UNIVERSITÀ DI MILANO BICOCCA

Dottorato di Ricerca in Matematica
Pura ed Applicata

NONTRIVIAL SOLUTIONS OF SEMILINEAR ELLIPTIC EQUATIONS WITH MEASURE DATA

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Introduction

Let Ω be a bounded open subset of $\mathbb{R}^n$, with $n \geq 2$, and let $a \in L^\infty(\Omega; \mathcal{M}_{n,n})$, where $\mathcal{M}_{n,n}$ denotes the space of $n \times n$ matrices. Assume that there exists $\nu > 0$ satisfying

$$(a(x)\xi) \cdot \xi \geq \nu\, |\xi|^2 \qquad \text{for a.e. } x \in \Omega \text{ and every } \xi \in \mathbb{R}^n$$

and denote by $A, A^* : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ the operators defined as $Au = -\mathrm{div}(a\nabla u)$, $A^*u = -\mathrm{div}(a^t\, \nabla u)$.

The regularity results of De Giorgi [12], Nash [23] and Stampacchia (see [27, 28]) ensure that every $v \in W_0^{1,2}(\Omega)$, with $A^*v \in W^{-1,q}(\Omega)$ for some $q > n$, is continuous and bounded on Ω. As observed in [27], this fact allows to define, by duality, a generalized solution u of

$$\text{(1)} \qquad \begin{cases} -\mathrm{div}(a\nabla u) = \mu & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

for any $\mu \in \mathcal{M}_b(\Omega)$, the space of (signed) Radon measures with bounded total variation. More precisely, for every $\mu \in \mathcal{M}_b(\Omega)$, there exists one and only one u satisfying

$$\text{(2)} \qquad \begin{cases} u \in \mathcal{D}'(\Omega)\,, \\ \langle u, A^*v \rangle = \displaystyle\int_\Omega v\, d\mu & \text{for every } v \in W_0^{1,2}(\Omega) \text{ with } A^*v \in C_c^\infty(\Omega)\,. \end{cases}$$

According to [27, Définition 9.1], it will be considered as *the generalized solution* of (1). Moreover, such a solution u satisfies

$$\begin{cases} u \in \displaystyle\bigcap_{p < \frac{n}{n-1}} W_0^{1,p}(\Omega) \subseteq \bigcap_{r < \frac{n}{n-2}} L^r(\Omega)\,, \\ \langle A^*v, u \rangle = \displaystyle\int_\Omega v\, d\mu & \text{for every } v \in W_0^{1,2}(\Omega) \text{ with } A^*v \in \displaystyle\bigcup_{q>n} W^{-1,q}(\Omega)\,. \end{cases}$$

In particular, u satisfies (2) if and only if

$$\text{(3)} \qquad \begin{cases} u \in L^1(\Omega)\,, \\ \displaystyle\int_\Omega u\, A^*v\, dx = \int_\Omega v\, d\mu & \text{for every } v \in W_0^{1,2}(\Omega) \text{ with } A^*v \in L^\infty(\Omega)\,. \end{cases}$$

If a and $\partial\Omega$ are smooth enough to guarantee that

$$\left\{v \in W_0^{1,2}(\Omega): \ A^*v \in C_c^\infty(\Omega)\right\} \subseteq \left\{v \in C^2(\overline{\Omega}): \ v = 0 \text{ on } \partial\Omega\right\} \subseteq \left\{v \in W_0^{1,2}(\Omega): \ A^*v \in L^\infty(\Omega)\right\},$$

then an equivalent reformulation of (3) is given by

$$\begin{cases} u \in L^1(\Omega)\,, \\ \displaystyle\int_\Omega u\, A^*v\, dx = \int_\Omega v\, d\mu \qquad \text{for every } v \in C^2(\overline{\Omega}) \text{ with } v = 0 \text{ on } \partial\Omega\,. \end{cases}$$

A first important development of this topic has concerned quasilinear problems of the form

$$\begin{cases} -\mathrm{div}(\alpha(x,\nabla u)) = \mu & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $\alpha : \Omega \times \mathbb{R}^n \longrightarrow \mathbb{R}^n$ satisfies assumptions of Leray-Lions type. In such a case, it is a challenging open question to give a definition of generalized solution which provides both existence and uniqueness for any $\mu \in \mathcal{M}_b(\Omega)$. Let us refer the reader to [2, 4, 11, 30] and references therein.

A second development has concerned semilinear problems of the form

$$(4)\qquad \begin{cases} -\mathrm{div}(a\nabla u) + g(u) = \mu & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

where $g : \mathbb{R} \to \mathbb{R}$ is a nondecreasing, continuous function, whose study started with the work of Brezis and Strauss [7], in the case $\mu \in L^1(\Omega)$, and will be the object of this thesis. First of all, u is said to be a generalized solution of (4) if

$$(5)\qquad \begin{cases} u \in L^1(\Omega)\,,\ g(u) \in L^1(\Omega)\,, \\ \displaystyle\int_\Omega u\, A^*v\, dx + \int_\Omega g(u)\, v\, dx = \int_\Omega v\, d\mu \\ \qquad\qquad\qquad\qquad \text{for every } v \in W_0^{1,2}(\Omega) \text{ with } A^*v \in L^\infty(\Omega)\,. \end{cases}$$

Let us mention that such a solution u is unique whenever $\mu \in \mathcal{M}_b(\Omega)$ and does exist if $\mu \in L^1(\Omega)$. If $\mu \in \mathcal{M}_b(\Omega)$, then subtle existence/nonexistence phenomena occur, as described in [1, 3, 18, 19]. Let us mention in particular [6], which provides also an overview on the whole subject.

Assume now that $a(x)$ is symmetric for a.e. $x \in \Omega$. In spite of the fact that (4) looks as the Euler-Lagrange equation of the functional

$$J(u) = \frac{1}{2}\int_\Omega (a\nabla u)\cdot\nabla u\, dx + \int_\Omega G(u)\, dx - \int_\Omega u\, d\mu\,, \qquad G(s) = \int_0^s g(t)\, dt\,,$$

the application of variational methods to (4) seems to be impossible, as in general the solution u is not expected to belong to $W^{1,2}(\Omega)$. However, in the recent papers [15, 16, 17], Ferrero and Saccon were able to find, by a clever change of variable, a direct variational approach which recovers, for instance, the (known) existence of a solution u when $g(s) = |s|^{p-1}s$ and $p < n/(n-2)$. Moreover, they also started the study of multiple solutions by variational methods, when g is not assumed to be nondecreasing. On the other hand, their approach seems to require an asymptotic growth estimate on g also when g is nondecreasing and $\mu \in L^1(\Omega)$, in contrast with the results of [7].

The purpose of this thesis is to propose a different variational approach, more in the line of [9], and then prove some existence and multiplicity results for the solutions of (4).

More precisely, we assume that $a(x)$ is symmetric, that g is nondecreasing, that $\mu \in \mathcal{M}_b(\Omega)$ and that there exists the solution u_0 of (4). Then we look for solutions $(\lambda, u) \in \mathbb{R} \times L^1(\Omega)$ of

$$\begin{cases} -\mathrm{div}(a\nabla u) + g(u) = \lambda(u - u_0) + \mu & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{6}$$

without assuming any growth estimate on g. Of course, (λ, u_0) is a solution of (6) for any $\lambda \in \mathbb{R}$, so that (6) admits the "trivial branch" of solutions $\{(\lambda, u_0) : \lambda \in \mathbb{R}\}$. Therefore both local and global questions can be raised for (6).

As a result of global type, we will show that (6) admits at least two nontrivial solutions provided that

$$\lim_{|s|\to\infty} \frac{G(s)}{s^2} = +\infty$$

and that λ is large enough. If g is of class C^1, then the condition on λ can be expressed in a more precise way by requiring that

$$\lambda > \inf \left\{ \int_\Omega (a\nabla v) \cdot \nabla v \, dx + \int_\Omega g'(u_0)v^2 \, dx \; : \; v \in W_0^{1,2}(\Omega)\,, \; \int_\Omega v^2 \, dx = 1 \right\}.$$

This result has already appeared in [14].

As a result of local type, we will prove an adaptation to our setting of a celebrated bifurcation theorem of Rabinowitz (see [25, Theorem 11.35]).

Chapter 1

Some auxiliary results

1 On the regularity of solutions defined by duality

From now on, Ω will denote a bounded open subset of $\mathbb{R}^n$, with $n \geq 2$, and $a \in L^\infty(\Omega; \mathcal{M}_{n,n})$ a map such that there exists $\nu > 0$ satisfying

$$(a(x)\xi) \cdot \xi \geq \nu\, |\xi|^2 \qquad \text{for a.e. } x \in \Omega \text{ and every } \xi \in \mathbb{R}^n\,.$$

Then we denote by $A, A^* : W_0^{1,2}(\Omega) \to W^{-1,2}(\Omega)$ the bijective maps defined as $Au = -\mathrm{div}(a\nabla u)$, $A^*u = -\mathrm{div}(a^t\, \nabla u)$.

When $1 \leq p \leq \infty$, $\|\ \|_p$ will denote the usual norm in $L^p(\Omega)$ and $L^p_c(\Omega)$ the subspace of u's in $L^p(\Omega)$ vanishing a.e. outside some compact subset of Ω. Finally, for every $s \in \mathbb{R}$, we set $s^\pm = \max\{\pm s, 0\}$ and define $T_k : \mathbb{R} \to \mathbb{R}$ by $T_k(s) = \min\{\max\{s, -k\}, k\}$.

The regularity results of De Giorgi [12], Nash [23] and Stampacchia (see [27, 28]) ensure that $(A^*)^{-1}\varphi$ is continuous and bounded on Ω for every $\varphi \in W^{-1,q}(\Omega)$ with $q > n$ and

$$\|(A^*)^{-1}\varphi\|_\infty \leq c(n, q, \Omega)\|\varphi\|_{W^{-1,q}}\,.$$

Therefore, for every $\mu \in \mathcal{M}_b(\Omega)$ and $1 < p < \frac{n}{n-1}$, we can define a linear and continuous function

$$U : W^{-1,p'}(\Omega) \to \mathbb{R}$$

as

$$\langle U, \varphi\rangle = \int_\Omega ((A^*)^{-1}\varphi)\, d\mu\,.$$

Since $W_0^{1,p}(\Omega)$ is reflexive, there exists one and only one $u \in W_0^{1,p}(\Omega)$ such that

$$\langle \varphi, u\rangle = \int_\Omega ((A^*)^{-1}\varphi)\, d\mu \qquad \text{for any } \varphi \in W^{-1,p'}(\Omega)\,,$$

namely

$$\langle A^*v, u\rangle = \int_\Omega v\,d\mu \qquad \text{for any } v \in W_0^{1,2}(\Omega) \text{ with } A^*v \in W^{-1,p'}(\Omega)\,.$$

In particular, we have $u \in \mathcal{D}'(\Omega)$ and

$$\langle u, A^*v\rangle = \int_\Omega v\,d\mu \qquad \text{for any } v \in W_0^{1,2}(\Omega) \text{ with } A^*v \in C_c^\infty(\Omega)$$

and this last formulation is enough to guarantee the uniqueness of u in $\mathcal{D}'(\Omega)$. Therefore u is independent of the choice of $p \in]1, n/(n-1)[$.

We conclude that, given $\mu \in \mathcal{M}_b(\Omega)$, there exists one and only one $u \in \mathcal{D}'(\Omega)$ such that

$$\langle u, A^*v\rangle = \int_\Omega v\,d\mu \qquad \text{for any } v \in W_0^{1,2}(\Omega) \text{ with } A^*v \in C_c^\infty(\Omega)\,.$$

Moreover $u \in \bigcap_{1<p<\frac{n}{n-1}} W_0^{1,p}(\Omega)$ and

$$\langle A^*v, u\rangle = \int_\Omega v\,d\mu \qquad \text{for any } v \in W_0^{1,2}(\Omega) \text{ with } A^*v \in \bigcup_{n<q<\infty} W^{-1,q}(\Omega)\,.$$

In particular, u can be also characterized by

$$\begin{cases} u \in L^1(\Omega)\,, \\ \displaystyle\int_\Omega u\,A^*v\,dx = \int_\Omega v\,d\mu \qquad \text{for every } v \in W_0^{1,2}(\Omega) \text{ with } A^*v \in L^\infty(\Omega)\,. \end{cases}$$

Recall also that, according to [11, Theorem 10.1 and Formula (2.22)], we have $T_k(u) \in W_0^{1,2}(\Omega)$ for every $k > 0$,

$$\nu \int_\Omega |\nabla T_k(u)|^2\,dx \le k|\mu|(\Omega) \qquad \forall k > 0 \tag{1.1.1}$$

and there exists a cap_2-quasi continuous function $\tilde{u} : \Omega \to \mathbb{R}$ such that $\tilde{u} = u$ a.e. in Ω, where cap_2 denotes the capacity as defined in [11]. Moreover, a standard summability result holds.

Theorem 1.1.2 *Let $g : \Omega \times \mathbb{R} \to \mathbb{R}$ be a Carathéodory function such that*

$$s\,g(x,s) \ge 0 \qquad \text{for a.e. } x \in \Omega \text{ and every } s \in \mathbb{R}\,. \tag{1.1.3}$$

Let $u \in L^1(\Omega)$ and $w \in L^p(\Omega)$ with $p > 1$ be such that $g(x,u) \in L^1(\Omega)$ and

$$\int_\Omega u\,A^*v\,dx + \int_\Omega g(x,u)\,v\,dx = \int_\Omega vw\,dx \quad \text{for every } v \in W_0^{1,2}(\Omega) \text{ with } A^*v \in L^\infty(\Omega)\,.$$

Then the following facts hold:

(a) *if* $n \geq 3$ *and* $p < 2n/(n+2)$*, we have* $u \in W_0^{1,np/(n-p)}(\Omega) \subseteq L^{np/(n-2p)}(\Omega)$ *and*

$$\|\nabla u\|_{\frac{np}{n-p}} \leq c(n,p,\nu)\|w\|_p\,;$$

(b) *if* $n \geq 3$ *and* $2n/(n+2) \leq p < n/2$*, we have* $u \in W_0^{1,2}(\Omega) \cap L^{np/(n-2p)}(\Omega)$*,*

$$\begin{aligned} \|\nabla u\|_2 &\leq c(n,p,\nu)\|w\|_{\frac{2n}{n+2}}\,, \\ \|u\|_{\frac{np}{n-2p}} &\leq c(n,p,\nu)\|w\|_p \end{aligned}$$

and

$$\int_\Omega (a\nabla u)\cdot\nabla v\,dx + \int_\Omega g(x,u)\,v\,dx = \int_\Omega vw\,dx \qquad \text{for every } v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)\,;$$

(c) *if* $n \geq 2$ *and* $p > n/2$*, we have* $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$*,*

$$\|\nabla u\|_2 + \|u\|_\infty \leq c(n,p,\nu,\Omega)\|w\|_p$$

and

$$\int_\Omega (a\nabla u)\cdot\nabla v\,dx + \int_\Omega g(x,u)\,v\,dx = \int_\Omega vw\,dx \qquad \text{for every } v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)\,.$$

(d) *if* $n \geq 2$ *and* $|w| \leq w_0(1+|u|)$ *with* $w_0 \in L^q(\Omega), q > n/2$*, then* $u \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$*,*

$$\|u\|_\infty \leq c(n,p,\nu,\Omega)\,\|w_0\|_q\,(1 + \|\nabla u\|_2)$$

and

$$\int_\Omega (a\nabla u)\cdot\nabla v\,dx + \int_\Omega g(x,u)\,v\,dx = \int_\Omega vw\,dx \qquad \text{for every } v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)\,.$$

Proof. Let $\vartheta : \mathbb{R} \to \mathbb{R}$ be a nondecreasing, locally Lipschitz function with $\vartheta(0) = 0$. According to [11, Definition 2.25 and Theorem 2.33], we have

$$\nu \int_\Omega \vartheta'(T_k(u))|\nabla T_k(u)|^2\,dx \leq \int_\Omega \vartheta'(T_k(u))(a\nabla T_k(u))\cdot\nabla T_k(u)\,dx = \int_\Omega w\,\vartheta(T_k(u))\,dx\,.$$

(a) Given $r \in]0,1[$ and $\varepsilon > 0$, let

$$\vartheta_\varepsilon(s) = \int_0^s \frac{1}{(\varepsilon + |t|)^r}\, dt\,.$$

If we set $p^* = np/(n-p)$, then $p^* < 2$ and, as in the proof of [24, Lemma 2.1], we have

$$\begin{aligned}\int_\Omega |\nabla T_k(u)|^{p^*}\, dx &\le \left(\int_\Omega \frac{|\nabla T_k(u)|^2}{(\varepsilon + |T_k(u)|)^r}\, dx\right)^{\frac{p^*}{2}} \left(\int_\Omega (\varepsilon + |T_k(u)|)^{\frac{rp^*}{2-p^*}}\, dx\right)^{\frac{2-p^*}{2}} \\ &\le \left(\frac{1}{\nu}\int_\Omega |w|\, |\vartheta_\varepsilon(T_k(u))|\, dx\right)^{\frac{p^*}{2}} \left(\int_\Omega (\varepsilon + |T_k(u)|)^{\frac{rp^*}{2-p^*}}\, dx\right)^{\frac{2-p^*}{2}}\,.\end{aligned}$$

Passing to the limit as $\varepsilon \to 0$ and applying Lebesgue's theorem, it follows

$$\int_\Omega |\nabla T_k(u)|^{p^*}\, dx \le \left(\frac{1}{\nu(1-r)}\int_\Omega |w|\, |T_k(u)|^{1-r}\, dx\right)^{\frac{p^*}{2}} \left(\int_\Omega |T_k(u)|^{\frac{rp^*}{2-p^*}}\, dx\right)^{\frac{2-p^*}{2}}\,.$$

Then the same argument of [24, Lemma 2.1] yields assertion (a).

