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ABSOLUTE PARALLELISMS ON ALMOST COMPLEX MANIFOLDS

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To my parents

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Contents

Introduction

An *almost complex manifold* is a differentiable manifold M endowed with an almost complex structure, i.e. an endomorphism J on the tangent bundle of M such that $J^2 = -id$. By a remarkable result by Newlander-Nirenberg, [22], an almost complex manifold is a complex manifold if and only if the Nijenhuis tensor, defined as

$$N_J(X,Y) := [JX, JY] - [X,Y] - J([JX,Y] + [X, JY]), \qquad \forall X, Y \in \Gamma(TM),$$

vanishes. In such a case, the complex manifolds are locally equivalent, in other words, they have the same complex structure locally, but we cannot say the same for the almost complex manifolds in general. In this thesis we have studied the almost complex manifolds whose image of the Nijenhuis tensor forms a non integrable bundle (that is, it is not closed with respect to the Lie brakets) and we found some results when the real dimension of M is 4.

In general, the almost complex manifolds are not parallelizzable, that is, there exist no a global frame $(X_1, X_2, ..., X_n)$ formed by n (the dimension of the manifold) globally defined vector fields which are linearly independent at every point (see [23], [37] and [30]). When (M^4, J) is an almost complex manifold of dimension 4 admitting a non-integrable subbundle given by the image of the Nijenhuis tensor, it is possible to give a double absolute parallelism on it. This means that, for any point $p \in M^4$, there are two adapted frames of the tangent space: $(X_1, X_2, X_3, X_4)_p \in T_pM$ and $(-X_1, -X_2, X_3, X_4)_p \in T_pM$. This provide a $\mathbb{Z}_2$-structure F on (M^4, J). As a consequence, it results that the group of the automorphisms $Aut(M^4, J)$ of (M^4, J) is a Lie group of dimension less or equal to 4, and its isotropy subgroup has at most two elements.

When (M^4, J) is locally homogeneous, one can define the Lie algebra $\mathfrak{g}$ associated to (M^4, J) thanks to the absolute parallelisms on (M^4, J). These facts allow us to make the classification of the almost complex structures related to (M^4, J). Such a classification is complete when the Lie algebra $\mathfrak{g}$ is not solvable (in this case it is reductive too), while several examples are shown to explain how to make the classification when the algebra $\mathfrak{g}$ is solvable (since the latter has more cases than the former).

Finally, we studied if it is possible to introduce an invariant Riemannian metric on (M^4, J) such that it becomes an almost Kähler manifold.

More precisely, this thesis is structured as following.

The **first chapter** is an introduction of the relevant material and the notations about (almost) complex manifolds, (almost) CR manifolds and (almost) Kähler manifolds. The last two sections of the chapter are focused on the theory of the principal fiber bundles and G-structures; here, the most important theorems and properties useful to set the general problem are given.

In the **second chapter** we recall the principal properties of $2n$-dimensional almost complex manifolds (M^{2n}, J) and we define the torsion bundle $\mathcal{V}$ of (M^{2n}, J) as the subbundle of TM obtained as image of the Nijenhuis tensor. Here, we focus our study on the manifolds (M^{2n}, J) with *non-degenerate* torsion bundle, that is, with non-integrable $\mathcal{V}$. In particular, when the almost complex manifold (M^4, J) is of dimension 4, such torsion bundle $\mathcal{V}$ induces a filtration of the tangent bundle TM given by

$$\mathcal{V} \subseteq \mathcal{V}_{-2} \subseteq \mathcal{V}_{-3} \subseteq \Gamma(TM),$$

where

$$\begin{aligned}
&\mathcal{V}_{-1} := \Gamma(\mathcal{V}),\\
&\mathcal{V}_{-2} := \Gamma(\mathcal{V}) + [\Gamma(\mathcal{V}), \Gamma(\mathcal{V})] \neq \Gamma(\mathcal{V}),\\
&\mathcal{V}_{-3} := \mathcal{V} + [\Gamma(\mathcal{V}), \Gamma(\mathcal{V})] + [\Gamma(\mathcal{V}), [\Gamma(\mathcal{V}), \Gamma(\mathcal{V})]].
\end{aligned}$$

The behavior of the filtration in any point $p \in M^4$ gives rise to two types of torsion bundles: we will say that $\mathcal{V}_p$ is *fundamental* when $\mathcal{V}_{-3|p} = T_pM$, otherwise $\mathcal{V}_p$ is *non-fundamental*. It results that there are two distinguished sections $\pm X$ in $\mathcal{V}$, for which the filtration can be refined by

$$\mathcal{V}_p^+ \subseteq \mathcal{V}_p \subseteq \mathcal{V}_{-2|p} \subseteq \mathcal{V}_{-3|p} \subseteq T_pM,$$

where $\mathcal{V}_p^+$ is the vector space generated by X_p. If $\pm X$ are the distinguished sections, we have that the bases $(\pm X_p, \pm JX_p, [X, JX]_p, J[X, JX]_p)$, for p in M^4, form a couple of adapted frames, where (X_p, JX_p) is a base of $\mathcal{V}_p$ and $(X_p, JX_p, [X, JX]_p)$ is a base of $\mathcal{V}_{-2|p}$. One of the main results is announced in this chapter (here we set $T^X := [X, JX]$ to be concise):

Theorem. *If (M^4, J) is an almost complex manifold of dimension 4 and with non-degenerate torsion bundle $\mathcal{V}$, then, for each point $p \in M$, there exists an unique pair of adapted frames*

$$(X_p, JX_p, T_p^X, JT_p^X),$$

$$(-X_p, -JX_p, T_p^X, JT_p^X),$$

where $\pm X$ are the distinguished sections of $\mathcal{V}^+$ in a neighborhood of p.

The theorem above leads to the following consequences: the set of all adapted frames forms a reduction F of the principal bundle of linear frames on (M^4, J) having structure group isomorphic to $\mathbb{Z}_2$; the dimension of the symmetry algebra $aut_p(M^4, J)$ is less or equal to four; moreover, one can define a natural metric and a norm on (M^4, J). In this way there is an $\mathbb{Z}_2$-structure on (M^4, J) and an $\{e\}$-structure on F. In particular, the $\mathbb{Z}_2$-structure on (M^4, J) allows to solve the problem of the locally equivalence between four dimensional almost complex manifolds with non-degenerate torsion bundle: we have that two almost complex manifolds are locally equivalent if and only if they have the same structure functions (up to sign) associated to the adapted frame. The remaining part of this chapter is devoted to graded Lie algebras associated to the filtration given above.

The **third chapter** is devoted to the classification of the connected locally homogeneous almost complex manifolds (M^4, J), with non-degenerate torsion bundle. Here, we expose how the adapted frame $(X_p, JX_p, T_p^X, JT_p^X)$ forms the Lie algebra $\mathfrak{g}$ associated to (M^4, J). It allows us to classify these manifolds depending on the types of compatible Lie algebras. Because of the existence of an $\mathbb{Z}_2$-structure on (M^4, J), it is possible to classify the almost complex structures J in such a way to study the locally equivalence of these manifolds. The classification is complete when $\mathfrak{g}$ is non-solvable and it is summarized in the following tables. We restrict our study to the generalization of some examples when $\mathfrak{g}$ is solvable, because of the large number of cases. Moreover, in the latter, we show with examples that there exist manifolds $M^4 = M^3 \times \mathbb{R}$ for which M^3 is locally equivalent to a 3-dimensional hypersphere, but also for which M^3 is not.

Theorem. *If (M^4, J) is a connected locally homogeneous almost complex manifold with a non-degenerate torsion bundle $\mathcal{V}$, and its associated Lie algebra $\mathfrak{g}$ is non-solvable, we have the following classification for the almost complex structures J:*

	$\mathcal{V}$ is non-fundamental	
	$\mathcal{V} \subseteq \mathfrak{L}$	$\mathcal{V} \not\subseteq \mathfrak{L}$
$\mathfrak{L} = \mathfrak{so}(3)$ $[e_1, e_2] = e_3$ $[e_2, e_3] = e_1$ $[e_3, e_1] = e_2$	$\xi = k_{a,b} e_1, \quad k_{a,b} \neq 0$ $J\xi = ae_1 + be_2, \quad ab \neq 0$ $\eta = k_{a,b} b e_3$ $J\eta = e_0 + c_{a,b} e_3$	not possible

	$\mathcal{V}$ is fundamental	
	$\mathcal{V} \subseteq \mathfrak{L}$	$\mathcal{V} \not\subseteq \mathfrak{L}$
$\mathfrak{L} = \mathfrak{so}(3)$ $[e_1, e_2] = e_3$ $[e_2, e_3] = e_1$ $[e_3, e_1] = e_2$	not possible	$\xi = e_1$ $J\xi = ae_1 + b(e_0 + e_2), \quad b \neq 0$ $\eta = be_3$ $J\eta = \frac{2yz(a^2+1)}{z^2+2abz-b^2} e_1 + ye_2 + ze_3 + \frac{z^2+2abz-b^2+y^2}{y} e_0,$ $z^2 + 2abz - b^2 \neq 0$

	$\mathcal{V}$ is non-fundamental	
	$\mathcal{V} \subseteq \mathfrak{L}$	$\mathcal{V} \not\subseteq \mathfrak{L}$
	$\mathcal{V}$ has all regular elements (except the null matrix)	
$\mathfrak{L} = sl(2,\mathbb{R})$ $[X, H] = X$ $[X, Y] = 2H$ $[H, Y] = Y$	$\xi = kH$ $J\xi = a_{k,t,b}X + bY + c_{k,t,b}H$ $\eta = k(-a_{k,t,b}X + bY)$ $J\eta = e_0 + t(-a_{k,t,b}X + bY)$ with $a_{k,t,b}bk \neq 0$	$\xi = H$ $J\xi = e_0 + aX + bH$ $\eta = -aX$ $J\eta = \frac{a}{2}(t+2b)X + yY + 2(b^2+1)H + te_0$ with $a \neq 0, y \neq 0$
	$\mathcal{V}$ has exactly two lines made by non-regular elements	
$\mathfrak{L} = sl(2,\mathbb{R})$ $[X, H] = X$ $[X, Y] = 2H$ $[H, Y] = Y$	$\xi = X$ $J\xi = aX + bY + cH$ $\eta = 2bH + cX$ $J\eta = e_0 + \frac{cz}{2b}X + zH$ with $b \neq 0$	

	$\mathcal{V}$ is fundamental	
	$\mathcal{V} \subseteq \mathfrak{L}$	$\mathcal{V} \not\subseteq \mathfrak{L}$
$\mathfrak{L} = sl(2,\mathbb{R})$ $[X,H] = X$ $[X,Y] = 2H$ $[H,Y] = Y$	not possible	v with negative determinant $\xi = H$ $J\xi = e_0 + aX + bY + cH$ $\eta = -aX + bY$ $J\eta = xX + yY + z_{x,y,a,b,c}H + t_{x,y,a,b,c}e_0$ with $(a,b) \neq (0,0)$ $ay + bx - 2abt_{x,y,a,b,c} \neq 0$
$\mathfrak{L} = sl(2,\mathbb{R})$ $[Z,H] = W$ $[Z,W] = 2H$ $[H,W] = Z$	not possible	v with positive determinant $\xi = H$ $J\xi = e_0 + aH + bW$ $\eta = bZ$ $J\eta = x_{a,b,y,z}H + yW + zZ + t_{a,b,y,z}e_0$ with $b \neq 0$ $bt_{a,b,y,z} \neq y$
$\mathfrak{L} = sl(2,\mathbb{R})$ $[X,H] = X$ $[X,Y] = 2H$ $[H,Y] = Y$	not possible	v with null determinant $\xi = X$ $J\xi = e_0 + aX + bY + cH$ $\eta = cX + 2bH$ $J\eta = x_{a,b,c,y,z}X + yY + zH + t_{a,b,c,y,z}e_0$ with $b \neq 0$ $bt_{x,y,a,b,c} \neq y$

In the last section of the third chapter, there are some significant examples showing that, when (M^4, J) is homogeneous, it is not possible to have two independent absolute parallelisms globally on (M^4, J), but this is possible only locally.

The **fourth chapter** is devoted to the metrics, invariant and not, which are compatible with the almost complex structure J of an almost complex manifold having non degenerate torsion bundle; the study is focused on some significant examples. In particular, when the manifold (M^4, J) is homogeneous, there is not an invariant metric on (M^4, J) (see [6]). Here, are touched only a few aspects of the theory; the topics we deal with are good starting points for further research.

Other possible developments can be oriented to the study of almost complex manifolds in six dimension. One can find some relevant material about this topic in [2].

Chapter 1

Preliminaries

In this chapter we give a brief exposition of special structures on differentiable manifolds. We set up notation and terminology summarizing, often without proofs, the relevant material (for more details see, for example, [7, 16, 37, 43]).

1.1 Almost complex manifolds and complex manifolds

Definition 1.1. An *almost complex structure* on a real differential manifold M is an endomorphism $J : TM \to TM$ of the tangent bundle TM such that $J^2 = -id$. A manifold with an almost complex structure is called *almost complex manifold* and it will be written as (M, J).

An almost complex structure has real even dimension $2n$ and the integer n is called *complex dimension* of M.

A typical example of an almost complex structure is the *standard complex structure* J_{st} on $\mathbb{C}^n$. If $z_j = x_j + iy_j$ (with $j = 1, ..., n$) are complex coordinates of $\mathbb{C}^n$, the standard complex structure J_{st} is defined by

$$J_{st}\left(\frac{\partial}{\partial x_j}(p)\right) = \left(\frac{\partial}{\partial y_j}(p)\right), \qquad j = 1, ..., n.$$

Definition 1.2. Let (M, J) and (M', J') be two almost complex manifolds. A differential mapping $F : M \to M'$ is called *(J,J')-holomorphic*, or briefly *holomorphic*, if its differential satisfies

$$dF \circ J = J' \circ dF.$$

If F is also a diffeomorphism, we say that F is *(J,J')-biholomorphic.*

The existence of holomorphic functions on manifolds is not granted in general, but there are some results on almost complex manifolds: see [20, 21, 12, 13].

Definition 1.3. A *complex manifold* is an almost complex manifold (M, J) such that every point $p \in M$ has a neighborhood (J, J')-biholomorphic to an open set of $\mathbb{C}^n$. In such a case, the structure J is called *complex structure*.

We recall the following

Theorem 1.1.1 (**Newlander-Nirenberg** (see [22])). *An almost complex structure J is a complex structure if and only if it is* integrable*, that is the* Nijenhuis tensor

$$N_J(X, Y) := [JX, JY] - [X, Y] - J([JX, Y] + [X, JY]) \tag{1.1}$$

vanishes $\forall X, Y \in \Gamma(TM)$.

We use Γ to indicate smooth sections of a fiber bundle.
Since N_J is a $(1, 2)$-tensor, we can define

$$N_J(X, Y) := [JX', JY'] - [X', Y'] - J([JX', Y'] + [X', JY'])$$

$\forall X, Y \in T_pM$ and $\forall p \in M$, where X', Y' are vector fields that coincide with X and Y at p respectively.

1.2 Almost CR manifolds and CR manifolds

Let M be a differentiable manifold endowed with a (almost) complex structure J and N any submanifold of M. The vector space defined as

$$H_pN := T_pN \cap J(T_pN), \qquad \forall p \in N$$

is the largest J-invariant subspace of the tangent space T_pM and it is called *holomorphic tangent space*. When the dimension of H_pN does not depend on the point p, then $HN := \bigcup_{p \in N} H_pN$ gives a fiber bundle which is called *holomorphic tangent bundle* and the manifold N is an *(immersed) almost CR manifold*.

Definition 1.4. A differentiable manifold M is an *(abstract) almost CR manifold* [1] if there exist a subbundle HM of TM and an *almost CR structure*, that is, an almost complex structure $J : HM \to HM$ with $J^2 = -id$, such that

$$[X, Y] - [JX, JY] \in \Gamma(HM), \qquad \forall X, Y \in \Gamma(HM), \tag{1.2}$$

or equivalently

$$[X, JY] + [JX, Y] \in \Gamma(HM), \qquad \forall X, Y \in \Gamma(HM). \tag{1.3}$$

[1] Almost CR manifolds are a generalization of almost complex manifolds with $HM \neq TM$.

The dimension of H_pM, with $p \in M$, is even.

Definition 1.5. If $rk_{\mathbb{R}}HM = 2n$, we say that n is the *CR dimension* of M and we indicate it as

$$\dim_{CR} M = n.$$

Writing the dimension of M as $m = 2n + k$, we say that k is the (real) *CR codimension* of M and we write

$$codim_{CR}M := \dim M - 2n.$$

When $k = 1$, we call M of *hypersurface* type.

***Remark* 1.1.** Let $A, B \in \Gamma(HM)$. By definition of almost CR manifold we have,

$$[A, B] - [JA, JB] \in \Gamma(HM) \quad \text{ and } \quad [JA, B] + [A, JB] \in \Gamma(HM),$$

so also

$$N_J(A, B) := [JA, JB] - [A, B] - J([JA, B] + [A, JB]) \in \Gamma(HM).$$

Moreover, the following properties hold:

(i) $N_J(A, JB) = -JN_J(A, B) = N_J(JA, B)$,

(ii) $N_J(JA, JB) = -N_J(A, B)$,

(iii) $N_J(A, B) = -N_J(B, A)$,

(iv) $N_J(A, A) = 0$.

Definition 1.6. An almost CR manifold (M, J) such that

$$N_J(X, Y) = 0, \qquad \forall X, Y \in \Gamma(HM)$$

is called *CR manifold*.

Lemma 1.2.1. *Let (M^m, HM, J) be an almost CR manifold endowed with a holomorphic tangent bundle, with CR dimension $\dim_{CR} M = 1$. Then (M^m, HM, J) is a CR manifold.*

Proof. If $X \in \Gamma(HM)$, also $JX \in \Gamma(HM)$ (by definition of J) and every $Y \in \Gamma(HM)$ can be written as $Y = aX + bJX$, for some differentiable functions a, b. Then, $\forall X, Y \in \Gamma(HM)$, we have

$$\begin{aligned} N_J(X, Y) &= N_J(X, aX + bJX) = \\ &= aN_J(X, X) + bN_J(X, JX) = \\ &= -bJN_J(X, X) = 0. \end{aligned}$$

□

***Remark* 1.2.** If $m = 2$, we have that $HM = TM$ and so (M^2, HM, J) is a complex manifold, in other words, (M^2, HM, J) is a Riemann surface.

1.3 Almost Kähler manifolds and Kähler manifolds

Definition 1.7. Let (M, J) be an almost complex manifold of dimension $2n$ with an almost complex structure J. We say that $(M, J, \mathcal{G})$ is an *almost Kähler manifold* if

(i) $\mathcal{G}$ is a Riemannian metric such that $\mathcal{G}(JX, JY) = \mathcal{G}(X, Y)$,

(ii) the fundamental 2-form Ω, defined as $\Omega(X, Y) = \mathcal{G}(X, JY)$, is such that $d\Omega = 0$.

An almost Kähler manifold is called *Kähler manifold* if the Nijenhuis tensor vanishes.

1.4 Principal fiber bundles and G-structures

In this section we recall some well-known notions on principal fiber bundles (see [37, 16, 30]).

Definition 1.8. Let M be a manifold and G be a Lie group. A *principal fiber bundle* on M, with structure Lie group G, is a manifold P on which is defined a right action

$$R : \begin{cases} G \times P & \to P \\ (g, p) & \mapsto p \cdot g \end{cases}$$

such that

1. the action R is free, that is, if $\exists p \in P$ such that $p \cdot g = p$, then $g = id_G$;
2. the canonical projection $\pi : P \to M$ is differentiable;
3. M is the orbit space P/G, that is, if $p, q \in P$ then
$$\pi(p) = \pi(q) \Leftrightarrow (\exists g \in G : p \cdot g = q);$$
4. P is locally trivial, that is, $\forall x \in M, \exists U$ neighborhood of x such that $\pi^{-1}(x)$ is G-equivalent to $U \times G$.

The manifold M is called *base* of the fiber P and the set $\pi^{-1}(x)$ (for all $x \in M$) is the *fiber on x*.

Definition 1.9. Given a principal fiber bundle P on a manifold M, a *local section* on U, open set on M, is a smooth application

$$\sigma : U \subseteq M \to P$$

such that $\pi \circ \sigma = id|_U$.

The fiber bundle P is called *trivial* when P is G-equivalent to $M \times G$, in other words, there exists a global section

$$\sigma : M \to P.$$

Definition 1.10. Let P_1 e P_2 be two fiber bundles with structure groups G_1 e G_2 respectively, such that $G_1 \subseteq G_2$.

We say that P_1 is a *subbundle* of P_2 if there exists a regular immersion $\phi : P_1 \to P_2$ which is G_1-equivariant.

We say that P_1 is a *reduction* of P_2 (with structure group G_1) if the induced regular immersion between the basis, $\tilde{\phi} : M_1 \to M_2$, is surjective ($\tilde{\phi}(M_1) = M_2$).

Definition 1.11. Let M be a manifold with dimension n. A *linear frame* f_x at a point $x \in M$ is a non singular linear map

$$f_x : \mathbb{R}^n \to T_x M.$$

If $(e_1, ..., e_n)$ is the canonical base of $\mathbb{R}^n$, a linear frame is equivalent to give a base of the tangent space $(X_1, ..., X_n)$ at a point $x \in M$ given by $f_x(e_i) = X_i$ for $i = 1, ..., n$.

The vector space $\mathbb{R}^n$ is called *model vector space*.

Proposition 1.4.1. *The set of all linear frames on a manifold M, denoted with $L(M)$, forms a principal bundle with structure group $GL(n, \mathbb{R})$.*

Definition 1.12. Let G be a Lie subgroup of $GL(n, \mathbb{R})$. A *G-structure* on a manifold M is a reduction P_G of the frame fiber $L(M)$ with structure group G. So P_G is a submanifold of $L(M)$ with the property

$$\forall p \in P_G, \forall g \in GL(n, \mathbb{R}) \text{ we have that } p \cdot g \in P_G \Leftrightarrow g \in G.$$

***Remark* 1.3.** When $G = \{e\}$, the neutral element of the group, there is a biunivocal correspondence between the $\{e\}$-structures on M and the fields $(X_1, X_2, ..., X_n)$ of linear frame on M. Hence, the following definition makes sense.

Definition 1.13. A manifold M is called *parallelizable* when it admits an $\{e\}$-structure (that is so called *absolute parallelism*).

Theorem 1.4.2 (Kobayashi, see [17] and [30])**.** *Let $G = Aut(\{X_i\})$ be the group of the automorphisms of an $\{e\}$-structure*

$$G = \{\varphi : M \to M; \quad \varphi_* X_i = X_i \quad i = 1, ..., n\}.$$

The group G is a Lie group of transformations of M and it has dimension

$$\dim G \leq \dim M.$$

In particular, for any point x on M, the map

$$\begin{cases} Aut(\{X_i\}) & \to & M \\ \varphi & \mapsto & \varphi(x) \end{cases}$$

is injective and its image $\{\varphi(x), \varphi \in Aut(\{X_i\})\}$ is a closed regular submanifold of M. The submanifold structure of the orbit $G(x)$ induced by M is compatible with the Lie group structure of G.

1.5 {e}-structures and structure functions

For more information about this section, see [30] and [37].

Let us fix an absolute parallelism $(X_1, ..., X_n)$ on the manifold M; it is the same that to fix a global section

$$\gamma : M \to L(M),$$

where the vector fields $(X_1, ..., X_n)$ are defined by

$$X_{i|p} = \gamma_p(e_i), \qquad i = 1, ..., n,$$

and $(e_1, ..., e_n)$ is the canonical base of $\mathbb{R}^n$.

We call *torsion of the $\{e\}$-structure* γ the application

$$c_\gamma : M \to Hom(\Lambda^2\mathbb{R}^n, \mathbb{R}^n)$$

$$c_\gamma(v_1 \wedge v_2)_{|p} = \gamma^{-1}([\gamma(v_1), \gamma(v_2)]_p).$$

If $w^1, ..., w^n$ are the 1-forms given by the dual bases of the bases defined by $X_i = \gamma(e_i)$, we can determine the components c^k_{ij} of c_γ with respect to the base $(e_1, ..., e_n)$ as following:

$$c_\gamma(e_i \wedge e_j) = \sum_k c^k_{ij} e_k = \sum_{k=1}^n w^k([X_i, X_j])e_k.$$

The functions $c^k_{ij} = w^k([X_i, X_j])$ are called *structure functions* of γ.

***Remark* 1.4.** When M is a Lie group G and the {e}-structure is given by the left invariant vector fields, the structure functions are constants and are called *structure constants* of G.

To prove the opposite implication we need the hypothesis of completeness on the vector fields defined on M, as we see in the following Lemma.

Lemma 1.5.1. *Let M^n be a connected differentiable manifold and let $X_1, ...,$ X_n be n complete independent fields at every point of M^n such that the structure functions are constant. Then it is defined a group structure G on M^n such that the Lie algebra $\mathfrak{g}$ of G is generated by $X_1, ..., X_n$.*

Proof. Let us fix a point x_0 on M^n and let x and y be two generic points of M^n. Since M^n is connected, there exist $\varphi_{t_1}^{X_{i(1)}}, \varphi_{t_2}^{X_{i(2)}}, ..., \varphi_{t_p}^{X_{i(p)}}$ and $\varphi_{s_1}^{Y_{j(1)}}$, $\varphi_{s_2}^{Y_{j(2)}}, ..., \varphi_{s_q}^{Y_{j(q)}}$, such that $x = \varphi_{t_1}^{X_{i(1)}} \circ \cdots \circ \varphi_{t_p}^{X_{i(p)}}(x_0)$ and $y = \varphi_{s_1}^{Y_{j(1)}} \circ \cdots \circ \varphi_{s_q}^{Y_{j(q)}}(x_0)$, where each $\varphi_{t_k}^{X_{i(k)}}$ is the flux of $X_{i(k)}$ with $t_k \in \mathbb{R}$ and $X_{i(p)}$, $Y_{i(q)}$ vary in $(X_1, ..., X_n)$.

