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From spinors to forms:
results on G-structures in supergravity and on topological field theories

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DARIO ROSA

Tutor: Prof. Alessandro Tomasiello

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Abstract

This thesis is divided in two parts, that can be read separately even if both use the possibility of replacing spinors with differential forms in theories with supersymmetry.

The first part explores some recent results that have been obtained by applying the G-structure approach to type II supergravities. Using generalized complex geometry it is possible to reformulate the conditions for unbroken supersymmetry in type II supergravity in terms of differential forms. We use this result to find a classification for AdS_7 and AdS_6 solutions in type II supergravity.

Concerning AdS_7 solutions we find that in type IIB no solutions can be found, whereas in massive type IIA many new $AdS_7 \times \mathcal{M}_3$ solutions are at disposal with the topology of the internal manifold $\mathcal{M}_3$ given by a three-sphere. We develop a classification for such solutions.

Concerning AdS_6 solutions, very few $AdS_6 \times \mathcal{M}_4$ supersymmetric solutions are known in literature: one in massive IIA, and two IIB solutions dual to it. The IIA solution is known to be unique. We obtain a classification for IIB supergravity, by reducing the problem to two PDEs on a two-dimensional space Σ. The four-dimensional space $\mathcal{M}_4$ is then given by a fibration of S^2 over Σ.

We also explore other two contexts in which the G-structure approach has revealed its usefulness: first of all we derive the conditions for unbroken supersymmetry for a Mink_2 (2,0) vacuum, arising from type II supergravity on a compact eight-dimensional manifold $\mathcal{M}_8$. When $\mathcal{M}_8$ enjoys $SU(4) \times SU(4)$ structure the resulting system is elegantly rewritten in terms of generalized complex geometry. Finally we rewrite the equations for ten-dimensional supersymmetry in a way formally identical to an analogous system in $N = 2$ gauged supergravity; this provides a way to look for lifts of BPS solutions without having to reduce the ten-dimensional action.

The second part is devoted to study some aspects of two different Chern-Simons like theories: holomorphic Chern-Simons theory on a six-dimensional Calabi-Yau space and three-dimensional supersymmetric theories involving vector multiplets (both with Yang-Mills and Chern-Simons terms in the action).

Concerning holomorphic Chern-Simons theory, we construct an action that couples the gauge field to off-shell gravitational backgrounds, comprising the complex structure and the (3,0)-form of the target space. Gauge invariance of this off-shell action is achieved by enlarging the field space to include an appropriate system of Lagrange multipliers, ghost and ghost-for-ghost fields. From this reformulation it is possible to uncover a twisted supersymmetric algebra for this model that strongly constrains the anti-holomorphic dependence of physical correlators.

Concerning three-dimensional theories, we will develop a new way of computing the

exact partition function of supersymmetric three-dimensional gauge theories, involving vector supermultiplet only. Our approach will reduce the problem of computing the exact partition function to the problem of solving an anomalous Ward identity. To obtain such a result we will describe the coupling of three-dimensional *topological* gauge theories to background topological gravity. The Seifert condition for manifolds supporting global supersymmetry is elegantly deduced from the topological gravity BRST transformations. We will show how the geometrical moduli that affect the partition function can be characterized cohomologically. In the Seifert context Chern-Simons topological (framing) anomaly is BRST trivial and we will compute explicitly the corresponding local Wess-Zumino functional. As an application, we obtain the dependence on the Seifert moduli of the partition function of three-dimensional supersymmetric gauge theory on the squashed sphere by solving the anomalous topological Ward identities, in a regularization independent way and without the need of evaluating any functional determinant.

Contents

Chapter 1

Introduction and Conclusions

The dream of obtaining a unique, quantum, description for all the known interactions continues to be the most important driving force to current research in high energy physics. At the moment, despite its age, string theory is still the most serious candidate to address such a problem and other related issues.

One of the most serious obstacles to obtain a better understanding of string theory is given by the lacking of a second-quantized formulation for this theory: quantum field theories are usually formulated in a so-called formalism of *second quantization*. To each particle of the spectrum a corresponding quantum field has to be introduced; the theory is then formulated in terms of a path integral on the field configurations. On the other hand, once one fixes a classical background, it is possible to compute — at the perturbative level — the scattering amplitudes among particles in terms of a *first-quantized* formalism. The main problem due to this, more trivial, first-quantized formulation is given by the fact that such an approach is intrinsically valid only at the perturbative level. Since the scattering amplitudes are computed starting from a classical background, it is possible to compute in this way only perturbations around this background, whereas it is impossible to study in this way non perturbative phenomena, which intrinsically require the second-quantized formulation.

For these reasons, a good second-quantized description would be mandatory. Unfortunately, even if some attempts to obtain such a second-quantized formulation have been done in the past (under the name of *string-field-theory* [1]), and even if some very impressive results have been obtained (for example in [2]), at the moment a satifying second-quantized description of string theory is still lacking. For sure it would be important to put further efforts toward such an ambitious problem; on the other hand there is an evident difficulty that cannot be avoided: the string theory spectrum contains an infinite tower of fields with higher and higher masses and spins. Therefore a second-quantized formulation for string theory would require an infinite number of fields,

one for each particle, and constructing such a theory turns out to be a formidable task. This property, and the consequent technical and conceptual difficulties connected with string-field-theory, forces us to make a choice: explore string theory only through a first-quantized formalism (and at the end of the day compute only scattering amplitudes using some world-sheet pictures), or introduce some simplifications and hope that non-trivial results are still at disposal even in the simplified setup.

This Thesis is devoted to explore and collect some results that have been recently obtained in two of such simplified situations: ten-dimensional type II supergravity and topological field theories (which are related to string theory via topological strings). From a conceptual point of view the two approaches can be related by simply saying that both are theories that consider only a subset of the full string theory spectrum.

The supergravity view-point is obtained by imposing that the masses of the higher-spin particles (i.e. particles with spin greater than 2) are so high that they can be ignored. By restricting us to consider the massless spectrum one can show that the closed string theory is then effectively described by ten-dimensional (or eleven-dimensional for M-theory) supergravity. The fact that supergravities are not, in general, renormalizable theories can be considered as a non serious problem exactly for this reason: ten-dimensional supergravities must be considered as effective theories, and they are valid only at energies much below the Planck scale.[1]

Topological string theories can be roughly thought of as some counterparts of type IIA and type IIB string theory in which the local symmetry has been enlarged. More precisely the local symmetry is so large that *all* the local degrees of freedom are gauge-trivial and so they are removed from the spectrum. In this way only few states of the string spectrum, which are connected to *global* properties of the space-time in which strings propagate, survive and this makes reasonable to hope to obtain a second-quantized formulation for these models. Indeed we will come back later to this point and we will see that this is exactly what happens in some cases.

As just explained, the supergravity approach and the topological strings approach are conceptually related via the fact that both can be seen as suitable simplifications of string theory. On the other hand another, more technical, point of contact between these two topics can be found, and this point of contact represents, as already pointed out in the Abstract, one of the main technical themes of this Thesis. In both cases we will use the possibility of replacing spinors with differential forms in theories with supersymmetry. Since differential calculus techniques are in general much more powerful by using differential forms rather than spinors, usually this replacement allows to obtain

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[1]As we will discuss in a moment, after the discovery of the holographic correspondence of AdS/CFT [3], and the various strong-weak coupling dualities, the importance and the interest on the supergravity approximation to string theory has been possibly increased.

results that are not accessible by using spinors.

In the rest of the Introduction we will describe what results will be obtained in this Thesis by using this technical possibility.

1.1   Supergravity results

In the first part of the Thesis we will address the problem of finding supersymmetric solutions of the equations of motion of type II supergravity by using differential forms. Before moving to discuss some technical issues related to this problem, let us point out some motivations for why it should be an interesting problem to look for such configurations.