The proof of assertions (b) and (c) is more standard and follows the same lines of the regularity results of [21, 22, 27, 28].

(d) Considered $u \in \bigcap_{r<\frac{n}{n-2}} L^r(\Omega)$, we deduce from (a) that there exists $q_0 > 1$ with $w_0 u \in L^{q_0}(\Omega)$ and

$$\|w\|_{q_0} \le \|w_0\|_q\, c(\Omega)\, \|u\|_r\,,$$

so that $w \in L^{q_0}(\Omega)$.

Then, from (a),(b), (c) and a standard bootstrap argument, the assertion follows.

■

2 Convex functionals

Throughout this section, we also assume that $a(x)$ is symmetric for a.e. $x \in \Omega$, so that $A^* = A$, and consider a Carathéodory function $g : \Omega \times \mathbb{R} \to \mathbb{R}$ such that:

(g_1) for a.e. $x \in \Omega$, the function $g(x,\cdot)$ is nondecreasing;

(g_2) for a.e. $x \in \Omega$, we have $g(x,0) = 0$.

We set $G(x,s) := \int_0^s g(x,t)\,dt$ and observe that $0 \le G(x,s) \le s\,g(x,s)$ for a.e. $x \in \Omega$ and every $s \in \mathbb{R}$. In particular, we can define a lower semicontinuous and convex functional

$$J : W_0^{1,2}(\Omega) \to]-\infty, +\infty]$$

by

$$J(u) = \frac{1}{2}\int_\Omega (a\nabla u)\cdot\nabla u\,dx + \int_\Omega G(x,u)\,dx\,.$$

Theorem 1.2.1 *Let $u \in W_0^{1,2}(\Omega)$ and $w \in W^{-1,2}(\Omega)$ with $J(u) < +\infty$ and $w \in \partial J(u)$. Then we have $g(x,u)\,u \in L^1(\Omega)$ and the following facts hold:*

(*a*) *if $w \in L^1_{loc}(\Omega)$, we have $g(x,u) \in L^1_{loc}(\Omega)$;*

(*b*) *if $w \in L^1(\Omega)$, we have $g(x,u) \in L^1(\Omega)$ and $\|g(x,u)\|_1 \le \|w\|_1$.*

Proof. First of all, it is standard that $G(x,u) \in L^1(\Omega)$ and

$$\begin{aligned} &g(x,u)(v-u) \in L^1(\Omega) \qquad \text{for every } v \in W_0^{1,2}(\Omega) \text{ with } G(x,v) \in L^1(\Omega)\,,\\ &\int_\Omega (a\nabla u)\cdot(\nabla v - \nabla u)\,dx + \int_\Omega g(x,u)(v-u)\,dx \ge \langle w, v-u\rangle \\ &\qquad\qquad \text{for every } v \in W_0^{1,2}(\Omega) \text{ with } G(x,v) \in L^1(\Omega) \end{aligned}$$

(see also [13, Corollary 2.2]). The choice $v = 0$ yields $g(x,u)\,u \in L^1(\Omega)$. Moreover, for every $\varphi \in W^{1,\infty}(\Omega)$ with $0 \le \varphi \le 1$ and every $k > 0$, we can also choose as test function

$$v = u - T_{1/k}(u)\varphi\,,$$

obtaining

$$\begin{aligned} &\int_\Omega T_{1/k}(u)(a\nabla u)\cdot\nabla\varphi\,dx + \int_\Omega g(x,u)T_{1/k}(u)\varphi\,dx \\ &\quad \le \int_\Omega T_{1/k}(u)(a\nabla u)\cdot\nabla\varphi\,dx + \int_\Omega \varphi\,T'_{1/k}(u)(a\nabla u)\cdot\nabla u\,dx + \int_\Omega g(x,u)T_{1/k}(u)\varphi\,dx \\ &\qquad \le \langle w, T_{1/k}(u)\varphi\rangle \le \frac{1}{k}\int_\Omega |w|\,\varphi\,dx\,, \end{aligned}$$

hence

$$\int_\Omega k\,T_{1/k}(u)(a\nabla u)\cdot\nabla\varphi\,dx + \int_\Omega g(x,u)\,k\,T_{1/k}(u)\varphi\,dx \le \int_\Omega |w|\,\varphi\,dx\,.$$

Passing to the limit as $k \to \infty$, from the Lebesgue and the monotone convergence theorem, we get

$$\int_\Omega (a\nabla|u|) \cdot \nabla\varphi \, dx + \int_\Omega |g(x,u)| \, \varphi \, dx \le \int_\Omega |w| \, \varphi \, dx$$
$$\text{for any } \varphi \in W^{1,\infty}(\Omega) \text{ with } 0 \le \varphi \le 1$$

and assertions (a) and (b) easily follow. ■

Now we are interested in ruling out the possibility that ∂J be multivalued. For this purpose, we add the assumption:

(g_3) for every compact subset K of Ω, every $S > 0$ and every $\varepsilon > 0$, there exists an open subset ω of Ω with $\mathrm{cap}_2(\omega, \Omega) < \varepsilon$ such that

$$\sup_{|s| \le S} |g(\cdot, s)| \in L^1(K \setminus \omega) \,.$$

Proposition 1.2.2 *Let $u_0 : \Omega \to \mathbb{R}$ be a cap_2-quasi continuous function and define $\hat{g} : \Omega \times \mathbb{R} \to \mathbb{R}$ by $\hat{g}(x,s) = g(x, u_0(x) + s) - g(x, u_0(x))$. Then $\hat{g}$ also is a Carathéodory function satisfying $(g_1) - (g_3)$.*

Assume moreover that $\{s \mapsto g(x,s)\}$ is of class C^1 for a.e. $x \in \Omega$ and that the Carathéodory function $D_s g$ satisfies (g_3). If we define $\breve{g} : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\breve{g}(x,s) = \left(\sup_{|t| \le 1} D_s g(x, t u_0(x)) \right) s \,,$$

then $\breve{g}$ also is a Carathéodory function satisfying $(g_1) - (g_3)$.

Proof. Of course, $\hat{g}$ is a Carathéodory function satisfying (g_1) and (g_2). In particular, for every $S > 0$ the function

$$\sup_{|s| \le S} |\hat{g}(\cdot, s)| = \sup_{\substack{|s| \le S \\ s \in \mathbb{Q}}} |\hat{g}(\cdot, s)| \qquad \text{a.e. in } \Omega$$

is measurable.

Given a compact subset K of Ω, $S > 0$ and $\varepsilon > 0$, let ω' be an open subset of Ω with $\mathrm{cap}_2(\omega', \Omega) < \varepsilon/2$ such that the restriction of u_0 to $\Omega \setminus \omega'$ is continuous. Let S' be the maximum of $|u_0|$ on $K \setminus \omega'$ and let ω'' be an open subset of Ω with $\mathrm{cap}_2(\omega'', \Omega) < \varepsilon/2$ such that

$$\sup_{|s| \le S' + S} |g(\cdot, s)| \in L^1(K \setminus \omega'') \,.$$

If we set $\omega = \omega' \cup \omega''$, then $\mathrm{cap}_2(\omega, \Omega) < \varepsilon$ and, for every $x \in K \setminus \omega$, we have

$$\sup_{|s|\leq S} |\hat{g}(x,s)| \leq \sup_{|s|\leq S'+S} |g(x,s)| + \sup_{|s|\leq S'} |g(x,s)| \leq 2 \sup_{|s|\leq S'+S} |g(x,s)|\,,$$

whence property (g_3).

The assertions concerning $\check{g}$ can be proved in a similar way. ■

Theorem 1.2.3 *For every $u \in W^{1,2}_{loc}(\Omega)$ and every $v \in W^{1,2}_0(\Omega)$, there exists a sequence (v_k) in $W^{1,2}_0(\Omega) \cap L^\infty_c(\Omega)$ converging to v in $W^{1,2}_0(\Omega)$ with*

$$-v^- \leq v_k \leq v^+ \quad \text{a.e. in } \Omega\,, \qquad u \in L^\infty\left(\{x \in \Omega :\ v_k(x) \neq 0\}\right)\,,$$
$$G(x, v_k) \in L^1(\Omega)\,, \qquad g(x,u)v_k \in L^1(\Omega)\,.$$

In particular,

$$\left\{v \in W^{1,2}_0(\Omega) \cap L^\infty_c(\Omega) :\ g(x,u)v \in L^1(\Omega)\right\}$$

is a dense linear subspace of $W^{1,2}_0(\Omega)$.

Proof. Given $u \in W^{1,2}_{loc}(\Omega)$, $v \in W^{1,2}_0(\Omega)$ and $\varepsilon > 0$, there exists a sequence $(\hat{z}_k)$ in $C^\infty_c(\Omega)$ converging to v in $W^{1,2}_0(\Omega)$. Then $z_k = \min\{\max\{\hat{z}_k, -v^-\}, v^+\}$ belongs to $W^{1,2}_0(\Omega) \cap L^\infty_c(\Omega)$, satisfies $-v^- \leq z_k \leq v^+$ and is still convergent to v in $W^{1,2}_0(\Omega)$. Let $k \in \mathbb{N}$ be such that $\|\nabla z_k - \nabla v\|_2 < \varepsilon$.

Let now $\vartheta : \mathbb{R} \to [0,1]$ be a C^∞-function with $\vartheta = 1$ on $[-1,1]$ and $\vartheta = 0$ outside $]-2,2[$. Then $z_{k,h} = \vartheta(u/h)z_k$ belongs to $W^{1,2}_0(\Omega) \cap L^\infty_c(\Omega)$, satisfies $-v^- \leq z_{k,h} \leq v^+$, $u \in L^\infty\left(\{z_{k,h} \neq 0\}\right)$ and is convergent to z_k in $W^{1,2}_0(\Omega)$. Let $h \in \mathbb{N}$ be such that $\|\nabla z_{k,h} - \nabla z_k\|_2 < \varepsilon$.

Finally, let $K = \mathrm{supt}\, z_{k,h}$, $S = 2h + \|z_{k,h}\|_\infty$ and, given $j \in \mathbb{N}$, let ω_j be an open subset of Ω with $\mathrm{cap}_2(\omega_j, \Omega) < 1/j$ such that

$$\sup_{|s|\leq S} |g(\cdot, s)| \in L^1(K \setminus \omega_j)\,.$$

Let $\psi_j \in W^{1,2}_0(\Omega)$ with $\|\nabla\psi_j\|_2 < 1/j$, $\psi_j = 1$ a.e. on ω_j and $\psi_j \leq 1$ a.e. on Ω. Then $z_{k,h,j} = \min\{\max\{z_{k,h}, -S(1-\psi_j)\}, S(1-\psi_j)\}$ belongs to $W^{1,2}_0(\Omega) \cap L^\infty_c(\Omega)$, satisfies $-v^- \leq z_{k,h,j} \leq v^+$, $u \in L^\infty\left(\{z_{k,h,j} \neq 0\}\right)$ and is convergent to $z_{k,h}$ in $W^{1,2}_0(\Omega)$. Let $j \in \mathbb{N}$ be such that $\|\nabla z_{k,h,j} - \nabla z_{k,h}\|_2 < \varepsilon$, so that $\|\nabla z_{k,h,j} - \nabla v\|_2 < 3\varepsilon$. Since

$$|G(x, z_{k,h,j})| \leq \left(\|z_{k,h}\|_\infty \sup_{|s|\leq \|z_{k,h}\|_\infty} |g(x,s)|\right) \chi_{K\setminus\omega_j}(x)\,,$$

$$|g(x,u)\,z_{k,h,j}| \leq \left(\|z_{k,h}\|_\infty \sup_{|s|\leq 2h} |g(x,s)|\right) \chi_{K\setminus\omega_j}(x)\,,$$

we also have $G(x, z_{k,h,j}) \in L^1(\Omega)$, $g(x,u)\,z_{k,h,j} \in L^1(\Omega)$ and the assertion follows. ∎

Now we can show the main consequences of assumption (g_3). Let us point out that the next assertion (b) is an adaptation to our setting of the result of [5].