For simplicity, we put $\varphi_t^X := \varphi_{t_1}^{X_{i(1)}} \circ \cdots \circ \varphi_{t_p}^{X_{i(p)}}$ and $\varphi_s^Y := \varphi_{s_1}^{Y_{j(1)}} \circ \cdots \circ \varphi_{s_q}^{Y_{j(q)}}$. We can define the product of x and y as

$$x \cdot y := R_y(x) = L_x(y) = \varphi_s^Y \circ \varphi_t^X(x_0).$$

It is easy to check that M^n, endowed with this product, is a Lie group. We show that $X_1, ..., X_n$ are left invariants vector fields for M^n. For every $g \in M^n$ we have that $L_{g*}X_\xi = X_\xi$ (in other words, $X_1, ..., X_n$ are left invariants vector fields) if and only if $L_g \circ \varphi_{t_\xi}^{X_\xi} = \varphi_{t_\xi}^{X_\xi} \circ L_g$, for all $\xi = 1, ..., n$.

Since

$$(L_g \circ \varphi_{t_\xi}^{X_\xi})(y) = L_g(\varphi_{t_\xi}^{X_\xi} \circ \varphi_s^Y(x_0)) = \varphi_{t_\xi}^{X_\xi} \circ \varphi_s^Y \circ \varphi_u^Z(x_0),$$

where $g = \varphi_u^Z(x_0) := \varphi_{u_1}^{Z_{k(1)}} \circ \varphi_{u_2}^{Z_{k(2)}} \circ \cdots \circ \varphi_{u_r}^{Z_{k(r)}}(x_0)$, and

$$(\varphi_{t_j}^{X_\xi} \circ L_g)(y) = \varphi_{t_\xi}^{X_\xi} \circ \varphi_s^Y \circ \varphi_u^Z(x_0)$$

are equal, we have that $X_1, ..., X_n$ are left invariants vector fields for M^n and so $\mathfrak{g} =< X_1, ..., X_n >$. □

The following Lemma will be useful later.

Lemma 1.5.2. *The Lie bracket between right and left invariant vector fields of any Lie group G is zero.*

Proof. If X is a left invariant vector field and $\overline{Y}$ is a right invariant vector field, we denote with φ_t^X the flux of X at time t and $\varphi_s^{\overline{Y}}$ the flux of $\overline{Y}$ at time s. Note that any diffeomorphism commutes with every left translation L_g, with $g \in G$, if and only if it is a right multiplication. Then, since X is a left invariant vector field if and only if $L_g \circ \varphi_t^X = \varphi_t^X \circ L_g$, there must exists a $h \in G$ such that $\varphi_t^X = R_h$, where R_h is the right multiplication of h. For the same reason there exists a $h' \in G$ such that $\varphi_s^{\overline{Y}} = L_{h'}$. So we have:

$$\varphi_t^X \circ \varphi_s^{\overline{Y}} - \varphi_s^{\overline{Y}} \circ \varphi_t^X = R_h \circ L_{h'} - L_{h'} \circ R_h = 0,$$

that is equivalent to (see [16] pag.16)

$$[X, \overline{Y}] = 0.$$

□

Chapter 2

Almost complex manifolds

2.1 Almost complex manifolds of dimension 2n

Let (M^{2n}, J) be an almost complex manifold of real dimension $2n$. For any $p \in M$, we set

$$\mathcal{V}_p = \{X \in T_pM : X = N_J(A, B) \text{ for some } A, B \in T_pM\}.$$

If (M^{2n}, J) is a complex manifold, then $\mathcal{V}_p = \{0\}$ and $\bigcup_{p\in M} \mathcal{V}_p$ is the trivial bundle.

Lemma 2.1.1. *$\mathcal{V}_p$ is J-invariant.*

Proof. It follows from (i) of Remark (1.1). □

Definition 2.1. Suppose that the rank of $\mathcal{V}_p$ is constant $\forall p \in M^{2n}$, we have that

$$\mathcal{V} := \bigcup_{p\in M} \mathcal{V}_p$$

forms a bundle called *torsion bundle.*

Definition 2.2. Let $\mathcal{V}$ be the torsion bundle, $\mathcal{V}$ is called *non-degenerate at* $p \in M^{2n}$ if we have

$$[X, Y]_p \notin \mathcal{V}_p,$$

for some $X, Y \in \Gamma(\mathcal{V})$. The torsion bundle $\mathcal{V}$ is called *non-degenerate* if it is non-degenerate at all $p \in M^{2n}$.

***Remark* 2.1.** If $\mathcal{V}$ is non-degenerate at a point p, then $\mathcal{V}$ is non-degenerate in a neighborhood of p.

Let us consider the complexification of the tangent bundle $\mathbb{C}TM := \mathbb{C} \otimes_{\mathbb{R}} TM$ and let J be the $\mathbb{C}$-linear extension to it, in such a way to have

$J : \mathbb{C}TM \to \mathbb{C}TM$. We have that J admits two eigenvalues i and $-i$, whose eigenspaces are

$$T^{1,0}M = \{X - iJX : X \in TM\} \quad \text{and} \quad T^{0,1}M = \{X + iJX : X \in TM\}$$

respectively. So, it is possible to decompose $\mathbb{C}TM$ in the direct sum:

$$\mathbb{C}TM = T^{1,0}M \oplus T^{0,1}M.$$

A section of $T^{1,0}M$ is called a $(1,0)$-*vector field.*

***Remark* 2.2.** The integrability (see Theorem 1.1.1) of an almost complex structure is equivalent to say that

$$[\Gamma(T^{1,0}M), \Gamma(T^{1,0}M)] \subseteq \Gamma(T^{1,0}M),$$

where we recall that Γ indicate the smooth sections of the fiber bundle $T^{1,0}M$.

***Remark* 2.3.** The Nijenhuis tensor (see (1.1)) of two $(1,0)$-vector fields is a $(0,1)$-vector field and vice versa; while the Nijenhuis tensor of two vector fields of different type is null. In particular, $\forall X \in \Gamma(TM)$ and $\forall Y \in \Gamma(TM)$ we have

(i) $N_J(X - iJX, Y - iJY) = 2(N_J(X,Y) + iJN_J(X,Y))$,

(ii) $N_J(X + iJX, Y + iJY) = 2(N_J(X,Y) - iJN_J(X,Y))$,

(iii) $N_J(X + iJX, Y - iJY) = 0$.

If we denote as $X^{1,0}$ the component in $T^{1,0}M$ of a vector field $X \in \Gamma(TM)$, we also have that

(iv) $[X - iJX, Y - iJY]^{0,1} = -\dfrac{1}{4}N_J(X - iJX, Y - iJY)$,

(v) $[X + iJX, Y + iJY]^{1,0} = -\dfrac{1}{4}N_J(X + iJX, Y + iJY)$.

2.1.1 Infinitesimal automorphisms

Definition 2.3. Let (M^{2n}, J) be an almost complex manifold[1]. An *infinitesimal automorphism* is a tangent vector field $V \in \Gamma(TM)$ that satisfies

$$[V, JX] = J[V, X], \qquad \forall X \in \Gamma(TM). \tag{2.1}$$

[1] This definition can be generalized to the case in which (M^{2n}, J) is an almost CR manifold: it is sufficient to consider the equation (2.1) with $X \in \Gamma(HM)$.

***Remark* 2.4.** The condition (2.1) is equivalent to

$$[V, Z] \in \Gamma(T^{1,0}M), \qquad \forall Z \in \Gamma(T^{1,0}M).$$

It is also equivalent to $\mathcal{L}_V J = 0$, where $\mathcal{L}_V$ is the Lie derivative of the vector field V. Therefore, an infinitesimal automorphism is a Lie derivative that commutes with J.

Definition 2.4. The *symmetry algebra* $aut_p(M, J)$ of an almost complex manifold (M, J) at $p \in M$ is the set of germs of infinitesimal automorphisms at p.

Definition 2.5. We denote with $aut^0_p(M^4, J)$ the set of infinitesimal automorphisms that fix the point $p \in M^4$, that is, when $\mathcal{A} \in aut^0_p(M^4, J)$ we have that $\mathcal{A}_p = 0$. We call $aut^0_p(M^4, J)$ the *isotropy algebra* of p.

For a compact almost complex manifold the following theorem, due to Boothby, Kobayashi and Wang, holds.

Theorem 2.1.2. *(see [8] and [17] Corollary 4.2 pag.19) The automorphism group of a* compact *almost complex manifold* (M, J) *is a Lie transformation group.*

2.2 Almost complex manifolds of dimension 4

Proposition 2.2.1. *Let* (M^4, J) *be an almost complex manifold with real dimension 4 and with* $N_J \neq 0$ *at* $p \in M$*. Then* $\mathcal{V}_p$ *has real dimension 2.*

Proof. Since $\mathcal{V}_p$ is J-invariant, $\mathcal{V}_p$ is even dimensional, so its dimension is 2 or 4. If (X, JX, Y, JY) forms a base of T_pM, we have that

$$N_J(X, X) = 0 \text{ and } N_J(X, JX) = -JN_J(X, X) = 0,$$

$$N_J(Y, Y) = 0 \text{ and } N_J(Y, JY) = -JN_J(Y, Y) = 0;$$

hence, $\mathcal{V}_p$ is locally generated by $N_J(X, Y)$ and $N_J(X, JY)$.

□

***Remark* 2.5.** Under the assumption of the Proposition 2.2.1, we have that

- $\forall X \in \mathcal{V}_p$, with $X \neq 0$, we have that (X, JX) gives a base of $\mathcal{V}_p$;
- there exists a neighborhood U of p such that $\forall q \in U$ the dimension of $\mathcal{V}_q$ is constant (and is 2).

***Remark* 2.6.** By Lemma 1.2.1, we have that $(M^4, \mathcal{V}, J_{|\mathcal{V}})$ is a CR manifold; in particular, if $A, B \in \mathcal{V}_p$, then $N_J(A, B) = 0$.

Let (M^4, J) be an almost complex manifold with real dimension 4, with $N_J \neq 0$ at p and such that $\mathcal{V}$ is non-degenerate (see Definition 2.2). Then

$$[\Gamma(\mathcal{V}), \Gamma(\mathcal{V})] \nsubseteq \Gamma(\mathcal{V}) \qquad (\mathcal{V} \text{ is non-degenerate})$$

and we set

$$\mathcal{V}_{-1} := \Gamma(\mathcal{V}),$$

$$\mathcal{V}_{-2} := \Gamma(\mathcal{V}) + [\Gamma(\mathcal{V}), \Gamma(\mathcal{V})] \neq \Gamma(\mathcal{V}).$$

From here on we always suppose that $\mathcal{V}$ is **non-degenerate** at a fixed point p, and so it is possible to construct the following vector spaces:

$$\mathcal{V}_{-1|p} := \mathcal{V}_p,$$

$$\mathcal{V}_{-2|p} := \mathcal{V}_p + [\Gamma(\mathcal{V}), \Gamma(\mathcal{V})]_p \neq \mathcal{V}_p.$$

Proposition 2.2.2. *If $X \in \Gamma(\mathcal{V}_{-1})$ is a vector field such that $X_p \neq 0$, we have that*

$$(X_p, JX_p, T_p)$$

is a base of $\mathcal{V}_{-2|p}$, where we set $T_p := [X, JX]_p$.

Proof. Let $X \in \Gamma(\mathcal{V}_{-1})$ be a vector field such that $X_p \neq 0$. Then (X_p, JX_p) forms a base of $\mathcal{V}_{-1|p}$.

Since we are considering the case in which $\mathcal{V}$ is non-degenerate at p, we have that $0 \neq T_p \notin \mathcal{V}_{-1|p}$ and in particular, the vectors (X_p, JX_p, T_p) form a base of $\mathcal{V}_{-2|p}$. So $\dim_{\mathbb{R}} \mathcal{V}_{-2|p} \geq 3$ and can not be $\dim_{\mathbb{R}} \mathcal{V}_{-2|p} = 4$, because the Lie bracket of two generic vector fields of $\mathcal{V}_{-1|p}$

$$[\alpha(p)X + \beta(p)JX, \gamma(p)X + \delta(p)JX]_p$$

(where $\alpha(p), \beta(p), \gamma(p), \delta(p)$ are functions depending on p) can be written as linear combination of (X_p, JX_p, T_p).

□

Iterating the Lie brackets, we have the following cases:

$$[\Gamma(\mathcal{V}), [\Gamma(\mathcal{V}), \Gamma(\mathcal{V})]]_p \subseteq \mathcal{V}_p + [\Gamma(\mathcal{V}), \Gamma(\mathcal{V})]_p = \mathcal{V}_{-2|p}, \tag{$*$}$$

$$[\Gamma(\mathcal{V}), [\Gamma(\mathcal{V}), \Gamma(\mathcal{V})]]_p \nsubseteq \mathcal{V}_p + [\Gamma(\mathcal{V}), \Gamma(\mathcal{V})]_p = \mathcal{V}_{-2|p}. \tag{$**$}$$

***Remark* 2.7.** If $p \in M$ is such that the second case $(**)$ happens, then the $(**)$ also holds in a neighborhood U of p.

Definition 2.6. When the case $(**)$ occurs, then $\mathcal{V}_p$ is called *fundamental*, otherwise $\mathcal{V}_p$ is called *non-fundamental*.

If we set

$$\mathcal{V}_{-3|p} := \underbrace{\mathcal{V}_p + [\Gamma(\mathcal{V}), \Gamma(\mathcal{V})]_p}_{= \mathcal{V}_{-2|p}} + [\Gamma(\mathcal{V}), [\Gamma(\mathcal{V}), \Gamma(\mathcal{V})]]_p,$$

we have that

$$\mathcal{V}_{-3|p} = \begin{cases} \mathcal{V}_{-2|p} & \text{if } (*) \text{ occurs,} \\ T_pM & \text{if } (**) \text{ occurs.} \end{cases}$$

In particular, the dimension of $\mathcal{V}_{-3|p}$ must be:

$$\dim_{\mathbb{R}} \mathcal{V}_{-3|p} = \begin{cases} 3 & \text{if } (*) \text{ occurs,} \\ 4 & \text{if } (**) \text{ occurs.} \end{cases}$$

In both cases we obtain a filtration of the tangent space T_pM:

$$\mathcal{V}_p := \mathcal{V}_{-1|p} \subseteq \mathcal{V}_{-2|p} \subseteq \mathcal{V}_{-3|p} \subseteq T_pM$$

(we have an equality between $\mathcal{V}_{-2|p}$ and $\mathcal{V}_{-3|p}$ when $(*)$ holds, while the equality is between $\mathcal{V}_{-3|p}$ and T_pM when $(**)$ holds).

2.3 Absolute parallelisms on (M^4, J)

In this and in the next section, we will consider (M^4, J) as an almost complex manifold with real dimension 4 and with non-degenerate torsion bundle $\mathcal{V}$.

***Remark* 2.8.** From Theorem 2.1 in [20] and Theorem 1.5 in [13], since (M^4, J) is an almost complex manifold having Nijenhuis tensor of rank 2, the number of independent holomorphic functions on (M^4, J) is at most 1. More precisely, because of the non degeneracy of the torsion bundle $\mathcal{V}$, there are not holomorphic functions on (M^4, J).

***Remark* 2.9.** When (M^4, J) is an almost complex manifold of real dimension 4, one can write the normal form of its non-integrable almost complex structures J (see [39]).

Definition 2.7. For $p \in M^4$ and $A \in \Gamma(\mathcal{V})$, with $A_p \neq 0$, we set

$$T_p^A := [A, JA]_p.$$

Note that $[A, JA]_p \notin \mathcal{V}_p$, because $\mathcal{V}$ is non-degenerate.

Proposition 2.3.1. *If A' is a vector field of $\mathcal{V}$ that coincides with A at p, we have*

$$T_p^A \equiv T_p^{A'} \mod \mathcal{V}_p.$$

Proof. It is clear that T_p^A depends on the field $A \in \Gamma(\mathcal{V})$ by definition. Since, by varying the point p in M, we have that (A_p, JA_p) forms a base of $\mathcal{V}_p$, we can write

$$A' = fA + gJA,$$

with f and g differentiable functions such that $f(p) = 1$ and $g(p) = 0$. Then we have

$$\begin{aligned}
[A', JA']_p = \; & [fA + gJA, fJA - gA]_p \\
= \; & \big(f(p)^2 + g(p)^2\big)[A, JA]_p + \\
& + \Big(\big(g(p) - f(p)\big)JA(f)_p - f(p)A(g)_p - g(p)JA(g)_p\Big)A_p + \\
& + \Big(f(p)A(f)_p + g(p)JA(f)_p - f(p)JA(g)_p - g(p)A(g)_p\Big)JA_p \\
= \; & [A, JA]_p - \big(A(g)_p + JA(f)_p\big)A_p + \big(A(f)_p - JA(g)_p\big)JA_p.
\end{aligned}$$

Hence, in general

$$[A', JA']_p \neq [A, JA]_p.$$

Quotienting modulo $\mathcal{V}_p$, we have

$$[A', JA']_p \equiv [A, JA]_p \mod \mathcal{V}_p.$$

$\square$

Definition 2.8. We call $\tau_{T_p^A}$ the linear application associated to T_p^A and defined by

$$\tau_{T_p^A} : \begin{cases} \mathcal{V}_p & \to \mathcal{V}_p \\ X_p & \mapsto N(X_p, T_p^A), \end{cases}$$

where T_p^A is a fixed vector which is not null, since $A_p \neq 0$ (it is a consequence of the previous proposition).

***Remark* 2.10.** The application $\tau_{T_p^A}$ is not the null application and we have

$$\tau_{T_p^A} \circ J_{|\mathcal{V}_p} = -J_{|\mathcal{V}_p} \circ \tau_{T_p^A}.$$

Since (A_p, JA_p) gives a base of $\mathcal{V}_p$, the matrix which represents $\tau_{T_p^A}$ with respect to this base is of the form

$$\begin{pmatrix} a & b \\ b & -a \end{pmatrix},$$

with $a, b \in \mathbb{R}$. The matrix is symmetric with null trace, hence it is diagonalizable and it has two not null opposite eigenvalues:

$$\lambda^+ = \sqrt{a^2 + b^2} \qquad \text{and} \qquad \lambda^- = -\sqrt{a^2 + b^2}.$$

If we set $\mathcal{V}_p^+$ and $\mathcal{V}_p^-$ the eigenspaces associated to the two eigenvalues λ^+ and λ^-, we have

$$J\mathcal{V}_p^+ = \mathcal{V}_p^-.$$

Proposition 2.3.2. *Varying the field $A \in \Gamma(\mathcal{V})$, with $A_p \neq 0$, the linear application $\tau_{T_p^A}$ varies of a positive constant, in other words: $\forall B \in \Gamma(\mathcal{V}), B_p \neq 0$, we have*

$$\tau_{T_p^B} = \alpha \tau_{T_p^A}, \qquad \textit{with} \qquad \alpha > 0.$$

Proof. Let $B \in \Gamma(\mathcal{V})$, $B_p \neq 0$, then $T_p^B = [B, JB]_p \notin \mathcal{V}_p$. Since $T_p^B \in \mathcal{V}_{-2|p}$ and (A_p, JA_p, T_p^A) gives a base of $\mathcal{V}_{-2|p}$, we can write T_p^B as linear combination of vectors of this base

$$T_p^B = \alpha T_p^A + \beta A_p + \gamma J A_p$$

for certain $\alpha, \beta, \gamma \in \mathbb{R}$.

Since B is a vector field of $\mathcal{V}$, it can be write as

$$B = fA + gJA,$$

for certain functions $f, g \in \mathcal{C}^\infty$, with $f(p)^2 + g(p)^2 \neq 0$. Hence, we have the following equalities, similarly to the calculation given in the proof of Proposition 2.3.1:

$$\begin{aligned} T_p^B &= [B, JB]_p \\ &= [fA + gJA, fJA - gA]_p \\ &= [fA, fJA]_p - [fA, gA]_p + [gJA, fJA]_p - [gJA, gA]_p \\ &= \underbrace{(f^2 + g^2)}_{=\alpha > 0}[A, JA]_p + \beta A_p + \gamma J A_p \end{aligned}$$

ans so α must necessarily be positive. Furthermore

$$\begin{aligned} \tau_{T_p^B}(X_p) &= N(X_p, T_p^B) \\ &= N(X_p, \alpha[A, JA]_p + \beta A_p + \gamma J A_p) \\ &= \alpha N(X_p, [A, JA]_p) + \beta N(X_p, A_p) + \gamma N(X_p, JA_p) \\ &= \alpha N(X_p, [A, JA]_p) \\ &= \alpha \tau_{T_p^A}(X_p), \end{aligned}$$

where $\beta N(X_p, A_p)$ and $\gamma N(X_p, JA_p)$ are zero because the Nijenhuis tensor of fields in $\mathcal{V}$ is always zero. $\square$

From the previous proof, we deduce that $f(p) = 1$ and $g(p) = 0$ when $B_p = A_p$, so the following corollary holds.

Corollary 2.3.3. *If a field B coincides with the field A at a point p, then we have that $\tau_{T_p^A} = \tau_{T_p^B}$. In other words, τ_{T_p} does not depend on the choice of the field A, but only from its value at the point.*

Corollary 2.3.4. *The eigenspaces $\mathcal{V}_p^\pm$ do not depend on the choice of the field A in the definition of $\tau_{T_p^A}$.*

Proof. We have that $\tau_{T_p^B}(X_p) = \alpha \tau_{T_p^A}(X_p)$ with $\alpha > 0$, so the matrix associated to $\tau_{T_p^B}$ is of the form

$$\alpha \begin{pmatrix} a & b \\ b & -a \end{pmatrix},$$

which has eigenvalues $\alpha\lambda^+$ and $\alpha\lambda^-$. Hence, the associated eigenspaces are

$$\mathcal{V}_p^{\pm}(\tau_{T_p^B}) = \mathcal{V}_p^{\pm}(\tau_{T_p^A}).$$

□

Let U and V be two open sets of M containing the point p and let $A \in \Gamma(U, \mathcal{V})$ and $B \in \Gamma(V, \mathcal{V})$ be two sections of $\mathcal{V}$ such that $A_q \neq 0$ and $B_q \neq 0$, for all $q \in U \cap V$. Because of the previous corollary, the eigenspace associated to the field B, $\mathcal{V}_q^{\pm}(\tau_{T_q^B})$, coincide with the eigenspace associated to the field A, $\mathcal{V}_q^{\pm}(\tau_{T_q^A})$, at each point q of $U \cap V$. Then the following notation makes sense.

We denote by

$$\mathcal{V}^{\pm} = \bigcup_{p \in M} \mathcal{V}_p^{\pm}$$

the fiber bundles obtained by varying $p \in M$.

***Remark* 2.11.** We have the following filtration of the fiber bundle of the almost complex manifold M:

$$\mathcal{V}^+ \subsetneqq \mathcal{V} \subsetneqq \mathcal{V}_{-2} \subsetneqq TM.$$

Proposition 2.3.5. *A distinguished section X is locally determined [2] in $\mathcal{V}^+$; such a section is unique up to sign and is such that $\tau_{T_q^X}$ has eigenvalue 1, for all q in an open neighborhood U of $p \in M$.*

Proof. Let U be a neighborhood of $p \in M$ such that $Z : U \to \mathcal{V}^+$ is a section of $\mathcal{V}^+$, with $Z_q \neq 0$ for all $q \in U$. We want to find another section $Y : U \to \mathcal{V}^+$, such that $\tau_{T_p^Y}$ has eigenvalue 1.

Computing $\tau_{T_p^Z}$, we have

$$\tau_{T_p^Z}(Z_p) = \lambda_p^+ Z_p,$$

for a certain positive eigenvalue $\lambda_p^+ \in \mathbb{R}^+$. If we set $Y_p = cZ_p$, with $c \in \mathbb{R}$, we obtain

[2] A similar result have been obtained independently by Kruglikov in [18].

$$\begin{aligned}\tau_{T_p^Y}(Z_p) &= N(Z_p, T_p^Y)\\ &= N(Z_p, [Y, JY]_p)\\ &= N(Z_p, [cZ, cJZ]_p)\\ &= c^2 N(Z_p, [Z, JZ]_p)\\ &= c^2 N(Z_p, T_p^Z)\\ &= c^2 \tau_{T_p^Z}(Z_p)\\ &= c^2 \lambda_p^+ Z_p.\end{aligned}$$

Choosing the constant c as

$$c = \pm \frac{1}{\sqrt{\lambda_p^+}},$$

we have that

$$\tau_{T_p^Y}(Z_p) = Z_p,$$

in other words, $Y : U \to \mathcal{V}^+$ is a section such that Z_p is an eigenvector associated to the eigenvalue 1. In particular, the section Y is univocally determined in $\mathcal{V}^+$ up to the sign:

$$Y_p = \pm \frac{1}{\sqrt{\lambda_p^+}} Z_p. \tag{2.2}$$

Let $q \in U$, then

$$\tau_{T_q^Y}(Z_q) = \lambda_q^+ Z_q,$$

for a certain $\lambda_q^+ \in \mathbb{R}^+$ depending on q in U and $\lambda_p^+ = 1$. We want to find a section $X : U \to \mathcal{V}^+$ such that the positive eigenvalue λ_q^+ of $\tau_{T_q^X}$ is 1 for every point $q \in U$. Such a section is univocally determined when the sign of Y in (2.2) is determined.

