient functions f and g being supersolution of the problem to which we will compare Ψ_ε. When (V^∞) is assumed with $\alpha = 2$, we introduce

$$f(x) = \frac{1}{|x|^\nu}, \qquad \text{for } x \in \mathbb{R}^3 \setminus B(0,R),$$

where $\nu > 1$ and $R > 1$ is taken large enough to have $\bar{\Lambda} \subset B(0,R)$ and

$$V(x) \geq \frac{c}{|x|^2}, \qquad \text{for } |x| > R.$$

We compute

$$-\varepsilon^2 \left(\Delta + H\right) f + (1-\mu)Vf \\ \geq |x|^{-\nu-2}\left(c + \varepsilon^2 \nu(\nu-1)\right) - \varepsilon^2\kappa |x|^{-\nu-2}\left(\log|x|\right)^{-\beta-1}.$$

For every $\varepsilon < \varepsilon_0$ small enough (ε_0 depending on ν and R), this expression is larger than 0. By the comparison principle, we obtain the estimate (3.49).

When we assume (V^∞) with $\alpha < 2$, the comparison function will be

$$f(x) = \exp\left(-\nu|x|^{\frac{2-\alpha}{2}}\right), \qquad \text{for } x \in \mathbb{R}^3 \setminus B(0,R),$$

where $\nu > 0$ and $R > 1$ is taken large enough to have $\bar{\Lambda} \subset B(0,R)$ and

$$V(x) \geq \frac{c}{|x|^\alpha}, \qquad \text{for } |x| > R.$$

In that case, we compute

$$-\varepsilon^2 \left(\Delta + H\right) f + (1-\mu)Vf \\ \geq \varepsilon^2 \frac{\nu}{4}(\alpha-2)(\alpha-4)|x|^{-\frac{\alpha+2}{2}} f - \varepsilon^2\kappa|x|^{-2}\left(\log|x|\right)^{-\beta-1} f \\ + |x|^{-\alpha}\left(c - \varepsilon^2\nu^2\left(\frac{2-\alpha}{2}\right)^2\right) f.$$

Thanks to the fact that $\alpha < 2$, for every $\varepsilon < \varepsilon_0$ small enough (ε_0 depending on ν and R), this expression is larger than 0. By the comparison principle, we obtain the estimate (3.50).

For the estimates at the origin, we proceed exactly in the same way. We consider $r < 1$ small enough to have $B(0,r) \cap \bar{\Lambda} = \emptyset$ and

$$V(x) \geq \frac{c}{|x|^\alpha}, \qquad \alpha \geq 2\,, \; 0 < |x| < r.$$

When we assume (V^0) with $\alpha = 2$, we introduce the comparison function

$$g(x) = |x|^\nu,$$

for $\nu > 0$ such that

$$-\varepsilon^2 \left(\Delta + H\right) g + (1-\mu)Vg \\ \geq |x|^{\nu-2}\left(c - \varepsilon^2\nu(\nu+1)\right) - \varepsilon^2\kappa|x|^{\nu-2}\left(\log\frac{1}{|x|}\right)^{-\beta-1},$$

which is greater than 0 for ε small enough. In the case (V^0) with $\alpha > 2$, the comparison function will be

$$g(x) = \exp\left(-\nu|x|^{\frac{2-\alpha}{2}}\right),$$

for $\nu > 0$. $\square$

Now, we construct a comparison function inside of Λ.

Lemma 3.6.4. *Consider $r > 0$. Let $(x_\varepsilon)_\varepsilon = (\rho_\varepsilon \cos\theta, \rho_\varepsilon \sin\theta, x_{3,\varepsilon})_\varepsilon \subset \Lambda$, $\theta \in [0, 2\pi)$, and $R > 0$ be such that $B_{cyl}(x_\varepsilon, R) \subset \Lambda$. We define*

$$\Phi_\varepsilon(x) = \cosh\left(\lambda \frac{R - \mathrm{d}_{cyl}(x, x_\varepsilon)}{\varepsilon}\right),$$

where $\lambda > 0$ is chosen such that

$$\inf_{\bar{\Lambda}} V > \frac{\lambda^2}{(1-\mu)}.$$

Then, there exists $\varepsilon_0 > 0$ such that, for every $\varepsilon < \varepsilon_0$,

$$-\varepsilon^2 (\Delta + H)\Phi_\varepsilon + (1-\mu)V\Phi_\varepsilon \geq 0 \quad \text{in } B_{cyl}(x_\varepsilon, R) \setminus B_{cyl}(x_\varepsilon, \varepsilon r).$$

