Scaling asymptotics of Szegö kernels under commuting Hamiltonian actions

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Declaration

I hereby declare that this thesis and the work reported herein was composed by and originated entirely from me. Information derived from the published and unpublished work of others has been acknowledged in the text and references are given in the list of sources.

Abstract

Let M be a connected d_M-dimensional complex projective manifold, and let A be a holomorphic positive Hermitian line bundle on M, with normalized curvature ω. Let G be a compact and connected Lie group of dimension d_G, and let T be a compact torus of dimension d_T. Suppose that both G and T act on M in a holomorphic and Hamiltonian manner, that the actions commute, and linearize to A. If X is the principal S^1-bundle associated to A, then this set-up determines commuting unitary representations of G and T on the Hardy space $H(X)$ of X, which may then be decomposed over the irreducible representations of the two groups. If the moment map for the T-action is nowhere zero, all isotypical components for the torus are finite-dimensional, and thus provide a collection of finite-dimensional G-modules. Given a non-zero integral weight ν_T for T, we consider the isotypical components associated to the multiples $k\nu_T$, $k \to +\infty$, and focus on how their structure as G-modules is reflected by certain local scaling asymptotics on X (and M). More precisely, given a fixed irreducible character ν_G of G, we study the local scaling asymptotics of the equivariant Szegö projectors associated to ν_G and $k\nu_T$, for $k \to +\infty$, investigating their asymptotic concentration along certain loci defined by the moment maps.

Contents

Chapter 1

Introduction

1.1 Preamble

The general framework of this thesis is the geometric quantization of compact Kähler manifolds or, more specifically, the local harmonic analysis of quantized Hamiltonian actions. Thus our basic ingredients will be a compact Hodge manifold (M, ω, J) that is a compact complex projective manifold with a Kähler metric and a positive ample line bundle (A, h) on it where, A is the Hermitian line bundle and h is the Hermitian metric, such that the curvature of the unique connection on A, denoted by ∇ is $-2i\omega$. One fundamental goal of geometric quantization is to associate to the phase space M a Hilbert space $H^{\hbar}$ (depending on a parameter $\hbar$) and to any real function $f \in \mathcal{C}^{\infty}(M)$ on it, viewed as classical observable, a collection of self-adjoint operators $T_f^{\hbar}$ on $H^{\hbar}$, and to relate the asymptotic properties of $H^{\hbar}$ and $T_f^{\hbar}$ for $\hbar \to 0^+$ to the symplectic geometry of M and the classical dynamics of the Hamiltonian flow of f.

The traditional Berezin-Toeplitz approach (see [RCG1], [RCG2], [BG1], [B], [KS], [Ch], [MM2], [GS2], [AE], [Z1], [E], [Sch], [Xu] and [BPU]) is to set $H^{1/k} = H^0(M, A^{\otimes k})$, the space of global holomorphic sections of A, and to take $T_f^{\hbar}(\sigma)$ to be the Toeplitz operator on it with multiplier f (see below). This essentially amount to decomposing a naturally associated Hardy space into isotypes with respect to the structure circle action of the circle. The local aspects of such decomposition, as decribed by the celebrated T-Y-Z expansion (see [C], [Z2], [Xu], [Loi] and [Lu]) and its near diagonal rescaled generalizations (see [BSZ], [SZ], [MM1] and [MZ]) have great importance in geometric quantization and in differential, complex and symplectic geometry. When additional symmetries are given, corresponding to quantizable (that is, linearizable) Hamiltonian actions on M, other decompositions are possible, leading to different natural and interesting decompositions into isotypes, whose local semiclassical behavior may be related to the underlying properties of the Hamiltonian action and the symplectic geometry of the phase space. We are interested here in one such situation, where we consider two concurring Hamiltonian actions which commute and play a different role in the asymptotics. We shall then study the asymptotic concentration of the Szegö projection and Toeplitz kernels associated to this picture.

Let us discuss our results in more detail.

1.2 The main Theorem

Let M be a connected d_M-dimensional, compact, complex projective manifold and (A, h) be an ample positive Hermitian line bundle on M. We may assume that the curvature form of the unique compatible connection ∇_A is $\Theta = -2i\omega$, where ω is a Kähler form. Let dV_M be the volume form $\frac{\omega^{\wedge d_M}}{d_M!}$ associated with (M, ω).

We put $h = g - i\omega$ where g is the induced Riemannian structure. Suppose given two connected compact Lie group G and T, with T a torus, of dimension d_G and d_T, respectively, and commuting holomorphic and Hamiltonian actions $\mu^G : G \times M \to M$ and $\mu^T : T \times M \to M$. Thus $\mu^G_g \circ \mu^T_t = \mu^T_t \circ \mu^G_g, \ \ \forall (g, t) \in G \times T$. Also, assume that both actions unitarily linearize to A, that is, that they admit metric preserving lifting $\tilde{\mu}^G, \tilde{\mu}^T$. Let $\widehat{G}$ be the collection of irreducible characters of G and for any $\nu_G \in \widehat{G}$ let V_{ν_G} be the corresponding irreducible unitary representation. The action of G on A dualizes to an action on the dual line bundle $A^\vee$ and the G invariant Hermitian metric h on A naturally induces an Hermitian metric on $A^\vee$ also denoted by h.

Let $X \subseteq A^\vee$ be the unit circle bundle, with projection $\pi : X \to M$. Then X is a contact manifold, with contact form given by the connection 1-form α. Since G and T preserve the Hermitian metric h on $A^\vee$, they act on X. Furthermore, as both linearized actions preserve the unique compatible connection ∇_A, both actions leave α invariant.

The actions of G and T on X preserves the volume form $dV_X = \alpha \wedge \pi^*\left(\frac{dV_M}{2\pi}\right)$ on X, whence they induce commuting unitary representations of G and T on $L^2(X)$, which preserve the Hardy space $H(X) = L^2(X) \cap \mathrm{Ker}(\overline{\partial}_b)$.

By virtue of the Peter-Weyl Theorem, we may then unitarily and equivariantly decompose $H(X)$ over the irreducible representations of G and T, respectively. For every $\nu_G \in \widehat{G}$ we define $H(X)^G_{\nu_G} \subseteq H(X)$ be the maximal subspace equivariantly isomorphic to a direct sum of copies of V_{ν_G}. In the same way we define $H(X)^T_{\nu_T}$. So decomposing the Hardy space of X unitarily and equivariantly over the irreducible representations of T and G, we have:

$$H(X) = \bigoplus_{\nu_T \in \mathbb{Z}^{d_T}} H(X)^T_{\nu_T} = \bigoplus_{\nu_G \in \widehat{G}} H(X)^G_{\nu_G}. \tag{1.1}$$

Similarly, under the previous assumptions there is an holomorphic Hamiltonian action of the product $P = G \times T$, and a corresponding unitary representation, so that we also have:

$$H(X) = \bigoplus_{\nu_G \in \widehat{G}, \nu_T \in \mathbb{Z}^{d_T}} H(X)^{G\times T}_{\nu_G,\nu_T}, \tag{1.2}$$

where $H(X)^{G\times T}_{\nu_G,\nu_T} = H(X)^G_{\nu_G} \cap H(X)^T_{\nu_T}$.

Under the assumption $\mathbf{0} \notin \Phi_T(M)$ we have that $\dim(H(X)^T_{\nu_T}) < +\infty$ for each $\nu_T \in \mathbb{Z}^{d_T}$.

Definition 1.2.1 *Given a pair of irreducible weights ν_G and ν_T for G and T, respectively, we shall denote by $\Pi_{\nu_G,\nu_T} : L^2(X) \to H(X)_{\nu_G,\nu_T}$ the orthogonal projector, and refer to its Schwartz kernel as the level (ν_G, ν_T)-Szegö projector of X (with the two actions understood). In terms of an orthonormal basis $\left\{ s_j^{(\nu_G,\nu_T)} \right\}_{j=1}^{N_{\nu_G,\nu_T}}$ of $H(X)_{\nu_G,\nu_T}$, it is given by:*

$$\Pi_{\nu_G,\nu_T}(x, y) = \sum_j \widehat{s}_j^{(\nu_G,\nu_T)}(x)\overline{\widehat{s}_j^{(\nu_G,\nu_T)}(y)}. \tag{1.3}$$

In this paper we shall consider the local asymptotics of the equivariant Szegö kernels $\widetilde{\Pi}_{\nu_G,k\nu_T}$, where the irreducible representation of T tends to infinity along a ray, and the irreducible representation of G is held fixed. To this end, we shall use a combination of the techniques in [P1] and [P4].

Observation 1.2.2 *Furthermore, the smooth function $x \mapsto \widetilde{\Pi}_{\nu_G,k\nu_T}(x, x)$ descends to a smooth function on M.*

Remark 1.2.3 *On the notation, we denote the Szegö kernel with*

1 $\widetilde{\Pi}_{\nu_G,k\nu_T}$ in the general case, under the action of $G \times T$;

2 $\widetilde{\Pi}_{k\nu_T}$ in the case of G trivial;

3 Π_k in the case of G trivial and $T = S^1$ with $\Phi_T = 1$;

4 $\Pi_{\nu_G,k}$ in the case of $T = S^1$ and $\Phi_T = 1$;

5 $\widetilde{\Pi}_{\nu_G,k}$ and $\widetilde{\Pi}_k$ also for the case $T = S^1$ and not necessarily $\Phi_T = 1$.

A key tool used in the proofs are the Heisenberg local coordinates centered at $x \in X$ (see [SZ]). We denote this system of coordinates by:

$$\gamma_x : (\theta, \mathrm{v}) \in (-\pi, +\pi) \times B_{2d_M}(\mathbf{0}, \delta) \mapsto x + (\theta, \mathrm{v}) \in X,$$

here $B_{2d_M}(\mathbf{0}, \delta)$ is the open ball of $\mathbb{R}^{2d_M}$ centered at the origin with radius $\delta > 0$. We have that θ is an angular coordinate along the circle fiber and v a local coordinate on M. We shall also set $x + \mathrm{v} = x + (\mathbf{0}, \mathrm{v})$. Given the choice of HLC centered at $x \in X$, there are induced unitary isomorphisms $T_x X \cong \mathbb{R} \oplus \mathbb{R}^{2d_M}$ and $T_m M \cong \mathbb{R}^{2d_M} \cong \mathbb{C}^{d_M}$.

Therefore, each equivariant Szegö kernel $\widetilde{\Pi}_{\nu_G, \nu_T}$ is a smoothing operator, with $\mathcal{C}^\infty$ Schwartz kernel given by (1.3).

We shall make the following three transversality assumptions on the moment maps:

1 $\mathbf{0} \notin \Phi_T(M)$ and Φ_T is transversal to the ray $\mathbb{R}_+ \cdot \nu_T$, so that $M^T_{\nu_T} = \Phi_T^{-1}(\mathbb{R}_+ \cdot \nu_T) \subseteq M$ is a compact, $G \times T$-invariant and connected submanifold of dimension $2d_M + 1 - d_T$. This is equivalent to requiring that the action of T on X be locally free on the inverse of $M^T_{\nu_T}$ (see [P4]);

2 $\mathbf{0} \in \mathfrak{g}^\vee$ is a regular value of Φ_G, so that $M^G_0 = \Phi_G^{-1}(\mathbf{0}) \subseteq M$ is a compact, $G \times T$-invariant and connected submanifold of dimension $2d_M - d_G$;

3 the two submanifolds $M^T_{\nu_T}$ and M^G_0 are mutually transversal.

These conditions imply the following (which is what we shall really be using). Since the two actions commute, they give rise to an action of the product group $P = G \times T$, which is also holomorphic and Hamiltonian, with moment map:

$$\Phi_P = (\Phi_G, \Phi_T) : M \to \mathfrak{g}^\vee \oplus \mathfrak{t}^\vee \cong \mathfrak{p}^\vee,$$

then $\mathbf{0} \notin \Phi_P(M)$, and Φ_P is transversal to the ray $\mathbb{R}_+ \cdot \big(\mathbf{0}, \nu_T\big)$.

Then

$$M_{0,\nu_T} = \Phi_P^{-1}\left(\mathbb{R}_+ \cdot (\mathbf{0}, \nu_T)\right)$$

is a smooth connected submanifold of M with codimension $d_G + d_T - 1$. Leaving aside that here G is not required to be a torus, these hypothesis are similar in nature to the hypothesis in [P4], applied however to P rather than T. Unlike [P4], where the local scaling asymptotics for representations along a ray $k\nu_T$ are considered, we shall study the local scaling asymptotics of doubly equivariant pieces of Π associated to pair of representations $(\nu_G, k\nu_T)$, where only one of the representations drifts to infinity, while the other is held fixed. Let us also remark that when $T = S^1$ and $\Phi_T = 1$, we are reduced to considering the isotypical components of the spaces of holomorphic global sections $H^0(M, A^{\otimes k})$ under the action of G, as in [P1] and [MM2]. We will find asymptotic expansions that generalize and combine the previous cases.

Let N_m we denote the normal bundle to $\Phi_P^{-1}\big(\mathbb{R}_+(\mathbf{0}, \nu_T)\big)$ then N_m is naturally isomorphic to $J_m\big(\operatorname{Ker}(\Phi_P(m))\big)$ (see [P4] section 2.2). The transversality condition is equivalent to require the injectivity of the evaluation map (see as before [P4]). Now for every $m \in M_{0,\nu_T}$ we have two Euclidean structures on

$$\operatorname{Ker}(\Phi_P(m)) = \mathfrak{g} \times \operatorname{Ker}(\Phi_T(m)) \subseteq \mathfrak{g} \oplus \mathfrak{t},$$

one induced from $\mathfrak{g} \oplus \mathfrak{t}$ and the second from $T_m M$. Let $D(m)$ the matrix representing the latter Euclidean product on N_m, with respect to an orthonormal basis. Then $D(m)$ is indipendent of the choice of an orthonormal basis for $\mathfrak{g} \times \operatorname{Ker}(\Phi_T(m))$, and it determines a positive smooth function on M_{0,ν_T}. As in [P4] we have the following definition.

Definition 1.2.4 *Define $\mathcal{D} \in \mathcal{C}^\infty(M_{0,\nu_T})$, with $m \in M_{0,\nu_T}$, by setting*

$$\mathcal{D}(m) = \sqrt{\det D(m)}.$$

We consider M_{0,ν_T} and for the decomposition of the tangent space $T_m M$ we have that:

$$T_m M = H_m \oplus V_m \oplus N_m \tag{1.4}$$

where, given $J_m : T_m M \to T_m M$ the complex structure, we have:

$$V_m = \mathfrak{g}_M(m) \oplus \operatorname{val}(\operatorname{Ker}(\Phi_T(m))), \quad N_m = J_m(V_m), \quad H_m = [V_m \oplus N_m]^\perp, \tag{1.5}$$

are respectively the vertical, the transversal and the horizontal part. Given $m \in M_{0,\nu_T}$ and $\mathrm{v} \in T_m M$ we can decompose v uniquely as $\mathrm{v} = \mathrm{v}_h + \mathrm{v}_v + \mathrm{v}_t$ with $\mathrm{v}_v \in V_m$, $\mathrm{v}_t \in N_m$ and $\mathrm{v}_h \in H_m$. The scaling asymptotics of the equivariant Szegö kernels, that we will see later, are controlled by a quadratic exponent in the components $\mathrm{v}_h, \mathrm{v}_v, \mathrm{v}_t$ of a tangent vector at a given $m \in M_{0,\nu_T}$ (viewed as a small displacement from m).

Definition 1.2.5 *Let $x \in X$ and $v_l = (\theta_l, \mathrm{v}_l) \in T_x X$ with $l = 1, 2$. We define $H : TX \oplus TX \to \mathbb{C}$ as*

$$H(v_1,v_2)=$$
$$\lambda_{\nu_T}\left(i[\omega_m(\mathrm{v}_{1v},\mathrm{v}_{1t})-\omega_m(\mathrm{v}_{2v},\mathrm{v}_{2t})]+i\omega_m\left(\frac{(\theta_2-\theta_1)}{\|\Phi_T(m)\|}\eta_{Mh}(m),\mathrm{v}_{1h}+\mathrm{v}_{2h}\right)-\right.$$
$$\left.-i\omega_m(\mathrm{v}_{1h},\mathrm{v}_{2h})-\|\mathrm{v}_{1t}\|^2-\|\mathrm{v}_{2t}\|^2-\frac{1}{2}\left\|\mathrm{v}_{1h}-\frac{(\theta_2-\theta_1)}{\|\Phi_T(m)\|}\eta_{Mh}(m)-\mathrm{v}_{2h}\right\|^2\right)$$

with $\eta_{Mh}(m)$ *the unitary generator of* $\mathrm{Ker}(\Phi_P(m))^\perp$ *such that* $\langle\eta,\Phi_P(m)\rangle=\|\Phi_T(m)\|$ *and* $\lambda_{\nu_T}=\frac{\|\nu_T\|}{\|\Phi_T(m)\|}$.

Theorem 1.2.6 (main Theorem) *Under the previous assumptions fix* $\nu_G\in\widehat{G}$ *and consider* $\nu_T\in\mathbb{Z}^{d_T}$, *assume that* Φ_P *is transversal to the ray* $\mathbb{R}_+\cdot(\mathbf{0},\nu_T)$. *We have:*

1) *If* $C,\delta>0$, *and*

$$\max\{\mathrm{dist}_M(\pi(x),M_{0,\nu_T}),\mathrm{dist}_M(\pi(y),M_{0,\nu_T})\}\geq Ck^{\delta-\frac{1}{2}},$$

then $\widetilde{\Pi}_{\nu_G,k\nu_T}=O(k^{-\infty})$.

2) *Uniformly in* $x\in X_{0,\nu_T}$ *and* $v_l\in T_xX$ *with* $\|\mathrm{v}_l\|\leq Ck^{\frac{1}{9}}$, *as* $k\to+\infty$ *we have:*

$$\widetilde{\Pi}_{\nu_G,k\nu_T}\left(x+\frac{\mathrm{v}_1}{\sqrt{k}},x+\frac{\mathrm{v}_2}{\sqrt{k}}\right)\sim\frac{1}{(\sqrt{2}\pi)^{d_T-1}}d_{\nu_G}2^{\frac{d_G}{2}}\cdot$$
$$\cdot\left(\frac{k}{\pi}\|\nu_T\|\right)^{d_M-\frac{d_P}{2}+\frac{1}{2}}\left(\sum_{j=1}^{N_x}\chi_{\nu_G}(g_j^{-1})e^{-ik\vartheta_j\nu_T}e^{H(v_1^j,v_2)}\right)\cdot\frac{e^{-i\sqrt{k}(\theta_2-\theta_1)\lambda_{\nu_T}}}{\mathcal{D}(m)}\cdot$$
$$\cdot\frac{1}{\|\Phi_T\|^{d_M+1-\frac{d_P}{2}+\frac{1}{2}}}\left(1+\sum_{l\geq1}R_{\nu_G,l}(m,v_1^j,v_2)k^{-\frac{l}{2}}\right)$$

where $d_P=d_G+d_T$, v_1^j, v^j *denote the monodromy representation* $F_x\to GL(T_mM)$, *such that for every* $j=1,\cdots,N_x$, $\mathrm{v}\in T_mM$ *we have* $p_j\mapsto d_m\mu^P_{p_j}(\mathrm{v})=\mathrm{v}^{(j)}\in T_mM$. *Where* F_x *is the stabilizator of* P *in* x *and* $R_{\nu_G,l}$ *are polynomials in* v_1^j,v_2 *with coefficients depending on* x,ν_G *and* ν_T.

3) *More in general, for every* $p_0\in P$, *denoting* $P\cdot x$ *the orbit of* $x\in X_{0,\nu_T}$, *then the following expansion holds for* $k\to+\infty$:

$$\widetilde{\Pi}_{\nu_G,k\nu_T}\left(x+\frac{u_1}{\sqrt{k}},p_0\cdot\left(x+\frac{u_2}{\sqrt{k}}\right)\right)\sim\frac{1}{(\sqrt{2}\pi)^{d_T-1}}d_{\nu_G}2^{\frac{d_G}{2}}\cdot \tag{1.6}$$
$$\cdot\left(\frac{k}{\pi}\|\nu_T\|\right)^{d_M-\frac{d_P}{2}+\frac{1}{2}}\sum_{j=1}^{N_x}\overline{\chi_{\nu_P}(p_jp_0^{-1})}e^{H(\mathrm{v}_1^j,\mathrm{v}_2)}\cdot\frac{e^{-i\sqrt{k}(\theta_2-\theta_1)\lambda_{\nu_T}}}{\mathcal{D}(m)}\cdot$$
$$\cdot\frac{1}{\|\Phi_T\|^{d_M+1-\frac{d_P}{2}+\frac{1}{2}}}\left(1+\sum_{l\geq1}R_{\nu_G,l}(m,\mathrm{v}_1^j,\mathrm{v}_2)k^{-\frac{l}{2}}\right),$$

where $p_j \in F_x$ *and* $u_j = (\theta_j, \mathrm{v}_j)$ *for* $j = 1, 2$.

The previous result describes the asymptotics of $\widetilde{\Pi}_{\nu_G, k\nu_T}$ in a shrinking neighborhood of the orbit $P \cdot x$, where $x \in X_{0,\nu_T}$. It is complemented by the following:

Proposition 1.2.7 *Suppose* $x \in X_{0,\nu_T}$ *and* $\varepsilon, D > 0$. *Then uniformly for* $\mathrm{dist}_X(y, P \cdot x) \geq D k^{\varepsilon - 1/2}$ *we have*

$$\widetilde{\Pi}_{\nu_G, k\nu_T}(x, y) = O(k^{-\infty}).$$

1.3 Special cases and relation to prior work

Before continuing our exposition, it is in order to digress on the relation of our results to prior work in this area. Let us focus on the following two special cases:

a) $T = S^1$ acts in the standard manner (with $\Phi_T = 1$);

b) G is trivial.