All the sections of $\mathcal{V}^+$ coinciding to Y at p are of the form $X = fY$, with $f \in \mathcal{C}^\infty(U)$, $f(p) = 1$ and $f(q) \neq 0$, $\forall q \in U$; in particular, $f > 0$.

We have that the eigenvalue of Z_q computed with respect to $\tau_{T_q^X}$ is 1 for all points $q \in U$ when

$$\begin{aligned}\tau_{T_q^X}(Z_q) = Z_q \quad &\Leftrightarrow \quad \tau_{T_q^{fY}}(Z_q) = Z_q\\ &\Leftrightarrow \quad f^2 \tau_{T_q^Y}(Z_q) = Z_q\\ &\Leftrightarrow \quad f^2 \lambda_q^+ Z_q = Z_q\\ &\Leftrightarrow \quad f = +\tfrac{1}{\sqrt{\lambda_q^+}} \qquad \text{(because } f > 0\text{)}.\end{aligned}$$

Therefore, the section X we are looking for is univocally determined (up to the sign) on varying q in U, and it is

$$X_q = \frac{1}{\sqrt{\lambda_q^+}} Y_q.$$

□

Definition 2.9. We say that the section X, which is univocally determined up to sign in Proposition 2.3.5, is the *distinguished section* of $\mathcal{V}^+$.

Definition 2.10. For any point $p \in M$, the isomorphisms

$$f_p : \begin{cases} (\mathbb{R}^4, J) & \to (T_pM, J) \\ (e_1, e_2, e_3, e_4) & \mapsto (X_p, JX_p, T_p^X, JT_p^X) \end{cases}$$

are called *adapted frames* at a point p, where (e_1, e_2, e_3, e_4) is the canonical base of $\mathbb{R}^4$ and $(X_p, JX_p, T_p^X, JT_p^X)$ is a base of T_pM, with X the distinguished section of $\mathcal{V}^+$.

***Remark* 2.12.** If X is the distinguished section of $\mathcal{V}^+$ relative to the positive eigenvalue, that is $N_J(X, T^X) = X$, with X in $\Gamma(\mathcal{V}^+)$, we note that JX is the distinguished section of $\mathcal{V}^-$ relative to the negative eigenvalue:

$$N_J(JX, T^{JX}) = N_J(JX, [JX, -X]) = -JN_J(X, [X, JX]) = -JX.$$

***Remark* 2.13.** Since $T^X = [X, JX] = [-X, -JX] = T^{-X}$, with X the distinguished section of $\mathcal{V}^+$, we have that T and JT do not depend on the sign, hence we have two adapted frames: $(X_p, JX_p, T_p^X, JT_p^X)$ and $(-X_p, -JX_p, T_p^X, JT_p^X)$.

From the Proposition 2.3.5 and the Remark 2.13 it follows:

Theorem 2.3.6. *If (M^4, J) is an almost complex manifold with dimension 4 and with non-degenerate torsion bundle $\mathcal{V}$, then, for each point $p \in M$, there exists an unique pair of adapted frames*

$$f'_p(e_1, e_2, e_3, e_4) = (X_p, JX_p, T_p^X, JT_p^X)$$

$$f''_p(e_1, e_2, e_3, e_4) = (-X_p, -JX_p, T_p^X, JT_p^X),$$

where X is one of the two distinguished sections of $\mathcal{V}^+$ in a neighborhood of p.

Corollary 2.3.7. *Let $\varphi : M \to M'$ be a diffeomorphism of (M^4, J) in (M'^4, J') such that $\varphi(p) = p'$, with $p \in M$ e $p' \in M'$. Then we have either*

$$d\varphi(X_p, JX_p, T_p^X, JT_p^X) = (X'_{p'}, JX'_{p'}, T_{p'}^{X'}, JT_{p'}^{X'}),$$

or

$$d\varphi(X_p, JX_p, T_p^X, JT_p^X) = (-X'_{p'}, -JX'_{p'}, T_{p'}^{X'}, JT_{p'}^{X'}),$$

that is, φ sends adapted frames into adapted frames.

In particular, when $(M^4, J) = (M'^4, J')$ and $p = p'$, we have either

$$d\varphi(X_p, JX_p, T_p^X, JT_p^X) = (X_p, JX_p, T_p^X, JT_p^X),$$

or

$$d\varphi(X_p, JX_p, T_p^X, JT_p^X) = (-X_p, -JX_p, T_p^X, JT_p^X).$$

***Remark* 2.14.** If two transformations φ and φ' of (M^4, J) fix a point p of M and have the differentiable that coincides at that point, that is $d\varphi(p) = d\varphi'(p)$, than $\varphi = \varphi'$.

Proposition 2.3.8. *The set of all adapted frames f_p, by varying the point p in M, forms a reduction F of the principal bundle of linear frames $L(M)$ on M which has structure group $G_0 \simeq \mathbb{Z}_2$.*

Proof. We have that F is a submanifold of $L(M)$.

Moreover, the matrices $A \in GL(4, \mathbb{R})$ that send bases of T_pM of the form $(X_p, JX_p, T_p^X, JT_p^X)$, with $0 \neq X_p \in \mathcal{V}_p$, into bases of the same form $(Y_p, JY_p, T_p^Y, JT_p^Y)$ are like

$$\begin{pmatrix} a & -b & c & -d \\ b & a & d & c \\ 0 & 0 & a^2+b^2 & 0 \\ 0 & 0 & 0 & a^2+b^2 \end{pmatrix},$$

with $a, b, c, d \in \mathbb{R}$.

Now, taking X_p as the distinguished vector of $\mathcal{V}_p^+$, we have that the matrices A, which send bases of the form $(X_p, JX_p, T_p^X, JT_p^X)$ into bases of the same form or of the opposite form $(-X_p, -JX_p, T_p^X, JT_p^X)$, become

$$I_4 \quad \text{and} \quad \begin{pmatrix} -I_2 & 0 \\ 0 & I_2 \end{pmatrix}.$$

So, if G_0 is the set of these matrices, from the definition of G-structure, it follows that F is a G_0-structure, with structure group $G_0 \simeq \mathbb{Z}_2$. $\square$

***Remark* 2.15.** From the proposition above it is clear that there is an $\{e^{\pm}\}$-structure on (M^4, J) (where $\{e^{\pm}\} = G_o$), given by the adapted frames

$$f'_p(e_1, e_2, e_3, e_4) = (X_p, JX_p, T_p^X, JT_p^X),$$

$$f''_p(e_1, e_2, e_3, e_4) = (-X_p, -JX_p, T_p^X, JT_p^X).$$

On the other hand, there is an $\{e\}$-structure on F. Indeed, if $\pi : F \to M$ is the canonical projection of F on M, for any point $f := f_p(e_1, e_2, e_3, e_4)$ in F, we have that the isomorphism

$$(\pi_{*|f})^{-1} \circ f : \mathbb{R}^4 \to T_f F$$

is an adapted frame for F. Hence, (F, J) has an absolute parallelism, where we denote the lift of J to F with the same letter.

Definition 2.11. Two almost complex manifolds (M, J) and (M', J') are *locally equivalent* at points $p \in M$ and $p' \in M'$, if there exist a neighborhood U of p, a neighborhood U' of p' and a (J, J')-biholomorphic map $\varphi : U \to U'$, such that $\varphi(p) = p'$ and $\varphi_*(J) = J'$.

***Remark* 2.16.** Since there is a double absolute parallelism on (M^4, J), the problem of local equivalence of two almost complex manifolds is solved in the case of four-dimensional almost complex manifolds (M^4, J) with non-degenerate torsion bundle. Indeed, it is sufficient to compute the adapted frames of the two manifolds, to find their structure functions (up to sign) and to compare them: if the structure functions are the same, the two manifolds are locally equivalent.

If we denote with $Aut(M, J)$ the group of the automorphisms of an almost complex manifold (M, J), we have the following results.

Corollary 2.3.9. *There exists a bijection between the automorphisms* $\varphi \in Aut(M, J)$ *and* $\tilde{\varphi} \in Aut(F, J)$*, where* $\tilde{\varphi}$ *is the lift of* φ *to* F*.*

Theorem 2.3.10. *The group of automorphisms* $Aut(M, J)$ *of* (M^4, J) *is a Lie group and it has dimension* $\dim Aut(M, J) \leq 4$ *and so the symmetry algebra of* (M^4, J) *at* p *also has dimension* $\dim aut_p(M, J) \leq 4$*. In particular, the group of automorphisms* $Aut^0(M, J)$ *of* (M^4, J) *that fix a point* p *has at most two elements and*

$$aut_p^0(M, J) = \{0\}.$$

Proof. Since, for Remark 2.15, there is an absolute parallelism on F, from Kobayashi's Theorem 1.4.2 it follows that $Aut(F, J)$ is a Lie group of dimension less or equal to 4. Moreover, the lifts of the automorphisms of (M^4, J) on (F, J) are automorphisms of (F, J) that conserve the absolute parallelism. Hence, for Corollary 2.3.9, we have that also $Aut(M, J)$ is a Lie group and that $\dim Aut(M, J) \leq 4$. As a consequence, the same holds for the dimension of $aut_p(M, J)$. For the Corollary 2.3.7, we deduce that $Aut^0(M, J)$ has at most two elements and $aut_p^0(M, J) = \{0\}$. $\square$

***Remark* 2.17.** When (M^4, J) is an almost complex manifold with non-degenerate torsion bundle, we have that Theorem 2.1.2 is true without the hypothesis of compactness.

Definition 2.12. We define a *metric* on M taking as orthogonal base of T_pM, with $p \in M$, the following

$$(X_p, JX_p, T_p^X, JT_p^X),$$

where X is the distinguished section of $\mathcal{V}^+$ in a neighborhood of p.

Corollary 2.3.11. *To fix a frame* $(X_p, JX_p, T_p^X, JT_p^X)$*, with* X *the distinguished section of* $\mathcal{V}^+$ *and* $p \in M$*, is equivalent to have an invariant canonical norm on* T_pM*. In particular, the* J*-holomorphic transformations are isometries of* M*.*

2.4 Graded Lie algebras of (M^4, J)

Now, we are going to apply the theory of prolongations developed by Tanaka in [38] and Alekseevsky and Spiro in [3] on the torsion bundle $\mathcal{V}$ of (M^4, J).

Let as consider the filtration

$$\mathcal{V}_p := \mathcal{V}_{-1|p} \subseteq \mathcal{V}_{-2|p} \subseteq \mathcal{V}_{-3|p} \subseteq T_pM$$

of the tangent space T_pM made in Section 2.2 (see consequences of the Definition 2.6). It induces a structure of graded Lie algebra $\mathfrak{M}$ associated to the filtered algebra (see [38] p. 9-10 and [3] p. 10). Indeed, quotienting we have

$$\begin{array}{llll}
\mathfrak{m}^{-1} := \mathcal{V}_{-1|p} & & \Rightarrow & \dim_{\mathbb{R}} \mathfrak{m}^{-1} = 2, \\
\mathfrak{m}^{-2} := \mathcal{V}_{-2|p}/\mathcal{V}_{-1|p} & & \Rightarrow & \dim_{\mathbb{R}} \mathfrak{m}^{-2} = 1, \\
\mathfrak{m}^{-3} := \mathcal{V}_{-3|p}/\mathcal{V}_{-2|p} = \mathcal{V}_{-2|p}/\mathcal{V}_{-2|p} = 0 & \text{if } (*) \text{ holds} & \Rightarrow & \dim_{\mathbb{R}} \mathfrak{m}^{-3} = 0, \\
\mathfrak{m}^{-3} := \mathcal{V}_{-3|p}/\mathcal{V}_{-2|p} & \text{if } (**) \text{ holds} & \Rightarrow & \dim_{\mathbb{R}} \mathfrak{m}^{-3} = 1,
\end{array}$$

from which we obtain

$$\mathfrak{M} := \mathfrak{m}^{-1} \oplus \mathfrak{m}^{-2} \oplus \mathfrak{m}^{-3},$$

where the term $\mathfrak{m}^{-3}$ is not trivial only when $(**)$ holds.

Definition 2.13. (see [38], p. 10) Let $\mathcal{V}$ be the torsion bundle and $\mathfrak{M}(p)$ be the graduated Lie algebra generated by $\mathcal{V}$ associated to the point p. The fiber bundle $\mathcal{V}$ is called *(strongly) regular* if the graduated algebras $\mathfrak{M}(p)$, $p \in M^4$, are all isomorphic one with respect to the other.

***Remark* 2.18.** (see [10]) If $\mathfrak{M} = \sum_{i=1}^{\mu} \mathfrak{m}^{-i}$ is a graded algebra (of μ-th kind), with $\dim \mathfrak{m}^{-1} = 2$ and $\dim \mathfrak{M} \leq 5$, then we have the following 5 cases:

- $\dim \mathfrak{M} = 2$, $\mu = 1$ ($\dim \mathfrak{m}^{-1} = 2$);
- $\dim \mathfrak{M} = 3$, $\mu = 2$ ($\dim \mathfrak{m}^{-1} = 2$, $\dim \mathfrak{m}^{-2} = 1$);
- $\dim \mathfrak{M} = 4$, $\mu = 3$ ($\dim \mathfrak{m}^{-1} = 2$, $\dim \mathfrak{m}^{-2} = \dim \mathfrak{m}^{-3} = 1$);
- $\dim \mathfrak{M} = 5$, $\mu = 3$ ($\dim \mathfrak{m}^{-1} = 2$, $\dim \mathfrak{m}^{-2} = 1, \dim \mathfrak{m}^{-3} = 2$);
- $\dim \mathfrak{M} = 5$, $\mu = 4$ ($\dim \mathfrak{m}^{-1} = 2$, $\dim \mathfrak{m}^{-2} = \dim \mathfrak{m}^{-3} = \dim \mathfrak{m}^{-4} = 1$).

Definition 2.14. We denote with $\mathfrak{gl}(T_pM)$ the algebra of the linear applications on the vector space T_pM. We say *linear representation of the isotropy* of $aut_p^0(M^4, J)$ the homomorphism of Lie algebras

$$\rho : \begin{cases} aut_p^0(M^4, J) & \to \mathfrak{gl}(T_pM) \\ \mathcal{A} & \mapsto \tilde{\mathcal{A}}, \end{cases}$$

such that $\tilde{\mathcal{A}}(Z_p) := [\mathcal{A}, Z]_p$, $\forall Z_p \in T_pM$, and Z is a germ of vector fields that in p is Z_p.

Proposition 2.4.1. *The definition of linear representation of the isotropy is well defined, that is, it does not depend on the choice of Z.*

Proof. Let Z and Z' be two vector fields of T_pM such that

$$Z_p = Z'_p,$$

then, $\forall q \in M^4$, Z' can be written as

$$Z' = Z + \alpha(q)W,$$

with $W \in T_pM$ and $\alpha(p) = 0$. We have

$$\begin{aligned} [\mathcal{A}, Z']_p &= [\mathcal{A}, Z]_p + [\mathcal{A}, \alpha(p)W]_p \\ &= [\mathcal{A}, Z]_p + \alpha(p)[\mathcal{A}, W]_p + \mathcal{A}(\alpha(p))W_p \\ &= [\mathcal{A}, Z]_p, \end{aligned}$$

where the term $\mathcal{A}(\alpha(p))W_p$ is zero because $\mathcal{A}_p = \sum_{i=0}^4 a_i \frac{\partial}{\partial x_i} = 0$, with $a_i = 0$. □

Since any linear application $\tilde{\mathcal{A}}$ preserve the filtration of T_pM, $\tilde{\mathcal{A}}$ induces a linear application on $\mathfrak{M}$ in a canonical way (that we also denote with $\tilde{\mathcal{A}}$). We denote $\tilde{\mathcal{A}}_{|\mathfrak{m}^{-i}}$, with $i = 1, 2, 3$, the application $\tilde{\mathcal{A}}$ restricted to the vectors of $\mathfrak{m}^{-i}$.

Proposition 2.4.2. *Let $\mathcal{A} \in aut_p^0(M^4, J)$ be a germ of infinitesimal automorphisms that fixes a point p. Then the linear application $\tilde{\mathcal{A}}$ associated to $\mathcal{A}$ preserve the degree of $\mathfrak{M}$, in other words, $\forall i = 1, 2, 3$, we have*

$$\tilde{\mathcal{A}}_{|\mathfrak{m}^{-i}} : \mathfrak{m}^{-i} \to \mathfrak{m}^{-i}.$$

In particular,

$$\tilde{\mathcal{A}} : \mathfrak{M} \to \mathfrak{M}.$$

Let us study how $\tilde{\mathcal{A}}$ can be written.

Since $\tilde{\mathcal{A}}$ sends elements of $\mathfrak{m}^{-1}$ in elements of $\mathfrak{m}^{-1}$ and it commutes with J, we have that $\tilde{\mathcal{A}}_{|\mathfrak{m}^{-1}}$ is of the form

$$\tilde{\mathcal{A}}_{|\mathfrak{m}^{-1}} = \alpha I + \beta J, \tag{2.3}$$

where $\alpha, \beta \in \mathbb{R}$ and I is the identity. If (X_p, JX_p) is a base of $\mathfrak{m}^{-1}$, called $T_p := [X, JX]_p$, we have that (T_p) (here, for simplicity of notation, we represents the class of equivalence $[T_p]$ of T_p without square brackets) forms a base of $\mathfrak{m}^{-2}$ because of the assumption of non-degeneracy of $\mathcal{V}$; hence, considering $\tilde{\mathcal{A}}$ restricted to $\mathfrak{m}^{-2}$, we have

$$\begin{aligned}\tilde{\mathcal{A}}_{|\mathfrak{m}^{-2}}(T_p) &= \tilde{\mathcal{A}}_{|\mathfrak{m}^{-2}}([X, JX]_p) = [\tilde{\mathcal{A}}_{|\mathfrak{m}^{-1}}X, JX]_p + [X, \tilde{\mathcal{A}}_{|\mathfrak{m}^{-1}}(JX)]_p = \\ &= [\alpha X + \beta JX, JX]_p + [X, \alpha JX - \beta X]_p = \\ &= \alpha[X, JX]_p + \alpha[X, JX]_p = \\ &= 2\alpha[X, JX]_p = \\ &= 2\alpha T_p.\end{aligned}$$

Proposition 2.4.3. *Let (M^4, J) be an almost complex manifold of real dimension 4, with $N_J \neq 0$ in $p \in M^4$ and such that $\mathcal{V}$ is non-degenerate.*

Then if $X \in \mathfrak{m}^{-1}$, setting $S = [X, [X, JX]]_p$ and $S' = [JX, [X, JX]]_p$, the following equalities hold

1. $\tilde{\mathcal{A}}(S) = 3\alpha S + \beta S'$,
2. $\tilde{\mathcal{A}}(S') = 3\alpha S' - \beta S$,

where α and β are defined as in (2.3).

Proof.

$$\begin{aligned}1.\ \tilde{\mathcal{A}}(S) &= \tilde{\mathcal{A}}([X, [X, JX]]_p) = \\ &= [\tilde{\mathcal{A}}_{|\mathfrak{m}^{-1}}X, [X, JX]]_p + [X, \tilde{\mathcal{A}}_{|\mathfrak{m}^{-2}}[X, JX]]_p = \\ &= [\alpha X + \beta JX, [X, JX]]_p + [X, 2\alpha[X, JX]]_p = \\ &= \alpha[X, [X, JX]]_p + \beta[JX, [X, JX]]_p + \\ &\quad +2\alpha[X, [X, JX]]_p = \\ &= 3\alpha[X, [X, JX]]_p + \beta[JX, [X, JX]]_p = \\ &= 3\alpha S + \beta S'.\end{aligned}$$

$$\begin{aligned}2.\ \tilde{\mathcal{A}}(S') &= \tilde{\mathcal{A}}([JX, [X, JX]]_p) = \\ &= [\tilde{\mathcal{A}}_{|\mathfrak{m}^{-1}}JX, [X, JX]]_p + [JX, \tilde{\mathcal{A}}_{|\mathfrak{m}^{-2}}[X, JX]]_p = \\ &= [\alpha JX - \beta X, [X, JX]]_p + [JX, 2\alpha[X, JX]]_p = \\ &= \alpha[JX, [X, JX]]_p - \beta[X, [X, JX]]_p + \\ &\quad +2\alpha[JX, [X, JX]]_p = \\ &= 3\alpha[JX, [X, JX]]_p - \beta[X, [X, JX]]_p = \\ &= 3\alpha S' - \beta S.\end{aligned}$$

□

We have the following proposition.

Proposition 2.4.4. *Let (M^4, J) be an almost complex manifold of real dimension 4, with $N_J \neq 0$ in $p \in M^4$ and such that $\mathcal{V}$ is non-degenerate. Then the linear representation of the isotropy is trivial, i.e.*

$$\rho(aut_p^0(M^4, J)) = 0.$$

This result is in agreement with the fact that the isotropy algebra fixing a point p is zero: $aut_p^0(M^4, J) = \{0\}$.

To prove the proposition we need to consider both cases: fundamental and non-fundamental $\mathcal{V}_p$.

2.4.1 Non-fundamental $\mathcal{V}_p$

When $\mathcal{V}_p$ is non-fundamental, we have

$$\mathcal{V}_p := \mathcal{V}_{-1|p} \subsetneqq \mathcal{V}_{-2|p} = \mathcal{V}_{-3|p} \subsetneqq T_pM,$$

and so quotienting

$$\mathfrak{M} := \mathfrak{m}^{-1} \oplus \mathfrak{m}^{-2}.$$

Since $\mathfrak{M} \ncong T_pM$ (because $\dim \mathfrak{M} = 3 \neq 4 = \dim T_pM$), there exists a "missing direction" $\mathfrak{n}$, such that

$$\mathfrak{M} := \mathfrak{m}^{-1} \oplus \mathfrak{m}^{-2} \oplus \mathfrak{n} \cong T_pM.$$

Such a $\mathfrak{n}$ can be defined by

$$\mathfrak{n} := T_pM/\mathcal{V}_{-2|p}.$$

Proposition 2.4.5. *We have that* $([JT])$ *forms a base of* $\mathfrak{n}$*, where* $[JT]$ *is the equivalence class whose representative is* $JT := \{JT + X : X \in \mathcal{V}_{-2|p}\}$.

Proof. Since the dimensions of $\mathfrak{m}^{-1}$ and $\mathfrak{m}^{-2}$ are respectively 2 and 1, and the dimension of T_pM is 4, then $\dim_{\mathbb{R}} \mathfrak{n} = 1$.

Moreover, for the Proposition 2.2.2, (X, JX, T) forms a base of $\mathcal{V}_{-2|p}$, and JT cannot be generated by (X, JX, T), otherwise there exist $a, b, c \in \mathbb{R}$ such that

$$JT = aX + bJX + cT;$$

so, appling J to both members of the equality, we have

$$T = -aJX + bX - cJT.$$

Comparing the two equalities we have that $a = b = 0$ and $c = -1 = +1$, a contradiction. $\square$

***Remark* 2.19.** When $\mathcal{V}_p$ is non-fundamental, S and S' defined in Proposition 2.4.3 belong to $\mathcal{V}_{-2|p}$.

If $\mathcal{A} \in aut_p^0(M^4, J)$ and $\tilde{\mathcal{A}}$ is the associated application, we already saw that

$$\tilde{\mathcal{A}}_{|\mathfrak{m}^{-1}} = \alpha I + \beta J, \qquad \alpha, \beta \in \mathbb{R};$$

moreover, if $X = N_J(A, B) \in \mathfrak{m}^{-1}$, we have

$$\tilde{\mathcal{A}}(N_J(A, B)) = N_J(\tilde{\mathcal{A}}(A), B) + N_J(A, \tilde{\mathcal{A}}(B)), \qquad \forall A, B \in T_pM.$$

***Remark* 2.20.** We note that, when we compute the Nijenhuis tensor of two vector fields, we can always suppose to take one vector in $\mathcal{V}_{-1|p}$ and the other vector outside of $\mathcal{V}_{-1|p}$. Indeed, if A and B were both in $\mathcal{V}_{-1|p}$, because of the Remark (2.6), we would have

$$N_J(A, B) = 0.$$

If A and B were both outside of $\mathcal{V}_{-1|p}$, we can take A as T, since T generates $\mathfrak{m}^{-2}$, and B as JT, since JT generates $\mathfrak{n}$; we would have

$$N_J(T, JT) = -JN_J(T, T) = 0.$$

Therefore, we can always take

$$A \in \mathcal{V}_{-1|p} \text{ and } B \notin \mathcal{V}_{-1|p}.$$

From the remark above, we can take $A \in \mathcal{V}_{-1|p}$ and $B \notin \mathcal{V}_{-1|p}$, in particular, we can choose $B = T$ (we have the same result choosing $B = JT$). Then

$$\begin{aligned} \tilde{\mathcal{A}}(N_J(A, T)) &= N_J(\tilde{\mathcal{A}}(A), T) + N_J(A, \tilde{\mathcal{A}}(T)) \\ &= N_J(\alpha A + \beta JA, T) + N_J(A, 2\alpha T)) \\ &= \alpha N_J(A, T) + \beta N_J(JA, T) + 2\alpha N_J(A, T) \\ &= \alpha N_J(A, T) - \beta J N_J(A, T) + 2\alpha N_J(A, T) \\ &= (\alpha I - \beta J) N_J(A, T) + 2\alpha N_J(A, T) \\ &= (3\alpha I - \beta J) N_J(A, T). \end{aligned}$$

On the other hand, we have

$$\tilde{\mathcal{A}}(N_J(A, T)) = (\alpha I + \beta J) N_J(A, T),$$

therefore, we must have necessarily

$$\alpha = 0, \quad \beta = 0.$$

In conclusion

$$\tilde{\mathcal{A}} = 0,$$

so we proved Proposition 2.4.4 with non-fundamental $\mathcal{V}_p$.