Proof. By simple calculation, we obtain that

$$\begin{aligned}&-\varepsilon^2 (\Delta + H)\Phi_\varepsilon + (1-\mu)V\Phi_\varepsilon = \\ &(\lambda^2 + (1-\mu)V)\Phi_\varepsilon - \varepsilon^2 H\Phi_\varepsilon + \varepsilon\lambda \frac{2\rho - \rho_\varepsilon}{\rho\, \mathrm{d}_{\mathrm{cyl}}(x, x_\varepsilon)} \sinh\left(\lambda \frac{R - \mathrm{d}_{\mathrm{cyl}}(x, x_\varepsilon)}{\varepsilon}\right) \geq 0,\end{aligned}$$

thanks to the assumption on λ and for ε small enough. $\square$

Thanks to Proposition 3.5.6 (ii), we remark that the assumption $B_{\mathrm{cyl}}(x_\varepsilon, R) \subset \Lambda$ is verified if ε is taken sufficiently small. From now, we will always consider that $\varepsilon_0 > 0$ is taken small enough to have this property.

With those two functions Ψ_ε and Φ_ε, we are ready to construct the barrier functions.

Lemma 3.6.5. *Take $r > r_0$ (r_0 introduced in Lemma 3.6.1). Let $\lambda > 0$ be as in Lemma 3.6.4 and $(x_\varepsilon)_\varepsilon$ be as in Proposition 3.5.6. Then, there exists $\varepsilon_0 > 0$ and a family $(w_\varepsilon)_\varepsilon \subset C^{1,\alpha}_{loc}(\mathbb{R}^3 \setminus (B_{cyl}(x_\varepsilon, \varepsilon r) \cup \{0\}))$ of barrier functions such that for $\varepsilon < \varepsilon_0$*

$$0 < w_\varepsilon(x) \leq C \exp\left(-\frac{\lambda}{\varepsilon} \frac{\mathrm{d}_{cyl}(x, x_\varepsilon)}{1 + \mathrm{d}_{cyl}(x, x_\varepsilon)}\right)(1 + |x|)^{-1}, \tag{3.55}$$

for all $x \in \mathbb{R}^3 \setminus (B_{cyl}(x_\varepsilon, \varepsilon r) \cup \{0\})$. Moreover, if we assume that

(V^∞) holds with $\alpha = 2$, then, for every $\nu > 1$ and for every large $R > 1$ with $\bar{\Lambda} \subset B(0, R)$, there exist $C > 0$ and ε_0 (eventually smaller than the previous one) such that, for all $\varepsilon < \varepsilon_0$,

$$0 < w_\varepsilon(x) \le C \exp\left(-\frac{\lambda}{\varepsilon}\frac{\mathrm{d}_{cyl}(x, x_\varepsilon)}{1 + \mathrm{d}_{cyl}(x, x_\varepsilon)}\right)|x|^{-\nu} \tag{3.56}$$

for every $x \in \mathbb{R}^3 \setminus B(0, R)$;

(V^∞) holds with $\alpha < 2$, then, for every $\nu > 0$ and for every large $R > 1$ with $\bar{\Lambda} \subset B(0, R)$, there exist $C > 0$ and $\varepsilon_0 > 0$ such that, for all $\varepsilon < \varepsilon_0$,

$$0 < w_\varepsilon(x) \le C \exp\left(-\frac{\lambda}{\varepsilon}\frac{\mathrm{d}_{cyl}(x, x_\varepsilon)}{1 + \mathrm{d}_{cyl}(x, x_\varepsilon)}\right)\exp\left(-\nu|x|^{\frac{2-\alpha}{2}}\right) \tag{3.57}$$

for every $x \in \mathbb{R}^3 \setminus B(0, R)$;