As already outlined in the beginning of this Introduction, the current understanding of string theory is based on a first-quantized formulation which requires therefore to know classical backgrounds around them to perform a perturbative analysis. It is therefore evident that the search for classical solutions of the equations of motion is an important step, in order to obtain many classical backgrounds and increase the possibility of a deeper understanding of string theory. Moreover, in the AdS/CFT context finding classical, asimptotically Anti deSitter, solutions of type II supergravity is of primary importance: indeed such solutions are often dual to some strong-interacting exotic conformal field theories living on the boundary of the (asimptotically) Anti deSitter space and, by studying the properties of the corresponding classical supergravity solutions, it is often possible to deduce many non-trivial properties of the dual CFTs that it would be very hard to obtain directly on the CFT side, (see for example [4]).[2] On both the situations just described, a potential source of troubles appears: in order to have a classical vacuum for string theory, one should take into account that type II supergravity is just a low energy approximation of string theory, and that at high energy many stringy corrections appear. It is therefore natural to ask how we can keep into account this problem in the supergravity approximation and this is the reason why we will be interested in solutions of the equations of motion that will be *supersymmetric*: the invariance under supersymmetry indeed will protect our solutions from quantum corrections and will make the results obtained through supergravity trustworthy. From a more practical point of view, the requirement of supersymmetry constitutes a great simplification in order to find solutions of the equations of motion, since it translates the problem from a system of differential equations of the second order to a system of the first order.

A systematic use of differential forms instead of spinorial quantities in the search for

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[2]The converse is also true and it is possible to find many non-trivial results in supergravity literature that have been suggested by studying the corresponding dual CFTs. Just to give an example the reader could see [5] and [6].

supersymmetric solutions in type II supergravity started in [7] but the use of Generalized Complex Geometry (GCG), a mathematical formalism introduced in [8] and [9], has been introduced starting from [10]. The main observation is the following: the presence of the SUSY parameters (which are nowhere vanishing spinorial quantities), implies a reduction of the *structure group*, which can be thought of as the group on the tangent bundle determined by the transition functions among different charts on the manifold. In other words, we can say that just the presence of the SUSY parameters implies some *topological* conditions on the manifolds which are allowed by supersymmetry. The supersymmetry conditions put then further restrictions on the allowed manifolds by imposing *differential* conditions on them. Given such an observation the idea goes as follows: we can try to replace the SUSY parameters with differential forms, by requiring that they determine the same reduction of the structure group. After a suitable set of differential forms has been identified, one can try to recast the SUSY conditions in terms of differential equations involving differential forms only. In this way the resulting system of equations is usually much easier to manage than the corresponding spinorial system.

From a technical point of view, in our discussion so far we have omitted an important source of difficulties: in type II supergravities we have two SUSY parameters. This is reflected in the fact that the structure group acquires a dependence on the points of the manifold, and this makes in general very hard the task of finding a description of the conditions for supersymmetry in terms of differential forms. This is the place where the use of GCG makes manifest its usefulness: in the GCG approach the structure group of interest is *not* the structure group on the tangent bundle, but the structure group on a more abstract space, which is called *generalized tangent bundle*, and it is roughly given by the direct sum of the tangent plus the cotangent bundle. It is indeed possible to show that, on this enlarged space, the structure group is independent on the points and this allows a much easier classifications of the supersymmetry conditions. In [10] GCG has been used for the first time to obtain the conditions for unbroken supersymmetry in terms of exterior differential calculus for the particular case of four-dimensional (both Minkowskian and AdS) vacua. The resulting system of equations is entirely rewritten in a very elegant form in terms of two polyforms.

Motivated by the nice result obtained in the case of four-dimensional vacua, in [11] the GCG approach has been applied to the conditions for unbroken supersymmetry for a *general* ten-dimensional configuration of type II supergravity, without putting any restriction on the fields and on the metric. The resulting system of equations is, unfortunately, less elegant than the corresponding four-dimensional counterpart of [10] and, beyond two equations that can be nicely understood in terms of a single polyform Φ, it includes two additional equations, called *pairing equations*, that involve two vectors that cannot be defined as bilinears of the supersymmetry parameters.

The study of the pairing equations, and the attempt to understand better their geometrical meaning, is one of the central themes of this first part of the Thesis. We will apply the ten-dimensional system to various kind of solutions and, beyond the intrinsic, physical, interest in studying such supersymmetric solutions, we will see how the pairing equations behave in the various cases. We will see that in some cases they will be simply redundant and can be dropped from the analysis (we will call such cases **lucky** cases), in other cases they can be recast in a very elegant form (these cases will be called **good** cases) and, unfortunately, in some cases they are not redundant and cannot be recast in some elegant forms (these are the **ugly** cases). Of course, it would be nice to have a simple criterion to determine in which cases a particular Ansatz falls. At the moment such a criterion has not be found yet, nevertheless it will become clear that the complexity of the pairing equations is somehow related to the complexity of the structure group on the generalized tangent bundle: the ugly cases are associated to structure groups which are much more complicated than in the other cases. Beyond the formal interest in understanding better the pairing equations, during the analysis we will discover many new examples of supersymmetric solutions in type II supergravity, that have shown their importance, for example, in holographic applications.

Let us now move to discuss how the chapters of this first part of the Thesis are organized.

In Chapter 2 we will give a very brief introduction to the elements of type II supergravity that will be relevant for what we will discuss in the following. In particolar we will introduce the fields of the theory, the SUSY transformations and we will also describe what kind of conceptual difficulties arise when one tries to understand the eleven-dimensional embedding of type IIA *massive* supergravity. Given these problems, we will treat the massive case as a separate case in the subsequent chapters.

In Chapter 3 an introduction to the GCG approach will be given. The aim of this Chapter is to be as concrete as possible. It is not the aim of the Chapter to give a rigorous introduction to GCG but to show that this is the natural language that one should use in order to rewrite the conditions for unbroken supersymmetry in terms of differential forms. A very concrete introduction to the generalized tangent bundle will be given, by using the case of four-dimensional vacua as an avatar. In this way we will introduce the conditions for four-dimensional vacua obtained in [10]. In the second part of the Chapter we will move to discuss the ten-dimensional system obtained in [11] (and here presented in (3.2.4)), also discussing the difficulties connected with the pairing equations.

Chapter 4 is based on the articles [12] and [13]. In this Chapter we will discuss AdS_7 and AdS_6 vacua, that represent two particular lucky cases. We will see that for such types of solutions the pairing equations are completely redundant and all the constraints on the allowed geometries come from the other equations. From a holographic point of view AdS_7

and AdS$_6$ solutions are very important since they are dual to higher dimensional ($d > 4$) SCFTs. It is well-known that higher-dimensional SCFTs are a very interesting class of quantum field theories (it is sufficient to think about the famous $(2,0)$ 6-dimensional theory or the 5-dimensional CFTs arising from D4 branes living on top of an O8/D8 system). On the other hand they are very difficult to study with traditional methods since they are intrinsically strongly coupled. For this reason the holographic approach is often one of the few possibilities at disposal to study such theories. In this Chapter, after a brief review of the results already at disposal in the literature, we will classifiy supersymmetric solutions in type II supergravity of the type AdS$_7 \times \mathcal{M}_3$ and AdS$_6 \times \mathcal{M}_4$. Concerning AdS$_7$ solutions, we will find that in type IIB no solutions can be found, on the other hand in massive type IIA a plethora of solutions is at disposal by including D8 branes. This probably represents the most important result of this first part of the Thesis, and it has represented the starting point to find a complete classification of $(1,0)$ six-dimensional SCFTs that can be constructed by considering intersecting branes in type IIA string theory (these brane setups have been discussed for the first time in [14] and [15] but the connection with the supergravity solutions has been obtained in [16]). Moving to AdS$_6$ the classification that we will find will be, unfortunately, less complete. Nevertheless we will be able to reduce the problem of finding AdS$_6$ solutions in type IIB supergravity to a system of 2 PDEs involving two functions on a plane. For some particular Ansätze we will reproduce the already known solutions, but finding the most general solution to this system of equations still represents a very interesting open problem which could suggest some intriguing new results concerning five-dimensional SCFTs.

Chapter 5 is based on [17]. In this Chapter we will describe the only known good case, i.e. the only known case which admits a way to rewrite the pairing equations in a very nice form. We will describe how the conditions for a Mink$_2$ $(2,0)$ vacuum are obtained using the system (3.2.4) and we will show that the paring equations, when specialized to this particular class of backgrounds, even if not redundant can be recast in a very elegant form, a result already noticed, with some crucial simplifying assumptions that here we will remove, in [18]. The way in which the pairing equations will take an elegant form is far from trivial, and it will require some more advanced notions on GCG (like the concept of "generalized Hodge diamond") that we will introduce in the Chapter. Of course it would be interesting to find new backgrounds that fall in the good cases. Hints that further "good" backgrounds really exist, can be found in the recent works [19]. It would be nice to find points of contact between that results and the system of equations (3.2.4).