Theorem 1.2.4 *Let $u \in W_0^{1,2}(\Omega)$ and $w \in W^{-1,2}(\Omega)$. Then the following facts hold:*

(a) *we have $J(u) < +\infty$ and $w \in \partial J(u)$ if and only if*

$$\int_\Omega ((a\nabla u)\cdot\nabla v + g(x,u)\,v)\;dx = \langle w, v\rangle$$
$$\text{for every } v \in W_0^{1,2}(\Omega) \text{ with } g(x,u)v \in L^1(\Omega)\,;$$

(b) *if $J(u) < +\infty$, $w \in \partial J(u)$, $v \in W_0^{1,2}(\Omega)$ and $(g(x,u)\,v)^- \in L^1(\Omega)$, then $g(x,u)\,v \in L^1(\Omega)$ and*

$$\int_\Omega ((a\nabla u)\cdot\nabla v + g(x,u)\,v)\;dx = \langle w, v\rangle\,;$$

(c) *if $J(u) < +\infty$, the set $\partial J(u)$ contains at most one element.*

Proof. Let $J(u) < +\infty$ and $w \in \partial J(u)$. As before, for every $v \in W_0^{1,2}(\Omega)$ with $G(x,v) \in L^1(\Omega)$, we have $g(x,u)(v-u) \in L^1(\Omega)$ and

$$\int_\Omega (a\nabla u)\cdot(\nabla v - \nabla u)\,dx + \int_\Omega g(x,u)(v-u)\,dx \geq \langle w, v-u\rangle\,,$$

namely, as $g(x,u)\,u \in L^1(\Omega)$ by Theorem 1.2.1,

$$\int_\Omega (a\nabla u)\cdot\nabla v\,dx + \int_\Omega g(x,u)\,v\,dx - \langle w,v\rangle \geq \int_\Omega (a\nabla u)\cdot\nabla u\,dx + \int_\Omega g(x,u)\,u\,dx - \langle w,u\rangle\,.$$

Now let $v \in W_0^{1,2}(\Omega)$ with $g(x,u)\,v \in L^1(\Omega)$ and let (v_k) be a sequence as in Theorem 1.2.3. Since

$$\int_\Omega (a\nabla u)\cdot\nabla v_k\,dx + \int_\Omega g(x,u)\,v_k\,dx - \langle w,v_k\rangle \geq \int_\Omega (a\nabla u)\cdot\nabla u\,dx + \int_\Omega g(x,u)\,u\,dx - \langle w,u\rangle \tag{1.2.5}$$

and $|g(x,u)\,v_k| \leq |g(x,u)\,v|$, we can pass to the limit as $k\to\infty$ in (1.2.5), obtaining

$$\int_\Omega (a\nabla u)\cdot\nabla v\,dx + \int_\Omega g(x,u)\,v\,dx - \langle w,v\rangle \geq \int_\Omega (a\nabla u)\cdot\nabla u\,dx + \int_\Omega g(x,u)\,u\,dx - \langle w,u\rangle\,.$$

Since $\left\{v \in W_0^{1,2}(\Omega): \; g(x,u)v \in L^1(\Omega)\right\}$ is a dense linear subspace of $W_0^{1,2}(\Omega)$, it follows

$$\int_\Omega (a\nabla u)\cdot\nabla v\,dx + \int_\Omega g(x,u)\,v\,dx = \langle w, v\rangle \quad \text{for every } v \in W_0^{1,2}(\Omega) \text{ with } g(x,u)\,v \in L^1(\Omega)$$

and $\partial J(u) = \{w\}$. In particular, the proof of assertion (c) is complete.

Consider now $v \in W_0^{1,2}(\Omega)$ with $(g(x,u)\,v)^- \in L^1(\Omega)$ and let (v_k) be a sequence as in Theorem 1.2.3. Since

$$\int_\Omega g(x,u)\,v_k\,dx = \langle w, v_k\rangle - \int_\Omega (a\nabla u)\cdot\nabla v_k\,dx$$

and $g(x,u)\,v_k \geq -\,(g(x,u)\,v)^-$, from Fatou's lemma we infer that $g(x,u)\,v \in L^1(\Omega)$ and assertion (b) also follows.

Finally, let us complete the proof of (a). Therefore, assume that $w \in W^{-1,2}(\Omega)$ satisfies

$$\int_\Omega \left((a\nabla u)\cdot\nabla v + g(x,u)\,v\right)\,dx = \langle w, v\rangle \qquad \text{for every } v \in W_0^{1,2}(\Omega) \text{ with } g(x,u)v \in L^1(\Omega)\,.$$

As before, we automatically have

$$\int_\Omega \left((a\nabla u)\cdot\nabla v + g(x,u)\,v\right)\,dx = \langle w, v\rangle \qquad \text{for every } v \in W_0^{1,2}(\Omega) \text{ with } (g(x,u)v)^- \in L^1(\Omega)\,.$$

In particular, from $g(x,u)\,u \geq 0$ we infer that $g(x,u)\,u \in L^1(\Omega)$, hence that $G(x,u) \in L^1(\Omega)$, namely $J(u) < +\infty$. Moreover, for every $v \in W_0^{1,2}(\Omega)$ with $G(x,v) \in L^1(\Omega)$, from

$$g(x,u)(u-v) \geq G(x,u) - G(x,v)$$

it follows

$$\int_\Omega \left((a\nabla u)\cdot(\nabla u - \nabla v) + g(x,u)\,(u-v)\right)\,dx = \langle w, u-v\rangle\,,$$

hence, by convexity,

$$J(v) \geq J(u) + \int_\Omega \left((a\nabla u)\cdot(\nabla v - \nabla u) + g(x,u)\,(v-u)\right)\,dx = J(u) + \langle w, v-u\rangle\,.$$

If $v \in W_0^{1,2}(\Omega)$ and $G(x,v) \notin L^1(\Omega)$, it is obvious that

$$J(v) \geq J(u) + \langle w, v-u\rangle\,.$$

Therefore $w \in \partial J(u)$ and the proof of assertion (a) is complete. ■

Corollary 1.2.6 *Let $u \in W_0^{1,2}(\Omega)$ and $w \in L^1(\Omega) \cap W^{-1,2}(\Omega)$. Then we have $J(u) < +\infty$ and $w \in \partial J(u)$ if and only if $g(x,u) \in L^1(\Omega)$ and*

$$\int_\Omega ((a\nabla u) \cdot \nabla v + g(x,u)\, v)\; dx = \langle w, v\rangle \qquad \text{for every } v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)\,.$$

Proof. If $J(u) < +\infty$ and $w \in \partial J(u)$, we infer from Theorems 1.2.1 and 1.2.4 that $g(x,u) \in L^1(\Omega)$ and

$$\int_\Omega ((a\nabla u) \cdot \nabla v + g(x,u)\, v)\; dx = \langle w, v\rangle \qquad \text{for every } v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)\,.$$

To prove the converse, consider $v \in W_0^{1,2}(\Omega)$ with $g(x,u)\, v \in L^1(\Omega)$. Let (v_k) be a sequence as in Theorem 1.2.3. Since

$$\int_\Omega ((a\nabla u) \cdot \nabla v_k + g(x,u)\, v_k)\; dx = \langle w, v_k\rangle$$

and $|g(x,u)\, v_k| \leq |g(x,u)\, v|$, we can pass to the limit, obtaining

$$\int_\Omega ((a\nabla u) \cdot \nabla v + g(x,u)\, v)\; dx = \langle w, v\rangle$$
$$\text{for every } v \in W_0^{1,2}(\Omega) \text{ with } g(x,u)v \in L^1(\Omega)\,.$$

From Theorem 1.2.4 we conclude that $J(u) < +\infty$ and $w \in \partial J(u)$. ■

3 Variational characterization

Throughout this section, we keep on Ω, a and g the same assumptions of Section 2. Moreover, we consider $\mu \in \mathcal{M}_b(\Omega)$ and assume that

$$\text{(1.3.1)} \qquad \begin{cases} \text{there exists } u_0 \in L^1(\Omega) \text{ such that } g(x,u_0) \in L^1(\Omega) \text{ and} \\ \displaystyle\int_\Omega u_0 Av\, dx + \int_\Omega g(x,u_0)\, v\, dx = \int_\Omega v\, d\mu \\ \text{for every } v \in W_0^{1,2}(\Omega) \text{ with } Av \in L^\infty(\Omega)\,. \end{cases}$$

We set $G(x,s) = \int_0^s g(x,t)\, dt$ and define $\hat{g}, \widehat{G} : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} \hat{g}(x,s) &= g(x, u_0(x) + s) - g(x, u_0(x))\,, \\ \widehat{G}(x,s) &= \int_0^s \hat{g}(x,t)\, dt = G(x, u_0(x) + s) - G(x, u_0(x)) - g(x, u_0(x))\, s\,. \end{aligned}$$

According to Proposition 1.2.2, also $\hat{g}$ is a Carathéodory function satisfying $(g_1)-(g_3)$. Finally, as in Section 2 we define a lower semicontinuous and convex functional

$$\widehat{J}: W_0^{1,2}(\Omega) \to]-\infty, +\infty]$$

by

$$\widehat{J}(u) = \frac{1}{2}\int_\Omega (a\nabla u)\cdot \nabla u\, dx + \int_\Omega \widehat{G}(x,u)\, dx\,.$$

The main result of the section is the next characterization.

Theorem 1.3.2 *For every $\lambda \in \mathbb{R}$ and $u \in L^1(\Omega)$, the following facts are equivalent:*

(*a*) *we have*

$$\begin{cases} g(x,u) \in L^1(\Omega)\,, \\ \displaystyle\int_\Omega uAv\, dx + \int_\Omega g(x,u)\, v\, dx = \lambda \int_\Omega (u-u_0)v\, dx + \int_\Omega v\, d\mu \\ \qquad\qquad\qquad \text{for every } v \in W_0^{1,2}(\Omega) \text{ with } Av \in L^\infty(\Omega)\,; \end{cases}$$

(*b*) *if we set $z = u - u_0$, we have*

$$\begin{cases} z \in W_0^{1,2}(\Omega)\,, \\ \displaystyle\widehat{J}(v) \geq \widehat{J}(z) + \lambda \int_\Omega z(v-z)\, dx \qquad \text{for every } v \in W_0^{1,2}(\Omega)\,. \end{cases}$$

Proof. If (a) holds, then $z \in L^1(\Omega)$, $\hat{g}(x,z) = g(x,u) - g(x,u_0) \in L^1(\Omega)$ and

$$\int_\Omega zAv\, dx + \int_\Omega \hat{g}(x,z)\, v\, dx = \lambda \int_\Omega zv\, dx \qquad \text{for every } v \in W_0^{1,2}(\Omega) \text{ with } Av \in L^\infty(\Omega)\,.$$

Then $z \in L^r(\Omega)$ for any $r < n/(n-2)$. By Theorem 1.1.2 and a standard bootstrap argument, it follows that $z \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ and

$$\int_\Omega (a\nabla z)\cdot \nabla v\, dx + \int_\Omega \hat{g}(x,z)\, v\, dx = \lambda \int_\Omega zv\, dx \qquad \text{for every } v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)\,.$$

By Corollary 1.2.6 we deduce that $\widehat{J}(z) < +\infty$ and $\lambda z \in \partial \widehat{J}(z)$, namely

$$\widehat{J}(v) \geq \widehat{J}(z) + \lambda \int_\Omega z(v-z)\, dx \qquad \text{for every } v \in W_0^{1,2}(\Omega)\,.$$

Conversely, assume that $z \in W_0^{1,2}(\Omega)$ and

$$\widehat{J}(v) \geq \widehat{J}(z) + \lambda \int_\Omega z(v-z)\, dx \qquad \text{for every } v \in W_0^{1,2}(\Omega)\,.$$

Then $\widehat{J}(z) < +\infty$ and by Corollary 1.2.6 we deduce that $\hat{g}(x,z) \in L^1(\Omega)$, namely $g(x,u) \in L^1(\Omega)$, and

$$\int_\Omega (a\nabla z)\cdot\nabla v\,dx + \int_\Omega \hat{g}(x,z)\,v\,dx = \lambda\int_\Omega zv\,dx \qquad \text{for every } v \in W_0^{1,2}(\Omega)\cap L^\infty(\Omega)\,.$$

In particular, for every $v \in W_0^{1,2}(\Omega)$ with $Av \in L^\infty(\Omega)$, we have

$$\int_\Omega zAv\,dx + \int_\Omega \hat{g}(x,z)\,v\,dx = \lambda\int_\Omega zv\,dx\,,$$

namely

$$\int_\Omega uAv\,dx + \int_\Omega g(x,u)\,v\,dx = \lambda\int_\Omega (u-u_0)v\,dx + \int_\Omega v\,d\mu$$

and assertion (a) follows. ∎

Corollary 1.3.3 *The function u_0 introduced in assumption (1.3.1) is unique.*

Proof. Let $\hat{u}_0$ be another function as in (1.3.1). If we apply Theorem 1.3.2 with $\lambda = 0$, we find that 0 and $\hat{u}_0 - u_0$ are two minima of the strictly convex functional $\widehat{J}$, whence $\hat{u}_0 = u_0$. ∎

4 Parametric minimization

Let X be a Banach space and $I : X \to [-\infty, +\infty]$ a convex function. Assume also that $X = X_- \oplus X_+$, with X_- finite dimensional and X_+ closed in X, and define $\varphi : X_- \to [-\infty, +\infty]$ as

$$\varphi(v) = \inf\left\{I(v+w):\; w \in X_+\right\}\,.$$

Finally, denote by $P : X \to X_-$ the projection associated to the direct decomposition and by $P' : {X_-}' \to X'$ the dual map defined as

$$\langle P'\alpha, u\rangle = \langle \alpha, Pu\rangle \qquad \forall\alpha \in {X_-}'\,,\; \forall u \in X\,.$$

Theorem 1.4.1 *The following facts hold:*

(a) *the function* φ *is convex;*

(b) *if* $v \in X_-$ *and* $w \in X_+$ *satisfy* $I(v+w) = \varphi(v) \in \mathbb{R}$, *then*

$$\partial I(v+w) \cap P'(X_-{}') = \{P'\alpha : \ \alpha \in \partial\varphi(v)\} \ ;$$

(c) *if* U *is an open subset of* X_- *and* $\varphi\big|_U$ *has values in* $\mathbb{R}$, *then* $\varphi\big|_U$ *is locally Lipschitz and* $\partial\varphi(v) \neq \emptyset$ *for any* $v \in U$; *if one also knows that* $\partial\varphi(v)$ *contains exactly one element for any* $v \in U$, *then* $\varphi\big|_U$ *is of class* C^1 *and* $\partial\varphi(v) = \{\varphi'(v)\}$ *for any* $v \in U$.

Proof. Let $(v_0, s_0), (v_1, s_1) \in X_- \times \mathbb{R}$ with $\varphi(v_j) \leq s_j$ and let $t \in]0,1[$. Let also $\varepsilon > 0$ and let $w_1, w_2 \in X_+$ be such that $I(v_j+w_j) < s_j+\varepsilon$. Then $(v_0+w_0, s_0+\varepsilon)$ and $(v_1+w_1, s_1+\varepsilon)$ belong to the epigraph of I, which is convex. It follows

$$\varphi((1-t)v_0 + tv_1) \leq I((1-t)(v_0+w_0) + t(v_1+w_1)) \leq (1-t)s_0 + ts_1 + \varepsilon \, ,$$

hence

$$\varphi((1-t)v_0 + tv_1) \leq (1-t)s_0 + ts_1$$

by the arbitrariness of ε. Therefore the epigraph of φ is convex, namely φ is convex.

If $\alpha \in \partial\varphi(v)$, for every $u \in X$ we have

$$\begin{aligned} I(u) &\geq \varphi(Pu) \geq \varphi(v) + \langle \alpha, Pu - v \rangle \\ &= I(v+w) + \langle \alpha, P(u-v-w) \rangle \\ &= I(v+w) + \langle P'\alpha, u-v-w \rangle \, , \end{aligned}$$

whence $P'\alpha \in \partial I(v+w)$.

On the other hand, if $P'\alpha \in \partial I(v+w)$, for every $u_- \in X_-$ and $u_+ \in X_+$ we have

$$I(u_- + u_+) \geq I(v+w) + \langle P'\alpha, u_- + u_+ - v - w \rangle = \varphi(v) + \langle \alpha, u_- - v \rangle \, ,$$

whence

$$\varphi(u_-) \geq \varphi(v) + \langle \alpha, u_- - v \rangle \, .$$

It follows $\alpha \in \partial\varphi(v)$.

Finally, if U is an open subset of X_- and $\varphi\big|_U$ has values in $\mathbb{R}$, it follows from [26, Corollary 2.36 and Example 9.14] that $\varphi\big|_U$ is locally Lipschitz with $\partial\varphi(v) \neq \emptyset$ for any $v \in U$. In particular, φ is strictly continuous at any $v \in U$. If $\partial\varphi(v)$ contains exactly one element for any $v \in U$, from [26, Theorems 9.18 and Corollary 9.19] it follows that $\varphi\big|_U$ is of class C^1 and $\partial\varphi(v) = \{\varphi'(v)\}$ for any $v \in U$. ∎

5 Abstarct bifurcation in finite dimension

First of all, let us recall [8, Theorem 5.1], which is in turn related to a celebrated bifurcation result of Rabinowitz [25, Theorem 11.35] (see also [20, Theorem 2]).