Let us first consider the case a), and to fix ideas let us start with the case where G is trivial. Let $\rho(\cdot, \cdot)$ be a system of Heisenberg local coordinates for X centered at x. We have for X centered at x, inducing a unitary isomorphism $(T_m M, h_m)$ with $\mathbb{C}^{d_M}$ the standard Hermitian structure. In Theorem 3.1 of [SZ] and in [BSZ], for $\mathrm{v}_1, \mathrm{v}_2 \in B(\mathbf{0}, 1) \subseteq \mathbb{C}^{d_M}$, $\theta \in (-\pi, \pi)$ and $k \to +\infty$ the following expansion has been determined for the level k of Szegö kernel Π_k (see also [MM1]):

$$\Pi_k\left(\rho\left(\theta, \frac{\mathrm{v}_1}{\sqrt{k}}\right), \rho\left(\theta', \frac{\mathrm{v}_2}{\sqrt{k}}\right)\right) \sim \left(\frac{k}{\pi}\right)^{d_M} \cdot e^{ik(\theta-\theta') + \psi_2(\mathrm{v}_1, \mathrm{v}_2)} \cdot \cdot \left(1 + \sum_{j \geq 1} a_j(x, \mathrm{v}_1, \mathrm{v}_2) k^{-\frac{j}{2}}\right) \tag{1.7}$$

where

$$\psi_2(\mathrm{v}_1, \mathrm{v}_2) = \mathrm{v}_1 \cdot \overline{\mathrm{v}_2} - \frac{1}{2}(\| \mathrm{v}_1 \|^2 + \| \mathrm{v}_2 \|^2)$$

and a_j are polynomials in v_1 and v_2.

Observation 1.3.1 *Another way to write* ψ_2 *is:*

$$\psi_2(\mathrm{v}_1, \mathrm{v}_2) = -i\omega_m(\mathrm{v}_1, \mathrm{v}_2) - \frac{1}{2}\| \mathrm{v}_1 - \mathrm{v}_2 \|^2,$$

in this form we can look directly the real and the imaginary part of ψ_2 *observing that it is responsible to the exponential decay near the diagonal.*

Now let us consider the Hamiltonian action of a compact Lie group G on M and suppose that $\mathbf{0} \in \mathfrak{g}^\vee$ is a regular value of the moment map. Then $\Pi_{\nu_G,k}(x,x)$ is rapidly decreasing away from $\Phi_G^{-1}(\mathbf{0})$, and assuming $\Phi_G(\pi(x)) = \mathbf{0}$, under the standard action of S^1 the following asymptotic expansion holds with $m = \pi(x)$:

$$\Pi_{\nu_G,k}\left(x + \frac{v_1}{\sqrt{k}}, x + \frac{v_2}{\sqrt{k}}\right) \tag{1.8}$$
$$\sim \left(\frac{k}{\pi}\right)^{d_M - \frac{d_G}{2}} e^{[Q(v_{1v}+v_{1t},v_{2v}+v_{2t})]} \sum_{g\in G_m} e^{\psi_2\left(v_{1gh},v_{2h}\right)} \cdot A_{\nu_G,k}(g,x) \cdot$$
$$\cdot \left(1 + \sum_{j\geq 1} R_{\nu_G,j}(m, v_{1g}, v_2) k^{-\frac{j}{2}}\right)$$

where $Q\left(v_{1v}+v_{1t}, v_{2v}+v_{2t}\right) = -\|v_{2t}\|^2 - \|v_{1t}\|^2 + i[\omega_m(v_{1v}, v_{1t}) - \omega_m(v_{2v}, v_{2t})]$, $G_m = \{g \in G : \mu_g(m) = m\}$, $R_{\nu_G,j}$ are polynomials in v_1, v_2 and

$$A_{\nu_G,k}(g,x) = 2^{\frac{d_G}{2}} \frac{\dim(V_{\nu_G})}{V_{\text{eff}}(\pi(x))} \frac{1}{|G_{\pi(x)}|} \chi_{\nu_G}(g) h_g^k,$$

where $V_{\text{eff}}(\pi(x))$ is the volume of the fiber above m in $\Phi_G^{-1}(\mathbf{0})$ (for more details on the effective potentials see [BG2]) and here we have set

$$v_{1g} = d_m \mu^G_{g_j^{-1}}(v_1)$$

with g_j in the stabilizator of G. Obviously (1.8) reduces to (1.7) for trivial G.

Let us consider case b). Thus assume that there is an holomorphic Hamiltonian action of a compact torus T, and that the moment map determining the linearization is nowhere zero. To fix ideas, let us first consider the case where T is one-dimensional. If ξ_M and ξ_X are vector fields on M and on X induced by μ^T and $\widetilde{\mu}^T$, we have that in the Heisenberg local coordinates $\xi_X(x) = (-\Phi_T(m), \xi_M(m))$ with $m = \pi(x)$. Let $\xi_X(x)^\perp \subseteq T_xX$ be the orthocomplement of $\xi_X(x)$. In view of Theorem 1 of [P4], again working in a system of HLC centered at x and that $v_l = (\theta_l, \mathrm{v}_l) \in T_xX \cong \mathbb{R} \times T_mM$ satisfying $\mathrm{v}_l \in \xi_X(x)^\perp$, $\|\mathrm{v}_l\| \leq Ck^{1/9}$, as $k \to +\infty$ we have:

$$\widetilde{\Pi}_k\left(x + \frac{\mathrm{v}_1}{\sqrt{k}}, x + \frac{\mathrm{v}_2}{\sqrt{k}}\right) \tag{1.9}$$
$$\sim \left(\frac{k}{\pi}\right)^{d_M} \Phi_T(m)^{-(d_M+1)} e^{i\sqrt{k}\frac{(\theta_1-\theta_2)}{\Phi_T(m)}} \cdot \left(\sum_{t\in T_m} t^k e^{E(d_x\tilde{\mu}^T_{t^{-1}}(\mathrm{v}_1),\mathrm{v}_2)}\right) \cdot$$
$$\cdot \left(1 + \sum_{j\geq 1} R_j(m, v_1, v_2) k^{-\frac{j}{2}}\right)$$

for certain smooth functions R_j, polynomial in the v_l's, with

$$E(v_1, v_2) = \frac{1}{\Phi_T(m)} \left\{ i \left[\frac{(\theta_2 - \theta_1)}{\Phi_T(m)} \omega_m(\xi_M(m), \mathrm{v}_1 + \mathrm{v}_2) - \omega_m(\mathrm{v}_1, \mathrm{v}_2) \right] - \right.$$
$$\left. -\frac{1}{2} \left\| \mathrm{v}_1 - \mathrm{v}_2 - \frac{(\theta_2 - \theta_1)}{\Phi_T(m)} \xi_M(m) \right\|^2 \right\}.$$

This last result can be generalize to a d_T-dimensional torus as in Theorem 2 of [P4]. In this Theorem we have a result similar to the previous but with the appearance of an additional important invariant, which plays a role analogous to the effective potential in (1.7). Suppose that Φ_T is transversal to the ray $\mathbb{R}_+ \cdot \nu_T$. Then the normal space to the inverse image N_m at any $m \in M_{\nu_T} = \Phi_T^{-1}(\mathbb{R}_+ \cdot \nu_T)$ is $N_m \cong J_m\left(\mathrm{Ker}(\Phi_T(m))\right)$ and the evaluation map val : $\mathrm{Ker}(\Phi_T(m)) \to T_m M$ is injective. Therefore, we have on $\mathrm{Ker}(\Phi_T(m))$ two Euclidean products, and given two orthonormal basis $\mathcal{B}_1, \mathcal{B}_2$ we can consider the matrix $D(m)$ whose determinant is independent of the choice of the basis. Thus we can let $\mathcal{D}(m) = \sqrt{\det D(m)}$. Considering $\nu_T \in \mathbb{Z}^{d_T}$, as $k \to +\infty$ we have:

$$\begin{aligned} &\widetilde{\Pi}_{k\nu_T}\left(x + \frac{\mathrm{v}_1}{\sqrt{k}}, x + \frac{\mathrm{v}_2}{\sqrt{k}}\right) \\ &\sim \left(\frac{1}{(\sqrt{2}\pi)^{d_T-1}}\right)\left(\|\nu_T\|\frac{k}{\pi}\right)^{d_M + \frac{1-d_T}{2}} \frac{1}{(\|\Phi_T\|)^{d_M+1+\frac{1-d_T}{2}}\mathcal{D}(m)} \\ &\quad e^{i\sqrt{k}\frac{(\theta_1-\theta_2)}{\Phi_T(m)}} \cdot \left(\sum_{t \in T_m} t^k e^{H_m(d_x\tilde{\mu}^T_{t^{-1}}(v_1), v_2)}\right) \cdot \\ &\quad \cdot \left(1 + \sum_{j \geq 1} R_j(m, v_1, v_2) k^{-\frac{j}{2}}\right) \end{aligned} \tag{1.10}$$

with

$$H_m(v_1, v_2) = \frac{\|\nu_T\|}{\|\Phi_T\|}\left[-i\omega_m(\mathrm{v}_1, \mathrm{v}_2) - \|\,\mathrm{v}_1\|^2 - \|\,\mathrm{v}_2\|^2\right].$$

In this paper, we shall pair these situations. More precisely, we shall assume given actions of G and T as above, compatible in the sense that they commute, and consider the resulting asymptotics relative to a pair $(\nu_G, k\nu_T)$ of irreducible characters, where ν_G is held fixed, and $k\nu_T \to +\infty$ along an integral ray.

We consider the case of a d_T-dimensional torus. We have shown at the beginning the fundamental result of this work. Now we present some observations.

Observation 1.3.2 *If G is trivial, $d_G = 0$ the leading term is:*

$$\frac{e^{-ik\vartheta_j\nu_T}}{(\sqrt{2}\pi)^{d_T-1}}\left(\frac{\|\nu_T\|k}{\pi}\right)^{d_M-\frac{d_T-1}{2}}\cdot\frac{1}{\|\Phi_T(m)\|^{d_M+1-\frac{d_T}{2}+\frac{1}{2}}\mathcal{D}(m)}e^{\lambda_{\nu_T}\left[-i\omega_m(\mathrm{v}_1^j,\mathrm{v}_2)-\frac{1}{2}\|\mathrm{v}_1^j-\mathrm{v}_2\|^2\right]}$$

and we're back to the equation (1.10) *when* $\theta_1 = \theta_2 = 0$.

Observation 1.3.3 *In the case $T = S^1$ with the standard action we have $\lambda_{\nu_T} = 1$, and when $\theta_1 = \theta_2 = 0$ the result is the formula* (1.8).

Now we present a Theorem that is the diagonal version, without scaling of the point 2) of the main Theorem.

Theorem 1.3.4 *Under the hypothesis of the main Theorem, without assumptions about the directions, for $m = \pi(x) \in M_{0,\nu_T}$ as $k \to +\infty$ we have:*

$$\begin{aligned}&\widetilde{\Pi}_{\nu_G,k\nu_T}(x,x) \qquad (1.11)\\ &\sim \frac{d_{\nu_G}2^{d_G/2}}{\left(\sqrt{2}\pi\right)^{d_T-1}}\cdot\left(\frac{\|\nu_T\|k}{\pi}\right)^{d_M+\frac{1-d_P}{2}}\cdot\sum_{j=1}^{N_x}\chi_{\nu_G}(g_j^{-1})e^{-ik\vartheta_j\nu_T}\\ &\cdot\frac{1}{\mathcal{D}(m)\|\Phi_T(m)\|^{d_M+1+\frac{1-d_P}{2}}}\cdot\left(1+\sum_{l\geq 1}B_l(m)k^{-l}\right)\end{aligned}$$

with B_l that are smooth functions on M_{0,ν_T}.

Corollary 1.3.5 *Under the assumptions of Theorem* 1.2.6*,*

$$\begin{aligned}\lim_{k\to+\infty}\left(\frac{\pi}{\|\nu_T\|k}\right)^{d_M-d_P+1}&\dim(H(X)_{\nu_G,k\nu_T}) =\\ &=\frac{d_{\nu_G}^2}{(2\pi)^{d_T-1}}\cdot\int_{M_{0,\nu_T}}\frac{\|\Phi_T(m)\|^{-(d_M+1)+d_P-1}}{\mathcal{D}(m)}dV_M(m).\end{aligned}$$

Observation 1.3.6 *We observe that the Theorem* 1.2.6 *implies something stronger, that is that the successive terms are of less degree.*

Observation 1.3.7 *This result about the dimension of the space of holomorphic sections is similar to results obtained by [DVP]. In fact about this last corollary, we observe that in the case of the standard action and when the line bundle A is very ample, we obtain the know result:*

$$\lim_{k\to+\infty}\left(\frac{1}{k^{d_M}}\right)\dim(H(X)_k) = \int_M\frac{c_1(A)^{d_M}}{d_M!},$$

where we have that $\frac{dV_M(m)}{\pi^{d_M}} = \frac{1}{d_M!}\left(\frac{\omega}{\pi}\right)^{d_M} = \frac{c_1(A)^{d_M}}{d_M!}$.

As a very special example, we observe that when $d_T = 1$ and $T^1 = S^1$ acts trivially on M with moment map $\Phi_T = 1$, we have $H(X)_k$ the k-th isotypical component for the standard S^1-action on X, which is naturally and unitarily isomorphic to $H^0(M, A^{\otimes k})$. In this case we have the celebrated Tian-Yau-Zelditch expansion. For this result we refer to the work of Zelditch in [Z2]. Let M be a compact complex manifold of dimension d_M (over $\mathbb{C}$) and $(L, h) \to M$ be a positive Hermitian holomorphic line bundle. Let g be the Kähler metric on M corresponding to the Kähler form ω_g defined as the normalized curvature of h.

Theorem 1.3.8 (Zelditch, 2000) *There exists a complete asymptotic expansion:*

$$\sum_{i=0}^{d_N} \|s_i^N(z)\|_{h^N}^2 = a_0 N^{d_M} + a_1(z) N^{d_M - 1} + \cdots \tag{1.12}$$

for certain smooth coefficients $a_j(z)$ with $a_0(z) = 1$. More precisely, for any N:

$$\left\| \sum_{i=0}^{d_N} \|s_i^N(z)\|_{h^N}^2 - \sum_{j<R} a_j(x) N^{d_M - j} \right\|_{\mathcal{C}^N} \leq C_{R,N} N^{d_M - R}. \tag{1.13}$$

1.4 Application to Toeplitz operator kernels

By way of application, motivated by the standard Berezin-Toeplitz quantization of a classical observable, let us consider the scaling asymptotics of the equivariant components of certain Toeplitz operators. Given $f \in \mathcal{C}^\infty(M)$ and assuming for simplicity that f is invariant under the action of the product group $P = G \times T$, we can consider the Toeplitz operators $T_{\nu_G, k\nu_T}[f] = \widetilde{\Pi}_{\nu_G, k\nu_T} \circ M_f \circ \widetilde{\Pi}_{\nu_G, k\nu_T}$, where M_f denotes multiplication by $f \circ \pi$. Then $T_{\nu_G, k\nu_T}[f]$ is a self-adjoint endomorphisms of $H(X)_{\nu_G, k\nu_T}$.

Given that $\mathbf{0} \notin \Phi_P$, the equivariant Toeplitz operator $T_{\nu_G, k\nu_T}[f]$ is smoothing, and its distributional kernel is given by the following two alternative expressions:

$$\begin{aligned} T_{\nu_G, k\nu_T}[f](x, x') &= \int_X \widetilde{\Pi}_{\nu_G, k\nu_T}(x, y) f(y) \widetilde{\Pi}_{\nu_G, k\nu_T}(y, x') dV_X(y) \\ &= \sum_j T_{\nu_G, k\nu_T}[f](s_j^k(x)) \overline{(s_j^k(x'))} \end{aligned} \tag{1.14}$$

with $x, x' \in X_{0, \nu_T}$ and s_j^k an orthonormal basis of $H(X)_{\nu_G, k\nu_T}$. We will see that $T_{\nu_G, k\nu_T}[f](x, x')$ has asymptotic expansions near the diagonal similar to the one for $\widetilde{\Pi}_{\nu_G, k\nu_T}$. Note that with $f(y)$ we denote $f(\pi(y))$ and that every $f \in \mathcal{C}^\infty(M)$ lifts to an invariant function $f(x)$ on X. For the sake of simplicity, we shall focus on points of the form $(x+n, x+n)$ (with rescaling), as usual, in a system of Heisenberg local coordinates centered at x, where n is a normal vector to the P-orbit of x and we shall make the extra assumption that the stabilizer of x in P is trivial. Notice that any point sufficiently close to $P \cdot x$ may be written in this manner, possibly replacing x with $p \cdot x$ for some $p \in P$.

Theorem 1.4.1 *Assume that $\mathbf{0} \notin \Phi_P$, $f \in \mathcal{C}^\infty(M_{0,\nu_T})$ is μ^P-invariant and that the stabilizer of P in x is trivial. Suppose $x \in X_{0,\nu_T}$ and fix a system of HLC centered at x. Let $m = \pi(x)$. Then we have:*

1) *If $C, \delta > 0$ and*

$$\max\left\{\mathrm{dist}_M(\pi(x), M_{0,\nu_T}), \mathrm{dist}_M(\pi(y), M_{0,\nu_T})\right\} \geq C k^{\delta - \frac{1}{2}},$$

then $T_{\nu_G, k\nu_T}[f](x, x') = O(k^{-\infty})$.

2) *Uniformly in $n_1 \in N_x^P = T_x(P \cdot x)^\perp$ as $k \to +\infty$:*

$$T_{\nu_G,k\nu_T}[f]\left(x + \frac{n_1}{\sqrt{k}}, x + \frac{n_1}{\sqrt{k}}\right) \tag{1.15}$$
$$\sim \frac{1}{(\sqrt{2}\pi)^{d_T - 1}} d_{\nu_G} 2^{\frac{d_G}{2}} \left(\frac{k}{\pi}\|\nu_T\|\right)^{d_M - \frac{d_P}{2} + \frac{1}{2}} f(m) e^{-2\lambda_{\nu_T}\|\mathrm{t}_1\|^2} \cdot$$
$$\cdot \frac{1}{\mathcal{D}(m)} \cdot \frac{1}{\|\Phi_T\|^{d_M + 1 - \frac{d_P}{2} + \frac{1}{2}}} \left(\sum_{l \geq 0} k^{-\frac{l}{2}} R_l(n_1, m)\right)$$

with $R_l(n_1, m)$ a polynomial in n_1 and $\mathrm{t}_1 \in N_m = J_m(\mathrm{val}_m(\mathrm{Ker}(\Phi_P(m))))$.

Corollary 1.4.2 *Under the assumptions of Theorem 1.4.1,*

$$\lim_{k \to +\infty} \left(\frac{\pi}{\|\nu_T\| k}\right)^{d_M - d_P + 1} \mathfrak{T}\left(T_{\nu_G, k\nu_T}[f]\right) =$$
$$= \frac{d_{\nu_G}^2}{(2\pi)^{d_T - 1}} \cdot \int_{X_{0,\nu_T}} \frac{f(\pi(x)) \|\Phi_T(\pi(x))\|^{-(d_M + 2 - d_P)}}{\mathcal{D}(\pi(x))} dV_X(x),$$

where $\mathfrak{T}\left(T_{\nu_G,k\nu_T}[f]\right)$ is the trace of the Toeplitz operator.

1.5 Examples

The main Theorem predicts that the diagonal restriction $\widetilde{\Pi}_{\nu_G, k\nu_T}(x,x)$ of the equivariant Szegö kernel (which descends to a function on M) is rapidly decreasing away from the locus M_{0,ν_T}, and grows like $k^{d_M + \frac{1-d_P}{2}}$ there. Let us illustrate this explicitly by two examples (cfr the computations in [P1]). Recall from [BSZ] that for $k = 1, 2, \cdots$ an orthonormal basis of $H^0(\mathbb{P}^n, \mathcal{O}_{\mathbb{P}^n}(k))$ is $\{s^k_J\}_{|J|=k}$, where:

$$s^k_J = \sqrt{\frac{(k+n)!}{\pi^n J!}} z^J \tag{1.16}$$

and where $J! = \prod_{l=0}^n j_l!$, $z^J = \prod_{l=0}^n z_l^{j_l}$. In the next example we consider a particular product action and we show that outside of M_{0,ν_T} we have the exponential decay of the Szegö kernel.

