Building blocks for designing arbitrarily smooth subdivision schemes with conic precision

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Abstract

Since subdivision schemes featured by high smoothness and conic precision are strongly required in many application contexts, in this work we define the building blocks to obtain new families of non-stationary subdivision schemes enjoying such properties. To this purpose, we firstly derive a non-stationary extension of the Lane-Riesenfeld algorithm, and we exploit the resulting class of schemes to design a non-stationary family of alternating primal/dual subdivision schemes, all featured by reproduction of $\{1, x, e^{tx}, e^{-tx}\}$, $t \in [0, \pi) \cup \mathrm{i}\mathbb{R}^+$. Then, we focus our attention on interpolatory subdivision schemes with conic precision, that can be obtained as a byproduct of the above classes. In particular, we present a novel construction of a family of non-stationary interpolatory $2n$-point schemes which generalizes the well-known Dubuc-Deslauriers family in such a way the n-th ($n \geq 2$) family member reproduces $\Pi_{2n-3} \cup \{e^{tx}, e^{-tx}\}$, $t \in [0, \pi) \cup \mathrm{i}\mathbb{R}^+$, and keeps the original smoothness of its stationary counterpart unchanged.

Keywords: Non-stationary subdivision; Exponential polynomial generation; Conic reproduction; Smoothness; Interpolation.

1. Introduction

Subdivision schemes are efficient tools for generating smooth curves and surfaces as the limit of an iterative algorithm based on simple refinement rules. More precisely, in the univariate case, for any given set of initial control points $\mathbf{P}^{(0)} := \{P_i^{(0)}, i \in \mathbb{Z}\}$, a linear subdivision scheme recursively produces denser sets of control points $\mathbf{P}^{(k+1)}$, for all $k \in \mathbb{N}_0 := \mathbb{N} \cup \{0\}$, by computing local linear combinations of points from the previous level. If the same refinement rules are used at all levels of refinement, then the scheme is called *stationary*, otherwise *non-stationary*.

In the stationary context, the Lane-Riesenfeld algorithm [17] defines the symbols associated to the family of B-spline schemes of order ℓ, with $\ell \in \mathbb{N}$. In literature, the use of these symbols as 'building blocks' to define both interpolatory schemes [7] and subdivision schemes with enhanced reproduction capabilities [14] has been recently shown. In fact, Conti and Romani observed that ℓ-point (with ℓ even) Dubuc-Deslauriers schemes [11] are characterized by a symbol containing the factor $\frac{(z+1)^\ell}{2^{\ell-1}}$, while Hormann and Sabin noticed that the same factor (with $\ell \in \mathbb{N}$) is also contained in the symbol of the family of subdivision schemes with cubic precision. The latter family is indeed defined by the product of the symbol of the Lane-Riesenfeld's family with a degree-2 polynomial, that they called *kernel*, tailored to increase the degree of polynomial reproduction of B-spline schemes from one to three. Moreover, in [10] it has been also recently illustrated that the first member of the Lane-Riesenfeld's family and that of the Hormann-Sabin's family can be combined together to give rise to a recursive formula defining the interpolatory $2n$-point Dubuc-Deslauriers schemes

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for all $n \geq 3$.
These observations prompted us to study the generalization of these two fundamental classes of schemes to the non-stationary setting. Our first contribution in this direction consists in the proposal of a level-dependent extension of the Lane-Riesenfeld algorithm, aimed at providing the symbols of normalized exponential B-splines. These symbols, together with a non-stationary version of Hormann-Sabin's kernels, are successively used as 'building blocks' to define a family of alternating primal/dual subdivision schemes reproducing conics. The first member of the resulting family, combined with the first one of the novel Lane-Riesenfeld's family, is shown to originate a three-term recurrence formula defining the symbols of the non-stationary interpolatory $2n$-point schemes reproducing the space $span\{1, x, ..., x^{2n-3}, e^{tx}, e^{-tx}\}$, where $t \in [0, \pi) \cup \mathrm{i}\mathbb{R}^+$ and $n \in \mathbb{N}$, $n \geq 3$. We highlight the fact that, non-stationary subdivision schemes enjoying properties like interpolation, conic precision and arbitrarily high smoothness, are considered wished tools both in geometric modelling and image segmentation. As to the latter, we recall that one of the most used tools for efficient image segmentation are active contours (snakes), i.e. 2D curves evolving through the image, capable of perfectly outlining elliptic objects and offering user-friendly models, versatile enough to provide a close smooth approximation of any closed polyline in the plane [9].

The remainder of the paper is organized as follows. In Section 2 we start by presenting all the fundamental notions about stationary and non-stationary subdivision schemes that are necessary to the development of the subsequent results. Section 3 is devoted to the stationary context. After recalling the basic formulations of the Lane-Riesenfeld algorithm and the Hormann-Sabin's family, we review the existing different formulations of the family of $2n$-point interpolatory Dubuc-Deslauriers schemes, and we show how to obtain its symbol exploiting the Lane-Riesenfeld's and Hormann-Sabin's families as building blocks. All remaining sections deal with the non-stationary setting and present original results. In particular, after presenting our extension of the Lane-Riesenfeld algorithm (Section 4), we construct a family of alternating primal/dual non-stationary subdivision schemes reproducing conics, which generalizes the well-known Hormann-Sabin's family (Section 5). Finally, in Section 6 we exploit a suitable perturbation of the symbols of the well-known Dubuc-Deslauriers schemes to define non-stationary interpolatory $2n$-point schemes which achieve the property of conic precision, without affecting the smoothness order of the original proposal.

2. Background notions

2.1. The stationary case

Let a_i, $i \in \mathbb{Z}$, be the coefficients appearing in the linear combination that defines at each iteration the new-level points. Then, for each $k \in \mathbb{N}_0$, the refinement rules are

$$P_{2i+h}^{(k+1)} = \sum_{j \in \mathbb{Z}} \mathrm{a}_{2j+h} \, P_{i-j}^{(k)}, \qquad h = 0, 1. \tag{2.1}$$

The set of coefficients $\{\mathrm{a}_i \in \mathbb{R}, i \in \mathbb{Z}\}$ appearing in (2.1) is called *subdivision mask* and is denoted by $\mathbf{a}$. The subdivision scheme with mask $\mathbf{a}$ is denoted by $\mathcal{S}_{\mathbf{a}}$ and can be equivalently seen as the repeated application of the subdivision matrix $M = \{M(i,j) = \mathrm{a}_{i-2j} : i, j \in \mathbb{Z}\}$ to the initial data $\mathbf{P}^{(0)}$.
Applying the z-transform, we can associate the mask $\mathbf{a}$ to the Laurent series

$$A(z) = \sum_{i \in \mathbb{Z}} \mathrm{a}_i z^i, \qquad z \in \mathbb{C} \setminus \{0\}, \tag{2.2}$$

which is called the *symbol* of the subdivision scheme. Since only a finite number of coefficients a_i are non-zero, the Laurent series $A(z)$ is indeed a Laurent polynomial.
The symbol $A(z)$ has been shown to be a convenient tool to investigate both convergence/smoothness and generation/reproduction properties of the subdivision scheme $\mathcal{S}_{\mathbf{a}}$.
We recall that a subdivision scheme is said to be *convergent* if, for any initial data $\mathbf{P}^{(0)} \in \ell^\infty(\mathbb{Z})$, there exists a function $\mathcal{F} \in C^0(\mathbb{R})$ such that for any compact set Ω in $\mathbb{R}$, $\lim_{k \to +\infty} \sup_{i \in \mathbb{Z} \cap 2^k \Omega} |P_i^{(k)} - \mathcal{F}(2^{-k} i)| = 0$, and $\mathcal{F}$ is not identically 0 for some initial data $\mathbf{P}^{(0)}$. In particular, for any convergent subdivision scheme, we denote

by Φ the limit function obtained from the initial sequence $\boldsymbol{\delta} = \{\delta_{i,0} : i \in \mathbb{Z}\}$ where $\delta_{i,0} = \begin{cases} 1, & \text{if } i = 0, \\ 0, & \text{otherwise.} \end{cases}$
Φ is usually called the *basic limit function* of the subdivision scheme.
Existing results on polynomial generation and reproduction properties of a stationary subdivision scheme are restricted to the class of non-singular schemes, i.e the ones that generate the zero function if and only if $\mathbf{P}^{(0)}$ is the zero sequence. In particular, in [6, Theorem 4.3], it was recently shown that for any convergent and non-singular subdivision scheme $\mathcal{S}_{\mathbf{a}}$, the polynomial generation and reproduction properties can be easily studied by looking at the values assumed by the symbol and its derivatives at $z = \pm 1$. More precisely, polynomial generation is guaranteed by the correct behaviour of the symbol $A(z)$ and its derivatives at $z = -1$, and if $A(z)$ and its derivatives also behave correctly at $z = 1$, then the scheme reproduces polynomials.

