UNIVERSITÀ DEGLI STUDI DI MILANO-BICOCCA
Scuola di Scienze
Scuola di Dottorato in Matematica Pura e Applicata

On the surface group conjecture

Relatore: Prof. Thomas WEIGEL

Tesi di dottorato di
Gianluca PONZONI
matricola n. 744874

Introduction

In this thesis we present some partial results on Melnikov's surface group conjecture. Melnikov conjectured that if G is a residually finite, non-free, non cyclic hereditary one-relator group, then G is a surface group.

In this original form the conjecture is not true. Baumslag-Solitar groups $BS(1,m) = \langle x, y | xy^m x^{-1} y^{-1} \rangle$ are residually finite, non-free and non-cyclic one-relator groups, all their subgroups of finite index are again one-relator groups, but they are not surface groups. The conjecture can thus be restated as follows.

Conjecture 1. *Let G be a residually finite, non-free, non cyclic hereditary one-relator group. Then G is either a surface group or a Baumslag-Solitar group $BS(1,m)$ for some $m \in \mathbb{Z}$.*

A group G is called a surface group if it is isomorphic to the fundamental group of a closed surface. Surface groups present some interesting properties. It is known that they admit a one-relator presentation, namely

$$\langle x_0, x_1, \ldots, x_g | x_0^2 x_1^2 x_2^2 \ldots x_g^2 \rangle$$

for non-orientable closed surfaces of genus g, and

$$\langle x_1, x_2, \ldots, x_{2g} | [x_1, x_2] \ldots [x_{2g-1}, x_{2g}] \rangle$$

for orientable closed surfaces of genus g. Moreover, every subgroup of finite index of a surface group is again a surface group, and consequently a one-relator group. In this work we will refer to a one-relator group in which all the subgroups of finite index are again one-relator groups as a hereditary one-relator group.

As a closed surface X is aspherical, it coincides with the classifying space $BG = K(G,1)$ of its fundamental group $G = \pi_1(X, x_0)$, i.e. the cohomology of a surface group and that of its associated surface are isomorphic. Thus surface groups have cohomological dimension 2 and they are duality groups.

In [3] G. Baumslag proved that surface groups are residually finite.

It is known that subgroups of infinite index of surface groups are free. In [8] Rosenberger et al. classified cyclically pinched and conjugacy pinched one-relator groups such that every subgroup of infinite index is free. Using this result they proved a modified form of the surface group conjecture, namely that if G is a finitely generated, non-free, freely indecomposable, fully residually free group such that every subgroup of infinite index of G is free, then G is a surface group.

A first approach to a positive solution of the problem is combinatorial. Some properties of one-relator groups are reflected by properties of the single relator r. For example, a one-relator group is torsion-free if and only if its relator is not a proper power. Using the theory of automorphisms of free groups it is possible to decide whether a given one-relator group is free or not, and whether it is isomorphic to a free product of a one-relator group with a free group.

Theorem 1. *Let $G = \langle x_1, \dots, x_n | r\rangle$ be a hereditary one-relator group such that r is a commutator involving every generator and not a proper power. Then G is non-free, torsion-free and is freely indecomposable.*

A deeper and more useful combinatorial result is Lyndon's Identity Theorem, which can be proved using the machinery of free differential calculus. Using this theorem it si possible to prove that if a one-relator group G is non-free, torsion-free and freely indecomposable, then it is a duality group. Furthermore, the Identity Theorem plays a central role in determining the structure of the dualizing module.

After this elementary combinatorial first step, we proceed with the analysis of hereditary one-relator groups using a result due to Bieri and al., that proved that Poincarè duality groups of dimension two are surface groups (see [9]).

Using Lyndon's identity theorem one knows that the dualizing module D_G of a one-relator duality group $G = \langle x_1, \dots, x_n | r\rangle$ can be written as

$$D_G =^{\times} \mathrm{H}^2(G, \mathbb{Z}G) = \frac{\mathbb{Z}G}{\sum \mathbb{Z}G \frac{\partial r}{\partial x_i}}$$

In fact, it is a quotient of $\mathbb{Z}G$ and we have a lifting

$$\begin{array}{ccc} D_G & & \\ \uparrow & \searrow^{\varepsilon_0} & \\ \mathbb{Z}G & \xrightarrow{\ \varepsilon\ } & \mathbb{Z} \end{array}$$

of the augmentation map $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ to a map $\varepsilon_0 : D_G \to \mathbb{Z}$ if and only if r is a commutator. If $K = \ker(\epsilon_0)$ is trivial then $D_G \simeq \mathbb{Z}$ and G is a surface group, so K can be seen as a measure of how distant G is to being a surface group. We refer the interested reader to chapter 3 for a more precise analysis of the properties of K, which satisfies an interesting duality relation in the context of derived categories with duality.

The hypothesis of the surface group conjecture have some striking similarities to some properties of Demushkin groups, which are one-relator pro-p

groups and Poincarè duality groups. Labute's classification of Demushkin groups shows that they admit a (pro-p) presentation that is quite similar to the presentation of a surface group of an orientable surface. We decided to pursue the possibility of a relation between the two situations.

Let G be a group that satisfies the hypothesis of the surface group conjecture and whose single relator r is a commutator. Our idea is to take the pro-p completion of G and study its structure. Since surface groups are in fact residually free we also require G to be residually free. Then we prove that G is p-good, that is that the natural isomorphism between G and its completion induces isomorphisms between the cohomology groups of the two groups. We use this to characterize the pro-p completion of G (see 2.6.5).

Theorem 2. *Let G be a residually finite non-free, non-cyclic hereditary one-relator group, suppose that the single relator r is a commutator. Then $\hat{G}_p$, the pro-p completion of G, is an oriented Demushkin group and thus coincides with the pro-p completion of a surface group.*

Using Labute's classification we then conclude that G must have an even number of generators and that r is not in the second derived subgroup of the free group whose quotient gives the presentation of G (see 2.6.1).

Also, if G has only two generators we have a positive answer to the conjecture.

Theorem 3. *Let G be a group such that*

i) G is a residually finite, non-free, non cyclic one-relator group;

ii) the single relator r is a commutator;

iii) G has only two generators as a one-relator group.

Then G is free abelian and thus a surface group.

The outline of this thesis is the following.

In the first chapter we recall some classical combinatorial results on one-relator groups. The aim of this chapter is to show that many properties of a one-relator group can be determined by the single relator word.

We present the basic theory on free groups and homomorphisms of free groups, in order to show that is decidable if a given word in a free group is a free generator. Then we define presentations and one-relator groups, citing Magnus' Freiheitssatz to prove some embedding results due to Lyndon. Then we introduce free differential calculus and use the results to prove Lyndon's Identity Theorem.

In the second chapter we define surface groups and duality groups and give a brief survey of results regarding the surface group conjecture. Then we focus on one-relator groups that satisfy some of the conditions of the conjecture and have a relator in the commutator subgroup. We show that such groups have a goodness property, using this and classification of Demushkin groups we are able to prove that the relator cannot be in the second derived subgroup and to prove the conjecture for the case of two generators and relator in the commutator.

In the third chapter we define derived categories with duality, that are the natural framework for the study of augmented duality groups, a class of groups that generalize duality groups. T. Weigel has proved that free groups are augmented duality. We prove here that one-relator groups with relator in the commutator are also augmented duality groups.

Chapter 1

Classical results on one relator groups

In this chapter we recall basic definitions about free groups and presentations and classical results on one relator groups, obtained mainly by Magnus and Lyndon. Then we delineate the technique of free differential calculus, which we use to prove Lyndon's Identity Theorem following [14]. We refer the reader to [13], [15], [21] and [20] for a more comprehensive treatment of these subjects.

1.1 Free groups

We give here the definition of free groups and recall how it is possible to construct a free group over any set X.

Definition. Let X be a set, F a group and $\psi : X \to F$ a function from X to F. F is a *free group* with basis X if for any group G and any function $f : X \to G$ there is a unique group homomorphism $\phi : F \to G$ such that $f = \phi \cdot \psi$, that is such that the diagram

$$\begin{array}{ccc} X & \xrightarrow{\psi} & F \\ & \searrow^{f} & \downarrow{\scriptstyle\phi} \\ & & G \end{array}$$

is commutative.

It follows from the definition that free groups with basis of the same cardinality are isomorphic. Given a set X, it is always possible to obtain a free group with basis X, we will provide here the classical construction of such a group.

Let X be a set, let X^{-1} be a disjoint set with a one-to-one correspondence to X, the element of X^{-1} corresponding to $x \in X$ will be denoted by x^{-1}. A *word* in X is a finite sequence of elements of $X \cup X^{-1}$, the sequence with no elements is the *empty word* ϵ.

The set $W(X)$ of the words in X with the operation given by concatenation is a monoid with identity element ϵ.

$w \in W(X)$ is a *reduced word* if it does not contain subsequences of the form xx^{-1} or $x^{-1}x$ for any $x \in X$. We say that two words $v, w \in W(X)$ are equivalent (writing $v \sim w$) if it is possible to obtain w from v adding (at any point) or removing subsequences of the form xx^{-1} or $x^{-1}x$ for some $x \in X$ in a finite number of steps; this is an equivalence relation and the reduced words are a system of representatives for the equivalence classes. The equivalence is compatible with the concatenation and the quotient $W(X)/\sim$ is a group under this operation. The inverse of a word $\prod_{i=1}^{l} x_i^{\varepsilon_i}$, $x_i \in X$, $\varepsilon_i = \pm 1$, is the word $\prod_{i=1}^{l} x_{l+1-i}^{-\varepsilon_i}$, $x_i \in X$.

This group, with the natural immersion of X in $W(X)/\sim$, is a free group with basis X and thus isomorphic to any free group with basis X. For this reason, we will often refer to the elements of any free group as words.

We give now some definitions that can be useful in describing the elements of a free group.

Let F be a free group with basis X. We say that a reduced word $r \in F$ involves $x \in X$ if x or x^{-1} appears in r.

If r is a reduced word, the *length* $l(r)$ of r is the number of symbols from $X \cup X^{-1}$ appearing in r. When we write $l(w)$ for some element $w \in F$, we indicate the length of the reduced word corresponding to r.

We say that a word r is *ciclically reduced* if it is reduced and the first and last symbol of its expression are not x and x^{-1}, or viceversa, for every $x \in X$.

For $x \in X$, we denote with $\#_x(r)$ the *number of occurrences* of x in r, that is the total number of the letters x and x^{-1} in r.

We denote with $\sigma_x(r)$ the *exponent sum* of x in r, that is the number of letters $x \in X$ in r minus the number of letters $x^{-1} \in X^{-1}$ in r.

1.2 Nielsen transformations

Let $B = (b_1, b_2, \ldots)$ be an ordered subset of a free group F. A *Nielsen transformation* is any finite product of the following transformations on the set of ordered subsets of F:

N1) substitute a b_i with b_i^{-1};

N2) substitute a b_i with $b_i b_j$ for some $j \neq i$;

N3) delete u_i if $u_i = 1_F$.

The three transformations above are called elementary Nielsen transformation.

A Nielsen transformation is called *regular* if it has no factor of type $N3$ and *singular* otherwise.

Since the inverse of a Nielsen transformations of type $N1$ and $N2$ is again a Nielsen transformation, the regular Nielsen transformations form a group.

The interest in Nielsen transformations lies in the fact that they bring sets of generators of any subgroup H of F in sets of generators of the same subgroup.

Proposition 1.2.1. *Let B be an ordered subset of F and C its image under a Nielsen transformation. Then the subgroup of F generated by B coincides with the subgroup of F generated by C.*

Proof. Since Nielsen transformations are compositions of elementary Nielsen transformations, it suffices to show that the theorem holds for elementary Nielsen transformations.

Let F_B be the subgroup of F generated by B and F_C the subgroup of F generated by C.

Let C be the image of B under an elementary Nielsen transformation of type $N1$. Since F_B is a group, it contains every b_i^{-1}, then $C \subseteq F_B$ and consequently $F_C \subseteq F_B$. Viceversa, F_C must contain the inverse of every element of C, so if $b_i^{-1} \in C$ then $(b_i^{-1})^{-1} = b_i \in F_C$, so $B \subseteq F_C$ and consequently $F_B \subseteq F_C$. We conclude that $F_B = F_C$.

Let C be the image of B uncer an elementary Nielsen transformation of type $N2$. Since F_B is a group, it contains every $b_i b_j$ for every $b_i, b_j \in B$, so $C \subseteq F_B$ and consequently $F_C \subseteq F_B$. Viceversa, F_C contains $b_i b_j$ and b_j, so it must contain $(b_i b_j) b_j^{-1} = b_i$, then $B \subseteq F_C$ and consequently $F_B \subseteq F_C$. We conclude that $F_B = F_C$.

Let C be the image of B under an elementary Nielsen transformation of type $N3$, then $C \subseteq F_B$ because C is contained in B as a (non-ordered) set, and $B \subseteq F_C$ because $1_F \in F_C$. Then we conclude that $F_B = F_C$. $\square$

The aim is to modify the set of generators of a given subgroup via Nielsen transformations in order to obtain a reduced set of generators, in the following sense.

Definition. Let $B = (b_1, b_2, \ldots)$ be an ordered subset of a free group F with basis X. We say that B is N-reduced if for every choice of elements β_1, β_2, β_3, where $\beta_i = b_j$ or $\beta_i = b_j^{-1}$ for $i = 1, 2, 3$ and some $b_j \in B$, we have:

1. $\beta_1 \neq 1_F$;
2. if $\beta_1\beta_2 \neq 1_F$, then $l(\beta_1\beta_2) \geq l(\beta_1)$ and $l(\beta_1\beta_2) \geq l(\beta_2)$;
3. if $\beta_1\beta_2 \neq 1_F$ and $\beta_2\beta_3 \neq 1_F$, then $l(\beta_1\beta_2\beta_3) > l(\beta_1) - l(\beta_2) + l(\beta_3)$.

The interest in N-reduced sets of words is that every subgroup of F generated by such a set is free.

Proposition 1.2.2. *Let B be a N-reduced set of words in F, then the subgroup of F generated by B is free.*

Proof. For any $b \in B^{\pm 1}$, $b = i_b t_n f_b$, where i_b is the longest initial subword of b that cancels in any product ab with $a \in B^{\pm 1}$, f_b is the longest terminal subword of b that cancels in any product ba with $a \in B^{\pm 1}$ and $t_b \neq 1_F$ because B satisfies the third condition for N-reduced sets.

Let $c = \prod_{i=1}^{n} b_i$ with $b_i \in B^{\pm 1}$ for $1 \leq i \leq n$, with $b_i b_{i+1} \neq 1$ for $1 \leq 1 \leq n-1$. Then by the observation above $c = \prod_{i=1}^{n} b_i'$ where b_i' is a subword of b_i containing t_{b_i} and there is no cancellation in $b_i' b_{i+1}'$ for $1 \leq i \leq n-1$. It follows that $l(c) \geq n$.

Let G be the free group with basis B, let $\phi : B \to F$ the immersion of B in F, then ϕ extends uniquely to a group homomorphism $\Phi : G \to F$ since G is free. Let $g \in G$, $g \neq 1$ be a reduced word in G, let $l(g) = n$, then also $l(\Phi(g)) = n$, so Φ is injective. Then $\Phi(G)$ is the subgroup generated by B and it is a free subgroup of F. $\square$

The following proposition ensures that it is always possible to bring a finite set of elements of F in a N-reduced one applying Nielsen transformations.

Proposition 1.2.3. *Let $B = (b_1, b_2, \ldots, b_n)$ be a finite ordered subset of F. Then there exists a Nielsen transformation h such that $h(B)$ is N-reduced.*

Proof. Let $A = (a_1, \ldots, a_n)$ be a finite ordered subset of F, we define $\mu(A) = \sum_{i=1}^{n} l(a_i)$.

Observe that utilizing elementary Nielsen transformations of type $N1$ and $N2$ we can obtain any permutation of the b_i and substitute any b_i with its inverse.

Suppose that B does not satisfy the second condition. Without loss of generality we can suppose that there are some b_i, b_j such that $b_i b_j \neq 1_F$ and $l(b_i b_j) < j$. Since $l(w^2) \geq l(w)$ for any $w \in F$, it is $i \neq j$. Using an elementary Nielsen transformation of type $N2$ we can substitute b_j with $b_i b_j$ obtaining the set B', observe that $\mu(B') < \mu(B)$. Then by induction we can obtain a set B'' such that that $\mu(B'')$ is minimum, hence B'' satisfy the second condition.

Applying an elementary Nielsen transformation of type $N3$ to B'', the resulting ordered set C satisfies the first condition.

Let $a, b, c \in C$ with $ab \neq 1_F$ and $bc \neq 1_F$. Since C satisfies the second condition for N-reduced sets, $l(ab) \geq l(a)$ and $l(bc) \geq l(c)$. Let u be the initial subword of b that is cancelled in the product ab and w the final subword of b that is cancelled in bc, then u and w have length less than or equal to $l(b)$. So it is $a = a'u^{-1}$, $b = ub'w$, $c = w^{-1}c'$ for some reduced words a', b', c', and $abc = a'b'c'$.

If $b' \neq 1_F$ there are no cancellations in the second product, so

$$l(abc) = l(a) - l(b) + l(c) + l(b')$$

since $l(b') \geq 1$ we have that a, b, c satisfy the third condition for N-reduced sets.

If $b' = 1_F$ then $abc = a'c'$ and

$$l(abc) \leq l(a) - l(b) + l(c),$$

in this case a, b, c do not satisfy the third condition.

Then take any well-ordering of the set $X \cup X^{-1}$, this induces a lexicographical well-ordering $<$ on the elements of F identified with the reduced words in $X \cup X^{-1}$. For every reduced word w let $L(w)$ be the reduced subword given by the initial $\frac{l(w)}{2}$ letters if $l(w)$ is even or the first $\frac{l(w)+1}{2}$ letters if $l(w)$ is odd. We now define a well-ordering $\prec$ on the set of the pairs (w, w^{-1}) with w reduced word in F. $(w, w^{-1}) \prec (u, u^{-1})$ if one of the following conditions is verified:

1. $\min(L(w), l(w^{-1}) < \min(L(u), l(u^{-1})$
2. $\min(L(w), l(w^{-1}) = \min(L(u), l(u^{-1})$ and

$$\max(L(w), l(w^{-1}) < \max(L(u), l(u^{-1})$$

If $u < w^{-1}$ in the lexicographical ordering, we have that

$$(bc, (bc)^{-1}) = (uc', (uc')^{-1}) \prec (c, c^{-1}) = (w^{-1}c', (w^{-1}c')^{-1})$$

since $L(uc')$ has u as an initial subword and $L(w^{-1}c')$ has w^{-1} as an initial subword and $L(c'^{-1}w) = L(c'^{-1}u^{-1})$ since C' satisfies the second condition for N-reduction. On the other hand, if $w^{-1} < u$ in the lexicographical ordering, we have that

$$(ab, (ab)^{-1}) = (a'w, (a'w)^{-1}) \prec (a, a^{-1}) = (a'u^{-1}, (a'u^{-1})^{-1})$$

We can substitute a with $(ab)^{-1} = w^{-1}a'^{-1}$ using elementary Nielsen transformations of type $N1$ and $N2$, or substitute c with $bc = uc'$. Let C' be the set obtained with this substitution. Note that in either case $\mu(V) = \mu(C')$, so the second condition is still satisfied. By induction we can use elementary Nielsen transformation to minimize the words in C' with respect to the relation $\prec$, then there are no triples a, b, c such that b cancels out completely in abc and the third condition is satisfied. □

This is enough to prove an important theorem about subgroups of free groups, at least for finitely generated subgroups.

Theorem 1.2.1. *Let F be a free group and U a finitely generated subgroup of F. Then U is a free group.*

Proof. Let X be a basis for F, let B be a finite (ordered) set of generators for U. By Proposition 1.2.3 above there is a Nielsen transformation f such that $f(A)$ is N-reduced. By Proposition 1.2.1 U is generated by $f(B)$, then by the proposition above U is free. □

1.3 Free generators

Let F be a free group with basis X. Then the image of the set X under every automorphism of F is a basis for F as a free group. Conversely, any one-to-one map between two basis of F can be extended to an automorphism of free groups.

For any $x \in X$, let α_x be the endomorphism under which the image of x is x^{-1} and that fixes $X \setminus \{x\}$. For any $x, y \in X$, with $x \neq y$, let β_{xy} be the endomorphisms under which the image of x is xy and that fixes $X \setminus \{x\}$.

Since the image of X under α_x and β_{xy} is a basis for F, they are automorphisms of F.

Proposition 1.3.1. *Let F be a free group with basis X, let $\mathrm{Aut}_f(F)$ be the subgroup of $\mathrm{Aut}(F)$ generated by the elementary Nielsen transformations. Then for any $\gamma \in \mathrm{Aut}(F)$ and for any $w_1, \dots, w_k \in F$, $k \in \mathbb{N}$, there is $\alpha \in \mathrm{Aut}_f(F)$ such that $\gamma(w_i) = \alpha(w_i)$ for $1 \leq i \leq k$.*

Proof. Let Y be the set of the elements of X involved in $w_1, \dots, w_k$, then $Y = \{y_1, \dots, y_t\}$, $t \in \mathbb{N}$, is a finite subset of X and $w_1, \dots, w_k$ are in the subgroup of F generated by Y.

$\alpha^{-1}(X)$ is a basis for F, so there is a finite subset $Z \subseteq X$ such that the group generated by $\alpha^{-1}(Z)$ contains Y, moreover we can suppose $Y \subseteq Z = \{y_1, \dots, y_t, y_{t+1}, \dots, y_m\}$, $m \in \mathbb{N}$.

Let $B = \{b_1, \dots, b_m\}$, with $b_i = \alpha^{-1} y_i$, $\leq i \leq m$. Then some Nielsen transformation β carries B into $\beta(B)$ reduced. The group generated by $\beta(B)$ coincides with the group generated by $B = \alpha^{-1}(Z)$, since this groups have rank m, β is regular. But Y is contained in the subgroup of F generated by B, so $Y \subseteq B^{\pm 1}$, and without loss of generality we may assume $B = (x_1, \dots, x_t, z_{t+1}, \dots, z_m)$ for some $z_1, \dots, z_m \in F$.

Z is an initial segment of X, so $\alpha^{-1}(Z)$ is an initial segment of $\alpha^{-1}(X)$. Since β is a composition of trasformations that involve only the first m components of a matrix, $\beta X \alpha^{-1} = \beta(B)$ coincides with the initial segment of length m of $\beta Z \alpha^{-1}$. But X is the matrix of the identity automorphism, so $\beta X \alpha^{-1} = \beta \alpha^{-1}$. Then we have $\alpha^{-1}(\beta(x_i)) = x_i$. □

Let w be a ciclically reduced word in a free group F over a set X, then any word conjugated to w is either not ciclically reduced or a cyclical permutation of the letters of w, hence we can identify the conjucacy classes of F with the sets of cyclic permutations of ciclically reduced words. This motivates the following definition.

Definition. A *cyclic word* of length n is a ciclically ordered set of n letters a_i, $i \in \mathbb{Z}_n$, such that $a_i a_{i+1} \neq 1$ for all $i \in \mathbb{Z}_n$.

Given a cyclic word w, we define the function $\gamma_w : X^{\pm 1} \times X^{\pm 1} \to \mathbb{Z}$, where $\gamma_w(x, y)$ is the number of subwords xy^{-1} or yx^{-1} in w. If the cyclic word is clearly stated we will write $x \cdot y$ instead of $\gamma_w(x, y)$. For $W_1, W_2 \subseteq X^{\pm 1}$ we define

$$W_1 \cdot W_2 = \sum_{w_1 \in W_1, w_2 \in W_2} w_1 \cdot w_2$$

A *Whitehead automorphism* of F is any automorphism of F that either permutes the elements of $X^{\pm 1}$ or carries each $x \in X^{\pm 1}$ into one of $x, xa, a^{-1}x$ or $a^{-1}xa$ for some fixed $a \in X^{\pm 1}$. If α is a Whitehead automorphisms of the second kind we define $\alpha = (A, a)$, where A is the set of all the $x \in X^{\pm 1}$ such that $\alpha(x) = xa$ or $\alpha(x) = a^{-1}xa$, including a but not a^{-1}. We denote the set of Whitehead automorphisms with Ω.