Example 1.5.1 *Let us make $M = \mathbb{P}^1$. Let us consider the actions of $G = T^1$ on M induced by the representation on $\mathbb{C}^2$ given by $\mu^G(z_0, z_1) = w \cdot (z_0, z_1) = (w^{-1}z_0, wz_1)$, and the action of $T = T^1$ induced by the representation given by $\mu^T(z_0, z_1) = (s^{-1}z_0, s^{-2}z_1)$. These actions are holomorphic and Hamiltonian, with moment maps:*

$$\Phi_G(z_0, z_1) = \frac{|z_0|^2 - |z_1|^2}{|z_0|^2 + |z_1|^2}$$

and

$$\Phi_T(z_0, z_1) = \frac{|z_0|^2 + 2|z_1|^2}{|z_0|^2 + |z_1|^2}.$$

Then we have:

$$\Phi_G^{-1}(0) = \{[z_0 : z_1] : |z_0| = |z_1|\}$$

and placing $X = S^3 \subseteq \mathbb{C}^2$ we have

$$X_0 = \pi^{-1}\left(\Phi_G^{-1}(0)\right) = \left\{(z_0, z_1) : |z_0| = |z_1| = \frac{1}{\sqrt{2}}\right\} \cong S^1 \times S^1$$

with a free action of S^1 on X_0. We have $\nu_T = 1 \in \mathbb{Z}$, $\Phi_P^{-1}(\mathbb{R}_+ \cdot (0,1)) = \Phi_G^{-1}(0) = \{(z_0, z_1) : |z_0| = |z_1|\}$ and the action of P is given by:

$$\mu^P([z_0 : z_1]) = (w, s) \cdot (z_0, z_1) = \left((ws)^{-1}z_0, ws^{-2}z_1\right).$$

If $|z_0| = |z_1|$ ($\neq 0$) and $(w, s) \cdot (z_0, z_1) = (z_0, z_1) \Rightarrow ws = 1, ws^{-2} = 1 \Rightarrow s = s^{-2}$ so $s = e^{\frac{2}{3}\pi ji}$ with $j = 0, 1, 2$ and $w = \frac{1}{s}$ then the action is locally free. We are in the hypothesis of the main Theorem. We have $s \cdot (z_0^a z_1^b) = (sz_0)^a(s^2z_1)^b = s^{a+2b}z_0^a z_1^b$ and then $\widetilde{H}^T(X)_k = \mathrm{span}\{z_0^a z_1^b : a + 2b = k\}$. In the other side we have $w \cdot (z_0^a z_1^b) = (wz_0)^a(w^{-1}z_1)^b = w^{a-b}z_0^a z_1^b$ and then $\widetilde{H}^G(X)_{\nu_G} = \mathrm{span}\{z_0^a z_1^b : a = b + \nu_G\}$. Thus

$$\widetilde{H}^P(X)_{\nu_G,k} = \mathrm{span}\left\{z_0^a z_1^b : a = b + \nu_G, a + 2b = k\right\}$$

then $a + 2b = k \Rightarrow b + \nu_G + 2b = k \Rightarrow 3b = k - \nu_G$ and

$$\dim\Big(\widetilde{H}^P(X)_{\nu_G,k}\Big) = \begin{cases} 0 & if \;\; k \equiv \nu_G \mod 3 \\ 1 & if \;\; k \not\equiv \nu_G \mod 3 \end{cases}.$$

If $k = \nu_G + 3b$ we have:

$$\widetilde{H}^P_{\nu_G,\nu_G+3b}(X) = \mathrm{span}\Big\{ z_0^{b+\nu_G} z_1^b \Big\},$$

the corresponding Szegö projector is:

$$\widetilde{\Pi}^P_{\nu_G,\nu_G+3b}\left((z_0,z_1),(u_0,u_1)\right) = \frac{(2b+\nu_G+1)!}{\pi(b+\nu_G)!b!}(z_0\overline{u}_0)^{b+\nu_G}(z_1\overline{u}_1)^b. \tag{1.17}$$

Now consider $z_j = u_j$ with $|z_0|^2+|z_1|^2 = 1$ and we set $x = |z_0|^2$, $y = y_b = \frac{b+\nu_G}{2b+\nu_G} \to \frac{1}{2}$ as $b \to +\infty$. Using Stirling approximation:

$$n! \sim \sqrt{2\pi n}\frac{n^n}{e^n}$$

and the projector:

$$\widetilde{\Pi}^P_{\nu_G,\nu_G+3b}\left((z_0,z_1),(z_0,z_1)\right) = \frac{(2b+\nu_G+1)!}{\pi(b+\nu_G)!b!}|z_0|^{2(b+\nu_G)}|z_1|^{2b}$$

we can find the following asymptotic for the coefficient:

$$\begin{aligned} \frac{(2b+\nu_G+1)!}{\pi(b+\nu_G)!b!} &\sim \sqrt{\frac{2\pi(2b+\nu_G+1)}{2\pi(\nu_G+b)2\pi b}}\frac{(2b+\nu_G+1)^{2b+\nu_G+1}}{(b+\nu_G)^{b+\nu_G}b^b}e^{-1} \\ &\sim_{[2b+\nu_G+1\sim 2b]} \frac{2\sqrt{b}}{\sqrt{\pi}}\frac{(2b+\nu_G+1)^{2b+\nu_G}}{(b+\nu_G)^{b+\nu_G}b^b}e^{-1} \\ &\sim \frac{2}{\sqrt{\pi}}\sqrt{b}\frac{(2b+\nu_G)^{2b+\nu_G}}{(b+\nu_G)^{b+\nu_G}b^b}e^{-1}\left\{\left[1+\frac{1}{2b+\nu_G}\right]^{2b+\nu_G} \to e\right\} \\ &\sim \frac{2}{\sqrt{\pi}}\sqrt{b}\left[\frac{2b+\nu_G}{b+\nu_G}\right]^{b+\nu_G}\left[\frac{2b+\nu_G}{b}\right]^b \\ &\sim \frac{2}{\sqrt{\pi}}\sqrt{b}\left(\frac{1}{y_b}\right)^{b+\nu_G}\left(\frac{1}{1-y_b}\right)^b \end{aligned} \tag{1.18}$$

and for the projector:

$$\begin{aligned} \widetilde{\Pi}^P_{\nu_G,\nu_G+3b}\left((z_0,z_1),(z_0,z_1)\right) &\sim \frac{2}{\sqrt{\pi}}\sqrt{b}\left(\frac{x}{y}\right)^{b+\nu_G}\left(\frac{1-x}{1-y}\right)^b = \\ &= 2\sqrt{\frac{b}{\pi}}\left(\frac{x}{y}\right)^{\nu_G}\left[\left(\frac{x}{y}\right)\cdot\left(\frac{1-x}{1-y}\right)\right]^b = 2\sqrt{\frac{b}{\pi}}\left(\frac{x}{y}\right)^{\nu_G} e^{bF(x,y)}, \end{aligned} \tag{1.19}$$

where we set $F(x,y) = \log x + \log(1-x) - \log y - \log(1-y) = f(x) - f(y)$ *with* $f(t) = \log t + \log(1-t)$ *and* $0 < t < 1$. *We observe that for* $t \to 0^+, 1^-$ *we obtain* $f(t) \to -\infty$ *and that the derivative:*

$$f'(t) = \frac{1}{t} - \frac{1}{1-t} = 0 \Leftrightarrow 1 - t = t \Leftrightarrow t = \frac{1}{2}$$

with $f(1/2) = -\log 4$. *Thus for* $b \gg 0$ *and* $y = y_b$ *we have* $f(y) = -\log 4 - \delta(b)$ *with* $\delta(b) > 0$ *and* $\delta(b) \to 0$ *as* $b \to +\infty$. *If* $x \neq \frac{1}{2}$ *we have* $f(x) = -\log 4 - \delta(x)$ *(with* $\delta(x) > 0$ *fixed). Then* $F(x, y_b) = -\delta(x) + \delta(b) \leq -\frac{\delta(x)}{2}$.

Now

$$\begin{aligned}\left|\widetilde{\Pi}^P_{\nu_G,\nu_G+3b}\left((z_0,z_1),(z_0,z_1)\right)\right| &\sim 2\sqrt{\frac{b}{\pi}}\left(\frac{x}{y}\right)^{\nu_G} e^{bF(x,y)} \\ &\leq 2\sqrt{\frac{b}{\pi}}\left(\frac{x}{y}\right)^{\nu_G} e^{-b\frac{\delta(x)}{2}} = O(b^{-\infty})\end{aligned} \tag{1.20}$$

but $x = \frac{1}{2}$ *that is* $|z_0| = |z_1| = \frac{1}{\sqrt{2}}$ *and we have:*

$$\begin{aligned}\left|\widetilde{\Pi}^P_{\nu_G,\nu_G+3b}\left((z_0,z_1),(z_0,z_1)\right)\right| &\sim 2\sqrt{\frac{b}{\pi}}\left(\frac{x}{y}\right)^{\nu_G} e^{bF(1/2,y_b)} \\ &\leq 2\sqrt{\frac{b}{\pi}}\left(\frac{1}{2y_b}\right)^{\nu_G} e^{b\delta(b)}\end{aligned} \tag{1.21}$$

and considering that we have for y_b*:*

$$\begin{aligned}y_b = \frac{b+\nu_G}{2b+\nu_G} = \frac{1+\frac{\nu_G}{b}}{2\left(1+\frac{\nu_G}{2b}\right)} = \frac{1}{2}\left(1+\frac{\nu_G}{b}\right)\left(1-\frac{\nu_G}{2b}+\frac{\nu_G^2}{4b^2}+\cdots\right) &= \\ = \frac{1}{2}+\frac{\nu_G}{4b}+O\left(\frac{1}{b^2}\right)&,\end{aligned} \tag{1.22}$$

then $f(y_b) = -\log 4 + O\left(\frac{1}{b^2}\right)$ *(because* $f'(1/2) = 0$*) and so follows that* $b\delta(b) = O\left(\frac{1}{b}\right) \to 0$ *as* $b \to +\infty$. *Thus*

$$\left|\widetilde{\Pi}^P_{\nu_G,\nu_G+3b}\left((z_0,z_1),(z_0,z_1)\right)\right| \sim 2\sqrt{\frac{b}{\pi}}. \tag{1.23}$$

Another possible variation similar to the previous is the following.

Example 1.5.2 *Let us make* $M = \mathbb{P}^2$. *Let us consider the actions of* $G = T^2$ *on* M *induced by the representation on* $\mathbb{C}^3$ *given by* $\mu^G(z_0,z_1,z_2) = (w_1^{-1}z_0, w_1w_2^{-1}z_1, w_2z_2)$, *and the action of* $T = T^1$ *induced by the representation given by* $\mu^T(z_0,z_1,z_2) = (s^{-1}z_0, s^{-2}z_1, s^{-3}z_2)$. *These actions are holomorphic and Hamiltonian, with moment maps:*

$$\Phi_G(z_0,z_1,z_2)=\left(\frac{|z_0|^2-|z_1|^2}{|z_0|^2+|z_1|^2},\frac{|z_1|^2-|z_2|^2}{|z_1|^2+|z_2|^2}\right)$$

and

$$\Phi_T(z_0,z_1,z_2)=\frac{|z_0|^2+2|z_1|^2+3|z_2|^2}{|z_0|^2+|z_1|^2+|z_2|^2}.$$

Then

$$\Phi_G^{-1}(\mathbf{0})=\{[z_0:z_1:z_2]:|z_0|=|z_1|=|z_2|\}$$

and placing $X=S^5\subseteq\mathbb{C}^3$ we have:

$$X_0=\pi^{-1}\left(\Phi_G^{-1}(\mathbf{0})\right)=\left\{(z_0,z_1,z_2):|z_0|=|z_1|=|z_2|=\frac{1}{\sqrt{3}}\right\}\cong S^1\times S^1\times S^1$$

with a free action of G on X_0. We have $\nu_T=1\in\mathbb{Z}$ and $\Phi_P^{-1}(\mathbb{R}_+\cdot(\mathbf{0},1))=\Phi_G^{-1}(\mathbf{0})=\{(z_0,z_1,z_2):|z_0|=|z_1|=|z_2|\}$ and the action of P is given by:

$$\mu^P([z_0:z_1:z_2])=(w,s)\cdot(z_0,z_1,z_2)=\left((w_1s)^{-1}z_0,w_1w_2^{-1}s^{-2}z_1,w_2s^{-3}z_2\right).$$

If $|z_0|=|z_1|=|z_2|$ ($\neq 0$) and $(w,s)\cdot(z_0,z_1,z_2)=(z_0,z_1,z_2)\Rightarrow w_1s=1,w_1w_2^{-1}s^{-2}=1,w_2s^{-3}=1\Rightarrow s^6=1$ so $s=e^{\frac{2}{6}\pi ji}$ with $j=0,1,2,4,5$ and $w_1=\frac{1}{s},w_2=\frac{1}{s^3}$ then the action is locally free. The hypothesis of the main Theorem are satisfied. We have $s\cdot(z_0^az_1^bz_2^c)=(sz_0)^a(s^2z_1)^b(s^3z_2)^c=s^{a+2b+3c}z_0^az_1^bz_2^c$ and then $\widetilde{H}^T(X)_k=\operatorname{span}\{z_0^az_1^bz_2^c:a+2b+3c=k\}$. In analogue way we obtain that

$$\widetilde{H}^G(X)_{\nu_G}=\operatorname{span}\{z_0^az_1^bz_2^c:(a-b,b-c)=(\nu_1,\nu_2)=\nu_G\}.$$

Thus

$$\widetilde{H}^P(X)_{\nu_G,k}=\operatorname{span}\left\{z_0^az_1^bz_2^c:(a-b,b-c)=(\nu_1,\nu_2),a+2b+3c=k\right\}$$

and

$$\dim\left(\widetilde{H}^P(X)_{\nu_G,k}\right)=\begin{cases}0 & \text{if } k\equiv\nu_1 \mod 6\\ 1 & \text{if } k\not\equiv\nu_1 \mod 6\end{cases}.$$

If $k=6c+\nu_1+3\nu_2$ we have:

$$\widetilde{H}^P(X)_{(\nu_1,\nu_2),\nu_1+3\nu_2+6c}=\operatorname{span}\left\{z_0^{c+\nu_1+\nu_2}z_1^{c+\nu_2}z_2^c\right\}.$$

Now consider $z_j=u_j$ with $|z_0|^2+|z_1|^2+|z_2|^2=1$ and we set $x=|z_0|^2$, $y=|z_1|^2$, $z=|z_2|^2=1-x-y$. As before, using Stirling approximation for the projector:

$$\widetilde{\Pi}^P_{\nu_G,6c+\nu_1+3\nu_2}\left((z_0,z_1,z_2),(z_0,z_1,z_2)\right) \\ = \frac{(3c+\nu_1+2\nu_2+2)!}{\pi^2(c+\nu_2+\nu_1)!(c+\nu_2)!c!}(|z_0|)^{2(c+\nu_2+\nu_1)}(|z_1|)^{2(c+\nu_2)}(|z_2|)^{2c}, \tag{1.24}$$

the following asymptotic holds:

$$\frac{(3c+\nu_1+2\nu_2+2)!}{\pi^2(c+\nu_2+\nu_1)!(c+\nu_2)!c!} \sim \\ \frac{1}{\pi^2}\sqrt{\frac{2\pi(3c+\nu_1+2\nu_2+2)}{2\pi(c+\nu_2+\nu_1)2\pi(c+\nu_2)2\pi c}}\frac{(3c+\nu_1+2\nu_2+2)^{(3c+\nu_1+2\nu_2+2)}}{(c+\nu_2+\nu_1)^{c+\nu_2+\nu_1}(c+\nu_2)^{(c+\nu_2)}c^c}e^{-2} \\ \sim \frac{9\sqrt{3}c}{2\pi^3}\left[\frac{3c+\nu_1+2\nu_2}{c+\nu_2+\nu_1}\right]^{c+\nu_2+\nu_1}\left[\frac{3c+\nu_1+2\nu_2}{c+\nu_2}\right]^{c+\nu_2}\left[\frac{3c+\nu_1+2\nu_2}{c}\right]^{c} \tag{1.25}$$

and, for the projector:

$$\widetilde{\Pi}^P_{\nu_G,6c+\nu_1+3\nu_2}\left((z_0,z_1,z_2),(z_0,z_1,z_2)\right) \sim \frac{9\sqrt{3}c}{2\pi^3}x^{\nu_1+\nu_2}y^{\nu_2}x^c y^c(1-x-y)^c = \\ = \frac{9\sqrt{3}c}{2\pi^3}x^{\nu_1+\nu_2}y^{\nu_2}e^{cF(x,y)}, \tag{1.26}$$

with $F(x,y) = \log x + \log y + \log(1-x-y)$, $0 < x, y$ *and* $x+y<1$. *Studing the partial derivatives we find that we have a critical point* $x = y = \frac{1}{3}$ *with* $F\left(\frac{1}{3},\frac{1}{3}\right) = -\log 27$. *So we have* $F(x,y) = -\log 27 - \delta(z)$ *with* $\delta(z) > 0$. *If* $x = y \neq \frac{1}{3}$ *we have:*

$$\left|\widetilde{\Pi}^P_{\nu_G,6c+\nu_1+3\nu_2}\left((z_0,z_1,z_2),(z_0,z_1,z_2)\right)\right| \\ \leq \frac{9\sqrt{3}c}{2\pi^3}x^{\nu_1+\nu_2}y^{\nu_2}e^{-c\delta(z)} = O(c^{-\infty}) \tag{1.27}$$

but $x = y = \frac{1}{3}$, $\delta(z) = 0$ *and*

$$\left|\widetilde{\Pi}^P_{\nu_G,6c+\nu_1+3\nu_2}\left((z_0,z_1,z_2),(z_0,z_1,z_2)\right)\right| \sim \frac{9\sqrt{3}c}{2\pi^3}\left(\frac{1}{3}\right)^{\nu_1+\nu_2}\left(\frac{1}{3}\right)^{\nu_2}. \tag{1.28}$$

Chapter 2

Preliminaries

2.1 Basic Objects

In this section, considering the setting of geometric quantization (see [GS1], [W] and [RCG1]) we define a basic quantum objects necessary to develope our results. Associated to the compact symplectic manifold (M, ω) we consider a prequantization triple (L, ∇, h), where L is an Hermitian line bundle, ∇ is a connection with curvature form given by

$$\Theta = -2i\omega, \tag{2.1}$$

and h an Hermitian inner product. Sometimes the condition (2.1) is called the prequantization condition. As quantum space, a good candidate is the space of holomorphic sections $H^0(M, L)$. These spaces, with the completition respect the norm:

$$\|s\|^2 = \int_M h(s,s)\frac{\omega^{d_M}}{d_M!} \tag{2.2}$$

are Hilbert spaces.

Other basic objects in the theory of geometric quantization are the circle bundle and the tensor power of the line bundle. Let L as before, considering the dual space $L^\vee$ we have the following definition:

Definition 2.1.1 (Circle Bundle) *A circle bundle associated to L is a subset X of $L^\vee$ defined as:*

$$X = \{(m, \lambda) : m \in M, \lambda \in L^\vee_m, h(\lambda, \lambda) = 1\}.$$

Observation 2.1.2 *Note that we have the restriction of $\pi : L^\vee \to M$ to X that for simplicity we denote in the same way.*

The circle bundle is the boundary of $D = \{(m, \lambda) : m \in M, \lambda \in L^\vee_m, h(\lambda, \lambda) \leq 1\}$ that is a strictly pseudoconvex domain in L. We denote with $\| \cdot \|_m$ the induced norm by h so we have that $D = \{\rho > 0\}$ where $\rho : L^\vee \to \mathbb{R}$ is defined as $\rho(m, \lambda) = 1 - \|\lambda\|^2_m = 1 - a(m)^{-1}|\alpha|^2$ where we write $\lambda = \alpha s^\vee_L$ and $a(m) = \|s^\vee_L\|^2_m$ is a smooth function over a $U \subset M$ and $s^\vee_L$ is a local coframe over U. We have a circle action on X denoted by $r_\theta : S^1 \times X \to X$ with infinitesimal generator $\frac{\partial}{\partial\theta}$. As in [Z2] we consider the holomorphic and respectively antiholomorphic subspaces $T'D, T''D \subset TD_\mathbb{C}$ and the corrisponding differentials $d'f = df|_{T'}, d''f = df|_{T''}$ for f smooth on D. $TD_\mathbb{C} = T'D \oplus T''D \oplus \mathbb{C}\frac{\partial}{\partial\theta}$ has a Cauchy Riemann structure and the vectors on D that are elements of $T'X$ (resp. $T''X$) are of the form $\sum_j a_j \frac{\partial}{\partial \bar{z}_j}$ (resp. $\sum_j a_j \frac{\partial}{\partial z_j}$). We can choose a basis for these vector spaces and consider the Cauchy Riemann operator $\overline{\partial}_b : \mathcal{C}^\infty(X) \to \mathcal{C}^\infty(X, (T''D)^\vee)$ defined as $\overline{\partial}_b = df|_{T''}$. If we define $\alpha = \frac{1}{i} d'\rho|_X$ and the Volume form $dV = \alpha \wedge (d\alpha)^{d_M}$ we have that (X, α) is a contact manifold. In the compact Kähler case from (2.1) we have that L is a positive line bundle and by the Kodaira embedding Theorem there exists a positive tensor power $L^{\otimes k}$ with $k \in \mathbb{N}$ and global holomorphic sections $\{s^k_i\}^{d_k}_{i=0}$ that give the following embedding:

$$\Phi : M \to \mathbb{P}^{d_k}(\mathbb{C}),$$

where $\Phi(z) = [s^k_0(z) : \cdots : s^k_{d_k}(z)]$. The set $\{s^k_i\}^{d_k}_{i=0}$ is a basis for $H^0(M, L^{\otimes k})$ the space of holomorphic sections of $L^{\otimes k}$. The $\dim(H^0(M, L^{\otimes k})) = 1 + d_k$.

2.2 The Hardy space and the Szegö kernel

In this section we give the basic definitions of the Hardy space and Szegö kernel.

Definition 2.2.1 *We define the Hardy space $H(X) = L^2(X) \cap \mathrm{Ker}(\overline{\partial}_b)$ that admit the following decomposition:*

$$H(X) = \bigoplus_k H(X)_k,$$

where the subspaces

$$H(X)_k = \{f \in \mathcal{C}^\infty(X) : f(e^{i\varphi}x) = e^{ik\varphi}f(x)\} \cap \mathrm{Ker}(\overline{\partial}_b)$$

are called k-Hardy spaces.

Observation 2.2.2 *On $H(X)_k$ we have the L^2-Hermitian product:*

$$\langle \rho, \sigma \rangle = \int_X \rho\overline{\sigma} dV_X$$

and on $H^0(M, L^{\otimes k})$ *we have the Hermitian product* $\langle \rho, \sigma \rangle = \int_M h_m(\rho(m), \sigma(m)) dV_M(m)$ *determined by the metric and the symplectic volume form. On the other hand, to a section* $s \in H^0(M, L^{\otimes k})$ *there is naturally associated a function* $\widehat{s} \in H(X)_k$ *given by*

$$\widehat{s}_k = \langle \lambda^{\otimes k}, s_k(z) \rangle.$$

This correspondence determines a unitary isomorphism:

$$H^0(M, L^{\otimes k}) \cong H(X)_k.$$

Definition 2.2.3 (Equivariant Szegö projector) *We define the equivariant Szegö projector* $\Pi_k : L^2(X) \to H(X)_k$ *where* $\forall f \in L^2(X)$ *as:*

$$\Pi_k(f) = \sum_j \langle f, \widehat{s}_j^{(k)} \rangle_{L^2(X)} \widehat{s}_j^{(k)}$$

where $(\widehat{s}_j^{(k)})_{j=1}^{d_k}$ *is an orthonormal basis of* $H(X)_k \cong H^0(M, L^{\otimes k})$.

The projector Π_k can be written in the form:

$$(\Pi_k(f))(x) = \int_X \sum_j \widehat{s}_j^{(k)}(x) \overline{\widehat{s}_j^{(k)}(y)} f(y) dV_X(y)$$

for any choice of the orthonormal basis $\{s_j^{(k)}(y)\}_{j=1}^{d_k}$. We denote the equivariant Szegö kernel as:

$$\Pi_k(x, y) = \sum_j \widehat{s}_j^{(k)}(x) \overline{\widehat{s}_j^{(k)}(y)}.$$

By a theorem of [BS] it is possible to represent the Szegö kernel as a complex Fourier integral operator (FIO representation).

Theorem 2.2.4 *Let* $\Pi(x, y)$ *the Szegö kernel of* X*, the boundary of a strictly pseudoconvex domain in* L*. Then there exists a symbol* $s \in S^{d_M}(X \times X \times \mathbb{R}_+)$ *that admit the following expansion:*

$$s(x, y, t) = \sum_{k=0}^{\infty} t^{d_M - k} s_k(x, y), \tag{2.3}$$

so that

$$\Pi(x, y) = \int_0^{+\infty} e^{it\psi(x,y)} s(x, y, t) dt, \tag{2.4}$$

where $\psi \in \mathcal{C}^\infty(L \times L)$ *such that* $\psi(x, x) = -i\rho(x)$ *(*ρ *is a defining function for* X*),* $d''_x \psi, d'_y \psi$ *vanish to infinite order along the diagonal and* $\psi(x, y) = -\overline{\psi(y, x)}$.