Theorem 1.5.1 *Let X be a finite dimensional normed space, let $\delta > 0$, $\hat{\lambda} \in \mathbb{R}$ and, for every $\lambda \in [\hat{\lambda} - \delta, \hat{\lambda} + \delta]$, let $\varphi_\lambda : B(0, \delta) \to \mathbb{R}$ be a function of class C^1. Assume that:*

(a) the maps $\{(\lambda, u) \mapsto \varphi_\lambda(u)\}$ and $\{(\lambda, u) \mapsto \varphi'_\lambda(u)\}$ are continuous on $[\hat{\lambda} - \delta, \hat{\lambda} + \delta] \times B(0, \delta)$;

(b) φ_λ has an isolated local minimum (maximum) at zero for every $\lambda \in]\hat{\lambda}, \hat{\lambda} + \delta]$ and an isolated local maximum (minimum) at zero for every $\lambda \in [\hat{\lambda} - \delta, \hat{\lambda}[$.

Then one at least of the following assertions holds:

(i) $u = 0$ is not an isolated critical point of $\varphi_{\hat{\lambda}}$;

(ii) for every $\lambda \neq \hat{\lambda}$ in a neighborhood of $\hat{\lambda}$ there is a nontrivial critical point of φ_λ converging to zero as $\lambda \to \hat{\lambda}$;

(iii) there is a one-sided (right or left) neighborhood of $\hat{\lambda}$ such that for every $\lambda \neq \hat{\lambda}$ in the neighborhood there are two distinct nontrivial critical points of φ_λ converging to zero as $\lambda \to \hat{\lambda}$.

For our purposes, the next adaptation is more suited.

Theorem 1.5.2 *Let X be a finite dimensional normed space, let $\delta > 0$, $\hat{\lambda} \in \mathbb{R}$ and, for every $\lambda \in [\hat{\lambda} - \delta, \hat{\lambda} + \delta]$, let $\varphi_\lambda : B(0, \delta) \to \mathbb{R}$ be a function of class C^2. Assume that:*

(a) $\varphi_\lambda(0) = 0$, $\varphi'_\lambda(0) = 0$ for every $\lambda \in [\hat{\lambda} - \delta, \hat{\lambda} + \delta]$, and the map $\{(\lambda, u) \mapsto \varphi''_\lambda(u)\}$ is continuous on $[\hat{\lambda} - \delta, \hat{\lambda} + \delta] \times B(0, \delta)$;

(b) Ker $\varphi''_{\hat{\lambda}}(0) \neq \{0\}$ *and there exist two linear maps $L, K : X \to X'$ such that*

$$
\begin{aligned}
&\langle Lu, v \rangle = \langle Lv, u \rangle\,, \qquad \langle Ku, v \rangle = \langle Kv, u \rangle\,, && \forall u, v \in X\,,\\
&\langle Ku, u \rangle > 0 && \forall u \neq 0\,,\\
&\varphi''_\lambda(0) = L - \lambda K && \forall \lambda \in [\hat{\lambda} - \delta, \hat{\lambda} + \delta]\,.
\end{aligned}
$$

Then one at least of the following assertions holds:

(i) $u = 0$ is not an isolated critical point of $\varphi_{\hat{\lambda}}$;

(ii) for every $\lambda \neq \hat{\lambda}$ in a neighborhood of $\hat{\lambda}$ there is a nontrivial critical point of φ_λ converging to zero as $\lambda \to \hat{\lambda}$;

(iii) there is a one-sided (right or left) neighborhood of $\hat{\lambda}$ such that for every $\lambda \neq \hat{\lambda}$ in the neighborhood there are two distinct nontrivial critical points of φ_λ converging to zero as $\lambda \to \hat{\lambda}$.

Proof. Consider in X the scalar product

$$(u|v) = \langle Ku, v\rangle\,,$$

which induces a compatible norm in X, as X is finite dimensional.
Let

$$X_0 = \mathrm{Ker}\varphi''_{\hat{\lambda}}(0),$$

$$X_1 = \{w \in X :\ \langle Kv, w\rangle = 0\ \forall v \in X_0\}\,,$$

so that

$$X = X_0 \oplus X_1.$$

On the other hand, if $v \in X_0$ and $w \in X_1$, we have

$$\langle Lv, w\rangle = \hat{\lambda}\,\langle Kv, w\rangle = 0.$$

Therefore

$$\langle \varphi''_\lambda(0)v, w\rangle = 0\ \ \forall \lambda \in [\hat{\lambda} - \delta, \hat{\lambda} + \delta],\ \forall v \in X_0,\ \forall w \in X_1.$$

By the implicit function theorem, we can define a C^1 map ψ_λ such that $\psi_\lambda(0) = 0$ and

$$\langle \varphi'_\lambda(v + \psi_\lambda(v)), w\rangle = 0\ \ \forall w \in X_1.$$

The map $\psi_\lambda(v)$ is defined for v in a neighborhood of zero in X_0 and for λ in a neighborhood of $\hat{\lambda}$ (possibly smaller than $[\hat{\lambda} - \delta, \hat{\lambda} + \delta]$). Moreover, $\varphi''_\lambda(0)$ is injective on X_1.

Proceeding by differentation we find

$$\langle \varphi''_\lambda(0)(v + \psi'_\lambda(0)v), w\rangle = 0\ \ \forall v \in X_0, \forall w \in X_1,$$

hence

$$\langle \varphi_\lambda''(0)\psi_\lambda'(0)v, w\rangle = 0 \ \ \forall v \in X_0,\, \forall w \in X_1.$$

From the previous statements, we have

$$\langle \varphi_\lambda''(0)\psi_\lambda'(0)v, u\rangle = 0 \ \ \forall v \in X_0,\, \forall u \in X,$$

then

$$\varphi_\lambda''(0)\psi_\lambda'(0)v = 0 \ \text{ in } X'.$$

It follows, from the injectivity of $\varphi_\lambda''(0)$, that

$$\psi_\lambda'(0)v = 0 \ \ \forall v \in X_0,$$

namely

$$\psi_\lambda'(0) = 0. \tag{1.5.3}$$

Let us introduce the function $\widetilde{\varphi}$ defined as

$$\widetilde{\varphi}_\lambda(v) = \varphi_\lambda(v + \psi_\lambda(v)).$$

Then $\widetilde{\varphi}_\lambda$ is of class C^1 with

$$\langle \widetilde{\varphi}_\lambda'(z), v\rangle = \langle \varphi_\lambda'(z + \psi_\lambda(z)), v\rangle .$$

Then $\widetilde{\varphi}_\lambda$ is of class C^2 with

$$\langle \widetilde{\varphi}_\lambda''(z)v, v\rangle = \langle \varphi_\lambda''(z + \psi_\lambda(z))(v + \psi_\lambda'(z)v), v\rangle .$$

Then it is easily seen that the function $\widetilde{\varphi}_\lambda$ satisfies the assumptions of theorem (1.5.1). In particular, we have

$$\langle \widetilde{\varphi}_\lambda''(0)v, v\rangle = \langle \varphi_\lambda''(0)v, v\rangle = \langle Lv, v\rangle - \lambda L\langle Kv, v\rangle = (\hat{\lambda} - \lambda)\langle Kv, v\rangle .$$

It follows that

(a) for $\lambda < \hat{\lambda}$, 0 is an isolated local minimum,

(b) for $\lambda > \hat{\lambda}$, 0 is an isolated local maximum.

From the Theorem (1.5.1), the assertion follows. ■

Chapter 2

The main results

1 Existence of nontrivial solutions

Throughout this section, we keep on Ω, a and g the same assumptions of Chapter 1, Section 2. More explicitly, Ω is a bounded open subset of $\mathbb{R}^n$, with $n \geq 2$, and $a \in L^\infty(\Omega; \mathcal{M}_{n,n})$ satisfies

$$\begin{aligned} &a(x) \text{ is symmetric} && \text{for a.e. } x \in \Omega\,, \\ &(a(x)\xi) \cdot \xi \geq \nu\,|\xi|^2 && \text{for a.e. } x \in \Omega \text{ and every } \xi \in \mathbb{R}^n \end{aligned}$$

for some $\nu > 0$.

Moreover, $g : \Omega \times \mathbb{R} \to \mathbb{R}$ is a Carathéodory function satisfying:

(g_1) for a.e. $x \in \Omega$, the function $g(x, \cdot)$ is nondecreasing;

(g_2) for a.e. $x \in \Omega$, we have $g(x, 0) = 0$;

(g_3) for every compact subset K of Ω, every $S > 0$ and every $\varepsilon > 0$, there exists an open subset ω of Ω with $\mathrm{cap}_2(\omega, \Omega) < \varepsilon$ such that

$$\sup_{|s|\leq S} |g(\cdot, s)| \in L^1(K \setminus \omega)\,.$$

Finally, we consider $\mu \in \mathcal{M}_b(\Omega)$ and assume that

$$\begin{cases} \text{there exists } u_0 \in L^1(\Omega) \text{ such that } g(x, u_0) \in L^1(\Omega) \text{ and} \\ \displaystyle\int_\Omega u_0 Av\,dx + \int_\Omega g(x, u_0)\,v\,dx = \int_\Omega v\,d\mu \\ \text{for every } v \in W^{1,2}_0(\Omega) \text{ with } Av \in L^\infty(\Omega)\,. \end{cases}$$

We consider the problem

$$\text{(2.1.1)} \qquad \begin{cases} -\mathrm{div}(a\nabla u) + g(x,u) = \lambda(u-u_0) + \mu & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases}$$

namely

$$\begin{cases} u \in L^1(\Omega)\,, \quad g(x,u) \in L^1(\Omega)\,, \\ \displaystyle\int_\Omega uAv\,dx + \int_\Omega g(x,u)\,v\,dx = \lambda \int_\Omega (u-u_0)v\,dx + \int_\Omega v\,d\mu \\ \qquad\qquad\qquad\qquad \text{for every } v \in W_0^{1,2}(\Omega) \text{ with } Av \in L^\infty(\Omega)\,, \end{cases}$$

which admits u_0 as solution for any $\lambda \in \mathbb{R}$, and look for other solutions u.

As before, we set $G(x,s) = \int_0^s g(x,t)\,dt$ and, throughout this section, suppose that

(g_4) we have

$$\lim_{|s|\to+\infty} \frac{G(x,s)}{s^2} = +\infty \qquad \text{for a.e. } x \in \Omega.$$

The first result we aim to prove is the next

Theorem 2.1.2 *There exists $\overline{\lambda} > 0$ such that, for every $\lambda > \overline{\lambda}$, problem (2.1.1) admits at least two other different solutions u_1 and u_2 with $u_1 \le u_0 \le u_2$ a.e. in Ω.*

Proof. If we define $\hat{g}, \widehat{G} : \Omega \times \mathbb{R} \to \mathbb{R}$ and $\widehat{J} : W_0^{1,2}(\Omega) \to]-\infty, +\infty]$ as in Chapter 1, Section 3, we already know that $\hat{g}$ satisfies $(g_1) - (g_3)$. It is also clear that $\widehat{G}$ satisfies (g_4). Define now $\hat{g}_+, \widehat{G}_+ : \Omega \times \mathbb{R} \to \mathbb{R}$ by $\hat{g}_+(x,s) = \hat{g}(x,s^+)$, $\widehat{G}_+(x,s) = \int_0^s \hat{g}_+(x,t)\,dt$ and consider the functionals $\widehat{J}_+, I : W_0^{1,2}(\Omega) \to]-\infty, +\infty]$ defined as

$$\widehat{J}_+(u) = \frac{1}{2}\int_\Omega (a\nabla u)\cdot\nabla u\,dx + \int_\Omega \widehat{G}_+(x,u)\,dx\,,$$
$$I(u) = \widehat{J}_+(u) - \frac{\lambda}{2}\int_\Omega (u^+)^2\,dx\,.$$

It is clear that also $\hat{g}_+$ satisfies $(g_1) - (g_3)$, so that $\widehat{J}_+$ is convex and lower semicontinuous, and that I is sequentially lower semicontinuous with respect to the weak topology of $W_0^{1,2}(\Omega)$.

Let us show that I is also coercive. Assume, for a contradiction, that (v_k) is a sequence in $W_0^{1,2}(\Omega)$ with $\|\nabla v_k\|_2 = 1$ and (ϱ_k) a sequence with $\varrho_k \to +\infty$ such that $I(\varrho_k v_k)$ is bounded from above. Up to a subsequence, (v_k) is convergent weakly in $W_0^{1,2}(\Omega)$ and a.e. on Ω to some v. It follows that

$$\liminf_k \frac{\int_\Omega \widehat{G}_+(x,\varrho_k v_k)\,dx}{\varrho_k^2} < +\infty\,,$$

hence, as $\widehat{G}_+(x,s) \geq 0$, that

$$\liminf_k \frac{\widehat{G}_+(x,\varrho_k v_k)}{\varrho_k^2} < +\infty \qquad \text{a.e. in } \Omega\,.$$

From (g_4) it follows that $v \leq 0$ a.e. in Ω, whence

$$\liminf_k \left(\frac{1}{2}\int_\Omega (a\nabla v_k)\cdot\nabla v_k\,dx\right) \leq \liminf_k \frac{I(\varrho_k v_k)}{\varrho_k^2} \leq 0\,,$$

in contradiction with $\|\nabla v_k\|_2 = 1$.

Since $I(0) = 0 < +\infty$, the functional I admits a minimum point $u \in W_0^{1,2}(\Omega)$, which satisfies $\lambda u^+ \in \partial\widehat{J}_+(u)$ (see e.g. [29]), namely

$$\widehat{J}_+(v) \geq \widehat{J}_+(u) + \lambda\int_\Omega u^+(v-u)\,dx \qquad \forall v \in W_0^{1,2}(\Omega)\,.$$

The choice $v = u^+$ yields

$$\frac{1}{2}\int_\Omega (a\nabla u^-)\cdot\nabla u^-\,dx \leq 0\,,$$

whence $u \geq 0$ a.e. in Ω. Therefore, we also have $\lambda u \in \partial\widehat{J}(u)$ and from Theorem 1.3.2 we infer that $u_0 + u$ is a solution of (2.1.1) with $u_0 \leq u_0 + u$ a.e. in Ω.

Now let us show that $I(u) < 0$, provided that λ is large enough, so that $u_0 + u$ is different from u_0. By Theorem 1.2.3 there exists $v \in W_0^{1,2}(\Omega)\setminus\{0\}$ with $v \geq 0$ a.e. in Ω and $\widehat{G}_+(x,v) \in L^1(\Omega)$. Then it is clear that

$$I(u) \leq I(v) = \frac{1}{2}\int_\Omega (a\nabla v)\cdot\nabla v\,dx + \int_\Omega \widehat{G}_+(x,v)\,dx - \frac{\lambda}{2}\int_\Omega (v^+)^2\,dx < 0\,,$$

provided that λ is large enough.

If we apply we same argument to $\hat{g}_-(x,s) = \hat{g}(x,-s^-)$, we find another solution u_1 different from u_0 with $u_1 \leq u_0$ a.e. in Ω. ■

Under further assumptions on g, an estimate of $\overline{\lambda}$ can be provided.

Theorem 2.1.3 *Assume also that $\{s \mapsto g(x,s)\}$ is of class C^1 for a.e. $x \in \Omega$ and that the Carathéodory function $D_s g$ satisfies (g_3). Then*

$$\lambda_1 := \inf\left\{\int_\Omega (a\nabla v)\cdot\nabla v\,dx + \int_\Omega D_s g(x,u_0)v^2\,dx \;:\; v \in W_0^{1,2}(\Omega)\,,\ \int_\Omega v^2\,dx = 1\right\} < +\infty$$

and, for every $\lambda > \lambda_1$, problem (2.1.1) admits at least two other different solutions u_1 and u_2 with $u_1 \leq u_0 \leq u_2$ a.e. in Ω.