If $\alpha = (A, a)$, then $\alpha^{-1} = ((A \setminus \{a\}) \cup \{a^{-1}\}, a^{-1})$. If $\bar{\alpha} = (A', a^{-1})$, where A' is the complement of X in A, then $\alpha^{-1} \circ \bar{\alpha}$ is the inner automorphism defined by conjugation by a. Then $\alpha^{-1} \circ \bar{\alpha}$ is the identity over the set of cyclic words, so $\alpha(w) = \bar{\alpha}(w)$ for every cyclic word w.

We give the following technical proposition about Whitehead transformations, that will be used to prove that the existance of an automorphism of F that brings one given word in another given word is always decidable.

Proposition 1.3.2. *Let v_1 and v_2 be cyclic words, let $v_2 = \alpha(v_1)$ for some $\alpha \in \mathrm{Aut}(F)$ and $l(v_2) \leq l(v_1)$. Then $\alpha = \prod_{i=1}^{n} \tau_i$, $n \geq 0$, $\tau_i \in \Omega$ for every i, $0 \leq i \leq n$, and $l((\prod_{i=1}^{j} \tau_i)(v_1)) \leq v_1$ for every j, $0 < j < n$. The equality holds if and only if $l(v_1) = l(v_2)$.*

Proposition 1.3.3. *Let F be a free group, $w_1, w_2 \in F$. Then it is decidable whether there is an automorphism α of F such that $\alpha(w_1) = w_2$.*

Proof. Since w_1, w_2 are contained in a finitely generated subgroup of F, we can suppose without loss of generality that F is finitely generated.

Let (w_1) and (w_2) be the cyclic words associated to w_1 and w_2. Since the Whitehead automorphisms are finite, we can replace (w_1) (risp. (w_2)) with a word that is of minimal length under Whitehead automorphisms. Then by Proposition 1.3.2 (w_1) and (w_2) have now minimal length under $\mathrm{Aut}(F)$.

If $l(w_1) \neq l(w_2)$ then no automorphism of F can bring w_1 in w_2. Suppose $l(w_1) = l(w_2) = n$. The set V of cyclic words of length n in F is finite because F is finitely generated. Let $\Gamma = (V, E)$ be a graph with V as the set of vertices and $E = \{(v_1, v_2) \in V \times V | \exists \alpha \in \Omega : \alpha(v_1) = v_2\}$ Then by Proposition 1.3.2 there is an automorphism that brings (w_1) in (w_2) if and only if there is a connected path in Γ from (w_1) to (w_2).

If there is such an automorphism α, then $\alpha(w_1)$ is conjugate to w_2 since they have the same cyclic word, so there is an automorphism that brings $\alpha(w_1)$ to w_2, and by composition there is an automorphism that brings w_1 to w_2. □

1.4 Presentations

Every group G is isomorphic to a quotient of some free group F over a normal subgroup K of F. If K is the normal closure of the subgroup of F generated by a subset $R \subseteq F$ and X is a basis for F then we write

$$G = \langle X | R \rangle$$

This is called a *presentation* for G. X is called a set of generators and R a set of defining relations for G. The elements of K are called *consequences* of R.

Definition. Let G be a group. G is *finitely presented* if $G = \langle X|R \rangle$ with X and R finite.

It should be noted that if G is a finitely presented group, while the subgroup generated by R is a finitely generated free subgroup of F, its closure K is not, in general, finitely generated.

A presentation determines a group uniquely (up to isomorphisms), but a group admits infinitely many different presentations. Furthermore, in general it is not possible to decide whether two different presentations define isomorphic groups.

It is clear that if we have a presentation $G = \langle X|R \rangle$ and we add a consequence of R to the set of relations, or add a generator $y \notin X$ and a relation that defines this new generator in terms of the elements of X, the resulting presentation defines a group isomorphic to G. On the other hand, we can omit superfluous relations and generators that are defined in terms of the others (substituting their expressions in all the relations in which they appear). We formalize these procedures.

Definition. A *Tietze transformation* is a passage from a presentation $\langle X|R \rangle$ to a presentation $\langle X'|R' \rangle$ in one of the following ways.

1. Given a consequence w of R, take $X' = X$ and $R' = R \cup \{w\}$.

2. If $w \in R$ is a consequence of $R \setminus \{w\}$, take $X = X'$ and $R' = R \setminus \{w\}$.

3. Given $y \notin X$, take $X' = X \cup \{y\}$ and $R' = R \cup \{y^{-1}t\}$ with t any word in X.

4. If $x \in X$ and there is only one $r \in R$ involving x and $r = x^{-1}t$ with t any word in $X \setminus \{x\}$, take $X' = X \setminus \{x\}$ and $R' = R \setminus \{r\}$.

The Tietze transformations are the only way to modify a finite presentation obtaining another presentation of the same group, in the sense of the following theorem.

Theorem 1.4.1. *Two finite presentations define isomorphic groups if and only if it is possible to obtain one from the other by a finite sequence of Tietze transformations.*

Proof. A presentation and one obtained from it by a Tietze transformation define isomorphic groups, so the same holds true for a finite sequence of Tietze transformations.

Viceversa, let $\langle X_1|R_1\rangle$ and $\langle X_2|R_2\rangle$ two presentations of the same group G with $X_1 \cap X_2 = \emptyset$, let ϕ_1 and ϕ_2 two homomorphisms respectively from the free group F_1 over X_1 and from the free group F_2 over X_2 in G with kernel the normal closure respectively of R_1 and R_2.

For any $x \in X_1$ let t_x be an element in F_2 such that $\phi_2(t_x) = \phi_1(x)$, and for any $y \in X_2$ let u_y be an element in F_1 such that $\phi_1(u_y) = \phi_1(y)$. Let $R_3 = \{x^{-1}t_x|x \in X_1\}$, $R_4 = \{y^{-1}u_y|y \in X_2\}$, then using Tietze transformations we obtain the presentation $\langle X_1 \cup X_2|R_1 \cup R_4\rangle$ from $\langle X_1|R_1\rangle$ and $\langle X_1 \cup X_2|R_2 \cup R_3\rangle$ from $\langle X_2|R_2\rangle$ in a finite number of steps.

ϕ_1 and ϕ_2 determine a unique group homomorphism ϕ from the free group F over $X_1 \cup X_2$ in G such that $\phi_1(x) = \phi(x)$ for every $x \in X$ and $\phi_2(y) = \phi(y)$ for every $y \in X_2$. Then $\phi(r) = 1_G$ for every $r \in R_1 \cup R_2 \cup R_3 \cup R_4$, but the kernel of ϕ is the normal closure both of $R_1 \cup R_4$ and of $R_2 \cup R_3$, so each set is composed of consequences of the other. Then we can pass from both the presentations $\langle X_1 \cup X_2|R_1 \cup R_4\rangle$ and $\langle X_1 \cup X_2|R_2 \cup R_3\rangle$ to the presentation $\langle X_1 \cup X_2|R_1 \cup R_2 \cup R_3 \cup R_4\rangle$ with a finite number of Tietze transformations, since every Tietze transformation is invertible we can then pass from $\langle X_1|R_1\rangle$ to $\langle X_2|R_2\rangle$. □

Tietze transformations are useful to simplify presentations and can be used in certain cases to show that two presentations define isomorphic groups. For a simple consequence of the theorem above, we observe that we can assume that every relation in a presentation is ciclically reduced, since every word in a free group is conjugated to a ciclically reduced word.

Finally we define the concept of a HNN extension of a group giving its presentation.

Definition. Let G be a group with a presentation $\langle X|R\rangle$ and $\alpha : H \to K$ be an isomorphism between two subgroups H and K of G. Let $t \notin X$, then the HNN-extension of G relative to α is the group defined by the presentation

$$G*_\alpha = \langle X, t|R, tht^{-1} = \alpha(t) \quad \forall h \in H\rangle$$

1.5 One-relator groups

We are particularly interested in the case where R has only one element.

Definition. A group G is a *one-relator group* if $G = \langle X|r\rangle$ for some set X and reduced word r. If X is finite then G is a finitely generated one-relator group.

For example, all finite cyclic groups are one-relator groups via the presentation $C_n = \langle x | x^n \rangle$, $n \in \mathbb{N}$. More interesting is the fact that all the fundamental groups of 2-manifolds (called surface groups) are one-relator groups.

One of the first results on one-relator groups is Magnus' Freiheitssatz.

Theorem 1.5.1. *[Magnus' Freiheitssatz] Let $G = \langle x_1, \dots, x_n | r \rangle$ a one-relator group with r cyclically reduced word, suppose that r involves x_1. Then the subgroup $\langle x_2, \dots, x_n \rangle$ is a free group with free generators $\{x_2, \dots, x_n\}$.*

The theorem is equivalent to stating that every non-trivial consequence of the reduced word r involves every generator involved in r.

Using this Lyndon proved that every one relator group can be embedded in an HNN extension of a one relator group with a shorter relation, providing a framework for induction on the length of the relator.

Theorem 1.5.2. *Let $G = \langle t, x_1, \dots, x_n | r \rangle$ a (non-free) one-relator group, with $n \geq 1$ and r a cyclically reduced word that involves at least two generators, one of which has exponent sum zero. Then G can be expressed as a HNN extension of a one-relator group with a shorter relation.*

Proof. Without loss of generality, we can assume that t is the generator involved in r with exponent sum zero and that r involves x_1 as well.

Let $x_{i,j} = t^{-j} x_i t^j$ for $j \in \mathbb{Z}$, $1 \leq i \leq n$. We can write r as a shorter word r' in terms of the $x_{i,j}$, replacing each occurrence of x_i in r with $x_{i,k}$, where k is the exponent sum of t in the subword of r preceding the given occurrence of x_i. Obviously, $l(r') = l(r) - \#_t(r)$.

Let m and M be respectively the smallest and greatest integers such that r' involves $x_{1,m}$ and $x_{1,M}$. Let H be the group with the following presentation:

$$H = \langle x_{1,m}, \dots, x_{1,M}, x_{i,j}, 2 \leq i \leq n, j \in \mathbb{Z} | r' \rangle.$$

By Magnus' Freiheitssatz, H has the two free subgroups

$$H_1 = \langle x_{1,m}, \dots, x_{1,M-1}, x_{i,j}, 2 \leq i \leq n, j \in \mathbb{Z} \rangle,$$
$$H_2 = \langle x_{1,m+1}, \dots, x_{1,M}, x_{i,j}, 2 \leq i \leq n, j \in \mathbb{Z} \rangle.$$

We define a homomorphism $\phi : H_1 \to H_2$ that takes every generator $x_{i,j}$ of H_1 in a generator of H_2 via $\phi(x_{i,j}) = x_{i,j+1}$.

It's easy to check that G is isomorphic to the HNN-extension $H *_\phi$. □

Theorem 1.5.3. *Let $G = \langle x_1, \dots, x_n | r(x_1, \dots, x_n) \rangle$ be a (non-free) one-relator group with relation r involving at least two generators. Then G can be embedded in a HNN extension of a one-relator group with a shorter relation.*

Proof. If at least one generator appears in r with exponent sum zero, then we can apply Theorem 1.5.2 and we are done. Suppose then that every generator that appears in r has non-zero exponent sum. Without loss of generality we can assume that r involves x_1 and x_2, with $\sigma_{x_1}(r) = k$ and $\sigma_{x_2}(r) = l$.

Let H be the amalgamated product of G with $F_1 = \langle y_1 \rangle \simeq \mathbb{Z}$, the free group of rank one, along the free group of rank one generated by x_2 in G and by y_1^k in F_1. Obviously there is an injection from G to H. We want to prove that G is a HNN extension of a one relator group with defining relation shorter than r.

We have

$$H = G *_{\mathbb{Z}} F_1 = \langle x_1, \dots, x_n, y_1 | r(x_1, \dots, x_n), y_1^k = x_2 \rangle,$$

using Tietze transformations we obtain the relations

$$\begin{aligned} H = \langle x_1, \dots, x_n, y_1 | r(x_1, y_1^k, x_3, \dots, x_n), y_1^k = x_2 \rangle = \\ = \langle x_1, y_1, x_3, \dots, x_n | r(x_1, y_1^k, x_3, \dots, x_n) \rangle = \\ = \langle x_1, y_1, x_3, \dots, x_n, y_2 | r(x_1, y_1^k, x_3, \dots, x_n), y_2 = x_1 y_1^l \rangle = \\ = \langle x_1, y_1, x_3, \dots, x_n, y_2 | r(x_1, y_1^k, x_3, \dots, x_n), x_1 = y_2 y_1^{-l} \rangle = \\ = \langle y_1, y_2, x_3, \dots, x_n | r(y_2 y_1^{-l}, y_1^k, x_3, \dots, x_n) \rangle. \end{aligned}$$

Note that $r(y_2 y_1^{-l}, y_1^k, x_3, \dots x_n)$ involves the generators y_1 and y_2 and that the exponent sum of y_1 is zero. Then by Theorem 1.5.2 H is a HNN extension of a one relator group H_1 with defining relation r' shorter than the relation $r(y_2 y_1^{-l}, y_1^k, x_3, \dots, x_n)$. We have

$$l(r(y_2 y_1^{-1}, y_1^k, x_3, \dots, x_n)) = l(r) + (k-1)\#_{x_2}(r) + \#_{x_1}(r),$$

then, as seen in the proof of Theorem 1.5.2, we can take r' such that

$$\begin{aligned} l(r') = l(r(y_2 y_1^{-1}, y_1^k, x_3, \dots, x_n)) - \sigma_{y_1}(r(y_2 y_1^{-1}, y_1^k, x_3, \dots, x_n)) = \\ = l(r) + (k-1)\#_{x_2}(r) + \#_{x_1}(r) - (k\#_{x_2}(r) + \#_{x_1}(r)) = l(r) - \#_{x_2}(r), \end{aligned}$$

so r' is shorter than r and we have the desired result. □

Finally, we note that it is always possible to know if a given one-relator group is in fact a free group.

Proposition 1.5.1. *Let $G = \langle x_1, \dots, x_n | r \rangle$ be a one-relator group. It is decidable whether G is a free group or not.*

Proof. G is free if and only if r is a free generator of F, the free group with basis X. r is part of a free basis of X if and only if there is an automorphism of F that sends an element of $X^{\pm 1}$ in r. By Proposition 1.3.3, since $X^{\pm 1}$ is finite, it is decidable if there is such an automorphism by checking every generator. □

1.6 Free differential calculus

Let G be a group. A *derivation* from G to a $\mathbb{Z}G$-module M is a map $d : G \to M$ such that $d(gh) = d(g) + g \cdot d(h)$ for all $g, h \in G$. Every derivation can be extended uniquely to a homomorphism d' of abelian groups from $\mathbb{Z}G$ to M such that $d'(\alpha\beta) = d'(\alpha)\epsilon(\alpha) + \alpha d'(\beta)$ for every $\alpha, \beta \in \mathbb{Z}G$, where $\epsilon : \mathbb{Z}G \to \mathbb{Z}$ is the augmentation homomorphism.

Since for any derivation d we have

$$d(1) = d(1 \cdot 1) = d(1) + 1 \cdot d(1) = d(1) + d(1),$$

it is $d(1) = 0$. Since for any $g \in G$ it is

$$0 = d(1) = d(g^{-1}g) = d(g^{-1}) + g^{-1} \cdot d(g),$$

we conclude that $d(g^{-1}) = -g^{-1} \cdot d(g)$

The set $\mathrm{Der}(G, M)$ of derivations from G to M has an obvious $\mathbb{Z}$-module structure. There is an isomorphism $\rho : \mathrm{Der}(G, M) \to \mathrm{Hom}_{\mathbb{Z}G}(\mathcal{G}, M)$, where $\mathcal{G}$ is the augmentation ideal of $\mathbb{Z}G$, given by $\rho(d)(x - 1) = d(x)$ for any $d \in \mathrm{Der}(G, M)$, $x \in G$.

Let $F = \langle x_1, \ldots, x_n \rangle$ be a free group. $\mathcal{F}$, the augmentation ideal of F, is a $\mathbb{Z}G$-free module on the set $\{x_i - 1\}_{i=1}^{n}$. Using the isomorphism above, a choice of n elements $\alpha_1, \ldots, \alpha_n \in M$ identifies a unique derivation $d : F \to M$ with $d(x_i) = \alpha_i$, $1 \leq i \leq n$.

Let $\frac{\partial}{\partial x_i} : F \to \mathbb{Z}F$ be the derivation of F to $\mathbb{Z}F$ such that $\frac{\partial}{\partial x_i}(x_j) = \delta_{i,j}$, we call this derivation a *partial derivative* with respect to x_i. It's easy to check that for any derivation $d : F \to M$ we have

$$d(g) = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} d(x_i) \qquad \forall g \in F.$$

In particular, if we consider the inner derivation $g \mapsto g - 1 \; \forall g \in F$ we obtain the relation

$$g - 1 = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i}(x_i - 1).$$

Let $d\mathbb{Z}F$ be the module of all linear forms $v = \sum_{i=1}^{n} v_i dx_i$ in the indeterminates dx_i with coefficients $v_i \in \mathbb{Z}F$; $d\mathbb{Z}F$ has an obvious F-module structure by taking $g \cdot v = \sum_{i=1}^{n} g \cdot v_i dx_i \ \forall g \in F$. We define a derivation

$$d^* : F \to d\mathbb{Z}F$$

setting

$$d^* g = \sum_{i=1}^{n} \frac{\partial g}{\partial x_i} dx_i.$$

By the observation above it's obvious that any derivation d of F in a F-module M factors in d^* followed by a F-module homomorphism γ from $d\mathbb{Z}F$ to M with $\gamma(dx_i) = d(x_i)$.

Now let $G = \frac{F}{K}$ be a quotient of F by the normal subgroup K. The projection from F to G induces a projection π from $\mathbb{Z}F$ to $\mathbb{Z}G$, whose kernel is the ideal $K - 1$ generated by the elements $k - 1 \in \mathbb{Z}F$ with $k \in K$. We say that two elements $a, b \in \mathbb{Z}F$ are equivalent modulo K, writing $a \equiv b \mod K$, if $\pi(a) = \pi(b)$ in $\mathbb{Z}G$. With this notation we can also define the quotient ring $d\mathbb{Z}G$ of $d\mathbb{Z}F$ by taking coefficients modulo K.

We state some easy properties of d^* with respect to the equivalence modulo K. For any $k_1, k_2 \in K$, $g \in F$, $\epsilon = \pm 1$, it is

$$\begin{gathered} d^*(k_1 k_2) = d^* k_1 + k_1 d^* k_2 \equiv d^* k_1 + d^* k_2 \mod K; \\ d^*(k_1^{-1}) = -k_1^{-1} d^* k_1 \equiv -d^* k_1 \mod K; \\ d^*(g k_1^{\epsilon} g^{-1}) = d^* g + g d^* k_1^{\epsilon} - g k_1^{\epsilon} g^{-1} d^* g \equiv \epsilon g d^* k_1. \end{gathered}$$

In the following theorems and in their proofs we will usually write $a \equiv b$, omitting $\mod K$.

Theorem 1.6.1. *$d^* u \equiv 0 \mod K$ if and only if $u \in [K, K]$.*

Proof. Let $u \in [K, K]$, then $u = \prod_{i=1}^{k} [a_i, b_i]$ with $a_i, b_i \in R$, so

$$d^* u = \sum_{i=0}^{k-1} \left(\left(\prod_{j=1}^{i} [a_j, b_j] \right) d^*[a_i, b_i] \right).$$

But $d^*[a_i, b_i] = d^* a_i + a_i d^* b_i + a_i b_i d^* a_i^{-1} + a_i b_i d a_i^{-1} d^* b_i^{-1} \equiv d^* a_i + d^* b_i - d^* a_i - d^* b_i = 0$, so $d^* u = 0$.

If $d^* u \equiv 0$, then $u - 1 \equiv 0$ and $u \in K$. Writing u in terms of the generators, $u = \prod_{i=1}^{m} x_{n_i}^{\epsilon_i}$ we see that from $d^* u \equiv 0$ follows that the indices can be paired in couples i, j with $x_{n_j} = x_{n_i}$, $\epsilon_i = -\epsilon_j$, and $\prod_{l=1}^{i-1} x_{n_l}^{\epsilon_l} \equiv \left(\prod_{t=1}^{j-1} x_{n_t}^{\epsilon_t} \right) x_j^{-\epsilon_j}$, so $\prod_{l=1}^{i} x_{n_l}^{\epsilon_l} \equiv \prod_{t=1}^{j-1} x_{n_t}^{\epsilon_t}$.

Let $q_k = \prod_{t=1}^{k} x_{n_t}^{\epsilon_t}$ and choose a representative $\bar{q}_k$ for q_k modulo K, with $\bar{q}_0 = \bar{q}_m = 1$. We define $r_k = \bar{q}_{k-1} x_{n_k}^{\epsilon_k} \bar{q}_k^{-1} \in K$, obviously $\prod_{k=1}^{m} r_k = u$. From the previously stated pairing of the indices we deduce that $\bar{q}_{i-1} = \bar{q}_j$, $\bar{q}_i = \bar{q}_{j-1}$. Since $x_{n_i}^{\epsilon_i} = x_{n_j}^{-\epsilon_j}$, it is $r_i = r_j^{-1}$. This means that $u \in [K, K]$. □

Corollary 1.6.1. *The coefficient sums of all the* $\frac{\partial u}{\partial x_i}$ *are zero if and only if* $u \in [F, F]$.

Proof. Taking $K = F$ in the previous theorem we have that $d^* u \equiv 0 \mod F$ if and only if $u \in [F, F]$. The kernel of the projection π from $\mathbb{Z}F$ to $\mathbb{Z}G = \mathbb{Z}\frac{F}{F} \simeq \mathbb{Z}$ is $\mathcal{F}$, the augmentation ideal of $\mathbb{Z}F$, so π is the augmentation homomorphism $\epsilon : \mathbb{Z}G \to \mathbb{Z}$, which sends any element in its coefficient sum. Since $d^* u \equiv 0 \mod F$ if and only if $\frac{\partial u}{\partial x_i} \equiv 0 \mod F$ for $1 \leq i \leq n$, we are done. □

Theorem 1.6.2. *Let* F_0 *be the group generated by a certain subset of the generators* x_i *of the free group* F*, let* K_0 *be the smallest normal subgroup of* F *containing* $F_0 \cap K$*. Then if*

$$v = \sum_{x_i \in F_0} v_i (x_i - 1) \equiv 0 \mod K$$

there exists $r \in K_0$ *such that* $\frac{\partial r}{\partial x_i} \equiv v_i$ *for all* x_i.

Proof. Each v_i is a sum of finitely many elements of F with coefficient ± 1. Since $-(x_i - 1) = x_i (x_i^{-1} - 1)$, we can write

$$v = \sum_{k=1}^{m} w_k (x_{i_k}^{\epsilon_k} - 1)$$

where $w_k \in F$, $\epsilon_k = \pm 1$. We make induction on m.

Suppose the thesis holds for any $l < m$. $v = \sum_{k=1}^{m} w_k (x_{i_k}^{\epsilon_k} - 1) \equiv 0$, so $\sum_{k=1}^{m} w_k x_{i_k}^{\epsilon k} \equiv \sum_{k=1}^{m} w_k$. Then for any index a there is an index b such that $w_a x_{i_a}^{\epsilon a} \equiv w_b$, but m is finite, so there exists an integer $1 \leq q \leq m$ such that, reindexing, we have

$$\begin{gathered} w_2 \equiv w_1 x_{i_1}^{\epsilon_1} \\ w_3 \equiv w_2 x_{i_2}^{\epsilon_2} \equiv w_1 x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \\ \cdots \\ w_{q+1} = w_1 \equiv w_q x_{i_q}^{\epsilon_n} \equiv w_1 x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \cdots x_{i_q}^{\epsilon_q}. \end{gathered}$$

We define $v_1 = \sum_{k=1}^{q} w_k (x_{i_k}^{\epsilon k} - 1)$, so $v_1 \equiv \sum_{k=1}^{q} (w_{k+1} - w_k) = 0$, and we have $v = v_1 + v_2$ with $v_2 = \sum_{k=q+1}^{m} w_k (x_{i_k}^{\epsilon k} - 1)$; since $v \equiv 0$ and $v_1 \equiv 0$, we have also $v_2 \equiv 0$. Collecting the terms of v_1 and v_2, we can write $v_1 = \sum_{x_j \in F_0} v_{1,j}(x_j - 1)$ and $v_2 = \sum_{x_j \in F_0} v_{2,j}(x_j - 1)$. Obviously $v_j = v_{1,j} + v_{2,j}$ for every $1 \leq j \leq n$.