Concerning the phase function ψ we have the following proposition.

Proposition 2.2.5 *There exists a constant $C > 0$ such that:*

$$\Im\psi(x,y) \geq C\left[\mathrm{dist}_X(x,X) + \mathrm{dist}_X(y,X) + |x-y|^2\right] + O(|x-y|^3)$$

for $x, y \in D$.

Proof. We have that:

$$\frac{1}{i}\left[\psi(x,y) + \psi(y,x) - \psi(x,x) - \psi(y,y)\right] = L_\rho(x-y) + O(|x-y|^3),$$

where $L_\rho = \sum_{j,k} \frac{\partial^2\rho}{\partial\overline{z}_j\partial z_k} z_j\overline{z}_k$ is the Levi Hermitian form. We consider the immaginary part of ψ, that is:

$$\Im\psi(x,y) = \frac{1}{2i}(\psi(x,y) - \overline{\psi(x,y)}) = \frac{1}{2i}(\psi(x,y) + \psi(y,x)).$$

Now on D we have that $\rho < 0$ and $d\rho \neq 0$ on X, then there exist $C' > 0$ such that:

$$\frac{1}{i}\psi(x,x) = -\rho(x) \geq C'\,\mathrm{dist}_X(x,X)$$

for $x \in D$. In a similar way there exist $C'' > 0$ such that:

$$\frac{1}{i}\psi(y,y) = -\rho(y) \geq C''\,\mathrm{dist}_X(y,X)$$

for $y \in D$.

This conclude the proof. □

The phase functions plays an important role in geometric quantization. Geometrically we have that ψ satisfies the Hamilton Jacobi equation and the image of the differential $d\psi$ is a Lagrangian submanifold of the phase space. So ψ can be taken as the phase function of a first order approximate solution of Schrödinger equation. From the point of view of physics, this is the crucial point that connect the classical description of mechanics and the modern vision of the quantum world.

Another important concept in microlocal analysis, is the concept of symbol of a vector field. We denote the space of symbols of order m with S^m following the notation introduced by [H]. Considering now a distribution $\mathcal{I}_{\Psi,s}$, depending by the pase function $\Psi \in C^\infty(U \times \mathbb{R}^N \setminus \{0\})$ and the symbol s, we have that the wave front set $WF(\mathcal{I})$ is contaided in the closed conic subset:

$$\Sigma = \{(x, d_x\Psi(x,\theta)) \in TU^\vee \setminus \{0\}, (x,\theta) \in \text{Esssup}(s), d_\theta\Psi(x,\theta) = 0\} \subseteq TU^\vee \setminus \{0\}.$$

Here Esssup(s) is the smallest conic subset of $U \times \mathbb{R}^N \setminus \{0\} \subset \mathbb{R}^n$ outside of which is of a class $S^{-\infty}$ (for more details see [D]).

In our case we consider as distribution the distributional kernel Π and as nondegenerate complex phase function $t\psi(x,y)$ in an open cone Σ in $T(X \times X)^\vee \times \mathbb{R}_+$. We have that Π is a Fourier integral operator with complex phase (see [MS]) and the corresponding almost complex canonical relation $\mathcal{C} \subset (TX)^\vee \times (TX)^\vee$ parametrized by the phase $t\psi(x,y)$ on $X \times X \times \mathbb{R}_+$. The condition for the parametrization of the Lagrangian submanifold $\mathcal{C}$ is that:

$$\frac{d(t\psi)}{dt} = 0.$$

Let $\alpha = -id'\rho$ and let $\Sigma = \{(x, r\alpha) : r \in \mathbb{R}_+\}$ the symplectic cone generated by the contact form α, the real points of $\mathcal{C}$ consist in the diagonal $\Sigma \times \Sigma$. We say that Π has a Toeplitz structure on the symplectic cone Σ.

Using this microlocal description of Π, Zelditch provided a quick proof in [Z2] of the celebrated Tian-Yau-Zelditch expansion:

Theorem 2.2.6 *Let M a compact complex projective manifold of dimension d_M, and let (A,h) a positive Hermitian holomorphic line bundle. Let g_J a Kähler metric on M and $-2i\omega = \Theta$ a Kähler form. For each $k \in \mathbb{N}$, h induces a Hermitian metric h_k on $L^{\otimes k}$. Let $\{s_i^k\}_{i=0}^{d_k}$ be any orthonormal basis of $H^0(M, L^{\otimes k})$ with $\dim(H^0(M, L^{\otimes k})) = 1 + d_k$. Then there exists a complete asymptotic expansion:*

$$\Pi_k(z,z) = \sum_j \|s_j^{(k)}(z)\|_{h_k}^2 = a_0 k^{d_M} + a_1(z) k^{d_M - 1} + \cdots \tag{2.5}$$

for some a_j smooth with $a_0 = 1$.

2.3 The geometric setting

Before taking a closer look at the geometric setting we need to indroduce some more pieces of notation. As is well-known, if G and T both act on a symplectic manifold M in an Hamiltonian fashion with moment maps Φ_G and Φ_T and these actions commute, then $P = G \times T$ act on M and the moment map is

$$\Phi_P = \Phi_G \oplus \Phi_T : M \to \mathfrak{p}^\vee,$$

with $\mathfrak{p}^\vee = \mathfrak{g}^\vee \oplus \mathfrak{t}^\vee$. We give an explicit expression for $H(X)_{\nu_G,\nu_T}$ as

$$H(X)_{\nu_G,\nu_T} = \left\{ s \in H(X)_{\nu_G} : s(\tilde{\mu}^T_{t^{-1}}(x)) = t^{\nu_T} s(x), \forall x \in X, \forall t \in T^{d_T}, \forall \nu_G \in \widehat{G} \right\},$$

where in general we have $\nu_T = (\nu_1, \cdots, \nu_{d_T}) \in \mathbb{Z}^{d_T}$, $t = (t_1, \cdots, t_{d_T}) \in T^{d_T}$ and $\chi_{\nu_T}(t) = t^{\nu_T} = t_1^{\nu_1} \cdots t_{d_T}^{\nu_{d_T}}$.

The geometrical setting is essentially the same illustrated in [P4] with $P = G \times T$ in place of T.

Now we want to fix our ideas about some important facts.

Before we remember that the matrix $D(m)$ represent the Euclidean product on N_m respect to an orthonormal basis. It determines a positive smooth function $\mathcal{D}$ on M_{0,ν_T} defined above. Note that there is a relation between the $D(m)$ matrix and the $C(m)$ matrix used in Theorem 1.2.6. We have in fact that $D(m) = C(m)^t \cdot C(m)$ and so $\mathcal{D}(m) = \sqrt{\det D(m)} = |\det C(m)|$.

About the product action (referring to [P4]), by the fact that $\mathbf{0}$ is a regular value for Φ_G, the group P acts freely on M_{0,ν_T} and then also on $X_{0,\nu_T} = \pi^{-1}(M_{0,\nu_T})$. We can view this in the same time as a conseguence of the assumption that Φ_P is transverse to $\mathbb{R}_+ \cdot (\mathbf{0}, \nu_T)$. After we have also that X_{0,ν_T} is invariant for $G \times T$. In fact we have that $\mathbb{R}_+ \cdot (\mathbf{0}, \nu_T)$ is invariant for the coadjoint action and, given that Φ_P is equivariant, we conclude.

As we have argued previously, for our purposes we need an additional hypothesis on the submanifolds $\Phi_G^{-1}(\mathbf{0})$ and $\Phi_T^{-1}(\mathbb{R}_+\nu_T)$ that is that they should be mutually transverse. In fact given two maps between manifolds $f_1 : M \to N_1$ and $f_2 : M \to N_2$ with $A_1 \subseteq N_1$ and $A_2 \subseteq N_2$ submanifolds, we have the following result.

Proposition 2.3.1 *$(f_1, f_2) \pitchfork A_1 \times A_2$ if and only if $f_1^{-1}(A_1) \pitchfork f_2^{-1}(A_2)$.*

In our case this means that Φ_P is transversal to $\mathbb{R}_+ \cdot (\mathbf{0}, \nu_T)$ if and only if $\Phi_G^{-1}(\mathbf{0}) \pitchfork \Phi_T^{-1}(\mathbb{R}_+\nu_T)$. So we are in analogy with the assumption of [P4] where Φ_T was transversal to $\mathbb{R}_+\nu_T$.

We recall the definition of symplectic cone $\Sigma \subseteq TX^\vee \setminus \{0\}$ sprayed by the connection form α:

$$\Sigma = \{(x, r\alpha_x) : x \in X, r > 0\}.$$

This cone is important for the microlocal description of Szegö kernel (as in [BS]) and in the theory of Toeplitz operators (see [BG1]). We have that the wave front set of Π is the anti-diagonal:

$$\Sigma^{\#} = \{(x, r\alpha_x, x, -r\alpha_x) : x \in X, r > 0\}.$$

Notice that $\Sigma \cong X \times \mathbb{R}_+$ in a natural manner (for every $r > 0$). Let ω_Σ the restriction to Σ of the symplectic structure on $TX^\vee$. Let r be the cone coordinate on Σ and θ be the circle coordinate on X, locally defined, and pulled-back to Σ. Then $\omega_\Sigma = d\lambda = dr \wedge \alpha + 2r\omega$, with $\lambda = r\alpha$. Let $\widetilde{\xi}_f$ be the contact lift to X of the Hamiltonian vector field ξ_f on $(M, 2\omega)$. Then the cotangent flow restricted to Σ is generated by $(\widetilde{\xi}_f, 0)$. Thus the cotangent flow on $\Sigma \cong X \times \mathbb{R}$ is $\phi^\Sigma_\tau = \phi^X_\tau \times id_\mathbb{R}$. It is follows that if f and g Poisson commute on M, then their flows on M, X and Σ also commute, and conversely.

2.4 Heisenberg local coordinates

Now we recall a basic tool from [SZ], Heisenberg local coordinates.

We start with the definition of preferred coordinate system and preferred frame. After, combining these two ingredients will follow the definition of Heisenberg coordinate chart.

Definition 2.4.1 *Let $m_0 \in M$, a coordinate system $z = (z_1, \cdots, z_{d_M})$ on a neighborhood U of m_0 is called preferred at m_0 if*

1 $\partial_{z_j}\big|_{m_0} \in T^{1,0}M$, *with* $1 \leq j \leq d_M$*;*

2 $\omega(m_0) = \omega_0$*;*

3 $g(m_0) = g_0$.

Where ω_0 is the standard symplectic form and g_0 is the Euclidean metric.

Let now L an Hermitian line bundle on M, we proceed to the next definition.

Definition 2.4.2 *A preferred frame for $L \to M$ at point $m_0 \in M$ is a local frame e_L in a neighborhood of m_0 such that:*

1 $\|e_L\|_{m_0} = 1$*;*

2 $\nabla e_L\big|_{m_0} = 0$*;*

3 $\nabla^2 e_L\big|_{m_0} = -(g + i\omega) \otimes e_L\big|_{m_0} \in TM^\vee \otimes TM^\vee \otimes L$.

Remark 2.4.3 *In the previous definition* 2) *and* 3) *are independent of the choice of connection on $TM^\vee$ used to define $\nabla : \mathcal{C}^\infty(M, L \otimes TM^\vee) \to \mathcal{C}^\infty(M, L \otimes TM^\vee \otimes TM^\vee)$.*

Remark 2.4.4 *Property* 3) *is a necessary condition for obtaining universal scaling asymptotics.*

Remark 2.4.5 *If* e_L *is a preferred frame at* m_0 *and if* $z = (z_1, \cdots, z_{d_M})$ *are preferred coordinates at* m_0*, then we compute the Hessian:*

$$\left(\nabla^2 \|e_L\|_h\right)_{m_0} = \Re\left(\nabla^2 e_L, e_L\right)_{m_0} = -g_0,$$

thus if the preferred coordinates are centered at m_0 *(i.e.* $m_0 = 0$*), we have:*

$$\|e_L\|_h = 1 - \frac{1}{2}|z|^2 + O(|z|^3).$$

Our next step is the definition of the Heisenberg chart on the S^1-bundle.

Definition 2.4.6 *A Heisenberg coordinate chart at a point* $x_0 \in X$ *is a coordinate chart* $\rho : U \approx V$ *with* $0 \in U \subset \mathbb{C}^{d_M} \times \mathbb{R}$ *and* $\rho(0) = x_0 \in V \subset X$ *of the form:*

$$\rho(z_1, \cdots, z_{d_M}, \theta) = e^{i\theta} a(z)^{-\frac{1}{2}} e_L^{\vee}(z), \tag{2.6}$$

where e_L *is a preferred local frame for* $L \to M$ *at* $m_0 = \pi(x_0)$*, and* $(z_1, \cdots, z_{d_M})$ *a preferred coordinates centered at* m_0*.*

Remark 2.4.7 *Suppose that* $s_k(z) = f e_L^{\otimes k}(z)$ *is a local section of* $L^{\otimes k}$*. Then by the previous definition and the expression of the lifted section* $\hat{s}_k(\lambda_x) = (\lambda_x^{\otimes k}, s_k(z))$*, with* $\lambda_x \in X$*, we have that:*

$$\hat{s}_k(z, \theta) = f(z) a(z)^{-\frac{1}{2}} e^{ik\theta}. \tag{2.7}$$

We denote briefly with $\gamma_x(\theta, \mathrm{v}) = x + (\theta, \mathrm{v})$ the system of Heisenberg local coordinates on X centered at x. We have the following facts:

1 The standard circle action $r : S^1 \times X \to X$ is expressed by translation in the following way:

$$r_{e^{i\vartheta}}(x + (\theta, \mathrm{v})) = x + (\vartheta + \theta, \mathrm{v}).$$

2 If $m \in M$ and $m = \pi(x)$, we set

$$m + \mathrm{v} = \pi(x + (0, \mathrm{v})),$$

that is a local coordinate chart centered at m, inducing a unitary isomorphism $\mathbb{C}^{d_M} \cong \mathbb{R}^{2d_M} \cong T_m M$.

3 γ_x induces at x an isomorphism $\mathbb{R} \oplus \mathbb{R}^{2d_M} \cong T_x X$ compatible with the decomposition in vertical and horizontal space.

4 Heisenberg local coordinates can be locally and smoothly deformed with the base point x. In other words, for any $x \in X$ exist an open neighborhood $x \in U \subseteq X$ and a smooth map

$$\Gamma : U \times (-\pi, +\pi) \times B_{2d_M}(\mathbf{0}, \delta) \to X,$$

such that $\gamma_y(\theta, \mathrm{v}) = \Gamma(y, \theta, \mathrm{v})$ is a system of Heisenberg local coordinates centered at y, for each $y \in U$.

Let us now locally express the action of P in terms of Heisenberg local coordinates. To fix ideas, let us first consider the case of a 1-dimensional torus and trivial stabilizer.

We want to find the scaling Heisenberg coordinates of:

$$\widetilde{\mu}^P_{\frac{p^{-1}}{\sqrt{k}}}(x_{1k}) = \tilde{\mu}^G_{e^{-\frac{\varsigma}{\sqrt{k}}}} \circ \tilde{\mu}^T_{-\frac{\vartheta}{\sqrt{k}}}(x_{1k}),$$

where $x_{1k} = x + \frac{(\theta_1, \mathrm{v}_1)}{\sqrt{k}}$, $p \in P$ and ϑ, ς are the linear coordinates respectively for G and T. Assume $x \in X$, $\Phi_G \circ \pi(x) = \mathbf{0}$, and fix a system of HLC centered at x. Let $\xi_M(m), \varsigma_M(m)$ the valutations of $\xi \in \mathfrak{t}, \varsigma \in \mathfrak{g}$ and assume that the stabilizer F_x of x in P is trivial. We have the following Lemma.

Lemma 2.4.8 *Under the previous assumptions there exist $\mathcal{C}^\infty$ functions $\widetilde{B}_3, \widetilde{B}_2 : \mathbb{R} \times \mathbb{C}^{d_M} \times \mathbb{R}^{d_G} \to \mathbb{C}^{d_M}$, vanishing at the origin to third and second order, respectively, such that as $k \to +\infty$ the Heisenberg local coordinates of:*

$$\tilde{\mu}^G_{e^{-\frac{\varsigma}{\sqrt{k}}}} \circ \tilde{\mu}^T_{-\frac{\vartheta}{\sqrt{k}}}(x_{1k})$$

are given by

$$\left(\frac{1}{\sqrt{k}} (\vartheta \Phi_T(m) + \theta_1) + \frac{1}{k} \omega_m (\vartheta \xi_M(m), \mathrm{v}_1) + \frac{1}{k} \omega_m (\varsigma_M(m), \mathrm{v}_1) + \widetilde{B}_3 \left(\frac{\vartheta}{\sqrt{k}}, \frac{\mathrm{v}}{\sqrt{k}}, \frac{\varsigma}{\sqrt{k}} \right), \right.$$
$$\left. , \frac{1}{\sqrt{k}} (\mathrm{v}_1 - \vartheta \xi_M(m) - \varsigma_M(m)) + \widetilde{B}_2 \left(\frac{\vartheta}{\sqrt{k}}, \frac{\mathrm{v}}{\sqrt{k}}, \frac{\varsigma}{\sqrt{k}} \right) \right).$$

Proof. For corollary 2.2 of [P4] we have:

$$\begin{aligned} &\tilde{\mu}^G_{e^{-\frac{\varsigma}{\sqrt{k}}}} \circ \tilde{\mu}^T_{-\frac{\vartheta}{\sqrt{k}}}(x_{1k}) \\ &= \tilde{\mu}^G_{e^{-\frac{\varsigma}{\sqrt{k}}}} \left(x + \left(\frac{1}{\sqrt{k}} (\vartheta \Phi_T(m) + \theta_1) + \frac{1}{k} \omega_m (\vartheta \xi_M(m), \mathrm{v}_1) + B_3 \left(\frac{\vartheta}{\sqrt{k}}, \frac{\mathrm{v}}{\sqrt{k}} \right), \right. \right. \\ &\qquad \left. \left. , \frac{1}{\sqrt{k}} (\mathrm{v}_1 - \vartheta \xi_M(m)) + B_2 \left(\frac{\vartheta}{\sqrt{k}}, \frac{\mathrm{v}}{\sqrt{k}} \right) \right) \right), \end{aligned}$$

where B_j denotes a smooth function from $\mathbb{R} \times \mathbb{C}^{d_M} \to \mathbb{C}^{d_M}$ vanishing to j-th order at the origin.

Applying lemma 4.3 of [P1] we have:

$$\left(\frac{1}{\sqrt{k}}\left(\vartheta\Phi_T(m)+\theta_1\right)+\frac{1}{k}\omega_m\left(\vartheta\xi_M(m)+\varsigma_M(m),\mathrm{v}_1\right)-\frac{1}{k}\omega_m\left(\varsigma_M(m),\vartheta\xi_M(m)\right)+Q\left(\frac{\mathrm{v}}{\sqrt{k}},\frac{\varsigma}{\sqrt{k}}\right)+\right.$$
$$\left.+B_3\left(\frac{\vartheta}{\sqrt{k}},\frac{\mathrm{v}}{\sqrt{k}}\right),\frac{1}{\sqrt{k}}\left(\mathrm{v}_1-\vartheta\xi_M(m)-\varsigma_M(m)\right)+T\left(\frac{\mathrm{v}}{\sqrt{k}},\frac{\varsigma}{\sqrt{k}}\right)+B_2\left(\frac{\vartheta}{\sqrt{k}},\frac{\mathrm{v}}{\sqrt{k}}\right)\right),$$

where Q, T are smooth functions from $\mathbb{C}^{d_M} \times \mathbb{R}^{d_G} \to \mathbb{C}^{d_M}$ vanishing at the origin to third and second order, respectively. Set $\widetilde{B}_2 = T + B_2$ and $\widetilde{B}_3 = Q + B_3$ remains to prove that $\omega_m\left(\varsigma_M(m), \vartheta\xi_M(m)\right) = 0$. To prove that $\omega_m\left(\varsigma_M(m), \xi_M(m)\right) = 0$, we can observe that G acts on $\mathfrak{g}$ via the adjoint representation and on vector fields on M. It follows that the mapping $\xi \to \xi_M$ is a G morphism. Since the map is uniquely determined by the relation between ξ and ξ_M, it follows that the map Φ_G is a G morphism, that is,

$$\Phi_G(gm) = g\Phi_G(m)$$

for all $g \in G$ and $m \in M$. The action used in the formula before is coadjoint representation of G on $\mathfrak{g}^\vee$

$$\langle g\Phi_G(m), \xi\rangle = \langle \Phi_G(m), \mathrm{ad}_{g^{-1}}\xi\rangle$$

if we take $g = e^{t\varsigma}$ and differentiate the above equation at $t = 0$ we have:

$$\left.\frac{d\langle e^{t\varsigma}\Phi_G(m), \xi\rangle}{dt}\right|_{t=0} = \left.\frac{d\langle \Phi_G(m), e^{-t\varsigma}\xi e^{t\varsigma}\rangle}{dt}\right|_{t=0}$$

so

$$\langle d_m\Phi_G(\varsigma_M(m)), \xi\rangle = -\langle \Phi_G(m), [\varsigma, \xi]\rangle,$$

then

$$\omega_m(\varsigma_M(m), \xi_M(m)) = -\langle \Phi_G(m), [\varsigma, \xi]\rangle = 0$$

by assumption. This complete the proof. □

Now we present a general version of the preceding lemma assuming non trivial F_x and that T is a d_T-dimensional torus. In this case have to find the scaling Heisenberg local coordinates of:

$$\widetilde{\mu}^P_{\frac{(pp_j)^{-1}}{\sqrt{k}}}(x_{1k}).$$

We remember that $\vartheta = (\vartheta_1, \cdots, \vartheta_{d_T})$ with $-\pi < \vartheta_i < \pi$ and we have that $\xi = \frac{\partial}{\partial \vartheta}\big|_0$. In addiction $\xi = (\xi_1, \cdots, \xi_{d_T})$ and $\Phi_T(m) = (\Phi_1(m), \cdots, \Phi_{d_T}(m))$ where:

$$\Phi_l = \langle \Phi, \xi_l \rangle,$$

and

$$\vartheta \cdot \xi = \sum_{l=1}^{d_T} \vartheta_l \xi_l.$$

We denote with v_1^j the monodromy representation of F_x defined as $\mathrm{v}_1^j = d_m \tilde{\mu}^P_{p_j^{-1}}(\mathrm{v}_1)$. Now under the assumption of Lemma 2.4.8 and the previous notations we have a more general Lemma for the Heisenberg coordinates with scaling. An adaptation of the previous argument then shows the following:

Lemma 2.4.9 *Let $P = G \times T$ with T a d_T-dimensional torus, under the assumption of the previous Lemma and that the stabilizer is $F_x = \{p_j : j = 1, \cdots, N_x\}$. Then there exist $\mathcal{C}^\infty$ functions $\widetilde{B}_3, \widetilde{B}_2 : \mathbb{R}^{d_T} \times \mathbb{C}^{d_M} \times \mathbb{R}^{d_G} \to \mathbb{C}^{d_M}$, vanishing at the origin to third and second order, respectively, such that for $k \to +\infty$ the Heisenberg local coordinates of:*

$$\tilde{\mu}^G_{e^{-\frac{\varsigma}{\sqrt{k}}}} \circ \tilde{\mu}^T_{-\frac{\vartheta}{\sqrt{k}}} \left(\tilde{\mu}^P_{p_j^{-1}}(x_{1k}) \right)$$

are given by

$$\left(\frac{1}{\sqrt{k}} (\vartheta \Phi_T(m) + \theta_1) + \frac{1}{k} \omega_m \left(\vartheta \xi_M(m), \mathrm{v}_1^j \right) + \frac{1}{k} \omega_m \left(\varsigma_M(m), \mathrm{v}_1^j \right) + \widetilde{B}_3 \left(\frac{\vartheta}{\sqrt{k}}, \frac{\mathrm{v}}{\sqrt{k}}, \frac{\varsigma}{\sqrt{k}} \right), \right.$$
$$\left. , \frac{1}{\sqrt{k}} \left(\mathrm{v}_1^j - \vartheta \xi_M(m) - \varsigma_M(m) \right) + \widetilde{B}_2 \left(\frac{\vartheta}{\sqrt{k}}, \frac{\mathrm{v}}{\sqrt{k}}, \frac{\varsigma}{\sqrt{k}} \right) \right).$$

Chapter 3

Proofs

3.1 Proof of the main Theorem

Proof.