Proof. By Proposition 1.2.2 and Theorem 1.2.3, there exists $v \in W_0^{1,2}(\Omega) \setminus \{0\}$ such that $D_s g(x, u_0)\, v^2 \in L^1(\Omega)$, whence $\lambda_1 < +\infty$. Then it is standard that the infimum which defines λ_1 is achieved. Let $\varphi \in W_0^{1,2}(\Omega) \setminus \{0\}$ be such that

$$\int_\Omega (a\nabla\varphi) \cdot \nabla\varphi \, dx + \int_\Omega D_s g(x, u_0)\varphi^2 \, dx = \lambda_1 \int_\Omega \varphi^2 \, dx \, .$$

By substituting φ with $|\varphi|$, we may assume that $\varphi \geq 0$ a.e. in Ω and, by choosing a suitable representative, that φ is cap_2-quasi continuous.

Now let $\lambda > \lambda_1$ and let $\hat{g}_+$, $\widehat{G}_+$, $\widehat{J}_+$ and I be as in the previous proof. We only have to show that there exists $v \in W_0^{1,2}(\Omega)$ with $I(v) < 0$.

Again by Proposition 1.2.2 and Theorem 1.2.3, there exists a sequence (φ_k) in $W_0^{1,2}(\Omega) \cap L^\infty_c(\Omega)$ converging to φ in $W_0^{1,2}(\Omega)$ with $0 \leq \varphi_k \leq \varphi$ and

$$\left(\sup_{|t| \leq 1} D_s g(x, t(|u_0| + \varphi))\right) \varphi_k^2 \in L^1(\Omega) \, .$$

Since $0 \leq D_s g(x, u_0)\varphi_k^2 \leq D_s g(x, u_0)\varphi^2$, by Lebesgue theorem there exists $k \in \mathbb{N}$ such that

$$\int_\Omega (a\nabla\varphi_k) \cdot \nabla\varphi_k \, dx + \int_\Omega D_s g(x, u_0)\varphi_k^2 \, dx < \lambda \int_\Omega \varphi_k^2 \, dx \, .$$

Since, for every $t \in]0,1[$, we have

$$0 \leq \frac{\widehat{G}_+(x, t\varphi_k)}{t^2} \leq \frac{1}{2} \left(\sup_{0<t<1} D_s g(x, u_0 + t\varphi_k)\right) \varphi_k^2 \leq \frac{1}{2} \left(\sup_{|t| \leq 1} D_s g(x, t(|u_0| + \varphi))\right) \varphi_k^2 \, ,$$

again by Lebesgue theorem we infer that

$$\lim_{t \to 0^+} \frac{\int_\Omega \widehat{G}_+(x, t\varphi_k) \, dx}{t^2} = \frac{1}{2} \int_\Omega D_s g(x, u_0)\, \varphi_k^2 \, dx \, ,$$

hence that

$$\lim_{t \to 0^+} \frac{I(t\varphi_k)}{t^2} = \frac{1}{2} \int_\Omega (a\nabla\varphi_k) \cdot \nabla\varphi_k \, dx + \frac{1}{2} \int_\Omega D_s g(x, u_0)\, \varphi_k^2 \, dx - \frac{\lambda}{2} \int_\Omega \varphi_k^2 \, dx < 0 \, .$$

For $t > 0$ small enough, we have $I(t\varphi_k) < 0$, whence the existence of $u_2 \geq u_0$.

Arguing on $\hat{g}_-(x, s) = \hat{g}(x, -s^-)$, one finds in a similar way $u_1 \leq u_0$. ∎

2 Bifurcation from trivial solutions

To avoid some technicalities, we will consider here a less general situation. More precisely, let Ω be a bounded open subset of $\mathbb{R}^n$, with $n \geq 2$, let $g : \mathbb{R} \to \mathbb{R}$ be a nondecreasing function of class C^1 with $g(0) = 0$ and let $\mu \in \mathcal{M}_b(\Omega)$. Assume that

$$\begin{cases} \text{there exists } u_0 \in L^1(\Omega) \text{ such that } g(u_0) \in L^1(\Omega) \text{ and} \\ \displaystyle -\int_\Omega u_0 \Delta v\,dx + \int_\Omega g(u_0)\,v\,dx = \int_\Omega v\,d\mu \\ \text{for every } v \in W_0^{1,2}(\Omega) \text{ with } \Delta v \in L^\infty(\Omega)\,, \end{cases}$$

so that (λ, u_0) is a solution of the problem

$$\text{(2.2.1)} \quad \begin{cases} u \in L^1(\Omega)\,, \quad g(u) \in L^1(\Omega)\,, \\ \displaystyle -\int_\Omega u\Delta v\,dx + \int_\Omega g(u)\,v\,dx = \lambda \int_\Omega (u - u_0)v\,dx + \int_\Omega v\,d\mu \\ \qquad \text{for every } v \in W_0^{1,2}(\Omega) \text{ with } \Delta v \in L^\infty(\Omega)\,, \end{cases}$$

for any $\lambda \in \mathbb{R}$.

As before, we set $G(s) = \int_0^s g(t)\,dt$ and define $\hat{g}, \widehat{G} : \Omega \times \mathbb{R} \to \mathbb{R}$ by

$$\begin{aligned} \hat{g}(x,s) &= g(u_0(x) + s) - g(u_0(x))\,, \\ \widehat{G}(x,s) &= \int_0^s \hat{g}(x,t)\,dt = G(u_0(x) + s) - G(u_0(x)) - g(u_0(x))\,s\,, \end{aligned}$$

and

$$\widehat{J} : W_0^{1,2}(\Omega) \to]-\infty, +\infty]$$

by

$$\widehat{J}(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\,dx + \int_\Omega \widehat{G}(x,u)\,dx\,.$$

Definition 2.2.2 *A real number $\hat{\lambda}$ is said to be of* bifurcation *for* (2.2.1) *if there exists a sequence (λ_h, w_h) of solutions of* (2.2.1) *with $w_h \neq u_0$ and $(\lambda_h, w_h) \to (\hat{\lambda}, u_0)$ in $\mathbb{R} \times L^1(\Omega)$.*

Theorem 2.2.3 *Let $\hat{\lambda}$ be a bifurcation value of* (2.2.1). *Then there exists $u \in W_0^{1,2}(\Omega) \setminus \{0\}$ such that $\sqrt{g'(u_0)}\,u \in L^2(\Omega)$ and*

$$\int_\Omega (\nabla u \cdot \nabla v + g'(u_0)uv)\,dx = \hat{\lambda}\int_\Omega uv\,dx$$
$$\textit{for every } v \in W_0^{1,2}(\Omega) \textit{ with } \sqrt{g'(u_0)}\,v \in L^2(\Omega)\,.$$

Proof. Let $u_h = w_h - u_0$, so that by Theorems 1.3.2 and 1.2.4 $u_h \in W_0^{1,2}(\Omega)$ satisfies

$$\int_\Omega \nabla u_h \cdot \nabla v\, dx + \int_\Omega \widehat{g}(x, u_h)v\, dx = \lambda_h \int_\Omega u_h v\, dx$$
$$\text{for every } v \in W_0^{1,2}(\Omega) \text{ with } \widehat{g}(x, u_h)v \in L^1(\Omega)\,.$$

By theorem (1.2.1) we have $\widehat{g}(x, u_h)u_h \in L^1(\Omega)$ and, as $u_h \to 0$ in $L^1(\Omega)$, also $\widehat{g}(x, u_h) \to 0$ in $L^1(\Omega)$, namely $g(w_h) \to g(u_0)$ in $L^1(\Omega)$.
From the definition of generalized solution, it follows that (w_h) is bounded in any $L^r(\Omega)$ with $r < \frac{n}{n-2}$, so that also (u_h) is bounded in any $L^r(\Omega)$ with $r < \frac{n}{n-2}$. From theorem (1.1.2), we infer, by a bootstrap argument, that $\nabla u_h \to 0$ in $L^2(\Omega)$.

Coming back to the equation

$$\int_\Omega \nabla u_h \cdot \nabla v\, dx + \int_\Omega \widehat{g}(x, u_h)v\, dx = \lambda_h \int_\Omega u_h v\, dx,$$

we set $\varrho_h = \|\nabla u_h\|_2$ and define $z_h = \dfrac{u_h}{\varrho_h}$.
Dividing both the sides of the previous equation by ϱ_h, we find

$$\int_\Omega \nabla z_h \cdot \nabla v\, dx + \int_\Omega \frac{\widehat{g}(x, \varrho_h z_h)}{\varrho_h}\, v\, dx = \lambda_h \int_\Omega z_h v\, dx,$$

for every $v \in W_0^{1,2}(\Omega)$ with $\widehat{g}(x, \varrho_h z_h)v \in L^1(\Omega)$.
Since z_h is bounded in $W_0^{1,2}(\Omega)$, up to a subsequence we have $z_h \rightharpoonup z$ in $W_0^{1,2}(\Omega)$ and

$$\int_\Omega |\nabla z_h|^2\, dx + \int_\Omega \frac{\widehat{g}(x, \varrho_h z_h)}{\varrho_h}\, z_h\, dx = \lambda_h \int_\Omega z_h^2\, dx,$$

whence

$$\lambda_h \int_\Omega z_h^2\, dx \geq 1$$

and, finally,

$$\hat{\lambda} \int_\Omega z^2\, dx \geq 1,$$

so that $z \neq 0$.
We also have by Fatou's lemma

$$\hat{\lambda} \int_\Omega z^2\, dx - 1 = \liminf_h \int_\Omega \frac{\widehat{g}(x, \varrho_h z_h)}{\varrho_h} z_h\, dx \geq \int_\Omega D_s\widehat{g}(x, 0) z^2\, dx\,,$$

whence $\sqrt{g'(u_0)}z = \sqrt{D_s\widehat{g}(x,0)}z \in L^2(\Omega)$.

Coming back to the equation satisfied by z_h, we introduce the function

$$\vartheta(s) = \begin{cases} 1 & \text{if } |s| \le 1, \\ 2 - |s| & \text{if } 1 < |s| < 2, \\ 0 & \text{if } |s| \ge 2, \end{cases}$$

and we test in $\vartheta\left(\frac{u_0}{k}\right) \cdot v \cdot \vartheta\left(u_h\right)$, with $v \in C_c^\infty(\Omega)$, which is strongly convergent to $\vartheta\left(\frac{u_0}{k}\right) v$ in $W_0^{1,2}(\Omega)$.
Since, from Lagrange theorem,

$$\frac{\widehat{g}(x, \varrho_h z_h)}{\varrho_h} = g'(u_0 + t_h \varrho_h z_h) z_h = g'(u_0 + t_h u_h) z_h,$$

with $0 < t_h < 1$, we have

$$\left| \frac{\widehat{g}(x, \varrho_h z_h)}{\varrho_h} \cdot \vartheta\left(\frac{u_0}{k}\right) \cdot v \cdot \vartheta(u_h) \right| \le \max_{|s| \le 2k+2} |g'(s)| \cdot |z_h| \cdot |v|$$

with $z_h \to z$ in $L^2(\Omega)$.
Passing to the limit as $h \to \infty$, we deduce that

$$\int_\Omega \nabla z \cdot \nabla \left[\vartheta\left(\frac{u_0}{k}\right) v\right] \, dx + \int_\Omega g'(u_0) z \vartheta\left(\frac{u_0}{k}\right) v \, dx = \hat{\lambda} \int_\Omega z \vartheta\left(\frac{u_0}{k}\right) v \, dx,$$

for every $v \in C_c^\infty(\Omega)$. An easy density argument shows that then we can take any $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$.
In particular, if $v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega)$ with $\sqrt{g'(u_0)}v \in L^2(\Omega)$, by (1.1.1) we can pass to the limit as $k \to \infty$, obtaining

$$\int_\Omega \nabla z \cdot \nabla v \, dx + \int_\Omega g'(u_0) z v \, dx = \hat{\lambda} \int_\Omega z v \, dx$$
$$\text{for every } v \in W_0^{1,2}(\Omega) \cap L^\infty(\Omega) \text{ with } \sqrt{g'(u_0)}\, v \in L^2(\Omega)\,.$$

Finally, given $v \in W_0^{1,2}(\Omega)$ with $\sqrt{g'(u_0)}\, v \in L^2(\Omega)$, consider $v_i = T_i(v)$.

Testing the previous equation in v_i and passing to the limit as $i \to \infty$, we obtain

$$\int_\Omega \nabla z \cdot \nabla v \, dx + \int_\Omega g'(u_0) z v \, dx = \hat{\lambda} \int_\Omega z v \, dx$$
$$\text{for every } v \in W_0^{1,2}(\Omega) \text{ with } \sqrt{g'(u_0)}\, v \in L^2(\Omega)$$

and the proof is complete. ∎

The previous result justifies the next notion.

Definition 2.2.4 *A real number $\hat{\lambda}$ is said to be an* eigenvalue *of the linearized problem*

$$\begin{cases} -\Delta u + g'(u_0) u = \lambda u & \text{in } \Omega\,, \\ u = 0 & \text{on } \partial\Omega\,, \end{cases} \tag{2.2.5}$$

if there exists $u \in W_0^{1,2}(\Omega) \setminus \{0\}$ such that $\sqrt{g'(u_0)}\, u \in L^2(\Omega)$ and

$$\int_\Omega \left(\nabla u \cdot \nabla v + g'(u_0) u v\right) dx = \hat{\lambda} \int_\Omega u v \, dx$$
$$\text{for every } v \in W_0^{1,2}(\Omega) \text{ with } \sqrt{g'(u_0)}\, v \in L^2(\Omega)\,.$$

Our main result is an adaptation to our setting of a celebrated bifurcation theorem of Rabinowitz (see e.g. [25, Theorem 11.35]).

Theorem 2.2.6 *Let $\hat{\lambda}$ be an eigenvalue of (2.2.5). Then one at least of the following assertions hold:*

(i) $(\hat{\lambda}, u_0)$ is not an isolated solution of (2.2.1) in $\left\{\hat{\lambda}\right\} \times L^1(\Omega)$;

(ii) for every $\lambda \neq \hat{\lambda}$ in a neighborhood of $\hat{\lambda}$ there is a nontrivial solution (λ, u_λ) of (2.2.1) with u_λ converging to u_0 in $L^1(\Omega)$ as $\lambda \to \hat{\lambda}$;

(iii) there is a one-sided (right or left) neighborhood of $\hat{\lambda}$ such that for every $\lambda \neq \hat{\lambda}$ in the neighborhood there are two distinct nontrivial solutions $(\lambda, u_\lambda^{(1)})$ and $(\lambda, u_\lambda^{(2)})$ of (2.2.1) with $u_\lambda^{(j)}$ converging to u_0 in $L^1(\Omega)$ as $\lambda \to \hat{\lambda}$.

To prove this result we observe that, given $\lambda \in \mathbb{R}$, by Theorem 1.3.2 we have that u is a solution of (2.2.1) if and only if $z = u - u_0$ satisfies

$$\begin{cases} z \in W_0^{1,2}(\Omega)\,, \\ \widehat{J}(v) \geq \widehat{J}(z) + \lambda \displaystyle\int_\Omega z(v - z)\, dx & \text{for every } v \in W_0^{1,2}(\Omega)\,. \end{cases} \tag{2.2.7}$$

Observe also that $(\lambda, 0)$ is a solution of (2.2.7) for any $\lambda \in \mathbb{R}$ and that

$$D_s\hat{g}(x,s) = g'(u_0(x)+s)\,.$$

Consider the space H defined as

$$H = \left\{u \in W_0^{1,2}(\Omega):\ \sqrt{D_s\widehat{g}(x,0)}u \in L^2(\Omega)\right\} \subseteq W_0^{1,2}(\Omega). \tag{2.2.8}$$

It is easily seen that H is a Hilbert space with respect to the scalar product

$$(u|v)_H := \int_\Omega \nabla u \cdot \nabla v dx + \int_\Omega D_s\widehat{g}(x,0)uv\,dx\,,$$

while

$$\left\{u \mapsto \int_\Omega u^2\,dx\right\}$$

is a smooth quadratic form on H with compact gradient.