Let $r_1 = x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \cdots x_{i_q}^{\epsilon_n} \in F_0$. Since $w_1 \equiv w_1 r_1$ and $w_1 \in F$, it is $r_1 \in K \cap F_0 \subseteq K_0$, so $w_1 r_1 w_1^{-1} \in K_0$. Computing the partial derivatives we obtain

$$\frac{\partial w_1 r_1 w_1^{-1}}{\partial x_j} = \frac{\partial w_1}{\partial x_j} + w_1 \frac{\partial r_1}{\partial x_j} + w_1 r_1 \frac{\partial w_1^{-1}}{\partial x_j} =$$
$$= \frac{\partial w_1}{\partial x_j} + w_1 \frac{\partial r_1}{\partial x_j} - w_1 r w_1^{-1} \frac{\partial w_1}{\partial x_j} \equiv w_1 \frac{\partial r_1}{\partial x_j},$$

on the other hand it is

$$w_1(r_1 - 1) = w_1 \sum_{j=1}^{n} \frac{\partial r_1}{\partial x_j}(x_j - 1) =$$
$$= w_1 \sum_{k=1}^{q} x_{i_1}^{\epsilon_1} x_{i_2}^{\epsilon_2} \cdots x_{i_{k-1}}^{\epsilon_{k-1}} (x_k^{\epsilon_k} - 1) \equiv \sum_{k=1}^{q} w_k (x_{i_k}^{\epsilon k} - 1) = v_1,$$

so $\frac{\partial w_1 r_1 w_1^{-1}}{\partial x_j} \equiv w_1 \frac{\partial r_1}{\partial x_j} \equiv v_{1,j}$ for every x_j. If $n = m$ we are done; this establish also the first step of the induction process (because if $m = 1$, then $n = m = 1$).

If $n < m$, then for induction hypothesis there exists $r_2 \in K_0$ such that $\frac{\partial r_2}{\partial x_j} \equiv v_{2,j}$ for every v_j. Lat $r = w_1 r_1 w_1^{-1} r_2$, then taking the partial derivatives we have

$$\frac{\partial r}{\partial x_j} = \frac{\partial w_1 r_1 w_1^{-1} r_2}{x_j} = \frac{\partial w_1 r_1 w_1^{-1}}{\partial x_j} + w_1 r_1 w_1^{-1} \frac{\partial r_2}{\partial x_j} \equiv v_{1,j} + v_{2,j} = v_j$$

so the theorem holds. □

Corollary 1.6.2. $\sum_{i=1}^{n} v_i(x_i - 1) \equiv 0 \mod K$ *if and only if there exists* $r \in K$ *such that* $\frac{\partial r}{\partial x_i} \equiv v_i$, $1 \leq i \leq n$.

Proof. If $r \in K$ and $v_i = \frac{\partial r}{\partial x_i}$ for every x_i, then $\sum_{i=1}^{n} v_i(x_i - 1) \equiv \sum_{i=1}^{n} \frac{\partial r}{\partial x_i}(x_i - 1) = r - 1 \equiv 0 \mod K$.

If $\sum_{i=1}^{n} v_i(x_i - 1) \equiv 0 \mod K$, taking $F_0 = F$ in the previous theorem we have that there exists $r \in K$ such that $\frac{\partial r}{\partial x_i} \equiv v_i \mod K$ for every x_i. □

We conclude this section with a result on zero divisors in $\mathbb{Z}G$.

Proposition 1.6.1. *Let $g \in G$ such that $g^q \equiv 1 \mod K$ for some $q \in \mathbb{N}$. Let $s = \sum_{i=0}^{q-1} g^q$.*

If $u(g-1) \equiv 0$, then $u \equiv vs$ for some v.

If $us \equiv 0$ then $u \equiv v(g-1)$ for some v.

Proof. For $x \in \mathbb{Z}G$, if $x = \sum_{i=1}^{n} a_i g_i$ with $a_i \in \mathbb{Z}$, $g_i \in G$, with $g_i \neq g_j$ if $i \neq j$, we define $|x| = \sum_{i=1}^{n} |a_i|$.

Let $u(g-1) \equiv 0$. If $|u| = 0$, then $u = 0 = 0 \cdot s$. Suppose the result holds for $|u| < m$, let $\bar{u} \in \mathbb{Z}G$ with $\bar{u} = m$ and $\bar{u}(g-1) \equiv 0$. We can write $\bar{u}$ as $\sum_{i=1}^{m} \epsilon_i g_i$ where $\epsilon_i = \pm 1$, $g_i \in G$. Since $\bar{u}(g-1) \equiv 0$, it is $\sum_{i=1}^{m} \epsilon_i g_i g = \sum_{i=1}^{m} \epsilon_i g_i$, then up to a reindexing, it must be

$$g_1 \equiv g_0 g \qquad g_2 \equiv g_1 g \qquad g_0 g^2 \qquad \dots \qquad g_{q-1} \equiv g_{q-2} g \equiv g_0 g^{q-1}$$

so $\bar{u} \equiv g_0 s + u'$, with $|u'| < |u| = m$, so by the induction hypothesis $u' \equiv v's$ for some s'. Taking $v = g_0 + v'$ it is $\bar{u} \equiv vs$.

Let $us \equiv 0$. We can write u as $\sum_{i=1}^{m} \epsilon_i g_i$ where $\epsilon_i = \pm 1$, $g_i \in G$ and $m = |u|$, so $us = \sum_{i=1}^{m} \sum_{j=1}^{q-1} \epsilon_i g_i g^j \equiv 0$. Then for every couple (i_1, j_1) there is a couple (i_2, j_2) such that $g_{i_1} g^{j_1} \equiv g_{i_2} g^{j_2}$, hence $g_{i_1} \equiv g_{i_2} g^{j_2 - j_1}$. Then we can decompose u as a sum of elements $\tilde{u}$ of $\mathbb{Z}G$ such that $\tilde{u} \equiv u' \sum_{k=1}^{t} n_k g^k$ with $n_k \in \mathbb{Z}$ and $u' \in G$, and $\tilde{u}s \equiv 0$. But $g^k s \equiv s$, so $\tilde{u}s \equiv \sum_{k=1}^{t} n_k s \equiv 0$, but then it must be $\sum_{k=1}^{t} n_k = 0$. Then the polynomial $\sum_{k=1}^{t} n_k g^k$ admits $g-1$ as a factor, so we can write $\tilde{u} \equiv u'\tilde{v}(g-1)$ for some $\tilde{v}$. Since u is sum of elements of this form, $u \equiv v(g-1)$ for some v. □

1.7 Identity Theorem

The aim of this section is to establish Lyndon's Identity Theorem (see [14]).

Throughout this section, let F be a free group on generators $x_1, \dots, x_{n+s}$, $y_1, \dots, y_m$ and $R_1, \dots, R_n$ be cyclically reduced words in $\langle x_1, \dots, x_{n+s} \rangle \subseteq F$ such that t and $t+s$ are respectively the least and greatest indices of the x_i involved in R_t, with $R_t = Q_t^{q_t}$ for q_t maximal. Let also K be the smallest normal subgroup of F containing all the R_t and $G = \frac{F}{K}$ the quotient group.

Theorem 1.7.1 (Identity Theorem)**.** *If $\prod_{i=1}^{m} T_i R_{t_i}^{\epsilon_i} T_i^{-1} = 1$ with $T_i \in F$, $\epsilon_i = 1$ and $1 \leq t_i \leq n$ for every $1 \leq i \leq m$, then the indices $1, \dots, m$ fall into pairs (i, j) such that $t_i = t_j$, $\epsilon_i = -\epsilon_j$ and there are $c_i \in \mathbb{Z}$ such that $T_i \equiv T_j Q_{t_i}^{c_i} \mod K$.*

If $n = 1$ we can drop the hypothesis of cyclical reduction of R_1, stating the Simple Identity Theorem.

Theorem 1.7.2 (Simple Identity Theorem)**.** *Let $R = Q^q$ for q maximal be a word in F free group, and K the smallest normal subgroup of F containing R. If $\sum_{i=1}^m T_i R^{\epsilon_i} T_i^{-1} = 1$, with $T_i \in F$ and $\epsilon_i = \pm 1$ for every i, then the indices can be grouped in pairs (i, j) such that $\epsilon_i = \epsilon_j$ and there is $c_i \in \mathbb{Z}$ such that $T_i \equiv T_j Q^{c_i} \mod K$.*

We will use the following consequence of the Freiheitssatz and some results on free products to reduce the Identity Theorem to the Simple Identity Theorem.

Proposition 1.7.1. *The Identity Theorem is equivalent to the theorem obtained taking the hypothesis*

$$\prod_{i=1}^m T_i R_{t_i}^{\epsilon_i} T_i^{-1} \in [K, K]$$

instead of

$$\prod_{i=1}^m T_i R_{t_i}^{\epsilon_i} T_i^{-1} = 1.$$

Proof. Obviously if the theorem holds for $\prod_{i=1}^m T_i R_{t_i}^{\epsilon_i} T_i^{-1} \in [K, K]$ then it holds for $\prod_{i=1}^m T_i R_{t_i}^{\epsilon_i} T_i^{-1} \in [K, K]$.

Vice versa, if $P = \prod_{i=1}^m T_i R_{t_i}^{\epsilon_i} T_i^{-1} \in [K, K]$ then we can write it as a product P' with indices paired so that corresponding T_i' are equal. Applying the identity theorem to PP'^{-1} we obtain a pairing on the indices of the product from which we get the pairing on P. □

Proposition 1.7.2. *Let $F_1 = \langle x_1, \ldots, x_{t+s} \rangle$, $F_2 = \langle x_{t+1}, \ldots, x_{n+s}, y_1, \ldots, y_m \rangle$, and $G_1 = \frac{F_1}{K \cap F_1}$, $G_2 = \frac{F_2}{K \cap F_2}$. Then $G \cong G_1 *_{G_0} G_2$ with G_0 free.*

Proof. By the Freiheitssatz, $F_1 \cap K$ is generated in F_1 by $R_1, \ldots, R_t$, while $F_2 \cap K$ is generated in F_2 by $R_{t+1}, \ldots, R_n$. $G_0 \simeq \frac{F_1 \cap F_2}{F_1 \cap F_2 \cap K}$, but $F_1 \cap F_2 = \langle x_{t+1}, \ldots, x_{t+s} \rangle$, so by the Freiheitssatz it is $F_1 \cap F_2 \cap K = \langle 1 \rangle$ and G_0 is free. □

Proposition 1.7.3. *Let $G = G_1 *_{G_0} G_2$ with G_0 free, with $G = \frac{F}{K}$ where F is a free group and $G_i = \frac{F_i}{K_i}$ where F_i is a subgroup of F and $K_i = F_i \cap K$, $i = 0, 1, 2$. Let (K, K) be the smallest normal subgroup of F containing $[K, K]$ and (K_1, K_1) be the smallest normal subgroup of F containing $[K_1, K_1]$. If*

$$P = \prod_{i=1}^m u_i R_i u_i^{-1} \in (K, K),$$

where $u_i \in F$ and $R_i \in K_1$ or $R_i \in K_2$, then there exist $v_i \equiv u_i \mod K$ such that

$$P' = \prod_{1 \le i \le m, R_i \in K_1} v_i R_i v_i^{-1} \in (K_1, K_1).$$

Proof. We can rearrange the factors of P to obtain a product $Q = P_1 P_2$ where P_1 is the product of the factors $u_i R_i u_i^{-1}$ with $R_i \in K_1$, while P_2 is the product of the factors with $R_i \in K_2$. Obviously it is $Q \in (K, K)$.

$d(P_1P_2) = dP_1 + P_1 dP_2 \equiv dP_1 + dP_2 \equiv 0 \mod K$, since

$$d(u_i R_i u_i^{-1}) = du_i + u_i dR_i - u_i R_i u_i^{-1} du_i \equiv du_i + u_i dR_i - du_i = u_i dR_i$$

we have that $\frac{\partial P_1}{\partial x_i} \equiv 0 \mod K$ for all x_i such that $x_i \notin F_0$. By Proposition 1.6.2 $dP_1 \equiv dP_0$ for some $P_0 \in K_0$, but G_0 is free so we can choose F such that $K_0 = \langle 1 \rangle$, thus it is $dP_1 \equiv 0$.

Fix a system α of representatives for the cosets of F modulo F_1, then any $u \in F$ can be written uniquely as $u = fw$ with $f \in \alpha$ and $w \in F_1$. Fixing also a system β of representatives for the cosets of F_2 modulo K_2 we obtain that every $f \in \alpha$ can be written uniquely as $f = gr$ with $g \in \beta$, $r \in F_2$, so every $u \in F$ can be written uniquely as $u = fgw$, with $f \in \alpha$, $g \in \beta$ and $w \in F_1$. Decomposing the u_i appearing in P_1 as $u_1 = f_i g_i w_i$ in this way and rearranging its factors so that those with the same g_i are grouped together, we obtain

$$\bar{P}_1 = \prod_h g_h (\prod_k r_{hk} w_{hk} R_{hk} w_{hk}^{-1} r_{hk}^{-1}) g_h^{-1}.$$

Since $r_{hk} \in K_2$ then $v_{hk} = g_h w_{hk} \equiv g_h r_{hk} w_{hk} = u_{hk}$. Define now

$$P_1' = \prod_h g_h (\prod_k w_{hk} R_{hk} w_{hk}^{-1}) g_h^{-1} = \prod_{hk} v_{hk} R_{hk} v_{hk}^{-1},$$

this prouct is of the form required, so we only need to show that it lies in (K_1, K_1).

Since $r_{hk} \in K$ it is $dP_1' \equiv \sum g_h \sum w_{hk} dR_{hk} \equiv dP_1 \equiv 0 \mod K$. But the g_h were taken as representatives of the cosets of F_2 mod $K_2 = F_2 \cap K$, so it must be $\sum w_{hk} dR_{hk} \equiv 0$ for any h. $w_{hk} \in F_1$ and $R_{hk} \in F_1$, so we have $\sum w_{hk} dR_{hk} \equiv 0 \mod K_1$ and by Proposition 1.6.1 we have $P_h = \prod_k w_{hk} R_{hk} w_{hk}^{-1} \in [K_1, K_1]$ for every h, so $P_1' = \sum_h g_h P_h g_h^{-1} \in (K_1, K_1)$. □

Proposition 1.7.4. *If the Simple Identity theorem holds for each R_t then the Identity Theorem holds for $R_1, \ldots, R_n$.*

Proof. Fix $t \in \mathbb{N}$, $1 \leq t \leq n$. By Proposition 1.7.2 we can decompose G as $G_1 *_{G_0} G_2$ with G_0 free, with $F_1 = \langle x_1, \ldots, x_{t+s} \rangle$, $K_1 = F_1 \cap K$ and $G_1 \cong \frac{F_1}{K_1}$. Let $\prod_{i=1}^m T_i R_{t_i}^{\epsilon_i} T_i^{-1} = 1$ with $T_i \in F$, by Proposition 1.7.3 there is a product

$$P_1' = \prod_{j=1}^{m'} v_j R_{t_j}^{\epsilon_j} v_j^{-1} \in (K_1, K_1),$$

where (K_1, K_1) is the smallest normal subgroup of F containing $[K_1, K_1]$, the index j runs through the i such that $t_i \leq t$ and $v_j \equiv T_j \mod K$.

We apply Proposition 1.7.2 again to obtain a decomposition of G as

$$G = G_1' *_{G_0'} G_2'$$

with G_0' free, where $G_1' = \frac{F_1'}{F_1' \cap K}$ and $F_1' = \langle x_1, \ldots, x_{t-1+s} \rangle$. Now we apply Proposition 1.7.3 to P_1', obtaining that there is a product

$$P_{12}'' = \prod_{k=1}^{m''} w_k R_{t_k}^{\epsilon_k} w_k^{-1} \in (K_t, K_t),$$

where $K_t = K \cap \langle R_t \rangle$ and (K_t, K_t) is the smallest normal subgroup of F containing $[K_t, K_t]$, the indiex k runs through the j such that $t_i = t$ and $w_k \equiv v_k \equiv T_k \mod K$.

By Proposition 1.7.1 if the Simple Identity Theorem holds for R_t we can apply it to P_{12}''. Then the indices $\bar{i}$ such that $t_{\bar{i}} = t$ admit a pairing (i, j) with $\epsilon_i = -\epsilon_j$ and such that there are $c_i \in \mathbb{Z}$ with $T_i \equiv w_i \equiv w_j Q_t^{c_i} \equiv T_j Q_t^{c_i}$ mod K.

Since we can find a pairing as above for every $1 \leq t \leq n$, the Identity Theorem holds. □

Proposition 1.7.5. *The Simple Identity Theorem holds for any power of a free generator of F.*

Proof. Without loss of generality, let $R = x_1^q$ with $q > 0$. It is $dR = (\sum_{j=1}^{q-1} x_1^j) dx_1$, or, in the notation of Proposition 1.6.1, $dR = s dx$. If we have an identity $\sum_{i=1}^m T_i R^{\epsilon_i} T_i^{-1} = 1$ then

$$d\left(\sum_{i=1}^m T_i R^{\epsilon_i} T_i^{-1} = 1\right) = \sum_{i=1}^m \epsilon_i T_i dR = \left(\sum_{i=1}^m \epsilon_i T_i\right) s dx_1 \equiv 0$$

so $(\sum_{i=1}^m \epsilon_i T_i)\, s \equiv 0$ and by Proposition 1.6.1 it is $\sum_{i=1}^m \epsilon_i T_i \equiv \sum_{i=j}^{m'} \epsilon_j T_j'$, $T_j' \in F$, where $\sum_{i=j}^{m'} \epsilon_j T_j'$ is divisible by $x_1 - 1$, so there is a pairing (j_1, j_2)

on the indices such that $\epsilon_{j_1} = -\epsilon_{j_2}$ and $T'_{j_1} \equiv T'_{j_2} x_1^{c_i}$ for some $c_i \in \mathbb{Z}$. Adding zeroes if necessary, we can take $m = m'$, and up to reindexing we have $T_i \equiv T'_i$, $1 \leq i \leq m$, so the pairing on the indices j induces the desired pairing on the indices i. □

In particular, the Simple Identity Theorem holds for any $R = Q^q \in F$, q maximal, such that $l(Q) = 1$. We will make induction on $k = l(Q)$, bearing in mind that if the Simple Identity Theorem holds for all R with $l(Q) < k$ then by Theorem 1.7.4 the Identity Theorem holds for any collection $R_1, \ldots, R_n$ with $l(Q_j) < k$, $1 \leq j \leq n$. Furthermore, by Theorem 1.7.5 we can suppose that Q involves at least two generators.

Proposition 1.7.6. *Suppose $l(Q) = k$ and that Q involves at least two free generators of F, one of which with exponent sum zero. Then under the induction hypothesis the Simple Identity Theorem holds for R.*

Proof. Without loss of generality let x_1 be the generator involved in Q with exponent sum zero.

The one-relator group $G = \langle F | R \rangle$ can be expressed by Theorem 1.5.2 as a HNN extension of a one-relator group H with a shorter relation $R' = Q'^{q'}$. x_1 plays the role of t in the proof of Theorem 1.5.2 and obviously any $f \in F$ can be written as a product $\bar{f} x_1^s$ for some $\bar{f} \in F_2$ and $s \in \mathbb{Z}$, where $F_2 \leq F$ is the free group on the generators $x_{j,k} = x_1^k x_j x_1^{-k}$, $y_{j,k} = x_1^k y_j x^{-k}$, whose quotient is H.

If we define $Q_i = x_1^i Q x_1^{-i}$ and $R_i = Q_i^q = x_1^i R x_1^{-i}$, then any identity $\sum_{i=1}^m T_i R^{\epsilon_i} T_i^{-1} = 1$ can be written as $\sum_{i=1}^m \bar{T}_i R_{j_i}^{\epsilon_i} \bar{T}_i^{-1} = 1$ with $\bar{T}_i \in F_2$. Since $\sigma_{x_1}(R_j) = 0$, every R_{j_i} is an element of F_2. Moreover, $K \triangleleft F_2$ and it is the smallest normal subgroup of F_2 containing every R_t.

Writing $R_0 = R$ in terms of the generators $x_{j,k}$ of F for every j let k_{j1} and k_{j2} be respectively the least and greatest integer such that R involves $x_{j,k_{js}}$, $s = 1, 2$, then for every j the least and greatest integers such that R_t involves $x_{j,k}$ are $k_{j,k_{j1}}$ and $k_{j,k_{j2}}$ respectively. Since the identity $\sum_{i=1}^m \bar{T}_i R_{j_i}^{\epsilon_i} \bar{T}_i^{-1} = 1$ contains only a finite number of R_t, we can reindex the R_t and rearrange the generators $x_{j,k}$ so that the Identity Theorem is applicable to the identity.

Since the length of any Q_t expressed in the free generators of F_2 is $l(Q) - \#_{x_i}(Q) < k$, by induction hypothesis the Identity Theorem holds for the relevant R_t, then we have a pairing (i_1, i_2) on the indices such that $\epsilon_{i_1} = -\epsilon_{i_2}$, $t_{i_1} = t_{i_2}$ and $\bar{T}_{i_1} \equiv \bar{T}_{i_2} Q_{t_{i_2}}^{c_{i_2}} \mod K$ for some $c_{i_2} \in \mathbb{Z}$. But then $T_{i_1} = \bar{T}_{i_1} x_1^{t_{i_1}} \equiv \bar{T}_{i_2} Q_{t_{i_2}}^{c_{i_2}} x_1^{t_{i_1}} = \bar{T}_{i_2} x_1^{t_{i_1}} Q^{c_{i_2}} = T_{i_2} Q^{c_{i_2}}$ and we have the desired pairing on the original identity. □

Proposition 1.7.7. *Suppose $l(Q) = k$ and that Q involves at least two free generators of F, but none of the free generators involved in Q has exponent sum zero. Then under the induction hypothesis the Simple Identity Theorem holds for R.*

Proof. Without loss of generality we can assume that Q involves both x_1 and x_2.

From Theorem 1.5.3 we have that G can be embedded in a HNN extension H of a one-relator group with defining relation $R' = Q'^q$ of length less than $k = l(R)$. Using letters z_1 and z_2 to express the generators y_1, y_2 of H in the proof of Theorem 1.5.3, we can write Q as a word Q' in $F_2 = \langle z_1, z_2, x_3, \ldots, x_n \rangle$ via the obvious injection τ of F in F_2.

Then $\sigma_{z_1}(Q') = 0$, $\#_{z_2}(Q') = \#_{x_2}(Q)$ and $\#_{x_i}(Q') = \#_{x_i}(Q)$ for every $i \neq 1, 2$, so by the previous Proposition the Simple Identity Theorem can be applied to Q' in F_2. Then for any identity $\sum_{i=1}^{m} T_i R^{\epsilon_i} T_i^{-1} = 1$ in F expressed in terms of $z_1, z_2, x_3, \ldots, x_n$ the indices fall into pairs (i_1, i_2) with $\epsilon_{i_1} = -\epsilon_{i_2}$ and there is $c_{i_1} \in \mathbb{Z}$ with $T_{i_1} \equiv T_{i_2} Q'^{c_{i_1}} \mod \bar{K}$, where $\bar{K}$ is the smallest normal subgroup of F_2 containing R. Since T_{i_1}, T_{i_2} and Q' are in the image of F in F_2 via τ, then we can conclude that $T_{i_1} \equiv T_{i_2} Q^{c_{i_1}} \mod K$, thus proving that the Simple Identity Theorem holds for R. □

Chapter 2

Surface group conjecture

In this chapter we state the surface group conjecture, recalling first the relevant definitions about duality groups. Then

2.1 Group homology

Let G be a group. The *integral group ring* $\mathbb{Z}G$ is the free $\mathbb{Z}$ module generated by the elements of G.

A left $\mathbb{Z}G$-module or *G-module*, consists of an abelian group A together with a homorphism from $\mathbb{Z}G$ to the ring of endomorphisms of A, or equivalently of an abelian group A together with an action of G on A.

Let $(P_n, \delta_n)_{n\in\mathbb{N}}$ be a projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$, that is an exact sequence

$$\ldots \to P_3 \xrightarrow{\delta_2} P_2 \xrightarrow{\delta_1} P_1 \xrightarrow{\delta_0} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

with P_i a projective $\mathbb{Z}G$-module for every $i \in \mathbb{N}$ and where $\varepsilon : \mathbb{Z}G \to \mathbb{Z}$ is the augmentation map.

We call $\mathrm{H}_i(G; M)$ the i-th homology group of the chain complex obtained by $(P_n, \delta_n)_{n\in\mathbb{N}}$ applying the functor $- \otimes_G M$. It is the left derived functor of the right exact functor that associates to a $\mathbb{Z}G$ module M the group M_G of co-invariants of M, that is the quotient of M by the additive subgroup generated by elements $gm - m$, $m \in M$, $g \in G$.