Proof of 1). We consider $(\rho_{\nu_G}, V_{\nu_G})$ an unitary irreducible representation of G and we define $\rho_{\nu_G,k\nu_T} : G \times T \to GL(V_{\nu_G})$ as $\rho_{\nu_G,k\nu_T}(g,t) = t^{k\nu_T}\rho_{\nu_G}(g)$. We have that $(\rho_{\nu_G,k\nu_T}, V_{\nu_G})$ is an unitary irreducible representation of $G \times T$ with character $\chi_{\nu_G,k\nu_T}(g,t) = \mathrm{Tr}(\rho_{\nu_G,k\nu_T}(g,t)) = t^{k\nu_T}\,\mathrm{Tr}(\rho_{\nu_G}(g)) = t^{k\nu_T}\chi_{\nu_G}(g)$.

Assuming that $\mathbf{0} \notin \Phi_T(M)$ we have that $H(X)_{k\nu_T}$ is finite dimensional, then $H(X)_{\nu_G,k\nu_T} \subseteq H(X)_{k\nu_T}$ and $\widetilde{\Pi}_{\nu_G,k\nu_T} \in \mathcal{C}^\infty(X \times X)$. We want study the asymptotic behavior of $\widetilde{\Pi}_{\nu_G,k\nu_T}$ with $k \to +\infty$. Since $\widetilde{\Pi}_{\nu_G,k\nu_T}$ is the the composition of $\Pi : L^2(X) \to H(X)$ and the orthogonal projector of $H(X)$ onto:

$$\widetilde{\Pi}_{\nu_G,k\nu_T}(x,y) = \frac{d_{\nu_G}}{(2\pi)^{d_T}} \int_G \int_T \chi_{\nu_G}(g^{-1}) t^{-k\nu_T} \Pi\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{t^{-1}}(x), y\right) dt dg, \tag{3.1}$$

where $d_{\nu_G} = \dim(V_{\nu_G})$ and dg, dt are the associated measure for G and T such that $\int_G dg = 1$ and $\int_T dt = 1$. We start considering the diagonal case, so we have:

$$\begin{aligned}\widetilde{\Pi}_{\nu_G,k\nu_T}(x,x) &= d_{\nu_G} \int_G \int_T \chi_{\nu_G}(g^{-1}) t^{-k} \Pi\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{t^{-1}}(x), x\right) dt dg \\ &= \frac{d_{\nu_G}}{2\pi} \int_G \int_{(-\pi,+\pi)^{d_T}} \chi_{\nu_G}(g^{-1}) e^{-ik\nu_T\cdot\vartheta} \Pi\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), x\right) d\vartheta dg,\end{aligned} \tag{3.2}$$

where $\vartheta \in (-\pi,\pi)^{d_T}$. For the moment suppose $x \in X$ generic and fixed, and denote $F_x \subseteq G \times T$ the stabilizer of x. For $\varepsilon > 0$ we set

$$A = \{(g,t) \in G \times T : \mathrm{dist}_{G\times T}((g,t), F_x) < 2\varepsilon\}$$

and

$$B = \{(g,t) \in G \times T : \mathrm{dist}_{G\times T}((g,t), F_x) > \varepsilon\}$$

so we have $G \times T = A \cup B$ and we can consider a partition of the unity $\gamma_1 + \gamma_2 = 1$ associated to the covering $\{A, B\}$. We observe that the function:

$$(g,t) \mapsto \gamma_2(g,t)\chi_{\nu_G}(g^{-1})\Pi\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), x\right) \tag{3.3}$$

is $\mathcal{C}^\infty$ because the singular support of Π is included in the diagonal of $X \times X$.

Then

$$t \mapsto \int_G \gamma_2(g,t)\chi_{\nu_G}(g^{-1})\Pi\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), x\right) dg \tag{3.4}$$

is infinitely smooth and the Fourier transform is rapidly decreasing. Thus the contribution coming from B is rapidly decreasing and we can multiply the integrand by γ_1. So we can only consider:

$$\widetilde{\Pi}_{\nu_G, k\nu_T}(x,x) \sim \frac{d_{\nu_G}}{2\pi} \int_G \int_{(-\pi,\pi)^{d_T}} \gamma_1(g,\vartheta)\chi_{\nu_G}(g^{-1})e^{-ik\nu_T\cdot\vartheta}\Pi\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), x\right) d\vartheta dg. \tag{3.5}$$

Now if $\gamma_2(g,\vartheta) \neq 0$ then $\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), x$ are near and we can represent Π as Fourier integral operator as in [BS]:

$$\Pi(y,y') = \int_0^{+\infty} e^{it\psi(y,y')} s(y,y',t)dt, \tag{3.6}$$

where $\Im(\psi) \geq 0$ and s is a semiclassical symbol admitting an asymptotic expansion $s(y,y',t) = \sum_{j=0}^{+\infty} t^{n-j} s_j(y,y')$. Inserting (3.6) in (3.5) we obtain:

$$\widetilde{\Pi}_{\nu_G, k\nu_T}(x,x) \sim \frac{d_{\nu_G}}{2\pi} \int_G \int_{(-\pi,+\pi)^{d_T}} \int_0^{+\infty} \gamma_1(g,\vartheta)\chi_{\nu_G}(g^{-1})e^{i[t\psi(\tilde{\mu}^G_{g^{-1}}\circ\tilde{\mu}^T_{-\vartheta}(x),x) - k\nu_T\cdot\vartheta]} \cdot$$
$$\cdot\, s\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), x, t\right) dt d\vartheta dg, \tag{3.7}$$

and performing the change of variables $t \to kt$, we get:

$$\widetilde{\Pi}_{\nu_G, k\nu_T}(x,x) \sim \frac{d_{\nu_G}}{2\pi} k \cdot \int_G \int_{(-\pi,+\pi)^{d_T}} \int_0^{+\infty} \gamma_1(g,\vartheta)\chi_{\nu_G}(g^{-1})e^{ik\Psi(t,g,\vartheta,x)} \cdot \tag{3.8}$$
$$\cdot\, s\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), x, kt\right) dt d\vartheta dg,$$

where we have set $\Psi(t,g,\vartheta,x) = t\psi(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), x) - \nu_T \cdot \vartheta$. We shall now use integration by parts in ϑ to prove that only a rapidly decreasing contribution to the asymptotic is lost, if the integrand in (3.8) is multiplied by a suitable cut-off function. In local coordinates we have $\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x) = x + O(\varepsilon)$ with $\varepsilon > 0$ very small, because $\left(g, e^{i\vartheta}\right) \in U$ with U a small neighborhood of F_x. Thus we have that:

$$d_{(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), x)}\psi = d_{(x,x)}\psi + O(\varepsilon) = (\alpha_x, -\alpha_x) + O(\varepsilon),$$

with $\partial_\vartheta \Psi = t\Phi_T(m) - \nu_T + O(\varepsilon)$. Therefore, since $\Phi_T(m) \neq \mathbf{0}$ and $\nu_T \neq \mathbf{0}$ we have for $t \gg 0$ that $\|\partial_\vartheta \Psi\| \geq Ct$, for some $C > 0$. In a similar way for $0 < t \ll 1$ we have $\|\partial_\vartheta \Psi\| \geq C_1 > 0$, for some $C_1 > 0$. Therefore by integration by parts in $d\vartheta$, we have that the asymptotics for $k \to +\infty$ is unchanged. We multiply the integrand by $\rho(t)$, where $\rho \in \mathcal{C}^\infty_0\left(\frac{1}{2D}, 2D\right)$ and $\rho \equiv 1$ on $\left(\frac{1}{D}, D\right)$, so that the integral in dt is now compactly supported. We shall now use integration by parts in dt to show that only a rapidly decreasing contribution is lost, if the integration in (g,ϑ) is restricted to a tubular neighborhood of F_x of radius $O(k^{\delta-\frac{1}{2}})$. We have that $\partial_t \Psi = \psi(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), x)$. If $\mathrm{dist}\left(\left(g, e^{i\vartheta}\right), F_x\right) \geq Ck^{\delta-\frac{1}{2}}$, then

$$\mathrm{dist}\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), x\right) \geq Ck^{\delta-\frac{1}{2}}$$

and so:

$$\left|\psi(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), x)\right| \geq \Im\psi(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), x) \geq C_2 k^{2\delta-1} \tag{3.9}$$

(see Corollary 2.3 of [BS]). Introducing the operator

$$L_t = \left[\psi(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), x)\right]^{-1} \partial_t,$$

we have that:

$$e^{ik\Psi} = -\frac{i}{k} L_t\left(e^{ik\Psi}\right).$$

We can now mimick the standard proof of the Stationary Phase Lemma: iteratively integrating by parts, we obtain at each step in view of (3.9) a factor of order $O\left(k^{-2\delta}\right)$, and then after N steps a factor of order $O\left(k^{-2N\delta}\right)$. This proves that the contribution to the asymptotics coming from the locus where $\mathrm{dist}\left(\left(g, e^{i\vartheta}\right), F_x\right) \geq Ck^{\delta-\frac{1}{2}}$ is rapidly decreasing. We can now prove that (3.8) is rapidly decreasing in k for $\mathrm{dist}_X(x, X_{0,\nu_T}) \geq Ck^{\delta-\frac{1}{2}}$. Now we consider a bump function $\rho_1 : P \to \mathbb{R}$ supported in a small neighborhood of F_x and $\equiv 1$ near to F_x. The function ρ_1 is defined as $\rho_1 = \rho_1(f,\xi)$ with $f \in F_x$ and ξ the normal coordinate to F_x. We can multiply the integrand of (3.8) by $\rho_1\left(f, k^{\frac{1}{2}-\delta}\xi\right)$ losing only an $O(k^{-\infty})$. Then if $\rho_1(g,t)\neq 0$ we have $\tilde{\mu}^G_{g^{-1}}\circ\tilde{\mu}^T_{-\vartheta}(x) = x + O\left(k^{\delta-\frac{1}{2}}\right)$. Therefore:

$$d_{\left(\tilde{\mu}^G_{g^{-1}}\circ\tilde{\mu}^T_{-\vartheta}(x),x\right)}\psi = d_{(x,x)}\psi + O\left(k^{\delta-\frac{1}{2}}\right) = (\alpha_x, -\alpha_x) + O(k^{\delta-\frac{1}{2}}),$$

and $\partial_{(\varsigma,\vartheta)}\psi = t\Phi_P(m) - \nu_T + O\left(k^{\delta-\frac{1}{2}}\right)$. Here (ς,ϑ) are local coordinates on P induced by the exponential map $\exp_P$. Then if $\mathrm{dist}_X(x, X_{0,\nu_T}) \geq C_3 k^{\delta-\frac{1}{2}}$ we have that:

$$\|\partial_{(\varsigma,\vartheta)}\psi\| \geq C_4 k^{\delta-\frac{1}{2}}.$$

Thus we find a differential operator $L_{\varsigma,\vartheta}$ with $|L_{\varsigma,\vartheta}| \geq C_5 k^{\delta-\frac{1}{2}}$ where $\mathrm{dist}_X(x, X_{0,\nu_T}) \geq O\left(k^{\delta-\frac{1}{2}}\right)$ such that $L_{\varsigma,\vartheta}\left(e^{ik\Psi}\right) = ike^{ik\Psi}$. Iterating the integration by parts, in view of the scaling factor we have at each step a factor $O\left(k^{-2\delta}\right)$. This proves that $\widetilde{\Pi}_{\nu_G,k\nu_T}(x,x) = O(k^{-\infty})$ for $\mathrm{dist}_X(x, X_{0,\nu_T}) \geq Ck^{\delta-\frac{1}{2}}$. Let us consider $(x,y)\in X\times X$ with

$$\max\left\{\mathrm{dist}_X(x, X_{0,\nu_T}), \mathrm{dist}_X(y, X_{0,\nu_T})\right\} \geq Ck^{\delta-\frac{1}{2}}$$

for every δ fixed and using the Cauchy-Schwarz inequality we have:

$$\left|\widetilde{\Pi}_{\nu_G,k\nu_T}(x,y)\right| \leq \sqrt{\widetilde{\Pi}_{\nu_G,k\nu_T}(x,x)}\cdot\sqrt{\widetilde{\Pi}_{\nu_G,k\nu_T}(y,y)}, \tag{3.10}$$

so $\widetilde{\Pi}_{\nu_G,k\nu_T}(x,y) = O(k^{-\infty})$. This complete the proof of 1).

Let us now consider the proof of 2). Now setting $x_{jk} = x + \frac{(\theta_j,\mathrm{v}_j)}{\sqrt{k}}$ for $j=1,2$, using FIO representation as before and changing variables $t\to kt$, we get:

$$\widetilde{\Pi}_{\nu_G,k\nu_T}(x_{1k},x_{2k}) = \frac{kd_{\nu_G}}{(2\pi)^{d_T}}\int_W \chi_{\nu_G}(g^{-1})e^{ik\Psi^{(1)}(t,\vartheta,x)}\cdot \tag{3.11}$$
$$\cdot\, s\left(\tilde{\mu}^G_{g^{-1}}\circ\tilde{\mu}^T_{-\vartheta}(x_{1k}), x_{2k}, kt\right)dV_W(w),$$

where

$$W = G\times(-\pi,\pi)^{d_T}\times(0,+\infty)\qquad dV_W(w) = dg\,d\vartheta\,dt, \tag{3.12}$$

and

$$\Psi^{(1)}(t,\vartheta,x) = t\psi\left(\tilde{\mu}^G_{g^{-1}}\circ\tilde{\mu}^T_{-\vartheta}(x_{1k}), x_{2k}\right) - \nu_T\cdot\vartheta. \tag{3.13}$$

Here $t=(t_1,\cdots,t_{d_T})=\left(e^{i\vartheta_1},\cdots,e^{i\vartheta_{d_T}}\right)=e^{i\vartheta}$. Let $F_m\subseteq P$, $F_m=\{p_j\}=\{(g_j,t_j)\}$ the finite stabilizer of $x\in X_{0,\nu_T}$. We introduce a bump function $\rho=\sum_{j=1}^{N_x}\rho_j$ with support of ρ_j in a neighborhood of $p_j=(g_j,t_j)$. As consequence we have:

$$\widetilde{\Pi}_{\nu_G,k\nu_T}\left(x_{1k},x_{2k}\right)\sim\sum_j \Pi_{\nu_G,k\nu_T}\left(x_{1k},x_{2k}\right)^{(j)}, \tag{3.14}$$

where each addend of (3.14) is given by (3.11) multiplied by ρ_j. In the support of each p_j we write $g=g_j\exp_G\frac{\gamma}{\sqrt{k}}$ and $t=t_je^{\frac{i\vartheta}{\sqrt{k}}}$, where with $\exp_G$ we denote the exponential map from $\mathfrak{g}\to G$ and γ,ϑ are coordinates on $\mathfrak{g}\cong\mathbb{R}^{d_G}$,$\mathfrak{t}\cong\mathbb{R}^{d_T}$ associated with the respective orthonormal basis. Omitting ρ_j in the integrand we have:

$$\widetilde{\Pi}_{\nu_G,k\nu_T}\left(x_{1k},x_{2k}\right)\sim\frac{k^{1-\frac{d_P}{2}}d_{\nu_G}}{(2\pi)^{d_T}}\int_{W'}\chi_{\nu_G}\left(g_j^{-1}\exp_G\left(-\frac{\gamma}{\sqrt{k}}\right)\right)t_j^{-1}e^{ik\Psi^{(2)}(t,\vartheta,x)}. \tag{3.15}$$

$$\cdot\, s\left(\tilde{\mu}^G_{-\frac{\gamma}{\sqrt{k}}}\circ\tilde{\mu}^T_{-\frac{\vartheta}{\sqrt{k}}}\left(x_{1k}\right),x_{2k},kt\right)dV_{W'}(w),$$

where

$$W'=\mathbb{R}^{d_G}\times\mathbb{R}^{d_T}\times(0,+\infty)\qquad dV_{W'}(w)=d\gamma d\vartheta dt \tag{3.16}$$

and

$$\Psi^{(2)}(t,\vartheta,x)=t\psi\left(\tilde{\mu}^G_{-\frac{\gamma}{\sqrt{k}}}\circ\tilde{\mu}^T_{-\frac{\vartheta}{\sqrt{k}}}\circ\tilde{\mu}^G_{g_j^{-1}}\circ\tilde{\mu}^T_{t_j^{-1}}\left(x_{1k}\right),x_{2k}\right)-\nu_T\cdot\frac{\vartheta}{\sqrt{k}}. \tag{3.17}$$

We write $x^j_{1k}=\tilde{\mu}^G_{g_j^{-1}}\circ\tilde{\mu}^T_{t_j^{-1}}\left(x_{1k}\right)=x+\frac{1}{\sqrt{k}}(\theta_1,\mathrm{v}_1^j)$, for a particular choice of σ adapted section in the definition of HLC. We assume that the orthonormal basis of $\mathfrak{t}$ is taken as $(w_1,\cdots,w_{d_T})$ with $(w_1,\cdots,w_{d_T-1})$ an orthonormal basis for $\mathrm{Ker}(\Phi_T(m))$ and $\langle\Phi_T(m),w_{d_T}\rangle=\|\Phi_T(m)\|$. So we have that, if $(v_1,\cdots,v_{d_G})$ is the orthonormal basis for $\mathfrak{g}$, an orthonormal basis for $\mathfrak{p}=\mathrm{Lie}(P)$ is of the form:

$$(v_1,\cdots,v_{d_G},w_1,\cdots,w_{d_T}=\eta).$$

We call $(a_1,\cdots,a_{d_P}=b)$ the corresponding linear coordinates on $\mathfrak{p}$ such that $a=(a_1,\cdots,a_{d_P}=b)\in\mathbb{R}^{d_P-1}\cong\mathrm{Ker}(\Phi_P(m))$ and $a_M(m)\in\mathbb{R}^{2d_M}\cong T_mM$ is his injective valutation. Considering (3.17), we write $(\gamma,\vartheta)\in\mathfrak{p}\cong\mathbb{R}^{d_P}\cong\mathbb{R}^{d_G}\times\mathbb{R}^{d_T}$ as

$$(\gamma,\vartheta)=(\gamma,\vartheta')+\vartheta\eta=a+b\eta, \tag{3.18}$$

remember that $\nu_T = \lambda \cdot \Phi_T(m) \Rightarrow (\mathbf{0}, \nu_T) = \lambda \Phi_P(m)$, then we have:

$$\nu_T \cdot \vartheta = (\mathbf{0}, \nu_T) \cdot (a + b\eta) = b\lambda \|\Phi_T(m)\|.$$

Here $\lambda = \lambda_{\nu_T}$ is such that $\nu_T = \lambda_{\nu_T} \cdot \Phi_T(m)$. Thus we have:

$$\Psi^{(2)}(t,b,x) = t\psi\left(\tilde{\mu}^P_{-\frac{a+b\eta}{\sqrt{k}}}\left(x^j_{1k}\right), x_{2k}\right) - \lambda_{\nu_T}\frac{\|\Phi_T(m)\|b}{\sqrt{k}}. \tag{3.19}$$

Now let $p = a + b\eta$ we have that $a \in \mathrm{Ker}(\Phi_P(m))$ and $\eta \in \mathrm{Ker}(\Phi_P(m))^\perp$, with $\|\eta\| = 1$, $\langle \Phi_P(m), \eta\rangle = \langle \Phi_T(m), \eta\rangle = \|\Phi_T(m)\|$. We get:

$$\tilde{\mu}^P_{-\frac{p}{\sqrt{k}}}\left(x^j_{1k}\right) = \tilde{\mu}^P_{-\frac{p}{\sqrt{k}}}\left(x + \frac{1}{\sqrt{k}}(\theta_1, \mathrm{v}^j_1)\right) =$$

$$= x + \Bigg(\frac{\theta_1 + \langle \Phi_T(m), p\rangle}{\sqrt{k}} + \frac{1}{k}\omega_m(a_M(m) + b\eta_M(m), \mathrm{v}^j_1) + \widetilde{B}_3\left(\frac{a_M(m)}{\sqrt{k}}, \frac{\mathrm{v}}{\sqrt{k}}, \frac{b}{\sqrt{k}}\right),$$
$$, \frac{\theta_1}{\sqrt{k}}(\mathrm{v}^j_1 - a_M(m) - b\eta_M) + \widetilde{B}_2\left(\frac{a_M(m)}{\sqrt{k}}, \frac{\mathrm{v}}{\sqrt{k}}, \frac{b}{\sqrt{k}}\right)\Bigg)$$

$$= x + (\mathcal{A}_{j,k}(\vartheta, v^j_1, v_2), \mathcal{B}_{j,k}(\vartheta, v^j_1, v_2)), \tag{3.20}$$

with $\widetilde{B}_2, \widetilde{B}_3$ that vanish at the origin to third and second order. We obtain that:

$$t\psi\left(\tilde{\mu}^P_{-\frac{p}{\sqrt{k}}}\left(x^j_{1k}\right), x_{2k}\right) - \frac{\vartheta}{\sqrt{k}}$$

$$= it\left[1 - e^{i\left(\mathcal{A}_{j,k}(\vartheta, v^j_1, v_2) - \frac{\theta_2}{\sqrt{k}}\right)}\right] - \frac{it}{k}\psi_2\left(\mathrm{v}^j_1 - a_M(m) - b\eta_M(m), \mathrm{v}_2\right) e^{i\left(\mathcal{A}_{j,k}(\vartheta, v^j_1, v_2) - \frac{\theta_2}{\sqrt{k}}\right)} -$$
$$- \frac{\vartheta}{\sqrt{k}} + itR^\psi_3\left(\frac{1}{\sqrt{k}}\left(\mathrm{v}^j_1 - \vartheta\xi_M(m) - \varsigma_M(m)\right), \frac{\mathrm{v}_2}{\sqrt{k}}\right) e^{i\left(\mathcal{A}_{j,k}(\vartheta, v^j_1, v_2) - \frac{\theta_2}{\sqrt{k}}\right)}, \tag{3.21}$$

where R^ψ_3 vanishes to third order at the origin and

$$\psi_2(r,s) = -i\omega_m(r,s) - \frac{1}{2}\|r - s\|^2 \quad (r, s \in \mathbb{C}^n).$$