2.2. The non-stationary case

Since the refinement rules are not the same at all levels of refinement, the mask at the k-th level is defined as the sequence $\mathbf{a}^{(k)} = \{\mathrm{a}_i^{(k)} \in \mathbb{R},\, i \in \mathbb{Z}\}$, and the associated k-level symbol is

$$A^{(k)}(z) = \sum_{i \in \mathbb{Z}} \mathrm{a}_i^{(k)} z^i, \qquad z \in \mathbb{C} \setminus \{0\}. \tag{2.3}$$

The resulting subdivision scheme is either denoted by $\{\mathcal{S}_{\mathbf{a}^{(k)}},\, k \in \mathbb{N}_0\}$ or by $\{A^{(k)}(z),\, k \in \mathbb{N}_0\}$, and can be seen as the successive application of the level-dependent matrices $M^{(k)} = \{M^{(k)}(i,j) = \mathrm{a}_{i-2j}^{(k)} : i,j \in \mathbb{Z}\}$, $k \in \mathbb{N}_0$, to the initial data $\mathbf{P}^{(0)}$.
The notion of convergence is analogous to the one seen in the stationary case, but, differently from the stationary case, one could start the subdivision process with a mask at level $m \in \mathbb{N}_0$, and get a family of subdivision schemes based on the masks $\{\mathbf{a}^{(m+k)},\, k \in \mathbb{N}_0\}$, $m \in \mathbb{N}_0$. For any convergent, non-stationary subdivision scheme, we can thus define *basic limit functions* $\Phi^{(m)}$, $m \in \mathbb{N}_0$ as

$$\Phi^{(m)} = \lim_{k \to +\infty} \mathcal{S}_{\mathbf{a}^{(m+k)}}\, \mathcal{S}_{\mathbf{a}^{(m+k-1)}} \cdots \mathcal{S}_{\mathbf{a}^{(m)}}\, \boldsymbol{\delta}, \qquad m \in \mathbb{N}_0.$$

In the following we denote by Φ the basic limit function obtained with $m = 0$.
The study of generation and reproduction properties of a non-stationary, convergent and non-singular subdivision scheme requires us to introduce a generalization of the well-known space of polynomials, given by the space of exponential polynomials

$$EP_{\mathcal{R},\mathcal{T}} = span\{x^s e^{t_\ell x},\, s = 0, \ldots r_\ell - 1,\, \ell = 1, \ldots, N\}, \tag{2.4}$$

with $\mathcal{R} = \{r_\ell \in \mathbb{N},\, \ell = 1, \ldots, N\}$ and $\mathcal{T} = \{t_\ell \in \mathbb{R} \cup \mathrm{i}\mathbb{R},\, \ell = 1, \ldots, N\}$.
The following definition characterizes the notions of generation and reproduction of a space of exponential polynomials $EP_{\mathcal{R},\mathcal{T}}$.

Definition 2.1 ($EP_{\mathcal{R},\mathcal{T}}$-Generation and Reproduction)**.** *The non-stationary subdivision scheme associated with the symbols* $\{A^{(k)}(z),\ k \in \mathbb{N}_0\}$ *is said to* generate $EP_{\mathcal{R},\mathcal{T}}$ *if it is convergent and for* $f \in EP_{\mathcal{R},\mathcal{T}}$ *there exists an initial sequence* $\mathbf{f}^{(0)} := \{\tilde{f}(t_i^{(0)}),\ i \in \mathbb{Z}\}$, $\tilde{f} \in EP_{\mathcal{R},\mathcal{T}}$ *such that* $\lim_{k \to +\infty} \mathcal{S}_{\mathbf{a}^{(k)}} \mathcal{S}_{\mathbf{a}^{(k-1)}} \cdots \mathcal{S}_{\mathbf{a}^{(0)}} \mathbf{f}^{(0)} = f$. *Moreover, it is said to* reproduce $EP_{\mathcal{R},\mathcal{T}}$ *if it is convergent and for* $f \in EP_{\mathcal{R},\mathcal{T}}$ *and for the initial sequence* $\mathbf{f}^{(0)} := \{f(t_i^{(0)}),\ i \in \mathbb{Z}\}$, $\lim_{k \to +\infty} \mathcal{S}_{\mathbf{a}^{(k)}} \mathcal{S}_{\mathbf{a}^{(k-1)}} \cdots \mathcal{S}_{\mathbf{a}^{(0)}} \mathbf{f}^{(0)} = f$.

For later use we also recall some results proven in [4, 8].

Proposition 2.2. *Let* $\zeta_\ell^{(k)} = e^{\frac{-t_\ell}{2^{k+1}}}$ *with* $t_\ell \in \mathcal{T}$. *If*

$$\frac{d^s A^{(k)}\big(-\zeta_\ell^{(k)}\big)}{dz^s} = 0, \qquad \ell = 1, \ldots, N, \quad s = 0, \ldots, r_\ell - 1, \quad r_\ell \in \mathcal{R}, \tag{2.5}$$

then the subdivision scheme $\{\mathcal{S}_{\mathbf{a}^{(k)}},\ k \in \mathbb{N}_0\}$ *generates the space of exponential polynomials* $EP_{\mathcal{R},\mathcal{T}}$ *in* (2.4). *Moreover, if there exists a parameter shift* $\tau \in \mathbb{R}$ *such that*

$$A^{(k)}\big(\zeta_\ell^{(k)}\big) = 2\big(\zeta_\ell^{(k)}\big)^\tau, \quad \ell = 1,\dots,N, \tag{2.6}$$

$$\frac{d^s A^{(k)}\big(\zeta_\ell^{(k)}\big)}{dz^s} = 2(\zeta_\ell^{(k)})^{\tau-s} \prod_{q=0}^{s-1} \big(\tau - q\big), \quad \ell = 1,\dots,N, \quad s = 1,\dots,r_\ell - 1, \quad r_\ell \in \mathcal{R}, \tag{2.7}$$

are satisfied together with (2.5), *then the non-stationary scheme reproduces* $EP_{\mathcal{R},\mathcal{T}}$.