2.4.2 Fundamental $\mathcal{V}_p$

When $\mathcal{V}_p$ is fundamental, we have

$$\mathcal{V}_p := \mathcal{V}_{-1|p} \subsetneqq \mathcal{V}_{-2|p} \subsetneqq \mathcal{V}_{-3|p} = T_pM,$$

and quotienting

$$\mathfrak{M} := \mathfrak{m}^{-1} \oplus \mathfrak{m}^{-2} \oplus \mathfrak{m}^{-3} \cong T_pM.$$

Since Remark 2.20 also holds when $\mathcal{V}_p$ is fundamental (with the difference that JT generates $\mathfrak{m}^{-3}$ instead of $\mathfrak{n}$), if $X = N_J(A, B) \in \mathfrak{m}^{-1}$, we can always suppose that

$$A \in \mathcal{V}_{-1|p} \quad \text{and} \quad B \notin \mathcal{V}_{-1|p};$$

in particular, choosing $B = T \in \mathfrak{m}^{-2}$ (it is the same if we take $B = JT$, but when $\mathcal{V}_p$ is fundamental, we have that $B = JT$ belongs to $\mathfrak{m}^{-3}$), we have the same result of the non-fundamental case:

$$\alpha = 0, \quad \beta = 0$$

and so

$$\tilde{\mathcal{A}} = 0.$$

When $\mathcal{V}_p$ is fundamental, this conclusion can be shown in an alternative way, using the following remark.

***Remark* 2.21.** When $\mathcal{V}_p$ is fundamental, there exists a field $Y \in \mathfrak{m}^{-1}$, with $Y \neq 0$, such that

$$[Y, [Y, JY]] \neq 0 \qquad \text{and} \qquad [JY, [Y, JY]] = 0.$$

Indeed, if $X \in \mathfrak{m}^{-1}$, $X \neq 0$, we have that the application

$$\phi : \begin{cases} \mathfrak{m}^{-1} & \to \mathfrak{m}^{-3} \\ X_p & \mapsto [X, T]_p \end{cases}$$

is linear and surjective, where we set $T = [X, JX]_p$ for brevity. Therefore, since

$$\dim_{\mathbb{R}}(\mathfrak{m}^{-1}) = \dim_{\mathbb{R}}(ker\phi) + \dim_{\mathbb{R}}(Im\phi),$$

we have that $\dim_{\mathbb{R}}(ker\phi) = 1$, this means that So there exists at least one $Y \in \mathfrak{m}^{-1}$ such that $[Y, T]_p = 0$. We note that it cannot be $[JY, T]_p = 0$, otherwise we would have $[Y, T]_p = [JY, T]_p = 0$, that is

$$[Y, [Y, JY]]_p = [JY, [Y, JY]]_p,$$

and so $\mathfrak{m}^{-3}$ should be null, against the assumption that $\mathcal{V}_p$ is fundamental.

By the preceding remark, we can choose an appropriate $X \in \mathfrak{m}^{-1}$ such that $S' = [JX, [X, JX]]_p = 0$ and $S = [X, [X, JX]]_p \neq 0$. Then, by the second equality of Proposition 2.4.3, we have

$$\underbrace{\tilde{\mathcal{A}}([JX, [X, JX]]_p)}_{=0} = \underbrace{3\alpha[JX, [X, JX]]_p}_{=0} - \beta[X, [X, JX]]_p$$

and, since $[X, [X, JX]]_p \neq 0$, it must be

$$\beta = 0.$$

Moreover, because of the first equality of Proposition 2.4.3, we have

$$\tilde{\mathcal{A}}([X,[X,JX]]_p) = 3\alpha[X,[X,JX]]_p.$$

So, $\tilde{\mathcal{A}}$ acts on the elements X, JX, T, S as follows:

$$\tilde{\mathcal{A}}X = \alpha X,$$

$$\tilde{\mathcal{A}}(JX) = \alpha JX,$$

$$\tilde{\mathcal{A}}T = 2\alpha T,$$

$$\tilde{\mathcal{A}}S = 3\alpha S.$$

In particular, since J commutes with $\tilde{\mathcal{A}}$, from $\tilde{\mathcal{A}}T = 2\alpha T$, we also have

$$\tilde{\mathcal{A}}(JT) = J\tilde{\mathcal{A}}(T) = J(2\alpha T) = 2\alpha JT.$$

Therefore, T and JT are both eigenvectors related to the eigenvalue 2α and they are linearly independent. So, the eigenspace related to 2α has dimension at least 2, against the hypothesis $\dim_{\mathbb{R}} \mathfrak{m}^{-2} = 1$. Necessarily it must be $\alpha = 0$ and so $\tilde{\mathcal{A}} = 0$.

In conclusion

$$\tilde{\mathcal{A}} = 0,$$

so we proved Proposition 2.4.4 with fundamental $\mathcal{V}_p$.

Chapter 3

Locally homogeneous almost complex manifolds

3.1 Lie algebras associated to locally homogeneous almost complex manifolds (M^4, J)

Let us consider a *connected* almost complex manifold (M^4, J) of dimension 4 with non-degenerate torsion bundle $\mathcal{V}$. In the previous chapter we gave an adapted frame (X, JX, T, JT) unique up to the sign of X and JX on (M^4, J). From Theorem 2.3.10, we obtain that $Aut(M, J)$ is a Lie group (having dimension $\dim Aut(M, J) \leq 4$), hence, we can consider the left action of the connected component G of $Aut(M, J)$ on (M^4, J) given by

$$l : G \times M \longrightarrow M,$$

such that $l(\varphi, p) = \varphi(p)$. (For more details about Lie groups and Lie algebras, see for example [1, 24, 25, 26, 27, 33, 40, 41].)

In this chapter we assume that (M^4, J) is *locally homogeneous* (with the exception of the last section or when specified).

Under this assumption, if $(\pm X_1, \pm X_2, X_3, X_4)$ are the adapted frames of (M^4, J), we have that the structure functions c_{ij}^k associated to $(\pm X_1, \pm X_2, X_3, X_4)$ (see Section 1.5) are locally constant, and are defined by

$$[X_i, X_j] = c_{ij}^k X_k, \qquad i, j = 1, ..., 4$$

(it is clear that some of these constants differ for the sign). We denote with $\mathfrak{g}$ the Lie algebra generated by $< X_1, X_2, X_3, X_4 >$. We note that the existence of two adapted frames on (M^4, J) imply that $l_{g*}(X_i) = X_i$, $\forall i = 1, ..., 4$, for any $g \in G$, that is, X_1, X_2, X_3 and X_4 are left invariant vector fields under the action l.

For conciseness, from now on we will say that the Lie algebras built above are *associated* to the (locally) homogeneous almost complex manifold.

Now we are going to study the Lie algebra $\mathfrak{g}$; in order to do this, we need some notations.

Given a Lie algebra $\mathfrak{a}$, we denote by $\mathfrak{a}^{(k)}$ its *derived algebras*

$$\mathfrak{a}^{(0)} := \mathfrak{a}, \quad \mathfrak{a}^{(k)} := [\mathfrak{a}^{(k-1)}, \mathfrak{a}^{(k-1)}],$$

for any $k = 0, 1, 2, ...$

We denote by $\mathfrak{a}^k$ the *descending central series* of $\mathfrak{a}$ given by

$$\mathfrak{a}^0 := \mathfrak{a}, \quad \mathfrak{a}^k := [\mathfrak{a}, \mathfrak{a}^{k-1}].$$

We recall that the algebra $\mathfrak{a}$ is *solvable* if there exists a j such that $\mathfrak{a}^{(j)} = 0$; the algebra $\mathfrak{a}$ is *nilpotent* if there exists a j such that $\mathfrak{a}^j = 0$.

We denote by $\mathfrak{z}(\mathfrak{a})$ the *center* of a Lie algebra $\mathfrak{a}$ given by

$$\mathfrak{z}(\mathfrak{a}) = \{Z \in \mathfrak{a} : [Z, A] = 0, \forall A \in \mathfrak{a}\}.$$

***Remark* 3.1.** We see at once that for the Lie algebra $\mathfrak{g}$, such that $\mathfrak{g} = \Gamma(TM)$, we have

$$\mathfrak{z}(\mathfrak{g}) \cap \Gamma(\mathcal{V}) = \{0\}.$$

Indeed, if we suppose by contradiction that there exists an element $Z \in \mathfrak{z}(\mathfrak{g}) \cap \Gamma(\mathcal{V})$, with $Z_p \neq 0$, we get that $\mathcal{V}_p =< Z_p, JZ_p >$, with $[Z, JZ]_p = 0$, since Z is in the center. This contradict the non-degeneracy of $\mathcal{V}$.

We have the following result for the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^{(1)} = \mathfrak{g}^1$. For simplicity of notation, we will use the same writing for $\mathfrak{g}$ when its vector fields are computed into the points p of M^4.

Proposition 3.1.1. *If (M^4, J) is a locally homogeneous almost complex manifold with non-degenerate torsion bundle, then the derived algebra $\mathfrak{g}^{(1)}$ of the Lie algebra $\mathfrak{g}$ associated to (M^4, J) has dimension*

$$2 \leq \dim \mathfrak{g}^{(1)} \leq 3.$$

In particular, if $\dim \mathfrak{g}^{(1)} = 2$ *the algebra* $\mathfrak{g}$ *is solvable.*

Proof. We note that the dimension of the derived algebra $\mathfrak{g}^{(1)}$ of $\mathfrak{g}$ cannot be zero, because the vector $0 \neq T_p \in \mathfrak{g}^{(1)}$ (since $\mathcal{V}_p \subseteq \mathfrak{g}$).

The dimension of $\mathfrak{g}^{(1)}$ cannot be 1 because, otherwise, we had $\mathfrak{g}^{(1)} =< T_p >$, from which

$$N_J(X, Y) = \underbrace{[JX, JY] - [X, Y]}_{\in <T>} - \underbrace{J([JX, Y] + [X, JY])}_{\in <JT>}.$$

In other words, we had that $\mathcal{V}_p =< T_p, JT_p >$, because $\mathcal{V}_p$ is J-invariant, and this is a contradiction.

We want to show that the dimension of $\mathfrak{g}^{(1)}$ cannot be 4.

If the dimension of $\mathfrak{g}^{(1)}$ was 4 with $\mathfrak{g}$ solvable, we had that $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}] \neq \mathfrak{g}$, a contradiction.

When $\mathfrak{g}$ is not solvable, for the Levi-Malcev Theorem (see [14], pag.103) we have

$$\mathfrak{g} = \mathfrak{L} \ltimes \mathfrak{r},$$

where $\mathfrak{L}$ is the Levi factor (semisimple Lie subalgebra of $\mathfrak{g}$) and $\mathfrak{r}$ is the radical of $\mathfrak{g}$ (maximal solvable ideal). Since the dimension of $\mathfrak{L}$ is not zero (because $\mathfrak{g}$ is not solvable), the semisimple factor $\mathfrak{L}$ must have dimension 3 and so the the radical $\mathfrak{r}$ has dimension 1 and is abelian, hence

$$\mathfrak{g}^{(1)} = [\mathfrak{L}, \mathfrak{L}] + [\mathfrak{L}, \mathfrak{r}] = \mathfrak{L} + [\mathfrak{L}, \mathfrak{r}].$$

We have that $[\mathfrak{L}, \mathfrak{r}] = 0$ (so $\mathfrak{L}$ is an ideal). Indeed, it is sufficient to construct the homomorphism of algebras

$$\varphi : \begin{cases} \mathfrak{L} & \to & Der(\mathfrak{r}) \\ X & \mapsto & ad_X|_{\mathfrak{r}}, \end{cases}$$

to see that φ cannot be injective, because of the dimension of $Der(\mathfrak{r})$. Hence, φ must be the null homomorphism, from which $ad_X|_{\mathfrak{r}} = 0$, that is, $[\mathfrak{L}, \mathfrak{r}] = 0$. In conclusion, when $\mathfrak{g}$ is not solvable, we have that $\mathfrak{g}^{(1)} = \mathfrak{L}$ and that $\mathfrak{r}$ is the center of $\mathfrak{g}$.

The proof above also implies that the dimension of $\mathfrak{g}^{(1)}$ cannot be 4 and that $\mathfrak{g}$ must be solvable when $\dim \mathfrak{g}^{(1)} = 2$. $\square$

***Remark* 3.2.** When $\mathfrak{g}$ is non-solvable, we have that the radical $\mathfrak{r}$ equals the center $\mathfrak{z}(\mathfrak{g})$ of $\mathfrak{g}$. This is equivalent to say that $\mathfrak{g}$ is a reductive algebra, that is, it is the direct sum of a semisimple Lie algebra $\mathfrak{L}$ and an abelian Lie algebra (see for example [14]). Hence, we can rewrite

$$\mathfrak{g} = \mathfrak{L} \oplus \mathfrak{r}.$$

Now we are going to make a complete classification of the almost complex structures related to the locally homogeneous almost complex manifold (M^4, J) having non degenerate torsion bundle when $\mathfrak{g}$ is non-solvable, and we will show with examples how to make the classification when $\mathfrak{g}$ is solvable (which have more cases than the non-solvable case). For a deeper discussion of invariant complex structure on solvable Lie groups see, for example, [28, 34, 35, 36].

3.2 (M^4, J) with non-solvable Lie algebras

Let us first consider the case in which the Lie algebra $\mathfrak{g}$, formed by the adapted frames of (M^4, J), is non-solvable. For Proposition 3.1.1, we have that the dimension of $\mathfrak{g}^{(1)}$ must be 3, so $\mathfrak{g}^{(1)}$ have to be isomorphic to $\mathfrak{so}(3)$ or $\mathfrak{sl}(2,\mathbb{R})$. Since $\mathfrak{g} = \mathfrak{L} \oplus \mathfrak{r}$ and $\mathfrak{g}^{(1)} = \mathfrak{L}$, we have

$$\mathfrak{g} \cong \mathfrak{so}(3) \oplus \mathfrak{r} \quad \text{or} \quad \mathfrak{g} \cong \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r},$$

where $\mathfrak{r}$ is the center of $\mathfrak{g}$.

From now on, we identify $\mathfrak{g}$ with $\mathfrak{so}(3) \oplus \mathfrak{r}$, or with $\mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}$, for simplicity.

Proposition 3.2.1. *Let $\mathfrak{g} = \mathfrak{L} \oplus \mathfrak{r}$ and suppose that $\mathcal{V}_p$ is non-fundamental, with $\mathcal{V}_p \nsubseteq \mathfrak{L}$. Then $\mathfrak{L} = \mathfrak{sl}(2,\mathbb{R})$.*

Proof. We have that $\mathfrak{L}$ is an ideal in $\mathfrak{g}$ and that $\mathcal{V}_{-2|p}$ is a 3-dimensional subalgebra of $\mathfrak{g}$. So, the intersection $\pi := \mathcal{V}_{-2|p} \cap \mathfrak{L}$ is a 2-dimensional subalgebra of $\mathfrak{L}$ (and an ideal in $\mathcal{V}_{-2|p}$), considering that $\mathcal{V}_p \nsubseteq \mathfrak{L}$ by assumption. Since $\mathfrak{so}(3)$ has no nontrivial subalgebras, $\mathfrak{L}$ must be $\mathfrak{sl}(2,\mathbb{R})$. □

Lemma 3.2.2. *Let $\mathfrak{g} = \mathfrak{L} \oplus \mathfrak{r}$, with $\mathcal{V}_p$ non-fundamental and $\mathcal{V}_p \nsubseteq \mathfrak{L}$. Then $\mathfrak{L}$ (which is equal to $\mathfrak{sl}(2,\mathbb{R})$) has no abelian subalgebras of dimension 2. In particular, $\pi := \mathcal{V}_{-2|p} \cap \mathfrak{L}$ is not an abelian subagebra.*

Proof. Suppose, by contradiction, that $\mathfrak{L}$ has an abelian subalgebra of dimension 2. If (e_1, e_2, e_3) is a base of $\mathfrak{L}$ such that (e_1, e_2) is a base of its abelian subalgebra, then we have $[e_1, e_2] = 0$, $f := [e_1, e_3]$ and $g := [e_3, e_2]$. Since $\mathfrak{L} = \mathfrak{L}^1$, it must be $< f, g >= \mathfrak{L}$, a contradiction, because $\mathfrak{L}$ is 3-dimensional. □

***Remark* 3.3.** Let $\mathfrak{g} = \mathfrak{L} \oplus \mathfrak{r}$ and $\mathcal{V}_p \subseteq \mathfrak{L}$. Then $\mathcal{V}_p$ is non-fundamental. Indeed, since $\mathfrak{L}$ is an algebra containing $\mathcal{V}_p$, we have that $\mathcal{V}_{-2|p}$ must be an algebra (in particular, it coincides with $\mathfrak{L}$).

First, let us classify the case in which $\mathfrak{L} = \mathfrak{so}(3)$, that is, $\mathfrak{g} = \mathfrak{so}(3) \oplus \mathfrak{r}$.

3.2.1 Classification for $\mathfrak{g} = \mathfrak{so}(3) \oplus \mathfrak{r}$

We can choose a base (e_0, e_1, e_2, e_3) of $\mathfrak{g}$ such that

$$\mathfrak{r} =< e_0 > \quad \text{and} \quad \mathfrak{so}(3) =< e_1, e_2, e_3 >,$$

and such that

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2, \quad [e_0, e_i] = 0, \quad \forall i = 1, 2, 3.$$

CASE A: $\mathfrak{g} = \mathfrak{so}(3) \oplus \mathfrak{r}$ and $\mathcal{V}_p$ non-fundamental

When $\mathcal{V}_p$ is non-fundamental, from Proposition 3.2.1 and Remark 3.3 we have that $\mathcal{V}_p \subseteq \mathfrak{L}$, so the base (e_1, e_2, e_3) of $\mathfrak{L}$ can be chosen such that it satisfies the conditions above and such that $\mathcal{V}_p$ is generated by two of the elements of the base (e_1, e_2, e_3), for example $\mathcal{V}_p =< e_1, e_2 >$.

Since $\mathcal{V}_p =< \xi, J\xi >$, for any $0 \neq \xi \in \mathcal{V}_p$, we can define the almost complex structure J in such a way that $\xi = ke_1$ (with $k \neq 0$) and $J\xi = ae_1 + be_2$, for some $a, b \in \mathbb{R}$. So, $\eta := [\xi, J\xi] = kbe_3$ (with $b \neq 0$, because $\mathcal{V}_p$ is not degenerate) and $J\eta = e_0 + xe_1 + ye_2 + ze_3$, for some $x, y, z \in \mathbb{R}$ (up to rescaling e_0 as base of $\mathfrak{r}$).

Computing the Nijenhuis tensor of ξ and η, we obtain

$$\begin{aligned} N_J(\xi, \eta) = \quad & \frac{(b^2 + a^2 - k^2)z - 2abk^2}{bk}\xi + \\ & -\frac{(2az + b^3 + a^2b - bk^2)}{b}J\xi + \\ & +\frac{ay - bx}{bk}\eta - \frac{ky}{bk}J\eta. \end{aligned}$$

Since N_J only depends on ξ and $J\xi$, the coefficients of η and $J\eta$ must be zero, hence $x = 0, y = 0$. Therefore,

$$N_J(\xi, \eta) = \frac{(b^2 + a^2 - k^2)z - 2abk^2}{bk}\xi - \frac{(2az + b^3 + a^2b - bk^2)}{b}J\xi. \quad (3.1)$$

To compute the distinguished vector field of $\mathcal{V}_p$, because of the arbitrariness in the choice of ξ in $\mathcal{V}_p$, it is sufficient to impose that $N_J(\xi, \eta) = \xi$, that is, the coefficient of ξ must be 1 and the coefficient of $J\xi$ must be 0 in formula (3.1), so

$$\begin{cases} \dfrac{(b^2 + a^2 - k^2)z - 2abk^2}{bk} = 1, \\ \dfrac{(2az + b^3 + a^2b - bk^2)}{b} = 0. \end{cases}$$

We observe that $a \neq 0$, otherwise, from the previous conditions, we have that $bk = 0$, against the assumptions. Computing the previous system, we obtain the parameters k and z as a function of a and b, and so

$$\begin{aligned} & \xi = k_{a,b}e_1, \quad k_{a,b} \neq 0, \\ & J\xi = ae_1 + be_2, \quad ab \neq 0, \\ & \eta = k_{a,b}be_3, \\ & J\eta = e_0 + z_{a,b}e_3. \end{aligned}$$

In conclusion, if (M^4, J) is a locally homogeneous almost complex manifold with non-fundamental and non-degenerate torsion bundle $\mathcal{V}$, and the Lie algebra $\mathfrak{g}$ associated to (M^4, J) is non solvable with $\mathfrak{g} = \mathfrak{so}(3) \oplus \mathfrak{r}$, we have the following classification for the almost complex structure J by varying the coefficients $a, b \in \mathbb{R}$:

	$\mathcal{V}_p$ is non-fundamental	
	$\mathcal{V}_p \subseteq \mathfrak{L}$	$\mathcal{V}_p \not\subseteq \mathfrak{L}$
$\mathfrak{L} = \mathfrak{so}(3)$ $[e_1, e_2] = e_3$ $[e_2, e_3] = e_1$ $[e_3, e_1] = e_2$	$\xi = k_{a,b} e_1, \quad k_{a,b} \neq 0$ $J\xi = ae_1 + be_2, \quad ab \neq 0$ $\eta = k_{a,b} b e_3$ $J\eta = e_0 + c_{a,b} e_3$	not possible

Table 3.1: CASE A: $\mathfrak{g} = \mathfrak{so}(3) \oplus \mathfrak{r}$ with non-fundamental $\mathcal{V}_p$.

Here, $(\xi, J\xi, \eta, J\eta)$ is the adapted frame and $k = k_{a,b}$ and $c = c_{a,b}$ are function depending on the parameters $a, b \in \mathbb{R}$ and are given as solutions of the system:

$$\begin{cases} (b^2 + a^2 - k^2)c - 2abk^2 - bk = 0, \\ 2ac + b^3 + a^2 b - bk^2 = 0. \end{cases}$$

CASE B: $\mathfrak{g} = \mathfrak{so}(3) \oplus \mathfrak{r}$ and $\mathcal{V}_p$ fundamental

Similar to what is done for the previous case, when $\mathcal{V}_p$ is fundamental, since it must be $\mathcal{V}_p \not\subseteq \mathfrak{L}$ (because of Proposition 3.2.1 and Remark 3.3), the bases (e_1, e_2, e_3) of $\mathfrak{L}$ and (e_0) of $\mathfrak{r}$ can be chosen such that $\mathcal{V}_p \cap \mathfrak{L} =< e_1 >$ and $\mathcal{V}_p =< e_1, e_0 + e_2 >$, without loss of generality. In general, here we cannot say that e_1 can be chosen as the distinguished field of $\mathcal{V}_p$. Then, if we define:

$$\begin{aligned} &\xi = e_1, \\ &J\xi = ae_1 + b(e_0 + e_2), \quad b \neq 0, \\ &\eta = be_3, \\ &J\eta = xe_1 + ye_2 + ze_3 + te_0, \end{aligned}$$

for appropriate coefficients a, b, c, x, y, z, t of $\mathbb{R}$, we have that $(\xi, J\xi, \eta, J\eta)$ is not (in general) the adapted frame. The coefficients a, b, c, x, y, z, t of $\mathbb{R}$ must be chosen such that the Nijenhuis tensor of ξ and η only depends on

ξ and $J\xi$, i.e. in the formula

$$\begin{aligned} N_J(\xi,\eta) = \; & \frac{(b^2y+abx+(-b^2-a^2+1)t)z-b^2x+2abt}{b(y-t)}\xi+ \\ & +\frac{(2at-bx)z-b^3y-ab^2x+(b^3+(a^2-1)b)t}{b(y-t)}J\xi+ \\ & +\frac{az^2-2bz+ay^2+(-bx-at)y+btx-ab^2}{b(y-t)}\eta+ \\ & +\frac{-z^2-2abz-y^2+ty+b^2}{b(y-t)}J\eta, \end{aligned}$$

the coefficients of η and $J\eta$ must be null, that is

$$\begin{cases} az^2-2bz+ay^2+(-bx-at)y+btx-ab^2=0, \\ -z^2-2abz-y^2+ty+b^2=0. \end{cases}$$

Obtaining x and t from the system, we have

$$\begin{cases} x = \dfrac{2yz(a^2+1)}{z^2+2abz-b^2}, \\ t = \dfrac{z^2+2abz-b^2+y^2}{y}. \end{cases}$$

This leads to have the following construction:

$$\begin{aligned} & \xi = e_1, \\ & J\xi = ae_1+b(e_0+e_2), \quad b\neq 0, \\ & \eta = be_3, \\ & J\eta = \frac{2yz(a^2+1)}{z^2+2abz-b^2}e_1+ye_2+ze_3+\frac{z^2+2abz-b^2+y^2}{y}e_0, \end{aligned}$$

with $z^2+2abz-b^2\neq 0$. The Nijenhuis tensor becomes

$$N_J(\xi,\eta) = \frac{1}{C}(A\xi+BJ\xi), \tag{3.2}$$

where

$$\begin{aligned} A = \; & (b^2+a^2-1)z^5+(4ab^3+(4a^3-6a)b)z^4+((a^2-1)y^2+ \\ & +(4a^2-2)b^4+(4a^4-14a^2+2)b^2)z^3+ \\ & +(-6aby^2-4ab^5+(8a-12a^3)b^3)z^2+ \\ & +((3-3a^2)b^2y^2+b^6+(9a^2-1)b^4)z+2ab^3y^2-2ab^5, \\ B = \; & 2az^5+(b^3+(9a^2-1)b)z^4+ \\ & +(2ay^2+4ab^4+(12a^3-8a)b^2)z^3+ \\ & +((3a^2-3)by^2+(4a^2-2)b^5+(4a^4-14a^2+2)b^3)z^2+ \\ & +(-6ab^2y^2-4ab^6+(6a-4a^3)b^4)z+ \\ & +(1-a^2)b^3y^2+b^7+(a^2-1)b^5, \\ C = \; & b(z^2+2abz-b^2)^2. \end{aligned}$$

To calculate the adapted frame (X, JX, T, JT), we have to find α and β in $\mathbb{R}$ such that the linear combination $\alpha\xi + \beta J\xi$ is the distinguished field X of $\mathcal{V}_p$, that is

$$N_J(\alpha\xi + \beta J\xi, [\alpha\xi + \beta J\xi, J(\alpha\xi + \beta J\xi)]) = \alpha\xi + \beta J\xi.$$

In conclusion, if (M^4, J) is a locally homogeneous almost complex manifold with fundamental and non-degenerate torsion bundle $\mathcal{V}$, and the Lie algebra $\mathfrak{g}$ associated to (M^4, J) is non solvable with $\mathfrak{g} = \mathfrak{so}(3) \oplus \mathfrak{r}$, we have the following classification for the almost complex structure J by varying the coefficients $a, b, y, z \in \mathbb{R}$:

	$\mathcal{V}_p$ is fundamental	
	$\mathcal{V}_p \subseteq \mathfrak{L}$	$\mathcal{V}_p \not\subseteq \mathfrak{L}$
$\mathfrak{L} = \mathfrak{so}(3)$ $[e_1, e_2] = e_3$ $[e_2, e_3] = e_1$ $[e_3, e_1] = e_2$	not possible	$\begin{aligned} \xi &= e_1 \\ J\xi &= ae_1 + b(e_0 + e_2) \\ \eta &= be_3 \\ J\eta &= \frac{2yz(a^2+1)}{z^2 + 2abz - b^2} e_1 + ye_2 + ze_3 + \\ &\quad + \frac{z^2 + 2abz - b^2 + y^2}{y} e_0 \end{aligned}$ with $b \neq 0$ and $z^2 + 2abz - b^2 \neq 0$

Table 3.2: CASE B: $\mathfrak{g} = \mathfrak{so}(3) \oplus \mathfrak{r}$ with fundamental $\mathcal{V}_p$.