(V^0) holds with $\alpha = 2$, then, for every $\nu > 0$ and for every $r < 1$ with $B(0, r) \cap \bar{\Lambda} = \emptyset$, there exist $C > 0$ and $\varepsilon_0 > 0$ such that, for all $\varepsilon < \varepsilon_0$,

$$0 < w_\varepsilon(x) \le C \exp\left(-\frac{\lambda}{\varepsilon}\frac{\mathrm{d}_{cyl}(x, x_\varepsilon)}{1 + \mathrm{d}_{cyl}(x, x_\varepsilon)}\right)|x|^{\nu} \tag{3.58}$$

for every $x \in B(0, r) \setminus \{0\}$;

(V^0) holds for $\alpha > 2$, then, for every $\nu > 0$ and for every $r > 1$ with $B(0, r) \cap \bar{\Lambda} = \emptyset$, there exist $C > 0$ and $\varepsilon_0 > 0$ such that, for all $\varepsilon < \varepsilon_0$,

$$0 < w_\varepsilon(x) \le C \exp\left(-\frac{\lambda}{\varepsilon}\frac{\mathrm{d}_{cyl}(x, x_\varepsilon)}{1 + \mathrm{d}_{cyl}(x, x_\varepsilon)}\right)\exp\left(-\nu|x|^{\frac{2-\alpha}{2}}\right) \tag{3.59}$$

for every $x \in B(0, r) \setminus \{0\}$.

Proof. First, we choose a open bounded set $U \subset \mathbb{R}^3$ such that $\bar{\Lambda} \subset U$, $0 \notin U$ and $\inf_{\bar{U}} V > 0$. This is always possible since $\inf_{\bar{\Lambda}} V > 0$. We consider a new function $\tilde{\Psi}_\varepsilon \in C^{1,\alpha}_{\mathrm{loc}}(\mathbb{R}^3 \setminus \{0\})$ such that $\tilde{\Psi}_\varepsilon = \Psi_\varepsilon$ in $\mathbb{R}^3 \setminus U$, for Ψ_ε defined in Lemma 3.6.3, and $\tilde{\Psi}_\varepsilon = 1$ in Λ. Thanks to the regularity of $\tilde{\Psi}_\varepsilon$, we have that $\|\tilde{\Psi}_\varepsilon\|_{L^\infty(U)} \le C$. Then, we choose $R > 0$ such that $B_{\mathrm{cyl}}(x_\varepsilon, R) \subset \Lambda$ for all $\varepsilon < \varepsilon_0$, which is possible thanks to Proposition 3.5.6. We construct the barrier function as follows. We first define

$$z_\varepsilon(x) = \begin{cases} \Phi_\varepsilon(x) & x \in B_{\mathrm{cyl}}(x_\varepsilon, R) \\ \tilde{\Psi}_\varepsilon(x) & x \in \mathbb{R}^3 \setminus B_{\mathrm{cyl}}(x_\varepsilon, R). \end{cases}$$

This function has the good regularity because $\tilde{\Psi}_\varepsilon$ and Φ_ε are both $C^{1,\alpha}_{\mathrm{loc}}$ and the junction on $\partial B_{\mathrm{cyl}}(x_\varepsilon, R)$ is also $C^{1,\alpha}_{\mathrm{loc}}$ (thanks to the choice of an cosh in Φ_ε). The property (i) of the barrier functions is verified in $B_{\mathrm{cyl}}(x_\varepsilon, R) \setminus B_{\mathrm{cyl}}(x_\varepsilon, \varepsilon r)$ and in $\mathbb{R}^3 \setminus U$. Indeed, Lemma 3.6.4 implies that Φ_ε is a supersolution of the operator in $B_{\mathrm{cyl}}(x_\varepsilon, R) \setminus B_{\mathrm{cyl}}(x_\varepsilon, \varepsilon r)$, while Lemma 3.6.3 says that $\tilde{\Psi}_\varepsilon$ is a solution of the operator in $\mathbb{R}^3 \setminus U$. In $U \setminus \Lambda$, we have

$$-\varepsilon^2 (\Delta + H)\, z_\varepsilon + (1-\mu)V z_\varepsilon = -\varepsilon^2 (\Delta + H)\, \tilde{\Psi}_\varepsilon + (1-\mu)V \tilde{\Psi}_\varepsilon \geq 0,$$

for ε small enough, since $V\tilde{\Psi}_\varepsilon > 0$ on U and $\tilde{\Psi}_\varepsilon$ is bounded in U. In $\Lambda \setminus B_{\mathrm{cyl}}(x_\varepsilon, R)$, we have