Finally, Chapter 6 will be devoted to study a particular example of ugly case, which has been constructed in [20]. We will discuss the conditions for unbroken supersymmetry for a configuration given by a product of an external four-dimensional space with an

internal six-dimensional manifold. No restrictions will be put on the shape of the external space (in this respect the situation is different, and much more involved, from the situation of four-dimensional vacua) and we will make a spinorial Ansatz for the SUSY parameters in which a *pair* of external spinors will be introduced. In this way we will study backgrounds that can be considered as embedding in type II supergravity of solutions of four-dimensional $N = 2$ supergravity. The interest towards such an embedding of four-dimensional supergravity solutions relies again on the AdS/CFT perspective. Indeed, it is well known that supergravities theories in dimensions lower than ten, usually, are not considered as good quantum theories, since they have ultraviolet divergences. On the other hand, as already discussed, type II supergravities are considered as good quantum theories thanks to their embedding in string theory. For these reasons, it is often important to know whether a particular solution, found in lower dimensional supergravity, can be consistently embedded in string theory. The usual strategy to address such a question is to start from type II supergravity and then performing a reduction procedure. However, a solution found in the lower dimensional theory obtained in this way is not guaranteed to have a consistent lift to type II supergravity, since most of the reductions are "not consistent", which means that not all the solutions of the reduced theory uplift to solutions of the higher dimensional theory. In this Chapter we will present an alternative and new approach to the problem of lifting solutions from four-dimensional $N = 2$ supergravity to type II supergravity: we will simply rewrite the conditions for unbroken supersymmetry in ten dimensions in a way which is formally identical to a corresponding system of equations obtained in four dimensions. In this way it becomes possible to study the problem of lifting solutions from four dimensions to ten dimensions without the necessity of reducing any action but simply by looking to the BPS conditions. The resulting system of equations in ten dimensions has some equations that do not have a four-dimensional counterparts, and such equations can be thought as obstructions, that the four-dimensional solution have to satisfy automatically in order to be embedded in ten dimensions. It would be interesting in the future to test explicitly this new approach on some particular cases.

1.2   Topological field theories results

Topological field theories (TFT) have been introduced by Witten in [21], with the aim of giving a quantum field theory interpretation to the theory of Donaldson's invariants and to other related issues (like Floer's theory). The peculiar feature of a TFT is given by the lacking of local observables that can be interpreted as propagating particles. Indeed all the observables of the theory have *global* character. The lacking of particles and the presence of a very restricted spectrum introduce an enormous amount of simplifications

into the theory, and this allows to obtain results that usually are not at disposal for other theories.

In this thesis we will be interested in discussing TFTs for two different reasons: as string-field-theory realizations of topological string theories and as counterparts of supersymmetric quantum field theories. Let us now explain in more details both these aspects.

As already remarked, topological strings can be thought as appropriate counterparts for type II string theories. Indeed they can be defined, in a first-quantized formalism, by taking a two-dimensional topological sigma model [3] and by coupling it to two-dimensional topological gravity [23]. The resulting theory describes the propagation of a topological string in a six-dimensional target-space that usually is taken to be a Calabi-Yau three-fold X.

Since topological strings are TFTs, they have a spectrum which is very small when compared to the spectrum of physical string theory. For this reason it is reasonable to hope that a string-field-theory description for such theory can be obtained. This turns out to be the case: it is shown in [24] that the open string-field-theory description of the A model, when the Calabi-Yau X is taken to be the cotangent space of a three-dimensional manifold M, is given by three-dimensional Chern-Simons theory (with gauge group $SU(N)$) on M, which describes a stack of N topological D branes in propagation on the total Calabi-Yau X; on the other hand the closed string-field-theory description is still mysterious and poorly understood (it goes under the name of *Kahler gravity* [25]). Moving to discuss the B model, it is known that the closed string-field-theory description si given by the so-called *Kodaira-Spencer gravity*, which has been introduced in [26] and that can be thought as a quantum field theory for the complex structure of the Calabi-Yau X. The open string-field-theory is instead given by a holomorphic variant of the usual Chern-Simons theory, which takes the name of *holomorphic Chern-Simons* theory (HCS). Such a theory is defined on the entire Calabi-Yau X, it describes a stack of N 5 branes living on X and its action is naturally coupled with the closed fields: the complex structure, that using the Beltrami parametrization is represented by the Beltrami differential μ, and the holomorphic three-form Ω.

In Chapter 7 we will put our attention to the results concerning HCS theory that have been obtained in [27]. As just remarked, HCS can be thought as a gauge theory for a gauge connection that, in a system of coordinates adapted to the complex structure put on the Calabi-Yau X, is a $(0,1)$-form. The theory is also explicitly coupled to the

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[3]Topological sigma models have been introduced in [22] and can be thought as twisted versions of supersymmetric two-dimensional sigma models. There are two inequivalent ways of performing the topological twist, leading to the so-called A model and B models.

holomorphic three-form Ω since its action writes

$$\Gamma = \int_X \Omega \wedge \text{Tr} \left(\frac{1}{2} A \, \bar{\partial}_Z \, A + \frac{1}{3} A^3 \right) \, , \tag{1.2.1}$$

where $\bar{\partial}_Z$ is the anti-holomorphic Dolbeault differential defined by the complex structure put on X.

To start with, we will rewrite the action (1.2.1) in a way where the coupling between the open field, represented by the gauge connection A, and the closed fields, represented by the gravitational backgrounds μ and Ω, will be explicitly. We will see that, in order to have gauge invariance, the closed backgrounds fields must satisfy their own equations of motion: the Kodaira-Spencer equation

$$\mathcal{F}^i \equiv \bar{\partial}\mu^i - \mu^j \partial_j \mu^i = 0 \, , \tag{1.2.2}$$

and the request of holomorphicity of the three-form Ω. However, from a string-field-theory point of view, such a request is not satisfactory: to obtain a *coupled* open-closed system, it would be necessary to write an action for HCS which is gauge invariant even when the closed backgrounds are *off-shell*, i.e. even when they do not satisfy their own equations of motion. Since the HCS term, in a string-field-theory framework, can be understood as describing a stack of topological D-branes on the closed string background, such a description in terms of off-shell backgrounds should be necessary to understand the backreaction of the branes on the vacuum, since the presence of the branes puts the backgrounds off-shell.

In Chapter 7 we will solve this problem: we will obtain a description of HCS theory which is gauge invariant for *generic*, off-shell values of the backgrounds fields. We will obtain such a formulation by enlarging the field space to include an appropriate set of Lagrange multiplier, ghost fields and ghost-for-ghost fields. The resulting action will take a nice formulation in terms of superfields (or polyforms). An interesting feature of the resulting final system will be the presence of a (twisted) $N = 2$ supersymmetric algebra for this model. In the later part of the Chapter we will make some preliminary observations concerning how such an algebra could give interesting constraints on the anti-holomorphic dependence of physical correlators. This supersymmetry algebra could also be a very important ingredient in order to study the renormalization properties of HCS. HCS is indeed a gauge theory in six dimensions, therefore it is not renormalizable for power counting. On the other hand, the correspondence with the topological strings should suggest that an ultraviolet description for this theory can be found. It would be very interesting to study better this problem in the future.

In Chapters 8 and 9 we will pass to discuss another aspect that makes topological field theories interesting: their relation with supersymmetric quantum field theories and some

nice features that their share together, with particular emphasis on *localization* properties. It is well-known that many TFTs can be obtained starting from a supersymmetric quantum field theory, by performing a topological twist. After the twist one of the supercharges of the supersymmetric theory becomes a scalar and therefore can be thought as a BRST operator for the resulting theory. It is known from a long time (see for example the references [28]) that topological field theories and supersymmetric field theories often enjoy localization properties, i.e. under some favourable conditions the semiclassical approximation becomes exact and it is then possible to compute exactly (and not at perturbative level only) some observables. More recently, starting from the work [29], localization has been used in supersymmetric theories without performing the topological twist, but by considering the original *spinorial* supersymmetry of the theory. Later this approach has been exported in other contexts and in other dimensions too, leading to a considerable amount of exact results for various theories and in various dimensions.