We call $\mathrm{H}^i(G; M)$ the i-th homology group of the cochain complex obtained by $(P_n, \delta_n)_{n\in\mathbb{N}}$ applying the functor $\mathrm{Hom}(-; M)$. It is the right derived functor of the left exact functor that associates to a $\mathbb{Z}G$ module M the group M^G of invariants of M under the action of G.

2.2 Ends

Definition (Cayley graph)**.** Let G be a finitely generated group and S a finite set of generators for G, $1_G \notin S$, $S = S^{-1}$. The *Cayley graph* $\Gamma_{G,S}$ of G with respect to S is the graph whose set of vertices is G and whose edges are given by (g, gs) for any $g \in G$, $s \in S$.

Since S is a finite set of generators, the Cayley graph is a connected and locally finite graph.

Theorem 2.2.1. *Let G be a finitely generated group, let S and T be two finite sets of generators for G, $1_G \notin S$, $1_G \notin T$, $S = S^{-1}$, $T = T^{-1}$. If Γ_S and Γ_T are the Cayley graphs of G with respect to S and T respectively, then there are maps $\phi_{TS} : \Gamma_T \to \Gamma_S$ and $\phi_{ST} : \Gamma_S \to \Gamma_T$ such that:*

1. *$\phi_{TS} \circ \phi_{ST}$ and $\phi_{ST} \circ \phi_{TS}$ induce the identity on the set of vertices of Γ_S and Γ_T respectively;*
2. *there is $N \in \mathbb{N}$ such that $\phi_{TS} \circ \phi_{ST}$ sends any edge $e = (g, h)$ of Γ_S in the ball $B(h, N)$ of Γ_S, and $\phi_{ST} \circ \phi_{TS}$ sends any edge $e' = (g', h')$ of Γ_T in the ball $B(h', N)$ of Γ_T.*

Proof. Let the maps ϕ_{TS} and ϕ_{ST} be the identity maps on the set of vertices, then the first condition is trivially met.

For every $s \in S$, let $w_s = \prod_{i=1}^{m} t_i$ be a word in the alphabet $T \cup T^{-1}$ that expresses s in the set of generators T. Let $e = (g, gs)$ be an edge of Γ_S, then the map ϕ_{ST} sends e to the path given by $(g, gt_1)(gt_1, gt_1t_2) \ldots (g \prod_{i=1}^{m-1} t_i, gs)$.

The map ϕ_{TS} is defined similarly, for every $t \in T$ let $w_t = \prod_{i=1}^{k} s_i$ be a word in the alphabet $S \cup S^{-1}$ that expresses t in the set of generators s. Then the map ϕ_{TS} sends an edge (g, gt) to the path from g to gt described by w_t.

Let M be the maximal length of the words w_s and w_t for $s \in S$, $t \in T$. Then the image of any edge in S under ϕ_{ST} and of every edge in T under ϕ_{TS} is a path of length at most M. Then M^2 can be taken as the constant for the second condition of the theorem. □

If Γ is a connected and locally finite graph, let $B_\Gamma(n)$ be the ball of radius n in Γ based on a fixed vertex. We call $C_\Gamma(n)$ the number of connected and unbonunded components of $\Gamma \setminus B_\Gamma(n)$.

Proposition 2.2.1. *Let Γ be a locally finite graph. If $n < m$ then $C_\Gamma(n) \leq C_\Gamma(m)$.*

Proof. Let Y be an unbounded connected component of $\Gamma \setminus B_\Gamma(n)$. Then $Y \setminus B_\Gamma(m)$ is unbounded and it is either connected (and thus still an unbounded connected component) or not connected, in this case it contains at least one unbounded connected component. Then the number of unbounded connected components of $\Gamma \setminus B_\Gamma(m)$ is at least $C_\Gamma(n)$. □

Definition (Ends of a graph)**.** Let Γ be a connected, locally finite graph. Let $B_\Gamma(n)$ be the ball of radius n based on a fixed vertex v of Γ. The number of ends $e(\Gamma)$ of Γ is defined as

$$e(\Gamma) = \lim_{n\to\infty} C_\Gamma(n)$$

The limit in the definition exists because $C(n)$ is a non-decreasing succession. Furthermore, the limit does not depend on the choice of vertex v. The number of ends can also be computed using finite subgraphs instead of balls of fixed radius.

We want to consider the number of ends of the Cayley graph. The first step is to establish the independence of the limit from the choice of system of generators.

Proposition 2.2.2. *Let G be a finitely generated group, let S and T be two finite sets of generators for G, $1_G \notin S$, $1_G \notin T$, $S = S^{-1}$, $T = T^{-1}$. If Γ_S and Γ_T are the Cayley graphs of G with respect to S and T respectively, let $B_S(n)$ and $B_T(n)$ be the balls of radius n in Γ_S and Γ_T respectively. Then there is a constant K such that if g and h are vertices in Γ_S that can be joined by an edge path outside of $B_S(Kn + K)$ then g and h are outside of $B_T(n)$ in Γ_T and can be joined by a path without edges in $B_T(n)$.*

Proof. Let Φ_{ST} be the map defined as in Theorem 2.2.1. Let λ be the length of the longest expression chosen to represent the generators in S as words in T, then the distance in Γ_T from g to h is at least equal to the distance in Γ_S from g to h divided by λ. Let $K = \lambda^2 + 1$.

Let $W = \{g, g_1, \ldots, g_{n-1}, h\}$ be the vertices that occur on an edge path from g to h contained in $\Gamma_S \setminus B_S(Kn + K)$. Then the distance in T between any two vertices in W is greater than $n+\lambda$, so they are all outside of $B_T(n+\lambda)$.

If e is an edge from g_i to g_{i+1} in Γ_S, then $\Phi_{ST}(e)$ is an edge path of length at most λ. Then $\Phi_{ST}(e)$ is outside $B_T(n)$, so Φ_{ST} takes the path from g to h in $\Gamma_S \setminus B_S(Kn + K)$ to a path from g to h in $\Gamma_T \setminus B_T(n)$. □

Theorem 2.2.2. *Let G be a finitely generated group, let S and T be two finite sets of generators for G, $1_G \notin S$, $1_G \notin T$, $S = S^{-1}$, $T = T^{-1}$. If Γ_S and Γ_T are the Cayley graphs of G with respect to S and T respectively, then $e(\Gamma_S) = e(\Gamma_T)$.*

Proof. By the previous proposition, if two vertices in Γ_S can be connected with a path in $\Gamma_S \setminus B_S(Kn + K)$ then that vertices can be connected in Γ_T with a path in $\Gamma_T \setminus B_T(n)$. Then the number of unbounded connected components of $\Gamma_S \setminus B_S(Kn+K)$ is at least equal to the number of unbounded connected components of $\Gamma_T \setminus B_T(n)$. We conclude that

$$\lim_{n\to\infty} C_{\Gamma_S}(n) \geq \lim_{n\to\infty} C_{\Gamma_T}(n)$$

so $e(\Gamma_S) \geq e(\Gamma_T)$.

Using Φ_{TS} instead of Φ_{ST}, we conclude that $e(\Gamma_T) \geq e(\Gamma_S)$, so the equality holds. □

This theorem proves that the number of ends doesn't depend on the choice of set of generators for G. Thus it is possible to define the concept of ends of a group recurring to the ends of its Cayley graph.

Definition (Ends of a group)**.** Let G be a finitely generated group. The number of ends of G, denoted with $e(G)$, is the number of ends of any of its Cayley graphs.

A finitely generated group can only have zero, one, two, or infinitely many ends. This is stated in the Freudenthal-Hopf Theorem.

Theorem 2.2.3 (Freudenthal-Hopf Theorem)**.** *Every finitely generated group has either zero, one, two, or infinitely many ends.*

Proof. Let G be a finite group, then its Cayley graph is a finite graph and obviously $e(G) = 0$. In fact, since a finitely generated group H with $e(H) = 0$ must have a bounded Cayley graph, H is finite. So a group has zero ends if and only if it is finite.

The free group F_1 with only one generator is isomorphic to $\mathbb{Z}$, its Cayley graph with respect to the free generator and its inverse is an unbounded sequence of edges. Then $e(F_1) = 2$, since removing any finite sequence of edges leaves two unbounded connected components.

The free abelian group A_2 with two generators is isomorphic to $\mathbb{Z} \oplus \mathbb{Z}$. Its Cayley graph with respect to the free generators and their inverses is a grid, removing a ball of finite radius leaves one unbounded connected component, so $e(A_2) = 1$.

The free group F_2 with two generators has a Cayley graph, with respect to the free generators and their inverses, that is a tree with valency 4 on each vertex. Then removing a ball of radius n gives a number of unbounded connected components strictly increasing when n increases, so $e(F_2) = \infty$.

Thus a finitely generated group can have zero, one, two or infinitely many ends. We need only to prove that there is no group G with $e(G)$ finite but $e(G) > 2$.

Suppose G is a finitely generated group with a Cayley graph Γ with k ends, $k \in \mathbb{N}$, $k \geq 3$. Then G is an infinite group since $e(G) \neq 0$ and there is a number n such that $\Gamma \setminus B(n)$ has k unbounded connected components.

Since G is infinite, there is an element $g \in G$ whose distance from 1_G is greater than $2n$, and that is a vertex of an unbounded connected component of $\Gamma \setminus B(n)$. Then $g \cdot B(n) \cap B(n) = \emptyset$, and $g \cdot B(n)$ is contained in an unbounded connected component of $\Gamma \setminus B(n)$. $g \cdot B(n)$ divides this component into at least k connected pieces, and at least $k-1$ of them must be unbounded. Then $B(n) \cup g \cdot B(n)$ is a finite subgraph of Γ whose removal leaves at least $2k-2$ unbounded connected components, so $e(\Gamma) \geq 2k-2 > k$ since $k \geq 3$, this contradicts $e(G) = k$. □

The number of ends is an invariant for subgroups of finite index. We refer to [17] for the proof.

Theorem 2.2.4. *Let G be a finitely generated group and H a subgroup of G of finite index. Then $e(G) = e(H)$.*

We have already seen that $e(G) = 0$ if and only if G is finite. We now want to characterize groups with 2 or infinite ends.

Proposition 2.2.3. *Let G be a finitely generated group with $e(G) = 2$, let C be a subgraph of a Cayley graph Γ of G such that $\Gamma \setminus C$ consists exactly of two unbounded connected components, let E be the set of vertices of one of the components. Then the subset*

$$H = \{g \in G | E \Delta gE \text{ is finite}\}$$

forms a subgroup of index at most 2, where Δ denotes the symmetric difference.

Proof. We need to prove that H is a subgroup.

Since $|E \Delta h^{-1}E| = |hE \Delta E|$, if $h \in H$ then $hE \Delta E$ is finite, so also $h^{-1} \in H$ and H is closed under inverses.

Let $h_1, h_2 \in H$, then $h_1 E \Delta E$ and $h_2 E \Delta E$ are finite. We have

$$\begin{aligned} E \Delta h_1 h_2 E = (E \Delta h_1 E) \Delta (h_1 E \Delta h_1 h_2 E) = \\ = (E \Delta h_1 E) \Delta h_1 (E \Delta h_2 E) \end{aligned}$$

so $h_1 h_2 \in H$ since the symmetric difference of two finite sets is finite. So H is closed under products and it is a subgroup of G.

Suppose $H \neq G$, let $g_1, g_2 \in G \setminus H$. Then it is

$$E\Delta g_1 g_2^{-1} E = (E\Delta g_1 E)\Delta(g_1 E\Delta g_1 g_2^{-1} E) =$$
$$= (E\Delta g_1 E)\Delta g_1(E\Delta g_2^{-1} E) = (E\Delta g_1 E)^c \Delta g_1 (E\Delta g_2^{-1} E)^c$$

Since G has two ends and $E\Delta g_1 E$ and $E\Delta g_2 E$ are infinite because $g_1, g_2 \notin H$, the sets $E\Delta g_1 E$ and $E\Delta g_2^{-1} E$ must be finite, so $E\Delta g_1 g_2^{-1} E$ is finite and the index of H in G is two. □

Theorem 2.2.5. *Let G be a finitely generated group. Then $e(G) = 2$ if and only if G contains a finite index subgroup isomorphic to $\mathbb{Z}$.*

Proof. Suppose $e(G) = 2$, let C be a subgraph of a Cayley graph Γ of G such that $\Gamma \setminus C$ consists exactly of two unbounded connected components, let E be the set of vertices of one of the components. Then $H = \{g \in G | E\Delta g E \text{ is finite}\}$ is a subgroup of G by the previous proposition. If $h \in H$, $E\Delta hE$ is finite, so $E \cap hE^c$ and $E^c \cap hE$ are finite. Let $\phi : H \to \mathbb{Z}$ be the function that sends each $h \in H$ to $|E \cap hE^c| - |E^c \cap hE|$

If $h' \in H$, then $E \cap hE^c$ is the disjoint union

$$E \cap hE^c = (E \cap hE^c \cap hh'E^c) \cup (E \cap hE^c \cap h'hE^c)$$

and $E^c \cap hE$ is the disjoint union

$$E^c \cap hE = (E^c \cap hE \cap hh'E) \cup (E^c \cap hE \cap h'hE)$$

so it is

$$\phi(h) = |E \cap hE^c \cap hh'E^c| + |E \cap hE^c \cap h'hE^c| - |E^c \cap hE \cap hh'E| - |E^c \cap hE \cap h'hE|$$

On the other hand

$$\phi(h') = |E \cap h'E^c| - |E^c \cap h'E| =$$
$$= |hE \cap hh'E^c| - |hE^c \cap hh'E|$$

We can divide $hE \cap hh'E^c$ and $hE^c \cap hh'E$ in the subsets of vertices that are in E and that in E^c, so it is

$$\phi(h') = |E \cap hE \cap hh'E^c| + |E^c \cap hE \cap hh'E^c| - |E \cap hE^c \cap hh'E^c| - |E^c \cap hE^c \cap hh'E^c|$$

Then we obtain

$$\phi(h) + \phi(h') =$$
$$= |E \cap hE^c \cap hh'E^c| + |E \cap hE \cap hh'E^c| - |E^c \cap hE \cap hh'E| - \left|E^c \cap hE^c \cap hh'E^E\right| =$$
$$= |E \cap hh'E^c| - |E^c \cap hh'E| = \phi(hh')$$

Thus ϕ is a group homomorphism from H to $\mathbb{Z}$.

C is a finite subgraph of Γ, so there are finitely many elements $h \in H$ such that $C \cap hC \neq \emptyset$. If $C \cap hC = \emptyset$, then either $E \cap hE^c = \emptyset$ and $E^c \cap hE \neq \emptyset$, or $E^c \cap hE = \emptyset$ and $E \cap hE^c \neq \emptyset$, so $\phi(h) \neq 0$. We conclude that the kernel of ϕ is finite.

Let $h \in H$ be an element such that $\phi(h) \neq 0$, then $H' = \langle h \rangle \simeq \mathbb{Z}$ and H' is of finite index in H. But either $G = H$ or $[G : H] = 2$, so H' is of finite index in G.

Conversely, let H' be a subgroup of finite index in G, $H' \simeq \mathbb{Z}$. Then $e(G) = e(H') = 2$, since $e(\mathbb{Z}) = 2$. □

John Stalling characterized in [22] finitely generated groups with more than one end.

Theorem 2.2.6. *Let G be a finitely generated group. Then $e(G) > 1$ if and only if one of the following conditions hold:*

1. *G splits as a free product with amalgamation $G = H *_C K$, where C is a finite group, $C \neq H$, $C \neq K$;*

2. *G splits as a HNN-extension $G = \langle H, t | C_1^t = C_2 \rangle$, where C_1 and C_2 are finite subgroups of H $C_1 \simeq C_2$.*

In particular, if G is a finitely generated torsion-free group, then $e(G) = \infty$ if and only if G admits a non-trivial free product decomposition.

2.3 Duality groups

We recall that the projective dimension of an A-module M is the length of the shortest projective resolution of M as an A-module.

Definition. The cohomology dimension of a group G, or $\operatorname{cd} G$, is the projective dimension of $\mathbb{Z}$ as a $\mathbb{Z}G$-module.

The projective dimension of a R-module M is n if and only if $\operatorname{Ext}_R^{n+1}(M, -) = 0$ and $\operatorname{Ext}_R^{n+1}(M, N) = 0$ for some R-module N, so it is

$$\operatorname{cd} G = \sup\{n : \mathrm{H}^n(G, M) \text{ for some } G\text{-module } M\}.$$

Definition. A group G is said to be of type FP_n if $\mathbb{Z}$ is of type FP_n as a $\mathbb{Z}G$-module, that is $\mathbb{Z}$ admits a partial projective resolution

$$P_n \to \ldots \to P_0 \to M \to 0$$

over $\mathbb{Z}G$ of finite type (i.e. each P_i is finitely generated as a $\mathbb{Z}G$-module).

A group G is said to be of type FP_∞ if it is of type FP_n for every n.

A group G is said to be of type FP if $\mathbb{Z}$ admits a finite projective resolution over $\mathbb{Z}G$.

Every group is of type FP_0 because $\mathbb{Z}G \to^{\varepsilon} \mathbb{Z} \to 0$ is a partial projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ of finite type.

A group is of type FP_1 if and only if $\mathbb{Z}$ is finitely presented as a $\mathbb{Z}G$ module, and this holds if and only if G is finitely generated.

Proposition 2.3.1. *Let G be a group. G is of type* FP_2 *if and only if it is almost finitely presented.*

Proof. If G is of type FP_2, then it is also of type FP_1 and therefore finitely generated.

Then there is an exact sequence of groups

$$0 \to K \to F \xrightarrow{\pi} G \to 0$$

where F is a finitely generated free group, $F = \langle x_1, \ldots, x_n \rangle$. Let $\mathcal{F}$ be the augmentation ideal of $\mathbb{Z}F$, it is a free $\mathbb{Z}F$-module of finite rank. Then we have the following free resolution of $\mathbb{Z}$ over $\mathbb{Z}F$:

$$0 \to \mathcal{F} \to \mathbb{Z}F \xrightarrow{\epsilon} \mathbb{Z} \to 0.$$

Applying $\mathbb{Z}G \otimes_{\mathbb{Z}F} -$ to the sequence we obtain, by definition of $\mathrm{H}_1(F; \mathbb{Z}G)$ and right exactness of the tensor product, the exact sequence

$$0 \to \mathrm{H}_1(F, \mathbb{Z}G) \to \mathbb{Z}G \otimes_{\mathbb{Z}F} \mathcal{F} \to \mathbb{Z}G \otimes_{\mathbb{Z}F} \mathbb{Z}F \to \mathbb{Z}G \otimes_{\mathbb{Z}F} \mathbb{Z} \to 0.$$

It is $\mathbb{Z}G \otimes_{\mathbb{Z}F} \mathbb{Z}F \simeq \mathbb{Z}G$ and $\mathbb{Z}G \otimes_{\mathbb{Z}F} \mathbb{Z} \simeq \mathbb{Z}$ because the action of G on $\mathbb{Z}$ is the trivial action induced by $\mathbb{Z}F$. $\mathbb{Z}G \otimes_{\mathbb{Z}F} \mathcal{F}$ is a free $\mathbb{Z}G$-module of finite rank, with generators $\{1 \otimes (x_i - 1)\}_{i=1}^n$. So we have an exact sequence of $\mathbb{Z}G$-modules

$$0 \to \mathrm{H}_1(F, \mathbb{Z}G) \to \mathbb{Z}G \otimes_{\mathbb{Z}F} \mathcal{F} \to \mathbb{Z}G \xrightarrow{\epsilon} \mathbb{Z} \to 0$$

and G is of type FP_2 if and only if $\mathrm{H}_1(F, \mathbb{Z}G)$ is a finitely generated $\mathbb{Z}G$-module. Since K is free,

$$\mathrm{H}_1(F, \mathbb{Z}G) \simeq \mathrm{H}_1(F, \mathrm{Ind}_K^F \mathbb{Z}) \simeq \mathrm{H}_1(K, \mathbb{Z}) \simeq \frac{K}{[K, K]}$$

so G is of type FP_2 if and only if $\frac{K}{[K,K]}$ is finitely generated as a $\mathbb{Z}G$-module, i.e., G is almost finitely presented. □

Remark. With the notation of the proof above, note that if $\frac{K}{[K,K]}$ is finitely generated and projective we have obtained a resolution for $\mathbb{Z}$ over $\mathbb{Z}G$, and G has cohomological dimension at most 2.

Theorem 2.3.1. *Let $G = \langle x_1, \dots, x_n | r \rangle$ be a non-free, torsion free one-relator group such that every generator is involved in r, let $F = \langle x_1, \dots, x_n \rangle$ and K be the normal closure of r in F. Then $\frac{K}{[K,K]} \simeq \mathbb{Z}G$.*

Proof. Let g be an element of K, then $g = \prod_{i=1}^{m} f_i r^{\varepsilon_i} f_i^{-1}$ with $\varepsilon_i = \pm 1$, $f_i \in F$ for all i. Let $\phi : \frac{K}{[K,K]} \to \mathbb{Z}G$ be the abelian group homomorphism such that $\phi(g[K,K]) = \sum_{i=1}^{m} \varepsilon_i \pi(f_i)$, where ϕ is the projection of F on G. By Proposition 1.7.1 and the Simple Identity Theorem, if $g \in [K,K]$ then the indices fall into pairs (i,j) with $\varepsilon_i = -\varepsilon_j$ and $\pi(f_i) = \pi(f_j)$, so $\phi([K,K]) = 0$ and ϕ is well defined.

Let $g[K,K] \in \operatorname{Ker} \phi$, then $\sum_{i=1}^{m} \varepsilon_i \pi(f_i)$, so the indices must fall in pairs (i,j) with $\varepsilon_i = -\varepsilon_j$ and $\pi(f_i) = \pi(f_j)$. We prove that $g \in [K,K]$ by induction on m.

If $m = 0$, then $g = k \in [K,K]$. Otherwise, let $g = \prod_{i=1}^{m} f_i r^{\varepsilon_i} f_i^{-1}$, then there is an index j, with $1 < j \leq m$, such that $\varepsilon_j = -\varepsilon_1$ and $f_j = f_1 k_1$ for some $k_i \in K$, let $S = \prod_{i=2}^{j-1} f_i r^{\varepsilon_i} f_i^{-1}$ and $T = \prod_{i=j+1}^{m} f_i r^{\varepsilon_i} f_i^{-1}$, we have

$$g = f_1 r^{\varepsilon_1} f_1^{-1} S f_1 k_1 r^{-\varepsilon_1} k_1^{-1} f_1^{-1} T =$$
$$f_1 r^{\varepsilon_1} f_1^{-1} S f_1 r^{-\varepsilon_1} f_1^{-1} S^{-1} S f_1 r^{\varepsilon} k_1 r^{-\varepsilon} k_1^{-1} f_1^{-1} T;$$

it is $n_1 = f_1 r^{\varepsilon_1} f_1^{-1} S f_1 r^{-\varepsilon_1} f_1^{-1} S^{-1} \in [K,K]$ because $S \in K$, and $n_2 = f_1 r^{\varepsilon} k_1 r^{-\varepsilon} k_1^{-1} f_1^{-1} \in [K,K]$ because $r^{\varepsilon} k_1 r^{-\varepsilon} k_1^{-1} \in [K,K]$ characteristic subgroup of K, so

$$g = n_1 S n_2 T = n_1 S n_2 S^{-1} n_2^{-1} n_2 S T \in [K,K]$$

since n_1, n_2, $S n_2 S^{-1} n_2^{-1}$ are in $[K,K]$, and also $ST \in [K,K]$ by induction hypothesis.

□

This proves that every finitely generated, non free, torsion free one relator group is of type FP with cohomological dimension 2.

Definition. A group G of type FP is a duality group if there is an integer n and a G-module D such that

$$\mathrm{H}^i(G, M) \simeq \mathrm{H}_{n-i}(G, D \otimes M)$$

for all G-modules M and all integers i.

Theorem 2.3.2. *Let G be a group of type FP. Then the following are equivalent:*

1. *G is a duality group.*

2. *There is an integer n such that*

$$\mathrm{H}^i(G, \mathbb{Z}G \otimes A) = 0$$

for all $i \neq n$ and every abelian group A.