$$-\varepsilon^2 (\Delta + H)\, z_\varepsilon + (1-\mu)V z_\varepsilon = -\varepsilon^2 H + (1-\mu)V \geq 0,$$

for ε small enough, since $\inf_{\bar{\Lambda}} V > 0$. Then, condition (i) is verified. Condition (ii) is also verified thanks to our specific construction of Ψ_ε. It remains us to show that (iii) is true. To this aim, we set

$$w_\varepsilon(x) = \frac{z_\varepsilon(x)}{\cosh\left(\lambda \frac{R-\varepsilon r}{\varepsilon}\right)}.$$

The conclusion follows immediatly.

Since we have the following asymptotic behaviour in $B_{\mathrm{cyl}}(x_\varepsilon, R) \setminus B_{\mathrm{cyl}}(x_\varepsilon, \varepsilon r)$

$$\frac{\cosh\left(\lambda \frac{R-\mathrm{d}_{\mathrm{cyl}}(x,x_\varepsilon)}{\varepsilon}\right)}{\cosh\left(\lambda \frac{R-\varepsilon r}{\varepsilon}\right)} \leq 2\, e^{\lambda r} \exp\left(-\frac{\lambda}{\varepsilon} \mathrm{d}_{\mathrm{cyl}}(x, x_\varepsilon)\right),$$

and estimates found in Lemma 3.6.3, we obtain (3.55)-(3.59).

□

Solutions of the initial problem

Thanks to Lemmas 3.6.1 and 3.6.5, we obtain an upper bound on $|u_\varepsilon|$.

Proposition 3.6.6. *Suppose the assumptions of Theorem 3.4.3 and Proposition 3.5.6 are satisfied. Let $\lambda > 0$ be as in Lemma 3.6.4, $(x_\varepsilon)_\varepsilon \subset \bar{\Lambda}$ be as in Proposition 3.5.6 and $(u_\varepsilon)_\varepsilon \subset \mathcal{X}_\varepsilon$ be the solutions found in Theorem 3.4.3. Then, there exists $C > 0$ and $\varepsilon_0 > 0$ such that, for all $\varepsilon < \varepsilon_0$,*

$$0 < |u_\varepsilon(x)| \leq C \exp\left(-\frac{\lambda}{\varepsilon} \frac{\mathrm{d}_{cyl}(x, x_\varepsilon)}{1 + \mathrm{d}_{cyl}(x, x_\varepsilon)}\right) (1 + |x|)^{-1}, \quad \forall x \in \mathbb{R}^3 \setminus \{0\}. \tag{3.60}$$

Moreover, (3.56)-(3.59) hold for $|u_\varepsilon|$ in place of w_ε if we make the same assumptions on V.

Proof. By Lemma 3.6.1, we know that $|u_\varepsilon|$ is a subsolution in $\mathbb{R}^3 \setminus B_{\text{cyl}}(x_\varepsilon, \varepsilon r)$, for some $r > r_0$. Furthermore, thanks to Lemma 3.5.2, we deduce that $\|u_\varepsilon\|_{L^\infty(B_{\text{cyl}}(x_\varepsilon,\varepsilon r))}$ is bounded for $\varepsilon < \varepsilon_0$. We can then say that $|u_\varepsilon| \leq \|u_\varepsilon\|_{L^\infty(B_{\text{cyl}}(x_\varepsilon,\varepsilon r))}$ on $\partial B_{\text{cyl}}(x_\varepsilon, \varepsilon r)$. With the comparison principle, we conclude that

$$0 < |u_\varepsilon(x)| \leq \|u_\varepsilon\|_{L^\infty(B_{\text{cyl}}(x_\varepsilon,\varepsilon r))} w_\varepsilon(x), \qquad \forall x \in \mathbb{R}^3 \setminus (B_{\text{cyl}}(x_\varepsilon, \varepsilon r) \cup \{0\}).$$

Finally, since $|u_\varepsilon|$ is bounded in $B_{\text{cyl}}(x_\varepsilon, \varepsilon r)$, we obtain the estimate (3.60) for all $x \in \mathbb{R}^3 \setminus \{0\}$. The other estimates follow by reasoning in the same way. □

We can now proof the main Theorem.