With the aim of obtaining more and more exact results, a preliminary question is to find the manifolds on which a supersymmetric theory can be put. Concerning this problem a concrete approach has been suggested in the work [30]. The recipe is very simple: couple the supersymmetric system in flat space to a suitable *off-shell* supergravity, and look for bosonic gravitational backgrounds (which include the metric and other background fields of the supergravity theory) which are invariant under supergravity transformations. By taking the resulting matter theory coupled to supergravity, and by "freezing" the supergravity backgrounds to their supersymmetric value, one obtains a *rigid* supersymmetric theory on a curved manifold, whose metric is determined by the supergravity equations.

As just remarked this paradigm, in which "physical", spinorial, supersymmetry is relevant, has been exported to various manifolds and various dimensions. For example, starting from the work [31], supersymmetric localization computations have been performed in three dimensions for Chern-Simons theories. Notice that Chern-Simons theory is a topological theory which does not have a supersymmetry invariance. Nevertheless usually supersymmetry is introduced "by hand", by completing the gauge field A to a supersymmetric vector multiplet which includes, beyond the gauge field, a Dirac spinor λ and two scalar fields D and σ; all these additional fields are non dynamical in the supersymmetric action and so the resulting supersymmetric theory is equivalent to the bosonic one.

In Chapter 9, which is based on the work [32], we will develop a new way to compute the supersymmetric partition function of gauge theories in 3 dimensions involving vector supermultiplets only.[4] We will study those theories by finding the coupling of three-

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[4]Chapter 8, which include the results of [33], can be thought as a warm-up, in which this new point of view will be developed in a simpler situation, namely for the case of two-dimensional Yang-Mills theory.

dimensional *topological* theories, both of the Yang-Mills type and of Chern-Simons type, to *topological* gravity, and not supergravity as in the common paradigm. This new approach has different advantages: first of all the supersymmetric backgrounds (which turn out to be Seifert manifolds) are straightforwardly identified from the BRST variations of topological gravity, without the necessity of passing through spinors. Moreover it will allow to give a cohomological characterization of the geometrical moduli on which the partition function of a three-dimensional supersymmetric theory depends. But the real payoff of this new approach is that it allows to compute, in an explicitly regularization-independent way and without the need of computing functional determinants, the explicit dependence from the geometrical moduli of the partition function for a three-dimensional theory involving vector supermultiplet only. The computation will be indeed reduced to finding the solution of an appropriate anomalous Ward identity determined by the well-known *framing anomaly* of Chern-Simons theory [34]. As an example, we will apply this approach to the squashed spheres of [35], finding complete agreement with the computation performed by following the standard localization recipe.

Of course, it would be very important try to extend this new approach to other theories, starting from three-dimensional models involving chiral matter multiplets and then moving to different dimensions. The long-term program could be to find a general recipe to reduce the computations based on functional determinants with finding solutions for appropriate anomalous Ward identities.

Chapter 2

Basics of type II supergravity

In this Chapter we will give a very short review of some features that type II supergravity possesses, with particular emphasis on the aspects that will be relevant in the next chapters.

2.1   Fields and supersymmetries

It is well-known that two different versions of type II supergravity exist: type IIA and type IIB. These two versions differ on the bosonic field content and on the chiralities of the fermionic fields. Both of them have four fermionic fields: two gravitini (ψ_M^a)[1] and two dilatini (λ^a), where $a = 1$, 2. All these fields are Majorana-Weyl spinors in ten dimensions and their chiralities are different in the two theories: type IIB is a chiral theory, with the gravitini having $+$ chirality and the dilatini $-$ chirality, whereas type IIA is not a chiral theory and in this case ψ_M^1 and λ^2 have $+$ chirality, ψ_M^2 and λ^1 have $-$ chirality. The Majorana condition implies that, chosen an appropriate basis for the ten-dimensional gamma matrices Γ^M, they satisfy the condition:

$$(\psi_M^a)^* = \psi_M^a \ , \qquad (\lambda^a)^* = \lambda^a \ . \tag{2.1.1}$$

The bosonic field content is given by the metric g_{MN} (that, as obvious, is a symmetric tensor), by a scalar ϕ (the dilaton, whose exponential $g_s = e^\phi$ gives the string coupling constant), by a two-form B_{MN} (the so-called B-field) and by a collection of differential forms (the RR-fields) $C_{M_1...M_p}$, where p is odd in type IIA and even in type IIB. Since they are differential forms, both the B-field and the RR-fields are antisymmetric in their

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[1]We will adopt the convention that eleven-dimensional (or ten-dimensional) indices will be indicated with M, N, $\ldots$, external indices will be indicated with μ, ν, $\ldots$ and internal indices will be indicated with m, n, $\ldots$.

indices. We will use very often the form notation:

$$B \equiv \frac{1}{2} B_{MN} dx^M \wedge dx^N , \qquad C_p \equiv \frac{1}{p!} C_{M_1 \ldots M_p} dx^{M_1} \wedge \cdots \wedge dx^{M_p} . \qquad (2.1.2)$$

The supersymmetry conditions that we will write in a moment are more naturally written in terms of the field-strengths (often called "fluxes")

$$H = dB , \qquad F_p = dC_{p-1} - H \wedge C_{p-3} . \qquad (2.1.3)$$

The fact that the physics depends on the potentials B and C_p through the field-strengths H and F_p only, gives rise to the gauge invariances

$$B \to B' = B + d\lambda_1 , \qquad C_p \to C'_p = C_p + d\lambda_{p-1} - H \wedge \lambda_{p-3} , \qquad (2.1.4)$$

where λ_p is a p-form; these gauge transformations can be seen as the natural higher-dimensional generalizations of the gauge transformations of Yang-Mills theories. The field-strengths F_p are not completely independent between each other: they must satisfy the conditions

$$F_p = (-1)^{\lfloor \frac{p}{2} \rfloor} * F_{10-p} , \qquad (2.1.5)$$

which in many cases are solved by going to the "electric" basis: with this choice one consider only F_0,[2] F_2 and F_4 in type IIA; and F_1, F_3 and F_5 in type IIB. The other, higher-degree forms, F_p are then determined via equation (2.1.5), whereas the self-duality of F_5 in type IIB is imposed by hand. In this thesis, however, we will use a different formulation of supersymmetry conditions, the "democratic" formulation of [36], where all the fluxes F_p appear on the same footing. It is convenient, in order to write formulas in a more elegant way, to group all the RR-fluxes and gauge potentials by introducing the polyforms F and C defined as

$$C \equiv C_1 + C_3 + C_5 + C_7 + C_9 , \quad F \equiv F_0 + F_2 + F_4 + F_6 + F_8 + F_{10} , \qquad (2.1.6)$$

in type IIA and

$$C \equiv C_0 + C_2 + C_4 + C_6 + C_8 , \quad F \equiv F_1 + F_3 + F_5 + F_7 + F_9 , \qquad (2.1.7)$$

in type IIB. In this way it is possible to rewrite the RR part of (2.1.3) in the compact form

$$F = d_H C , \qquad d_H \equiv d - H \wedge , \qquad (2.1.8)$$

whereas the self-duality condition (2.1.5) takes the form

$$F = \lambda * F , \qquad (2.1.9)$$

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[2]About the somewhat exotic zero-form field-strength (which takes the name of "Romans-mass") we will give some more details later, since it will appear in many places in our discussions about higher-dimensional AdS vacua.

where we have defined the λ operator which acts on a k-form ω_k as

$$\lambda \omega_k \equiv (-1)^{\lfloor \frac{k}{2} \rfloor} \omega_k \ . \tag{2.1.10}$$

Let us discuss the supersymmetry transformation properties. Both theories contain two SUSY parameters ϵ^a, $a = 1, 2$ and, accordingly with the fermionic fields ψ_M^a and λ^a, both of them are Majorana-Weyl spinors in ten dimensions. The chiralities can be deduced from the SUSY variations of the fermionic fields and in particular we have that, in IIA, ϵ_1 has chirality $+$, ϵ_2 has chirality $-$. On the other hand in IIB both of them have positive chirality.