Since $\hat{\lambda}$ is an eigenvalue of (2.2.5), there exist three linear subspaces H_-, H_0 and H_+ of H such that:

(a) we have

$$H = H_- \oplus H_0 \oplus H_+ \subseteq W_0^{1,2}(\Omega)$$

with $\dim H_- < \infty$, $1 \leq \dim H_0 < \infty$, and the decomposition is orthogonal with respect to both the scalar product of $L^2(\Omega)$ and the scalar product $(\ |\ )_H$;

(b) there exist $\underline{\lambda} < \hat{\lambda} < \overline{\lambda}$ such that

$$\begin{aligned}
&\int_\Omega |\nabla v|^2 + D_s\widehat{g}(x,0)v^2\,dx \quad\ \leq \underline{\lambda}\int_\Omega v^2\,dx && \forall v \in H_-\,,\\
&\int_\Omega \nabla u\cdot\nabla v + D_s\widehat{g}(x,0)uv\,dx = \hat{\lambda}\int_\Omega uv\,dx && \forall u \in H_0\,,\ \forall v \in H\,,\\
&\int_\Omega |\nabla w|^2 + D_s\widehat{g}(x,0)w^2\,dx \quad \geq \overline{\lambda}\int_\Omega w^2\,dx && \forall w \in H_+\,.
\end{aligned}$$

Since $D_s\widehat{g}(x,0) \geq 0$, by standard regularity results, we have $H_- \oplus H_0 \subseteq L^\infty(\Omega)$. We set

$$\widehat{Y} := \left\{u \in L^1(\Omega):\ \int_\Omega uv\,dx = 0 \text{ for every } v \in H_- \oplus H_0\right\}.$$

Then $H_+ \subseteq \widehat{Y}$, $\widehat{Y}$ is closed in $L^1(\Omega)$ and we have

$$L^1(\Omega) = H_- \oplus H_0 \oplus \widehat{Y}\,.$$

Let $\widehat{P} : L^1(\Omega) \to H_- \oplus H_0$ the associated projection.
We also have

$$W_0^{1,2}(\Omega) = H_- \oplus H_0 \oplus Y\,,$$

where $Y = \widehat{Y} \cap W_0^{1,2}(\Omega)$, and $P = \widehat{P}\big|_{W_0^{1,2}(\Omega)}$: $W_0^{1,2}(\Omega) \to H_- \oplus H_0$ is the associated projection, which is continuous with respect to the $L^1(\Omega)$ topology.

Given $\lambda \in \mathbb{R}$, introduce the functional $\widehat{I}_\lambda : W_0^{1,2}(\Omega) \to]-\infty, +\infty]$ defined as

$$\widehat{I}_\lambda(u) = \frac{1}{2}\int_\Omega |\nabla u|^2\, dx + \int_\Omega \widehat{G}(x,u)\, dx - \frac{\lambda}{2}\int_\Omega u^2\, dx = \widehat{J}(u) - \frac{\lambda}{2}\int_\Omega u^2\, dx.$$

We also set

$$D(r_1, r_2) = \left\{u \in W_0^{1,2}(\Omega) : \|\nabla(Pu)\|_2 \le r_1,\ \|\nabla(u - Pu)\|_2 \le r_2\right\}$$

Lemma 2.2.9 *There exists $r_+ > 0$ and $\varepsilon > 0$ such that*

$$\widehat{I}_\lambda\left(\frac{1}{2}w_0 + \frac{1}{2}w_1\right)$$
$$\le \frac{1}{2}\widehat{I}_\lambda(w_0) + \frac{1}{2}\widehat{I}_\lambda(w_1) - \varepsilon\,\|\nabla(w_0 - Pw_0) - \nabla(w_1 - Pw_1)\|_2^2 + \frac{1}{\varepsilon}\,\|(Pw_0) - (Pw_1)\|_2^2$$

whenever $\left|\lambda - \hat{\lambda}\right| \le r_+$ and $w_0, w_1 \in D(r_+, r_+)$.

Proof. By contradiction, let's consider $w_{0,k}$ and $w_{1,k}$ such that $w_{0,k}, w_{1,k} \to 0$ in $W_0^{1,2}(\Omega)$ and $\lambda_k \to \hat{\lambda}$ such that

$$\widehat{I}_{\lambda_k}\left(\frac{1}{2}w_{0,k} + \frac{1}{2}w_{1,k}\right) > \frac{1}{2}\widehat{I}_{\lambda_k}(w_{0,k}) + \frac{1}{2}\widehat{I}_{\lambda_k}(w_{1,k}) +$$
$$-\frac{1}{k}\,\|\nabla(w_{0,k} - Pw_{0,k}) - \nabla(w_{1,k} - Pw_{1,k})\|_2^2 + k\,\|(Pw_{0,k}) - (Pw_{1,k})\|_2^2\,.$$

Let us set

$$u_k = \frac{1}{2}w_{0,k} + \frac{1}{2}w_{1,k}\,,$$

$$v_k = \frac{1}{2}(w_{1,k} - w_{0,k})\,,$$

so that

$$\hat{I}_{\lambda_k}(u_k) > \frac{1}{2}\hat{I}_{\lambda_k}(u_k - v_k) + \frac{1}{2}\hat{I}_{\lambda_k}(u_k + v_k) - \frac{4}{k}\left\|\nabla(v_k - Pv_k)\right\|_2^2 + 4k\left\|Pv_k\right\|_2^2,$$

namely

$$\int_\Omega \widehat{G}(x,u_k)\,dx > \frac{1}{2}\int_\Omega \widehat{G}(x,u_k - v_k)\,dx + \frac{1}{2}\int_\Omega \widehat{G}(x,u_k + v_k)\,dx + \left(\frac{1}{2} - \frac{4}{k}\right)\left\|\nabla v_k\right\|_2^2$$

$$-\frac{4}{k}\left\|\nabla P v_k\right\|_2^2 + \frac{8}{k}(\nabla v_k|\nabla P v_k)_2 - \frac{\lambda_k}{2}\int_\Omega v_k^2\,dx + 4k\left\|Pv_k\right\|_2^2.$$

Introduced $\varrho_k = \|\nabla v_k\|_2$ and $z_k = \frac{v_k}{\varrho_k}$, up to a subsequence we have $z_k \rightharpoonup z$ in $W_0^{1,2}(\Omega)$. Dividing both the sides by $\dfrac{1}{2}\varrho_k^2$, from the convexity of $\widehat{G}(x,\cdot)$ we obtain

$$0 \ge \frac{\displaystyle\int_\Omega \widehat{G}(x,u_k)\,dx - \frac{1}{2}\int_\Omega \widehat{G}(x,u_k - \varrho_k z_k)\,dx - \frac{1}{2}\int_\Omega \widehat{G}(x,u_k + \varrho_k z_k)\,dx}{\frac{1}{2}\varrho_k^2} >$$

$$> \left(1 - \frac{8}{k}\right) - \frac{8}{k}\left\|\nabla P z_k\right\|_2^2 + \frac{16}{k}\left(\nabla z_k|\nabla P z_k\right)_2 - \lambda_k\|z_k\|_2^2 + 8k\left\|Pz_k\right\|_2^2.$$

First of all it follows that $Pz_k \to 0$ and, since $Pz_k \to Pz$, we infer that $Pz = 0$, namely $z \in Y$.

From the inequality

$$0 \ge \frac{\displaystyle\int_\Omega \widehat{G}(x,u_k)\,dx - \frac{1}{2}\int_\Omega \widehat{G}(x,u_k - \varrho_k z_k)\,dx - \frac{1}{2}\int_\Omega \widehat{G}(x,u_k + \varrho_k z_k)\,dx}{\frac{1}{2}\varrho_k^2} >$$

$$> \left(1 - \frac{8}{k}\right) - \frac{8}{k}\left\|\nabla P z_k\right\|_2^2 + \frac{16}{k}\left(\nabla z_k|\nabla P z_k\right)_2 - \lambda_k\|z_k\|_2^2$$

and from Fatou's lemma and De l'Hopital theorem, we have

$$-\int_\Omega D_s\hat{g}(x,0)z^2\,dx \ge 1 - \hat{\lambda}\int_\Omega z^2\,dx \ge \int_\Omega |\nabla z|^2\,dx - \hat{\lambda}\int_\Omega z^2\,dx.$$

Then

$$\int_\Omega |\nabla z|^2 \, dx + \int_\Omega D_s\hat{g}(x,0)z^2 \, dx \le \hat{\lambda} \int_\Omega z^2 \, dx \, .$$

On the other hand, since $z \in Y \setminus \{0\}$, we have

$$\int_\Omega |\nabla z|^2 \, dx + \int_\Omega D_s\hat{g}(x,0)z^2 \, dx \ge \overline{\lambda} \int_\Omega z^2 \, dx \, ,$$

whence

$$\overline{\lambda} \le \hat{\lambda},$$

that is an absurd. ■

Lemma 2.2.10 *There exist* $r_+ > 0$ *and* $\varepsilon > 0$ *such that, for every* $\lambda \in \mathbb{R}$ *with* $\left|\lambda - \hat{\lambda}\right| \le r_+$*, the functional*

$$\left\{u \mapsto \widehat{I}_\lambda(u) - \varepsilon \left\|\nabla(u - Pu)\right\|_2^2 + \frac{1}{\varepsilon} \left\|Pu\right\|_2^2\right\}$$

is convex on $D(r_+, r_+)$.

Proof. Since the functional is lower semicontinuous, it is enough to verify convexity on convex combinations $(1-t)w_0 + tw_1$ with $t = m2^{-n}$. Then the assertion follows from lemma 2.2.9. ■

It follows that, for every $\lambda \in \mathbb{R}$ with $\left|\lambda - \hat{\lambda}\right| \le r_+$ and every $v \in H_- \oplus H_0$ with $\|\nabla v\|_2 \le r_+$, there exists one and only one minimum $\psi_\lambda(v)$ of $\left\{w \mapsto \widehat{I}_\lambda(v+w)\right\}$ on $\{w \in Y : \ \|\nabla w\|_2 \le r_+\}$. Moreover, we have $\psi_\lambda(0) = 0$. We set also

$$\varphi_\lambda(v) := \widehat{I}_\lambda(v + \psi_\lambda(v)) = \min\left\{\widehat{I}_\lambda(v+w) : \ w \in Y \, , \ \|\nabla w\|_2 \le r_+\right\} \, .$$

To investigate the properties of ψ_λ and φ_λ, we introduce an auxiliary decomposition, with better properties of the finite dimensional part at the expenses of the orthogonality of the decomposition itself.

Let $\{e_1, e_2, ..., e_m\}$ be a base of H_- and $e_{m+1}, ..., e_k$ a base of H_0.
Introduce the spaces

$$H_-^h, \; H_0^h,$$

defined as:

$$H_-^h = \text{span}\left\{\vartheta\left(\frac{u_0}{h}\right)e_1, \ldots, \vartheta\left(\frac{u_0}{h}\right)e_m\right\},$$
$$H_0^h = \text{span}\left\{\vartheta\left(\frac{u_0}{h}\right)e_{m+1}, \ldots, \vartheta\left(\frac{u_0}{h}\right)e_k\right\}.$$

Taking into account (1.1.1), it is easily seen that $\left\|\vartheta\left(\frac{u_0}{h}\right)e_j - e_j\right\|_H \to 0$ as $h \to +\infty$. Therefore $H_-^h \oplus H_0^h$ is a finite dimensional subspace of $H \cap L^\infty(\Omega)$ and, if h is large enough, we have

$$L^1(\Omega) = H_-^h \oplus H_0^h \oplus \widehat{Y}\,,$$

$$W_0^{1,2}(\Omega) = H_-^h \oplus H_0^h \oplus Y\,,$$

$$H = H_-^h \oplus H_0^h \oplus H_+\,.$$

Accordingly, we denote by $\widetilde{P} : W_0^{1,2}(\Omega) \to H_-^h \oplus H_0^h$ the associated projection, which is again continuous with respect to the $L^1(\Omega)$ topology.

The advantage is that, for every $v \in H_-^h \oplus H_0^h$, we have $|u_0(x)| \le 2h$ where $v(x) \neq 0$.

Lemma 2.2.11 *There exists $r_- \in]0, r_+]$ such that*

$$\widehat{I}_\lambda(u) > \widehat{I}_\lambda(z)$$

whenever $\left|\lambda - \hat{\lambda}\right| \le r_+$, and $u, z \in D(r_-, r_+)$ with $\|\nabla(u - Pu)\|_2 = r_+$ and $z \in H_-^h \oplus H_0^h$. In particular, we have $\|\nabla(z - Pz)\|_2 < r_+$ and $\|\nabla\psi_\lambda(v)\|_2 < r_+$ whenever $\|\nabla v\|_2 \le r_-$.

Proof. By contradiction, consider u_k with $Pu_k \to 0$ and $\|\nabla(u_k - Pu_k)\|_2 = r_+$, $z_k \in H_-^h \oplus H_0^h$ with $Pz_k \to 0$ and $\lambda_k \to \hat{\lambda}$ such that

$$\widehat{I}_{\lambda_k}(u_k) \le \widehat{I}_{\lambda_k}(z_k).$$

Up to a subsequence, $z_k \to z$ and $u_k \rightharpoonup u$. It follows $Pz = 0$ namely $z \in Y$, whence $z = 0$. Therefore, we have $z_k \to 0$. Since $|u_0(x)| \le 2h$ where $z_k(x) \neq 0$, it follows that $\widehat{I}_{\lambda_k}(z_k) \to 0$.
Moreover, $u \in Y$ and $\|\nabla u\|_2 \le r_+$.

Passing to the lower limit in $\widehat{I}_{\lambda_k}(u_k) \le \widehat{I}_{\lambda_k}(z_k)$, we obtain $\widehat{I}_{\hat\lambda}(u) \le 0$, hence, from the strict convexity on Y, $u = 0$.
Since $\widehat{G}(x,s) \ge 0$, it easily follows that

$$\limsup_k \int_\Omega |\nabla u_k|^2\,dx \le 0,$$

so that $u_k \to 0$, that is an absurd. ■

Now we set

$$U = \{v \in H_- \oplus H_0 : \ \|\nabla v\|_2 < r_-\} \,.$$

Theorem 2.2.12 *For every $\lambda \in \mathbb{R}$ with $\left|\lambda - \hat\lambda\right| \le r_+$ and every $v \in U$, we have*

$$\psi_\lambda(v) \in L^\infty(\Omega)\,,$$
$$\widehat{g}(x, v + \psi_\lambda(v)) \in L^1(\Omega)$$

and

$$\int_\Omega \nabla(v + \psi_\lambda(v)) \cdot \nabla w\,dx + \int_\Omega \widehat{g}(x, v+\psi_\lambda(v)) w\,dx = \lambda \int_\Omega (v + \psi_\lambda(v)) w\,dx$$
$$\text{for any } w \in Y \text{ with } \widehat{g}(x, v + \psi_\lambda(v)) w \in L^1(\Omega)\,.$$

Moreover, $\|\psi_\lambda(v)\|_\infty$ is bounded by a uniform constant and the function φ_λ is of class C^1 on U with

$$\langle \varphi'_\lambda(z), v\rangle = \int_\Omega \nabla(z + \psi_\lambda(z)) \cdot \nabla v\,dx + \int_\Omega \widehat{g}\,(x, z + \psi_\lambda(z))\, v\,dx - \lambda \int_\Omega (z + \psi_\lambda(z))\, v\,dx.$$

In particular, $\varphi'_\lambda(0) = 0$.