Proof of Theorem 3.1.1. It remains us to prove that u_ε is in fact a solution of the initial problem (3.1). For this, we need to show that

$$f(|u_\varepsilon|^2) = |u_\varepsilon|^{p-2} \leq \varepsilon^2 H(x) + \mu V(x), \qquad \forall x \in \mathbb{R}^3 \setminus \Lambda.$$

We prove this for example in the case where we make no assumptions on V (then $p > 4$). We use Proposition 3.6.6 to say that

$$|u_\varepsilon(x)|^{p-2} \leq C e^{-\frac{\lambda}{\varepsilon}(p-2)}(1 + |x|)^{-(p-2)} \leq \varepsilon^2 H(x) + \mu V(x),$$

for small ε. The last inequality is verified since we considered $p > 4$. Indeed, for $|x|$ large, the right hand side behaves as $1/\left(|x|^2 \log |x|\right)$. The left hand side decays then faster since it behaves as $1/|x|^{p-2}$. For $|x|$ small, the left hand side behaves as a constant while the right hand side is unbounded. The other cases may be treated in a similar way. □

Remark 3.6.7. *In addition of theorem 3.1.1, we may also prove that estimates* (3.56)-(3.59) *hold for $|u_\varepsilon|$ instead of w_ε if we make the corresponding assumptions on V.*

3.7. Another class of symmetric solutions

When A is equal to the Lorentz potential, i.e. $A = (-x_2, x_1, 0)$ or has the slightly more general form

$$A(\rho, \theta, x_3) = c(\rho)(-\sin\theta, \cos\theta, 0), \tag{3.61}$$

Esteban and Lions have proposed in [56, Section 4.3] the class of solutions

$$u_k := C_k \left(\frac{x_2 + i x_1}{\rho}\right)^k v_k,$$

where $k \in \mathbb{Z}$, $C_k \in \mathbb{R}_0$ and v_k are real and cylindrically symmetric solutions of an auxiliary problem. One can check easily that the functions u_k solve

$$(i\varepsilon\nabla + A)^2\, u_k + V(\rho, x_3)u_k = |u_k|^{p-2}u_k, \qquad x \in \mathbb{R}^3, \tag{3.62}$$

if and only if the v_k are real solutions of

$$-\,\varepsilon^2\Delta v_k + \left(\left(\frac{k\varepsilon}{\rho} + c(\rho)\right)^2 + V(\rho, x_3)\right) v_k = C_k^{p-2}|v_k|^{p-2}v_k, \qquad x \in \mathbb{R}^3. \tag{3.63}$$

The limit equation in $\mathbb{R}^2$ has the form

$$-\Delta w_k + \left(c^2(\rho_0) + V(\rho_0, x_{3,0})\right) w_k = C_k^{p-2}|w_k|^{p-2}w_k, \tag{3.64}$$

where $(\rho_0, x_{3,0})$ is such that the concentration function

$$\mathcal{M}(\rho, x_3) = \rho\left(c^2(\rho) + V(\rho, x_3)\right)^{\frac{2}{p-2}} \tag{3.65}$$

is locally minimized at this point.

Observe that this reduction to a real valued problem allows us to use directly the arguments from [95] without much modifications. One can then consider several cases according to the properties of c and V. We do not address all these cases in details. We will focus on the special case which for instance allows to consider a critical frequency.

3.7.1. Existence at the critical frequency

Remember that the potential V stands for $U - E$, where U is the electrical potential and E is the frequency of the standing wave $\psi(x,t) = e^{-i\frac{E}{\hbar}t}\,u(x)$. When $E = \inf_{\mathbb{R}^N} U(x)$, we say that E is the critical frequency. When $A = 0$, the critical frequency was studied by many authors, starting with the contribution of Byeon and Wang [63, 64] and followed by many others.

Byeon and Wang have shown that there exists a standing wave which is trapped in a neighbourhood of the isolated minimum points of V and whose amplitude goes to 0 as $\hbar \to 0$. Moreover, depending upon the local behaviour of the potential function V near the minimum points, the limiting profile of the standing wave solutions was shown to exhibit quite different characteristic features. This is in striking contrast with the non-critical frequency case ($\inf U(x) > E$) where the solution develops a spike in the semiclassical limit.