3. *There is an integer n such that $\mathrm{H}^i(G, \mathbb{Z}G) = 0$ for all $i \neq n$ and $\mathrm{H}^n(G, \mathbb{Z}G)$ is torsion-free as an abelian group.*

4. *There are natural isomorphisms*

$$\mathrm{H}^i(G, -) \simeq \mathrm{H}_{n-i}(G, D \otimes -)$$

where $n = \operatorname{cd} G$ and $D = \mathrm{H}^n(G, \mathbb{Z}G)$, compatible with the connecting homomorphisms in the long exact homology and cohomology sequences associated to a short exact sequence of modules.

Proof.

1. ⇒ 2. Let A be an abelian group, then $M = \mathbb{Z}G \otimes A$ is an induced module, as also $D \otimes M$ is. For $i \neq n$ it is

$$\mathrm{H}^i(G, M) \simeq \mathrm{H}_{n-i}(G, D \otimes M) = 0$$

because induced modules are H_*-acyclic.

2. ⇒ 3. Let $A = \mathbb{Z}$, then $\mathbb{Z}G \otimes A \simeq \mathbb{Z}$, so $\mathrm{H}^i(G, \mathbb{Z}G) = 0$ for $i \neq n$.

Let $A = \mathbb{Z}_k$, $k \in \mathbb{N}$, then we have the short exact sequence

$$0 \to \mathbb{Z}G \to^{\cdot k} \mathbb{Z}G \to \mathbb{Z}G \otimes \mathbb{Z}_k \to 0$$

so applying the long exact sequence for cohomology we have

$$0 = \mathrm{H}^{n-1}(G, \mathbb{Z}G \otimes \mathbb{Z}_k) \to \mathrm{H}^n(G, \mathbb{Z}G) \to^{\cdot k} \mathrm{H}^n(G, \mathbb{Z}G)$$

Then $\mathrm{H}^n(G, \mathbb{Z}G)$ has no k torsion, since k is arbitrary, $\mathrm{H}^n(G, \mathbb{Z}G)$ is torsion-free.

3. $\Rightarrow$ 4 Since G is of type FP there is a finite projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ of length n. Consider the dual complex $\bar{P} = \mathcal{H}om_G(P, \mathbb{Z})$, this provides a projective resolution for $D = \mathrm{H}^n(G, \mathbb{Z}G)$ since $\mathrm{H}^i(H, \mathbb{Z}G) = 0$ for $i \neq n$.

   Using the duality isomorphism $\mathcal{H}om_G(P, M) \simeq \bar{P} \otimes_G M$ we have

$$\mathrm{H}^i(G, M) \simeq \mathrm{H}_{-i}(\bar{P} \otimes_G M) = \mathrm{H}_{n-i}(\Sigma^n \bar{P} \otimes_G M) = \mathrm{Tor}^G_{n-i}(D, M)$$

   for any G-module M. Since D is torsion-free it is

$$\mathrm{Tor}^G_{n-i}(D, M) \simeq \mathrm{H}_{n-i}(G, D \otimes M).$$

   Since all the isomorphisms are natural and compatible with the connecting homomorphisms, we have proved the implication.

4. $\Rightarrow$ 1. Trivial. □

Theorem 2.3.3. *Let G be a duality group with $\mathrm{cd}(G) > 1$. Then G cannot be decomposed as a free product of non-trivial groups.*

Proof. Suppose $G = H_1 * H_2$, with H_1 and H_2 non-trivial. Then H_1 and H_2 are torsion-free and of type FP_∞ (since they have finite cohomological dimension). Using the Mayer-Vietoris exact sequence, we have

$$0 \to \mathrm{H}^0(1, \mathbb{Z}G) \to \mathrm{H}^1(G, \mathbb{Z}G) \to \mathrm{H}^1(H_1, \mathbb{Z}G) \oplus \mathrm{H}^1(H_2, \mathbb{Z}G)$$

Since $\mathrm{H}^0(1, \mathbb{Z}G) \simeq \mathbb{Z}G$, it is $\mathrm{H}^1(G, \mathbb{Z}G) \neq 0$, and G is not a duality group, against our hypothesis. □

Theorem 2.3.4. *Any finitely presented group G of cohomological dimension 2 not freely decomposable is a duality group.*

Proof. G is finitely presented and $\mathrm{cd}(G) = 2$, so it is of type FP.

Let A be an abelian group, we have to prove that $\mathrm{H}^i(G, \mathbb{Z}G \otimes A) = 0$ for $i < 2$.

For $i = 0$ it is obvious since $\mathrm{H}^0(G, M) = M^G$.

For $i = 1$, $\mathrm{H}^1(G, \mathbb{Z}G)$ is a free abelian group of rank $e - 1$, where e is the number of ends of G, and G has only one by Stalling's theorem. Since $\mathrm{H}^1(G, \mathbb{Z}G \otimes A) \simeq \mathrm{H}^1(G, \mathbb{Z}G) \otimes A$, we have the thesis. □

In particular, finitely generated, non free, torsion free and freely indecomposable one relator groups are duality groups.

For finitely generated one relator groups we can find the dualizing module through the explicit resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

Theorem 2.3.5. *Let G be a one-relator group with presentation*

$$G = \langle x_1, x_2, \dots, x_n | r \rangle$$

Then $\mathrm{H}^2(G, \mathbb{Z}G) = \frac{\mathbb{Z}G}{\sum \mathbb{Z}G \frac{\partial r}{\partial x_i}}$

Proof. Let F be the free group on $X = \{x_1, \dots, x_n\}$ and K the normal closure of the subgroup generated by r in F. Then we have an exact sequence of groups

$$0 \to K \to F \xrightarrow{\pi} G \to 0$$

and a free resolution of $\mathbb{Z}$ over $\mathbb{Z}F$

$$0 \to \mathcal{F} \to \mathbb{Z}F \xrightarrow{\varepsilon_F} \mathbb{Z} \to 0$$

where $\mathcal{F}$ is the augmentation ideal of $\mathbb{Z}F$ and $\varepsilon_F : \mathbb{Z}F \to F$ is the augmentation map.

Applying the functor $- \otimes_{\mathbb{Z}K} \mathbb{Z}$ we obtain the free resolution of $\mathbb{Z}$ over $\mathbb{Z}F$

$$\mathcal{F} \otimes_{\mathbb{Z}K} \mathbb{Z} \to \mathbb{Z}F \otimes_{\mathbb{Z}K} \mathbb{Z} \overset{\varepsilon_F \otimes_{\mathbb{Z}K} \mathrm{id}}{\to} \mathbb{Z} \otimes_{\mathbb{Z}K} \mathbb{Z} \to 0$$

$\mathcal{F} \simeq \sum_{i=1}^n \mathbb{Z}F\langle x_i - 1\rangle$, so $\mathcal{F} \otimes_{\mathbb{Z}K} \mathbb{Z} \simeq \sum_{i=1}^n \mathbb{Z}G\langle x_i - 1\rangle$, $\mathbb{Z}F \otimes_{\mathbb{Z}K} \mathbb{Z} \simeq \mathbb{Z}G$ and $\mathbb{Z} \otimes_{\mathbb{Z}K} \mathbb{Z} \simeq \mathbb{Z}$. Then we can rewrite the exact sequence as

$$\sum_{i=1}^n \mathbb{Z}G\langle x_i - 1\rangle \xrightarrow{\delta_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where $\delta_1(\sum i = 1^n g_i \langle x_i - 1\rangle) = \sum_{i=1}^n g_i(x_i - 1)$ for $g_i \in \mathbb{Z}G$, $i = 1, \dots, n$, and ε is the augmentation map from $\mathbb{Z}G$ to $\mathbb{Z}$.

Since the homology of the complex above would be $\mathrm{H}_*(K, \mathbb{Z})$, the kernel of the morphism δ_1 is $\mathrm{H}_1(K, \mathbb{Z}) \simeq \frac{K}{[K,K]} \simeq \mathbb{Z}G$ by Prop. 2.3.1. The map $\delta_2 : \mathbb{Z}G \to \sum_{i=1}^n \mathbb{Z}G$ given by $\delta_2(f) = \sum_{i=1}^n f \frac{\partial r}{\partial x_i}\langle x_i - 1\rangle$ for any $f \in \mathbb{Z}G$ yields an exact sequence

$$0 \to \mathbb{Z}G \xrightarrow{\delta_2} \sum_{i=1}^n \mathbb{Z}G\langle x_i - 1\rangle \xrightarrow{\delta_1} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

which is a resolution of $\mathbb{Z}$ over $\mathbb{Z}G$.

Applying the functor $\mathrm{Hom}_G(-, \mathbb{Z}G)$ to the resolution above we obtain the complex

$$0 \to \mathrm{Hom}_G(\mathbb{Z}G, \mathbb{Z}G) \xrightarrow{\odot\delta_1} \mathrm{Hom}_G(\sum_{i=1}^n \mathbb{Z}G, \mathbb{Z}G) \xrightarrow{\odot\delta_2} \mathrm{Hom}_G(\mathbb{Z}G, \mathbb{Z}G) \to 0$$

from which we can compute $\mathrm{H}^*(G, \mathbb{Z}G)$. In particular, since $\mathrm{Hom}_G(\mathbb{Z}G, \mathbb{Z}G) \simeq \mathbb{Z}G$ and $\mathrm{Hom}_G(\sum_{i=1}^n \mathbb{Z}G, \mathbb{Z}G) \simeq \sum_{i=1}^n \mathbb{Z}G$, by the injectivity of the composition by δ_2 we get $\mathrm{H}^2(G, \mathbb{Z}G) = \frac{\mathbb{Z}G}{\sum \mathbb{Z}G \frac{\partial r}{\partial x_i}}$. □

2.4 Surface group conjecture

Definition. A group G is called a *surface group* if it is isomorphic to the fundamental group $\pi_1(X)$ for some closed surface X of genus at least 1.

For an orientable closed surface of genus $g \geq 1$, the fundamental group admits the presentation

$$\pi_1(X) = \langle x_1, x_2, \ldots, x_{2g} | [x_1, x_2] \ldots [x_{2g-1}, x_{2g}] \rangle$$

while for a non-orientable closed surface of genus $g \geq 1$ we have a presentation

$$\pi_1(X) = \langle x_0, x_1, \ldots, x_g | x_0^2 x_1^2 \ldots x_g^2 \rangle$$

Consequently, surface groups are one-relator groups. Since in either case the relation is not a free generator or a proper power, they are non-free and torsion-free.

Since covering spaces of closed surfaces are again closed surfaces, we have that every subgroup of a surface group is again a surface group.

Furthermore, since a closed surface is aspherical, the cohomology of X and of its fundamental group are isomorphic. For orientable closed surfaces this means that $\pi_1(X)$ is a Poincarè duality group of dimension 2.

Melnikov conjectured that if G is a residually finite, non free and non-cyclic hereditary one-relator group, then G is a surface group.

In this original form the conjecture is not true. Baumslag-Solitar groups $BS(1, m) = \langle x, y | xy^m x^{-1} y^{-1} \rangle$ are residually finite, non-free and non-cyclic one-relator groups, all their subgroups of finite index are again one-relator groups, but they are not surface groups. The conjecture can thus be restated as follows.

Conjecture 2. *Let G be a residually finite, non-free, non cyclic hereditary one-relator group. Then G is either a surface group or a Baumslag-Solitar group $BS(1, m)$ for some $m \in \mathbb{Z}$.*

It is known that subgroups of infinite index of surface groups are free. In [8] Rosenberger et al. classified cyclically pinched and conjugacy pinched one-relator groups such that every subgroup of infinite index is free. Using this result they proved a modified form of the surface group conjectur.

Conjecture 3. *Let G be a finitely generated, non-free, freely indecomposable, fully residually free group such that every subgroup of infinite index of G is free, then G is a surface group.*

In this thesis we are particularly interested in hereditary one-relator groups where the single relator is in the commutator subgroup. If the surface group conjecture is true for this groups, then they must be isomorphic to the fundamental group of an orientable surface. Bieri and Eckmann proved that Poincarè duality groups of dimension 2 are surface groups, so it would suffice to prove that such groups are Poincarè duality groups.

We give here the outline of Bieri and Eckmann's result.

First we stare a couple of propositions used in their proof. The first is due to Strebel and is proved in [23].

Proposition 2.4.1. *Let G be a Poincarè duality group of dimension 2. If H is a torsion-free subgroup of G with $[G : H] = \infty$ then H is a free group.*

Definition (Kaplansky rank)**.** Let N be a finitely generated projective $\mathbb{Z}G$-module. Let M be a $\mathbb{Z}G$-module such that $N \oplus M$ is a finitely generated free $\mathbb{Z}G$-module, let $\phi : N \oplus M \to N \oplus M$ be the endomorphism $\phi = \mathrm{id}_N \oplus M$. Given a basis β for $N \oplus M$, the trace $t = \mathrm{tr}_\beta(\phi) \in \mathbb{Z}G$ has a coefficient α_1 for $1 \in G$ that does not depend on the choice of M and of β. The *Kaplansky rank* is defined as $k(N) = \alpha_1$.

If N is a free module then $k(N) = \mathrm{rank}_{\mathbb{Z}G}(N)$.

We make use of the following theorem, proved in [16].

Theorem 2.4.1. *If $N \neq 0$ then $k(N) > 0$.*

Proposition 2.4.2. *Let G be a Poincarè duality group of dimension 2. Then the first Betti number $\beta_i(G) = \mathrm{rank}\, \mathrm{H}_i(G, \mathbb{Z})$ is not 0.*

Proof. Since G is a group of dimension 2, there is a finite projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$

$$0 \to P_2 \to P_1 \to P_0 \to \mathbb{Z} \to 0$$

with P_i finitely generated projective $\mathbb{Z}G$-modules, $i = 0, 1, 2$. Then the first three Betti numbers $\beta_i(G)$, $i = 0, 1, 2$, are equal to the ranks of the three abelian groups $\mathbb{Z} \otimes_{\mathbb{Z}G} P_i$, $i = 0, 1, 2$.

If G is orientable, then $\beta_0(G) = \beta_2(G) = 1$ and $\beta_1(G)$ must be even. If G is non-orientable, $\beta_0(G) = 1$ and $\beta_2(G) = 0$. Since the Euler characteristic of G is $\chi(G) = \beta_0(G) - \beta_1(G) + \beta_2(G)$, we have to prove that $\chi(G) \leq 0$.

If G is non-orientable, then it has an orientable subgroup H of index 2. Since G is of type FP, it is $\chi(H) = |H|\, \chi(G)$, so if $\chi(H) < 0$ then also $\chi(G) < 0$. Consequently, we need to prove the proposition only for the orientable case.

Since G is a Poincarè duality group of dimension 2, it admits a resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ of the form

$$0 \to P \to \mathbb{Z}G^d \xrightarrow{\delta} \mathbb{Z}G \xrightarrow{\varepsilon} \mathbb{Z} \to 0$$

where P is a finitely generated projective module over $\mathbb{Z}G$. Applying $\mathrm{Hom}_G(-, \mathbb{Z}G)$, since $\mathrm{Hom}_G(\mathbb{Z}G, \mathbb{Z}G) \simeq \mathbb{Z}G$ we get the sequence

$$0 \to \mathbb{Z}G \to \mathbb{Z}G^d \xrightarrow{\delta^*} P^* \to 0$$

where $P^* = \mathrm{Hom}_G(P, \mathbb{Z}G)$ is a finitely generated projective $\mathbb{Z}G$-module. Since $\mathrm{H}^i(G, \mathbb{Z}G) = 0$ for $i \neq 2$ and $\mathrm{H}^2(G, \mathbb{Z}G) = \mathbb{Z}$ we obtain another finite resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ with finitely generated projective $\mathbb{Z}G$-modules

$$0 \to \mathbb{Z}G \to \mathbb{Z}G^d \xrightarrow{\delta^*} P^* \to \mathbb{Z} \to 0$$

Then $\chi(G) = r - d + 1$, where r is the rank of $\mathbb{Z} \otimes_{\mathbb{Z}G} P^*$, and $\beta_1(G) = 2 - \chi(G) = 1 + d - r$.

Since $P^*/\delta^*(\mathbb{Z}G^d) \simeq \mathbb{Z}G/\delta(\mathbb{Z}G^d) \simeq \mathbb{Z}$, we have $P^* \oplus \delta(\mathbb{Z}G^d) \simeq \mathbb{Z}G \oplus \delta^*(\mathbb{Z}G^d)$, so there is a surjective map $\alpha : \mathbb{Z}G^{d+1} \to P^* \oplus \delta(\mathbb{Z}G^d)$. Since $\delta(\mathbb{Z}G^d) \neq 0$, we have also a surjective map $\bar{\alpha} : \mathbb{Z}G^{d+1} \to P^*$, if $\ker(\bar{\alpha}) = N$ then $\mathbb{Z}G^{d+1} \simeq P^* \oplus N$. N is a finitely generated projective $\mathbb{Z}G$-module and we have

$$\mathrm{rank}(\mathbb{Z} \otimes_{\mathbb{Z}G} P^*) + \mathrm{rank}(\mathbb{Z} \otimes_{\mathbb{Z}G} N) = d + 1$$

so $r = d + 1 - \mathrm{rank}(\mathbb{Z} \otimes_{\mathbb{Z}G} N)$. Then $\beta_1(G) = \mathrm{rank}(\mathbb{Z} \otimes_{\mathbb{Z}G} N)$.

If P is a free $\mathbb{Z}G$ module, then also P^* is a free $\mathbb{Z}G$-module then, since $P^* \oplus N$ is free, $k(N) = \mathrm{rank}(\mathbb{Z} \otimes_{\mathbb{Z}G} N) = \beta_1(G)$, so by Proposition 2.4.1 it is $\beta_1(G) > 0$.

If P and P^* are not free $\mathbb{Z}G$-modules, by a criterion of Bass in [2] if $k(N) \neq \mathrm{rank}(\mathbb{Z} \otimes_{\mathbb{Z}G} N)$ then G contains a subgroup H isomorphic to the additive group $\mathbb{Z}[\frac{1}{p}]$ for some prime p. If $[G : H] < \infty$, then H should be a Poincarè duality group of dimension 2, but it is not; if $[G : H] = \infty$ then by Proposition 2.4.1 H should be free, but it is not. We conclude that $k(N) = \mathrm{rank}(\mathbb{Z} \otimes_{\mathbb{Z}G} N) = \beta_1(G)$, so by Proposition 2.4.1 it is $\beta_1(G) > 0$. □

The results above will be used to prove the following theorem.

Theorem 2.4.2. *Let G be a Poincarè duality group of dimension 2. Then G is a surface group.*

We will need to define first the splitting of a group over a subgroup and relative homology.

Definition. Let G be a group and H a subgroup of G. We say that G *splits* over K if either G is an amalgamated free product $H = K *_H L$ with $K \neq H$ and $L \neq H$, or G is a HNN-extension $G = K*_{H,t} = \langle K, t | h^t = \phi(h), h \in H\rangle$ for some injective group homomorphism $\phi : H \to K$.

Let G be a group and H a finite subgroup of G. Then by Stalling's structure theorem G splits over H if and only if $\mathrm{H}^1(G, \mathbb{Z}G) \neq 0$.

Definition. A *group pair* $(G; \{S_j\}_{j\in J})$ consists of a group G and a family $\{S_j\}_{j\in J}$ of subgroups of G, not necessarily distinct.

For a subgroup S of G, let $\mathbb{Z}G/S$ be the G-module whose underlying Abelian group is freely generated by the cosets xS with G-action by left multiplication.

Let $(G; \{S_j\}_{j\in J})$ be a group pair, we define $\Delta = \{\oplus_j \mathbb{Z}G/S_j \xrightarrow{\epsilon} \mathbb{Z}\}$ where $\epsilon(xS_j) = 1$ for all $x \in G$, $j \in J$. Then we use this module to define the relative cohomology of the pair $(G; \{S_j\}_{j\in J})$:

$$\mathrm{H}_i(G; \{S_j\}_{j\in J}; A) = \mathrm{H}_{i-1}(G; \Delta \otimes A)$$
$$\mathrm{H}^i(G; \{S_j\}_{j\in J}; A) = \mathrm{H}^{i-1}(G; \mathrm{Hom}(\Delta, A))$$

for a G-module A, where $\otimes$ and Hom are equipped with diagonal G-action.

A duality pair of dimension n with dualizing module D is a group pair $(G; \{S_j\}_{j\in J})$ such that:

1. $\mathrm{H}^i(G; A) \simeq \mathrm{H}_{n-i}(G; \{S_j\}_{j\in J}; D \otimes A)$
2. $\mathrm{H}^i(G; \{S_j\}_{j\in J}; A) \simeq \mathrm{H}_{n-i}(G; D \otimes A)$

for every G-module A.

We say that a group pair $(G; \{S_j\}_{j\in J})$ is an orientable Poincarè duality pair of dimension n if $D = \mathbb{Z}$ and

$$\mathrm{H}^i(G; A) \simeq H_{n-i-1}(G; \Delta \otimes A)$$

and the second isomorphism is implied by the first.

$(G; \{S_j\}_{j\in J})$ is an orientable Poincarè duality pair of dimension n if and only if G is a duality group of dimension $n - 1$ with dualizing module D. Furthermore, relative exact sequences show that $\{S_j\}_{j\in J}$ must be a finite family of Poincarè duality groups of dimension $n - 1$.

Definition. Let G be a group and H a finitely generated subgroup of G. Let $\{x_v\}$ be a set of coset representatives of $G \mod H$. Let

$$r : \mathrm{H}^1(G, \mathbb{Z}G) \to \oplus_v \mathrm{H}^1(H, \mathbb{Z}H)x_v$$

be the composition of the restriction map

$$\text{res} : \mathrm{H}^1(G, \mathbb{Z}G) \to \mathrm{H}^1(H, \mathbb{Z}G)$$

with the isomorphism between $\mathrm{H}^1(H, \mathbb{Z}G)$ and $\mathrm{H}^1(H, \mathbb{Z}H)x_v$.

The minimal number of non-zero components of $r(c)$ for all c in $N(G; S_1, \ldots, S_m)$, $c \neq 0$, is called the *weight* $n(H)$ of H with respect to G and $S_1, \ldots, S_m$.

$n(H) = 0$ if and only if $N(G; H, S_1, \ldots, S_m) \neq 0$, that is if and only if G splits over H.

H. Muller established in [18] the simultaneous splitting theorem, as a corollary of this theorem we can classify G and H for $n(H) = 1$ and $n(H) = 2$. We are interested in the particular cases outlined in the following proposition.

Proposition 2.4.3. *Let G be a a group and H a subgroup of G.*

If G is torsion free and $n(H) = 1$ then G and H must be one of the following:

1. *$G = G_1 * G_2$ for some subgroups G_1, G_2 of G, with $H = H_1 * H_2$, $H_1 < G_1$, $H_2 < G_2$;*
2. *$G = G_1 * \langle t \rangle$ for some subgroup G_1 of G, with $H = H_1 * H_2^t$, $H_1 < G_1$, $H_2 < G_2$;*
3. *$G = \langle t \rangle = H$ and $S_1 = S_2 = \ldots = S_m = \langle 1_G \rangle$ or $m = 0$.*

If G is torsion-free, H is infinite cyclic and $n(H) = 2$ then G and H must be one of the following:

1. *$G = G_1 * G_2$ for some subgroups G_1, G_2 of G, with $H = \langle g_1 g_2 \rangle$, $g_i \in G_i$, $g_i \neq 1_G$, $i = 1, 2$;*
2. *$G = G_1 * \langle t \rangle$ for some subgroup G_1 of G, with $H = \langle g_1^t g_2 \rangle$, $H g_1, g_2 \in G_1$;*
3. *$G = \langle t \rangle$, $H = \langle t^2 \rangle$, and $S_1 = S_2 = \ldots = S_m = \langle 1_G \rangle$ or $m = 0$.*

The following proposition, proved by Bieri and Eckmann in [6], establish that a free product of Poincarè duality group pairs of dimension n with an amalgamated boundary component yiels again a Poincarè duality group pair of dimension n.

Proposition 2.4.4. *Let $(G_1; S_0, \ldots, S_m)$ and $(G_2; S'_0 \ldots, S'_{m'})$ be group pairs, let T_i be a subgroup of G_i for $i = 1, 2$, with $T_1 \simeq T_2$. Let G be the amalgamated product $G_1 *_T G_2$ with $T = T_1 = T_2$, then, identifying the subgroups of G_1, G_2 with subgroups of G, there is a group pair $(G; S_0, \ldots, S_m, S'_0, \ldots, S'_{m'})$.*

1. *If* $(G_1; S_0, \dots, S_m)$ *and* $(G_2; S'_0 \dots, S'_{m'})$ *are Poincarè duality group pairs of dimension* n *then also* $(G; S_0, \dots, S_m, S'_0, \dots, S'_{m'})$ *is a Poincarè duality group pair of dimension* n.