We further recall that if, for some $\ell \in \{1,...,N\}$, we have $\zeta_\ell^{(k)} = 1$ and $r_\ell \geq 2$, then we can conveniently compute $\tau = \frac{1}{2}\frac{dA^{(k)}(1)}{dz}$. If $\tau \in \mathbb{Z}$, the subdivision scheme with k-level symbol $A^{(k)}(z)$ is said *primal*, whereas if $\tau \in \frac{\mathbb{Z}}{2}$ it is called *dual*. Additionally, when $N = 1$, $t_1 = 0$ (i.e. $\zeta_1^{(k)} = 1$) and $r_1 = g+1$ where $g \in \mathbb{N}_0$, then (2.5)-(2.6) are nothing but the conditions for generation of degree g polynomials and reproduction of constants. They coincide with the *sum rules* of order $g+1$, which can be equivalently written as

$$A^{(k)}(1) = 2 \quad \text{and} \quad \max_{s=0,\dots,g} \left| \frac{d^s A^{(k)}(-1)}{dz^s} \right| = 0. \tag{2.8}$$

Differently, a non-stationary subdivision scheme is said to satisfy the *approximate sum rules* of order $g+1$, $g \in \mathbb{N}_0$, if the sequences

$$\mu_k := \left| A^{(k)}(1) - 2 \right| \quad \text{and} \quad \delta_k := \max_{s=0,\dots,g} 2^{-ks} \left| \frac{d^s A^{(k)}(-1)}{dz^s} \right|$$

satisfy

$$\sum_{k=0}^{\infty} \mu_k < +\infty \quad \text{and} \quad \sum_{k=0}^{\infty} 2^{kg}\, \delta_k < +\infty, \tag{2.9}$$

as recently shown in [3, Definition 5]. Therefore, sum rules are a special case of approximate sum rules, obtained when $\mu_k = \delta_k = 0$.
Finally, we conclude by recalling that a stationary subdivision scheme $\mathcal{S}_{\mathbf{a}}$ and a non-stationary one $\{\mathcal{S}_{\mathbf{a}^{(k)}},\ k \in \mathbb{N}_0\}$, are termed *asymptotically similar* (see [3, Definition 6] and [5]) if their masks satisfy

$$\lim_{k \to +\infty} \mathbf{a}^{(k)} = \mathbf{a}. \tag{2.10}$$

This definition allows us to check convergence and smoothness of a non-stationary scheme by comparison with a stationary one whose convergence, regularity and polynomial generation properties are known. More precisely, in [3, Theorem 4] the following result was given.

Proposition 2.3. *If a non-stationary subdivision scheme* $\{\mathcal{S}_{\mathbf{a}^{(k)}},\ k \in \mathbb{N}_0\}$ *satisfies approximate sum rules of order* $g+1$, $g \in \mathbb{N}_0$, *and is asymptotically similar to a convergent stationary subdivision scheme* $\mathcal{S}_{\mathbf{a}}$ *with a stable basic limit function in* $C^g(\mathbb{R})$, *then the basic limit function of* $\{\mathcal{S}_{\mathbf{a}^{(k)}},\ k \in \mathbb{N}_0\}$ *is also of class* C^g.

From Proposition 2.3 the following result follows straightforwardly.

Corollary 2.4. *If a non-stationary subdivision scheme* $\{\mathcal{S}_{\mathbf{a}^{(k)}},\ k \in \mathbb{N}_0\}$ *satisfies approximate sum rules of order* $g+1$, $g \in \mathbb{N}_0$, *and is asymptotically similar to a convergent stationary subdivision scheme* $\mathcal{S}_{\mathbf{a}}$ *with a stable basic limit function of class* C^ℓ, $\ell \leq g$, *then the non-stationary subdivision scheme* $\{\mathcal{S}_{\mathbf{a}^{(k)}},\ k \in \mathbb{N}_0\}$ *has the same integer smoothness.*

3. The stationary setting: review of known results

3.1. Lane-Riesenfeld algorithm

Refine-and-Smooth algorithms are characterized by a refine step which introduces new points on the initial control polygon, and a following smoothing step, which modifies the obtained points using simple local averaging rules. More smoothing steps provide limit curves of wider support as well as of higher smoothness [2]. One of the simplest Refine-and-Smooth algorithms is the well-known Lane-Riesenfeld algorithm, which generates polynomial uniform B-splines of degree $n+1$ for all $n \in \mathbb{N}_0$ [17]. We remind that this algorithm is defined using a smoothing operator described by a symbol of the form

$$S(z) = \frac{z+1}{2},$$

and a refine operator defined as

$$R(z) = 1 + S(z^2)z^{-1} = \frac{(z+1)^2}{2z},$$

which is well-known to reproduce Π_1 [2].
The Lane-Riesenfeld algorithm is obtained by applying the smoothing operator S n times, after one application of the refine operator R. This mechanism provides the symbol

$$A_n(z) = z^{-\lceil \frac{n}{2} \rceil} \Big(S(z)\Big)^n R(z) = \frac{(z+1)^{n+2}}{2^{n+1} z^{\lceil \frac{n}{2} \rceil + 1}}, \qquad n \in \mathbb{N}_0, \tag{3.1}$$

which is indeed the symbol of the degree-$(n+1)$ polynomial B-spline. We notice that the schemes defined by the symbol in (3.1) generate $\Pi_{n+1} = span\{1, x, x^2, \ldots, x^{n+1}\}$, but reproduce only Π_1.

3.2. Hormann-Sabin's family

In order to increase the degree of polynomial reproduction of B-spline schemes from one to three, the family of stationary subdivision schemes with cubic precision, hereinafter denoted by $\{F_n(z)\}_{n\geq 2}$, was proposed by Hormann and Sabin [14]. Its symbol can be written as

$$\begin{gathered} F_n(z) = A_n(z)\, K_n(z), \qquad n \in \mathbb{N}\backslash\{1\}, \\ \text{where} \quad A_n(z) = \frac{(z+1)^{n+2}}{2^{n+1} z^{\lceil \frac{n}{2} \rceil + 1}} \quad \text{and} \quad K_n(z) = -\frac{n+2}{8z} + \frac{n+6}{4} - \frac{n+2}{8} z. \end{gathered} \tag{3.2}$$

We remind that the scheme with symbol $F_1(z)$ is the dual three-point scheme which reproduces quadratics but not cubics, and hence it is not considered a member of the family. On the other hand, $F_2(z)$, $F_3(z)$ and $F_4(z)$ are respectively the symbols of the Dubuc-Deslauriers interpolatory four-point scheme, the dual four-point scheme and a relaxation of the interpolatory four-point scheme (see [14]).

3.3. The family of interpolatory $2n$-point Dubuc-Deslauriers schemes

The interpolatory $2n$-point ($n \in \mathbb{N}$, $n \geq 1$) Dubuc-Deslauriers scheme [11] is identified by the symbol (see, e.g, [7, 13])

$$I_{2n}(z) = A_{2n-2}(z) \sum_{\ell=0}^{n-1} (-1)^\ell\, 2^{-2\ell} \binom{n-1+\ell}{\ell} \frac{(1-z)^{2\ell}}{z^\ell} \qquad \text{where} \qquad A_{2n-2}(z) = \frac{(z+1)^{2n}}{2^{2n-1} z^n}, \tag{3.3}$$

which satisfies the interpolatory condition $I_{2n}(z) + I_{2n}(-z) = 2$ and reproduces Π_{2n-1}.
In [10] it was recently proven that for all $n \in \mathbb{N}$, $n \geq 2$ the subdivision schemes with symbols $\{I_{2n}(z)\}_{n\geq 2}$ satisfy the two-term recurrence relation

$$I_{2n}(z) = I_{2n-2}(z) + \frac{(-1)^{n-1}}{2^{4(n-1)}} \binom{2n-3}{n-1} \left(z - \frac{1}{z}\right)^{2n-2} \left(z + \frac{1}{z}\right), \tag{3.4}$$

starting from

$$I_2(z) = \frac{(z+1)^2}{2z},$$

which is also the first member of the Lane-Riesenfeld's family $\{A_{2n-2}(z)\}_{n\geq 1}$.
From (3.4) the following three-term recurrence relation

$$I_{2n}(z) = I_{2n-2}(z) - \beta_n \left(z - \frac{1}{z}\right)^2 \Big(I_{2n-2}(z) - I_{2n-4}(z)\Big) \qquad \text{where} \quad \beta_n = \frac{2n-3}{8(n-1)}, \tag{3.5}$$

defining the symbols of all interpolatory $2n$-point Dubuc-Deslauriers schemes with $n \geq 3$, can be also easily worked out [10]. The last recurrence is clearly based on the knowledge of the first member of the Lane-Riesenfeld's family, $I_2(z)$, and the first one in Hormann-Sabin's family, i.e.,

$$I_4(z) = F_2(z) = (z+1)^4 \frac{(-z^2+4z-1)}{16z^3}.$$

4. A non-stationary Lane-Riesenfeld algorithm

In the stationary setting we looked for a Refine-and-Smooth algorithm capable of defining the symbols of degree-$(n+1)$ polynomial B-splines for all $n \in \mathbb{N}_0$, and we observed that all the resulting schemes are featured by reproduction of Π_1. Here, instead of Π_1, we consider the 2-dimensional space

$$\mathit{span}\{e^{tx},\, e^{-tx}\}, \quad \text{with} \quad t \in [0,\pi) \cup \mathrm{i}\mathbb{R}^+. \tag{4.1}$$