Here we show that even if the frequency is critical, the presence of an external magnetic field allows for the existence of a solution concentrating on a circle and whose amplitude does not vanish in the semiclassical limit so that this solution is a spike type solution.

Let $p > 2$ and $k \in \mathbb{Z}$. Let $V \in C(\mathbb{R}^3 \setminus \{0\})$ be nonnegative and such that $V(gx) = V(x)$ for every $g \in G$. Assume $A \in C^1(\mathbb{R}^3, \mathbb{R}^3)$ is of the form (3.61) and such that $c(\rho) > 0$ for every $\rho > 0$ and

$$\liminf_{\rho \to \infty} c(\rho)\rho > 0.$$

With those assumptions, the assumption $\liminf_{|x|\to+\infty} W(x)|x|^2 > 0$ holds for the potential

$$W = \left(\frac{k\varepsilon}{\rho} + c(\rho)\right)^2 + V(\rho, x_3)$$

and ε small. Moreover this potential is nonnegative everywhere and for every $0 < \theta < 1$ and $\varepsilon > 0$, there exists $\rho_{\theta,\varepsilon} > 0$ such that

$$\left(\frac{k\varepsilon}{\rho} + c(\rho)\right)^2 + V(\rho, x_3) \geq \theta c(\rho)^2 + V(\rho, x_3),$$

for $\rho \geq \rho_{\theta,\varepsilon}$. Clearly $\rho_{\theta,\varepsilon} \to 0$ as $\varepsilon \to 0$ for any fixed θ.

The proof of the following theorem can be easily recovered from [95] with straightforward modifications.

Theorem 3.7.1. *With the above conditions on c, V, k and p, assume there exists a bounded smooth set $\Lambda \subset \mathbb{R}^3$ verifying (3.7), such that (3.8) is verified with $\mathcal{M}$ now defined by (3.65) and (3.9) is verified for $\theta c^2 + V$, $\theta \in (0,1)$. If $\varepsilon > 0$ is small enough, the equation (3.63) has a solution $v_{k,\varepsilon}$ such that $v_{k,\varepsilon}(gx) = v_{k,\varepsilon}(x)$ for all $g \in G$, $v_{k,\varepsilon}$ attains its maximum at some $x_{k,\varepsilon} = (\rho_{k,\varepsilon}\cos\theta, \rho_{k,\varepsilon}\sin\theta, x_{3,k,\varepsilon}) \in \Lambda$ such that*

(ii) $\liminf_{\varepsilon\to 0} |v_{k,\varepsilon}(x_{k,\varepsilon})| > 0;$

(iii) $\lim_{\varepsilon\to 0} \mathcal{M}(x_{k,\varepsilon}) = \inf_{\Lambda\cap\mathcal{H}^\perp} \mathcal{M};$

(iv) $\limsup_{\varepsilon\to 0} \dfrac{\mathrm{d}_{cyl}(x_{k,\varepsilon}, \mathcal{H}^\perp)}{\varepsilon} < +\infty$ *, that is* $x_{3,k,\varepsilon} \to 0;$

(v) $\liminf_{\varepsilon\to 0} \mathrm{d}_{cyl}(x_{k,\varepsilon}, \partial\Lambda) > 0.$

Finally, there exists $C_k \in \mathbb{R}_0$ such that

$$u_{k,\varepsilon} = C_k \left(\frac{x_2 + ix_1}{\rho}\right)^k v_{k,\varepsilon}$$

solves (3.62) *and for every* $\nu > 1$*, the asymptotic estimate holds*

$$0 < |u_{k,\varepsilon}(x)| \leq C \exp\left(-\frac{\lambda}{\varepsilon}\frac{\mathrm{d}_{cyl}(x, x_{k,\varepsilon})}{1 + \mathrm{d}_{cyl}(x, x_{k,\varepsilon})}\right)|x|^{-\nu}, \qquad \forall x \in \mathbb{R}^3 \setminus \{0\}.$$

As previously discussed, we can consider the critical frequency as one can allow V to vanish at the local minimum point of $\mathcal{M}$ in Λ.

Observe also that the ansatz fixes the concentration as the concentration set is the same for any choice of $k \in \mathbb{Z}$.