2. *If* $(G; S_0, \dots, S_m, S'_0, \dots, S'_{m'})$ *is a Poincarè duality group pair of dimension* n *and* T *is a Poincarè duality group pair of dimension* $n-1$, *then* $(G_1; S_0, \dots, S_m, T)$ *and* $(G_2; S'_0 \dots, S'_{m'}, T)$ *are Poincarè duality group pairs of dimension* n

Poincarè duality group pairs of dimension 2 are also completely classified.

Proposition 2.4.5. *Let* $(G; S_0, \dots, S_m)$ *be a Poincarè duality group pair of dimension* 2. *Then it is one of the following:*

1. G *is a free group generated by* $t_1, \dots, t_m, x_1, \dots, x_{2g}$, *with* $m + g > 0$, $S_1, \dots, S_m$ *are generated by conjugates of* $t_1, \dots, t_m$ *and* S_0 *is generated by* $\prod_{i=1}^m t_i \prod_{j=1}^g [x_{2j-1}, x_{2j}]$;

2. G *is a free group generated by* $t_1, \dots, t_m, x_0, \dots, x_g$, $m \geq 0$, $g \geq 0$, $S_1, \dots, S_m$ *are generated by conjugates of* $t_1, \dots, t_m$ *and* S_0 *is generated by* $\prod_{i=1}^m t_i \prod_{j=0}^g x_j^2$.

Proposition 2.4.6. *Let* $(G; S_0, \dots, S_m)$ *be a Poincarè duality group pair of dimension* 2. *Then* $(G; S_0, \dots, S_m)$ *is a surface group pair.*

Proof. Let n be the rank of the finitely generated free group G. We make induction on n.

If $n = 1$, then G is infinite cyclic, $G = \langle g \rangle$, and $\mathrm{H}^1(G; \mathbb{Z}G) = \mathbb{Z}$, since G is a duality group of dimension 1, the dualizing module D is isomorphic to $\mathbb{Z}$. From the exact sequence

$$0 \to D \to \oplus_{i=1}^m \mathbb{Z}G/S_i \to \mathbb{Z} \to \mathbb{Z}$$

we obtain that $\oplus_{i=1}^m \mathbb{Z}G/S_i \simeq \mathbb{Z} \oplus \mathbb{Z}$, this can happen if and only if either $m = 1$ and $S_1 = S_2 = G$ or $m = 0$ and $G = \langle g^2 \rangle$. Then the group pair is respectively the lowest orientable case for $g = 0, m = 1$ and the lowest non-orientable case for $g = 0, m = 0$ of the presentation list of surface group pairs.

Suppose $n > 1$, by Proposition 2.4.3 G splits in one of the following ways:

1. $G = G_1 * G_2$ with $S_0 = \langle g_1 g_2 \rangle$, $g_i \in G_i$, $g_i \neq 1_G$ for $i = 1, 2$, while S_j is conjugate to a subgroup of G_1 or G_2 for $j > 0$, we can suppose that S_j is conjugate to a subgroup of G_1 for $1 \leq j \leq k$ and to a subgroup of G_2 for $k + 1 \leq j \leq m$;

2. $G = G_1 * \langle t \rangle$ with $S_0 = \langle g_1^t g_2 \rangle$, $g_i \in G_1$ for $i = 1, 2$, and S_j conjugate to a subgroup of G_1.

In the first case (the second case is analogue), $G = (G_1 * \langle g_2 \rangle) *_{\langle g_2 \rangle} G_2$, so $S_0 \subseteq G_1 * \langle g_2 \rangle$. Then the group pair $(G_2; \langle g_2 \rangle, S_{k+1}, \ldots, S_m)$ is a Poincarè duality group pair of dimension 2 by Proposition 2.4.4. Similarly we have that also $(G_1; \langle g_1 \rangle, S_, \ldots, S_k)$ is a Poincarè duality group pair of dimension 2.

Since the rank of G_1 and G_2 is less than n, by induction $(G_1; \langle g_1 \rangle, S_, \ldots, S_k)$ and $(G_2; \langle g_2 \rangle, S_{k+1}, \ldots, S_m)$ are surface group pairs, so $(G; S_0, \ldots, S_m)$ is a surface group pair. □

Let $(G; S_0, \ldots, S_m)$ be a Poincarè duality group pair of dimension 2 with $m \geq 0$ and S_i infinite cyclic for $1 \leq i \leq m$ and $\operatorname{rank} G > 1$. The exact relative cohomology sequence of the group pair is

$$0 \to \mathrm{H}^1(G; S_0, \ldots, S_m; \mathbb{Z}G) \to \mathrm{H}^1(G; \mathbb{Z}G) \xrightarrow{r}$$
$$\oplus_{i=1}^m \mathrm{H}^1(S_i; \mathbb{Z}G) \to^\delta \mathrm{H}^2(G; S_0, \ldots, S_m; \mathbb{Z}G) \to 0$$

where r is the map $(\mathrm{res}_1, \ldots, \mathrm{res}_m)$. We have that

$$\mathrm{H}^1(G; S_0, \ldots, S_m; \mathbb{Z}G) = 0$$

and

$$\mathrm{H}^2(G; S_0, \ldots, S_m; \mathbb{Z}G) \simeq \mathbb{Z}$$

since $(G; S_0, \ldots, S_m)$ is a Poincarè duality group pair of dimension 2. If we omit S_0, the last term becomes 0 and the first must be non zero, that is $N = N(G; S_1, \ldots, S_m)$, the intersection of the kernels of the res_i, $1 \leq i \leq m$, is non zero. The weigth $n(S_0)$ is the minimal number of components in $\mathrm{H}^1(S_0; \mathbb{Z}G) \simeq \oplus_{v \in V} \mathrm{H}^1(S_0; \mathbb{Z}S_0)x_v$, where $\{x_v\}_{v \in V}$ is a set of representatives for the cosets of G modulo S_0 of $\mathrm{res}_0(c)$ for all $c \in N$, $c \neq 0$, and $\ker \mathrm{res}_0 \cap N = 0$. $r(N) = (\mathrm{res}_0, 0, \ldots, 0) = (\mathrm{H}^1(S_0, \mathbb{Z}G), 0, \ldots, 0) \cap \ker \delta$ and δ restricted to any summand $\mathbb{Z}x_v$ of $\mathrm{H}^1(S_0, \mathbb{Z}G)$ is bijective, so the minimum number of components of non zero elements in $\mathrm{res}_0(N)$ is 2. We conclude that $r(N) = 2$.

G is a Poincarè duality group of dimension 2 with $G = G_1 *_L G_2$ where L is free, $\operatorname{rank} L > 1$. Consider the Mayer-Vietoris exact sequence

$$\ldots \to 0 \to \mathrm{H}^1(G_1; \mathbb{Z}G) \oplus \mathrm{H}^1(G_2; \mathbb{Z}G) \xrightarrow{\mathrm{res}_1 - \mathrm{res}_2} \mathrm{H}^1(L; \mathbb{Z}G) \xrightarrow{\delta} \mathrm{H}^2(G; \mathbb{Z}G) \to \ldots$$

res_1 and res_1 are injective, so $n(L) > 0$ with respect to the group pair $(G_1; \emptyset)$ and $(G_2; \emptyset)$. Since L is free of rank greater than 1, $\mathrm{H}^1(L; \mathbb{Z}L)$ is free abelian of infinite rank. Then the restriction of δ to $\mathrm{H}^1(L; \mathbb{Z}L)$ is not injective since $\mathrm{H}^2(G; \mathbb{Z}G) \simeq \mathbb{Z}$, this implies that $\mathrm{im}(\mathrm{res}_1 - \mathrm{res}_2) \cap \mathrm{H}^1(L; \mathbb{Z}L) \neq 0$.

If $n(L)$ were greater than 1 with respect to both $(G_1;\emptyset)$ and $(G_2;\emptyset)$, then the image of any $(c_1, c_2) \in \mathrm{H}^1(G_1;\mathbb{Z}G) \oplus \mathrm{H}^1(G_2;\mathbb{Z}G)$, $(c_1, c_2) \neq (0, 0)$, through $\mathrm{res}_1 - \mathrm{res}_2$ cannot lie in $\mathrm{H}^1(L;\mathbb{Z}L)$, so $n(L) = 1$ with respect to at least one of the two group pairs.

Proposition 2.4.7. *Let G be a Poincarè duality group of dimension 2. Then G is a surface group.*

Proof. Since $\mathrm{H}^1(G;\mathbb{Z}G) \neq 0$, by Stalling's structure theorem G splits over a finite subgroup H, since G is torsion-free it must be $H = \langle 1_G \rangle$.

By Proposition 2.4.3 it is either $G = G_1 *_L G_2$ or $G = G_1 *_L \langle t \rangle$ with L finitely generated. Since $\mathrm{H}^1(G;\mathbb{Z}G) = 0$ it is $L \neq 1$. $[L : G] = \infty$, so by Strebel's theorem $\mathrm{cd}\, L \leq 1$, so L is a finitely generated free group. We will analyze the first case, the second is similar.

If $G = G_1 *_L G_2$ with $\mathrm{rank}\, L > 1$ then we have one of the following splittings:

1. $G_1 = H_1 * H_2$, $L = L_1 * L_2$ with $L_i \subseteq H_i$ for $i = 1, 2$;

2. $G_1 = H * \langle t \rangle$, $L = L_1 * L_2^t$ with $L_i \subseteq H$ for $i = 1, 2$.

In the first case, $G = H_1 *_{L_1} (H_2 *_{L_2} G_2)$. If $L_1 \neq H_1$ then G splits over L_1, if $L_1 = H$ then G splits over L_2, in both cases G splits over a free subgroup of rank less than that of L.

In the second case, $G = (H *_{L_1} G_2) *_{L_2,t}$, so G splits over L_2 whose rank is less than that of L.

By induction, we can suppose that G splits over an infinite cyclic subgroup, so $\mathrm{rank}\, L = 1$. Then L is a Poincarè duality group of dimension 2 and by Proposition the group pairs (G_1, L) and (G_2, L) in the first case and $(G_1, \{L, L^t\})$ in the second case are Poincarè duality group pairs of dimension 2. Then by Proposition 2.4.6 these are surface group pairs corresponding to closed surfaces with one disk or two disks removed respectively.

In the first case, $G = G_1 *_L G_2$ is the fundamental group of the closed surface obtained identifying the boundary circles; in the second case G is the fundamental group of the closed surface obtained by joining the two boundary circles by a tube. □

2.5 Demushkin groups

The hypothesis of the surface group conjecture have some striking similarities to some properties of Demushkin groups, which are one-relator pro-p groups and Poincarè duality groups.

Definition (Direct systems)**.** Let $(I; \leq)$ be a partially ordered set. A *direct system* of groups over I is a collection $\{G_i\}_{i\in I}$ of groups together with a collection of homomorphisms $\phi_{ij} : G_i \to G_j$ for every $i \leq j$, such that the diagram

$$\begin{array}{ccc} G_i & \xrightarrow{\phi_{ij}} & G_j \\ & \searrow^{\phi_{ik}} & \downarrow{\scriptstyle \phi_{jk}} \\ & & G_k \end{array}$$

commutes for every $i \leq j \leq k$ and $\phi_{ii} = \mathrm{id}_{G_i}$ for every $i \in I$.

Definition (Compatible homorphisms)**.** Let $(\{G_i\}_{i\in I}; \{\phi_{ij}\}_{ij\in I})$ be a direct systems of groups indexed over the partially ordered set I. Let H be a group and $\{\psi_i\}_{i\in I}$ be a collection of group homomorphisms with $\psi_i : G_i \to H$. $\{\psi_i\}_{i\in I}$ is called a collection of *compatible homomorphisms* if the diagram

$$\begin{array}{ccc} G_i & \xrightarrow{\phi_{ij}} & G_j \\ & \searrow^{\psi_i} & \downarrow{\scriptstyle \psi_j} \\ & & H \end{array}$$

commutes for every $i \leq j$.

Definition (Profinite and pro-p groups)**.** Let $(\{G_i\}_{i\in I}; \{\phi_{ij}\}_{ij\in I})$ be a direct systems of group indexed over the partially ordered set I. A group G together with a collection of compatible group homomorphisms $\psi_i : G_i \to G$, $i \in I$, is called a *direct limit* of the direct system if for any group H, together with compatible group homomorphisms $\lambda_i : G_i \to H$, there is a unique homomorphism $\gamma : G \to H$ such that the diagram

$$\begin{array}{ccc} G_i & \xrightarrow{\psi_i} & G \\ & \searrow^{\lambda_i} & \downarrow{\scriptstyle \gamma} \\ & & H \end{array}$$

commutes for every $i \in I$. We write $G = \overset{\lim}{\leftarrow} G_i$.

If G is the direct limit of a direct system of finite groups, G is said to be a *profinite group.*

If G is the direct limit of a direct system of finite p groups, G is said to be a *pro-p group.*

If we equip the groups of the direct system with the discrete topology, then the direct limit G inherits a topology and it is a topological group.

The kernels of the projection homomorphisms from G to G_i, $i \in I$, is a fundamental system of open neighborhoos of 1_G. A subgroup U of G is open if and only if U is closed of finite index.

Given a profinite group G and a profinite ring R, the *complete group algebra* $[[RG]]$ is the inverse limit

$$[[RG]] = \lim_{\leftarrow} R\frac{G}{U}$$

where $R\frac{G}{U}$ is the ordinary group algebra and U ranges over the open normal subgroups of G. $[[RG]]$ is a profinite ring.

If G is a profinite group, we can consider the projective resolution of $[[\hat{Z}G]]$ over $\hat{Z}$ and use this resolution to define a profinite homology $\hat{\mathrm{H}}_n(G,M)$ and profinite cohomology $\hat{\mathrm{H}}^n(G,M)$, where M is a profinite $[[\hat{Z}G]]$-module.

If G is a pro-p group, then $\hat{\mathrm{H}}^n(G,\mathbb{F}_p)$ has a natural structure of vector space over the finite field with p elements $\mathbb{F}_p$. For $n = 1, 2$ this structure is linked to the presentation of G as a quotient of a pro-p free group.

Theorem 2.5.1. *Let G be a pro-p group. Then $\dim_{\mathbb{F}_p} \hat{\mathrm{H}}^1(G,\mathbb{F}_p) = d(G)$, where $d(G)$ is the minimal cardinality of a set of generators of G converging to 1_G, which is the minimal dimension of free pro-p groups with quotient G.*

Proof. Let X be a set such that $|X| = \dim_{\mathbb{F}_p} \hat{\mathrm{H}}^1(G,\mathbb{F}_p)$, let F be the free pro-p group on the set X. We have

$$\hat{\mathrm{H}}^1(F,\mathbb{F}_p) = \mathrm{Hom}(F,\mathbb{F}_p) = \oplus_{x\in X}\mathbb{F}_p,$$

so $\dim_{\mathbb{F}_p} \hat{\mathrm{H}}^1(G,\mathbb{F}_p) = |X|$. Then there is an isomorphism

$$\alpha : \hat{\mathrm{H}}^1(G,\mathbb{F}_p) \to \hat{\mathrm{H}}^1(F,\mathbb{F}_p),$$

so there exists a continuos homomorphism $\phi : F \to G$ that induces α and $d(G) \leq |X|$.

Let Y be a set such that $|Y| = d(G)$, let F_2 be the free pro-p group on the set Y. Then there is a continuos epimorphism $\psi : F_2 \to G$ and this epimorphism induces an injective homorphism $\beta : \hat{\mathrm{H}}^1(G,\mathbb{F}_p) \to \hat{\mathrm{H}}^1(F_2,\mathbb{F}_p)$, so we have

$$\dim_{\mathbb{F}_p} \hat{\mathrm{H}}^1(G,\mathbb{F}_p) \leq \hat{\mathrm{H}}^1(F_2,\mathbb{F}_p) = |Y| = d(G).$$

We conclude $\dim_{\mathbb{F}_p} \hat{\mathrm{H}}^1(G,\mathbb{F}_p) = d(G)$. □

The next theorem gives another property of the first cohomology group.

Theorem 2.5.2. *Let G be a pro-p group and K a closed normal subgroup of G. Then the smallest cardinality of a generating set of K as a closed subgroup of G is equal to $\dim_{\mathbb{F}_p} \hat{\mathrm{H}}^1(G, \mathbb{F}_p)^G$, where $\hat{\mathrm{H}}^1(K, \mathbb{F}_p)^G$ is the fixed submodule of $\hat{\mathrm{H}}^1(K, \mathbb{F}_p)$ under the action of G.*

If G is a finitely generated pro-p group, let $G = \langle X|R \rangle$ be a presentation of G as a pro-p group such that $|X| = d(G)$. Then R generates a normal subgroup K of F, the free pro-p group on the set X. We define $r(G)$, the *relation rank* of G, as the smallest cardinality of a generating set of K as a normal subgroup of F. This is the smallest cardinality for a subset R' of F such that $\langle X|R' \rangle = G$.

Theorem 2.5.3. *Let G be a finitely generated pro-p group. Then*

$$\dim_{\mathbb{F}_p} \hat{\mathrm{H}}^2(G, \mathbb{F}_p) = rr(G).$$

Proof. Let X be a set such that $|X| = d(G)$, let F be the free pro-p group on the set X, let $\langle X|R \rangle$ be a presentation of G and K the normal subgroup of F generated by R. We have an exact sequence of groups

$$1 \to K \to F \to G \to 1$$

This exact sequence induces a five term exact sequence

$$0 \to \hat{\mathrm{H}}^1(G, \mathbb{F}_p) \to \hat{\mathrm{H}}^1(F, \mathbb{F}_p) \to \hat{\mathrm{H}}^1(K, \mathbb{F}_p)^F \to \hat{\mathrm{H}}^2(G, \mathbb{F}_p) \to \hat{\mathrm{H}}^2(F, \mathbb{F}_p)$$

By Theorem 2.5 we have that $\hat{\mathrm{H}}^1(G, \mathbb{F}_p)$ and $\hat{\mathrm{H}}^1(F, \mathbb{F}_p)$ are $\mathbb{F}_p$-vector space of the same dimension, so the injective homomorphism between them is also an isomorphism.

F is a free pro-p group, so its cohomological dimension over $\mathbb{F}_p$ is 1. Then $\hat{\mathrm{H}}^2(G, \mathbb{F}_p) \to \hat{\mathrm{H}}^2(F, \mathbb{F}_p) = 0$.

We conclude that the morphism between $\hat{\mathrm{H}}^1(K, \mathbb{F}_p)^F$ and $\hat{\mathrm{H}}^2(G, \mathbb{F}_p)$ in the five term exact sequence is an isomorphism.

Since by Theorem 2.5.2 $\dim_{\mathbb{F}_p} \hat{\mathrm{H}}^1(K, \mathbb{F}_p)^F$ is the smallest cardinality of a generating set of K as a normal subgroup of G, it is $\dim_{\mathbb{F}_p} \hat{\mathrm{H}}^2(G, \mathbb{F}_p) = rr(G)$.

The interest in the properties of low cohomology groups is motivated by the definition of Demushkin groups.

Definition. A pro-p group is a Demushkin group if

1. $\dim_{\mathbb{F}_p} \hat{\mathrm{H}}^2(G, \mathbb{F}_p) = 1$;

2. $\dim_{\mathbb{F}_p} \hat{\mathrm{H}}^1(G, \mathbb{F}_p) < \infty$;

3. $\hat{\mathrm{H}}^i(G, \mathbb{F}_p) = 0$ for all $i > 2$;

4. the cup product $\cup : \hat{\mathrm{H}}^1(G, \mathbb{F}_p) \times \hat{\mathrm{H}}^1(G, \mathbb{F}_p) \to \hat{\mathrm{H}}^2(G, \mathbb{F}_p)$ is a non-degenerate bilinear form.

Since the first cohomology group of a Demushkin group has finite dimension and the second cohomology group has dimension 1, Demushkin groups are finitely generated one-relator pro-p groups. Since the definition implies that $\hat{\mathrm{H}}^2(G, \mathbb{F}_p) \simeq \mathbb{F}_p$, they can be seen as the pro-p analogue of Poincarè duality groups of dimension 2. A subgroup of finite index of a Demushkin group is again a Demushkin group, so Demushkin groups are hereditary one-relator groups.

Labute completed in [12] the classification of all Demushkin groups. There are two important inveriants associated to Demushkin groups. The first is d, the minimal number of generators of G. The quotient group $\frac{G}{[G,G]}$ is either a free abelian pro-p group of rank d or the direct product of a finite cyclic group of order p^m for some m, and a free abelian pro-p group of rank $d-1$; we define the invariant q as ∞ in the first case and p^m in the latter. We will need the classification only for $q \neq 2$.

Theorem 2.5.4. *Let G be a Demushkin group with invariants d and q, suppose $q \neq 2$. Then d is even and G admits a presentation $\langle x_1, x_2, \ldots, x_d | r \rangle$ where*

$$r = [x_1, x_2][x_3, x_4] \cdots [x_{d-1}, x_d]$$

if $q = \infty$ and

$$r = x_1^q [x_1, x_2][x_3, x_4] \cdots [x_{d-1}, x_d]$$

if q is finite.

2.6 Goodness

Given a group G, the set of all the quotients $\frac{G}{H}$ where H is a normal subgroup of G of finite index, together with the projection homomorphisms, is a direct system of finite groups. The inverse limit of this direct system is a profinite group, called the *profinite completion* $\hat{G}$ of G.

Similarly, the set of all the quotients $\frac{G}{H}$ where H is a normal subgroup of G with $[G : H] = p^\alpha$ for a fixed prime p, together with the projection homomorphisms, is a direct system of p-groups. The inverse limit of this direct system is a pro-p group, called the *pro-p completion* G_p of G.

Definition. Let G be a group and $\hat{G}$ its profinite completion (risp. pro-p completion). G is called *good* (risp. p-good) if the natural homomorphism $G \to \hat{G}$ induces isomorphisms between the cohomology groups $\hat{\mathrm{H}}^i(\hat{G}, M)$ and $\mathrm{H}^i(G, M)$ for every finite G-module M.

Proposition 2.6.1. *Let G be a residually free group and $\hat{G}$ its profinite completion. Then the following properties are equivalent.*

1. $\hat{\mathrm{H}}^i(\hat{G}, M) \to \mathrm{H}^i(G, M)$ *are bijective for $i \leq n$ and injective for $i = n+1$ for every finite module M;*

2. $\hat{\mathrm{H}}^i(\hat{G}, M) \to \mathrm{H}^i(G, M)$ *are surjective for $i \leq n$ for every finite module M;*

3. *for all $x \in \mathrm{H}^i(G, M)$, $1 \leq i \leq n$, and for every finite module M, there is a $\hat{G}$-module N such that M is isomorphic to a G-submodule of N and the morphism $\mathrm{H}^i(G, M) \to \mathrm{H}^i(G, N)$ sends x in 0;*

4. *for all $x \in \mathrm{H}^i(G, M)$, $1 \leq i \leq n$, and for every finite module M, there is a subgroup H of G, with $[G : H] < \infty$, such that x induces zero in $\mathrm{H}^i(H, M)$.*

Proof. We prove some implications.

$1 \Rightarrow 2$) Trivial.

$2 \Rightarrow 3$) Since the category of $\hat{G}$-modules has enough injectives, there is N injective $\hat{G}$-module such that M injects in N. Then we have a commutative square

$$\begin{array}{ccc} \hat{\mathrm{H}}^i(\hat{G}, M) & \longrightarrow & \mathrm{H}^i(G, M) \\ \downarrow & & \downarrow \\ \hat{\mathrm{H}}^i(\hat{G}, N) & \longrightarrow & \mathrm{H}^i(G, N) \end{array}$$

for every $i \in \mathbb{N}$. Since N is injective as a $\hat{G}$-module, it is $\hat{\mathrm{H}}^i(\hat{G}, N) = 0$ for $i \geq 1$. Since for $1 \leq i \leq n$ we have that $\hat{\mathrm{H}}^i(\hat{G}, M) \to \mathrm{H}^i(G, M)$ is surjective, the morphism $\mathrm{H}^i(G, M) \to \mathrm{H}^i(G, N)$ is the zero morphism, proving the implication.