To write out the supersymmetry transformations we need a further ingredient: the Clifford map. To start with we note that, given a differential form ω_k, we can define a natural action of ω_k on the SUSY parameters ϵ^a, by simply replacing all the differentials dx^M with the corresponding gamma matrices Γ^M. In this way one maps a differential (poly)form to a "bispinor":

$$\text{Clifford map} : \alpha \equiv \sum_k \frac{1}{k!} \alpha_{M_1 \dots M_k} dx^{M_1} \wedge \cdots \wedge dx^{M_k} \longleftrightarrow \slashed{\alpha} \equiv \sum_k \frac{1}{k!} \alpha_{M_1 \dots M_k} \Gamma^{M_1 \dots M_k}_{\alpha\beta} \ , \tag{2.1.11}$$

where α and β are spinorial indices. In this way the differential form α obtains two spinorial indices and it can act on the SUSY parameters ϵ^a. Notice that what we have just described is nothing but a generalization of the usual definition of the Dirac operator $\slashed{\partial}$ of electrodynamics. In what follows, since we will make a heavy use of the Clifford map, we will systematically drop the slash symbol on (2.1.11) and we will simply write α whenever it should not lead to confusion.

We will be interested in finding supersymmetric, *bosonic* configurations; for this reason we put all the fermions to zero

$$\psi_M^a = 0 \ , \qquad \lambda^a = 0 \ , \tag{2.1.12}$$

and look to bosonic configurations which preserve supersymmetry. Thanks to (2.1.12), SUSY transformations are much simpler than in the general case: the SUSY variations of the bosons automatically vanish and the SUSY variations of the fermions get simplified, giving the system

$$\delta \psi_M^1 = \left(D_M - \frac{1}{4} H_M \right) \epsilon_1 + \frac{e^\phi}{16} F \Gamma_M \epsilon_2 = 0 \ ,$$

$$\delta \psi_M^2 = \left(D_M + \frac{1}{4} H_M \right) \epsilon_2 - \frac{e^\phi}{16} \lambda(F) \Gamma_M \epsilon_1 = 0 \ ,$$

$$\Gamma^M \delta \psi_M^1 - \delta \lambda^1 = \left(D - \frac{1}{4} H - \partial\phi \right) \epsilon_1 = 0 \ ,$$

$$\Gamma^M \delta \psi_M^2 - \delta \lambda^2 = \left(D + \frac{1}{4} H - \partial\phi \right) \epsilon_2 = 0 \ , \tag{2.1.13}$$

where we have introduced the expression $H_M \equiv \frac{1}{2} H_{MNP} \Gamma^{NP}$ and the Clifford map is used everywhere.

The equations in (2.1.13) are necessary and sufficient to supersymmetry: by solving them we ensure that our bosonic configuration is supersymmetric, i.e. it is invariant under some supersymmetry variations. However, we are interested in supersymmetric, bosonic configurations which solve the equations of motion of type II SUGRA too. In principle, in order to write out the equations of motion, one should write down the Lagrangian, derive the equations of motion and try to solve them; in general this problem could be very hard to solve, since we are dealing with partial differential equations of the second order. Fortunately the situation is much simpler when we have supersymmetry: it can be shown [37], [38] that almost all the equations of motion follow automatically if one imposes, beyond the supersymmetry equations (2.1.12), also the Bianchi identities for the fluxes

$$(d - H\wedge)F = 0 \ , \qquad dH = 0 \qquad \text{(almost everywhere)} \tag{2.1.14}$$

where we have specified "almost everywhere" since there could be sources; we will see later that in many cases the presence of sources is crucial in order to have interesting higher-dimensional AdS vacua.

2.1.1 The troubles with eleven-dimensional supergravity and Romans mass

To conclude this introductory chapter we want to recall some crucial properties connected with the presence of a non-zero Romans mass in type IIA supergravity. It is well-known that, if one consider among the RR field-strengths of type IIA supergravity only F_2 and F_4 (and their Hodge duals F_6 and F_8), type IIA supergravity on $\mathbb{R}^{10}$ is related, via a circle reduction, to eleven-dimensional supergravity on a S^1 fibration over $\mathbb{R}^{10}$. Let us briefly recall such a construction and what kind of problems arises when one adds the Romans mass in the framework.

The presence of an extra *hidden* circular dimension in type IIA supergravity can be inferred by studying D0 branes: to this end we recall that, starting from the effective action for a Dp brane, one can deduce a simple formula for the mass of a given Dp brane

$$m_{Dp} \propto \frac{1}{g_s l_s^{p+1}} \text{vol}(Dp) \ , \tag{2.1.15}$$

where, as already written, $g_s = e^\phi$ is the string coupling constant and l_s is the string length. We can see that, when $g_s \to \infty$, D branes become light. Another observation that can be done by looking at (2.1.15) is that a D0 has *finite* mass

$$m_{D0} \propto \frac{1}{l_s g_s} \ , \tag{2.1.16}$$

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and so it can be considered as an asymptotic state in the theory. Now, consider an asymptotic state given by k D0 branes. Its mass is given by

$$m_k \propto \frac{k}{l_s g_s} ,$$ (2.1.17)

but this looks like a Kaluza-Klein tower of states, whose masses depend on the length of an extra-dimensional circle of size

$$L_{11} = 2\pi l_s g_s ,$$ (2.1.18)

we see that the length of the circle is very small when g_s is small, but it becomes macroscopic when g_s gets increased. This argument leads us to conclude that, at strong g_s, type IIA supergravity approaches an eleven dimensional theory and it can be shown

$$\lim_{g_s \to \infty} \text{IIA} = \text{11d supergravity} .$$ (2.1.19)

To understand the problems connected with the Romans mass (and with the presence of D8 branes), let us briefly review how the other branes and fluxes of type IIA supergravity can be understood in the eleven-dimensional perspective. We know that M-theory (which reduces to eleven dimensional SUGRA in the low energy limit) has two types of branes: The M2 branes and the M5 branes. By considering a M2, it is very simple to understand that the fundamental string of type IIA (F1) is given by a M2 which wraps the eleven-dimensional circle, whereas a D2 brane is given by a M2 which is extended in ten dimensions. In the same way it is straightforward to realize that D4 branes are obtained from M5 branes wrapping the internal circle and NS5 branes are given by M5 branes extended along the ten dimensional space. Passing to discuss the fluxes, in eleven-dimensional supergravity we have the antisymmetric potential A_3, which is a three-form

$$A_3 \equiv \frac{1}{6} A_{MNP} dx^M \wedge dx^N \wedge dx^P ,$$ (2.1.20)

and that, in ten-dimensional language, writes

$$A_3 = B \wedge e^{11} + C_3 ,$$ (2.1.21)

where e^{11} is the unit vector along the eleventh direction y which has periodicity equal to L_{11}.

We want now describe the eleven dimensional origin of the RR-potential C_1: since its electric source is given by D0 branes, and that such branes have to do with the Kaluza-Klein momentum along the circle direction, it is reasonable that this potential has something to do with the geometry of the fibration. We can make this statement more precise by recalling that an S^1-fibration is classified by the *periods* of the curvature

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F_2, if we identify the curvature with the RR-field F_2 we conclude that the gauge potential C_1 enters into the eleven-dimensional metric via the formula

$$ds_{11}^2 = e^{-2\phi/3}(ds_{10}^2 + e^{2\phi}(dy + C_1)^2) ; \qquad (2.1.22)$$

the fact that C_1 is described, in the eleven-dimensional perspective, as part of the geometry suggests that also the D6 branes, that are magnetic sources for C_1 can be described in terms of the eleven-dimensional geometry; and one can show that this is exactly the case.

It remains to describe only the Romans mass and its magnetic sources: the D8 branes. This is a great mystery! At the moment it is not clear how to understand D8 branes and Romans mass in eleven-dimensional terms, it is also possible that simply the eleven-dimensional construction is not a good description when the Romans mass is turned on.

Anyway, in this thesis we will not devote our attention to this issue and we will apply a conservative point of view: given the great difference between the massless case and the massive case in type IIA (the case with $F_0 = 0$ and $F_0 \neq 0$ respectively), with the massive case that cannot be lifted to eleven-dimensional supergravity, in our discussion about higher dimensional AdS vacua we will split explicitly our considerations in three case: the type IIB case, the massive type IIA case and the massless type IIA case. It should be clear from our discussion that, by studying the massless type IIA case, we will have also the higher-dimensional AdS vacua in eleven dimensional supergravity.