Appendix A

Appendix of Chapter 2

A.1. Domains with conical singularities

The proofs of the main theorems can be partially adapted to the case when Ω presents isolated conical singularities, as in the numerical simulations which appear in the Introduction. The results are qualitatively the same as for the smooth domain, but the rate of convergence of the eigenvalues depends on the aperture of the cone. We can interpret this fact in the following way: the zero boundary conditions on an acute angle of $\partial\Omega$ play the same role as the nodal lines of the eigenfunction. The tighter is the angle, the faster is the convergence.

Consider the following conical domain of aperture α, for some $0 < \beta < \pi$

$$\Omega = \left\{ (r, \theta) : \, r \in (0,1), \, \theta \in \left(-\frac{\beta}{2}, \frac{\beta}{2} \right) \right\}. \tag{A.1}$$

The counterpart of Theorem 2.1.11 holds.

Theorem A.1.1. *Let $\alpha = 1/2$. Let Ω be as in (A.1) and let p satisfy (2.4). Suppose that λ_k is simple and that there exists an eigenfunction φ_k associated to λ_k having a zero of order $h/2 \geq 2$ at the origin (at least one nodal line). Then there exists $C > 0$, not depending on a, such that*

$$\lambda_k^a \leq \lambda_k - C|a|^{h\frac{\pi}{\beta}} \quad \textit{for } a \to 0 \textit{ along a nodal line of } \varphi_k.$$

As for the analogous of Theorem 2.1.14, we can prescribe the behaviour of the eigenvalues only in case the pole approaches the vertex of the cone along the angle bisector. This restriction is related to the open problem presented in Remark 2.1.16 (i).

Theorem A.1.2. *Let $\alpha = 1/2$. Let Ω be as in (A.1) and let p satisfy (2.4). Suppose that λ_k is simple and that φ_k has a zero of order 1 at the origin (no* *nodal lines). Then there exists $C > 0$, not depending on a, such that*

$$\lambda_k^a \geq \lambda_k + C a_1^{2\frac{\pi}{\beta}} \quad \textit{for } a = (a_1, 0),\ a_1 \to 0.$$

The strategy of proof consists in applying the conformal map $x^{\frac{\pi}{\beta}}$, so that the conical domain is transformed into the regular half ball $D_1^+(0)$. We end up with a singular equation of the following type

$$(i\nabla + A_a)^2 \varphi_k^a = \lambda_k^a \left(\frac{\beta}{\pi}\right)^2 \frac{p(x)}{|x|^{2-\frac{2\beta}{\pi}}} \varphi_k^a \qquad \text{in } D_1^+(0).$$

The singular potential $|x|^{-2+\frac{2\beta}{\pi}}$ belongs to the Kato class, which allows to adapt the proofs of the previous sections. In particular, the following Hardy inequality holds: for every $\varepsilon > 0$ there exists a positive constant C such that

$$\frac{C}{r^\varepsilon} \int_{D_r(0)} \frac{|u|^2}{|x|^{2-\varepsilon}} \,\mathrm{d}x \leq \int_{D_r(0)} |\nabla |u||^2 \,\mathrm{d}x + \frac{1}{r} \int_{\partial D_r(0)} |u|^2 \,\mathrm{d}\sigma, \tag{A.2}$$

for every $u \in H^1(D_r(0), \mathbb{C})$ and for every $r > 0$ (see [109]). By combining with the diamagnetic inequality

$$\int_{D_r(0)} |\nabla |u||^2 \,\mathrm{d}x \leq \int_{D_r(0)} |(i\nabla + A_a)u|^2 \,\mathrm{d}x,$$

we obtain the counterpart of the Poincaré inequality (2.67).

Concerning Proposition 2.2.9, its validity in case of a singular potential belonging to the Kato class is stated in [7, Theorem 1.3].