$2 \Rightarrow 4$) Let $x \in \mathrm{H}^i(G, M)$, with $1 \leq i \leq n$, let $\hat{x} \in \hat{\mathrm{H}}^i(\hat{G}, M)$ such that its image in $\mathrm{H}^i(G, M)$ is x. Since $\hat{x} \in \mathrm{H}^i(\hat{G}, M)$ there is a normal subgroup of finite index $\hat{U}$ of $\hat{G}$ such that there is an element $\bar{x} \in \hat{\mathrm{H}}^i(\frac{\hat{G}}{\hat{U}}, M)$ which is sent to $\hat{x}$ by the inflation map $\hat{\mathrm{H}}^i(\frac{\hat{G}}{\hat{U}}, M) \to \hat{\mathrm{H}}^i(\hat{G}, M)$, furthermore I can assume $\hat{U}$ acts trivially on M.

From the Hochschild-Serre spectral sequence for cohomology we have that the composition

$$\hat{\mathrm{H}}^i(\frac{\hat{G}}{\hat{U}}, M) \to \hat{\mathrm{H}}^i(\hat{G}M) \to \hat{\mathrm{H}}^i(\hat{U}, M)$$

is the zero morphism, so $\hat{x}$ induces 0 in $\hat{\mathrm{H}}^i(\hat{U}, M)$. Taking U as the inverse image of $\hat{U}$ under the natural morphism from G to $\hat{G}$, we obtain a commutative diagram

$$\begin{array}{ccc} \hat{\mathrm{H}}^i(\hat{G}, M) & \longrightarrow & \mathrm{H}^i(G, M) \\ \downarrow & & \downarrow \\ \hat{\mathrm{H}}^i(\hat{U}, M) & \longrightarrow & \mathrm{H}^i(U, M) \end{array}$$

and this proves the thesis.

$4 \Rightarrow 3)$ Take $x \in \mathrm{H}^i(G, M)$ and let H be the subgroup of finite index such that x maps to 0 in $\mathrm{H}^i(H, M)$. Then M injects in the module $N = \mathrm{Coind}_G^H M$ and there is a natural isomorphism between $\mathrm{H}^i(H, M)$ amd $\mathrm{H}^i(G, N)$, so x maps to 0 in $\mathrm{H}^i(G, N)$.

$4 \Rightarrow 1)$ For $n = 0$, we have $\hat{\mathrm{H}}^0(\hat{G}, M) \simeq \mathrm{H}^0(G, M)$ because $M^{\hat{G}} \simeq M^G$, and $\hat{\mathrm{H}}^1(\hat{G}, M) \to \mathrm{H}^1(G, M)$ is injective because G is dense in $\hat{G}$.

If the implication is true for $j < i$, then we have only to prove that $\hat{\mathrm{H}}^i(\hat{G}, M) \to \mathrm{H}^i(G, M)$ is surjective and that $\hat{\mathrm{H}}^{i+1}(\hat{G}, M) \to \mathrm{H}^{i+1}(G, M)$ is injective.

For any $x \in \mathrm{H}^i(G, M)$ there is U of finite index such that the image of x in $\mathrm{H}^i(U, M)$ is 0, then by the Hochschild-Serre spectral sequence there is $y \in \mathrm{H}^i(\frac{G}{U}, M)$ whose image is x, since $\hat{\mathrm{H}}^i(\hat{G}, M)$ is the limit of the i-th cohomology groups of the finite quotients of G, it follows that $\hat{\mathrm{H}}^i(\hat{G}, M) \to \mathrm{H}^i(G, M)$ is surjective.

Let $x \in \hat{\mathrm{H}}^{n+1}(\hat{G}, M)$ such that its image in $\mathrm{H}^{n+1}(G, M)$ is 0 Let I_M be an injective $\hat{G}$-module such that M injects in I_M. Then I_M is the direct limit of all the finite G-submodules of I_M containing the image of M. Since $\mathrm{H}^{n+1}(G; I_M) = 0$ there is a finite G-module M', $M \subseteq M' \subseteq I_M$, such that the image of x in $\mathrm{H}^{n+1}(G, M') = 0$. Take the short exact sequence

$$0 \to M \to M' \to X \to 0$$

For the long exact sequence for cohomology and naturality we have

$$\begin{array}{ccccccc}
\hat{\mathrm{H}}^n(\hat{G}, M') & \longrightarrow & \hat{\mathrm{H}}^n(\hat{G}, X) & \longrightarrow & \hat{\mathrm{H}}^{n+1}(\hat{G}, M) & \longrightarrow & \hat{\mathrm{H}}^{n+1}(\hat{G}, M') \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\mathrm{H}^n(G, M') & \longrightarrow & \mathrm{H}^n(G, X) & \longrightarrow & \mathrm{H}^{n+1}(G, M) & \longrightarrow & \mathrm{H}^{n+1}(G, M')
\end{array}$$

Since the image of x in $\hat{\mathrm{H}}^{n+1}(\hat{G}, M')$ is 0, by exactness there is $y \in \hat{\mathrm{H}}^n(\hat{G}, X)$ whose image is x, but $\hat{\mathrm{H}}^n(\hat{G}, X) \simeq \mathrm{H}^n(G, X)$ because X is finite, let $\bar{y} \in \mathrm{H}^n(G, X)$ be the image of y. By commutativity the image of $\bar{y}$ in $\mathrm{H}^{n+1}(G, M')$ is 0, so by exactness again there is $\bar{z} \in \mathrm{H}^n(G, M')$ whose image is $\bar{y}$, since again $\hat{\mathrm{H}}^n(\hat{G}, M') \simeq \mathrm{H}^n(G, M)$ let z be counterimage of $\bar{z}$. Then by commutativity y is in the image of $\hat{\mathrm{H}}^n(\hat{G}, M')$ in $\hat{\mathrm{H}}^n(\hat{G}, X)$ and by exactness it is $x = 0$, proving the injectivity of $\mathrm{H}^{n+1}(\hat{G}; M) \to \mathrm{H}^{n+1}(G, M)$.

□

Proposition 2.6.2. *Let $G = \langle x_1, \ldots, x_n | r \rangle$ be a torsion-free non-free one-relator group, then $\mathrm{H}^2(G, \mathbb{F}_p) \simeq \mathbb{F}_p$ if and only if $r \in F^p[F, F]$.*

Proof. Let $0 \to K \to F \to G \to 0$ be a presentation for G. For the five-term exact sequence we have

$$0 \to \mathrm{H}^1(G, \mathbb{F}_p) \to \mathrm{H}^1(F, \mathbb{F}_p) \to \mathrm{H}^1(K, \mathbb{F}_p)^G \to \mathrm{H}^2(G, \mathbb{F}_p) \to \mathrm{H}^2(F, \mathbb{F}_p)$$

$\mathrm{H}^2(F, \mathbb{F}_p) = 0$ because F is a free group.

$\mathrm{H}^1(K, \mathbb{F}_p) \simeq \mathbb{F}_p[G]^*$ by a corollary of Lyndon's theorem, so $\mathrm{H}^1(K, \mathbb{F}_p)^G \simeq \mathbb{F}_p$.

Then $\mathrm{H}^2(G, \mathbb{F}_p) \simeq \mathbb{F}_p$ if and only if $\mathrm{H}^1(G, \mathbb{F}_p) \to \mathrm{H}^1(F, \mathbb{F}_p)$ is an isomorphism, that is if and only if $\frac{G}{G^p[G,G]} \simeq \frac{F}{F^p[F,F]}$. But this happens if and only if $r \in F^p[F, F]$. □

Proposition 2.6.3. *Let p be a prime. G is p-good if and only if for all $x \in \mathrm{H}^i(G, \mathbb{Z}_p)$, $1 \leq i \leq n$, there is a subgroup H of G, with $[G : H] < \infty$, such that x induces zero in $\mathrm{H}^i(H, \mathbb{Z}_p)$.*

Proof. Obviously if G is p-good then one of the implications is 4) above for $M = \mathbb{Z}_p$.

For the other implication, let M be a finite G-module, we make induction on the length of the composition series of M. The first step is our hypothesis.

Let $0 \to M' \to M \to \mathbb{Z}_p \to 0$ be an exact sequence where the length of the composition series of M' is shorter than that of M. By applying the long exact sequence for cohomology and naturality we obtain the commutative diagram

$$\begin{array}{ccccccccc} \mathrm{H}^{n-1}(\hat{G},\mathbb{Z}_p) & \longrightarrow & \mathrm{H}^{n}(\hat{G},M') & \longrightarrow & \mathrm{H}^{n}(\hat{G},M) & \longrightarrow & \mathrm{H}^{n}(\hat{G},\mathbb{Z}_p) & \longrightarrow & \mathrm{H}^{n+1}(\hat{G},M') \\ \downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\ \mathrm{H}^{n-1}(G,\mathbb{Z}_p) & \longrightarrow & \mathrm{H}^{n}(G,M') & \longrightarrow & \mathrm{H}^{n}(G,M) & \longrightarrow & \mathrm{H}^{n}(G,\mathbb{Z}_p) & \longrightarrow & \mathrm{H}^{n+1}(G,M') \end{array}$$

By induction hypothesis the first two and the last two vertical morphisms are isomoprhisms, so for the five lemma $\mathrm{H}^n(\hat{G}, M) \simeq \mathrm{H}^n(G, M)$.

□

Proposition 2.6.4. *Let G be a finitely generated one-relator group such that every subgroup of finite index is again a non-free one-relator group and with relator in $[F, F]$ involving every generator of F. Then G is p-good for every prime p.*

Proof. We only need to prove that for every p prime $\mathrm{H}^2(G, \mathbb{Z}_p)$ maps to 0 in $\mathrm{H}^2(U, \mathbb{Z}_p)$ for some U of finite index in G.

Since $r \in [F, F]$ we can choose U of index p by taking the inverse image of the subgroup of index p of the abelianization of G. U is a torsion-free one-relator group of finite index in G, so $\mathrm{H}^2(U, \mathbb{Z}_p) \simeq \mathbb{Z}_p$.

For any $x \in \mathrm{H}^2(G, \mathbb{Z}_p)$ it is $\mathrm{cor}_U^G \mathrm{res}_G^U x = [G : U]x$, but $[G : U] = p$, so it is the zero morphism.

The corestriction $\mathrm{cor}_U^G : \mathrm{H}^2(U, \mathbb{Z}_p) \to \mathrm{H}^2(G, \mathbb{Z}_p)$ is surjective, so we have that the restriction must be the zero morphism, thus proving our claim. □

Proposition 2.6.5. *Let G be a non-free, torsion-free one-relator group such that every subgroup of finite index is again a non-free one-relator group and with relator in $[F, F]$ involving every generator of F. Then $\hat{G}_p$, the pro-p completion of G, is a Demushkin group.*

Proof. G is p-good, so we have that $\hat{G}_p$ has cohomological dimension 2.

For every maximal subgroup U of $\hat{G}_p$ we have that U is the pro-p completion of $U \cap G$, where G is the immersion of G in $\hat{G}_p$, but $U \cap G$ has finite index in G, so it is a one-relator group. Thus $\hat{U}$ is a one-relator pro-p group.

We can conlude that G is a Demushkin group. □

Proposition 2.6.6. *Let G be a non-free, torsion-free one-relator group such that every subgroup of finite index is again a non-free one-relator group and with relator r in $[F, F]$ involving every generator of F. Then $r \notin \gamma_3(F)$.*

Proof. Suppose $r \in \gamma_3(F)$, then, for every p, $\hat{G}_p$ should be an orientable Demushkin group, so $\hat{G}_p$ would be a pro-p group one-relator group with relator not in γ_3, which is absurd. $\square$

We have an interesting result for the case with only two generators.

Theorem 2.6.1. *Let $G = \langle x, y | r\rangle$ be a residually finite, hereditary one relator group with only two generators, suppose $r \in [F, F]$.*

Then G is a surface group.

Proof. Let p be a prime, the pro-p completion $\hat{G}_p$ of G is an orientable Demushkyn group, so $r = [x, y]r'$ for some $r' \in \gamma_3(F)$.

Let P be a p-Sylow of $\hat{G}$, we know that $\mathrm{H}^2(\hat{G}, \mathbb{F}_p) \to \mathrm{H}^2(P, \mathbb{F}_p)$ is an isomorphism because G is p-good.

Since $\hat{G}_p$ has two generators, P must have at least two generators. If P had three or more generators, then there would be an open subgroup U of $\hat{G}$ such that $\dim_{F_p} \mathrm{H}^1(U, \mathbb{F}_p) \geq 3$, a contradiction. Then $\mathrm{H}^1(\hat{G}, \mathbb{F}_p) \to \mathrm{H}^1(P, \mathbb{F}_p)$ is an isomorphism.

We can conclude that $\hat{G}$ is p-nilpotent for every p, and that the p-Sylow of G is isomorphic to $\mathbb{Z}_p \oplus \mathbb{Z}_p$, so $\hat{G} \simeq \hat{Z} \oplus \mathbb{Z}_p$ is abelian. Since G is residually finite, G is also abelian and thus a surface group. $\square$

Chapter 3

Augmented duality groups

In this chapter we want to establish the concept of augmented duality group and show that finitely generated free groups and some one-relator groups are augmented duality groups. We will follow the unpublished papers of T. Weigel.

3.1 Triangulated categories

We recall here briefly the definition of triagulated category, following mainly [19].

Let $\mathcal{C}$ be an additive category with an additive and invertible endofunctor Σ. We will write $X[n]$ for $\Sigma^n X$. A *candidate triangle* in $\mathcal{C}$ is a diagram of the form

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

where $v \circ u$, $w \circ v$ and $u[1] \circ w$ are zero morphisms. A morphism of candidate triangles is a commutative diagram whose rows are candidate triangles.

Definition. A *triangulated category* $\mathcal{T}$ is an additive category, together with an additive and invertible endofunctor Σ called *suspension functor*, and a class $\mathcal{T}\nabla\rangle(\mathcal{T})$ of candidate triangles called *distinguished triangles* that satisfy the following conditions:

[T1] Any candidate triangle which is isomorphic to a disinguished triangle is a triangle.

[T2] For any object X in $\mathcal{T}$ the candidate triangle

$$X \xrightarrow{1} X \longrightarrow 0 \longrightarrow X[1]$$

is a distinguished triangle.

[T3] For any morphism $f : X \to Y$ in $\mathcal{T}$ there exists a distinguished triangle of the form

$$X \xrightarrow{f} Y \longrightarrow Z \longrightarrow X[1]$$

The object Z is called a *mapping cone* on the morphism f.

[T4] If the candidate triangle

$$X \xrightarrow{u} Y \xrightarrow{v} Z \xrightarrow{w} X[1]$$

is a disitnguished triangle, the candidate triangles

$$Y \xrightarrow{-v} Z \xrightarrow{-w} X[1] \xrightarrow{-u[1]} Y[1]$$

and

$$Z[-1] \xrightarrow{-w[-1]} X \xrightarrow{-u} Y \xrightarrow{-v} Z$$

are distinguished triangles (we say that this distinguished triangles are obtained rotating respectively forward and back the original distinguished triangle).

[T5] For any commutative diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & & & \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

whose rows are distinguished triangles there exists a morphism $h : Z \to Z'$ such that

(a) the diagram

$$\begin{array}{ccccccc} X & \xrightarrow{u} & Y & \xrightarrow{v} & Z & \xrightarrow{w} & X[1] \\ \downarrow{\scriptstyle f} & & \downarrow{\scriptstyle g} & & \downarrow{\scriptstyle h} & & \downarrow{\scriptstyle f[1]} \\ X' & \xrightarrow{u'} & Y' & \xrightarrow{v'} & Z' & \xrightarrow{w'} & X'[1] \end{array}$$

is commutative;

(b) the mapping cone of the previous morphis of distinguished triangles

$$Y \oplus X' \xrightarrow{\begin{pmatrix} -v & 0 \\ g & u' \end{pmatrix}} Z \oplus Y' \xrightarrow{\begin{pmatrix} -w & 0 \\ h & v' \end{pmatrix}} X[1] \oplus Z' \xrightarrow{\begin{pmatrix} -u[1] & 0 \\ f[1] & w' \end{pmatrix}} Y[1] \oplus X'[1]$$

is again a distinguished triangle.

Let $\mathcal{A}$ be an abelian category, we call $\mathcal{K}(\mathcal{A})$ the category whose objects are chain complexes $A = (A_k, \delta_k)$ in $\mathcal{A}$ and whose morphisms are homotopy equivalence classes of morphisms of cahin complexes. $\mathcal{K}(\mathcal{A})$ is again an additive category. We can define the endofunctorfunctor $\Sigma : \mathcal{K}(\mathcal{A}) \to \mathcal{K}(\mathcal{A})$ with $\Sigma A_k = A_{k+1}$ and $\delta_k^{\Sigma A} = -\delta_k^A$, it is invertible and additive. $\mathcal{K}(\mathcal{A})$ is an example of triangulated category, with the mapping cone on a morphism $A \to B$ given by $C_f^n = A^{n+1} \oplus B^n$ and $\delta_n^{C_f} = \begin{pmatrix} -\delta_{n+1}^A & 0 \\ f_{n+1} & \delta_n^B \end{pmatrix}$..

The *derived category* $\mathcal{D}(\mathcal{A})$ is the localization of $\mathcal{K}(\mathcal{A})$ with respect to quasi-isomorphisms, that is morphisms of chain complexes that induce isomorphisms on the cohomology of the chain complexes. $\mathcal{D}(\mathcal{A})$ is again a triangulated category, inheriting the structure from $\mathcal{K}(\mathcal{A})$.

3.2 Triangulated categories with duality

We define here the concept of duality in the contest of triangulated categories, introduced by P. Balmer in [1].

Definition (Categories with duality)**.** Let $\mathcal{C}$ be a category. A pair $(\_^\sharp, \omega)$, where $\_^\sharp : \mathcal{C}^{op} \to \mathcal{C}$ is a controvariant functor and $\omega : \mathrm{id}_\mathcal{C} \to \_^{\sharp\sharp}$ is a natural isomorphism, is called a *duality* if

$$\omega(C)^\sharp \circ \omega(C^\sharp) = \mathrm{id}_{C^\sharp}$$

for all $C \in obj(\mathcal{C})$.

An easy example of an additive category with duality is a ring with antipode.

If $(\mathcal{C}, \_^\sharp, \omega)$ is a category with duality, then any map $\alpha : A \to B^\sharp$ in $\mathcal{C}$ has an adjoint $\alpha_\omega^\sharp : B \to A^\sharp$ given by $\alpha^\sharp \circ \omega(B)$. Analogously, any map $\beta : A^\sharp \to B$ in $\mathcal{C}$ has an adjoint ${}_\omega\beta^\sharp : B^\sharp \to A$ given by $\omega(A)^{-1} \circ \beta^\sharp$. From the definition of duality follows that $(\alpha_\omega^\sharp)_\omega^\sharp = \alpha$ and ${}_\omega({}_\omega\beta^\sharp)^\sharp = \beta$.

We say that a map α from $A^\sharp$ to A (respectively from A to $A^\sharp$) is *self-adjoint* if $\alpha_\omega^\sharp = \alpha$ (respectively ${}_\omega\alpha^\sharp = \alpha$).

Proposition 3.2.1. *Let $(\mathcal{C}, \_^\sharp, \omega)$ be a category with duality, let α be a self-adjoint isomorphism. Then α^{-1} is also a self-adjoint isomorphism.*

Proof. Suppose $\alpha : A \to A^\sharp$, the case $\alpha : A^\sharp \to A$ is analogous.

Since α is a self-dual, $\alpha = \alpha_\omega^\sharp = \alpha^\sharp \circ \omega(A)$. ${}_\omega(\alpha^{-1})^\sharp = \omega(A)^{-1} \circ (\alpha^{-1})^\sharp$ by definition, so

$$\begin{aligned} {}_\omega(\alpha^{-1})^\sharp \circ \alpha &= (\omega(A)^{-1} \circ (\alpha^{-1})^\sharp) \circ \alpha_\omega^\sharp = \\ &= \omega(A)^{-1} \circ (\alpha^{-1})^\sharp \circ (\alpha^\sharp \circ \omega(A)) = \mathrm{id}_{A^\sharp} \end{aligned}$$

and

$$\alpha \circ_\omega (\alpha^{-1})^\sharp = \alpha^\sharp_\omega \circ (\omega(A)^{-1} \circ (\alpha^{-1})^\sharp) =$$
$$= (\alpha^\sharp \circ \omega(A)) \circ \omega(A)^{-1} \circ (\alpha^{-1})^\sharp = \mathrm{id}_A$$

We conclude that ${}_\omega(\alpha^{-1})^\sharp = \alpha^{-1}$. □

We will call a self-adjoint isomorphism a *self-duality*.

Now let $(\mathcal{C}, \mathcal{T}(\mathcal{C})$ and $(\mathcal{D}, \mathcal{T}(\mathcal{D})$ be triangulated categories. A controvariant functor $F : \mathcal{C}^{op} \to \mathcal{D}$ satisfying $F(C[n]) = F(C)[-n]$ for all $n \in \mathbb{Z}$, $C \in \mathrm{obj}(\mathcal{C})$ is called δ-*exact*, $\delta = \pm 1$, if for every distinguished triangle in $\mathcal{T}(\mathcal{C})$

$$A \xrightarrow{\alpha} B \xrightarrow{\beta} C \xrightarrow{\gamma} A[1]$$

the candidate triangle

$$F(C) \xrightarrow{F(\beta)} F(B) \xrightarrow{F(\alpha)} F(A) \xrightarrow{\delta F(\gamma)[1]} F(C)[1]$$

is also a distinguished triangle in $\mathcal{T}(\mathcal{D})$. If F is δ-exact then $F(\_)[n]$ is $(-1)^n \cdot \delta$-exact.

A trangulated category with δ-*duality* is a triangulated category $(\mathcal{C}, \mathcal{T}(\mathcal{C})$ together with a δ-exact controvariant functor $\_^\sharp : \mathcal{C}^{op} \to \mathcal{C}$ and a natural isomorphism $\omega : \mathrm{id}_\mathcal{C} \to \_^{\sharp\sharp}$ such that

$$\omega(C)^\sharp \circ \omega(C^\sharp) = \mathrm{id}_{C^\sharp}$$

and

$$\omega(C[n]) = \omega(C)[n]$$

for all $C \in obj(\mathcal{C})$, $n \in \mathbb{Z}$.

If $(\mathcal{C}, \mathcal{T}(\mathcal{C}, \_^\sharp, \omega)$ is a triangulated category with δ-duality, then $(\mathcal{T}(\mathcal{C}, \_^\sharp, \omega)$ with $\_^\sharp$ and ω defined in the obvious way is a category with duality. A self-duality in this category is an isomorphism of distinguished triangles

$$\begin{array}{ccccccc}
A & \xrightarrow{\alpha} & B & \xrightarrow{\beta} & C & \xrightarrow{\gamma} & A[1] \\
\downarrow f & & \downarrow g & & \downarrow h & & \downarrow f[1] \\
C^\sharp & \xrightarrow{\beta^\sharp} & (B^\sharp & \xrightarrow{\alpha^\sharp} & A^\sharp & \xrightarrow{\delta\gamma^\sharp[1]} & C^\sharp[1]
\end{array}$$

with g self-duality and f and h isomorphisms satisfying $h = f^\sharp_\omega$.

3.3 Derived categories with duality

Let R be a commutative ring and (A, σ) an associative R-algebra with antipode σ.

If P is a finitely generated projective left A-module then twisting the action through the antipode σ we obtain a finitely generated projective right A-module $P^\times$ and vice versa.

The *σ-dual* of a finitely generated projective left A module P will be defined as

$$P^* = \mathrm{Hom}_A^\sigma(P, A) = \{F \in \mathrm{Hom}_\mathbb{Z}(P, A) | f(a \times p) = f(p) \times \sigma(a) \forall p \in P, a \in A\}$$

P^* is a finitely generated projective left A-module. The map

$$\omega : P \to (P*) * p \to \omega(p)$$

where $\omega(p)(x*) = \omega(x * (p))$ for all $x* \in P*$ is an isomorphism of left A-modules. If $\alpha : P \to Q$ is a homomorphism of finitely generated projective left A-modules then the adjoint map $\alpha^* : Q^* \to P^*$ is given by $\alpha * (q^*)(p) = q^*(\alpha(p))$.

We call $\mathcal{D}(^\times A)$ (respectively $\mathcal{D}(A^\times)$) the full subcategory of $\mathcal{D}_b({}_A\mathrm{Mod})$ (resp. $\mathcal{D}_b(\mathrm{Mod}_A)$), the bounded derived category of chain complexes of left (resp. right) A-modules, whose objects are the finite chain complexes of finitely generated projective left (resp. right) A-modules. $\mathcal{D}(^\times A)$ (resp. $\mathcal{D}(A^\times)$) is a triangulated category.