Chapter 3

Basics of GCG and its application to supergravity

In this section we will review some basic facts about Generalized Complex Geometry (GCG) and its relevance to the search for supergravity solutions. It is not the aim of this section to give a detailed review of the argument, which would require a much longer chapter; instead our aim will be just to review the basic ideas behind this argument and how it can represent a powerful instrument in finding supersymmetric solutions in type II supergravity. From a more concrete point of view, our aim in this section is to give a global understanding of the system of equations (3.2.4) that we will use extensively in later chapters. Some more technical details concerning GCG will be necessary in chapter 5 when we will discuss two-dimensional $N = (2, 0)$ vacua; we will discuss such details in that chapter.

3.1 From spinors to forms: the 4d vacua example

To explain the principal ideas and the general philosophy behind the GCG approach we will use the particolar example of four-dimensional (Minkowski or AdS) vacua. From a historical point of view, this kind of vacua has been the first example in which GCG has been applied to supergravity [10]. However, we start from this example since it should be more familiar to the reader: most of the concepts that we will introduce can be seen as generalizations of the well-known framework which appears by studying Calabi-Yau vacua.

To start with, we need to impose the requirement that the solutions that we are looking for are four-dimensional vacua: the word *vacuum* suggests that in our solution we do not have particles, and indeed this idea can be formalized by requiring that our solution respects completely the maximal symmetry of the four-dimensional space.

This requirement enforces the metric to take the following block-form

$$ds_{10}^2 = e^{2A(y)} ds^2_{\text{Mink}_4/\text{AdS}_4}(x) + ds^2_{\mathcal{M}_6}(y) , \qquad (3.1.1)$$

where x are the coordinates on Mink_4 or AdS_4 (that we will call the external space) and y are the coordinates on the internal space $\mathcal{M}_6$, the function $A(y)$ is the so-called *warping* function which is a function that depends on the internal coordinate only and it gives the most general form of the metric (3.1.1) which is compatible with the maximal symmetry of the external space. The requirement of maximal symmetry imposes also some constraints on the other fields that compose our supergravity multiplet (the dilaton and the RR and NSNS fluxes F and H) that at the moment are not crucial for our purposes and that we will describe case by case in the subsequent chapters. More important at the moment is to discuss what kind of implications follows on the supersymmetry parameters ϵ_1 and ϵ_2 from the requirement of maximal external symmetry.

Given that the metric (3.1.1) has been factorized in a four-dimensional part and a six-dimensional part, it becomes reasonable to impose that the gamma-matrices get factorized in a similar fashion:

$$\Gamma_\mu = \gamma_\mu^{(4)} \otimes 1^{(6)} , \qquad \Gamma_m = \gamma^{(4)} \otimes \gamma_m^{(6)} , \qquad (3.1.2)$$

where $\gamma_\mu^{(4)}$ and $\gamma_m^{(6)}$ are four-dimensional and six-dimensional gamma matrices respectively, $\gamma^{(4)}$ is the chiral gamma matrix in four dimensions which has been introduced to ensure the right anticommutation relations between the ten-dimensional gamma-matrices Γ_μ and Γ_m. The same decomposition just described for the gamma matrices can be applied to the SUSY parameters ϵ_1 and ϵ_2 that take the form

$$\epsilon_1 = \zeta_+ \otimes \eta_+^1 + \text{c.c.} , \qquad \epsilon_2 = \zeta_+ \otimes \eta_\mp^2 + \text{c.c.} , \qquad (3.1.3)$$

where ζ_+ is a four-dimensional spinor of positive chirality and $\eta_\mp^{1,2}$ are a pair of six-dimensional spinors of positive or negative chiralities depending on the signs (the upper sign is for IIA whereas the lower sign is for IIB).[1] Finally we added the complex conjugates to ensure that the ten-dimensional spinors $\epsilon_{1,2}$ are Majorana-Weyl in ten dimensions.

We can now move to the crucial consequence that the assumption of maximal external symmetry introduces: if we found a solution to the supersymmetry equations for a specific external spinor ζ_+, the maximal symmetry of the external background would be broken. Indeed, starting from ζ_+ one can easily construct a vector $v_\mu = \zeta_+^t \gamma_\mu^{(4)} \zeta_+$ and, of course, such a vector would break the maximal symmetry. To overcome this problem we must assume that our configuration is a solution of the supersymmetry equations for a *general*

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[1] One could consider also more general Ansatzes in which one consider more than one ζ_+ and more than two internal spinors but for our purposes the Ansatz that we did is sufficient.

choice of the external supersymmetry spinor ζ_+. In Minkowski spaces a basis for the spinors is simply given by the constant spinors, whereas in AdS spaces it can be shown [39] that a basis for the spinors can be constructed by considering *Killing* spinors. In four dimensions, a Killing spinor ζ_+ satisfies the equation

$$D_\mu \zeta_+^c = \frac{1}{2} \mu \gamma_\mu \zeta_+ \ , \tag{3.1.4}$$

where the spinor ζ_+^c is just the complex conjugate of ζ_+ (which in four Lorentzian dimensions has negative chirality) and μ is a number related to the cosmological constant Λ via the relation $\Lambda = -|\mu|^2$.

The upshot of all this discussion is the following: given that the external spinor ζ_+ is fixed to be constant or to obey the Killing spinor equation (3.1.4), the supersymmetry equations get translated into differential equations involving the *internal* parts of the SUSY parameters $\eta_\pm^{1,2}$ only, and so we conclude that to find supersymmetric four-dimensional vacua we need to solve a system of differential equations for them. The resulting system looks like

$$\left(D_m - \frac{1}{4} H_m \right) \eta_+^1 \mp \frac{e^\phi}{8} f \gamma_m \eta_\mp^2 = 0 \ ,$$

$$\left(D_m + \frac{1}{4} H_m \right) \eta_\mp^2 - \frac{e^\phi}{8} \lambda(f) \gamma_m \eta_+^1 = 0 \ ,$$

$$\mu e^{-A} \eta_-^1 + \partial A \eta_+^1 - \frac{e^\phi}{4} f \eta_\mp^2 = 0 \ ,$$

$$\mu e^{-A} \eta_\pm^2 + \partial A \eta_\mp^2 - \frac{e^\phi}{4} \lambda(f) \eta_+^1 = 0 \ ,$$

$$2\mu e^{-A} \eta_-^1 + D\eta_+^1 + \left(\partial(2A - \phi) + \frac{1}{4} H \right) \eta_+^1 = 0 \ ,$$

$$2\mu e^{-A} \eta_\pm^2 + D\eta_\mp^2 + \left(\partial(2A - \phi) - \frac{1}{4} H \right) \eta_\mp^2 = 0 \ , \tag{3.1.5}$$

where the constant μ vanishes for Minkowski vacua and with f we mean the *internal* part of the RR flux which, to respect the maximal symmetry, must take a form like

$$F = f + \mathrm{vol}_4 \wedge \lambda(*_6 f) \ . \tag{3.1.6}$$

Being η_+^1 and $\eta_\mp^2$ spinors and not differential forms, the system (3.1.5) is usually very complicated to analyze; for these reasons often one has to do some Ansatzes on the solutions in order to solve them, weheras it is very hard to perform a general analysis of the conditions for supersymmetry.

The basic idea of the G-structures approach (in which GCG enters) is very simple: one replaces the spinorial SUSY parameters (that in the case of 4d vacua are η_+^1 and $\eta_\mp^2$) with *differential forms*, and in this way one obtains differential equations for them; such

equations are much more elegant than the original spinorial equations and, in some cases, it is possible to perform a complete and general analysis of the geometry of the solutions.