Now

$$i\left(\mathcal{A}_{j,k}(\vartheta, v^j_1, v_2) - \frac{\theta_2}{\sqrt{k}}\right) =$$
$$\frac{i}{\sqrt{k}}(\theta_1 - \theta_2 + b\|\Phi_T\|) + \frac{i}{k}\omega_m(a_M + b\eta_M, \mathrm{v}^j_1) + \widetilde{B}'_3\left(\frac{a_M(m)}{\sqrt{k}}, \frac{\mathrm{v}}{\sqrt{k}}, \frac{b}{\sqrt{k}}\right),$$

then

$$
\begin{aligned}
&1-e^{i\left(\mathcal{A}_{j,k}(\vartheta,v_1^j,v_2)-\frac{\theta_2}{\sqrt{k}}\right)}= \\
&=1-\left\{1+\frac{i}{\sqrt{k}}(\theta_1-\theta_2+b\|\Phi_T\|)+\frac{i}{k}\omega_m\left(a_M(m)+b\eta_M(m),\mathrm{v}_1^j\right)-\right. \\
&\qquad\left.-\frac{1}{2k}(\theta_1-\theta_2+b\|\Phi_T\|)^2+\widetilde{B}_3''\left(\frac{a_M(m)}{\sqrt{k}},\frac{\mathrm{v}}{\sqrt{k}},\frac{b}{\sqrt{k}}\right)\right\} \\
&\quad=-\frac{i}{\sqrt{k}}(\theta_1-\theta_2+b\|\Phi_T\|)-\frac{i}{k}\omega_m\left(a_M(m)+b\eta_M(m),\mathrm{v}_1^j\right)+ \\
&\qquad\qquad+\frac{1}{2k}(\theta_1-\theta_2+b\|\Phi_T\|)^2+\widetilde{B}_3'''\left(\frac{a_M(m)}{\sqrt{k}},\frac{\mathrm{v}}{\sqrt{k}},\frac{b}{\sqrt{k}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&it\left[1-e^{i\left(\mathcal{A}_{j,k}(\vartheta,v_1^j,v_2)-\frac{\theta_2}{\sqrt{k}}\right)}\right]= \\
&\quad=\frac{t}{\sqrt{k}}(\theta_1-\theta_2+b\|\Phi_T\|)+\frac{1}{k}t\omega_m\left(a_M(m)+b\eta_M(m),\mathrm{v}_1^j\right)+ \\
&\qquad\qquad+\frac{i}{2k}t(\theta_1-\theta_2+b\|\Phi_T\|)^2+it\widetilde{B}_3'''\left(\frac{a_M(m)}{\sqrt{k}},\frac{\mathrm{v}}{\sqrt{k}},\frac{b}{\sqrt{k}}\right).
\end{aligned}
$$

Thus as conseguence:

$$
\begin{aligned}
&\Psi^{(2)}(t,b,x)=\frac{t}{\sqrt{k}}(\theta_1-\theta_2+b\|\Phi_T\|)+ \\
&+\frac{1}{k}t\omega_m\left(a_M(m)+b\eta_M(m),\mathrm{v}_1^{(j)}\right)+\frac{i}{2k}t(\theta_1-\theta_2+b\|\Phi_T\|)^2-\lambda_{\nu_T}\frac{\|\Phi_T(m)\|b}{\sqrt{k}}- \\
&\qquad-\frac{it}{k}\psi_2\left(\mathrm{v}_1^j-a_M(m)-b\eta_M(m),\mathrm{v}_2\right)+it\widetilde{B}_3'''\left(\frac{a_M(m)}{\sqrt{k}},\frac{\mathrm{v}}{\sqrt{k}},\frac{b}{\sqrt{k}}\right)
\end{aligned}
$$

$$
\begin{aligned}
&=\frac{1}{\sqrt{k}}\left(t(\theta_1-\theta_2+b\|\Phi_T\|)-\lambda_{\nu_T}\|\Phi_T(m)\|b\right)+ \\
&\quad+\frac{1}{k}\left[t\omega_m\left(a_M(m)+b\eta_M(m),\mathrm{v}_1^{(j)}\right)+\frac{i}{2}t(\theta_1-\theta_2+b\|\Phi_T\|)^2-\right. \\
&\quad\left.-it\psi_2\left(\mathrm{v}_1^j-a_M(m)-b\eta_M(m),\mathrm{v}_2\right)\right]+it\widetilde{B}_3'''\left(\frac{a_M(m)}{\sqrt{k}},\frac{\mathrm{v}}{\sqrt{k}},\frac{b}{\sqrt{k}}\right)
\end{aligned}
$$

and

$$
\begin{aligned}
&ik\Psi^{(2)}(t,b,x)\\
&\qquad = i\sqrt{k}\left(t(\theta_1-\theta_2+b\|\Phi_T\|)-\lambda_{\nu_T}\|\Phi_T(m)\|b\right)+\\
&+\left[it\omega_m\left(a_M(m)+b\eta_M(m),\mathrm{v}_1^{(j)}\right)-\frac{1}{2}t(\theta_1-\theta_2+b\|\Phi_T\|)^2+\right.\\
&\left.+t\psi_2\left(\mathrm{v}_1^j-a_M(m)-b\eta_M(m),\mathrm{v}_2\right)\right]-kt\widetilde{B}_3'''\left(\frac{a_M(m)}{\sqrt{k}},\frac{\mathrm{v}}{\sqrt{k}},\frac{b}{\sqrt{k}}\right).
\end{aligned}
$$

Continuing calculations we can rewrite the j-term as:

$$
\begin{aligned}
&\widetilde{\Pi}_{\nu_G,k\nu_T}(x_{1k},x_{2k})^{(j)} \\
&\sim k^{1-\frac{d_P}{2}}\frac{d_{\nu_G}}{(2\pi)^{d_T}}\cdot\int_{\mathbb{R}^{d_P-1}}da\cdot\left[\int_0^{+\infty}dt\int_{-\infty}^{+\infty}db e^{i\sqrt{k}\Upsilon(t,b)}e^{A(m,\theta,\mathrm{v},v,a)}B(j)\right],
\end{aligned}
\tag{3.22}
$$

where

$$
\Upsilon(t,b)=t\left(b\|\Phi_T(m)\|+\theta_1-\theta_2\right)-\lambda\|\Phi_T\|b, \tag{3.23}
$$

$$
\begin{aligned}
&A(m,\theta,\mathrm{v},v,a)\\
&\quad=-\frac{t}{2}(b\|\Phi_T(m)\|+\theta_1-\theta_2)^2+it\omega_m(a_M(m)+b\eta_M(m),\mathrm{v}_1^j)+\\
&\qquad\qquad+t\psi_2(\mathrm{v}_1^j-a_M(m)-b\eta_M(m),\mathrm{v}_2)e^{i\mathcal{A}_{j,k}(\vartheta,v_1^j,v_2)}
\end{aligned}
\tag{3.24}
$$

and

$$
B(j)=\chi_{\nu_G}(g_j^{-1})e^{-ik\vartheta_j\nu_T}. \tag{3.25}
$$

The internal integral in (3.22) is oscillatory in $\sqrt{k}$ with phase Υ. The phase has critical points $(t_0,b_0)=\left(\lambda_{\nu_T},\frac{\theta_2-\theta_1}{\|\Phi_T(m)\|}\right)$. The Hessian is

$$
H(\Upsilon)(P_0)=\begin{pmatrix}0 & \|\Phi_T(m)\|\\ \|\Phi_T(m)\| & 0\end{pmatrix}
$$

and

$$
\sqrt{\left|\det\frac{\sqrt{k}H}{2\pi}\right|}=\frac{\sqrt{k}}{2\pi}\|\Phi_T(m)\|.
$$

Using the Stationary Phase Lemma [H] we have:

$$\widetilde{\Pi}_{\nu_G,k\nu_T}(x_{1k},x_{2k})^{(j)} \sim \frac{d_{\nu_G}}{(2\pi)^{d_T-1}}B(j)k^{\frac{1}{2}-\frac{d_P}{2}}\cdot\int_{\mathbb{R}^{d_P-1}} e^{-i\sqrt{k}(\theta_2-\theta_1)\lambda_\nu}\cdot \cdot e^{A(a,t_0,b_0)}\cdot\frac{1}{\|\Phi_T(m)\|}\left(\frac{k\lambda_{\nu_T}}{\pi}\right)^{d_M} da, \tag{3.26}$$

where

$$A(a,t_0,b_0) = \lambda_{\nu_T}\left[i\omega_m\left(a_M(m)+b_0\eta_M,\mathrm{v}_1^j\right)+ +\psi_2\left(\mathrm{v}_1^j-a_M(m)-b_0\eta_M(m),\mathrm{v}_2\right)\right],$$

and we have to evaluate the integral $\int_{\mathbb{R}^{d_P-1}} dae^{A(a,t_0,b_0)}$. In order to do this we define the following spaces:

$$V_m = \mathrm{val}_m(\mathfrak{g}\oplus\mathrm{Ker}(\Phi_T(m))) = \mathrm{val}_m(\mathrm{Ker}(\Phi_P(m)))\subseteq T_mM,$$

$$N_m = J_m(V_m),$$

$$H_m = [V_m\oplus N_m]^\perp,$$

where V_m, H_m are complex subspaces of T_mM and N_m is the normal space to M_{0,ν_T}. Decomposing $\mathrm{v}\in T_mM$ as $\mathrm{v}=\mathrm{v}_h+\mathrm{v}_v+\mathrm{v}_t$, with respectively $\mathrm{v}_h\in H_m$, $\mathrm{v}_v\in V_m$ and $\mathrm{v}_t\in N_m$, we have that $a_M=a_{Mv}$, $\eta_M=\eta_{Mh}+\eta_{Mv}$ and

$$\begin{aligned}
&i\omega_m\left(a_M(m)+b_0\eta_M(m),\mathrm{v}_1^j\right)+i\psi_2\left(\mathrm{v}_1^j-a_M(m)-b_0\eta_M(m),\mathrm{v}_2\right)\\
&\ = i\omega_m(a_M(m),\mathrm{v}_{1t}^j)+ib_0\omega_m(\eta_{Mh}(m),\mathrm{v}_{1h}^j+\mathrm{v}_{2h})+ib_0\omega_m(\eta_{Mv}(m),\mathrm{v}_{1t}^j)-i\omega_m(\mathrm{v}_{1h}^j,\mathrm{v}_{2h})+\\
&\ +i\omega_m(a_M(m),\mathrm{v}_{2t})+ib_0\omega_m(\eta_{Mv}(m),\mathrm{v}_{2t})-\frac{1}{2}\|\,\mathrm{v}_{1h}^j-b_0\eta_{Mh}(m)-\mathrm{v}_{2h}\|^2-i\omega_m(\mathrm{v}_{1v}^j,\mathrm{v}_{2t})-\\
&\qquad -\frac{1}{2}\|\,\mathrm{v}_{1t}^j-\mathrm{v}_{2t}\|^2-\frac{1}{2}\|\,\mathrm{v}_{1v}-a_M(m)-b_0\eta_{Mv}(m)-\mathrm{v}_{2v}\|^2-i\omega_m(\mathrm{v}_{1t}^j,\mathrm{v}_{2v})
\end{aligned}$$

$$\begin{aligned}
&= i\omega_m(a_M(m),\mathrm{v}_{1t}^j+\mathrm{v}_{2t})-\frac{1}{2}\|\,\mathrm{v}_{1v}-a_M(m)-b_0\eta_{Mv}(m)-\mathrm{v}_{2v}\|^2\\
&\quad +ib_0\omega_m(\eta_{Mh}(m),\mathrm{v}_{1h}^j+\mathrm{v}_{2h})+ib_0\omega_m(\eta_{Mv}(m),\mathrm{v}_{1t}^j+\mathrm{v}_{2t})-i\omega_m(\mathrm{v}_{1h}^j,\mathrm{v}_{2h})+\\
&\quad -\frac{1}{2}\|\,\mathrm{v}_{1h}^j-b_0\eta_{Mh}(m)-\mathrm{v}_{2h}\|^2-i\omega_m(\mathrm{v}_{1v}^j,\mathrm{v}_{2t})-\frac{1}{2}\|\,\mathrm{v}_{1t}^j-\mathrm{v}_{2t}\|^2-i\omega_m(\mathrm{v}_{1t}^j,\mathrm{v}_{2v}).
\end{aligned} \tag{3.27}$$

We define $r_M(m)\in\mathbb{R}^{d_P-1}$ translated by $a_M(m)$ such that:

$$r_M(m) = \mathrm{v}_{1v}^j-a_M(m)-b_0\eta_{Mv}(m)-\mathrm{v}_{2v},$$

so $a_M(m) = \mathrm{v}_{1v}^j - r_M(m) - b_0\eta_{Mv}(m) - \mathrm{v}_{2v}$ and

$$\begin{aligned}&\omega_m(a_M(m), \mathrm{v}_{1t}^j + \mathrm{v}_{2t})\\ &\quad = \omega_m(\mathrm{v}_{1v}^j - \mathrm{v}_{2v}, \mathrm{v}_{1t}^j + \mathrm{v}_{2t}) - \omega_m(r_M(m), \mathrm{v}_{1t}^j + \mathrm{v}_{2t}) - \omega_m(b_0\eta_{Mv}(m), \mathrm{v}_{1t}^j + \mathrm{v}_{2t}).\end{aligned} \tag{3.28}$$

We get:

$$\begin{aligned}\omega_m(a_M(m), \mathrm{v}_{1t}^j + \mathrm{v}_{2t}) = \omega_m(\mathrm{v}_{1v}^j, \mathrm{v}_{1t}^j) + \omega_m(\mathrm{v}_{1v}^j, \mathrm{v}_{2t}) - \omega_m(\mathrm{v}_{2v}, \mathrm{v}_{1t}^j) - \omega_m(\mathrm{v}_{2v}, \mathrm{v}_{2t}) -\\ - \omega_m(r_M(m), \mathrm{v}_{1t}^j + \mathrm{v}_{2t}) - \omega_m(b_0\eta_{Mv}(m), \mathrm{v}_{1t}^j + \mathrm{v}_{2t}).\end{aligned} \tag{3.29}$$

Putting (3.29) in (3.28) and deleting the opposite terms, we obtain:

$$\begin{aligned}&i\omega_m\left(a_M(m) + b_0\eta_M(m), \mathrm{v}_1^j\right) + i\psi_2\left(\mathrm{v}_1^j - a_M(m) - b_0\eta_M(m), \mathrm{v}_2\right)\\ &\qquad = i\left[\omega_m(\mathrm{v}_{1v}^j, \mathrm{v}_{1t}^j) - \omega_m(\mathrm{v}_{2v}, \mathrm{v}_{2t})\right] - \frac{1}{2}\| \mathrm{v}_{1t}^j - \mathrm{v}_{2t}\|^2 +\\ &+ ib_0\omega_m(\eta_{Mh}(m), \mathrm{v}_{1h}^j + \mathrm{v}_{2h}) - i\omega_m(\mathrm{v}_{1h}^j, \mathrm{v}_{2h}) - \frac{1}{2}\left\|\mathrm{v}_{1h}^j - b_0\eta_{Mh}(m) - \mathrm{v}_{2h}\right\|^2 -\\ &\qquad - i\omega_m(r_M(m), \mathrm{v}_{1t}^j + \mathrm{v}_{2t}) - \frac{1}{2}\|r_M(m)\|^2.\end{aligned}$$

Let C the matrix of $\mathrm{val}_m : \mathrm{Ker}(\Phi_T(m)) \oplus \mathfrak{g} \to V_m$. Changing variable $r' = Cr$ we have that $dr = \det C^{-1} dr'$. We have to evaluate:

$$\frac{1}{\det C}\int_{\mathbb{R}^{d_P-1}} e^{\lambda_{\nu_T}\left[-i\omega_m(r', \mathrm{v}_{1t}^j + \mathrm{v}_{2t}) - \frac{1}{2}\|r'\|^2\right]} dr'. \tag{3.30}$$

We make the substitution $s = \sqrt{\lambda_{\nu_T}} r'$, so we obtain:

$$= \frac{\lambda_{\nu_T}^{-\frac{1}{2}(d_P-1)}}{\det C}\int_{\mathbb{R}^{d_P-1}} e^{\left[-i\omega_m(s, \sqrt{\lambda_{\nu_T}}(\mathrm{v}_{1t}^j + \mathrm{v}_{2t})) - \frac{1}{2}\|s\|^2\right]} ds = \frac{(2\pi)^{\frac{d_P-1}{2}}}{\lambda^{\frac{1}{2}(d_P-1)}} e^{-\frac{1}{2}\lambda_{\nu_T}\| \mathrm{v}_{1t}^j + \mathrm{v}_{2t}\|^2}. \tag{3.31}$$

Thus the exponential factor in the asymptotic expansion is $e^{H(v_1^j, v_2)}$ with

$$\begin{aligned}H(v_1^j, v_2) = \lambda_{\nu_T}\Bigg(-i\omega_m\left(\mathrm{v}_{1h}^j, \mathrm{v}_{2h}\right) - \| \mathrm{v}_{1t}\|^2 - \| \mathrm{v}_{2t}\|^2 + i\frac{(\theta_2 - \theta_1)}{\|\Phi_T(m)\|}\omega_m\left(\eta_{Mh}(m), \mathrm{v}_{1h}^j + \mathrm{v}_{2h}\right)\\ + i[\omega_m\left(\mathrm{v}_{1v}^j, \mathrm{v}_{1t}^j\right) - \omega_m\left(\mathrm{v}_{2v}, \mathrm{v}_{2t}\right)] - \frac{1}{2}\left\|\mathrm{v}_{1h}^j - \frac{(\theta_2 - \theta_1)}{\|\Phi_T\|}\eta_{Mh}(m) - \mathrm{v}_{2h}\right\|^2\Bigg),\end{aligned} \tag{3.32}$$

and the principal term is of the form:

$$\frac{d_{\nu_G} 2^{\frac{d_G}{2}}}{(\sqrt{2}\pi)^{d_T-1}} \left(\frac{k\lambda_{\nu_T}}{\pi}\right)^{d_M-\frac{d_G}{2}-\frac{(d_T-1)}{2}} \cdot \chi_{\nu_G}(g_j^{-1}) e^{-ik\vartheta_j \nu_T} \cdot \cdot \frac{e^{i\sqrt{k}(\theta_2-\theta_1)\lambda_\nu} \cdot e^H}{\|\Phi_T(m)\| \mathcal{D}(m)}. \tag{3.33}$$

This complete the proof of 2).

Let us now describe the necessary changes to the previous argument to prove 3). Instead of considering a neighborhood of (x, x) we consider the asymptotic in a neighborhood of $(x, p_0 \cdot x)$. We set $y = p \cdot x$ and we assume given the local system of Heisenberg coordinates in a neighborhood of x and y. We may also assume without loss that the Heisenberg coordinate system centered at y is obtained from the one centered at x by a p-translation, that is,

$$y + (\theta, \mathrm{v}) = p \cdot \big(x + (\theta, \mathrm{v})\big).$$

We must evaluate:

$$\widetilde{\Pi}_{\nu_G, k\nu_T} (x_{1k}, y_{2k}),$$

where $x_{1k} = x + \frac{u_1}{\sqrt{k}}$, $y_{2k} = y + \frac{u_2}{\sqrt{k}}$ and $u_j = (\theta_j, \mathrm{v}_j)$. Now with the preceding interpretation we have that $y_{2k} = p_0 \left(x + \frac{u_2}{\sqrt{k}}\right) = p_0 x_{2k}$. Proceeding as in the previous case we have:

$$\widetilde{\Pi}_{\nu_G, k\nu_T} (x_{1k}, y_{2k}) = \frac{d_{\nu_G}}{(2\pi)^{d_T}} \int_G \int_T \chi_{\nu_G}(g^{-1}) t^{-k\nu_T} \cdot \Pi\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{t^{-1}}(x), p_0 \cdot x_{2k}\right) dt dg, \tag{3.34}$$

and due to the unitary action:

$$\widetilde{\Pi}_{\nu_G, k\nu_T} (x_{1k}, y_{2k}) = \frac{d_{\nu_G}}{(2\pi)^{d_T}} \int_G \int_T \chi_{\nu_G}(g^{-1}) t^{-k\nu_T} \cdot \Pi\left(\tilde{\mu}^P_{p_0^{-1}} \circ \tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{t^{-1}}(x), x_{2k}\right) dt dg. \tag{3.35}$$

Let $p_0 = (g_0, t_0) = \left(g_0, e^{i\vartheta_0}\right) \in P$, then

$$\tilde{\mu}^P_{p_0^{-1}} \circ \tilde{\mu}^P_{(g,t)^{-1}}(x_{1k}) = \tilde{\mu}^G_{g_0^{-1} g^{-1}} \circ \tilde{\mu}^T_{t_0^{-1} t^{-1}}(x_{1k}),$$

and changing variables $g' = gg_0$, $t' = tt_0$, we have $g = g'g_0^{-1}$, $t = t't_0^{-1}$ and $\vartheta = \vartheta' - \vartheta_0$. Then

$$\widetilde{\Pi}_{\nu_G,k\nu_T}(x_{1k}, y_{2k}) = \frac{d_{\nu_G}}{(2\pi)^{d_T}} \int_G \int_T \overline{\chi_{\nu_G}(g'g_0^{-1})} e^{-ik\nu_T(\vartheta'-\vartheta_0)} \Pi\left(\tilde{\mu}^P_{p_0^{-1}} \circ \tilde{\mu}^G_{g'^{-1}} \circ \tilde{\mu}^T_{t'^{-1}}(x), x_{2k}\right) dt'dg', \tag{3.36}$$

that will be the same as before with the difference that in the j-addendum we will make the substitution:

$$\overline{\chi_{\nu_G}(g_j)} t_j^{-k\nu_T} = \overline{\chi_{\nu_g,k\nu_T}(p_j)} \mapsto \overline{\chi_{\nu_G}(g_j g_0^{-1})} (t_j t_0^{-1})^{-k\nu_T} = \overline{\chi_{\nu_G,k\nu_T}(p_j p_0^{-1})}.$$

This complete the proof of 3) and complete the proof of the main Theorem. □

3.2 Proof of Theorem 1.3.4

Proof.