Given $P = (P_k, \partial_k) \in \mathrm{obj}(\mathcal{D}(^\times A))$ finite chain completx of finitely generated projective left A-modules, the *σ-dual chain complex* $P^\circledast = (P_k^\circledast, \partial_k^\circledast)$ is defined by $P_k^\circledast = P_{-k}^*$ and $\partial^{\circledast_k}(p*_k)(p_{1-k}) = p *_k (\partial_{1-k}(p_{1-k}))$

Proposition 3.3.1. *Let (A, σ) be an associative R-algebra with antipode. Then:*

1. *The functor $\_^\circledast : \mathcal{D}(^\times A)^{op} \to \mathcal{D}(^\times A)$ is a controvariant $+1$-exact functor and $P^\circledast[n] = (P[-n])^\circledast$ for all P object in $\mathcal{D}(^\times A)^{op}$ and $n \in \mathbb{Z}$.*

2. *The natural morphism $\omega : id_{\mathcal{D}(^\times A)} \to \_^{\circledast\circledast}$, defined for $P = (P_k, \partial_k)$ object in $\mathcal{D}(^\times A)$ by $w_k(p_k)(q_{-k}^*) = \sigma(q_{-k}^*(p_k))$ where $p_k \in P_k$, is a natural isomorphism of covariant additive functors that satisfyies the identities $\omega(P)^\circledast \circ \omega(P^\circledast) = id_{P^\circledast}$ and $\omega(P[n]) = \omega(P)[n]$ for all $n \in \mathbb{N}$.*

Proof. 1. Let $f : P \to Q$ be a morphism of degree 0 of finite chain complexes of finitely generated projective left A-modules.

Consider the diagram

$$
\begin{array}{ccccccc}
P[1]^{\circledast} & \xrightarrow{-\delta^{\circledast}} & C(f)^{\circledast} & \xrightarrow{\pi^{\circledast}} & Cyl(f)^{\circledast} & \xrightarrow{\overline{f}^{\circledast}} & P^{\circledast} \\
\| & & \downarrow{\scriptstyle \beta(-\delta^{\circledast})} & & \| & & \| \\
P[1]^{\circledast} & \xrightarrow{-\delta^{\circledast}} & Cyl(-\delta^{\circledast}) & \xrightarrow{\pi(-\delta^{\circledast})} & C(-\delta^{\circledast}) & \xrightarrow{\delta(-\delta^{\circledast})} & P^{\circledast}
\end{array}
$$

where the chain elements and chain morphisms are defined in the following way:

$$
\begin{gathered}
P[1]^{\circledast}_{-k} = P^*_{k+1} \\
\partial_k^{P^{\circledast}[1]}(p^*_{k-1}) = -\partial_{k-1}^{P^{\circledast}}(p^*_{k-1}) \\
C(f)^{\circledast}_{-k} = P^*_{k+1} \oplus Q^*_k \\
\partial_{-k}^{C(f)^{\circledast}}(p^*_{k+1}, q^*_k) = (\partial_{-k}^{P[1]^{\circledast}} + f^*_k, \partial_{-k}^{Q^{\circledast}}) \\
Cyl(f)^{\circledast}_{-k} = P^*_k \oplus P^*_{k+1} \oplus Q^*_k \\
\partial_{-k}^{Cyl(f)^{\circledast}}(p^*_k, p^*_{k+1}, q^*_k) = (\partial_{-k}^{P^{\circledast}}(p^*_k), \partial_{-k}^{P[1]^{\circledast}}(p^*_{k+1}) - p^*_k + f^*_k(q^*_k), \partial_{-k}^{Q^{\circledast}}(q^*_k)) \\
C(-\delta^{\circledast})_{-k} = P^*_k \oplus P^*_{k+1} \oplus Q^*_k \\
\partial_{-k}^{C(-\delta^{\circledast})}(p^*_k, p^*_{k+1}, q^*_k) = (\partial_{-k}^{P^{\circledast}}(p^*_k), \partial_{k}^{P[1]^{\circledast}}(p^*_{k+1}) - p^*_k + f^*_k(q^*_k), \partial_{-k}^{Q^{\circledast}}(q^*_k)) \\
Cyl(-\delta^{\circledast})_{-k} = P^*_{k+1} \oplus P^*_k \oplus P^*_{k+1} \oplus Q^*_k \\
\partial_{-k}^{Cyl(\delta^{\circledast})}(r^*_{k+1}, p^*_k, p^*_{k+1}, q^*_k) = (\partial_{-k}^{P[1]^{\circledast}}(r^*_{k+1}) - p^*_k, \partial_{-k}^{P^{\circledast}}(p^*_k), \partial_{-k}^{P[1]^{\circledast}}(p^*_{k+1}) - p^*_k + f^*_k(q^*_k).\partial_k^{Q^{\circledast}}(q^*_k))
\end{gathered}
$$

for $p^*_k \in P^*_k$, $p^*_{k-1} \in P^*_{k-1}$, $r^*_{k+1}, p^*_{k+1} \in P^*_{k+1}$, $q^*_k \in Q^*_k$.
The maps are given by

$$
\begin{gathered}
-\delta^{\circledast}_k(p^*_{k+1}) = (-p^*_{k+1}, 0) \\
\pi^{\circledast}_k(p^*_{k+1}, q^*_k) = (0, p^*_{k+1}, q^*_k) \\
\overline{f}^{\circledast}_k(p^*_k, p^*_{k+1}, q^*_k) = p^*_k \\
-\overline{\delta}^{\circledast}_k(r^*_{k+1}) = (r^*_{k+1}, 0, 0) \\
\pi(-\delta^{\circledast})_k(r^*_{k+1}, p^*_k, p^*_{k+1}, q^*_k) = (p^*_k, p^*_{k+1}, q^*_k) \\
\delta(-\delta^{\circledast})_k(p^*_k, p^*_{k+1}, q^*_k) = p^*_k \\
\beta(-\delta^{\circledast})_k(p^*_{k+1}, q^*_k) = (0, 0, p^*_{k+1}, q^*_k)
\end{gathered}
$$

The two rightmost suqares of the diagram commute as maps of chain complexes. The leftmost square commutes as a mapping of chain complexes modulo chain homotopies (cf). Sinche $\beta(-\delta^{\circledast})$ is a quasi-isomorphism, the diagram is an isomorphism of triangles, but the second row is a distinguished triangle, hence

$$P[1]^{\circledast} \to C(f)^{\circledast} \to Cyl(f)^{\circledast} \to \overline{f}^{\circledast} P^{\circledast}$$

is also a distinguished triangle.

2. For $q^*_{1-k} \in P^*_{1-k}$, $p_k \in P_k$ we have that

$$(\partial_k^{P^{\circledast\circledast}}(\omega_k(p_k)))(q^*_{1-k}) = \sigma(\partial_{1-k}^{P^{\circledast}}(q^*_{1-k})(p_k)) = \\ = \sigma(q^*_{1-k}(\partial_k^P(p_k)))(q^*_{1-k}),$$

Then ω is a mapping of chain complexes of degree 0 and a natural isomorphism for hhh.

For $p^*_k \in P^*_k$, $q_{-k} \in P_{-k}$ it is

$$(\omega^P)_k^{\circledast}(\omega_k^{P^{\circledast}}(p^*_k))(q_{-k}) = \omega_k^{P^{\circledast}}(p^*_k)(\omega_{-k}^P(q_{-k})) = \sigma((\omega_{-k}^p(q_{-k})(p^*_k))) = \\ = \sigma^2(p^8_k(q_{-k})) = p^*_k(q_{-k})$$

Form this we conclude that $(\omega^P)^{\circledast} \circ \omega^{P^{\circledast}} = id_{P^{\circledast}}$.

The identity $\omega(P)^{\circledast} \circ \omega(P^{\circledast}) = id_{P^{\circledast}}$ and $\omega(P[n]) = \omega(P)[n]$ is obvious.

□

The proposition above shows that $(\_^{\circledast}, \omega)$ is a $+1$-duality on the triangulated category $\mathcal{D}(^{\times}A)$ that depends only on the antipode σ. We will therefore write $(\mathcal{D}(^{\times}A), \sigma)$ instead of $(\mathcal{D}(^{\times}A), \_^{\circledast}, \omega)$ to underline this dependance.

3.4 Augmented duality groups

Let G be a duality group of cohomological dimension d. Its integral group algebra $\mathbb{Z}[G]$ admits an antipode σ obtained by the standard antipode on G, $g \to g^{-1}$, by twisting with a linear character.Then $\mathcal{D}(^{\times}A)$ has a structure of triangulated category with duality induced by σ.

Definition. Let $P \in obj(\mathcal{D}(^{\times}A))$ a finite and finitely generated projective resolution of the trivial left $\mathbb{Z}[G]$ module concentrated in degree 0. (G, σ) is called an *augmented duality group* of dimension d if there exists a mapping $\zeta : P^{\circledast}[d] \to P$ such that:

1. the morphism $\mathrm{H}_0(\zeta) :^{\times} D \to \mathbb{Z}$ is surjective;

2. ζ is $(-1)^d$-symmetric in $(\mathcal{D}(^{\times}A), \sigma)[d]$;

3. let $C = C(\zeta)[1]$ be the 1-shifted cone of ζ, then there is a $(-1)^{d-1}$ self duality $\eta : C \to C^{\circledast}[d-1]$ in $(\mathcal{D}(^{\times}A), \sigma)[d-1]$ that gives the isomorphism of distinguished triangles

$$\begin{array}{ccccccc} P[-1] & \xrightarrow{a} & C & \xrightarrow{b} & P^{\sharp_1} & \xrightarrow{\zeta} & P \\ \| & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle (-1)^d \operatorname{id}_{P^{\sharp_1}}} & & \| \\ P[-1] & \xrightarrow{-b^{\sharp_2}_{\omega_1}} & C^{\sharp_2} & \xrightarrow{a^{\sharp_2}} & P^{\sharp_1} & \xrightarrow{{}_{\omega_1}\zeta^{\sharp_1}} & P \end{array}$$

where $(\_^{\sharp_1}, \omega_1)$ denotes the duality in $(\mathcal{D}(^{\times}A), \sigma)[d]$ and $(\_^{\sharp_2}, \omega_2)$ the duality in $(\mathcal{D}(^{\times}A), \sigma)[d-1]$, a and b are canonical maps and $b^{\sharp_2}_{\omega_2} = b^{\sharp_2} \circ \omega_2(P[-1])$, ${}_{\omega_1}\zeta^{\sharp_1} = \omega_1(P)^{-1} \circ \zeta^{\sharp_1}$

Theorem 3.4.1. *Let F be a finitely generated free group. Then F is an augmented duality group of dimension* 1*.*

Proof. Let F be a finitely generated free group over the basis X. Let $P = (P_k, \delta_k)$ be the chain complex of left $\mathbb{Z}F$ modules given by

$$P_0 = \mathbb{Z}F\langle 1\rangle, P_1 = \prod_{x \in X} \mathbb{Z}F\langle x\rangle, P_k = 0 (k \neq 0, 1)$$
$$\delta_1(\langle x\rangle\langle 1\rangle) = (x-1) \quad \forall x \in X$$

Then $\varepsilon : P \to \mathbb{Z}[[0]]$, $\varepsilon_0(a\langle 1\rangle) = \varepsilon(a)$ for $a \in \mathbb{Z}F$, where ε is the augmentation map from $\mathbb{Z}F$ to $\mathbb{Z}$, $\varepsilon_k = 0$ for $k \neq 0$, is a projective resolution of the trivial left $\mathbb{Z}F$ module $\mathbb{Z}$ concentrated in degree 0, that is an isomorphism in the derived category of bounded chain complexes of left $\mathbb{Z}F$-modules.

P is a finite complex of finitely generated projective left $\mathbb{Z}F$-modules and it is possible to give an explicit description of the dual of the standard complex shifted by 1, that is $P^{\circledast}[1]$.

$$P^{\circledast}[1]_0 = \prod_{x \in X} \mathbb{Z}F\langle x^*\rangle, P^{\circledast}[1]_1 = \mathbb{Z}F\langle 1^*\rangle, P^{\circledast}[1]_k = 0 (k \neq 0, 1)$$
$$\delta_1^{\circledast}(\langle 1^*\rangle) = \sum_{x \in X}(1 - x^{-1})\langle x^*\rangle$$

where $\langle 1^*\rangle \in (\mathbb{Z}F\langle 1\rangle)^*$ with $\langle 1^*\rangle(\langle 1\rangle) = 1$ and $\langle x^*\rangle \in (\mathbb{Z}F\langle x\rangle)^*$ with $\langle x^*\rangle(\langle x\rangle) = 1$ for every $x \in X$.

Since F is a finitely generated free group, it is a duality group of dimension 1, so we have that $\mathrm{H}^0(P^{\circledast}[1]) =^{\times} D$ where D denotes the right dualizing of F and $^{\times}D$ is the corresponding left $\mathbb{Z}F$-module via the antipode σ, while $\mathrm{H}^k(P^{\circledast}[1]) = 0$ for $k \neq 0$.

Now let $\zeta : P^{\circledast}[1] \to P$ be the map given by $\zeta_0(\langle x^* \rangle) = x\langle 1 \rangle$, $\zeta_1(\langle 1* \rangle) = \sum_{x \in X} \langle x \rangle$ and $\zeta_k = 0$ for $k \neq 0, 1$, it is a mapping of chain complexes of degree 0 and it induces a surjective map

$$\mathrm{H}^0(\zeta) : \mathrm{H}^0(P^{\circledast}[1]) \to \mathrm{H}^0(P)$$

The mapping $\varepsilon_D :^{\times} D \to \mathbb{Z}$ is the unique map such that the diagram

$$\begin{array}{ccc} \mathrm{H}^0(P^{\circledast}[1]) & \xrightarrow{\mathrm{H}^0(\zeta)} & \mathrm{H}^0(P) \\ \| & & \downarrow{\scriptstyle \mathrm{H}^0(\varepsilon)} \\ {}^{\times}D & \xrightarrow{\varepsilon_D} & \mathbb{Z} \end{array}$$

commutes. Let $(-^{\sharp}, \bar{\omega})$ be the duality in the triangulated category with duality $(\mathcal{D}(^{\times}\mathbb{Z}F), -^{\circledast}, \omega)[1]$. We have that $-^{\sharp}$ is (-1)-exact and $\bar{\omega} = -\omega$. The mapping of chain complexes $\bar{\omega}(P)^{-1} \circ \zeta^{\sharp} =_{\bar{\omega}} \zeta^{\sharp} : P^{\sharp} \to P$ is then given by

$${}_{\bar{\omega}}\zeta_0^{\sharp}(\langle x^* \rangle) = -\langle 1 \rangle, \quad {}_{\bar{\omega}}\zeta_1^{\sharp}(\langle 1^* \rangle) = \sum_{x \in X} -x^{-1}\langle x \rangle$$

and ${}_{\bar{\omega}}\zeta_k^{\sharp} = 0$ for $k \neq 0, 1$.

Let $s_k : P_k^{\sharp} \to P_{k+1}$ defined by $s_0(\langle x^* \rangle) = \langle x \rangle$ and $s_k = 0$ for $k \neq 0$, then it is $({}_{\bar{\omega}}\zeta^{\sharp} + \zeta)_k = \delta_{k+1} \circ s_k + s_{k-1} \circ \delta_k$, that is ${}_{\bar{\omega}}\zeta^{\sharp}$ is homotopy equivalent to $-\zeta$. Thus ζ is (-1)-symmetric in $(\mathcal{D}(^{\times}\mathbb{Z}F), -^{\circledast}, \omega)[1]$.

Let $C = C(\zeta)[1]$ be the 1-shifted cone of ζ, it is defined by the following:

$$C_{-1} = P_0, C_0 = P_1^* \oplus P_1, C_1 = P_0^*, C_k = 0 (k \neq \pm 1, 0)$$
$$\delta_1^C(\langle 1^* \rangle) = \sum_{x \in X} (1 - x^{-1})\langle x^* \rangle - \sum_{x \in X} \langle x \rangle$$
$$\delta_0^C(\langle x^* \rangle) = -x\langle 1 \rangle$$
$$\delta_0^C(\langle x \rangle) = (1 - x)\langle 1 \rangle$$

Then we have that the σ-dual chain complex $C^{\circledast}$ is defined as follows:

$$C_{-1}^{\circledast} = P_0^{**}, C_0^{\circledast} = P_1^{**} \oplus P_1^*, C_1^{\circledast} = P_0^*, C_k^{\circledast} = 0 (k \neq \pm 1, 0)$$
$$\delta_1^{C^{\circledast}}(\langle 1^* \rangle) = -\sum_{x \in X} x^{-1}\langle x^{**} \rangle + \sum_{x \in X} (1 - x^{-1})\langle x^* \rangle$$
$$\delta_0^{C^{\circledast}}(\langle x^{**} \rangle) = (1 - x)\langle 1^{**} \rangle$$
$$\delta_1^{C^{\circledast}}(\langle x^* \rangle) = -\langle 1^{**} \rangle$$

where $\langle x^{**} \rangle(\langle x^* \rangle) = 1$ for $x \in X$ and $\langle 1^{**} \rangle(\langle 1^* \rangle) = 1$.

We have a mapping of chain complexes $\eta : C \to C^{\circledast}$ defined by

$$\begin{aligned} \eta_{-1}(\langle 1\rangle) &= -\langle 1^{**}\rangle, \\ \eta_0(\langle x\rangle) &= -\langle x^{**}\rangle, \\ \eta_0(\langle x^*\rangle) &= -(\langle x^*\rangle + \langle x^{**}\rangle), \\ \eta_1(\langle 1^*\rangle) &= -\langle 1^*\rangle \end{aligned}$$

and it is $\eta^{\circledast} \circ \omega = \eta$, so it is a self-duality in $(\mathcal{D}({}^{\times}\mathbb{Z}F), -^{\circledast}, \omega)[1]$.

Let $a : P[-1] \to C$ and $C \to P^{\circledast}[1]$ be the canonical maps, consider the diagram

$$\begin{array}{ccccccc} P[-1] & \xrightarrow{a} & C & \xrightarrow{b} & P^{\sharp_1} & \xrightarrow{\zeta} & P \\ \| & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle (-1)^d \mathrm{id}_{P^{\sharp_1}}} & & \| \\ P[-1] & \xrightarrow{-b^{\sharp_2}_{\omega_1}} & C^{\sharp_2} & \xrightarrow{a^{\sharp_2}} & P^{\sharp_1} & \xrightarrow{{}_{\omega_1}\zeta^{\sharp_1}} & P \end{array}$$

where ${}_{\omega_1}\zeta^{\sharp_1} = \bar{\omega}(P)^{-1} \circ \zeta^{\circledast}[1]$.

The first square is commutative because

$$\begin{aligned} \eta_{-1}(a_{-1}(\langle 1\rangle)) = \eta(\langle 1\rangle) = -\langle 1\rangle^{**} = -b^{\sharp_2}_{-1}(\langle 1\rangle^{**}) = \\ = -b^{\sharp_2}_{-1}(\omega(P)_0)(\langle 1\rangle)) \\ \eta_0(a_0(\langle x\rangle)) = \eta_0(\langle x\rangle) = -\langle x^{**}\rangle = -b^{\sharp_2}_0(\langle x^{**}\rangle) = -b^{\sharp_2}_0(\omega(P)_1(\langle x\rangle)) \\ \eta_1 \circ a_1 = 0 = -b^{\sharp_2}_1 \circ \omega(P)_2 \end{aligned}$$

In particular we have that

$$a^{\sharp_2}_1 \circ \eta^{\sharp_2} = -\omega(P)^{\sharp_1} \circ b^{\sharp_2\sharp_2} = -b \circ \omega(C)^{-1},$$

since $\eta^{\sharp_1} \circ \omega = \eta$ we have that the central square is also commutative. Finally, the third square is commutative because because ζ is -1-symmetric.

We conclude that the diagram is an isomorphism of distinguished triangles and conclude that G is an augmented duality group. □

Theorem 3.4.2. *Let $G = \langle X|r\rangle$ be a non free and torsion free finitely generated one-relator group, such that the relation r (minimal under isomorphisms of the free group generated by X) involves every generator with exponential sum 0. Then G is an augmented duality group.*

Proof. Let $G = \langle X|r\rangle$ be a non free and torsion free finitely generated one-relator group, such that the relation r (minimal under isomorphisms of the free group generated by X) involves every generator with exponential sum 0. We have already seen that the dualizing module $\mathrm{H}^2(G, \mathbb{Z}G)$ is a quotient of $\mathbb{Z}G$ that inherits the augmentation. If K is the kernel of the inherited augmentation, we have a short exact sequence of left $\mathbb{Z}G$-modules

$$0 \longrightarrow K \longrightarrow \mathrm{H}^2(G, \mathbb{Z}G) \longrightarrow \mathbb{Z} \longrightarrow 0$$

If P is the projective resolution of $\mathbb{Z}$ over $\mathbb{Z}G$ then $P^{\circledast}[2]$ is a projective resolution of $\mathrm{H}^2(G, \mathbb{Z}G)$ over $\mathbb{Z}G$. The morphism $\mathrm{H}^2(G, \mathbb{Z}G) \to \mathbb{Z}$ induces a morphism $\zeta : P^{\circledast}[2] \to P$ in $\mathcal{D}(^{\times}A)$ such that $\mathrm{H}^0(\eta) : \mathrm{H}^2(G, \mathbb{Z}G) \to \mathbb{Z}$ is the inherited augmentation.

Since

$$0 \longrightarrow K \longrightarrow \mathrm{H}^2(G, \mathbb{Z}G) \longrightarrow \mathbb{Z} \longrightarrow 0$$

is an exact sequence, if Q is a projective resolution of K over $\mathbb{Z}G$ we have an induced distinguished triangle

$$P[-1] \xrightarrow{\ \alpha\ } Q \xrightarrow{\ \beta\ } P^{\circledast}[2] \xrightarrow{\ \zeta\ } P$$

Let $(-^{\sharp}, \bar{\omega})$ be the duality in the triangulated category with duality $(\mathcal{D}(^{\times}\mathbb{Z}F), -^{\circledast}, \omega)[2]$. We have that $-^{\sharp}$ is 1-exact and $\bar{\omega} = \omega$, so ζ is 1-symmetric.

Consider the commutative diagram

$$\begin{array}{ccccccc}
P[-1] & \xrightarrow{\ \alpha\ } & Q & \xrightarrow{\ \beta\ } & P^{\circledast}[2] & \xrightarrow{\ \zeta\ } & P \\
\| & & & & \downarrow{\scriptstyle \mathrm{id}_{P^{\circledast}[2]}} & & \| \\
P[-1] & \xrightarrow{-\beta^{\sharp_2}_{\omega_1}} & Q^{\circledast}[2] & \xrightarrow{\alpha^{\omega_2}} & P^{\circledast}[2] & \xrightarrow{{}^{\omega_1}\zeta^{\sharp_1}} & P
\end{array}$$

The lines are distinguished triangles, so there is a unique morphism $\eta : Q \to Q^{\circledast}[2]$ (up to homotopy equivalence) such that the diagram

$$\begin{array}{ccccccc}
P[-1] & \xrightarrow{\ \alpha\ } & Q & \xrightarrow{\ \beta\ } & P^{\circledast}[2] & \xrightarrow{\ \zeta\ } & P \\
\| & & \downarrow{\scriptstyle \eta} & & \downarrow{\scriptstyle \mathrm{id}_{P^{\circledast}[2]}} & & \| \\
P[-1] & \xrightarrow{-\beta^{\sharp_2}_{\omega_1}} & Q^{\circledast}[2] & \xrightarrow{\alpha^{\omega_2}} & P^{\circledast}[2] & \xrightarrow{{}^{\omega_1}\zeta^{\sharp_1}} & P
\end{array}$$

is commutative. η is a -1-self duality in $(\mathcal{D}(^{\times}\mathbb{Z}F), -^{\circledast}, \omega)[1]$, since $\bar{\omega}$ is -1-exact.

Since the distinguished triangle

$$P[-1] \xrightarrow{\ \alpha\ } Q \xrightarrow{\ \beta\ } P^{\circledast}[2] \xrightarrow{\ \zeta\ } P$$

is isomorphic to the distinguished triangle

$$P[-1] \xrightarrow{\ a\ } C \xrightarrow{\ b\ } P^{\circledast}[2] \xrightarrow{\ \zeta\ } P$$

where C is the mapping cone of ζ, we have that G is an augmented duality group. □

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Contents