Here, $(\xi, J\xi, \eta, J\eta)$ is not the adapted frame in general and the Nijenhuis tensor of ξ and η is given by (3.2).

3.2.2 Classification for $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{r}$

For the classification when $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{r}$, we first need some notation and terminology.

Definition 3.1. If $\mathfrak{L}$ is a Lie algebra of dimension n, we recall that for any $X \in \mathfrak{L}$, the characteristic polynomial of adX is of the form

$$\sum_{i=0}^{n} a_i(X)t^i, \qquad \text{with } a_i(X) \in \mathbb{R}.$$

If k is the smallest integer such that $a_k \neq 0$, an element is called *regular* when $a_k(X) \neq 0$.

***Remark* 3.4.** (see [9], ch. 7, sec. 2.2, p.16) Let $\mathfrak{L} = \mathfrak{sl}(2, \mathbb{R})$, then the set of its regular elements is formed by the matrices with not null determinant (and the smallest integer k in the definition is 1).

Proposition 3.2.3. *Let $\mathfrak{L} = \mathfrak{sl}(2, \mathbb{R})$. If (H, X, Y) is a base of $\mathfrak{L}$ such that*

$$[X, H] = X, \quad [X, Y] = 2H, \quad [H, Y] = Y, \tag{3.3}$$

then X and Y are non-regular elements and H is a regular element. If (H, W, Z) is a base of $\mathfrak{L}$ such that

$$[H, Z] = -W, \quad [H, W] = Z, \quad [Z, W] = 2H, \tag{3.4}$$

then H, W, Z are regular elements.

Vice versa, if X is a non-regular element of $\mathfrak{L}$, there exists a base (X, Y, H) containing X such that (3.3) *holds; if H is a regular element of $\mathfrak{L}$, we have one and only one of the following cases:*

1. *there exists a base (H, X, Y) of $\mathfrak{L}$ containing H, such that* (3.3) *holds and such that X, Y are non-regular elements, whereas H is a regular element (iff $\det H < 0$);*

2. *there exists a base (H, W, Z) of $\mathfrak{L}$ containing H, such that* (3.4) *holds and such that H, Z, W are regular elements (iff $\det H > 0$).*

Proof. Let (H, X, Y) be a base of $\mathfrak{L}$ such that (3.3) holds, then the matrix of adX with respect to this base is of the form

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 1 & 0 & 0 \end{pmatrix}.$$

It has the characteristic polynomial given by $p_\lambda(adX) = -\lambda^3$, which has the coefficient of the term of first degree equal to zero. So X is non-regular. Similarly, we obtain the same result for Y.

The matrix of adH with respect to this base above is of the form

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

that has the characteristic polynomial given by $p_\lambda(adX) = -\lambda^3 - \lambda$. Since the term of the first degree is not null, the element H is regular.

Vice versa, let X be a non-regular element of $\mathfrak{L}$, then its rank is one (since the regular elements of $\mathfrak{sl}(2, \mathbb{R})$ are matrices with not null determinant).

Hence, X must be similar to the matrix $\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$, that is, there exists a matrix A, with $\det A \neq 0$, such that

$$X = A \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} A^{-1}.$$

It is sufficient to consider Y and H as

$$Y = -A \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} A^{-1}, \qquad H = -\frac{1}{2} A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A^{-1},$$

to satisfy the properties (3.3).

Let H be a regular element of $\mathfrak{L}$, then its rank is two. Hence H must be similar to one of these two matrices: $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$, $\begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$.

If H is similar to the first one, then there exists a matrix A, with $\det A \neq 0$ such that

$$H = -\frac{1}{2} A \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} A^{-1}.$$

Then it is sufficient to consider X and Y as

$$X = A \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} A^{-1}, \qquad Y = -A \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} A^{-1},$$

to satisfy the properties (3.3).

If H is similar to the second one, then there exists a matrix A, with $\det A \neq 0$, such that

$$H = \frac{1}{2} A \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} A^{-1}.$$

Then it is sufficient to consider Z and W as

$$Z = \frac{1}{2} A \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix} A^{-1}, \qquad W = -\frac{1}{2} A \begin{pmatrix} 1 & -1 \\ -1 & -1 \end{pmatrix} A^{-1},$$

to satisfy the properties (3.4). $\square$

The proof of the following proposition is straightforward.

Proposition 3.2.4. *If $\mathfrak{L} = \mathfrak{sl}(2, \mathbb{R})$, then*

1. *every plane in $\mathfrak{L}$ contains a regular element;*
2. *every plane in $\mathfrak{L}$ contains a unique line made by non-regular elements if and only if it is a subalgebra of $\mathfrak{L}$;*
3. *a plane of $\mathfrak{L}$, which is not a subalgebra, either contains all regular elements (except the null matrix), or it has exactly two distinct lines made by non-regular elements.*

Now that we have stated the useful properties, we can proceed with the classification.

CASE C: $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}$, $\mathcal{V}_p$ non-fundamental and $\mathcal{V}_p \not\subseteq \mathfrak{sl}(2,\mathbb{R})$

Proposition 3.2.5. *Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}$. If $\mathcal{V}_p$ is non-fundamental with $\mathcal{V}_p \not\subseteq \mathfrak{L}$, then there exists a base (H, X, Y) of $\mathfrak{L}$ such that* (3.3) *holds and*

$$\mathcal{V}_{-2|p} =< e_0, X, H >,$$

$$\mathcal{V}_{-2|p} \cap \mathfrak{L} =< X, H >,$$

where (e_0) is a base of $\mathfrak{r}$.

Proof. Since $\mathcal{V}_{-2|p}$ and $\mathfrak{L}$ are subalgebras, the plane defined by $\pi := \mathcal{V}_{-2|p} \cap \mathfrak{L}$ is a subagebra of $\mathfrak{L}$, so, for Proposition 3.2.4, π contains a unique line made by non-regular elements. If X is an element of this line, for Proposition 3.2.3 there exists a base (X, Y, H) of $\mathfrak{L}$ such that Y is non-regular, H is regular, and (3.3) holds. Hence $\pi =< X, aY + bH >$, for certain $a, b \in \mathbb{R}$. Since π is a subalgebra, it must be $a = 0$, so that we can rewrite $\pi =< X, H >$. From $\pi \subseteq \mathcal{V}_{-2|p}$ and $\mathcal{V}_p \not\subseteq \mathfrak{L}$, we have $\mathcal{V}_{-2|p} =< e_0 + cY, X, H >$, for some $c \in \mathbb{R}$, but since $\mathcal{V}_{-2|p}$ is a subalgebra, the coefficient c must be zero, hence $\mathcal{V}_{-2|p} =< e_0, X, H >$. □

Corollary 3.2.6. *Let $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}$. If $\mathcal{V}_p$ is non-fundamental, with $\mathcal{V}_p \not\subseteq \mathfrak{L}$, then $< v >= \mathcal{V}_p \cap \mathfrak{L}$ is a line made by regular elements. In particular, there exists a base (H, X, Y) of $\mathfrak{L}$ such that* (3.3) *holds and*

$$\mathcal{V}_p \cap \mathfrak{L} =< H >,$$

$$\mathcal{V}_p =< H, e_0 + aX >, \quad a \neq 0.$$

Proof. From Proposition 3.2.5, we have $< v >:= \mathcal{V}_p \cap \mathfrak{L} \subseteq \pi := \mathcal{V}_{-2|p} \cap \mathfrak{L} =< X, H >$. If v is a non-regular element, we have $< v >=< X >$ because $< X >$ is the unique non-regular line of π (see Proposition 3.2.4). Therefore, we can write $\mathcal{V}_p =< X, e_0 + bH >$, for some $b \in \mathbb{R}$, since, for Proposition 3.2.5, (e_0, X, H) is a base of $\mathcal{V}_{-2|p}$ and $\mathcal{V}_p \not\subseteq \mathfrak{L}$. In this way $\mathcal{V}_p$ is degenerate, against our assumption. Hence v must be a regular element and the base (X, Y, H) of $\mathfrak{L}$ can be chosen such that $< v >=< H >$. As a consequence, $\mathcal{V}_p =< H, e_0 + aX >$, where a is not null because of the non degeneracy of $\mathcal{V}_p$. □

When $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}$ and $\mathcal{V}_p$ is non-fundamental, with $\mathcal{V}_p \not\subseteq \mathfrak{L}$, the Corollary 3.2.6 asserts that there exists a base (H, X, Y) of $\mathfrak{L}$ such that (3.3) holds and $\mathcal{V}_p =< H, e_0 + aX >$, with $a \neq 0$. Hence a base $(\xi, J\xi)$ of $\mathcal{V}_p$ can be defined taking $\xi = H$ and $J\xi = e_0 + aX + bH$, for some $a, b \in \mathbb{R}$, with $a \neq 0$. Now, since $\eta = [\xi, J\xi]$, we have that $\eta = -aX$, while $J\eta$ is the linear combination of X, Y, H, e_0, for certain $x, y, z, t \in \mathbb{R}$ such that $N_J(\xi, \eta)$ is the linear combination of ξ and $J\xi$. Then we have

$$\begin{aligned}
&\xi = H, \\
&J\xi = e_0 + aX + bH, \quad a \neq 0, \\
&\eta = -aX, \\
&J\eta = xX + yY + zH + te_0, \quad y \neq 0,
\end{aligned}$$

where a, b, x, y, z, t are appropriate coefficients in $\mathbb{R}$ such that the calculation of $N_J(\xi, \eta)$ only depends on ξ and $J\xi$, that is, since we have

$$\begin{aligned}
N_J(\xi, \eta) = \quad & (2ay + b^2t - bz - t)\xi + (z - 2bt)J\xi + \\
& + \left(\frac{2bx}{a} - z - bt + 2\right)\eta + \left(t + 2b - \frac{2x}{a}\right)J\eta,
\end{aligned}$$

it must be

$$\begin{cases} \dfrac{2bx}{a} - z - bt + 2 = 0, \\ t + 2b - \dfrac{2x}{a} = 0. \end{cases}$$

This gives

$$\begin{cases} z = 2(b^2 + 1), \\ x = \dfrac{a}{2}(t + 2b), \end{cases}$$

so that we obtain

$$\begin{aligned}
&\xi = H, \\
&J\xi = e_0 + aX + bH, \quad a \neq 0, \\
&\eta = -aX, \\
&J\eta = \frac{a}{2}(t + 2b)X + yY + 2(b^2 + 1)H + te_0, \quad y \neq 0,
\end{aligned}$$

and

$$N_J(\xi, \eta) = (2ay + b^2t - t - 2b^3 - 2b)\xi + 2(b^2 + 1 - bt)J\xi. \tag{3.5}$$

To calculate the adapted frame (X, JX, T, JT), we have to find α and β in $\mathbb{R}$ such that the linear combination $\alpha\xi + \beta J\xi$ is the distinguished field X of $\mathcal{V}_p$, that is

$$N_J(\alpha\xi + \beta J\xi, (\alpha^2 + \beta^2)\eta) = \alpha\xi + \beta J\xi.$$

In conclusion, if (M^4, J) is a locally homogeneous almost complex manifold with non-fundamental and non-degenerate torsion bundle $\mathcal{V}$, and the Lie algebra $\mathfrak{g}$ associated to (M^4, J) is non solvable with $\mathfrak{g} = sl(2, \mathbb{R}) \oplus \mathfrak{r}$, we have the following classification for the almost complex structure J by varying the coefficients $a, b, t \in \mathbb{R}$:

Here $(\xi, J\xi, \eta, J\eta)$ is not the adapted frame in general and the Nijenhuis tensor of ξ and η is given by (3.5).

	$\mathcal{V}$ is non-fundamental
	$\mathcal{V} \not\subseteq \mathfrak{L}$
$\mathfrak{L} = sl(2,\mathbb{R})$ $[X,H] = X$ $[X,Y] = 2H$ $[H,Y] = Y$	$\xi = H$ $J\xi = e_0 + aX + bH$ $\eta = -aX$ $J\eta = \frac{a}{2}(t+2b)X + yY + 2(b^2+1)H + te_0$ with $a \neq 0, y \neq 0$

Table 3.3: CASE C: $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}$, with non-fundamental $\mathcal{V}_p$ and with $\mathcal{V}_p \not\subseteq \mathfrak{sl}(2,\mathbb{R})$.

CASE D: $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}$, $\mathcal{V}_p$ non-fundamental and $\mathcal{V}_p \subseteq \mathfrak{sl}(2,\mathbb{R})$

Consider the case in which $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}$ and $\mathcal{V}_p$ is non-fundamental, with $\mathcal{V}_p \subseteq \mathfrak{L}$. Since $\mathcal{V}_p$ is a plane of $\mathfrak{sl}(2,\mathbb{R})$ which is not a subalgebra (because we are assuming that $\mathcal{V}_p$ is non-degenerate at any point p of (M^4, J)), from the Proposition 3.2.4 we have that $\mathcal{V}_p$ either contains all regular elements (except the null matrix) or it has exactly two distinct lines made by non regular elements. Hence, we have two cases: CASE D1, when $\mathcal{V}_p$ contains all regular elements, and CASE D2, when $\mathcal{V}_p$ has exactly two distinct lines made by non regular elements.

CASE D1: $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}$, $\mathcal{V}_p$ non-fundamental and $\mathcal{V}_p \subseteq \mathfrak{sl}(2,\mathbb{R})$

One can prove that if a plane have all regular elements, they all have negative determinant. By the Proposition 3.2.3, this implies that for any regular element H of $\mathcal{V}_p$ there exist a base of $\mathfrak{sl}(2,\mathbb{R})$ of type (H, X, Y) having H as regular element and X, Y as non-regular elements of $\mathfrak{sl}(2,\mathbb{R})$.

If H is such that the base of $\mathfrak{sl}(2,\mathbb{R})$ is of the type (H, X, Y) and (3.3) holds, then we have

$$\begin{aligned}
&\xi = kH, \quad k \neq 0, \\
&J\xi = aX + bY + cH, \quad ab \neq 0, \\
&\eta = k(-aX + bY), \\
&J\eta = e_0 + xX + yY + zH,
\end{aligned}$$

for certain k, a, b, c, x, y, z in $\mathbb{R}$ taken such that the Nijenhuis tensor of ξ and

η only depends on ξ and $J\xi$, that is, since the calculation gives

$$
\begin{aligned}
N_J(\xi,\eta) = \quad & \frac{((-ak^2+ac^2-4a^2b)y+(bk^2-bc^2+4ab^2)x-4abck^2)}{2abk}\xi+ \\
& +\frac{(-2acky+2bckx+2abk^3+(8a^2b^2-2abc^2)k)}{2abk}J\xi+ \\
& +\frac{(2abz-acy-bcx)}{2abk}\eta+\frac{(aky+bkx)}{2abk}J\eta,
\end{aligned}
$$

and since the coefficients of η and $J\eta$ must be zero, we obtain

$$
\begin{cases} 2abz-acy-bcx=0, \\ aky+bkx=0, \end{cases}
$$

that is

$$
\begin{cases} ay+bx=0, \\ z=0. \end{cases}
$$

As a consequence, obtaining x from the system and replacing it into the formula of $N_J(\xi,\eta)_p$, we have

$$
N_J(\xi,\eta) = \frac{(k^2-c^2+4ab)y+2bck^2}{bk}\xi - \frac{2cky-bk^3+(bc^2-4ab^2)k}{bk}J\xi.
$$

Another way to interpret the first equation of the system is as a proportion between $-aX+bY$ and $xX+yY$ in the definition of η and $J\eta$. This leads to have

$$
\begin{aligned}
&\xi = kH, \quad k\neq 0, \\
&J\xi = aX+bY+cH, \quad ab\neq 0, \\
&\eta = k(-aX+bY), \\
&J\eta = e_0+t(-aX+bY),
\end{aligned}
$$

for any t in $\mathbb{R}$ and the Nijenhuis tensor becomes

$$
N_J(\xi,\eta) = \frac{((k^2-c^2+4ab)t+2ck^2)}{k}\xi + (2ct-k^2+c^2-4ab)J\xi. \tag{3.6}
$$

Here $(\xi, J\xi, \eta, J\eta)$ is not the adapted frame in general, but since the distinguished field of $\mathcal{V}_p$ must have negative determinant (because all the elements of $\mathcal{V}_p$ are regular with negative determinant), we can compute k, a, b, c, t such that H is exactly the distinguished field. In other words, we impose that $N_J(\xi,\eta)=\xi$:

$$
\begin{cases} \dfrac{(k^2-c^2+4ab)t+2ck^2}{k} = 1, \\ 2ct-k^2+c^2-4ab=0. \end{cases}
$$

It gives, for example, a and c as functions of t, k, b:

$$\begin{cases} a = \dfrac{-k(4kt^4 - 4t^3 + 8k^3t^2 - 4k^2t + 4k^5 - k)}{16b(t^2+k^2)^2}, \\[2ex] c = \dfrac{k}{2(t^2+k^2)}. \end{cases}$$

In conclusion, if (M^4, J) is a locally homogeneous almost complex manifold with non-fundamental and non-degenerate torsion bundle $\mathcal{V}$ such that $\mathcal{V}_p$ is a plane containing all regular elements (except the null matrix), for any p in M, and the Lie algebra $\mathfrak{g}$ associated to (M^4, J) is non solvable with $\mathfrak{g} = sl(2,\mathbb{R}) \oplus \mathfrak{r}$, we have the following classification for the almost complex structure J by varying the coefficients $t, k, b \in \mathbb{R}$:

	$\mathcal{V}$ is non-fundamental
	$\mathcal{V} \subseteq \mathfrak{L}$
	$\mathcal{V}$ all is made by regular elements (except the null matrix)
$\mathfrak{L} = sl(2,\mathbb{R})$ $[X, H] = X$ $[X, Y] = 2H$ $[H, Y] = Y$	$\xi = kH$ $J\xi = a_{k,t,b}X + bY + c_{k,t,b}H$ $\eta = k(-a_{k,t,b}X + bY)$ $J\eta = e_0 + t(-a_{k,t,b}X + bY)$ with $a_{k,t,b}bk \neq 0$

Table 3.4: CASE D1: $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}$, with non-fundamental $\mathcal{V}_p$ and with $\mathcal{V}_p \subseteq \mathfrak{sl}(2,\mathbb{R})$.

Here $(\xi, J\xi, \eta, J\eta)$ is the adapted frame and $a_{k,t,b}$ are functions depending on the parameters t, k, b in $\mathbb{R}$ and are given by the solutions of the system:

$$\begin{cases} a = \dfrac{-k(4kt^4 - 4t^3 + 8k^3t^2 - 4k^2t + 4k^5 - k)}{16b(t^2+k^2)^2}, \\[2ex] c = \dfrac{k}{2(t^2+k^2)}. \end{cases}$$

CASE D2: $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}$, $\mathcal{V}_p$ non-fundamental and $\mathcal{V}_p \subseteq \mathfrak{sl}(2,\mathbb{R})$

When $\mathcal{V}_p$ is a plane having exactly two distinct lines made by non-regular elements, by the Proposition 3.2.3, given a non-regular element X of $\mathcal{V}_p$ there exist a base of $\mathfrak{sl}(2,\mathbb{R})$ of type (H, X, Y), with H as regular element and X, Y as non-regular elements of $\mathfrak{sl}(2,\mathbb{R})$ such that (3.3) holds.

We can take a base $(\xi, J\xi)$ of $\mathcal{V}_p$ and $\eta = [\xi, J\xi], J\eta$ as (up to rescaling e_0)

$$\begin{aligned}
&\xi = X,\\
&J\xi = aX + bY + cH \quad b \neq 0,\\
&\eta = 2bH + cX,\\
&J\eta = e_0 + xX + yY + zH,
\end{aligned}$$

for certain a, b, c, x, y, z in $\mathbb{R}$ taken such that the Nijenhuis tensor of ξ and η, given by

$$\begin{aligned}
N_J(\xi, \eta) = \;& \frac{(4ab^2 - bc^2)z + (c^3 - 4abc)y - 8b^3}{2b^2}\xi + \\
&+ \frac{-4b^2 z + 4bcy + 2b^2c^2 - 8ab^3}{2b^2} J\xi + \\
&+ \frac{bcz - (c^2 - 2ab)y - 2b^2 x}{2b^2}\eta - \frac{y}{b} J\eta,
\end{aligned}$$

only depends on ξ and $J\xi$; this means that the coefficients of η and $J\eta$ must be zero:

$$\begin{cases} bcz - (c^2 - 2ab)y - 2b^2 x = 0, \\ \\ y = 0, \end{cases}$$

so that we obtain

$$\begin{cases} x = \dfrac{cz}{2b}, \\ \\ y = 0. \end{cases}$$

Because of this system, we can redefine

$$\begin{aligned}
&\xi = X,\\
&J\xi = aX + bY + cH \quad b \neq 0,\\
&\eta = 2bH + cX,\\
&J\eta = e_0 + \frac{cz}{2b}X + zH,
\end{aligned}$$

and the Nijenhuis tensor of ξ and η becomes

$$N_J(\xi, \eta) = \left(-\frac{c^2 z}{2b} + 2a - 4b \right) \xi + (c^2 - 2z - 4ab) J\xi. \tag{3.7}$$

Here $(\xi, J\xi, \eta, J\eta)$ is not the adapted frame in general. To calculate the distinguished field, it is sufficient to find the opportune linear combination of ξ and $J\xi$ such that $N_J(\alpha\xi + \beta J\xi, [\alpha\xi + \beta J\xi, \alpha J\xi - \beta\xi]) = \alpha\xi + \beta J\xi$ (the calculation is similar to that of the previous cases). There is no reason why $\xi = X$, which is a non-regular element, must be the distinguished field. Indeed, one can find examples in which the distinguished field of the plane $\mathcal{V}_p$ is regular or non-regular.

In conclusion, if (M^4, J) is a locally homogeneous almost complex manifold with non-fundamental and non-degenerate torsion bundle $\mathcal{V}$, such that $\mathcal{V}_p$ is a plane containing exactly two distinct lines made by non-regular elements for any p in M, and the Lie algebra $\mathfrak{g}$ associated to (M^4, J) is non solvable with $\mathfrak{g} = sl(2, \mathbb{R}) \oplus \mathfrak{r}$, we have the following classification for the almost complex structure J by varying the coefficients $a, b, c, z \in \mathbb{R}$:

	$\mathcal{V}$ is non-fundamental
	$\mathcal{V} \subseteq \mathfrak{L}$
	$\mathcal{V}$ has exactly two lines made by non-regular elements
$\mathfrak{L} = sl(2, \mathbb{R})$ $[X, H] = X$ $[X, Y] = 2H$ $[H, Y] = Y$	$\xi = X$ $J\xi = aX + bY + cH$ $\eta = 2bH + cX$ $J\eta = e_0 + \frac{cz}{2b}X + zH$ with $b \neq 0$

Table 3.5: CASE D2: $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{r}$, with non-fundamental $\mathcal{V}_p$ and with $\mathcal{V}_p \subseteq \mathfrak{sl}(2, \mathbb{R})$.

Here $(\xi, J\xi, \eta, J\eta)$ is not the adapted frame in general and the Nijenhuis tensor is given by (3.7).

CASE E: $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{r}$, $\mathcal{V}_p$ fundamental and $\mathcal{V}_p \not\subseteq \mathfrak{sl}(2, \mathbb{R})$

When $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{r}$ and $\mathcal{V}_p$ is fundamental, it must be $\mathcal{V}_p \not\subseteq \mathfrak{sl}(2, \mathbb{R})$ (as a consequence of Remark 3.3). As we are going to see, in this context it is useful to consider the plane π, defined as the intersection of $\mathcal{V}_{-2|p}$ and $\mathfrak{sl}(2, \mathbb{R})$, and the line $< v >$, defined as the intersection of $\mathcal{V}_p$ and $\mathfrak{sl}(2, \mathbb{R})$. There are three cases, CASE E1, CASE E2 and CASE E3, depending on the vector v: when v is regular of negative determinant, v is regular of positive determinant, and v is non-regular (of null determinant) respectively.