Moreover, we define $\forall k \in \mathbb{N}_0$

$$v^{(k)} := \frac{1}{2}\left(e^{\mathrm{i}\frac{t}{2^{k+1}}} + e^{-\mathrm{i}\frac{t}{2^{k+1}}}\right) = \cos\left(\frac{t}{2^{k+1}}\right), \qquad t \in [0,\pi) \cup \mathrm{i}\mathbb{R}^+. \tag{4.2}$$

Note that, after choosing an arbitrary $v^{(0)} \in (0,+\infty)$ defined as

$$v^{(0)} := \cos\left(\frac{t}{2}\right) = \begin{cases} \cos\left(\frac{s}{2}\right) \in (0,1) & \text{if } \ t = s, \quad s \in (0,\pi), \\ 1 & \text{if } \ t = 0, \\ \cosh\left(\frac{s}{2}\right) \in (1,+\infty) & \text{if } \ t = \mathrm{i}s, \quad s \in \mathbb{R}^+, \end{cases}$$

we can equivalently compute the value of $v^{(k)}$ in (4.2) via the recursive formula (see [1, Proposition 2])

$$v^{(k+1)} = \sqrt{\frac{v^{(k)}+1}{2}}, \qquad \forall\, k \in \mathbb{N}_0. \tag{4.3}$$

In view of [1, Remark 3], we also remind that

$$\lim_{k\to+\infty} v^{(k)} = 1. \tag{4.4}$$

Following the stationary case, we define the k-level symbols of the smoothing and refine operators as follows.

Definition 4.1. *Let $v^{(k)}$ be as in (4.2). For all $k \in \mathbb{N}_0$ we define*

$$S^{(k)}(z) := \frac{z+1}{2v^{(k+1)}}, \tag{4.5}$$

and

$$R^{(k)}(z) := 1 + \frac{v^{(k+1)}}{v^{(k)}} S^{(k)}(z^2) z^{-1},$$

to be the k-level symbols of the smoothing and refine operators, respectively.

Lemma 4.2. *The refine operator in Definition 4.1, explicitly described by the k-level symbol*

$$R^{(k)}(z) = \frac{z + 2v^{(k)} + z^{-1}}{2v^{(k)}}, \tag{4.6}$$

reproduces the 2-dimensional space in (4.1).

Proof: Since $R^{(k)}(z)$ fulfills the conditions $R^{(k)}(-e^{\pm\frac{t}{2^{k+1}}}) = 0$, $R^{(k)}(e^{\pm\frac{t}{2^{k+1}}}) = 2$, then, in view of Proposition 2.2, it reproduces the 2-dimensional space in (4.1) with respect to the parameter shift $\tau = 0$. ■

The non-stationary Lane-Riesenfeld algorithm, obtained by one application of the refine operator and n successive applications of the smoothing operator, is thus performed by the k-level symbol

$$A_n^{(k)}(z) = z^{-\lceil \frac{n}{2} \rceil} \left(S^{(k)}(z)\right)^n R^{(k)}(z) = \frac{(z+1)^n \, (z + 2v^{(k)} + z^{-1})}{2v^{(k)} \, \left(2(v^{(k)}+1)\right)^{\frac{n}{2}} z^{\lceil \frac{n}{2} \rceil}}, \quad n \in \mathbb{N}_0, \tag{4.7}$$

where $v^{(k)}$ is the level-dependent parameter in (4.2).

Proposition 4.3. *For all $n \in \mathbb{N}_0$ the subdivision scheme related to the symbols $\{A_n^{(k)}(z), k \in \mathbb{N}_0\}$ in* (4.7) *generates span$\{1, x, \ldots, x^{n-1}, e^{tx}, e^{-tx}\}$ and reproduces the 2-dimensional subspace span$\{e^{tx}, e^{-tx}\}$ with $t \in [0, \pi) \cup \mathrm{i}\mathbb{R}^+$.*

Proof: We start by observing that, for all $n \in \mathbb{N}_0$, $A_n^{(k)}(-e^{\pm\frac{t}{2^{k+1}}}) = 0$ and, whenever $n \geq 1$, $(A_n^{(k)})^{(r)}(-1) = 0$ for all $r = 0, \ldots, n-1$. Thus, recalling conditions in Proposition 2.2, the generation of the space *span*$\{1, x, \ldots, x^{n-1}, e^{tx}, e^{-tx}\}$ is proven. Moreover, we notice that $S^{(k)}(e^{\pm\frac{t}{2^{k+1}}}) = e^{\pm\frac{1}{2}\frac{t}{2^{k+1}}}$, while from Lemma 4.2 we have $R^{(k)}(e^{\pm\frac{t}{2^{k+1}}}) = 2$. Thus the conditions $A_n^{(k)}(e^{\pm\frac{t}{2^{k+1}}}) = 2(e^{\pm\frac{t}{2^{k+1}}})^{\tau}$ with

$$\tau = \begin{cases} 0 & \text{if } n \text{ even}, \\ -\frac{1}{2} & \text{if } n \text{ odd}, \end{cases} \tag{4.8}$$

are satisfied too. Hence reproduction of *span*$\{e^{tx}, e^{-tx}\}$ is guaranteed for all values of $n \in \mathbb{N}_0$. ■

Remark 4.4. *Note that, when $v^{(0)} = 1$, $A_n^{(k)}(z)$ reduces to the symbol of the degree-$(n+1)$ polynomial B-spline in* (3.1)*, namely the non-stationary Lane-Riesenfeld algorithm in* (4.7) *gets back to its stationary counterpart.*

We conclude by observing that the proposed non-stationary extension of the Lane-Riesenfeld algorithm offers an alternative definition of the symbols of normalized exponential B-splines recently introduced in [15, 16].

5. A family of alternating primal/dual subdivision schemes reproducing conics

Aim of this section is to show that, using the symbols of the non-stationary extension of the Lane-Riesenfeld's family, we can define a family of non-stationary subdivision schemes reproducing the 4-dimensional space of exponential polynomials

$$span\{1, x, e^{tx}, e^{-tx}\}, \quad t \in [0, \pi) \cup \mathrm{i}\mathbb{R}^+, \tag{5.1}$$

as shown in the following proposition.

Proposition 5.1. *Let $v^{(k)}$ be defined as in* (4.2)*. The family of non-stationary subdivision schemes with k-level symbol*

$$F_n^{(k)}(z) = A_n^{(k)}(z)\, K_n^{(k)}(z), \tag{5.2}$$

with $A_n^{(k)}(z)$ *in* (4.7) *and*

$$K_n^{(k)}(z) = u_n^{(k)} z + (1 - 2u_n^{(k)} v^{(k)}) + u_n^{(k)} z^{-1}, \qquad u_n^{(k)} = \frac{1}{2(v^{(k)}-1)} - v^{(k)} \frac{\left(\frac{v^{(k)}+1}{2}\right)^{\frac{n}{2}}}{(v^{(k)})^2 - 1},$$

reproduces the 4-dimensional space in (5.1) *for all* $n \in \mathbb{N}\backslash\{1\}$*, with respect to the parameter shift* τ *in* (4.8).