On the diagonal we have:

$$\begin{aligned} \widetilde{\Pi}_{\nu_G,k\nu_T}(x,x) &= d_{\nu_G} \int_G \int_T \chi_{\nu_G}(g^{-1}) t^{-k\nu_T} \Pi\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{t^{-1}}(x), x\right) dtdg \\ &= \frac{d_{\nu_G}}{(2\pi)^{d_T}} \int_G \int_{(-\pi,\pi)^{d_T}} \chi_{\nu_G}(g^{-1}) e^{-ik\nu_T\cdot\vartheta} \Pi\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), x\right) d\vartheta dg, \end{aligned} \tag{3.37}$$

where $\vartheta = (\vartheta_1, \cdots\cdots, \vartheta_{d_T})$, $\nu_T = (\nu_1, \cdots, \nu_{d_T}) \in \mathbb{Z}^{d_T}$ and $\nu_T \cdot \vartheta = \sum_{j=1}^{d_T} \nu_{Tj}\vartheta_j$. We consider $F_x = \{(g_1, t_1), \cdots, (g_{N_x}, t_{N_x})\}$ the stabilizer, with $|F_x| = N_x$. Let $\varepsilon > 0$ and we consider the following open subsets of $P = G \times T$:

$$A = \{(g,t) \in G \times T : \mathrm{dist}_{G\times T}((g,t), F_x) < 2\varepsilon\}$$

$$B = \{(g,t) \in G \times T : \mathrm{dist}_{G\times T}((g,t), F_x) > \varepsilon\}.$$

Then $P = A \cup B$, and we have choose a partition of unity $\gamma_1 + \gamma_2 = 1$ subordinate to the open cover $\{A, B\}$. Then for $(g,t) \in \mathrm{supp}(\gamma_2)$ we have

$$\mathrm{dist}_X\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{t^{-1}}(x), x\right) \geq C\varepsilon$$

for some constant $C > 0$. Therefore, the map

$$(g,t) \in P \mapsto \gamma_2(g,t)\chi_{\nu_G}(g^{-1})\Pi\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{t^{-1}}(x), x\right)$$

is $\mathcal{C}^\infty$ because the singular support of Π is included in the diagonal $X \times X$. As consequence the function:

$$t \in T \mapsto \int_G \gamma_2(g,t)\chi_{\nu_G}(g^{-1})\Pi\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{t^{-1}}(x), x\right) dg$$

is $\mathcal{C}^\infty$ and so its Fourier transform evaluated at $k\nu_T$ is rapidly decreasing for $k \to +\infty$, since by assumption $\nu_T \neq 0$. We set $\gamma_1(g,\vartheta) = \sum_{j=1}^{N_x} \rho_j(g,\vartheta)$, with each ρ_j supported in a neighborhood of (g_j, ϑ_j), and consider

$$\widetilde{\Pi}_{\nu_G,k\nu_T}(x,x) \sim \sum_{j=1}^{N_x} \widetilde{\Pi}_{\nu_G,k\nu_T}(x,x)^{(j)}$$

where

$$\begin{aligned} &\widetilde{\Pi}_{\nu_G,k\nu_T}(x,x)^{(j)} \\ &\quad\sim \frac{d_{\nu_G}}{(2\pi)^{d_T}} \int_G \int_{(-\pi,\pi)^{d_T}} \rho_j(g,\vartheta)\chi_{\nu_G}(g^{-1})e^{-ik\nu_T\cdot\vartheta}\Pi\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), x\right) d\vartheta dg. \end{aligned} \tag{3.38}$$

Let us now examine the asymptotics of each integrand separately. On the support of γ_1, $\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{t^{-1}}(x)$ is close to x, and therefore we may replace Π by its representation as a Fourier integral, perhaps after disregarding a smoothing term which contributes negligibly to the asymptotics. After rescaling in t we have:

$$\begin{aligned} &\widetilde{\Pi}_{\nu_G,k\nu_T}(x,x)^{(j)} \\ &\qquad\sim \frac{d_{\nu_G}}{(2\pi)^{d_T}} k \int_G \int_{(-\pi,\pi)^{d_T}} \int_0^{+\infty} \chi_{\nu_G}(g^{-1})e^{ik\Psi(x,t,g,\vartheta)} \\ &\qquad\qquad \cdot s\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), x, kt\right) \gamma_j(g,\vartheta) dt d\vartheta dg \end{aligned} \tag{3.39}$$

with

$$\Psi(x,t,g,\vartheta) = t\psi\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), x\right) - \nu_T \cdot \vartheta. \tag{3.40}$$

Let us regard 3.39 as an oscillatory integral with a complex phase Ψ of positive type. Let us look for critical points of Ψ. We have that $\partial_t\Psi(x,t,g,\vartheta) = \psi\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), x\right) = 0$ if and only if $\vartheta = \vartheta_j$ and $g = g_j$. We set $\vartheta = \eta + \vartheta_j$

with $\eta \sim \mathbf{0}$. In the neighborhood of g_j, we can write $g = g_j \exp_G \xi$, where $\xi \in \mathfrak{g}$ is close to the origin. With abuse of notation we shall write $\tilde{\mu}^G_{-\varsigma} \circ \tilde{\mu}^T_{-\vartheta}(x) = \tilde{\mu}^G_{-\xi} \circ \tilde{\mu}^T_{-\eta}(x)$. Upon choosing orthonormal basis for the Lie algebras, we shall identify them with $\mathbb{R}^{d_T}$ and $\mathbb{R}^{d_P}$, respectively. We consider now $\partial_{(\varsigma,\vartheta)}\Psi(x,t,\varsigma,\vartheta)\big|_{\xi=0,\eta=0} = t\Phi_T - \nu_T$. Thus in the new coordinates at any critical point $\xi = 0$, $\eta = 0$. The critical points are of the form:

$$P_0 = (t_0, (\varsigma_0, \vartheta_0)) = \left(\frac{\|\nu_T\|}{\|\Phi_T(m)\|}, (\varsigma_j, \vartheta_j) \right).$$

Considering the second derivatives, $\partial^2_{tt}\Psi = 0$, and using Heisenberg coordinates we write:

$$\begin{aligned}\Psi(x,t,\xi,\eta) = t\psi\big(x + \big(\xi\Phi_G(m) + \eta\Phi_T(m) + O(\|(\xi,\eta)\|^3), \\ , -\xi_M(m) - \eta_M(m) + O(\|(\xi,\eta)\|^2)\big), x\big) - \nu_T \cdot \eta - \nu_T \cdot \vartheta_j.\end{aligned} \tag{3.41}$$

Here with abuse of language we have identified $\eta_M(m)$ and $\xi_M(m)$ with their representation in local coordinates. We note that $m \in M_{0,\nu_T}$ so

$$\begin{aligned}\Psi(x,t,\xi,\eta) &= t\psi\big(x + \big(\eta\Phi_T(m) + O(\|(\xi,\eta)\|^3), \\ &\quad , -\xi_M(m) - \eta_M(m) + O(\|(\xi,\eta)\|^2)\big), x\big) - \nu_T \cdot \eta - \nu_T \cdot \vartheta_j \\ &= -\nu_T \cdot \eta - \nu_T \cdot \vartheta_j + it\Big\{\big[1 - e^{i\eta\Phi_T(m)}\big] \\ &\quad + \frac{1}{2}\|\eta_M(m) + \xi_M(m)\|^2 e^{i\eta\Phi_T(m)} + O(\|(\xi,\eta)\|^3)\Big\}\end{aligned} \tag{3.42}$$

and setting $\gamma = (\gamma_1, \cdots, \gamma_{d_P}) = (\xi_1, \cdots, \xi_{d_G}, \eta_1, \cdots, \eta_{d_T})$ and $\sigma = (\varsigma, \vartheta)$, we have $\sigma_0 = (\varsigma_0, \vartheta_0)$ and

$$\partial^2_{t\sigma}\Psi(x,t,\sigma)\big|_{(t_0,\sigma_0)} = \Phi_T(m). \tag{3.43}$$

Thus, we have:

$$\partial^2_{\sigma_l\sigma_k}\Psi(x,t,\sigma)\big|_{(t_0,\sigma_0)} = i\lambda_{\nu_T}\left[\Phi_l(m)\Phi_k(m) + \langle\gamma_l,\gamma_k\rangle_m\right], \tag{3.44}$$

with $\lambda_{\nu_T} = t_0$, we are in the same case of [P4] in the proof of Theorem 2, so we have that:

$$\det H(t_0,\sigma_0) = i^{d_P+1}\lambda_{\nu_T}^{d_P-1}\|\Phi_T(m)\|^2 \det C(m). \tag{3.45}$$

Where $C(m)$ is a scalar product on $\mathrm{Ker}(\Phi_P(m))$ and

$$\det\left[\frac{k}{2\pi i}H(t_0,\sigma_0)\right] = \left(\frac{k}{2\pi}\right)^{d_P+1}\lambda_{\nu_T}^{d_P-1}\|\Phi_T(m)\|^2\mathcal{D}(m)^2. \tag{3.46}$$

The principal term is:

$$\frac{d_{\nu_G}e^{-ik\vartheta_j\cdot\nu_T}2^{d_G/2}\chi_{\nu_G}(g_j^{-1})}{\left(\sqrt{2}\pi\right)^{d_T-1}}\cdot\left(\frac{\|\nu_T\|k}{\pi}\right)^{d_M+\frac{1-d_P}{2}}\cdot\frac{1}{\mathcal{D}(m)\|\Phi_T(m)\|^{d_M+1+\frac{1-d_P}{2}}}. \tag{3.47}$$

This complete the proof of the Theorem. □

3.3 Proof of Corollary 1.3.5

Proof.

We start considering the dimension of $H(X)_{\nu_G,k\nu_T}$:

$$\dim(H(X)_{\nu_G,k\nu_T}) = \int_X \widetilde{\Pi}_{\nu_G,k\nu_T}(x,x)dV_X(x).$$

Now let us observe that $\widetilde{\Pi}_{\nu_G,k\nu_T}(x,x)$ is naturally S^1-invariant as a function of x, and therefore descends to a function on M, that we shall denote by $\widetilde{\Pi}_{\nu_G,k\nu_T}(m,m)$ with abuse of language. Thus by integrating first along the fibers the previous integral may be naturally interpreted as an integral over M, that we shall write in the form:

$$\dim(H(X)_{\nu_G,k\nu_T}) = \int_M \widetilde{\Pi}_{\nu_G,k\nu_T}(m,m)dV_M(m).$$

Now by the above $\widetilde{\Pi}_{\nu_G,k\nu_T}(m,m)$ is rapidly decreasing away from a shrinking neighborhood of M_{0,ν_T}. So, using a smoothly varying system of adapted coordinates centered at points $m\in M_{0,\nu_T}$, we can locally parametrize a neighborhood U of M_{0,ν_T} in the form $m+\mathrm{v}$, where $m\in M_{0,\nu_T}$ and $\mathrm{v}\in N_m$. This parametrization is only valid locally in m, since we may not expect to find a single $\mathcal{C}^\infty$ family of adapted coordinates γ_m ($m\in M_{0,\nu_T}$). Hence to make this argument strictly rigorous we should introduce a partition of unity on M_{0,ν_T} subordinate to an appropriate open cover. However, we shall simplify notation and leave this point implicit.

$$\dim(H(X)_{\nu_G,k\nu_T}) = \int_{M_{0,\nu_T}}\int_{\mathbb{R}^{d_P-1}}\widetilde{\Pi}_{\nu_G,k\nu_T}(m+\mathrm{v},m+\mathrm{v})\,d\mathrm{v}dV_M(m).$$

In view of Theorem 1.2.6 the asymptotics of the previous integral are unchanged, if the integrand is multiplied by a cut-off of the form $\varrho\big(k^{\frac{7}{18}}\|\mathrm{v}\|\big)$, where $\varrho \in \mathcal{C}_0^\infty(\mathbb{R})$ is identically equal to 1 in some neighborhood of 0.

$$\dim(H(X)_{\nu_G,k\nu_T}) = \int_{M_{0,\nu_T}} \int_{\mathbb{R}^{d_P-1}} \widetilde{\Pi}_{\nu_G,k\nu_T}(m+\mathrm{v}, m+\mathrm{v})\,\varrho\big(k^{\frac{7}{18}}\|\mathrm{v}\|\big)\,d\mathrm{v}\,dV_M(m).$$

Let us now operate the rescaling $\mathrm{v} = \frac{\mathrm{u}}{\sqrt{k}}$. We can now make use of the asymptotic expansion in Theorem 1.3.4, with $\mathrm{u} = \mathrm{u}_t$ (that is, $\mathrm{u}_v = \mathrm{u}_h = 0$). We obtain:

$$\begin{aligned}\dim(H(X)_{\nu_G,k\nu_T}) &= k^{-\frac{d_P-1}{2}} \int_{M_{0,\nu_T}} \int_{\mathbb{R}^{d_P-1}} \widetilde{\Pi}_{\nu_G,k\nu_T}\left(m+\frac{\mathrm{u}}{\sqrt{k}}, m+\frac{\mathrm{u}}{\sqrt{k}}\right)\varrho\big(k^{-\frac{1}{9}}\|\mathrm{u}\|\big)\,d\mathrm{u}\,dV_M \\ &= k^{-\frac{d_P-1}{2}} \cdot \int_{M_{0,\nu_T}} \frac{2^{\frac{d_G}{2}} d_{\nu_G}^2}{(\sqrt{2})^{d_T-1}\pi^{d_T-1}} \left(\frac{\|\nu_T\|k}{\pi}\right)^{d_M-\frac{d_P-1}{2}} \cdot \\ &\quad \cdot \int_{\mathbb{R}^{d_P-1}} \frac{1}{\|\Phi_T\|^{d_M-\frac{d_P-1}{2}+1}\det C(m)} e^{-\lambda_{\nu_T}2\|\mathrm{u}\|^2}\varrho\big(k^{-\frac{1}{9}}\|\mathrm{u}\|\big)\,d\mathrm{u}\,dV_M(m) + \cdots,\end{aligned} \tag{3.48}$$

where $d_P = d_G + d_T$ and the dots denote lower order terms. Now we evaluate the Gaussian integral, let us operate the change of variables $q = \sqrt{2\lambda_{\nu_T}}\,\mathrm{u}$, we have that:

$$\begin{aligned}\int_{\mathbb{R}^{d_P-1}} e^{-2\lambda_{\nu_T}\|\mathrm{u}\|^2} d\mathrm{u} &= \frac{1}{\lambda_{\nu_T}^{\frac{d_P-1}{2}}(\sqrt{2})^{d_P-1}} \int_{\mathbb{R}^{d_P-1}} e^{-\|q\|^2} dq \\ &= \frac{\|\Phi_T(m)\|^{\frac{d_P-1}{2}}\pi^{\frac{d_P-1}{2}}}{(\sqrt{2})^{d_P-1}\|\nu_T\|^{\frac{d_P-1}{2}}}\end{aligned} \tag{3.49}$$

and substituting in (3.48) we obtain the following expression:

$$\begin{aligned}\dim(H(X)_{\nu_G,k\nu_T}) &= \frac{d_{\nu_G}^2}{2^{d_T-1}\pi^{d_T-1}}\left(\frac{\|\nu_T\|k}{\pi}\right)^{d_M-d_P+1} \cdot \\ &\quad \cdot \int_{M_{0,\nu_T}} \frac{\|\Phi_T(m)\|^{-d_M+d_P-2}}{\det C(m)} dV_M(m) + \cdots.\end{aligned}$$

The proof is complete. □

3.4 Proof of Proposition 1.2.7

Proof.

We assume that $\mathrm{dist}_X(y, p \cdot x) \geq Dk^{\varepsilon-1/2}$, for every $p \in P$. The method consist to use iteratively integration by parts to deduce the rapidly decreasing behavior of the kernel. We start following the previous situations using the standard representation as Fourier integral operator. So, by performing the change of variables $t \mapsto kt$, we obtain the following expression for the Szegö kernel:

$$\widetilde{\Pi}_{\nu_G, k\nu_T}(x, y) = \frac{k d_{\nu_G}}{(2\pi)^{d_T}} \int_G \int_T \int_0^{+\infty} \chi_{\nu_G}(g^{-1}) e^{ik\Psi(t,\vartheta,x)} \cdot \\ \cdot s\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), y, kt\right) dt dg d\vartheta, \tag{3.50}$$

where

$$\Psi(t, \vartheta, x) = t\psi\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), y\right) - \nu_T \cdot \vartheta \tag{3.51}$$

is the phase of the oscillatory integral. First we observe that:

$$\|\partial_\vartheta \Psi\| = \|t\psi\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), y\right) - \nu_T\| \geq C$$

for $0 < t \ll 1$ and $C > 0$. In a similar way

$$\|\partial_\vartheta \Psi\| \geq C_1 t$$

for $t \gg 0$ and $C_1 > 0$. Using integration by parts in $d\vartheta$, the asymptotics for $k \to +\infty$ is unchanged. We multiply the integrand by $\gamma(t)$, where $\gamma \in \mathcal{C}^\infty_0\left(\frac{1}{2D}, 2D\right)$, $\gamma \equiv 1$ on $\left(\frac{1}{D}, D\right)$ and $\gamma \equiv 0$ outside of $\left(\frac{1}{2D}, 2D\right)$. The integral now is compactly supported in dt. Taking the partial derivative respect t, we deduce that:

$$\partial_t \Psi(t, \vartheta, x) = \psi\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), y\right)$$

and by the assumption we find that:

$$\|\partial_t \Psi(t, \vartheta, x)\| = \left|\psi\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), y\right)\right| \geq \Im\psi\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), y\right) \geq D' k^{2\varepsilon-1}.$$

Now we introduce the differential operator:

$$L_t = \left[\psi\left(\tilde{\mu}^G_{g^{-1}} \circ \tilde{\mu}^T_{-\vartheta}(x), y\right)\right]^{-1} \partial_t$$

and observing that $e^{ik\Psi} = -\frac{i}{k} L_t\left(e^{ik\Psi}\right)$ we can apply iteratively the integration by parts. So step by step we obtain a factor of order $O(k^{-2N\varepsilon})$.

The proof is complete. □

3.5 Proof of Theorem 1.4.1

Proof.

In view of the equality on the first line of (1.14) we observe that 1) follows immediately from the point 1) of the main Theorem. Let us now consider the proof of 2).

Let $f \in C^{\infty}(M)$, we consider the associated Toeplitz operator:

$$\begin{aligned} &T_{\nu_G,k\nu_T}[f]\left(x+\frac{\mathrm{v}}{\sqrt{k}},x+\frac{\mathrm{v}}{\sqrt{k}}\right) \\ &\quad= \int_X \widetilde{\Pi}_{\nu_G,k\nu_T}\left(x+\frac{\mathrm{v}}{\sqrt{k}},y\right) f(y)\widetilde{\Pi}_{\nu_G,k\nu_T}\left(y,x+\frac{\mathrm{v}}{\sqrt{k}}\right) dV_X(y) \end{aligned} \tag{3.52}$$

with $x \in X_{0,\nu_T}$, where $f(y) = f(\pi(y))$. Now in view of Proposition 1.2.7 only a shrinking neighborhood of the orbit $P \cdot x$ contributes non-negligibly to the asymptotics. Therefore, the asymptotics are unchanged if the integrand in (3.52) is multiplied by a cut-off function $\varrho_k(y)$, where $\varrho_k = 1$ for $\mathrm{dist}_X(y, P \cdot x) \leq Dk^{\delta-1/2}$ (for example concretely δ equal 1/9) and $\varrho_k = 0$ for $\mathrm{dist}_X(y, P\cdot x) \geq 2Dk^{\delta-1/2}$. We shall make a more explicit choice of ϱ_k below.