A.2. Green's function for a perturbation of the Laplacian

Lemma A.2.1. *Consider the set of equations (depending on the parameter ε) $-\Delta f = \varepsilon c(x) f$ in $\Omega \subset \mathbb{R}^2$ bounded, with $c \in L^\infty(\Omega)$. For $\varepsilon > 0$ sufficiently small there exists a Green's function $G(x, y)$ such that the following representation formula holds for $x \in \Omega$*

$$f(x) = -\int_{\partial\Omega} f \partial_\nu G(x, \cdot) \,\mathrm{d}\sigma(y).$$

Moreover, for every $1 \leq p < \infty$ there exists C independent from ε such that we have

$$\|\partial_{x_i} G(x, \cdot) - \partial_{x_i} \Phi(x, \cdot)\|_{W^{1,p}(\partial\Omega)} \leq C\varepsilon,$$

for $x \in \Omega$, where Φ is the Green function of the Laplacian with homogeneous Dirichlet boundary conditions in Ω.

Proof. We define the Green function as $G(x, y) = \Gamma(y - x) + L(x, y)$, where $\Gamma(x) = -\frac{1}{2\pi} \log |x|$ is the fundamental solution of the Laplacian in $\mathbb{R}^2$ and $L(x, \cdot)$ solves, for $x \in \Omega$,

$$\begin{cases} -\Delta L(x,y) - \varepsilon c(y) L(x,y) = \varepsilon c(y) \Gamma(y-x) & y \in \Omega \\ L(x,y) = -\Gamma(y-x) & y \in \partial\Omega. \end{cases}$$

Notice that this equation admits a solution for ε small because the quadratic form

$$\int_\Omega (|\nabla v|^2 - \varepsilon c(x) v^2) \, \mathrm{d}x \tag{A.3}$$

is coercive for $v \in H_0^1(\Omega)$, and moreover $\Gamma \in L^2(\Omega)$.

The validity of the representation formula is standard. Indeed, the following identity holds (see for example [110], equation (25) in paragraph 2.2.4)

$$f(x) = -\int_\Omega \Gamma(y-x) \Delta f(y) \, \mathrm{d}y + \int_{\partial\Omega} (\Gamma(y-x) \partial_\nu f(y) - f(y) \partial_\nu \Gamma(y-x)) \, \mathrm{d}\sigma(y).$$

By using the Green formula

$$\int_\Omega (\Delta L \, f - L \Delta f) \, \mathrm{d}y = \int_{\partial\Omega} (\partial_\nu L \, f - L \partial_\nu \, f) \, \mathrm{d}\sigma(y)$$

and the equation satisfied by $L(x, \cdot)$, we obtain the representation formula for f.

In order to estimate $\partial_{x_i}(G - \Phi)$, we write $\Phi(x, y) = \Gamma(y - x) + H(x, y)$, with

$$\begin{cases} -\Delta H(x,y) = 0 & y \in \Omega \\ H(x,y) = -\Gamma(y-x) & y \in \partial\Omega, \end{cases}$$

so that $\partial_{x_i}(G - \Phi) = \partial_{x_i}(L - H) =: u$ solves

$$\begin{cases} -\Delta u - \varepsilon c(y) u = \varepsilon c(y) \partial_{x_i} \Phi(x, \cdot) & y \in \Omega \\ u = 0 & y \in \partial\Omega. \end{cases}$$

We apply Poincaré inequality and the positivity of the quadratic form in (A.3) as follows

$$\begin{aligned} \|u\|_{H^1(\Omega)} &\le C \|\nabla u\|_{L^2(\Omega)} \\ &\le C \left(\int_\Omega (|\nabla u|^2 - \varepsilon c(y) u^2) \, \mathrm{d}y \right)^{1/2} \\ &= C \left(\int_\Omega \varepsilon c(y) \partial_{x_i} \Phi(x,y) u \, \mathrm{d}y \right)^{1/2}. \end{aligned}$$

Since $\partial_{x_i}\Phi(x,\cdot) \in L^q(\Omega)$ for $1 \le q < 2$, we can apply the Hölder inequality and the Sobolev embedding to obtain

$$\|u\|_{H^1(\Omega)} \le C\varepsilon^{1/2}\left(\|\partial_{x_i}\Phi\|_{L^{3/2}(\Omega)}\|u\|_{L^3(\Omega)}\right)^{1/2} \le C\varepsilon^{1/2}\|u\|_{H^1(\Omega)}^{1/2}.$$

Finally, using again the Sobolev embeddings and a bootstrap argument, we obtain that $u \in W^{2,q}(\Omega)$ for every $1 \le q < 2$ and

$$\|u\|_{W^{2,q}(\Omega)} \le C\varepsilon. \tag{A.4}$$

$\square$

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