Proof. We set

$$\check{I}_\lambda(u) = \begin{cases} \widehat{I}_\lambda(u) + \frac{1}{\varepsilon}\|Pu\|_2^2 & \text{if } u \in D(r_-, r_+)\,, \\ +\infty & \text{otherwise}\,, \end{cases}$$

$$\check{\varphi}_\lambda(v) = \min_{v+Y} \check{I}_\lambda = \check{I}_\lambda(v + \psi_\lambda(v))\,,$$

so that $\check{I}_\lambda$ is convex by lemma 2.2.10 and

$$\varphi_\lambda(v) = \check{\varphi}_\lambda(v) - \frac{1}{\varepsilon}\|v\|_2^2\,.$$

Moreover, $\check{\varphi}_\lambda$ is finite by lemma 2.2.11, so that by theorem 1.4.1 $\check{\varphi}_\lambda\big|_U$ is convex and locally Lipschitz with $\partial\check{\varphi}_\lambda(v) \neq \emptyset$ for any $v \in U$. If $\alpha \in (H_- \oplus H_0)'$, for every $u \in W_0^{1,2}(\Omega)$ we have

$$|\langle P'\alpha, u\rangle| = |\langle \alpha, Pu\rangle| \leq \|\alpha\|\|Pu\| \leq C\|\alpha\|\|u\|_1 \,.$$

It follows that $P'\alpha \in L^\infty(\Omega)$ with

$$\|P'\alpha\|_\infty \leq C\|\alpha\| \,.$$

If $\alpha \in \partial\check{\varphi}_\lambda(v)$, we have $P'\alpha \in \partial\check{I}_\lambda\,(v + \psi_\lambda(v))$, hence

$$P'\alpha \in \partial\left\{u \mapsto \widehat{I}_\lambda(u) + \frac{1}{\varepsilon}\,\|Pu\|_2^2\right\}_{u=v+\psi_\lambda(v)} \,,$$

as $\|\nabla v\|_2 < r_-$ and $\|\nabla\psi_\lambda(v)\|_2 < r_+$. From theorems 1.2.1 and 1.2.4 we infer that $\widehat{g}(x, v + \psi_\lambda(v)) \in L^1(\Omega)$ and

$$\int_\Omega \nabla(v + \psi_\lambda(v)) \cdot \nabla w\, dx + \int_\Omega \widehat{g}(x, v + \psi_\lambda(v))w\, dx = \lambda \int_\Omega (v + \psi_\lambda(v))w\, dx$$
$$+ \langle \alpha, Pw\rangle - \frac{2}{\varepsilon}\,(v|Pw)_2 \qquad \text{for any } w \in W_0^{1,2}(\Omega) \text{ with } \widehat{g}(x, v + \psi_\lambda(v))w \in L^1(\Omega) \,.$$

whence

$$\int_\Omega \nabla(v + \psi_\lambda(v)) \cdot \nabla w\, dx + \int_\Omega \widehat{g}(x, v + \psi_\lambda(v))w\, dx = \lambda \int_\Omega (v + \psi_\lambda(v))w\, dx$$
$$\text{for any } w \in Y \text{ with } \widehat{g}(x, v + \psi_\lambda(v))w \in L^1(\Omega) \,.$$

Moreover, we have $(v + \psi_\lambda(v)) \in L^\infty(\Omega)$, hence $\psi_\lambda(v) \in L^\infty(\Omega)$, by theorem (1.1.2). Since $\partial\widehat{I}_\lambda\,(v + \psi_\lambda(v))$ contains at most one element by theorem 1.2.4, also $\partial\check{I}_\lambda\,(v + \psi_\lambda(v))$ does the same.

From the injectivity of the map $P' : (H_- \oplus H_0)' \to W^{-1,2}(\Omega)$, it follows that also $\partial\check{\varphi}_\lambda(v)$ contains at most one element.

We deduce from theorem (1.4.1) that $\check{\varphi}_\lambda$ is of class C^1, so that also φ_λ is of class C^1.

In particular we have

$$\langle \varphi_\lambda'(z), v\rangle = \langle \check{\varphi}_\lambda'(z), v\rangle - \frac{2}{\varepsilon}(z|v)_2,$$

i.e.

$$\langle \varphi_\lambda'(z), v\rangle = \int_\Omega \nabla(z + \psi_\lambda(z))\nabla v\, dx + \int_\Omega \widehat{g}\,(x, z + \psi_\lambda(z))\, v\, dx - \lambda \int_\Omega (z + \psi_\lambda(z))\, v\, dx.$$

■

We set

$$\widetilde{U} = \left\{ v \in H^h_- \oplus H^h_0 : \ \|\nabla P v\|_2 < r_- \right\}$$

and we define $\widetilde{\psi}_\lambda : \widetilde{U} \to Y \cap L^\infty(\Omega)$ as $\widetilde{\psi}_\lambda(v) := \psi_\lambda(Pv) - (v - Pv)$. It holds $v + \widetilde{\psi}_\lambda(v) = Pv + \psi_\lambda(Pv)$.

Theorem 2.2.13 *The map $\{(\lambda, v) \mapsto \widetilde{\psi}_\lambda(v)\}$ is continuous and the map $\widetilde{\psi}_\lambda$ is Lipschitz continuous uniformly with respect to λ, when Y is endowed with the $W^{1,2}_0(\Omega)$ metric.*

Proof. We have

$$\begin{aligned}\int_\Omega \nabla\left(z + \widetilde{\psi}_\lambda(z)\right) \cdot \nabla\left(\widetilde{\psi}_\lambda(z+v) - \widetilde{\psi}_\lambda(z)\right)\, dx \\ + \int_\Omega \widehat{g}\left(x, z + \widetilde{\psi}_\lambda(z)\right)\left(\widetilde{\psi}_\lambda(z+v) - \widetilde{\psi}_\lambda(z)\right)\, dx \\ = \lambda \int_\Omega \left(z + \widetilde{\psi}_\lambda(z)\right)\left(\widetilde{\psi}_\lambda(z+v) - \widetilde{\psi}_\lambda(z)\right)\, dx\end{aligned}$$

and

$$\begin{aligned}\int_\Omega \nabla\left(z + v + \widetilde{\psi}_\lambda(z+v)\right) \cdot \nabla\left(\widetilde{\psi}_\lambda(z+v) - \widetilde{\psi}_\lambda(z)\right)\, dx \\ + \int_\Omega \widehat{g}\left(x, z + v + \widetilde{\psi}_\lambda(z+v)\right)\left(\widetilde{\psi}_\lambda(z+v) - \widetilde{\psi}_\lambda(z)\right)\, dx \\ = \lambda \int_\Omega \left(z + v + \widetilde{\psi}_\lambda(z+v)\right)\left(\widetilde{\psi}_\lambda(z+v) - \widetilde{\psi}_\lambda(z)\right)\, dx\,.\end{aligned}$$

We deduce that

$$\begin{aligned}\int_\Omega \nabla v \cdot \nabla\left[\widetilde{\psi}_\lambda(z+v) - \widetilde{\psi}_\lambda(z)\right]\, dx + \int_\Omega \left|\nabla\left(\widetilde{\psi}_\lambda(z+v) - \widetilde{\psi}_\lambda(z)\right)\right|^2\, dx \\ + \int_\Omega \left[\widehat{g}\left(x, z + v + \widetilde{\psi}_\lambda(z+v)\right) - \widehat{g}\left(x, z + \widetilde{\psi}_\lambda(z)\right)\right]\left(\widetilde{\psi}_\lambda(z+v) - \widetilde{\psi}_\lambda(z)\right)\, dx \\ = \lambda \int_\Omega v\left(\widetilde{\psi}_\lambda(z+v) - \widetilde{\psi}_\lambda(z)\right)\, dx + \lambda \int_\Omega \left(\widetilde{\psi}_\lambda(z+v) - \widetilde{\psi}_\lambda(z)\right)^2.\end{aligned}$$

By lemma 2.2.10 we obtain

$$
\begin{aligned}
&\lambda \int_\Omega v \left(\widetilde{\psi}_\lambda(z+v) - \widetilde{\psi}_\lambda(z)\right) dx - \int_\Omega \nabla v \cdot \nabla \left(\widetilde{\psi}_\lambda(z+v) - \widetilde{\psi}_\lambda(z)\right) dx \\
&- \int_\Omega \left[\widehat{g}\left(x, z+v+\widetilde{\psi}_\lambda(z)\right) - \widehat{g}\left(x, z+\widetilde{\psi}_\lambda(z)\right)\right] \left(\widetilde{\psi}_\lambda(z+v) - \widetilde{\psi}_\lambda(z)\right) dx = \\
&= \int_\Omega \left|\nabla \left(\widetilde{\psi}_\lambda(z+v) - \widetilde{\psi}_\lambda(z)\right)\right|^2 dx \\
&+ \int_\Omega \left(\widehat{g}\left(x, z+v+\widetilde{\psi}_\lambda(z+v)\right) - \widehat{g}\left(x, z+v+\widetilde{\psi}_\lambda(z)\right)\right) \left(\widetilde{\psi}_\lambda(z+v) - \widetilde{\psi}_\lambda(z)\right) dx \\
&-\lambda \int_\Omega \left(\widetilde{\psi}_\lambda(z+v) - \widetilde{\psi}_\lambda(z)\right)^2 dx \geq 2\varepsilon \left\|\nabla \left(\widetilde{\psi}_\lambda(z+v) - \widetilde{\psi}_\lambda(z)\right)\right\|_2^2 .
\end{aligned}
$$

There exists $0 < \sigma < 1$ such that

$$
\begin{aligned}
&\widehat{g}\left(x, z+v+\widetilde{\psi}_\lambda(z)\right) - \widehat{g}\left(x, z+\widetilde{\psi}_\lambda(z)\right) \\
&= g\left(u_0+z+v+\widetilde{\psi}_\lambda(z)\right) - g\left(u_0+z+\widetilde{\psi}_\lambda(z)\right) = g'\left(u_0+z+\widetilde{\psi}_\lambda(z)+\sigma v\right) v \,,
\end{aligned}
$$

whence

$$
\begin{aligned}
&\left|\widehat{g}\left(x, z+v+\widetilde{\psi}_\lambda(z)\right) - \widehat{g}\left(x, z+\widetilde{\psi}_\lambda(z)\right)\right| \\
&\qquad \leq \max\{g'(s) :\ |s| \leq 2h + \|z+\widetilde{\psi}_\lambda(z)\|_\infty + \|v\|_\infty\}|v| \,.
\end{aligned}
$$

It follows

$$
2\varepsilon \left\|\nabla \left(\widetilde{\psi}_\lambda(z+v) - \widetilde{\psi}_\lambda(z)\right)\right\|_2^2 \leq C \left\|\nabla \left(\widetilde{\psi}_\lambda(z+v) - \widetilde{\psi}_\lambda(z)\right)\right\|_2 \|\nabla v\|_2 \,,
$$

so that the map $\widetilde{\psi}_\lambda$ is Lipschitz continuous.

Now, to prove that the map $\{(\lambda, v) \mapsto \widetilde{\psi}_\lambda(v)\}$ is continuous, it is enough to show that $\{\lambda \mapsto \widetilde{\psi}_\lambda(v)\}$ is continuous for any v, which is easy to verify. ■

Given $z \in \widetilde{U}$ and $v \in H_-^h \oplus H_0^h$, we have

$$
D_s\widehat{g}(x, z+\widetilde{\psi}_\lambda(z))v^2 \in L^1(\Omega)
$$

and there is one and only one η in Y with $D_s\widehat{g}(x, z+\widetilde{\psi}_\lambda(z))\eta^2 \in L^1(\Omega)$ and

$$
\begin{aligned}
&\int_\Omega \nabla(v+\eta) \cdot \nabla w \, dx + \int_\Omega D_s\hat{g}(x, z+\widetilde{\psi}_\lambda(z))(v+\eta)w \, dx \\
&\qquad = \lambda \int_\Omega (v+\eta)w \, dx \quad \text{for any } w \in Y \text{ with } D_s\widehat{g}(x, z+\widetilde{\psi}_\lambda(z))w^2 \in L^1(\Omega) \,,
\end{aligned}
$$

as

$$\int_\Omega \nabla\eta \cdot \nabla w\, dx + \int_\Omega D_s\hat{g}(x, z + \widetilde{\psi}_\lambda(z))\eta w\, dx - \lambda \int_\Omega \eta w\, dx$$

is a Hilbert scalar product on

$$\left\{ w \in Y : \; D_s\widehat{g}(x, z + \widetilde{\psi}_\lambda(z))w^2 \in L^1(\Omega) \right\} .$$

Moreover, the map $\{v \mapsto \eta\}$ is linear and continuous from $H_-^h \oplus H_0^h$ into $W_0^{1,2}(\Omega)$. We set $L_z v = \eta$.

Theorem 2.2.14 *If (λ_k) is a sequence convergent to λ in $[\hat{\lambda} - r_+, \hat{\lambda} + r_+]$, (z_k) is a sequence convergent to z in $\widetilde{U}$ and (v_k) is a sequence convergent to* 0 *in $H_-^h \oplus H_0^h$, we have*

$$\lim_k \frac{\widetilde{\psi}_{\lambda_k}(z_k + v_k) - \widetilde{\psi}_{\lambda_k}(z_k) - L_z v_k}{\|v_k\|} = 0$$

in the weak topology of $W_0^{1,2}(\Omega)$.

Proof. Since $\widetilde{\psi}_{\lambda_k}$ is uniformly locally Lipschitz, we have that, up to a subsequence,

$$\frac{\widetilde{\psi}_{\lambda_k}(z_k + v_k) - \widetilde{\psi}_{\lambda_k}(z_k) - L_z v_k}{\|v_k\|} \rightharpoonup \xi$$

in the weak topology of $W_0^{1,2}(\Omega)$. We know that $\xi \in Y$ and we have to prove that $\xi = 0$.