Proof: Recalling Proposition 2.2 it can be easily verified that conditions for generation of the 4-dimensional space in (5.1) are fulfilled for all $n \in \mathbb{N}\backslash\{1\}$. Moreover, for all $n \in \mathbb{N}\backslash\{1\}$

$$F_n^{(k)}(1) = 2, \quad (F_n^{(k)})'(1) = 2\tau, \quad F_n^{(k)}(e^{\pm\frac{t}{2^{k+1}}}) = 2(e^{\pm\frac{t}{2^{k+1}}})^\tau, \quad \text{with } \tau \text{ in (4.8)},$$

thus proving the claim. ■

Remark 5.2. *We emphasize that the family proposed in* (5.2) *is a subcase of the family of fourth-order exponential quasi-splines presented in [16], which can be obtained by specifying the fourth-order space of exponential polynomials as in* (5.1).

Lemma 5.3. *For all* $n \in \mathbb{N}\backslash\{1\}$ *and for all* $v^{(0)} \in (0, +\infty)$*, the parameter* $u_n^{(k)}$ *in Proposition 5.1 verifies*

$$\lim_{k\to+\infty} u_n^{(k)} = -\frac{n}{8} - \frac{1}{4}.$$

Proof: The claimed result follows from (4.4) and De l'Hopital theorem. ■

Corollary 5.4. *For all* $n \in \mathbb{N}\backslash\{1\}$ *and for all* $v^{(0)} \in (0, +\infty)$*, the symbol in* (5.2) *is such that*

$$\lim_{k\to+\infty} F_n^{(k)}(z) = F_n(z), \tag{5.3}$$

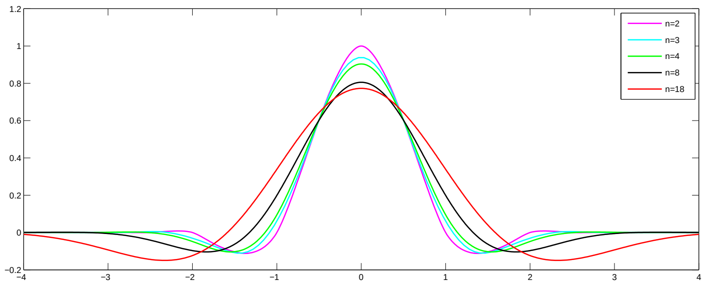
with $F_n(z)$ *in* (3.2). *Thus, the non-stationary subdivision scheme with k-level symbol* $F_n^{(k)}(z)$ *is asymptotically similar to the stationary scheme with symbol* $F_n(z)$.

Proof: The claimed result follows from (4.4) and Lemma 5.3. ■

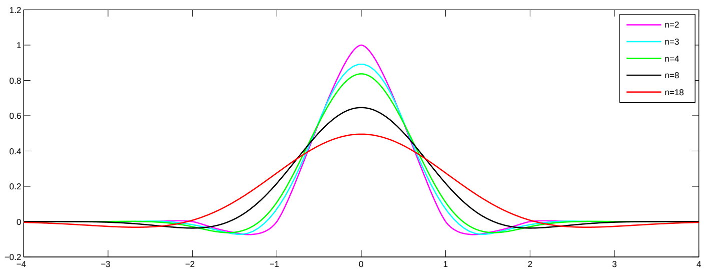
Proposition 5.5. *Let* Φ_n *be the basic limit function of the non-stationary subdivision scheme with k-level symbol* $F_n^{(k)}(z)$*,* $n \in \mathbb{N}\backslash\{1\}$ *in* (5.2). *Then the support of* Φ_n *is* $J_n = \left[-\frac{n+4}{2}, \frac{n+4}{2}\right]$.

Proof: By definition (see Section 2), the basic limit function Φ_n is obtained as the limit function of the non-stationary subdivision scheme with k-level symbol $F_n^{(k)}(z)$, when applied to the initial data $P_i^{(0)} = \delta_{i,0}$, $i \in \mathbb{Z}$. Thus, introducing the notation $D^{(k)} = \{\frac{i}{2^k} \,|\, i \in \mathbb{Z}\}$, we have that, at the initial level $k = 0$, the restriction of the basic limit function Φ_n to $D^{(0)}$ vanishes everywhere except at $i = 0$. Then, by equation (5.2) we get that, at refinement step $k = 1$, the restriction of the basic limit function Φ_n to $D^{(1)}$ vanishes outside the interval $J_n^{(1)} = [-\frac{n+4}{4}, \frac{n+4}{4}] \subset J_n$ and, at each successive step $k > 1$, the width of the interval $J_n^{(k)}$, where the restriction of Φ_n to $D^{(k)}$ does not vanish, is obtained by extending the left and right hand side of $J_n^{(k-1)}$ by a factor of $\frac{n+4}{2}\frac{1}{2^k}$. Hence, at the N-th subdivision step, the restriction of the basic limit function Φ_n to $D^{(N)}$ vanishes outside the interval

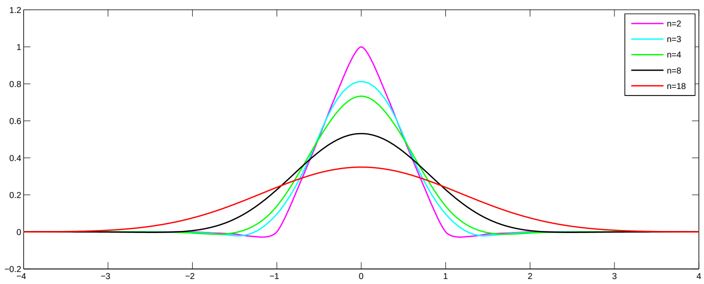
$$J_n^{(N)} = \left[-\frac{n+4}{4} - \sum_{k=2}^{N} \frac{n+4}{2}\frac{1}{2^k}, \frac{n+4}{4} + \sum_{k=2}^{N} \frac{n+4}{2}\frac{1}{2^k}\right] = \left[-\frac{n+4}{2}\left(1 - \frac{1}{2^N}\right), \frac{n+4}{2}\left(1 - \frac{1}{2^N}\right)\right]$$



(a) $v^{(0)} = 0$



(b) $v^{(0)} = 1$



(c) $v^{(0)} = 15$

Figure 1: Basic limit function of the subdivision scheme having symbol $F_n^{(k)}(z)$ with $n = 2, 3, 4, 8, 18$ and $v^{(0)} = 0$ (a), 1 (b), 15 (c). In each picture the function with the highest peak at 0 corresponds to $n = 2$, and as n increases, the height of the peak decreases.

and from the inequality $\left(1 - \frac{1}{2^N}\right) < 1$, it follows $J_n^{(N)} \subset J_n$ for all $N \in \mathbb{N}$. Since the support J_n of the basic limit function Φ_n is given by $\lim_{N\to+\infty} J_n^{(N)}$, the claimed result follows straightforwardly. $\blacksquare$

In Figure 1 we plot the basic limit function Φ_n obtained when varying the value of $n \in \mathbb{N}\backslash\{1\}$ and of the initial tension parameter $v^{(0)} \in (0, +\infty)$. Note that the x-axis has been reduced to $[-4, 4]$ even if the supports of Φ_8 and Φ_{18} are larger.

The following proposition analyzes the smoothness properties of the family of non-stationary subdivision schemes in (5.2).