Proposition 3.2.7. *Let $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{r}$. If $\mathcal{V}_p$ is fundamental (so $\mathcal{V}_p \not\subseteq \mathfrak{L}$), then we have one of the following cases.*

1. *There exists a base (X, Y, H) of $\mathfrak{L}$ such that (3.3) holds and $\mathcal{V}_p =< H, e_0 + aX + bY >$, $ab \neq 0$ and $\mathcal{V}_p \cap \mathfrak{L} =< H >$. In particular, $\pi := \mathcal{V}_{-2|p} \cap \mathfrak{L}$ is plane which is not a subalgebra of $\mathfrak{L}$ and*
   - *if $ab > 0$, then all elements of π are regular;*

- *if $ab < 0$, then π has exactly two lines made by non-regular elements.*

2. *There exists a base (K, W, Z) of $\mathfrak{L}$ such that (3.4) holds and $\mathcal{V}_p =< K, e_0 + cW >, \quad c \neq 0$ and $\mathcal{V}_p \cap \mathfrak{L} =< K >$. In particular, $\pi := \mathcal{V}_{-2|p} \cap \mathfrak{L}$ is plane which is not a subalgebra of $\mathfrak{L}$ and π has exactly two lines made by non-regular elements.*

3. *There exists a base (X, Y, H) of $\mathfrak{L}$ such that (3.3) holds and $\mathcal{V}_p =< X, e_0 + aY + bH >, \quad a \neq 0$ and $\mathcal{V}_p \cap \mathfrak{L} =< X >$. In particular, $\pi := \mathcal{V}_{-2|p} \cap \mathfrak{L}$ is a subalgebra of $\mathfrak{L}$ and π has exactly one line made by non-regular elements.*

Proof. Since $\mathcal{V}_p$ is fundamental, we have that $\mathcal{V}_{-2|p}$ is not a subalgebra of $\mathfrak{g}$, but $\pi := \mathcal{V}_{-2|p} \cap \mathfrak{L}$ can be or cannot be a subalgebra of $\mathfrak{g}$.

Let us first consider that π is a plane which is not a subalgebra. Then, for the Proposition 3.2.4, we have the following two possibilities.

(a) All the elements of π are regular. Since $< v >:= \mathcal{V}_p \cap \mathfrak{L}$ is in π, then v is regular. So, according to Proposition 3.2.3, there exist two kinds of basis of $\mathfrak{L}$, one of the type (X, Y, H), such that (3.3) holds and such that $< v >=< H >$ and $\mathcal{V}_p =< H, e_0 + aX + bY >$ (since $\mathcal{V}_p$ is fundamental and not degenerate, we have $ab \neq 0$; from the fact that π only contains regular elements, an easy calculation shows that $ab > 0$). The other base is of the type (K, W, Z), such that (3.4) holds and such that $< v >=< K >$ and $\mathcal{V}_p =< K, e_0 + cW + dZ >$. An easy calculation shows that this last case never happens, since π only contains regular elements.

(b) The palne π has exactly two lines of non-regular elements. If $< v >:= \mathcal{V}_p \cap \mathfrak{L}$ is non-regular, from the Proposition 3.2.3 there exist a base of $\mathfrak{L}$ of the type (X, Y, H), such that (3.3) holds and such that $< v >=< X >$ and $\mathcal{V}_p =< X, e_0 + aY + bH >$. Hence, $\mathcal{V}_{-2|p} =< X, e_0 + aY + bH, H >$, from which $\pi =< X, H >$. Since π has exactly two lines of non-regular elements, a calculation shows that this case never happens. This implies that $< v >$ is regular, so, there exist two kinds of basis, one of the type (X, Y, H), such that (3.3) holds and such that $< v >=< H >$ and $\mathcal{V}_p =< H, e_0 + aX + bY >$ (since $\mathcal{V}_p$ is fundamental and not degenerate, we have $ab \neq 0$; from the fact that π has exactly two lines of non-regular elements, an easy calculation shows that $ab < 0$). The second base is of the type (K, W, Z), such that (3.4) holds and such that $< v >=< K >$ and $\mathcal{V}_p =< K, e_0 + cW + dZ >$ (since $\mathcal{V}_p$ is fundamental and not degenerate, we have $cd \neq 0$); in particular, we can chose a base of this type such that $\mathcal{V}_p =< K, e_0 + cW >$, with $c \neq 0$ (an easy calculation shows that effectively π has exactly two lines of non-regular elements).

Now let us consider that π is a subalgebra 2-dimensional of $\mathfrak{L}$, then π has a unique line of non-regular elements. There exist a base (X, Y, H) such that (3.3) holds and such that $\pi =< X, H + aY >$, but the constant a is zero, since π is a subalgebra.

If $< v >$ is non-regular, then $< v >=< X >$ (since $< X >$ is the unique line of non-regular elements) and so $\mathcal{V}_p =< X, e_0 + aY + bH >$ (with $a \neq 0$ because $\mathcal{V}_p$ is not degenerate and fundamental).

If $< v >$ is regular, then $< v >=< H + \mu X >$ and the base (X, Y, H) can be chosen such that $\mu = 0$ and $\mathcal{V}_p =< H, e_0 + aX + bY >$ (with $ab \neq 0$ because $\mathcal{V}_p$ is not degenerate and fundamental), so that $\pi =< X, H >$. We have $\mathcal{V}_{-2|p} =< H, e_0 + aX + bY, -aX + bY >$, from which $\pi =< H, -aX + bY >$. As a consequence, $b = 0$ and this shows that such a case cannot happen.

$\square$

CASE E1: $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}$, $\mathcal{V}_p$ fundamental, $\mathcal{V}_p \not\subseteq \mathfrak{sl}(2,\mathbb{R})$ and v regular of negative determinant

Since v is regular of negative determinant, there exists a base (H, X, Y) of $\mathfrak{L}$ such that (3.3) holds and $v = H$, then, for Proposition 3.2.7, we have

$$\begin{aligned}
&\xi = H,\\
&J\xi = e_0 + aX + bY + cH, \quad ab \neq 0,\\
&\eta = -aX + bY,\\
&J\eta = te_0 + xX + yY + zH,
\end{aligned}$$

for certain a, b, c, x, y, z, t in $\mathbb{R}$ taken such that the Nijenhuis tensor of ξ and η only depends on ξ and $J\xi$. The calculation gives

$$\begin{aligned}
N_J(\xi,\eta) = \quad & \frac{1}{2abt - ay - bx}\Bigg\{\Big(acyz - bcxz - 2abz - 2a^2y^2 - ac^2ty + 4a^2bty +\\
&+ aty + 2b^2x^2 + bc^2tx - 4ab^2tx - btx + 4abct\Big)\xi +\\
&+\Big(ayz - bxz + 2abcz - 2acty - 4a^2by + 2bctx - 4ab^2x +\\
&-2abc^2t + 8a^2b^2t + 2abt\Big)J\xi +\\
&+\Big(ayz + bxz - 2abtz - 2cxy + acty - 2ay + bctx + 2bx - 2abc\Big)\eta +\\
&+\Big(2xy - aty - 2acy + 2bcx - btx + 2ab\Big)J\eta\Bigg\},
\end{aligned}$$

where the denominator $2abt - ay - bx$ is not zero because it is equivalent to the condition of linear independence between $\xi, J\xi, \eta, J\eta$. We impose that the coefficients of η and $J\eta$ must be zero:

$$\begin{cases}
ayz + bxz - 2abtz - 2cxy + acty - 2ay + bctx + 2bx - 2abc = 0,\\[2mm]
2xy - aty - 2acy + 2bcx - btx + 2ab = 0,
\end{cases}$$

that is, solving in z and t:

$$\begin{cases} z = \dfrac{2(c^2+1)(ay-bx)(ay+bx)}{a^2y^2 - 2abxy + 4a^2bcy + b^2x^2 - 4ab^2cx - 4a^2b^2}, \\ t = \dfrac{2(xy - acy + bcx + ab)}{ay+bx}. \end{cases}$$

Here $(\xi, J\xi, \eta, J\eta)$ is not the adapted frame in general. To calculate the distinguished frame it is sufficient to do the same calculation seen for some of the previous cases.

In conclusion, if (M^4, J) is a locally homogeneous almost complex manifold with non-fundamental and non-degenerate torsion bundle $\mathcal{V}$, and the Lie algebra $\mathfrak{g}$ associated to (M^4, J) is non solvable with $\mathfrak{g} = sl(2,\mathbb{R}) \oplus \mathfrak{r}$, such that $< v >:= \mathcal{V}_p \cap \mathfrak{sl}(2,\mathbb{R})$ is a negative determinant regular element (for p in M), we have the following classification for the almost complex structure J by varying the coefficients $a, b, c, x, y \in \mathbb{R}$:

	$\mathcal{V}$ is fundamental	
	$\mathcal{V} \subseteq \mathfrak{L}$	$\mathcal{V} \not\subseteq \mathfrak{L}$
		v with negative determinant
$\mathfrak{L} = sl(2,\mathbb{R})$ $[X,H] = X$ $[X,Y] = 2H$ $[H,Y] = Y$	not possible	$\xi = H$ $J\xi = e_0 + aX + bY + cH$ $\eta = -aX + bY$ $J\eta = xX + yY + z_{x,y,a,b,c}H + t_{x,y,a,b,c}e_0$ with $(a,b) \neq (0,0)$ with $ay + bx - 2abt_{x,y,a,b,c} \neq 0$

Table 3.6: CASE E1: $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}$, with fundamental $\mathcal{V}_p$ and with $\mathcal{V}_p \not\subseteq \mathfrak{sl}(2,\mathbb{R})$. The vector v, defined by $< v >= \mathfrak{sl}(2,\mathbb{R}) \cap \mathcal{V}_p$, is regular of negative determinant.

Here $z_{x,y,a,b,c}$ and $t_{x,y,a,b,c}$ are functions depending on x, y, a, b, c, given by the system:

$$\begin{cases} z_{x,y,a,b,c} = \dfrac{2(c^2+1)(ay-bx)(ay+bx)}{a^2y^2 - 2abxy + 4a^2bcy + b^2x^2 - 4ab^2cx - 4a^2b^2}, \\ t_{x,y,a,b,c} = \dfrac{2(xy - acy + bcx + ab)}{ay+bx}. \end{cases} \tag{3.8}$$

and $(\xi, J\xi, \eta, J\eta)$ is not the adapted frame in general.

CASE E2: $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R})\oplus\mathfrak{r}$, $\mathcal{V}_p$ fundamental, $\mathcal{V}_p \not\subseteq \mathfrak{sl}(2,\mathbb{R})$ and v regular of positive determinant

Since v is regular of positive determinant, there exists a base (H, W, Z) of $\mathfrak{L}$ such that (3.4) holds and $v = H$, then, for Proposition 3.2.7, we have

$$\begin{aligned}
&\xi = H,\\
&J\xi = e_0 + aH + bW, \quad b \neq 0,\\
&\eta = bZ,\\
&J\eta = xH + yW + zZ + te_0,
\end{aligned}$$

for certain a, b, x, y, z, t in $\mathbb{R}$ taken such that the Nijenhuis tensor of ξ and η only depends on ξ and $J\xi$. The calculation gives

$$\begin{aligned}
N_J(\xi,\eta) = \;& \frac{2byz - axz - 2b^2tz + a^2tz - tz + bx - 2abt}{bt - y}\xi +\\
&+\frac{xz - 2atz - 2b^2y + abx + 2b^3t - a^2bt + bt}{bt - y}J\xi +\\
&+\frac{ab^2 - az^2 + 2bz - ay^2 + bxy + abty - b^2tx}{b(bt - y)}\eta +\\
&+\frac{z^2 + 2abz + y^2 - bty - b^2}{b(bt - y)}J\eta,
\end{aligned}$$

where the denominator $bt - y$ is not zero because it is equivalent to the condition of linear independence between $\xi, J\xi, \eta, J\eta$. We impose that the coefficients of η and $J\eta$ must be zero:

$$\begin{cases}
ab^2 - az^2 + 2bz - ay^2 + bxy + abty - b^2tx = 0,\\[2mm]
z^2 + 2abz + y^2 - bty - b^2 = 0,
\end{cases}$$

that is, solving in x and t:

$$\begin{cases}
x = \dfrac{2yz(a^2+1)}{z^2 + 2abz - b^2},\\[3mm]
t = \dfrac{z^2 + 2abz - b^2 + y^2}{by}.
\end{cases}$$

This gives the following Nijenhuis tensor of ξ and η:

$$N_J(\xi,\eta) = \frac{1}{C}(A\xi + BJ\xi), \tag{3.9}$$

where

$$\begin{aligned}
A = \;& (-2b^2+a^2-1)z^5+((4a^3-6a)b-8ab^3)z^4+\\
&+((a^2-1)y^2+(4-8a^2)b^4+(4a^4-14a^2+2)b^2)z^3+\\
&+(-6aby^2+8ab^5+(8a-12a^3)b^3)z^2+\\
&+((3-3a^2)b^2y^2-2b^6+(9a^2-1)b^4)z+2ab^3y^2-2ab^5,\\
B = \;& -2az^5+(2b^3+(1-9a^2)b)z^4+(-2ay^2+8ab^4+(8a-12a^3)b^2)z^3+\\
&+((3-3a^2)by^2+(8a^2-4)b^5+(-4a^4+14a^2-2)b^3)z^2+\\
&+(2a(3b^2y^2-4b^6)+(4a^3-6a)b^4)z+(a^2-1)b^3y^2+(2b^2+1-a^2)b^5,\\
C = \;& b(z^2+2abz-b^2)^2.
\end{aligned}$$

Here $(\xi, J\xi, \eta, J\eta)$ is not the adapted frame in general. To calculate the distinguished frame it is sufficient to do the same calculation seen for some of the previous cases.

In conclusion, if (M^4, J) is a locally homogeneous almost complex manifold with non-fundamental and non-degenerate torsion bundle $\mathcal{V}$, and the Lie algebra $\mathfrak{g}$ associated to (M^4, J) is non solvable with $\mathfrak{g} = sl(2,\mathbb{R}) \oplus \mathfrak{r}$, such that $< v >:= \mathcal{V}_p \cap \mathfrak{sl}(2,\mathbb{R})$ is a positive determinant regular element (for p in M), we have the following classification for the almost complex structure J by varying the coefficients $a, b, y, z \in \mathbb{R}$:

	$\mathcal{V}$ is fundamental	
	$\mathcal{V} \subseteq \mathfrak{L}$	$\mathcal{V} \not\subseteq \mathfrak{L}$
		v with positive determinant
$\mathfrak{L} = sl(2,\mathbb{R})$ $[Z,H] = W$ $[Z,W] = 2H$ $[H,W] = Z$	not possible	$\xi = H$ $J\xi = e_0 + aH + bW$ $\eta = bZ$ $J\eta = x_{a,b,y,z}H + yW + zZ + t_{a,b,y,z}e_0$ with $b \neq 0$ with $bt_{a,b,y,z} - y \neq 0$

Table 3.7: CASE E2: $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}$, with fundamental $\mathcal{V}_p$ and with $\mathcal{V}_p \not\subseteq \mathfrak{sl}(2,\mathbb{R})$. The vector v, defined by $< v >= \mathfrak{sl}(2,\mathbb{R}) \cap \mathcal{V}_p$, is regular of positive determinant.

Here $x_{a,b,y,z}$ and $t_{a,b,y,z}$ are functions depending on a, b, y, z, obtained by the system:

$$\begin{cases} x_{a,b,y,z} = \dfrac{2yz(a^2+1)}{z^2+2abz-b^2}, \\[2ex] t_{a,b,y,z} = \dfrac{z^2+2abz-b^2+y^2}{by}. \end{cases} \tag{3.10}$$

The Nijenhuis tensor is given by (3.9) and $(\xi, J\xi, \eta, J\eta)$ given above is not the adapted frame in general.

CASE E3: $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}$, $\mathcal{V}_p$ fundamental, $\mathcal{V}_p \not\subseteq \mathfrak{sl}(2,\mathbb{R})$ and v non-regular

Since v is non-regular, there exists a base (H, X, Y) of $\mathfrak{L}$ such that (3.3) holds and $v = X$, then, for Proposition 3.2.7, we have

$$\begin{aligned}
&\xi = X,\\
&J\xi = e_0 + aX + bY + cH, \quad bc \neq 0,\\
&\eta = cX + 2bH,\\
&J\eta = xX + yY + zH + te_0,
\end{aligned}$$

for certain a, b, c, x, y, z, t in $\mathbb{R}$ taken such that the Nijenhuis tensor of ξ and η only depends on ξ and $J\xi$. The calculation gives

$$\begin{aligned}
N_J(\xi,\eta) = \;& \frac{1}{2b(bt-y)}\Bigg\{\Bigg(bcz^2 - c^2yz - 2abyz - 2b^2xz - bc^2tz + 4ab^2tz +\\
&+2acy^2 + 2bcxy + c^3ty - 4abcty + 4b^2y - 8b^3t\Bigg)\xi + 2\Bigg(byz +\\
&-2b^2tz - b^2cz - cy^2 + 2bcty + 2ab^2y + 2b^3x + b^2c^2t - 4ab^3t\Bigg)J\xi +\\
&+\Bigg(-bz^2 + cyz + bctz - 2ay^2 + 2bxy - c^2ty + 2abty - 2b^2tx +\\
&+4b^3\Bigg)\eta + 2\Bigg(2b^2z + y^2 - bty - 2bcy\Bigg)J\eta\Bigg\},
\end{aligned}$$

where the denominator $2b(bt - y)$ is not zero because it is equivalent to the condition of linear independence between $\xi, J\xi, \eta, J\eta$. We impose that the coefficients of η and $J\eta$ must be zero

$$\begin{cases}
-bz^2 + cyz + bctz - 2ay^2 + 2bxy - c^2ty + 2abty - 2b^2tx + 4b^3 = 0,\\
2b^2z + y^2 - bty - 2bcy = 0,
\end{cases}$$

that is, solving in x and t:

$$\begin{cases}
x = \dfrac{1}{4b^3(cy-bz)}\Bigg(b^2yz^2 - 2b^3cz^2 - 2bcy^2z + 4b^2c^2yz - 4ab^3yz +\\
\qquad + c^2y^3 - 2bc^3y^2 + 4ab^2cy^2 - 4b^4y\Bigg),\\
t = \dfrac{2b^2z + y^2 - 2bcy}{by}.
\end{cases}$$

This gives the following Nijenhuis tensor of ξ and η:

$$N_J(\xi,\eta) = \frac{A}{C}\xi + \frac{B}{D}J\xi, \tag{3.11}$$

where

$$
\begin{aligned}
A = \; & b^2y^2z^2 - 4b^4c^2z^2 + 16ab^5z^2 - 2bcy^3z + 8b^3c^3yz + \\
& -32ab^4cyz - 32b^6z + c^2y^4 - 4b^2c^4y^2 + 16ab^3c^2y^2 + \\
& -12b^4y^2 + 32b^5cy, \\
B = \; & -8b^4z^3 - 3b^2y^2z^2 + 24b^3cyz^2 + 4b^4c^2z^2 + \\
& -16ab^5z^2 + 6bcy^3z - 24b^2c^2y^2z - 8b^3c^3yz + \\
& +32ab^4cyz - 3c^2y^4 + 8bc^3y^3 + 4b^2c^4y^2 + \\
& -16ab^3c^2y^2 + 4b^4y^2, \\
C = \; & 8b^4(bz - cy), \\
D = \; & 4b^2(bz - cy)^2.
\end{aligned}
$$

Here $(\xi, J\xi, \eta, J\eta)$ is not the adapted frame in general. To calculate the distinguished field, it is sufficient to do the same calculation seen for some of the previous cases.

In conclusion, if (M^4, J) is a locally homogeneous almost complex manifold with non-fundamental and non-degenerate torsion bundle $\mathcal{V}$, and the Lie algebra $\mathfrak{g}$ associated to (M^4, J) is non solvable with $\mathfrak{g} = sl(2,\mathbb{R}) \oplus \mathfrak{r}$, such that $< v >:= \mathcal{V}_p \cap \mathfrak{sl}(2,\mathbb{R})$ is a non-regular element (for p in M), we have the following classification for the almost complex structure J by varying the coefficients $a, b, c, y, z \in \mathbb{R}$:

	$\mathcal{V}$ is fundamental	
	$\mathcal{V} \subseteq \mathfrak{L}$	$\mathcal{V} \nsubseteq \mathfrak{L}$
		v with null determinant
$\mathfrak{L} = sl(2,\mathbb{R})$ $[X, H] = X$ $[X, Y] = 2H$ $[H, Y] = Y$	not possible	$\xi = X$ $J\xi = e_0 + aX + bY + cH$ $\eta = cX + 2bH$ $J\eta = x_{a,b,c,y,z}X + yY + zH + t_{a,b,c,y,z}e_0$ with $b \neq 0$ with $b(bt_{x,y,a,b,c} - y) \neq 0$

Table 3.8: CASE E3: $\mathfrak{g} = \mathfrak{sl}(2,\mathbb{R}) \oplus \mathfrak{r}$, with fundamental $\mathcal{V}_p$ and with $\mathcal{V}_p \nsubseteq \mathfrak{sl}(2,\mathbb{R})$. The vector v, defined by $< v >= \mathfrak{sl}(2,\mathbb{R}) \cap \mathcal{V}_p$, is non-regular of null determinant.

Here $x_{a,b,c,y,z}$ and $t_{a,b,c,y,z}$ are functions depending on a, b, c, y, z, obtained by the system:

$$\begin{cases} x = & \dfrac{1}{4b^3(cy-bz)}\Big(b^2yz^2 - 2b^3cz^2 - 2bcy^2z + 4b^2c^2yz + \\ & -4ab^3yz + c^2y^3 - 2bc^3y^2 + 4ab^2cy^2 - 4b^4y\Big), \\ t = & \dfrac{2b^2z + y^2 - 2bcy}{by}. \end{cases} \tag{3.12}$$

The Nijenhuis tensor is given by (3.11) and $(\xi, J\xi, \eta, J\eta)$ given above is not the adapted frame in general.

3.2.3 Table of classification when $\mathfrak{g}$ is non-solvable

In conclusion we have the following classification when $\mathfrak{g}$ is non-solvable:

Theorem 3.2.8. *If (M^4, J) is a locally homogeneous almost complex manifold with a non-degenerate torsion bundle $\mathcal{V}$, and its associated Lie algebra $\mathfrak{g}$ is non-solvable, we have the following classification for the almost complex structure J:*

	$\mathcal{V}$ is non-fundamental	
	$\mathcal{V} \subseteq \mathfrak{L}$	$\mathcal{V} \not\subseteq \mathfrak{L}$
$\mathfrak{L} = \mathfrak{so}(3)$ $[e_1, e_2] = e_3$ $[e_2, e_3] = e_1$ $[e_3, e_1] = e_2$	$\xi = k_{a,b}e_1, \quad k_{a,b} \neq 0$ $J\xi = ae_1 + be_2, \quad ab \neq 0$ $\eta = k_{a,b}be_3$ $J\eta = e_0 + c_{a,b}e_3$	*not possible*

	$\mathcal{V}$ is fundamental	
	$\mathcal{V} \subseteq \mathfrak{L}$	$\mathcal{V} \not\subseteq \mathfrak{L}$
$\mathfrak{L} = \mathfrak{so}(3)$	*not possible*	$\xi = e_1$ $J\xi = ae_1 + b(e_0 + e_2), \quad b \neq 0$ $\eta = be_3$ $J\eta = \dfrac{2yz(a^2+1)}{z^2 + 2abz - b^2}e_1 + ye_2 + ze_3 + \dfrac{z^2 + 2abz - b^2 + y^2}{y}e_0,$ $z^2 + 2abz - b^2 \neq 0$

	$\mathcal{V}$ *is non-fundamental*	
	$\mathcal{V} \subseteq \mathfrak{L}$	$\mathcal{V} \not\subseteq \mathfrak{L}$
	$\mathcal{V}$ *has all regular elements (except the null matrix)*	
$\mathfrak{L} = sl(2,\mathbb{R})$ $[X,H] = X$ $[X,Y] = 2H$ $[H,Y] = Y$	$\xi = kH$ $J\xi = a_{k,t,b}X + bY + c_{k,t,b}H$ $\eta = k(-a_{k,t,b}X + bY)$ $J\eta = e_0 + t(-a_{k,t,b}X + bY)$ *with* $a_{k,t,b}bk \neq 0$	$\xi = H$ $J\xi = e_0 + aX + bH$ $\eta = -aX$ $J\eta = \frac{a}{2}(t+2b)X + yY$ $+2(b^2+1)H + te_0$ *with* $a \neq 0, y \neq 0$
	$\mathcal{V}$ *has exactly two lines made by non-regular elements*	
$\mathfrak{L} = sl(2,\mathbb{R})$ $[X,H] = X$ $[X,Y] = 2H$ $[H,Y] = Y$	$\xi = X$ $J\xi = aX + bY + cH$ $\eta = 2bH + cX$ $J\eta = e_0 + \frac{cz}{2b}X + zH$ *with* $b \neq 0$	

	$\mathcal{V}$ *is fundamental*	
	$\mathcal{V} \subseteq \mathfrak{L}$	$\mathcal{V} \not\subseteq \mathfrak{L}$
$\mathfrak{L} = sl(2,\mathbb{R})$ $[X,H] = X$ $[X,Y] = 2H$ $[H,Y] = Y$	*not possible*	*v with negative determinant* $\xi = H$ $J\xi = e_0 + aX + bY + cH$ $\eta = -aX + bY$ $J\eta = xX + yY + z_{x,y,a,b,c}H + t_{x,y,a,b,c}e_0$ *with* $(a,b) \neq (0,0)$ $ay + bx - 2abt_{x,y,a,b,c} \neq 0$ *and* (3.8) *holds*
$\mathfrak{L} = sl(2,\mathbb{R})$ $[Z,H] = W$ $[Z,W] = 2H$ $[H,W] = Z$	*not possible*	*v with positive determinant* $\xi = H$ $J\xi = e_0 + aH + bW$ $\eta = bZ$ $J\eta = x_{a,b,y,z}H + yW + zZ + t_{a,b,y,z}e_0$ *with* $b \neq 0$ $bt_{a,b,y,z} \neq y$ *and* (3.10) *holds*
$\mathfrak{L} = sl(2,\mathbb{R})$ $[X,H] = X$ $[X,Y] = 2H$ $[H,Y] = Y$	*not possible*	*v with null determinant* $\xi = X$ $J\xi = e_0 + aX + bY + cH$ $\eta = cX + 2bH$ $J\eta = x_{a,b,c,y,z}X + yY + zH + t_{a,b,c,y,z}e_0$ *with* $b \neq 0$ $bt_{x,y,a,b,c} \neq y$ *and* (3.12) *holds*

3.3 (M^4, J) with solvable Lie algebras

Now, let us consider the case in which the Lie algebra $\mathfrak{g}$ is solvable. We restric our study to the generalizaton of several examples, since the classification of solvable Lie algebras provide a large number of cases.