To see how such a program can be realized in the context of 4d vacua we start by observing that SUSY conditions require the existence of a well-defined, everywhere non-vanishing, six-dimensional spinors η^1_+ and $\eta^2_{\mp}$. Let us consider for the moment only one of these spinors, that we call η and we drop the chirality symbol, and let us see what kind of geometrical structures are defined by it. Simply by requiring that such a spinor η is everywhere non-vanishing one obtains a reduction of the *structure group*: on each chart of the six-dimensional manifold $\mathcal{M}_6$, one can put η to take a fixed expression, like for example $(1, 0, 0, 0)^t$. Since this procedure can be done for all the charts, this means that the *transition functions* beetwen two charts must leave invariant η; but this requires that the structure group, which can be thought as the group formed by the various transition functions, cannot be completely general but must be reduced to the *stabilizer* of η, in this case $SU(3)$. On the other hand, it is well-known that a $SU(3)$ structure can be described also in an alternative way, by requiring that on $\mathcal{M}_6$ are defined a *real* two-form J and a *complex*, decomposable, non-degenerate three-form Ω; such that they satisfy the additional conditions

$$J \wedge \Omega = 0 \ , \qquad J^3 = \frac{3}{4}i\Omega \wedge \bar{\Omega}. \tag{3.1.7}$$

Summarizing, we have seen that the central geometric ingredient defined by a spinor η (together with a metric, since without a metric one cannot define the gamma matrices) is the presence of a $SU(3)$ structure. Such a $SU(3)$ structure can be also described by giving a real two-form J and a complex three-form Ω such that they satisfy the constraints (3.1.7), and the two approaches are related in this way: starting from η one can construct the bilinears

$$J_{mn} \propto \eta^\dagger \gamma_{mn}\eta \ , \qquad \Omega_{mnp} \propto \eta^t \gamma_{mnp}\eta \ , \tag{3.1.8}$$

and they satisfy relations like (3.1.7), as can be shown by Fierzing. Equation (3.1.8) gives another way of seeing the constraints (3.1.7): they express the fact that J and Ω are not arbitrary forms on $\mathcal{M}_6$, they are differential forms that can be written as spinorial bilinears. Later on this section we will rephrase this concept by saying that J and Ω are *compatible*, which means equivalently that they satisfy relations like (3.1.7) or that they can be written as spinorial bilinears as in (3.1.8). We stress this point because it is important to keep in mind that SUSY equations, when rewritten in terms of differential forms, are not equations for arbitrary differential forms but these forms are forced to be compatible, and we will see case by case as the compatibility constraints can be solved.

Since we have been able to translate the geometrical content of η to a couple of differential forms J and Ω, obeying the constraints (3.1.7), it is conceivable that one can rephrase the SUSY equations (2.1.13), rewritten for a four-dimensional vacuum and under

the simplifying assumption of $\eta_1 = \eta_2$, by using J and Ω instead of η. If such a procedure can be made, one has obtained the aim of translating the original SUSY equations, which are *spinorial* equations, into a set of differential equations for *differential forms*, obtaining in this way a drastic simplification of the problem. For four-dimensional vacua this is exactly what it happens: in [10] it is shown, in the much more general context of $\eta_1 \neq \eta_2$ that we will discuss in a moment, that the conditions for supersymmetry for a four-dimensional vacuum can be *entirely* rewritten in terms of differential forms; the resulting system of equations is much more elegant than the original one and this has allowed to obtain, in later years, many results which would be very hard to deduce starting with the original spinorial system.

Until now our discussion lives in *ordinary* complex geometry rather than in *generalized* one: we have just taken a six dimensional spinor η, we have seen that it implies a reduction of the structure group to $SU(3)$ and we have translate all the data into two differential forms J and Ω. In the case of vanishing fluxes one obtains the well-known Calabi-Yau condition

$$d\Omega = 0 \; , \qquad dJ = 0 \; . \tag{3.1.9}$$

The generalized approach comes out when one relaxes the condition $\eta_1 = \eta_2$. To understand what difficulties arise when one relaxes such a condition we can proceed as follow: moving around the internal manifold $\mathcal{M}_6$ there will be some points where $\eta_1 \propto \eta_2$, and some other points where $\eta_1 \not\propto \eta_2$. From the point of view of the structure group reduction the two situations are completely different: when the two spinors are proportional we have, of course, a $SU(3)$-structure with our differential forms J and Ω, but in the points where the two spinors η_1 and η_2 are not proportional we have a further reduction of the structure-group to $SU(2)$; this can be understood since with $\eta_1 \not\propto \eta_2$ we can construct an additional vector $v_m = \eta_1^t \gamma_m \eta_2$. Since we have realized that the reduction of the structure group is at the core of the possibility of rephrasing the spinorial differential system with a system of equations on differential forms, we have that this possibility becomes very complicated to implement, since we have a structure group that depends on the points of the manifold.

Summarizing, if we remove the assumption $\eta_1 = \eta_2$ we have that the structure group acquires a dependence from the points, and this makes very difficult to rephrase the spinorial system of equations into a system of equations on forms using ordinary complex geometry.

Generalized Complex Geometry allows us to overcome all these difficulties at the price of extending our space of interest, from the tangent bundle T to the *generalized* tangent bundle $T \oplus T^*$: on this more abstract bundle we have that the structure group is *independent* from the points and it is equal to $SU(3) \times SU(3)$. Let us go to see what this

statement means: by $T \oplus T^*$ we mean the direct sum of the tangent plus the cotangent bundle and so our geometrical objects are not differential forms or vectors separately but a sum of them

$$X = \omega + v \; , \tag{3.1.10}$$

where ω is a one-form and v is a vector field; we will call such an object a *generalized vector*. Notice that, given a pair of generalized vectors X and Y, there is a natural definition of scalar product between $X = \omega_1 + v_1$ and $Y = \omega_2 + v_2$:

$$(X, Y) = (\omega_1 + v_1, \omega_2 + v_2) = v_2(\omega_1) + v_1(\omega_2) \; . \tag{3.1.11}$$

Given a generalized vector field X it is easy to understand that it has not a well-defined action on differential forms of *fixed* degree; on the other hand the situation is completely different if one consider, instead of differential forms of fixed degree, polyforms: given a generalized vector field X and a polyform α we can consider the action

$$X(\alpha) = \big(v + \omega\big)(\alpha) \equiv v(\alpha) + \omega \wedge \alpha \; , \tag{3.1.12}$$

that maps α to another polyform β.

It is therefore natural to look for a way of mapping the spinors η_1 and η_2 to polyforms. This can be done by considering a new kind of objects: *bispinors*. Let us define the bispinors

$$\Phi_+ \equiv \eta_1 \otimes \eta_2^\dagger \equiv \frac{1}{8} \sum_k \eta_2^\dagger \gamma_{m_1 \ldots m_k} \eta_1 \, \gamma^{m_k \ldots m_1} \; ,$$

$$\Phi_- \equiv \eta_1 \otimes \eta_2^t \equiv \frac{1}{8} \sum_k \eta_2^t \gamma_{m_1 \ldots m_k} \eta_1 \, \gamma^{m_k \ldots m_1} \; , \tag{3.1.13}$$

that, after a Clifford map (2.1.11), give rise to polyforms. Notice that in the particular case of $\eta_1 \propto \eta_2$, Φ_+ and Φ_- just defined become $\Phi_+ = \exp(J)$ and $\Phi_- = \Omega$ but in general they acquire additional parts.

Once that we have defined Φ_+ and Φ_- as in (3.1.13), one can get a rough idea of what the sentence "$SU(3) \times SU(3)$-structure" means in practice:[2] when we say that a globally defined spinor η_1 defines a $SU(3)$ structure, basically we are saying that it defines a decomposition between the six gamma matrices γ^m in three gamma matrices "holomorphic" $\gamma^{i_1} \ldots \gamma^{i_3}$ and three gamma matrices "anti-holomorphic" $\gamma^{\bar{i}_1} \ldots \gamma^{\bar{i}_3}$; the distinction between the holomorphic gamma matrices and the anti-holomorphic ones is given by the fact that the holomorphic matrices annihilate η. Notice that η is annihilated by exactly *half* of the gamma matrices: often this property is rephrased by saying that η is a *pure* spinor. It is possible to show that, in dimensions ≤ 6, a Weyl spinor is

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[2]For a precise definition of this concept one could read the original literature [8], [9].

automatically pure, but in greater dimensions this is not true and a pure spinor is a Weyl spinor which satisfies additional *algebraic* constraints. We will see an example of such a phenomenon in chapter 5 when we will study Mink$_2$ $N = (2,0)$ vacua.