Proposition 5.6. *The family of non-stationary subdivision schemes with k-level symbols $\{F_n^{(k)}(z)\}_{n\geq 2}$ in (5.2) has the same integer smoothness as the family described by the stationary symbols $\{F_n(z)\}_{n\geq 2}$ in (3.2).*

Proof: In view of Remark 5.2, the result follows from [16, Theorem 3.5]. ■

For the sake of completeness, we close this section by showing the refinement rules of the subdivision scheme with k-level symbol $F_1^{(k)}(z)$ (that we have excluded from the family since it does not reproduce the 4-dimensional space in (5.1)) and the first three members of $\{F_n^{(k)}(z)\}_{n\geq 2}$ corresponding to $n = 2, 3, 4$, in order to connect them to existing results from the literature.

5.1. $n = 1$: the non-stationary dual 3-point scheme

The subdivision scheme with k-level symbol $F_1^{(k)}(z)$ reproduces only $span\{1, e^{tx}, e^{-tx}\}$. In fact

$$F_1^{(k)}(-1) = 0, \quad F_1^{(k)}(-e^{\pm\frac{t}{2^{k+1}}}) = 0, \quad F_1^{(k)}(1) = 2, \quad F_1^{(k)}(e^{\pm\frac{t}{2^{k+1}}}) = 2(e^{\pm\frac{t}{2^{k+1}}})^{-\frac{1}{2}},$$

but

$$(F_1^{(k)})^{(1)}(-1) = \frac{4}{\left(e^{\frac{t}{2^{k+1}}} + e^{-\frac{t}{2^{k+1}}}\right)\left(e^{\frac{t}{2^{k+2}}} + e^{-\frac{t}{2^{k+2}}}\right)} - 1 \neq 0, \quad \text{for all} \quad t \neq 0.$$

The subdivision rules of this scheme are

$$\begin{aligned}
P_{2i}^{(k+1)} &= \tfrac{1}{8v^{(k+1)}v^{(k)}((v^{(k)})^2-1)}\Bigg(\left((v^{(k)}+1)(2v^{(k)}-1) - 2v^{(k)}v^{(k+1)}\right)P_{i-1}^{(k)} \\
&+ \left(4v^{(k)}(2(v^{(k)})^2-1)v^{(k+1)} - 2v^{(k)}(v^{(k)}+1)\right)P_i^{(k)} + \left(v^{(k)}+1-2v^{(k)}v^{(k+1)}\right)P_{i+1}^{(k)}\Bigg), \\
P_{2i+1}^{(k+1)} &= \tfrac{1}{8v^{(k+1)}v^{(k)}((v^{(k)})^2-1)}\Bigg(\left(v^{(k)}+1-2v^{(k)}v^{(k+1)}\right)P_{i-1}^{(k)} \\
&+ \left(4v^{(k)}(2(v^{(k)})^2-1)v^{(k+1)} - 2v^{(k)}(v^{(k)}+1)\right)P_i^{(k)} + \left((v^{(k)}+1)(2v^{(k)}-1) - 2v^{(k)}v^{(k+1)}\right)P_{i+1}^{(k)}\Bigg).
\end{aligned}$$

5.2. $n = 2$: the interpolatory 4-point scheme reproducing conics

The subdivision scheme with k-level symbol $F_2^{(k)}(z)$ coincides with the scheme proposed in [1], having refinement rules

$$\begin{aligned}
P_{2i}^{(k+1)} &= P_i^{(k)}, \\
P_{2i+1}^{(k+1)} &= \tfrac{1}{8v^{(k)}(v^{(k)}+1)}\left(-P_{i-1}^{(k)} + (2v^{(k)}+1)^2 P_i^{(k)} + (2v^{(k)}+1)^2 P_{i+1}^{(k)} - P_{i+2}^{(k)}\right).
\end{aligned}$$

5.3. $n = 3$: the dual 4-point scheme reproducing conics

$F_3^{(k)}(z)$ is the k-level symbol associated to the subdivision scheme with refinement rules

$$\begin{aligned}
P_{2i}^{(k+1)} &= \tfrac{1}{32v^{(k)}((v^{(k)})^2-1)(v^{(k+1)})^3}\cdot\Big(\left(2(v^{(k)})^2 + 3v^{(k)} + 1 - 6v^{(k)}(v^{(k+1)})^3\right)P_{i-1}^{(k)} \\
&+ \left((12(v^{(k)})^2-7)v^{(k)}(v^{(k)}+1)v^{(k+1)} - 4(v^{(k)})^2 - 5v^{(k)} - 1\right)P_i^{(k)} \\
&+ \left(2(v^{(k)})^2 + v^{(k)} - 1 + 2(4(v^{(k)})^2-5)v^{(k)}(v^{(k+1)})^3\right)P_{i+1}^{(k)} \\
&+ (v^{(k)}+1)(1 - v^{(k)}v^{(k+1)})P_{i+2}^{(k)}\Big), \\
P_{2i+1}^{(k+1)} &= \tfrac{1}{32v^{(k)}((v^{(k)})^2-1)(v^{(k+1)})^3}\cdot\Big((v^{(k)}+1)(1 - v^{(k)}v^{(k+1)})P_{i-1}^{(k)} \\
&+ \left(2(v^{(k)})^2 + v^{(k)} - 1 + 2(4(v^{(k)})^2-5)v^{(k)}(v^{(k+1)})^3\right)P_i^{(k)} \\
&+ \left((12(v^{(k)})^2-7)v^{(k)}(v^{(k)}+1)v^{(k+1)} - 4(v^{(k)})^2 - 5v^{(k)} - 1\right)P_{i+1}^{(k)} \\
&+ \left(2(v^{(k)})^2 + 3v^{(k)} + 1 - 6v^{(k)}(v^{(k+1)})^3\right)P_{i+2}^{(k)}\Big),
\end{aligned}$$

which has been recently proposed in [4].

5.4. $n=4$: a relaxation of the interpolatory 4-point scheme reproducing conics

The subdivision rules of the non-stationary scheme with k-level symbol $F_4^{(k)}(z)$ are

$$
\begin{aligned}
P_{2i}^{(k+1)} &= \tfrac{1}{32v^{(k)}(v^{(k)}+1)^2}\Bigg(-(2+v^{(k)})P_{i-2}^{(k)} + \left(4(v^{(k)})^2(2+v^{(k)})\right)P_{i-1}^{(k)} \\
&+ 2\left(12(v^{(k)})^3+24(v^{(k)})^2+17v^{(k)}+2\right)P_i^{(k)} + \left(4(v^{(k)})^2(2+v^{(k)})\right)P_{i+1}^{(k)} - (2+v^{(k)})P_{i+2}^{k}\Bigg),\\
P_{2i+1}^{(k+1)} &= \tfrac{1}{8v^{(k)}(v^{(k)}+1)}\Bigg(-P_{i-1}^{(k)} + (2v^{(k)}+1)^2P_i^{(k)} + (2v^{(k)}+1)^2P_{i+1}^{k} - P_{i+2}^{(k)}\Bigg).
\end{aligned}
$$

According to [14, Section 2.4] we call this family member a relaxation of the 4-point scheme since it shares with the interpolatory 4-point scheme (given by symbol $F_2^{(k)}(z)$) the same odd-point rule.