In [15], Kim and Lee give an example (from now on we will denote it as **KL** for simplicity) of an almost complex manifold (M^4, J) with non-degenerate torsion bundle and with symmetry algebra $aut_p(M, J)$ of dimension 4.

3.3.1 Kim and Lee example

If (z_1, z_2) are complex coordinates of $\mathbb{C}^2$ ($z_j = x_j + iy_j$ for $j = 1, 2$) and

$$\frac{\partial}{\partial z_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} - i\frac{\partial}{\partial y_j}\right), \quad \frac{\partial}{\partial \bar{z}_j} = \frac{1}{2}\left(\frac{\partial}{\partial x_j} + i\frac{\partial}{\partial y_j}\right),$$

choosing

$$Z_1 = \frac{\partial}{\partial z_1} - 2\bar{z}_1 i\frac{\partial}{\partial z_2},$$

$$Z_2 = (z_1 - \bar{z}_1)\frac{\partial}{\partial z_1} + (z_1 - \bar{z}_1)\bar{z}_1\frac{\partial}{\partial \bar{z}_1} + (-2i - z_1^2 + \bar{z}_1^2)\frac{\partial}{\partial z_2} + (-z_1^2 + \bar{z}_1^2)\frac{\partial}{\partial \bar{z}_2},$$

they define an almost complex structure J on $\mathbb{R}^4$ as $(1, 0)$ vector fields. A base of germs of the infinitesimal automorphisms V_1,
V_2, V_3, V_4 of $(\mathbb{R}^4, J)$ is given by

$$\begin{aligned}
&V_1 = Z_2 + \bar{Z}_2,\\
&V_2 = Z_1 + \bar{Z}_1 - i(z_1 - \bar{z}_1)V_1,\\
&V_3 = i(Z_1 - \bar{Z}_1) - (z_1 + \bar{z}_1)V_1 - (2z_1\bar{z}_1 + z_2 + \bar{z}_2)V_2,\\
&V_4 = i(Z_2 - \bar{Z}_2) - 2i(z_1 - \bar{z}_1)V_2 - (z_1 - \bar{z}_1)^2 V_1.
\end{aligned}$$

They generate the symmetry algebra $aut_p(\mathbb{R}^4, J)$ for any point p.

We want to generalize this example to a particular class of examples.

***Remark* 3.5.** We observe that the Lie algebra associated to the manifold $(\mathbb{R}^4, J)$, as we will see, is the solvable Lie algebra $\mathbf{A_{4.1}}$. Moreover, it is interesting to study this algebra because it is the unique nilpotent algebra of dimension 4 which is not decomposable (see [27]).

***Remark* 3.6.** The manifold $(\mathbb{R}^4, J)$ is homogeneous and its associated Lie algebra is formed by the left invariant vector fields (see the beginning of the Section 3.1).

One can compute the distinguished vector field X, that is the vector field such that $N_J(X, [X, JX]) = X$ (the eigenvalue is positive), hence the adapted frame is given by:

$$
\begin{aligned}
X &= \frac{1}{\sqrt{2}} \operatorname{Im} Z_1, \\
JX &= \frac{1}{\sqrt{2}} \operatorname{Re} Z_1, \\
T &= \frac{1}{2} \operatorname{Re} Z_2, \\
JT &= -\frac{1}{2} \operatorname{Im} Z_2,
\end{aligned}
$$

where

$$
\begin{aligned}
\operatorname{Re} Z_1 &= \frac{1}{2}\frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + y_1 \frac{\partial}{\partial y_2}, \\
\operatorname{Im} Z_1 &= -\frac{1}{2}\frac{\partial}{\partial y_1} + x_1 \frac{\partial}{\partial y_2} + y_1 \frac{\partial}{\partial x_2}, \\
\operatorname{Re} Z_2 &= -\frac{\partial}{\partial y_2}, \\
\operatorname{Im} Z_2 &= 2y_1 \frac{\partial}{\partial x_1} - (4x_1 y_1 + 1)\frac{\partial}{\partial x_2}.
\end{aligned}
$$

Moreover, an easy calculation gives that

$$
[\operatorname{Re} Z_1, \operatorname{Im} Z_1] = -\operatorname{Re} Z_2,
$$

$$
[\operatorname{Im} Z_1, \operatorname{Im} Z_2] = -2 \operatorname{Re} Z_1,
$$

and all the other Lie brakets are null. Hence, for the fields of the adapted frame, it holds

$$
[X, JX] = T, \quad [X, JT] = JX,
$$

and all the other Lie brakets are null.

We also note that the torsion bundle $\mathcal{V}$ in **KL** is non-fundamental, since $\mathcal{V}_{-2|p} =< X, JX, T >$ is a subalgebra of the Lie algebra $\mathfrak{g}$, defined by $\mathfrak{g} =< X, JX, T, JT >$.

Note that $\mathfrak{g}^{(1)} = \mathfrak{g}^1 =< JX, T >$ and that its derived algebra $\mathfrak{g}^{(2)}$ is null, while the descending central series is $\mathfrak{g}^2 =< T >$. This is the well-known solvable algebra $\mathbf{A_{4.1}}$ (see [31], [5] and [29]).

One can check that the infinitesimal automorphisms V_1, V_2, V_3, V_4, found by Kim and Lee in **KL**, generate the Lie algebra $\bar{\mathfrak{g}}$ of the right invariant vector fields as expected (it is sufficient to check that $[V_j, X_j] = 0$, for all $j = 1, ..., 4$, where the X_j are the fields of the distinguished frame) and that

$$
[V_2, V_3] = -2V_1, \qquad [V_3, V_4] = 4V_2,
$$

with all the other Lie brackets null. Moreover, there is an isomorphism from $\mathfrak{g}$ to $\bar{\mathfrak{g}}$ given by

$$\begin{cases} -2V_1 & \mapsto & T \\ V_2 & \mapsto & JX \\ V_3 & \mapsto & -X \\ V_4 & \mapsto & -4JT. \end{cases}$$

***Remark* 3.7.** The vector fields V_1, V_2, V_3 are the infinitesimal automorphisms of the Heienberg group $H_t = \{Rez_2 + |z_1|^2 = t\}$, with $t \in \mathbb{R}$, which is a 3-dimensional CR manifold. Moreover, V_1, V_2, V_3 act transitively on each H_t and V_4 is transversal to H_t. So the manifold $(\mathbb{R}^4, J)$ of **KL** is homogeneous and it is foliated with spherical hypersurfaces.

3.3.2 Generalizations of Kim and Lee example

Let us consider a locally homogeneous almost complex manifold (M^4, J) of real dimension 4 with non-degenerate torsion bundle $\mathcal{V}$ and with the associated Lie algebra $\mathfrak{g}$ given by the left invariant vector fields of T_pM, for any $p \in M^4$. Suppose that $\mathfrak{g}$ is the Lie algebra $\mathbf{A_{4.1}}$ defined by $\mathfrak{g} =< e_1, e_2, e_3, e_4 >$, with

$$[e_2, e_4] = e_1, \quad [e_3, e_4] = e_2.$$

Here, we are going to study the non equivalent almost complex structures J on M^4 having the same Lie algebra $\mathbf{A_{4.1}}$.

We have that $\mathfrak{g}^{(1)} = \mathfrak{g}^1 =< e_1, e_2 >$, $\mathfrak{g}^{(2)} = 0$ and $\mathfrak{g}^2 =< e_1 >$. It is clear that the center of $\mathfrak{g}$ is $\mathfrak{z}(\mathfrak{g}) =< e_1 >$ and that $\mathcal{V}_p \neq \mathfrak{g}^{(1)} =< e_1, e_2 >$, otherwise $\mathcal{V}_p$ would be degenerate. So, we have two possibilities: $\dim(\mathcal{V}_p \cap \mathfrak{g}^{(1)})$ is 1 or 0.

Proposition 3.3.1. *If $\mathfrak{g}$ is the Lie algebra $\mathbf{A_{4.1}}$ associated to a connected locally homogeneous almost complex manifold (M^4, J) with non-degenerate torsion bundle $\mathcal{V}$, then the following facts are equivalent:*

(a) $\dim(\mathcal{V}_p \cap \mathfrak{g}^1) = 1$,

(b) $\mathcal{V}_p$ *is non-fundamental,*

(c) $\mathfrak{g}^1 \subseteq \mathcal{V}_{-2|p}$,

(d) $(\mathcal{V}_p \cap \mathfrak{g}^1) = \mathcal{V}_p^{\pm}$.

Proof. $(a) \Rightarrow (b)$. Let us suppose that the $\dim(\mathcal{V}_p \cap \mathfrak{g}^1) = 1$, then there exist $k, h \in \mathbb{R}$, with $(k, h) \neq (0, 0)$, such that $(\mathcal{V}_p \cap \mathfrak{g}^1) =< ke_1 + he_2 >$, hence $\mathcal{V}_p =< ke_1 + he_2, ae_1 + be_2 + ce_3 + de_4 >$ for some $a, b, c, d \in \mathbb{R}$ (with $(c, d) \neq (0, 0)$, otherwise the vector filed $ae_1 + be_2 + ce_3 + de_4$ is in $\mathfrak{g}^1$). In

this way we have that $\mathcal{V}_{-2|p} =< e_1, e_2, ce_3 + de_4 >$, so that $\mathcal{V}_{-2|p}$ is a Lie subalgebra of $\mathfrak{g}$, i.e. $\mathcal{V}_p$ is non-fundamental.
$(b) \Rightarrow (c)$. Since $\mathcal{V}_p$ is non-fundamental, $\mathcal{V}_{-2|p}$ is a 3-dimensional Lie subalgebra of $\mathfrak{g}$, hence there exist $a, b, c, d, x, y, z, t \in \mathbb{R}$ such that $\mathcal{V}_{-2|p}$ is generated by

$$\begin{aligned}
&\xi := ae_1 + be_2 + ce_3 + de_4,\\
&J\xi := xe_1 + ye_2 + ye_3 + te_4,\\
&\eta := [\xi, J\xi] = (bt - dy)e_1 + (ct - dz)e_2,
\end{aligned}$$

with $(ct - dz, bt - dy) \neq (0, 0)$ and $(d, t) \neq (0, 0)$; moreover

$$[\xi, \eta] = -d(ct - dz)e_1 \in \mathcal{V}_{-2|p},$$

$$[J\xi, \eta] = -t(ct - dz)e_1 \in \mathcal{V}_{-2|p}.$$

If $(ct - dz)(bt - dy) \neq 0$, we have $e_1 \in \mathcal{V}_{-2|p}$ and hence $e_2 \in \mathcal{V}_{-2|p}$. If $(ct - dz) = 0$ and $(bt - dy) \neq 0$, $e_1 \in \mathcal{V}_{-2|p}$ and $ce_3 + de_4$ is proportional to $ye_3 + te_4$ so that $e_2 \in \mathcal{V}_{-2|p}$. If $(ct - dz) \neq 0$ and $(bt - dy) = 0$, $< e_1, e_2 >\subseteq \mathcal{V}_{-2|p}$. In all cases $\mathfrak{g}^1 \subseteq \mathcal{V}_{-2|p}$.
$(c) \Rightarrow (d)$. Since $< e_1, e_2 >= \mathfrak{g}^1 \subseteq \mathcal{V}_{-2|p}$, there esist $k, h \in \mathbb{R}$, not both null, such that $ke_1 + he_2 \in \mathcal{V}_p$. Let us define

$$\begin{aligned}
&\xi = ke_1 + he_2, \quad (k, h) \neq (0, 0),\\
&J\xi = ae_1 + be_2 + ce_3 + de_4, \quad (c, d) \neq (0, 0),\\
&\eta = [\xi, J\xi] = hde_1, \quad hd \neq 0,\\
&J\eta = xe_1 + ye_2 + ze_3 + te_4.
\end{aligned}$$

We have that $< \xi, J\xi >= \mathcal{V}_p$ and that the Nijenhuis tensor of ξ and η is

$$N_J(\xi, \eta) = \frac{1}{h}(ct - dz)\xi + \left(-\frac{k}{h^2 d}(ct - dz) + \frac{1}{hd}(bt - dy)\right)\eta - \frac{t}{d}J\eta,$$

and since the coefficients of η and $J\eta$ must be zero, we obtain

$$\begin{cases} t = 0 \\ y = \dfrac{k}{h}z, \end{cases}$$

from which

$$N_J(\xi, \eta) = -\frac{dz}{h}\xi.$$

This means that ξ is the distinguished direction that generates $\mathcal{V}_p^+$ or $\mathcal{V}_p^-$ (it depends on the sign of the coefficients d, h, z appearing ahead ξ in the calculation of N_J). We have actually proved (d).
$(d) \Rightarrow (a)$ is trivial. □

Proposition 3.3.2. *If $\mathfrak{g}$ is the Lie algebra $\mathbf{A_{4.1}}$ associated to a connected locally homogeneous almost complex manifold (M^4, J) with non-degenerate torsion bundle $\mathcal{V}$, then the following facts are equivalent:*

(a') $\dim(\mathcal{V}_p \cap \mathfrak{g}^1) = 0$,

(b') $\mathcal{V}_p$ *is fundamental,*

(c') $\mathfrak{g}^1 \nsubseteq \mathcal{V}_{-2|p}$,

(d') $(\mathcal{V}_p \cap \mathfrak{g}^1) = \{0\}$.

Proof. From the previous proposition it is clear that $(d') \Rightarrow (a') \Rightarrow (b') \Rightarrow (c') \Rightarrow (d')$. □

***Remark* 3.8.** **KL** is of the same type of Proposition 3.3.1.

***Remark* 3.9.** We note that since $\mathfrak{z}(\mathfrak{g}) \cap \mathcal{V}_p = \{0\}$ (see Remark 3.1) and $e_1 \in \mathfrak{z}(\mathfrak{g})$, the space $< e_1 >$ is never contained in $\mathcal{V}_p$ (in agreement to the fact that h, which appears into the proof of Proposition 3.3.1, is never zero).

Case $\mathbf{A_{4.1}}$ with non-fundamental $\mathcal{V}_p$

We first analyze the case of Proposition 3.3.1. Since $\mathcal{V}_p =< \xi, J\xi >$ for any ξ not null in $\mathcal{V}_p$, we can choose ξ such that it is one of the two distinguished vector fields of $\mathcal{V}_p$, for example such that $\mathcal{V}_p^+ =< \xi >$. From the proof of Proposition 3.3.1 we obtain the following

Corollary 3.3.3. *If* $\mathbf{A_{4.1}}$ *is the Lie algebra* $\mathfrak{g}$ *associated to a locally homogeneous almost complex manifold with non-fundamental and non-degenerate torsion bundle* $\mathcal{V}$*, taking* ξ *as the distinguished vector field of* $\mathcal{V}_p$ *in* p*, we have*

$$\begin{aligned}
&\xi = ke_1 - dze_2,\\
&J\xi = ae_1 + be_2 + ce_3 + de_4, \quad d \neq 0,\\
&\eta = [\xi, J\xi] = -d^2 ze_1, \quad z \neq 0\\
&J\eta = xe_1 - \frac{k}{d}e_2 + ze_3,
\end{aligned}$$

with $k, h, a, b, c, d, x, z \in \mathbb{R}$.

Proof. From the proof of Proposition 3.3.1 we obtain the following writings for any $\xi \in \mathcal{V}_p^+$.

$$\begin{aligned}
&\xi = ke_1 + he_2, \quad h \neq 0,\\
&J\xi = ae_1 + be_2 + ce_3 + de_4, \quad d \neq 0,\\
&\eta = [\xi, J\xi] = hde_1,\\
&J\eta = xe_1 + \frac{kz}{h}e_2 + ze_3, \quad z \neq 0,
\end{aligned}$$

with $k, h, a, b, c, d, x, z \in \mathbb{R}$.

Since we want that ξ is the distinguished field in p, that is $N_J(\xi, \eta) = \xi$, computing the Nijenhuis tensor we have that $N_J(\xi, \eta) = -\dfrac{dz}{h}\xi$, so we have to put $\dfrac{dz}{h} = -1$. □

We make another example, which is equivalent to **KL**, having the same associated Lie algebra $\mathbf{A_{4.1}}$, formed by the left invariant vector fields.

We choose $h = d = 1$, $z = -1$ and $k = a = b = c = x = 0$, hence we have

$$\begin{aligned}
&\xi = e_2, \\
&J\xi = e_4, \\
&\eta = [\xi, J\xi] = e_1, \\
&J\eta = -e_3,
\end{aligned}$$

and $N_J(\xi, \eta) = \xi$ (observe that $Je_2 = e_4$ and $Je_1 = -e_3$). If we consider the following realization of the algebra $\mathfrak{g} = \mathbf{A_{4.1}}$

$$\begin{aligned}
&e_1 = \partial_1, \\
&e_2 = \partial_2, \\
&e_3 = \partial_3, \\
&e_4 = x_1\partial_1 + x_3\partial_2 + \partial_4,
\end{aligned}$$

we can take the following $(1,0)$-vector fields

$$Z_1 = e_1 - iJe_1 = \partial_1 + i\partial_3,$$

$$Z_2 = e_2 - iJe_2 = -ix_1\partial_1 + (1 - ix_3)\partial_2 - i\partial_4,$$

they define an almost complex structure J. In this way the adapted frame is exactly given by $(\xi, J\xi, \eta, J\eta)$ and it is easy to check that the structure functions are constants.

Now we are going to find the infinitesimal automorphisms on $\mathfrak{g}$. Taking a generic infinitesimal automorphism as $W = \alpha\xi + \beta J\xi + \gamma\eta + \delta J\eta$, with $\alpha, \beta, \gamma, \delta$ differentiable functions, it is sufficient to compute the right invariant vector fields imposing $[\xi, W] = 0, [J\xi, W] = 0, [\eta, W] = 0, [J\eta, W] = 0$. We define

$$\begin{aligned}
W_1 &= \frac{1}{2}Z_1 + \frac{1}{2}\bar{Z}_1, \\
W_2 &= \frac{1}{2}x_4Z_1 + \frac{1}{2}x_4\bar{Z}_1 + \frac{1}{2}Z_2 + \frac{1}{2}\bar{Z}_2, \\
W_3 &= \frac{1}{2}x_1Z_1 + \frac{1}{2}x_1\bar{Z}_1 - \frac{1}{2}(x_3 - i)Z_2 - \frac{1}{2}(x_3 + i)\bar{Z}_2, \\
W_4 &= \left(\frac{1}{4}x_4^2 + \frac{1}{2}i\right)Z_1 + \left(\frac{1}{4}x_4^2 - \frac{1}{2}i\right)\bar{Z}_1 + \frac{1}{2}x_4Z_2 + \frac{1}{2}x_4\bar{Z}_2,
\end{aligned}$$

so that the algebra $\bar{\mathfrak{g}}$ of the right invariant vector fields is given by $\bar{\mathfrak{g}} =< W_1, W_2, W_3, W_4 >$; an easy calculation gives

$$[W_2, W_3] = -W_1, \quad [W_3, W_4] = W_2,$$

with null all the other Lie brackets. We thus get $\bar{\mathfrak{g}}^1 = \bar{\mathfrak{g}}^{(1)} =< W_1, W_2 >$, $\bar{\mathfrak{g}}^2 =< W_1 >$ and $\bar{\mathfrak{g}}^{(2)} = 0$.

***Remark* 3.10.** There are vector fields that are both left and right invariant vector fields, for example $\eta = W_1$ (they are vector fields belonging to the center of the algebra).

Infinitesimal automorphisms of $\mathfrak{g}$ send $\mathcal{V}_p^{\pm}$ in itself, as a consequence $\mathcal{V}_p$ is sent in $\mathcal{V}_p$, $\mathcal{V}_{-2|p}$ in $\mathcal{V}_{-2|p}$ and $\mathcal{V}_{-3|p}$ in $\mathcal{V}_{-3|p}$, but it is not possible to find a canonical isomorphism between $\bar{\mathfrak{g}}$ and $\mathfrak{g}$.

It should be noted, too, that since $\mathcal{V}_{-2|p}$ is a subalgebra of $\mathfrak{g}$, there is a foliation of (M^4, J) such that every vector field in $\mathcal{V}_{-2|p}$ acts on the leaves having $\mathcal{V}_{-2|p}$ as Lie algebra associated. All these leaves are locally equivalent to a sphere (because of Remark 3.7). In particular, these vectors send the points of any leaf in points of the same leaf; while the vector field W_4 send the points of a leaf in points of another leaf.

Case $\mathbf{A_{4.1}}$ with fundamental $\mathcal{V}_p$

It is of interest to know whether it is possible to find almost complex structures J on the Lie algebra $\mathbf{A_{4.1}}$ that are non equivalent to **KL** example.

For simplicity, in this section we will study only an example because its generalization involves the use of too many parameters (although this generalization can be done).

If we take

$$\begin{aligned} &\xi = e_3, \\ &J\xi = e_4, \\ &\eta = [\xi, J\xi] = e_2, \\ &J\eta = e_1 - e_3, \end{aligned}$$

we obtain that $N_J(\xi, \eta) = J\xi$. An easy calculation gives that the adapted frame is

$$\begin{aligned} X &= \frac{1}{\sqrt{2}}(e_3 + e_4), \\ JX &= \frac{1}{\sqrt{2}}(e_4 - e_3), \\ T &= \frac{2}{\sqrt{2}}e_2, \\ JT &= \frac{2}{\sqrt{2}}(e_1 - e_3). \end{aligned}$$

Here the Lie algebra associated is again $\mathbf{A_{4.1}}$, but the almost complex structure J is non equivalent to that of the previous example and to **KL**: it is easy to check that not only $(\mathcal{V}_p \cap \mathfrak{g}^1) = \{0\}$ holds, but $\mathcal{V}_p$ is fundamental too.

3.3.3 Solvable Lie algebra $\mathfrak{g}$ associated to a locally homogeneous almost complex manifold $M^3 \times \mathbb{R}$ with M^3 non-equivalent to a hypersphere

Unlike the case of **KL**, we are interested in finding some examples of 4-dimensional locally homogeneous almost complex manifold with non-degenerate torsion bundle and with foliation $M^3 \times \mathbb{R}$ such that M^3 is non-equivalent to a hypersphere of dimension 3 (see [11]). Our propose is to extend this kind of examples to a class of equivalent examples. We will consider the 3-dimensional Lie algebra $\mathbf{A_{3.2}}$ defined by

$$[e_1, e_3] = e_1, \quad [e_2, e_3] = e_1 + e_2, \quad [e_1, e_2] = 0,$$

and will assume that the almost complex structure J, defined on the holomorphic tangent space H_pM^3 in $\mathbf{A_{3.2}}$, is given by

$$Je_1 = e_3 + e_4,$$

$$Je_2 = e_3,$$

where e_4 is not in $\mathbf{A_{3.2}}$; we also assume that

$$[e_4, e_j] = 0, \quad j = 1, 2, 3,$$

to get the Lie algebra $\mathbf{A_{3.2}} \oplus \mathbf{A_1}$ (for the notation see [31]). It gives $\mathfrak{z}(\mathbf{A_{3.2}} \oplus \mathbf{A_1}) =< e_4 >= \mathbf{A_1}$.

By a simple calculation we have that $N_J(e_1, e_2) = e_3$ and, since $\mathcal{V}_p$ is J-invariant, we get $\mathcal{V}_p =< e_2, e_3 >$. Observe that $\mathcal{V}_p$ is non-degenerate, that $\mathcal{V}_p = H_pM^3$ and that $\mathcal{V}_{-2|p} =< e_1, e_2, e_3 >= \mathbf{A_{3.2}}$. By construction of $M^3 \times \mathbb{R}$ and $\mathbf{A_{3.2}} \oplus \mathbf{A_1}$, we have

$$T_p\mathbb{R} =< e_4 > .$$

A simple calculation gives

$$\begin{aligned}
\xi &= \frac{1}{\sqrt{2}}(e_2 - e_3),\\
J\xi &= \frac{1}{\sqrt{2}}(e_2 + e_3),\\
\eta &= e_1 + e_2,\\
J\eta &= 2e_3 + e_4,
\end{aligned}$$

as adapted frame.