Let $x + (\theta, \mathrm{v})$ be a system of Heisenberg local coordinates on X centered at x. This determines for every $p \in P$ a system of HLC centered at $p \cdot x$, by setting

$$p \cdot x + (\theta, \mathrm{v}) = p \cdot \big(x + (\theta, \mathrm{v})\big).$$

In this manner we have a unique smoothly varying family of HLC systems centered at points of $P \cdot x$, and identifications $T_{p\cdot x}X \cong T_xX \cong \mathbb{R} \times \mathbb{R}^{2d_M}$, $T_{p\cdot m}M \cong T_mM \cong \mathbb{R} \times \mathbb{R}^{2d_M}$. Furthermore, the action of P preserves the contact and CR structures of M, and the decomposition of the tangent spaces in h-, v-, and t-components. This means that the corresponding decomposition is preserved under the identification $T_{p\cdot m}M \cong T_mM$. If $y \in P \cdot x$, let N_y^P be the normal space to $P \cdot x$ in X at y; then we have natural unitary isomorphisms $N_y^P \cong N_x^P$. With this identification implicit and some abuse of language, we may then parametrize a suitably small open neighborhood of $P \cdot x$ by the map

$$(p, n) \in P \times N_x^P \mapsto p \cdot x + n.$$

We set $y = p \cdot (x + n)$, where $p = (g, t) \in P$ and n is a tangent vector normal to the orbit. For simplicity we suppose that the stabilizator of x in P is trivial. We have a diffeomorphism $P \times N(\varepsilon) \to X'$, with X' a ε-tubular neighborhood of $P \cdot x$ in

X and $N(\varepsilon)$ is a ball of radious ε in the normal space of the orbit in x (a real vector space of dimension $2d_M + 1 - d_P$). This diffeomorphism doesn't preserve the volume form. Thus in coordinates $(p, n) \in P \times N_x^P$ (where N_x^P denote the normal space to $P \cdot x$ in x):

$$dV_X(y) = \mathfrak{D}(p, n) dV_P(p) dn, \tag{3.53}$$

where $dV_P(p) = \frac{d\vartheta}{(2\pi)^{d_T}} dV_G(g)$ is the Haar measure of P, $dn = d\mathcal{L}(n)$ the Lebesgue measure on N_x^P (unitarily identified with $\mathbb{R}^{2d_M+1-d_P}$) and $\mathfrak{D}(p, 0) = \mathfrak{R}_x(p)$ with $\mathfrak{R}_x : P \to \mathbb{R}_{>0}$ a distortive function defined as follow. Let $\mathcal{B}_0$ an orthonormal basis of $\mathfrak{p}$ and let $\mathrm{val}_x : \mathfrak{p} \to T_x X$ the valutation map. Let $D_x(p)$ the matrix associated to $\mathrm{val}_x^\vee(g_X) : \mathfrak{p} \times \mathfrak{p} \to \mathbb{R}$ respect to $\mathcal{B}_0$ and

$$\mathfrak{R}_x(p) = \sqrt{\det D_x(p)}, \tag{3.54}$$

here g_X is the Riemannian metric on X and $\mathrm{val}_x^\vee(g_X)$ is the pull back of such metric to $\mathfrak{p} = \mathrm{Lie}(P)$ using the valutation $\mathrm{val}_x : \gamma \mapsto \gamma_X(x)$. We observe that $\mathfrak{R}_x(p)$ is constant along the orbit, because P acts by isometries. We put $\mathfrak{R}_x(p) = r_x$. We obtain that:

$$T_{\nu_G, k\nu_T}[f]\left(x + \frac{\mathrm{v}}{\sqrt{k}}, x + \frac{\mathrm{v}}{\sqrt{k}}\right) \tag{3.55}$$

$$= \int_X \widetilde{\Pi}_{\nu_G, k\nu_T}\left(x + \frac{\mathrm{v}}{\sqrt{k}}, y\right) f(y) \widetilde{\Pi}_{\nu_G, k\nu_T}\left(y, x + \frac{\mathrm{v}}{\sqrt{k}}\right) \varrho_k(y) dV_X(y)$$

$$= \int_P \int_{N_x} \widetilde{\Pi}_{\nu_G, k\nu_T}\left(x + \frac{\mathrm{v}}{\sqrt{k}}, p\,(x + n)\right) f\,(p \cdot (x + n)) \cdot$$

$$\cdot\, \widetilde{\Pi}_{\nu_G, k\nu_T}\left(p\,(x + n)\,, x + \frac{\mathrm{v}}{\sqrt{k}}\right) \mathfrak{D}(p, n) \varrho_k\,(p \cdot (x + n))\, dV_P(p) dn.$$

Rescaling $n \mapsto \frac{n}{\sqrt{k}}$, we observe that $\mathrm{Rank}(N) = \mathrm{dim}X - \mathrm{dim}P = 2d_M + 1 - d_P = 2\left[d_M + \frac{1-d_P}{2}\right]$ from which we have that $dn \to k^{-\left[d_M + \frac{1-d_P}{2}\right]} dn$ and we obtain that:

$$T_{\nu_G,k\nu_T}[f]\left(x+\frac{\mathrm{v}}{\sqrt{k}},x+\frac{\mathrm{v}}{\sqrt{k}}\right)$$

$$\begin{aligned}
&= k^{-\left[d_M+\frac{1-d_P}{2}\right]}\int_X \widetilde{\Pi}_{\nu_G,k\nu_T}\left(x+\frac{\mathrm{v}}{\sqrt{k}},y\right)f(y)\widetilde{\Pi}_{\nu_G,k\nu_T}\left(y,x+\frac{\mathrm{v}}{\sqrt{k}}\right)\varrho_k(y)dV_X(y)\\
&= k^{-\left[d_M+\frac{1-d_P}{2}\right]}\int_P\int_{N_x}\widetilde{\Pi}_{\nu_G,k\nu_T}\left(x+\frac{\mathrm{v}_1}{\sqrt{k}},p\left(x+\frac{n}{\sqrt{k}}\right)\right)f\left(p\left(x+\frac{n}{\sqrt{k}}\right)\right)\cdot\\
&\cdot\widetilde{\Pi}_{\nu_G,k\nu_T}\left(p\left(x+\frac{n}{\sqrt{k}}\right),x+\frac{\mathrm{v}}{\sqrt{k}}\right)\mathfrak{D}\left(p,\frac{n}{\sqrt{k}}\right)\varrho_k\left(p\cdot\left(x+\frac{n}{\sqrt{k}}\right)\right)dV_P(p)dn.
\end{aligned} \tag{3.56}$$

Here $\varrho_k(p\cdot x+n)=\varrho_k\left(p\cdot x+k^{1/2-\varepsilon}n\right)$ and $\varepsilon=1/9$. $n\in N_x^P$ by construction and $v\in N_x^P$ by assumption. We have that:

$$N_x^P=[\mathrm{val}_x(\mathrm{Ker}(\Phi_P(m)))\oplus\mathrm{span}(\eta_X(x))]^\perp.$$

Let $\eta\in\mathfrak{t}$ be the unique element such that

$$\eta\in\ker\Phi_T(m)^\perp,\quad\langle\Phi_T(m),\eta\rangle=\|\Phi_T(m)\|.$$

In particular we have $\eta\in\mathfrak{t}$, and η has unit norm. Then

$$\eta_X(x)=(\|\Phi_P(m)\|,-\eta_M(m)). \tag{3.57}$$

So in terms of the isomorphism $T_xX\cong\mathbb{R}\times T_mM$ we have:

$$p_X(x)=(\{0\}\times\mathrm{val}_m(\mathrm{Ker}(\Phi_P(m))))\oplus\mathrm{span}_{\mathbb{R}}((\|\Phi(m)\|,-\eta_M(m))),$$

and so in the notation of (1.4) and (1.5)

$$\begin{aligned}
p_X(x)^\perp=&(\{0\}\times V_m)^\perp\cap\mathrm{span}_{\mathbb{R}}(\|\Phi(m)\|,-\eta_M(m))^\perp\\
&=[\mathbb{R}\times(H_m\oplus N_m)]\cap[\mathrm{span}_{\mathbb{R}}((\|\Phi(m)\|,-\eta_M(m)))]^\perp,
\end{aligned}$$

with $V_m=\mathrm{val}_m(\mathrm{Ker}(\Phi_P(m)))$, $N_m=J_m(V_m)$ and $H_m=(V_m\oplus N_m)^\perp$. We have that $\eta_{Mt}(m)=0$, because M_{0,ν_T} is P-invariant. Thus if $(\lambda,\mathrm{h}+\mathrm{t})\in\mathbb{R}\times(H_m\oplus N_m)$ we have:

$$g_X((\lambda,\mathrm{h}+\mathrm{t}),(\|\Phi(m)\|,-\eta_M(m)))=\lambda\|\Phi_P(m)\|-g_X(\mathrm{h},\eta_{Mh}(m)).$$

Denoting $\Phi=\Phi_P(m)$ and $\eta_M=\eta_M(m)$ we have:

$$N_x^P = \left\{ (\lambda, \mathrm{h}+\mathrm{t}) \in \mathbb{R} \times (H_m \oplus N_m) : \lambda = \frac{g_X(\mathrm{h}, \eta_{Mh})}{\|\Phi\|} \right\} \tag{3.58}$$
$$= [\{0\} \times N_m] \oplus \left\{ \left(\frac{1}{\|\Phi\|} g_X(\mathrm{h}, \eta_{Mh}), \mathrm{h} \right) : \mathrm{h} \in H_m \right\}.$$

We write $v = n_1 = \left(\frac{1}{\|\Phi\|} g_X(\mathrm{h}_1, \eta_{Mh}), \mathrm{h}_1 + \mathrm{t}_1\right)$, $n = \left(\frac{1}{\|\Phi\|} g_X(\mathrm{h}, \eta_{Mh}), \mathrm{h} + \mathrm{t}\right)$. Recalling that $\Pi(x_1, x_2) = \overline{\Pi(x_2, x_1)}$ we obtain:

$$T_{\nu_G, k\nu_T}[f]\left(x + \frac{n_1}{\sqrt{k}}, x + \frac{n_1}{\sqrt{k}}\right) = \tag{3.59}$$
$$k^{-\left[d_M + \frac{1-d_P}{2}\right]} \int_P \int_{N_x} \left| \widetilde{\Pi}_{\nu_G, k\nu_T}\left(x + \frac{n_1}{\sqrt{k}}, p\left(x + \frac{n}{\sqrt{k}}\right)\right)\right|^2 f\left(p\left(x + \frac{n}{\sqrt{k}}\right)\right) \cdot$$
$$\cdot \mathfrak{D}\left(p, \frac{n}{\sqrt{k}}\right) \varrho_k\left(p \cdot \left(x + \frac{n}{\sqrt{k}}\right)\right) dV_P(p) dn.$$

Let us now make use of the asymptotic expansion of $\widetilde{\Pi}_{\nu_G, k\nu_T}$ from point 3) of the Theorem 1.2.6 and, using the Taylor expansion for $f\left(p \cdot x + \frac{n}{\sqrt{k}}\right)$ and for $\mathfrak{D}$

$$T_{\nu_G, k\nu_T}[f]\left(x + \frac{n_1}{\sqrt{k}}, x + \frac{n_1}{\sqrt{k}}\right) \sim k^{\left[d_M + \frac{1-d_P}{2}\right]} r_x \cdot f(p \cdot x)\, C(m, \nu_P) \tag{3.60}$$
$$\int_P \int_{N_x} |\chi_{\nu_P}(p)|^2 \cdot e^{H(n_1, n) + H(n, n_1)} \varrho_k\left(p \cdot \left(x + \frac{n}{\sqrt{k}}\right)\right) dV_P(p) dn (1 + \cdots)$$

where we have set:

$$C(m, \nu_P) = \left[\frac{d_{\nu_G} 2^{\frac{d_G}{2}}}{(\sqrt{2}\pi)^{d_T - 1}} \cdot \frac{\left(\frac{\|\nu_T\|}{\pi}\right)^{d_M + \frac{1-d_P}{2}}}{\mathcal{D}(m) \|\Phi_T\|^{d_M + 1 + \frac{1-d_P}{2}}} \right]^2$$

and the dots stand for terms of less degree.

Here the exponent is as follows: let

$$n = \mathrm{v} + \mathrm{t} + \mathrm{h}, \quad n_1 = \mathrm{v}_1 + \mathrm{t}_1 + \mathrm{h}_1$$

be the decomposition as in (1.4) and (1.5). We call $R_x = N_x^P$ and r, r_1 in place of n_1, n. Then

$$H(r_1, r) + H(r, r_1) = -2\lambda_{\nu_T}\left(\|\mathrm{t}_1\|^2 + \|\mathrm{t}\|^2\right) - \lambda_{\nu_T} \left\| \mathrm{h}_1 - \mathrm{h} - \frac{g_X(\mathrm{h} - \mathrm{h}_1, \eta_{Mh})}{\|\Phi_T(m)\|} \eta_{Mh} \right\|^2. \tag{3.61}$$

We evaluate the Gaussian integral $\int_{R_x} e^{H(r_1,r)+H(r,r_1)}dr$. So we have that:

$$\int_{R_x} e^{H(r_1,r)+H(r,r_1)}dr = e^{-2\lambda_{\nu_T}\|\,\mathrm{t}_1\|^2}\int_{N_m} e^{-2\lambda_{\nu_T}\|\,\mathrm{t}\|^2}d\,\mathrm{t}\cdot\int_{H_m} e^{-\lambda_{\nu_T}\left\|\mathrm{h}_1-\mathrm{h}-\frac{g_X(\mathrm{h}-\mathrm{h}_1,\eta_{Mh})}{\|\Phi_T(m)\|}\eta_{Mh}\right\|^2}d\,\mathrm{h}. \tag{3.62}$$

Let $d_{N_m} = d_P - 1$ the dimension of N_m. Let us first consider the first Gaussian integral in (3.62).

$$\int_{N_m} e^{-2\lambda_{\nu_T}\|\,\mathrm{t}\|^2}d\,\mathrm{t} = \frac{1}{\left(2\sqrt{\lambda_{\nu_T}}\right)^{d_{N_m}}}\int_{N_m} e^{-\frac{1}{2}\|s\|^2}ds = \left(\frac{\pi}{2\lambda_{\nu_T}}\right)^{(d_P-1)/2}. \tag{3.63}$$

To compute the second Gaussian integral in (3.62), let us operate the change of variable

$$-\,\mathrm{h}_1 + \mathrm{h} + \frac{g_X(\mathrm{h}-\mathrm{h}_1,\eta_{Mh})}{\|\Phi_T(m)\|}\eta_{Mh} = \mathrm{w}. \tag{3.64}$$

We differentiate the previous expression $d\,\mathrm{h} = \det\left(\frac{\partial\,\mathrm{w}}{\partial\,\mathrm{h}}\right)^{-1}d\,\mathrm{w}$ and in order to determine $\det\left(\frac{\partial\,\mathrm{w}}{\partial\,\mathrm{h}}\right)$ we have the following Lemma:

Lemma 3.5.1 *Under the change of coordinates* (3.64) *we have that:*

$$\det\left(\frac{\partial\,\mathrm{w}}{\partial\,\mathrm{h}}\right) = 1 + \frac{\|\eta_{Mh}\|^2}{\|\Phi_T\|}.$$

Proof.

We consider an orthonormal basis of H_m that include η_{Mh}. So the Jacobian matrix is of the form:

$$\left(\frac{\partial\,\mathrm{w}}{\partial\,\mathrm{h}}\right) = \begin{bmatrix} I & 0 \\ 0 & 1+\frac{\|\eta_{Mh}\|^2}{\|\Phi_T\|}\end{bmatrix}$$

where I is the identity matrix and the Lemma is proved. □

So we have:

$$\int_{H_m} e^{-\lambda_{\nu_T}\left\|\mathrm{h}_1-\mathrm{h}-\frac{g_X(\mathrm{h}-\mathrm{h}_1,\eta_{Mh})}{\|\Phi_T(m)\|}\eta_{Mh}\right\|^2}d\,\mathrm{h}$$
$$= \frac{\|\Phi_T\|}{\|\Phi_T\|+\|\eta_{Mh}\|^2}\int_{H_m} e^{-\frac{1}{2}\|s\|^2}\frac{ds}{(2\lambda_{\nu_T})^{\frac{d_{H_m}}{2}}} = \frac{\|\Phi_T\|}{\|\Phi_T\|+\|\eta_{Mh}\|^2}\left(\frac{\pi}{\lambda_{\nu_T}}\right)^{d_{H_m}/2}, \tag{3.65}$$

with $d_{H_m} = 2(d_M + 1 - d_P)$ the dimension of H_m. Inserting (3.65) and (3.63) in (3.62), we obtain

$$\int_{R_x} e^{H(r_1,r)+H(r,r_1)} dr \tag{3.66}$$
$$= e^{-2\lambda_{\nu_T} \| \mathrm{t}_1 \|^2} \cdot \frac{\|\Phi_T\|}{\|\Phi_T\| + \|\eta_{Mh}\|^2} \left(\frac{\pi}{\lambda_{\nu_T}} \right)^{\frac{d_{H_m}+d_{N_m}}{2}} 2^{-\frac{d_{N_m}}{2}}.$$

Let us now insert this in (3.60). We obtain a leading term depending on r_x and $f(x)$. We can then determine r_x using that for $f = 1$ this must reduce to the leading term in 2) of Theorem 1.2.6. Thus, recalling that $\int_P |\chi_{\nu_P}(p)|^2 \, dV_P(p) = 1$ (see [BD] Theorem 4.11) and noting that $d_{H_m} + d_{N_m} = 2d_M + 1 - d_P$ we obtain that the leading order term in (3.60) is given by:

$$k^{\left[d_M + \frac{1-d_P}{2}\right]} \cdot C(m, \nu_P) \cdot \left(\frac{\pi}{\lambda_{\nu_T}} \right)^{\frac{d_{H_m}+d_{N_m}}{2}} \cdot 2^{-\frac{d_{N_m}}{2}}$$
$$\cdot e^{-2\lambda_{\nu_T} \| \mathrm{t}_1 \|^2} \cdot \frac{\|\Phi_T\|}{\|\Phi_T\| + \|\eta_{Mh}\|^2} \cdot r_x = \frac{1}{(\sqrt{2}\pi)^{d_T - 1}} d_{\nu_G} 2^{\frac{d_G}{2}} \left(\frac{k}{\pi} \|\nu_T\| \right)^{d_M - \frac{d_P}{2} + \frac{1}{2}} e^{-2\lambda_{\nu_T} \| \mathrm{t}_1 \|^2} \cdot$$
$$\cdot \frac{1}{\mathcal{D}(m)} \cdot \frac{1}{\|\Phi_T\|^{d_M + 1 - \frac{d_P}{2} + \frac{1}{2}}}, \tag{3.67}$$

and we find that

$$r_x = \frac{\pi^{d_T - 1} \mathcal{D}(m) (\|\Phi_T\| + \|\eta_{Mh}\|^2)}{d_{\nu_G} \left(\sqrt{2}\right)^{-d_G - d_T + 1 - d_{N_m}}}. \tag{3.68}$$

The leading term become:

$$\frac{1}{(\sqrt{2}\pi)^{d_T - 1}} d_{\nu_G} 2^{\frac{d_G}{2}} \left(\frac{k}{\pi} \|\nu_T\| \right)^{d_M - \frac{d_P}{2} + \frac{1}{2}} f(m) e^{-2\lambda_{\nu_T} \| \mathrm{t}_1 \|^2} \cdot \tag{3.69}$$
$$\cdot \frac{1}{\mathcal{D}(m)} \cdot \frac{1}{\|\Phi_T\|^{d_M + 1 - \frac{d_P}{2} + \frac{1}{2}}}.$$

This complete the proof of 2) and of the Theorem. □

3.6 Proof of Corollary 1.4.2

Proof.

We start considering the trace of $T_{\nu_G,k\nu_T}[f]$:

$$\mathfrak{T}\left(T_{\nu_G,k\nu_T}[f]\right) = \int_X T_{\nu_G,k\nu_T}[f](x,x)dV_X(x).$$

Now we observe that $T_{\nu_G,k\nu_T}(x,x)$ is rapidly decreasing away from a shrinking neighborhood of X_{0,ν_T}. So, using a smoothly varying system of adapted coordinates centered at points $x \in X_{0,\nu_T}$, we can locally parametrize a neighborhood U of X_{0,ν_T} in the form $x + \mathrm{t}$, where $x \in X_{0,\nu_T}$ and $\mathrm{t} \in N_m$. This parametrization is only valid locally in x. We introduce a partition of unity on X_{0,ν_T} subordinate to an appropriate open cover and we simplify the notation leaving this point implicit.

$$\mathfrak{T}\left(T_{\nu_G,k\nu_T}[f]\right) = \int_{X_{0,\nu_T}} \int_{\mathbb{R}^{d_P-1}} T_{\nu_G,k\nu_T}[f]\left(x+\mathrm{t}, x+\mathrm{t}\right) d\,\mathrm{t} dV_X(x).$$

In view of Theorem 1.4.1 the asymptotics of the previous integral are unchanged, if the integrand is multiplied by a cut-off of the form $\varrho\big(k^{\frac{7}{18}}\|\,\mathrm{t}\|\big)$, where $\varrho \in \mathcal{C}_0^\infty(\mathbb{R})$ is identically equal to 1 in some neighborhood of 0.

$$\mathfrak{T}\left(T_{\nu_G,k\nu_T}[f]\right) = \int_{X_{0,\nu_T}} \int_{\mathbb{R}^{d_P-1}} T_{\nu_G,k\nu_T}\left(m+\mathrm{v}, m+\mathrm{v}\right) \varrho\big(k^{\frac{7}{18}}\|\,\mathrm{t}\|\big) d\,\mathrm{t} dV_X(x).$$

Let us now operate the rescaling $\mathrm{t} = \frac{\mathrm{u}}{\sqrt{k}}$. We can now make use of the asymptotic expansion in Theorem 1.4.1, with $n_1 = \mathrm{u}$. We obtain:

$$\begin{aligned}
\mathfrak{T}\left(T_{\nu_G,k\nu_T}[f]\right) &= k^{-\frac{d_P-1}{2}} \int_{X_{0,\nu_T}} \int_{\mathbb{R}^{d_P-1}} T_{\nu_G,k\nu_T}\left(x+\frac{\mathrm{u}}{\sqrt{k}}, x+\frac{\mathrm{u}}{\sqrt{k}}\right) \varrho\big(k^{-\frac{1}{9}}\|\,\mathrm{u}\|\big) d\,\mathrm{u} dV_X \\
&= k^{-\frac{d_P-1}{2}} \cdot \frac{2^{\frac{d_G}{2}} d_{\nu_G}^2}{(\sqrt{2})^{d_T-1}\pi^{d_T-1}} \left(\frac{\|\nu_T\|k}{\pi}\right)^{d_M-\frac{d_P-1}{2}} \cdot \\
\int_{X_{0,\nu_T}} \int_{\mathbb{R}^{d_P-1}} &\frac{f(\pi(x))}{\|\Phi_T\|^{d_M-\frac{d_P-1}{2}+1}\mathcal{D}(\pi(x))} e^{-\lambda_{\nu_T}2\|\,\mathrm{u}\|^2} \varrho\big(k^{-\frac{1}{9}}\|\,\mathrm{u}\|\big) d\,\mathrm{u} dV_X(x) + \cdots,
\end{aligned} \tag{3.70}$$

where $d_P = d_G + d_T$ and the dots denote lower order terms. Now we evaluate the Gaussian integral, that is the same of (3.49) in the Corollary 1.3.5. Substituting the result in (3.70) we obtain the following expression:

$$\begin{aligned}
\mathfrak{T}\left(T_{\nu_G,k\nu_T}[f]\right) = \frac{d_{\nu_G}^2}{2^{d_T-1}\pi^{d_T-1}} \left(\frac{\|\nu_T\|k}{\pi}\right)^{d_M-d_P+1} \cdot \\
\cdot \int_{X_{0,\nu_T}} \frac{f(\pi(x))\|\Phi_T(\pi(x))\|^{-d_M+d_P-2}}{\mathcal{D}(\pi(x))} dV_X(x) + \cdots.
\end{aligned}$$

The proof is complete. □

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