Remark 5.7. *It is interesting to observe that all members of the family $\{F_n^{(k)}(z)\}_{n\geq 2}$ corresponding to odd values of n, being dual, are characterized by k-level refinement rules involving the parameters $v^{(k)}$ and $v^{(k+1)}=\sqrt{\frac{v^{(k)}+1}{2}}$. This is a direct consequence of the definition of $A_n^{(k)}(z)$ with n odd.*

6. A family of non-stationary interpolatory $2n$-point schemes reproducing conics

A family of non-stationary interpolatory $2n$-point subdivision schemes with the capability of reproducing the space of exponential polynomials $span\{1,\, x,\, x^2,\, ...,\, x^{2n-3},\, e^{tx},\, e^{-tx}\}$, $t\in[0,\pi)\cup \mathrm{i}\mathbb{R}^+$, has been already introduced in [12]. Unlike [12], where the refinement rules of the family members are derived by means of an auxiliary orthogonal scheme, we here propose an alternative construction that lays the foundations for deriving a corresponding family of surface subdivision schemes interpolating quadrilateral meshes. More precisely, we here derive a three-term recurrence formula defining the k-level symbol of each family member as a function of the symbols of the two preceding members. As already shown in [10] for the family of (stationary) interpolatory Dubuc-Deslauriers $2n$-point schemes, this kind of recursion establishes the starting point to extend the tensor-product version of $2n$-point interpolatory schemes to quadrilateral meshes with arbitrary topology. However, differently from the stationary case where the three-term recurrence in (3.5) can be easily worked out, in the non-stationary setting more computational efforts are required to relate the k-level symbols $I_{2n}^{(k)}(z)$, $I_{2n-2}^{(k)}(z)$ and $I_{2n-4}^{(k)}(z)$. More precisely, to get the sought recurrence the following preliminary results are needed.

Proposition 6.1. *Let $I_{2n-2}(z)$ denote the Laurent polynomial of the $(2n-2)$-point Dubuc-Deslauriers scheme in (3.3). Let also $v^{(k)}$ be as in (4.2) and consider the space*

$$span\{1,\, x,\, x^2,\, ...,\, x^{2n-3},\, e^{tx},\, e^{-tx}\} \quad with \quad t\in[0,\pi)\cup \mathrm{i}\mathbb{R}^+. \tag{6.1}$$

For all $n\in\mathbb{N}$, $n\geq 2$, the non-stationary subdivision scheme with k-level symbol

$$I_{2n}^{(k)}(z) = I_{2n-2}(z) + (-1)^{n-1}\frac{\gamma_{n-2}^{(k)}}{2^{3(n-1)}v^{(k)}(v^{(k)}+1)^{n-1}}\left(z-\frac{1}{z}\right)^{2n-2}\left(z+\frac{1}{z}\right) \tag{6.2}$$

where

$$\gamma_{n-2}^{(k)} = \sum_{\ell=0}^{n-2} 2^{-\ell}\binom{n-2+\ell}{\ell}(v^{(k)}+1)^{\ell}, \tag{6.3}$$

is interpolatory and reproduces the exponential polynomial space in (6.1).

Proof. To simplify notation, we define the Laurent polynomial

$$G_n^{(k)}(z) := (-1)^{n-1} \frac{\gamma_{n-2}^{(k)}}{2^{3(n-1)} v^{(k)} (v^{(k)}+1)^{n-1}} \left(z - \frac{1}{z}\right)^{2n-2} \left(z + \frac{1}{z}\right), \tag{6.4}$$

such that $I_{2n}^{(k)}(z)$ can be simply written as $I_{2n}^{(k)}(z) = I_{2n-2}(z) + G_n^{(k)}(z)$. Since $G_n^{(k)}(z)$ verifies $G_n^{(k)}(z) + G_n^{(k)}(-z) = 0$ and $I_{2n-2}(z)$ fulfills the interpolatory condition $I_{2n-2}(z) + I_{2n-2}(-z) = 2$, it clearly follows that $I_{2n}^{(k)}(z) + I_{2n}^{(k)}(-z) = 2$ and hence the non-stationary $2n$-point scheme is also interpolatory.
Moreover, from the polynomial reproduction properties of the $(2n-2)$-point interpolatory Dubuc-Deslauriers scheme we know that

$$(I_{2n-2})^{(r)}(-1) = 0, \qquad r = 0, \ldots, 2n-3.$$

Taking into account that the symbol $I_{2n-2}(z)$ also satisfies

$$I_{2n-2}(-e^{\frac{t}{2^{k+1}}}) = I_{2n-2}(-e^{-\frac{t}{2^{k+1}}}) = \frac{(-1)^{n-1}}{2^{2n-3}} \frac{(e^{\frac{t}{2^{k+1}}}-1)^{2n-2}}{(e^{\frac{t}{2^{k+1}}})^{n-1}} \sum_{\ell=0}^{n-2} 2^{-2\ell} \binom{n-2+\ell}{\ell} \frac{(e^{\frac{t}{2^{k+1}}}+1)^{2\ell}}{(e^{\frac{t}{2^{k+1}}})^{\ell}},$$

while the Laurent polynomial $G_n^{(k)}(z)$ in (6.4) is such that

$$(G_n^{(k)})^{(r)}(-1) = 0, \qquad r = 0, \ldots, 2n-3,$$

and

$$G_n^{(k)}(-e^{\frac{t}{2^{k+1}}}) = G_n^{(k)}(-e^{-\frac{t}{2^{k+1}}}) = \frac{(-1)^{n}}{2^{2n-3}} \frac{(e^{\frac{t}{2^{k+1}}}-1)^{2n-2}}{(e^{\frac{t}{2^{k+1}}})^{n-1}} \sum_{\ell=0}^{n-2} 2^{-2\ell} \binom{n-2+\ell}{\ell} \frac{(e^{\frac{t}{2^{k+1}}}+1)^{2\ell}}{(e^{\frac{t}{2^{k+1}}})^{\ell}},$$

we can conclude that

$$(I_{2n}^{(k)})^{(r)}(-1) = 0, \;\; r = 0, \ldots, 2n-3 \quad \text{and} \quad I_{2n}^{(k)}(-e^{\pm\frac{t}{2^{k+1}}}) = 0.$$

Therefore, in view of Proposition 2.2, the scheme with k-level symbol in (6.2) generates the exponential polynomial space in (6.1) for all $n \in \mathbb{N}$, $n \geq 2$, and thus, being interpolatory, it also reproduces such space. ■

Remark 6.2. *Since $\gamma_{n-2}^{(k)}$ in* (6.3) *verifies $\lim_{k\to+\infty} \gamma_{n-2}^{(k)} = \binom{2n-3}{n-1}$, the family of non-stationary interpolatory $2n$-point schemes with k-level symbol* (6.2) *is asymptotically similar to the family of $2n$-point interpolatory Dubuc-Deslauriers schemes with symbol in* (3.4).

Proposition 6.3. *For all $n \in \mathbb{N}$ the non-stationary subdivision scheme with k-level symbol $I_{2n}^{(k)}(z)$ has the same integer smoothness as the stationary $2n$-point interpolatory Dubuc-Deslauriers scheme with symbol $I_{2n}(z)$.*

Proof: From [11], for all $n \in \mathbb{N}$ the stationary $2n$-point Dubuc-Deslauries scheme is C^L continuous with $L \leq n-1$, and from Remark 6.2 the non-stationary scheme with k-level symbol $I_{2n}^{(k)}(z)$ is asymptotically similar to $I_{2n}(z)$. Moreover, $I_{2n}^{(k)}(z)$ satisfies the sum rules of order n. In fact, $I_{2n}^{(k)}(1) = 2$ for all $n \in \mathbb{N}$ and $k \in \mathbb{N}_0$ and since, in view of Proposition 6.1 $I_{2n}^{(k)}(z)$ generates Π_{2n-3}, we have that

$$\max_{s=0,\ldots,n-1} \left| \frac{d^s I_{2n}^{(k)}(-1)}{dz^s} \right| = 0 \quad \text{for all } n \in \mathbb{N},\; k \in \mathbb{N}_0.$$

Thus, from Corollary 2.4, the claim is proven. ■

As seen in the stationary case, we conclude the section by showing that the first member of the non-stationary Lane-Riesenfeld's family, $A_0^{(k)}(z)$, and that of the corresponding Hormann-Sabin's family, $F_2^{(k)}(z)$, can be used as building blocks to obtain the family of non-stationary interpolatory $2n$-point schemes with k-level symbols $\{I_{2n}^{(k)}(z)\}_{n\geq 2}$ by means of two- and three-term recurrence relations.