If we take

$$\begin{aligned}
e_1 &= \partial_1,\\
e_2 &= \partial_2,\\
e_3 &= (x_1 + x_2)\partial_1 + x_2\partial_2 + \partial_3,\\
e_4 &= \partial_4,
\end{aligned}$$

as realization of the given algebra $\mathbf{A_{3.2}} \oplus \mathbf{A_1}$, we can choose

$$Z_1 = (1 - ix_1 + ix_2)\partial_1 - ix_2\partial_2 - i\partial_3 - i\partial_4,$$

$$Z_2 = -i(x_1 + x_2)\partial_1 + (1 - ix_2)\partial_2 - i\partial_3$$

as $(1,0)$-vector fields. Hence the Lie algebra $\bar{\mathfrak{g}}$ of the right invariant vector fields is given by

$$\begin{aligned}
W_1 &= \frac{1}{2}e^{x_3}(Z_1 + \bar{Z}_1),\\
W_2 &= \frac{1}{2}x_3e^{2x_3}Z_1 + \frac{1}{2}x_3e^{2x_3}\bar{Z}_1 + \frac{1}{2}e^{x_3}Z_2 + \frac{1}{2}e^{x_3}\bar{Z}_2,\\
W_3 &= -\frac{1}{2}(x_1 + x_2)Z_1 - \frac{1}{2}(x_1 + x_2)\bar{Z}_1 - \frac{1}{2}(x_2 + i)Z_2 - \frac{1}{2}(x_2 - i)\bar{Z}_2,\\
W_4 &= -\frac{1}{2}i(Z_1 - \bar{Z}_1 - Z_2 + \bar{Z}_2).
\end{aligned}$$

***Remark* 3.11.** Since $\mathcal{V}_{-2|p}$ is a subalgebra of $\mathbf{A_{3.2}} \oplus \mathbf{A_1}$ ($\mathcal{V}_{-2|p} = \mathbf{A_{3.2}}$) and $\mathbf{A_1}$ is the center of $\mathbf{A_{3.2}} \oplus \mathbf{A_1}$, there is a foliation of (M^4, J) such that the action given by the vector fields of $\mathcal{V}_{-2|p}$ sends points of a leaf, having $\mathcal{V}_{-2|p}$ as Lie algebra, in points of the same leaf, and the field e_4 sends points of a leaf in points of another leaf.

Note that we have actually obtained a manifold M^3 which is a non-degenerate CR manifold, in particular, M^3 is not diffeomorphic to a hypersphere of dimension 3. For more details we refer the reader to [11] pag.70.

3.3.4 Generalization of $M^3 \times \mathbb{R}$

A more complete theory may be obtained by a generalization of the examples seen in the previous section. In this section we will make the following assumptions: $\mathfrak{g} =< e_1, e_2, e_3, e_4 >$ is the Lie algebra $\mathbf{A_{3.2}} \oplus \mathbf{A_1}$, hence $\mathfrak{g}^1 =< e_1, e_2 >$, $\mathfrak{g}^{(2)} = 0$, $\mathfrak{g}^2 = \mathfrak{g}^1$. We see at once that $\mathcal{V}_p \neq \mathfrak{g}^1$, which is clear because $\mathcal{V}_p$ is non-degenerate, so $\dim(\mathcal{V}_p \cap \mathfrak{g}^1) = 1$ or $\dim(\mathcal{V}_p \cap \mathfrak{g}^1) = 0$.

Proposition 3.3.4. *If* $\mathfrak{g} =< e_1, e_2, e_3, e_4 >$ *is the Lie algebra* $\mathbf{A_{3.2}} \oplus \mathbf{A_1}$ *associated to a connected locally homogeneous almost complex manifold* (M^4, J) *with non-degenerate torsion bundle* $\mathcal{V}$*, then the following facts are equivalent:*

(a) $\dim(\mathcal{V}_p \cap \mathfrak{g}^1) = 1$,

(b) $\dim(\mathcal{V}_{-2|p} \cap \mathfrak{g}^1) = 2$.

As a consequence, the following facts are equivalent too:

(a') $\dim(\mathcal{V}_p \cap \mathfrak{g}^1) = 0$,

(b') $\dim(\mathcal{V}_{-2|p} \cap \mathfrak{g}^1) = 1$.

Proof. $(a) \Rightarrow (b)$. Let us assume, by contradiction, that $\dim(\mathcal{V}_{-2|p} \cap \mathfrak{g}^1) = 1$, we have that $(\mathcal{V}_{-2|p} \cap \mathfrak{g}^1) =< ae_1 + be_2 >$, for certain $a, b \in \mathbb{R}$ with $(a, b) \neq (0, 0)$, and $\mathcal{V}_p =< ae_1 + be_2, xe_1 + ye_2 + ze_3 + te_4 >$, for certain $x, y, z, t \in \mathbb{R}$. As a consequence, $\mathcal{V}_{-2|p} =< e_1, e_2, ze_3 + te_4 >$, a contradiction. The proof of $(b) \Rightarrow (a)$ is straightforward (it is sufficient to use Grassmann theorem on the dimension of $(\mathcal{V}_{-2|p} + \mathfrak{g}^1)$ considering that $\mathfrak{g}^1 \subseteq \mathcal{V}_{-2|p}$). $\square$

Case $\mathbf{A_{3.2} \oplus A_1}$ with $\dim(\mathcal{V}_p \cap \mathfrak{g}^1) = 1$

When $\dim(\mathcal{V}_p \cap \mathfrak{g}^1) = 1$, we can proceed analogously to the construction of

$$\begin{aligned}
&\xi = ke_1 + e_2,\\
&J\xi = ae_1 + be_2 + ce_3 + de_4, \quad c \neq 0,\\
&\eta = [\xi, J\xi] = c(k+1)e_1 + ce_2,\\
&J\eta = (2ac + (k-1)(y - 2bc))e_1 + ye_2 + 2c^2e_3 + te_4, \quad t \neq 2cd,
\end{aligned}$$

with $k, a, b, c, d, x, y, t \in \mathbb{R}$. Here we can assume that the coefficient of e_2 in the definition of ξ is 1, because it is not null: if it was null, $\xi, J\xi$ and η would be dependent one each other (ξ would be proportional to η); the condition $t \neq 2cd$ arises from the independence of $\xi, J\xi, \eta, J\eta$. A calculation gives that

$$N_J(\xi, \eta) = c(2bc - y)\xi - c^2 J\xi.$$

We note that the condition $\dim(\mathcal{V}_p \cap \mathfrak{g}^1) = 1$ implies that $\mathcal{V}_p$ is non-fundamental, but it is just a sufficient condition: in general the opposite implication does not hold.

Case $\mathbf{A_{3.2} \oplus A_1}$ with $\dim(\mathcal{V}_p \cap \mathfrak{g}^1) = 0$

When $\dim(\mathcal{V}_p \cap \mathfrak{g}^1) = 0$, we just consider examples to lighten this case, since the generalization involves a lot of parameters.

If we take the example

$$\begin{aligned}
&\xi = e_3,\\
&J\xi = e_1 + e_4,\\
&\eta = [\xi, J\xi] = -e_1,\\
&J\eta = -e_2 + e_4,
\end{aligned}$$

we have that $N_J(\xi, \eta) = \xi$, that is, ξ is the distinguished vector field of $\mathcal{V}_p$ and $(\xi, J\xi, \eta, J\eta)$ is the adapted frame.

We note that $\mathcal{V}_p$ is non-fundamental, but, conversely from the previous case, the condition $\dim(\mathcal{V}_p \cap \mathfrak{g}^1) = 0$ does not imply that $\mathcal{V}_p$ is non-fundamental in general. Indeed, we have the following counterexample:

$$\begin{aligned}
&\xi = e_3,\\
&J\xi = e_2 + e_4,\\
&\eta = [\xi, J\xi] = -e_1 - e_2,\\
&J\eta = e_2 + 2e_3.
\end{aligned}$$

The Nijenhuis tensor is $N_J(\xi,\eta) = -2\xi$ (hence $<\xi>= \mathcal{V}_p^-$) and $\mathcal{V}_p$ is fundamental.

3.4 Lie algebras of homogeneous almost complex manifolds of dimension 4

In this section we assume that (M, J) is a four dimensional *connected, homogeneous* almost complex manifold with non degenerate torsion bundle (see also [42] for the group of automorphisms of a homogeneous almost complex manifold). This means that the action of the connected component G of the Lie group $Aut(M, J)$, given by the automorphisms of (M, J), on (M, J) is transitive and the following theorem holds (see [40] Theorem 2.9.4, p. 77).

Theorem 3.4.1. *Let G be a Lie group, H a closed Lie subgroup. Then there exist exactly one analytic structure on G/H which converts it into an analytic manifold such that the natural action of G on G/H is analytic. If M is any analytic manifold on which G acts analytically and transitively, x_0 in M, and G_{x_0} is the isotropy subgroup at x_0, then the map*

$$\begin{cases} G/G_{x_0} & \longrightarrow & M \\ gG_{x_0} & \mapsto & g\cdot x_0 \end{cases}$$

is an analytic diffeomorphism of G/G_{x_0} in M.

As a consequence, we have that there exists a diffeomorphism such that

$$M \cong G/G_p,$$

with G_p the isotropy subgroup of G fixing a point p of M.

Let us now consider the double covering F of (M, J) given by

$$\varphi : F \to M, \tag{3.13}$$

where F is endowed with an $\{e\}$-structure and M is endowed with an $\{e^{\pm}\}$-structure (see Remark 2.15). Now, we have two possibilities based on the fact that F is connected or non-connected.

Proposition 3.4.2. *Given a connected homogeneous almost complex manifold (M, J) of dimension 4 with non-degenerate torsion bundle, if (F, φ) is the double covering of (M, J) given by (3.13) and G is the connected component of $Aut(M, J)$, we have that:*

(i) when F is connected, then $F \cong G$ and it has subgroups of order 2; in particular, M is a Lie group if and only if the isotropy subgroup G_p of G is in the center of G;

(ii) when F is non-connected, then $F \cong G \times \{0, 1\}$ and $M \cong G$.

Proof. (i) Since the action of G on F is an immersion, from the connection of G we have that $F \cong G$ when F is connected (because the only connected subgroups of F are the trivial ones). Hence, from $M \cong G/G_p$ and $M \cong F/\mathbb{Z}_2$, we obtain $M \cong G/\mathbb{Z}_2$. This means that in this case there must be subgroups of G of order 2, so we can write $G_p =< id_G, h >$, with $h^2 = id_G$ and $h \neq id_G$. In general, $M \cong G/G_p$ is not a Lie group, but just a manifold. We have that $M \cong G/G_p$ is a Lie group if and only if either $g^{-1}hg = id_G$ or $g^{-1}hg = h$ holds, for all $g \in G$. Since $h \neq id_G$, the latter holds, i.e. $G_p \subseteq Z(G)$, where $Z(G)$ is the center of G. So, G/G_p is a Lie group if and only if $G_p \subseteq Z(G)$.

(ii) When F is not connected, every sheet of F is isomorphic to G and, since $F \cong G \times \{0, 1\}$, we have that $M \cong G$. In this case the isotropy subgroup of G must be the trivial one. □

Let us analyze the **KL** example in consideration of the previous propreties. A matrix representation of the Lie algebra $\mathbf{A_{4.1}}$ is (for instance see [4], pag.30)

$$\mathfrak{g} = \left\{ \begin{pmatrix} 0 & x & 0 & t \\ 0 & 0 & x & z \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \end{pmatrix} : \quad x, y, z, t \in \mathbb{R} \right\}.$$

An easy computation of the exponential of $\mathfrak{g}$ gives that a Lie group $\tilde{G}$ associated to $\mathbf{A_{4.1}}$ is

$$\tilde{G} = \left\{ \begin{pmatrix} 1 & a & \frac{a^2}{2} & d \\ 0 & 1 & a & c \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix} : \quad a, b, c, d \in \mathbb{R} \right\}.$$

This group is connected and simply connected since it is isomorphic (as topologic manifold) to $\mathbb{R}^4$; in particular, it is equivalent to the manifold studied by Kim and Lee in their example. Moreover, $\tilde{G}$ is a nilpotent group (since also its algebra is) and its center is

$$Z(\tilde{G}) = \left\{ \begin{pmatrix} 1 & 0 & 0 & \alpha \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \quad \alpha \in \mathbb{R} \right\}.$$

It is easy to check that $\tilde{G}$ has not subgroups of order two, that is, there are not non-trivial solution of

$$\begin{pmatrix} 1 & a & \frac{a^2}{2} & d \\ 0 & 1 & a & c \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix}^2 = id_{\tilde{G}},$$

for some $a, b, c, d \in \mathbb{R}$.

So, if M is any homogeneous almost complex manifold isomorphic to $\tilde{G}$ and F is its double covering, for Proposition 3.4.2, we have that F has to be not connected.

Now, we want to build an example in which the double covering F of (M, J) is connected. Such example is obtained from the Lie group $\tilde{G}$ studied above.

Since $\tilde{G}$ is connected and simply connected, all the other Lie groups having $\mathbf{A_{4.1}}$ as Lie algebra are of the form $\bar{G} \cong \tilde{G}/N$ (see [25]), where N is a (normal) dicrete subgroup contained into the center of $\tilde{G}$.

A normal discrete subgroup of $\tilde{G}$ is of the form

$$N_t = \left\{ \begin{pmatrix} 1 & 0 & 0 & tk \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \quad k \in \mathbb{Z} \right\},$$

with $t \in \mathbb{R}$. We have that $\tilde{G}/N_2$ is connected and it is a double covering of $\tilde{G}/N_1$. Moreover, $\tilde{G}/N_2$ has non-trivial normal discrete subgroups of order two as expected: if $[N_2]$ and $[gN_2]$ are the classes of equivalence of the elements N_2 and gN_2 of $\tilde{G}/N_2$, with

$$g = \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix},$$

we have that $< [N_2], [gN_2] = [N_1] >$ is a normal subgroup of order two of $\tilde{G}/N_2$.

Since there is an isomorphism between N_t and $N_{t'}$ for any t, t' in $\mathbb{R}$, we have the following result.

Proposition 3.4.3. *Any homogeneous almost complex manifold (M, J), with non-degenerate torsion bundle and with the connected component of the group of automorphisms $Aut(M, J)$ isomorphic to $\mathbf{A_{4.1}}$, is equivalent to one of these Lie groups:*

(i)

$$\tilde{G} = \left\{ \begin{pmatrix} 1 & a & \frac{a^2}{2} & d \\ 0 & 1 & a & c \\ 0 & 0 & 1 & b \\ 0 & 0 & 0 & 1 \end{pmatrix} : \quad a, b, c, d \in \mathbb{R} \right\},$$

which has a non-connected double covering $F \cong \tilde{G} \times \{0, 1\}$;

(ii)

$$\tilde{G}/N_t,$$

where $N_t = \left\{ \begin{pmatrix} 1 & 0 & 0 & tk \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} : \quad k \in \mathbb{Z} \right\}$, *which has a connected double covering $F \cong \tilde{G}/N_t$.*

***Remark* 3.12.** From the examples we have just analyzed, we obtain the following considerations holding globally on (M, J). When M is any homogeneous almost complex manifold isomorphic to $\tilde{G}$, the double covering F is non-connected, hence there are two independent absolute parallelisms globally on (M, J). Whereas, when M is any homogeneous almost complex manifold isomorphic to $\tilde{G}/N_t$, the double covering F is connected, hence there is no a global double absolute parallelism on (M, J). This fact implies that, in general, there are two absolute parallelisms locally but not globally on (M, J).

We can find other examples with connected F. Let us consider the Lie algebra $\mathfrak{so}(3) \oplus \mathfrak{r}$, given by the matrix of the form

$$\begin{pmatrix} 0 & a & b & 0 \\ -a & 0 & c & 0 \\ -b & -c & 0 & 0 \\ 0 & 0 & 0 & d \end{pmatrix}.$$

If (e_1, e_2, e_3, e_4) is a base of $\mathfrak{so}(3) \oplus \mathfrak{r}$ such that

$$[e_1, e_2] = e_3, \quad [e_2, e_3] = e_1, \quad [e_3, e_1] = e_2, \quad [e_0, e_i] = 0, \forall i = 1, 2, 3,$$

it s sufficient to take, for example, $\mathcal{V}_p =< e_1, e_2 >$ and the almost complex stucture defined by $Je_1 = e_1 + e_2$ and $Je_3 = e_4$. An easy calculation gives that $S := SO(3) \times \mathbb{R}^+$ is a connected Lie group having $\mathfrak{so}(3) \oplus \mathfrak{r}$ as Lie algebra. The element

$$s = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$$

of S is such that $s^2 = id_S$, so $< id_S, s >$ is a discrete subgroup of S, then

$$S \to S/ < id_S, s >$$

is a covering of $S/ < id_S, s >$. We found that the group S (having subgroups of order 2) is a double covering of the almost complex manifold given by $S/ < id_S, s >$ with J defined as above; in particular, S is connected as wanted.

Chapter 4

Almost Kähler manifolds of dimension 4

Let us consider an almost complex manifold (M^4, J) of dimension 4 on which is possible to introduce a Riemannian metric $\mathcal{G}$ such that $\mathcal{G}(X,Y) = \mathcal{G}(JX, JY)$, for any $X, Y \in \Gamma(TM)$, and a fundamental 2-form Ω, defined by $\Omega(X,Y) = \mathcal{G}(X, JY)$, in a way that $(M^4, J, \mathcal{G})$ becomes an almost Kähler manifold, that is $d\Omega = 0$. In [6], Blair shows that there are no almost Kähler manifolds of constant curvature except in the case of the constant equal to zero, and then the manifold is Kählerian. In the previous chapter we developed the theory about the existence of two adapted frames on (M^4, J) and we gave the way to find them. Our intention is to endowed these almost complex manifolds with a metric induced by such adapted frames.

Let us suppose that (M^4, J) is an almost complex manifold of real dimension 4 and that (X, JX, T, JT) is one of the two adapted frames of (M^4, J). When we consider (X, JX, T, JT) as an orthogonal base of the tangent space T_pM of (M^4, J), we have

$$\mathcal{G}(X,X) = a, \quad \mathcal{G}(JX,JX) = b, \quad \mathcal{G}(T,T) = c, \quad \mathcal{G}(JT,JT) = d,$$

for some real differentiable functions a, b, c, d and all the other $\mathcal{G}$ are zero. For every $A, B \subseteq T_pM$, we want that $\mathcal{G}(JA, JB) = \mathcal{G}(A, B)$, so if we put $A = \alpha X_p + \beta JX_p + \gamma T_p + \delta JT_p$ and $B = \alpha' X_p + \beta' JX_p + \gamma' T_p + \delta' JT_p$, with $\alpha, \beta, \gamma, \delta, \alpha', \beta', \gamma', \delta'$ real constants, we obtain that

$$\begin{cases} a = b \\ c = d. \end{cases}$$

The formula for $d\Omega$ is

$$\begin{aligned} d\Omega(X,Y,Z) = \; & \frac{1}{3}\{X\Omega(Y,Z) + Y\Omega(Z,X) + Z\Omega(X,Y) \\ & -\Omega([X,Y],Z) - \Omega([Z,X],Y) - \Omega([Y,Z],X)\}, \end{aligned}$$

hence, since we want that $d\Omega = 0$, a simple calculation gives

$$\begin{cases} T(a) - \mathcal{G}([T, X], X) + \mathcal{G}([JX, T], JX) = 0 \\ JT(a) - c - \mathcal{G}([JT, X], X) + \mathcal{G}([JX, JT], JX) = 0 \\ X(c) - \mathcal{G}([X, T], T) + \mathcal{G}([JT, X], JT) + \mathcal{G}([T, JT], JX) = 0 \\ JX(c) - \mathcal{G}([JX, T], T) + \mathcal{G}([JT, JX], JT) - \mathcal{G}([T, JT], X) = 0. \end{cases} \tag{4.1}$$

***Remark* 4.1.** It is known that any almost complex manifold of dimension 4 has the local symplectic property (it was proved in Lemma A.1 [32] in an incomplete way, but it is completely proved in [19]), i.e., given an almost complex structure J on M^4 there exists a symplectic form which is compatible with J in a neighborhood of each point of (M^4, J), that is $\Omega(JX, JY) = \Omega(X, Y)$, for any $X, Y \in \Gamma(TM)$.

***Remark* 4.2.** From Theorem 4.2 in [20], the manifold (M^4, J) can not be nearly Kähler, since (M^4, J) is of type 0 (that is, according to the definition of Muskarov, the maximal number of independent holomorphic functions on (M^4, J) is zero).

***Remark* 4.3.** When (M^4, J) is a homogeneous almost complex manifold, it is not possible to give on it a structure of almost Kähler manifold. Indeed, since any diffeomorphism of (M^4, J) sends an adapted frame into an adaped frame, it is an isometry, hence it conserves the almost complex structure J and the metric $\mathcal{G}$, so it also conserve the curvature. Now, because of [6], we know that (M^4, J) must be a (non-almost) Kähler manifold.

4.1 Metric on Kim and Lee example

We want to analyze the **KL** example in order to find some solutions of (4.1). From the definition given above of $\mathcal{G}$, we have

$$\mathcal{G}(X, X) = a, \quad \mathcal{G}(JX, JX) = a, \quad \mathcal{G}(T, T) = c, \quad \mathcal{G}(JT, JT) = c;$$

hence, when a and c are constants $T(a), JT(a), X(c)$ and $JX(c)$ are zero in (4.1), and since in **KL** we have

$$[X, JX] = T \qquad \text{and } [JT, JX] = X,$$

the system (4.1) becomes

$$\begin{cases} \mathcal{G}(0, X) + \mathcal{G}(0, JX) = 0 \\ -c - \mathcal{G}(0, X) + \mathcal{G}(-X, JX) = 0 \\ -\mathcal{G}(0, T) + \mathcal{G}(0, JT) + \mathcal{G}(0, JX) = 0 \\ -\mathcal{G}(0, T) + \mathcal{G}(X, JT) - \mathcal{G}(0, X) = 0, \end{cases}$$

that is, $c = 0$ for any a.

So, when a and c are constants we can not introduce a metric compatible with J on (M^4, J).

There are different results when a and c are differentiable functions: the system becomes

$$\begin{cases} T(a) = 0 \\ JT(a) - c = 0 \\ X(c) = 0 \\ JX(c) = 0, \end{cases}$$

$$\begin{cases} \operatorname{Re} Z_2(a) = 0 \\ \frac{1}{2} \operatorname{Im} Z_2(a) + c = 0 \\ \operatorname{Re} Z_1(c) = 0 \\ \operatorname{Im} Z_1(c) = 0, \end{cases}$$

$$\begin{cases} \frac{\partial}{\partial y_2}(a) = 0 \\ \frac{1}{2}\left(2y_1 \frac{\partial}{\partial x_1} - (4x_1 y_1 + 1)\frac{\partial}{\partial x_2}\right)(a) + c = 0 \\ \left(\frac{1}{2}\frac{\partial}{\partial x_1} - x_1 \frac{\partial}{\partial x_2} + y_1 \frac{\partial}{\partial y_2}\right)(c) = 0 \\ \left(-\frac{1}{2}\frac{\partial}{\partial y_1} + x_1 \frac{\partial}{\partial y_2} + y_1 \frac{\partial}{\partial x_2}\right)(c) = 0. \end{cases}$$

A solution of this system is

$$\begin{cases} a = e^{x_1^2 + x_2 + y_1^2} \\ c = \frac{1}{2} e^{x_1^2 + x_2 + y_1^2}, \end{cases} \tag{4.2}$$

hence, with a and c differentiable functions chosen as in (4.2), $(M^4, J, \mathcal{G})$ becomes a Kähler manifold.

4.2 Metric on manifolds having non-solvable Lie algebra

We want to see if it is possible to find a metric on manifolds having non-solvable Lie algebra $\mathfrak{g}$. For example, we can consider $\mathfrak{g} = \mathfrak{sl}(2, \mathbb{R}) \oplus \mathfrak{r}$ with non-fundamental $\mathcal{V} \subseteq \mathfrak{sl}(2, \mathbb{R})$ such that it has exactly two lines made by non-regular elements (CASE D2). We can take

$$\begin{aligned} \xi &= \tilde{X}, \\ J\xi &= Y, \\ \eta &= 2H, \\ J\eta &= e_0, \end{aligned}$$

where $[\tilde{X}, Y] = 2H, [\tilde{X}, H] = \tilde{X}, [H, Y] = Y$. An easy calculation gives

$$\begin{aligned} X &= \frac{1}{2}Y, \\ JX &= -\frac{1}{2}\tilde{X}, \\ T &= \frac{1}{2}H, \\ JT &= \frac{1}{4}e_0, \end{aligned}$$

as adapted frame (indeed we have $N_J(X, T) = X$). If a and c are constants, it is easy to check that it is not possible to introduce a metric on the manifold; taking a and c as differentiable functions, the system (4.1) becomes

$$\begin{cases} T(a) = 0 \\ JT(a) - c = 0 \\ X(c) = 0 \\ JX(c) = 0. \end{cases}$$

Taking a representation of this algebra as in [31], we obtain

$$\begin{cases} \left(x_1 \dfrac{\partial}{\partial x_1} + x_2 \dfrac{\partial}{\partial x_2}\right)(a) = 0 \\ \dfrac{1}{4}\dfrac{\partial}{\partial x_4}(a) = c \\ \left(x_1^2 \dfrac{\partial}{\partial x_1} + 2x_1 x_2 \dfrac{\partial}{\partial x_2} + x_2 \dfrac{\partial}{\partial x_3}\right)(c) = 0 \\ \dfrac{\partial}{\partial x_1}(c) = 0. \end{cases}$$

A solution of this system is given by

$$\begin{cases} a = 4 \exp^{x_4} \\ c = \exp^{x_4}. \end{cases}$$

***Remark* 4.4.** In **KL** and in this example, we have $a = kc$, with $k \in \mathbb{R}$.

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