If we set $\eta_k = L_z v_k$, for every $w \in Y$ with

$$\widehat{g}\left(x, z_k + v_k + \widetilde{\psi}_{\lambda_k}(z_k + v_k)\right) w \in L^1(\Omega)\,,\; \widehat{g}\left(x, z_k + \widetilde{\psi}_{\lambda_k}(z_k)\right) w \in L^1(\Omega)\,,$$
$$D_s\widehat{g}(x, z + \widetilde{\psi}_\lambda(z))w^2 \in L^1(\Omega)\,,$$

we have

$$\int_\Omega \nabla\left[z_k + v_k + \widetilde{\psi}_{\lambda_k}(z_k + v_k)\right] \cdot \nabla w\, dx + \int_\Omega \widehat{g}\left(x, z_k + v_k + \widetilde{\psi}_{\lambda_k}(z_k + v_k)\right) w\, dx$$
$$-\lambda_k \int_\Omega \left(z_k + v_k + \widetilde{\psi}_{\lambda_k}(z_k + v_k)\right) w\, dx = 0,$$

$$\int_\Omega \nabla\left[z_k + \widetilde{\psi}_{\lambda_k}(z_k)\right] \cdot \nabla w\, dx + \int_\Omega \widehat{g}\left(x, z_k + \widetilde{\psi}_{\lambda_k}(z_k)\right) w\, dx - \lambda_k \int_\Omega \left(z_k + \widetilde{\psi}_{\lambda_k}(z_k)\right) w\, dx = 0,$$

$$\int_\Omega \nabla (v_k + \eta_k) \cdot \nabla w \, dx + \int_\Omega D_s \widehat{g}\left(x, z + \widetilde{\psi}_\lambda(z)\right) (v_k + \eta_k) \, w \, dx - \lambda \int_\Omega (v_k + \eta_k) \, w \, dx = 0.$$

In particular, for every $w \in Y$ such that $u_0 \in L^\infty(\{w \neq 0\})$, it follows

$$\int_\Omega \nabla \left[\widetilde{\psi}_{\lambda_k}(z_k + v_k) - \widetilde{\psi}_{\lambda_k}(z_k) - \eta_k\right] \cdot \nabla w \, dx$$
$$+ \int_\Omega \left[\widehat{g}\left(x, z_k + v_k + \widetilde{\psi}_{\lambda_k}(z_k + v_k)\right) - \widehat{g}\left(x, z_k + \widetilde{\psi}_{\lambda_k}(z_k)\right) - D_s\widehat{g}(x, z + \widetilde{\psi}_\lambda(z))(v_k + \eta_k)\right] w \, dx$$
$$-\lambda_k \int_\Omega \left(\widetilde{\psi}_{\lambda_k}(z_k + v_k) - \widetilde{\psi}_{\lambda_k}(z_k) - \eta_k\right) w \, dx - (\lambda_k - \lambda) \int_\Omega (v_k + \eta_k) \, w \, dx = 0.$$

On the other hand, by Lagrange theorem we have

$$\widehat{g}\left(x, z_k + v_k + \widetilde{\psi}_{\lambda_k}(z_k + v_k)\right) - \widehat{g}\left(x, z_k + \widetilde{\psi}_{\lambda_k}(z_k)\right) - D_s\widehat{g}(x, z + \widetilde{\psi}_\lambda(z))(v_k + \eta_k)$$
$$= D_s\widehat{g}(x, \varrho_k) \left[v_k + \widetilde{\psi}_{\lambda_k}(z_k + v_k) - \widetilde{\psi}_{\lambda_k}(z_k)\right] - D_s\widehat{g}(x, z + \widetilde{\psi}_\lambda(z))(v_k + \eta_k)$$
$$= D_s\widehat{g}(x, \varrho_k) \left[\widetilde{\psi}_{\lambda_k}(z_k + v_k) - \widetilde{\psi}_{\lambda_k}(z_k) - \eta_k\right] + \left[D_s\widehat{g}(x, \varrho_k) - D_s\widehat{g}(x, z + \widetilde{\psi}_\lambda(z))\right] (v_k + \eta_k),$$

where

$$\varrho_k = z_k + \widetilde{\psi}_{\lambda_k}(z_k) + \sigma_k \left(v_k + \widetilde{\psi}_{\lambda_k}(z_k + v_k) - \widetilde{\psi}_{\lambda_k}(z_k)\right),$$

with $\sigma_k \in]0, 1[$.
After dividing both sides by $\|v_k\|$ and passing to the limit as $k \to +\infty$, we obtain

$$\int_\Omega \nabla\xi \cdot \nabla w \, dx + \int_\Omega D_s\widehat{g}(x, z + \widetilde{\psi}_\lambda(z))\xi w \, dx - \lambda \int_\Omega \xi w \, dx = 0.$$

Now we choose as test function $\left[\vartheta(\frac{u_0}{h})\xi - \widetilde{P}\left(\vartheta(\frac{u_0}{h})\xi\right)\right]$. Consider, in particular,

$$\int_\Omega D_s\widehat{g}(x, z + \widetilde{\psi}_\lambda(z))\xi \left[\vartheta(\frac{u_0}{h})\xi - \widetilde{P}(\vartheta(\frac{u_0}{h})\xi)\right] \, dx,$$
$$= \int_\Omega D_s\widehat{g}(x, z + \widetilde{\psi}_\lambda(z))\vartheta(\frac{u_0}{h})\xi^2 \, dx - \int_\Omega D_s\widehat{g}(x, z + \widetilde{\psi}_\lambda(z))\xi \widetilde{P}\left[\vartheta(\frac{u_0}{h})\xi\right] \, dx.$$

Passing to the limit as $h \to +\infty$ and taking into account (1.1.1) we get, from Beppo Levi and Lebesgue theorem,

$$\int_\Omega D_s\widehat{g}(x, z + \widetilde{\psi}_\lambda(z))\xi^2 \, dx.$$

Therefore, we have $D_s\widehat{g}(x, z + \widetilde{\psi}_\lambda(z))\xi^2 \in L^1(\Omega)$ and

$$\int_\Omega |\nabla \xi|^2 \, dx + \int_\Omega D_s\widehat{g}(x, z + \widetilde{\psi}_\lambda(z))\xi^2 \, dx - \lambda \int_\Omega \xi^2 \, dx = 0.$$

We deduce that $\xi = 0$. ∎

Now we define also $\widetilde{\varphi} : \widetilde{U} \to \mathbb{R}$ as

$$\widetilde{\varphi}_\lambda(v) = \varphi_\lambda(Pv)\,.$$

Theorem 2.2.15 *$\widetilde{\varphi}_\lambda$ is of class C^1 with*

$$\langle \widetilde{\varphi}_\lambda'(z), v\rangle = \int_\Omega \nabla(z + \widetilde{\psi}_\lambda(z)) \cdot \nabla v \, dx + \int_\Omega \widehat{g}(x, z + \widetilde{\psi}_\lambda(z)) v \, dx - \lambda \int_\Omega (z + \widetilde{\psi}_\lambda(z)) v \, dx\,.$$

In particular, $\widetilde{\varphi}_\lambda'(0) = 0$.

Proof. Since $v - Pv \in Y \cap L^\infty(\Omega)$, we have

$$\langle \widetilde{\varphi}_\lambda'(z), v\rangle = \langle \varphi_\lambda'(Pz), Pv\rangle$$
$$= \int_\Omega \nabla(Pz + \psi_\lambda(Pz)) \cdot \nabla Pv \, dx + \int_\Omega \widehat{g}(x, Pz + \psi_\lambda(Pz)) Pv \, dx - \lambda \int_\Omega (Pz + \psi_\lambda(Pz)) Pv \, dx$$
$$= \int_\Omega \nabla(Pz + \psi_\lambda(Pz)) \cdot \nabla v \, dx + \int_\Omega \widehat{g}(x, Pz + \psi_\lambda(Pz)) v \, dx - \lambda \int_\Omega (Pz + \psi_\lambda(Pz)) v \, dx$$
$$= \int_\Omega \nabla(z + \widetilde{\psi}_\lambda(z)) \cdot \nabla v \, dx + \int_\Omega \widehat{g}(x, z + \widetilde{\psi}_\lambda(z)) v \, dx - \lambda \int_\Omega (z + \widetilde{\psi}_\lambda(z)) v \, dx.$$

∎

Theorem 2.2.16 *The function $\widetilde{\varphi}_\lambda$ is of class C^2 with*

$$\langle \widetilde{\varphi}_\lambda{}''(z)v, \hat{v}\rangle = \int_\Omega \nabla(v + L_z v) \cdot \nabla \hat{v} \, dx + \int_\Omega D_s\widehat{g}(x, u)(v + L_z v)\hat{v} \, dx - \lambda \int_\Omega (v + L_z v)\hat{v} \, dx,$$

where $u = z + \widetilde{\psi}_\lambda(z)$. Moreover the map $\{(\lambda, z) \mapsto \widetilde{\varphi}_\lambda{}''(z)\}$ is continuous.

Proof. Define

$$\widetilde{L}_z : H^h_- \oplus H^h_0 \to (H^h_- \oplus H^h_0)'$$

as

$$\langle \widetilde{L}_z v, \hat{v}\rangle = \int_\Omega \nabla(v + L_z v)\cdot\nabla\hat{v}\,dx + \int_\Omega D_s\widehat{g}(x,u)(v+L_z v)\hat{v}\,dx - \lambda\int_\Omega (v+L_z v)\hat{v}\,dx\,.$$

Then $\widetilde{L}_z$ is linear and of course continuous.

Fix $z \in \widetilde{U}$ and $\hat{v} \in H^h_- \oplus H^h_0$. Then consider a sequence (z_k) convergent to z in $\widetilde{U}$ and a sequence (v_k) convergent to 0 in $H^h_- \oplus H^h_0$. If we set $\eta_k = L_z v_k$, we have

$$\frac{\langle \widetilde{\varphi}'_\lambda(z_k+v_k) - \widetilde{\varphi}'_\lambda(z_k) - \widetilde{L}_z v_k, \hat{v}\rangle}{\|v_k\|}$$

$$= \frac{\int_\Omega \nabla\left[\widetilde{\psi}_\lambda(z_k+v_k) - \widetilde{\psi}_\lambda(z_k) - \eta_k\right]\cdot\nabla\hat{v}\,dx}{\|v_k\|} - \frac{\lambda\int_\Omega \left[\widetilde{\psi}_\lambda(z_k+v_k) - \widetilde{\psi}_\lambda(z_k) - \eta_k\right]\hat{v}\,dx}{\|v_k\|}$$

$$+\frac{\int_\Omega \left[\widehat{g}\left(x, z_k+v_k+\widetilde{\psi}_\lambda(z_k+v_k)\right) - \widehat{g}\left(x, z_k+\widetilde{\psi}_\lambda(z_k)\right) - D_s\widehat{g}(x,u)\,(v_k+\eta_k)\right]\hat{v}\,dx}{\|v_k\|}.$$

By theorem 2.2.14 we have

$$\lim_k \frac{\int_\Omega \nabla\left[\widetilde{\psi}_\lambda(z_k+v_k) - \widetilde{\psi}_\lambda(z_k) - \eta_k\right]\cdot\nabla\hat{v}\,dx}{\|v_k\|} = \lim_k \frac{\int_\Omega \left[\widetilde{\psi}_\lambda(z_k+v_k) - \widetilde{\psi}_\lambda(z_k) - \eta_k\right]\hat{v}\,dx}{\|v_k\|} = 0\,.$$

On the other hand, by Lagrange theorem there exists ϱ_k such that

$$\left[\widehat{g}\left(x, z_k+v_k+\widetilde{\psi}_\lambda(z_k+v_k)\right) - \widehat{g}(x, z_k+\widetilde{\psi}_\lambda(z_k))\right] =$$

$$= D_s\widehat{g}(x,\varrho_k)\left(v_k+\widetilde{\psi}_\lambda(z_k+v_k) - \widetilde{\psi}_\lambda(z_k)\right) =$$

$$= D_s\widehat{g}(x,\varrho_k)\,(v_k+\eta_k) + D_s\widehat{g}(x,\varrho_k)\left(\widetilde{\psi}_\lambda(z_k+v_k) - \widetilde{\psi}_\lambda(z_k) - \eta_k\right).$$

Since u_0 is bounded where $\hat{v} \neq 0$ and since $\widetilde{\psi}_\lambda$ is also bounded in $L^\infty(\Omega)$, we get

$$\lim_k \frac{\int_\Omega D_s\widehat{g}(x,\varrho_k)\left(\widetilde{\psi}_\lambda(z_k+v_k) - \widetilde{\psi}_\lambda(z_k) - \eta_k\right)\hat{v}\,dx}{\|v_k\|} = 0\,,$$

$$\lim_k \frac{\displaystyle\int_\Omega [D_s\widehat{g}(x,\varrho_k) - D_s\widehat{g}(x,u)]\,(v_k+\eta_k)\,\hat{v}\,dx}{\|v_k\|} = 0,$$

and the assertion follows. ■

Now we come back to the decompositions

$$H = H_- \oplus H_0 \oplus H_+ = H_-^h \oplus H_0^h \oplus H_+\,.$$

Theorem 2.2.17 *The function φ_λ is of class C^2 with*

$$\langle \varphi_\lambda\,''(0)v, v\rangle = \int_\Omega |\nabla v|^2\,dx + \int_\Omega D_s\widehat{g}(x,0)v^2\,dx - \lambda\int_\Omega v^2\,dx\,.$$

Moreover the map $\{(\lambda, z) \mapsto \varphi_\lambda\,''(z)\}$ is continuous.

Proof. Observe that

$$\varphi_\lambda(v) = \tilde{\varphi}_\lambda(\widetilde{P}v)\,,$$

so that φ_λ is of class C^2 with

$$\langle \varphi_\lambda''(z)v, v\rangle = \left\langle \widetilde{\varphi}_\lambda''(\widetilde{P}z)\widetilde{P}v, \widetilde{P}v\right\rangle.$$

If we set $v_+ = v - \widetilde{P}v$ and $\tilde{v} = \widetilde{P}v$, we have

$$\begin{aligned}
\langle \varphi_\lambda''(0)v, v\rangle &= \langle \widetilde{\varphi}_\lambda''(0)\tilde{v}, v - v_+\rangle \\
&= \int_\Omega \nabla(\tilde{v} + L_0\tilde{v})\cdot\nabla(v - v_+)\,dx + \int_\Omega D_s\widehat{g}(x,0)(\tilde{v}+L_0\tilde{v})(v-v_+)\,dx \\
&\quad - \lambda\int_\Omega (\tilde{v}+L_0\tilde{v})(v-v_+)\,dx \\
&= \int_\Omega \nabla(\tilde{v}+L_0\tilde{v})\cdot\nabla v\,dx + \int_\Omega D_s\widehat{g}(x,0)(\tilde{v}+L_0\tilde{v})v\,dx - \lambda\int_\Omega(\tilde{v}+L_0\tilde{v})v\,dx \\
&= \int_\Omega \nabla(v - v_+ + L_0\tilde{v})\cdot\nabla v\,dx + \int_\Omega D_s\widehat{g}(x,0)(v-v_++L_0\tilde{v})v\,dx \\
&\quad - \lambda\int_\Omega (v - v_+ + L_0\tilde{v})v\,dx \\
&= \int_\Omega |\nabla v|^2\,dx + \int_\Omega D_s\widehat{g}(x,0)v^2\,dx - \lambda\int_\Omega v^2\,dx\,.
\end{aligned}$$

■

We can now define the linear maps

$$L, K : \; H_- \oplus H_0 \to (H_- \oplus H_0)'$$

such that

$$\langle Lu, v\rangle = \int_\Omega \nabla u \cdot \nabla v \, dx + \int_\Omega D_s\widehat{g}(x,0)uv\, dx,$$

$$\langle Ku, v\rangle = \int_\Omega uv\, dx.$$

The maps L and K satisfy the assumption (b) of theorem (1.5.2) and

$$\varphi_\lambda''(0) = L - \lambda K.$$

On the other hand, if $\varphi_\lambda'(z) = 0$, we have

$$\int_\Omega \nabla(z+\psi_\lambda(z))\cdot\nabla v\, dx + \int_\Omega \widehat{g}(x, z+\psi_\lambda(z))v\, dx = \lambda\int_\Omega (z+\psi_\lambda(z))v\, dx \quad \forall v \in H_-\oplus H_0$$

and also

$$\int_\Omega \nabla(z+\psi_\lambda(z))\cdot\nabla w\, dx + \int_\Omega \widehat{g}(x, z+\psi_\lambda(z))w\, dx = \lambda\int_\Omega (z+\psi_\lambda(z))w\, dx \quad \forall w \in Y\cap L^\infty(\Omega)\,,$$

whence

$$\int_\Omega \nabla(z+\psi_\lambda(z))\cdot\nabla v\, dx + \int_\Omega \widehat{g}(x, z+\psi_\lambda(z))v\, dx = \lambda\int_\Omega (z+\psi_\lambda(z))v\, dx \quad \forall v \in W^{1,2}_0(\Omega)\cap L^\infty(\Omega)\,.$$

If we set $u = z + \psi_\lambda(z)$, from Corollary 1.2.6 we infer that

$$\widehat{J}(v) \geq \widehat{J}(u) + \lambda\int_\Omega u(v-u)\, dx \qquad \text{for every } v \in W^{1,2}_0(\Omega)\,,$$

namely that $u_0 + u$ is a solution of (2.2.1).

Moreover, if $z \neq 0$ we have $u \neq 0$ and if $z \to 0$ we have $u \to 0$ in $W^{1,2}_0(\Omega)$.

Then Theorem (2.2.6) follows from theorem (1.5.2).

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