Lemma 6.4. *For all $n \in \mathbb{N}$, $n \geq 3$, the factors $\gamma_{n-2}^{(k)}$ in (6.3) and*

$$\gamma_{n-3}^{(k)} = \sum_{\ell=0}^{n-3} 2^{-\ell} \binom{n-3+\ell}{\ell} (v^{(k)}+1)^{\ell}, \tag{6.5}$$

satisfy the relation

$$\gamma_{n-3}^{(k)} = \frac{1-v^{(k)}}{2}\gamma_{n-2}^{(k)} + v^{(k)} \left(\frac{v^{(k)}+1}{2}\right)^{n-2} \binom{2n-5}{n-2}.$$

Proof: After rewriting $\gamma_{n-3}^{(k)}$ in the following equivalent form,

$$\gamma_{n-3}^{(k)} = \sum_{\ell=0}^{n-2} 2^{-\ell}\binom{n-3+\ell}{\ell}(v^{(k)}+1)^{\ell} - 2^{-(n-2)}\binom{2n-5}{n-2}(v^{(k)}+1)^{n-2},$$

by using the well-known relation $\binom{n-2+\ell}{\ell} = \binom{n-3+\ell}{\ell} + \binom{n-3+\ell}{\ell-1}$ on binomial coefficients, we get

$$\begin{aligned}
\gamma_{n-3}^{(k)} &= \sum_{\ell=0}^{n-2} 2^{-\ell}\binom{n-2+\ell}{\ell}(v^{(k)}+1)^{\ell} - \frac{v^{(k)}+1}{2}\sum_{\ell=0}^{n-3} 2^{-\ell}\binom{n-2+\ell}{\ell}(v^{(k)}+1)^{\ell} \\
&- 2^{-(n-2)}\binom{2n-5}{n-2}(v^{(k)}+1)^{n-2} \\
&= \gamma_{n-2}^{(k)} - \frac{v^{(k)}+1}{2}\left(\gamma_{n-2}^{(k)} - 2^{-(n-2)}\binom{2n-4}{n-2}(v^{(k)}+1)^{n-2}\right) - 2^{-(n-2)}\binom{2n-5}{n-2}(v^{(k)}+1)^{n-2} \\
&= \frac{1-v^{(k)}}{2}\gamma_{n-2}^{(k)} + 2^{-(n-2)}(v^{(k)}+1)^{n-2}\left(\frac{v^{(k)}+1}{2}\binom{2n-4}{n-2} - \binom{2n-5}{n-2}\right).
\end{aligned}$$

Finally, using the fact that $\frac{1}{2}\binom{2n-4}{n-2} = \binom{2n-5}{n-2}$, the claimed result is obtained. ■

Proposition 6.5. *Let $v^{(k)}$ be as in (4.2) and $I_2^{(k)}(z) = A_0^{(k)}(z) = \frac{z^2+2v^{(k)}z+1}{2v^{(k)}z}$. For all $n \in \mathbb{N}$, $n \geq 2$, the non-stationary subdivision scheme with k-level symbol $I_{2n}^{(k)}(z)$ in (6.2) satisfies the two-term recurrence relation*

$$I_{2n}^{(k)}(z) = I_{2n-2}^{(k)}(z) + (-1)^{n-1}\left(z-\frac{1}{z}\right)^{2n-4}\left(z+\frac{1}{z}\right)\left(z^2 - (4(v^{(k)})^2-2) + \frac{1}{z^2}\right)\frac{\gamma_{n-2}^{(k)}}{2^{3(n-1)}v^{(k)}(v^{(k)}+1)^{n-1}}, \tag{6.6}$$

where $\gamma_{n-2}^{(k)}$ is defined as in (6.3).

Proof. From equations (3.3) and (6.2) we obtain

$$I_{2n}^{(k)}(z) - I_{2n-2}^{(k)}(z) = \frac{(-1)^{n-2}}{2^{4(n-2)}}\binom{2n-5}{n-2}\left(z-\frac{1}{z}\right)^{2n-4}\left(z+\frac{1}{z}\right) + G_n^{(k)}(z) - G_{n-1}^{(k)}(z),$$

with $G_n^{(k)}(z)$ in (6.4). Introducing the explicit expression of $G_n^{(k)}(z) - G_{n-1}^{(k)}(z)$ and simplifying the result, we have that

$$I_{2n}^{(k)}(z) - I_{2n-2}^{(k)}(z) = \tfrac{(-1)^{n-1}}{2^{3(n-1)}}\left(z-\tfrac{1}{z}\right)^{2n-4}\left(z+\tfrac{1}{z}\right)\left(-\tfrac{1}{2^{n-5}}\tbinom{2n-5}{n-2} + \tfrac{\gamma_{n-2}^{(k)}}{v^{(k)}(v^{(k)}+1)^{n-1}}\left(z-\tfrac{1}{z}\right)^2 + \tfrac{8\gamma_{n-3}^{(k)}}{v^{(k)}(v^{(k)}+1)^{n-2}}\right).$$

Finally, using Lemma 6.4, we can write

$$\frac{8\gamma_{n-3}^{(k)}}{v^{(k)}(v^{(k)}+1)^{n-2}} = \frac{4(1-v^{(k)})}{v^{(k)}(v^{(k)}+1)^{n-2}}\,\gamma_{n-2}^{(k)} + \frac{1}{2^{n-5}}\binom{2n-5}{n-2},$$

and hence the claim is obtained. ■

The following corollary is a straightforward consequence of the result in Proposition 6.5.

Corollary 6.6. *Let*

$$I_2^{(k)}(z) = A_0^{(k)}(z) = \frac{z^2+2v^{(k)}z+1}{2v^{(k)}z},$$

and

$$I_4^{(k)}(z) = F_2^{(k)}(z) = \frac{(z+1)^2(z^2+2v^{(k)}z+1)(-z^2+2(v^{(k)}+1)z-1)}{8v^{(k)}(v^{(k)}+1)z^3}.$$

For all $n \in \mathbb{N}$, $n \geq 3$, the symbol $I_{2n}^{(k)}(z)$ of Proposition 6.5 satisfies the three-term recurrence relation

$$I_{2n}^{(k)}(z) = I_{2n-2}^{(k)}(z) - \frac{(z^2-1)^2}{8(v^{(k)}+1)z^2}\,\frac{\gamma_{n-2}^{(k)}}{\gamma_{n-3}^{(k)}}\left(I_{2n-2}^{(k)}(z) - I_{2n-4}^{(k)}(z)\right),$$

with $\gamma_{n-2}^{(k)}$ in (6.3) *and $\gamma_{n-3}^{(k)}$ in* (6.5).

Remark 6.7. *The subdivision scheme with symbol $I_6^{(k)}(z)$, obtained from the family $\{I_{2n}^{(k)}(z)\}_{n\geq 1}$ when setting $n = 3$, coincides with the interpolatory 6-point scheme proposed in [18, Section 4.1].*

7. Conclusions

The representation of the wide variety of shapes required in many applications, ranging from geometric modelling to image segmentation, can be obtained by using non-stationary subdivision schemes with the properties of high smoothness and conic reproduction. To this end, in this paper we have presented new families of approximating and interpolating non-stationary subdivision schemes enjoying such properties. The definition of a non-stationary extension of the well-known Lane-Riesenfeld algorithm has been shown to be the fundamental element of this study, since its symbol is used as building block to design a non-stationary family of alternating primal/dual schemes as well as a family of non-stationary $2n$-point interpolatory schemes, both featured by members with increasing smoothness. In particular, the two families have been shown to be useful generalizations of the Hormann-Sabin's family [14] and the Dubuc-Deslauriers's family [11], respectively, and they have been proven to keep the smoothness of their stationary counterparts unchanged.

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