When a second spinor η_2 is introduced, obviously it defines a different splitting of the gamma matrices in $\gamma^{j_1} \ldots \gamma^{j_3}$ and $\gamma^{\bar{j}_1} \ldots \gamma^{\bar{j}_3}$. In general the two decompositions will be compatible in some points and not compatible in other points: when the two spinors are proportional we have of course a single decomposition and a $SU(3)$ structure, when the two spinors are not proportional that structure gets further reduced to $SU(2)$. This argument gives another way of understanding, in terms of gamma-matrices, why when we consider a pair of spinors η_1 and η_2 we have that the structure group acquires a dependence from the points.

When we consider, instead of the pure spinors η_1 and η_2, the bispinors Φ_+ and Φ_- we can overcome this difficulty. To see how this happens, we note that in both Φ_+ and Φ_- we have η_1 always on the left, and η_2 always on the right; we can therefore consider separately the action of the gamma matrices on the left or on the right of the bispinors

$$\gamma^m \Phi_\pm = [(dx^m \wedge + g^{mn} \iota_n) \Phi_\pm] , \qquad \Phi_\pm \gamma^m = \pm[(dx^m \wedge - g^{mn} \iota_n) \Phi_\pm] , \qquad (3.1.14)$$

where, on the rhs, we have considered $\Phi_\pm$ as polyforms and we have translated the action of the gamma matrices on them as wedges and contractions. Since, as we just said, η_1 sits always on the left and η_2 sits always on the right, we see that we can use the $SU(3)$ decomposition determined by η_1 when the corresponding gamma matrix acts on the left, and the $SU(3)$ decomposition corresponding to η_2 when the gamma matrix acts on the right. In this way we see that the two structure groups are not in competition but, together, they determine a $SU(3) \times SU(3)$ structure on the generalized tangent bundle. We see also that the naive intuition that η^1 defines a $SU(3)$ structure on (saying) T and η^2 defines a $SU(3)$ structure on T^* is not correct and must be refined: the actions on the left and on the right of the gamma matrices both contain a wedge and a contraction, and so it is the combination of η^1 and η^2 together that defines the $SU(3) \times SU(3)$ structure on $T \oplus T^*$, and that this structure cannot be decomposed into separate structures on T and T^* separately.

Summarizing, we have obtained a way to describe the structure group without any dependence on the points, at the price of enlarging our space of interest from T to $T \oplus T^*$. On this more abstract space the pure spinors η^1 and η^2 have been replaced by the bispinors $\Phi_\pm$ and on them we can act by wedge and contraction (when we think them as polyforms), or by the action of the gamma matrices γ^m on the left or on the right when we consider them as bispinors. We can now show how the purity property of the spinors η^1 and η^2 gets translated on Φ_+ and Φ_-. To this end notice that wedge product

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and contraction satisfy 12-dimensional Clifford algebra relations like

$$\{dx^m \wedge,\ dx^n \wedge\} = 0\ , \qquad \{dx^m \wedge,\ \iota_n\} = \delta^m_{\ n}\ , \qquad \{\iota_m,\ \iota_n\} = 0\ . \qquad (3.1.15)$$

Now, given the map (3.1.14), it is straightforward to realize that the purity of η^1 and η^2 is translated in the purity of the bispinors $\Phi_\pm$ with respect to the Clifford algebra (3.1.15). For this reason we will call $\Phi_\pm$ pure spinors, keeping in mind that they are not usual pure spinors on the tangent bundle but they are pure spinors on the generalized tangent bundle and that their Clifford algebra is not given by the ordinary Clifford algebra Cl(6) but by the Clifford algebra Cl(6,6) defined in (3.1.15).

Having introduced the bispinors Φ_+ and Φ_-, it has been shown in [10] that they can be used to rephrase the SUSY equations in terms of differential forms. The final form of the system of equations is very elegant, and we quote it here just to give a flavour of how much simpler it is with respect to the original spinorial system (3.1.5). For example, for an AdS$_4$ vacuum of type IIB it writes

$$d_H \Phi_- = -2\mu e^{-A} \mathrm{Re}\Phi_+\ , \qquad d_H(e^A \mathrm{Im}\Phi_+) + 3\mu \mathrm{Im}\Phi_- = e^{4A} * \lambda f\ , \qquad (3.1.16)$$

where the differential d_H has been introduced in (2.1.8), the constant μ is given in (3.1.4) and f are the internal RR-fluxes. Moreover, we have to impose that $\Phi_\pm$ are not arbitrary differential forms on $\mathcal{M}_6$, but they must be obtained as spinorial bilinears. Recall that the requirement that $\Phi_\pm$ can be written as bispinors can be also described through some compatibility relations. Even if we will not use in the following this second point of view, let us recall the compatibility conditions for an AdS$_4$ vacuum

$$(\Phi_-,\ X \cdot \Phi_+) = (\bar{\Phi}_-,\ X \cdot \Phi_+) \qquad X \in T \oplus T^*\ , \qquad (3.1.17)$$

where (,) is the six-dimensional Chevalley-Mukai pairing of forms that, in d dimensions, is defined by

$$(\alpha, \beta) = (\alpha \wedge \lambda(\beta))_d\ , \qquad (3.1.18)$$

where $_d$ means that we keep only the d-form part, α and β are two (poly)-forms and the λ operator acts on a k-form α_k as in (2.1.10). Moreover, the notation $X \cdot$ means $(X \wedge + \iota_X)$.

3.2   A ten-dimensional system of equations

The GCG approach has been extended in [11] to include *all* the ten-dimensional configurations which preserve some amounts of supersymmetry, no matter whether they are vacuum solutions or not. In other words, in [11] a differential system which is completely

equivalent to (2.1.13) has been obtained, without making any preliminar Ansatz regarding the nature of the solution (in the preceding section we have made the only Ansatz that our solution was a four-dimensional vacuum). Our aim in this section is to review the construction of [11] and to underline the main properties of the final system.

Using the ten-dimensional SUSY parameters ϵ_i , $i = 1, 2$ we can construct two different vectors (or equivalently one-forms)

$$K_i^M \equiv \frac{1}{32}\bar{\epsilon}_i\Gamma^M\epsilon_i \ , \qquad K \equiv \frac{1}{2}(K_1 + K_2) \ , \qquad \tilde{K} \equiv \frac{1}{2}(K_1 - K_2) \ . \qquad (3.2.1)$$

We can also consider the polyform

$$\Phi = \epsilon_1\bar{\epsilon}_2 \ , \qquad (3.2.2)$$

defining many different G-structures on the ten-dimensional tangent bundle, all of them corresponding to a single structure on the generalized ten-dimensional tangent bundle $T_{10} \oplus T_{10}^*$.[3] The situation would appear to be completely analogous to what happens for four-dimensional $N = 1$ vacua, where, as just explained, the pure spinors Φ_+ and Φ_- define together a $SU(3) \times SU(3)$ structure on the generalized tangent bundle of the internal manifold $T_6 \oplus T_6^*$, however one can show that Φ is *not* a pure spinor and as a consequence of this fact it is not sufficient to fully reconstruct the metric and the B-field (contrary to the situation for four dimensional vacua where Φ_+ and Φ_- do determine a metric and a B-field). This feature forces us to introduce additional geometrical data and indeed in [11] two additional vectors e_{+1} and e_{+2} satisfying

$$e_{+i}^2 = 0 \ , \qquad e_{+i} \cdot K_i = \frac{1}{2} \ , \qquad i = 1, 2 \ , \qquad (3.2.3)$$

are introduced. We will see in a moment that these additional vectors are a source of difficulties: since they are not defined as intrinsic geometrical objects starting from the SUSY parameters ϵ_1 and ϵ_2 but they must be defined by hand, the equations involving them are much more involved than the other. Nevertheless they are necessary in general and we must keep them into account.

We can now reformulate the conditions for unbroken supersymmetry in terms of the geometrical data $(K, \tilde{K}, \Phi, e_{+i})$ just discussed, obtaining the following system

$$L_K g = 0 \ , \qquad d\tilde{K} = \iota_K H \ ; \qquad (3.2.4a)$$

$$d_H(e^{-\phi}\Phi) = -(\tilde{K} \wedge +\iota_K)F \ ; \qquad (3.2.4b)$$

$$(e_{+1} \cdot \Phi \cdot e_{+2}, \ \Gamma^{MN}[\pm d_H(e^{-\phi}\Phi \cdot e_{+2}) + \frac{1}{2}e^\phi d^\dagger(e^{-2\phi}e_{+2})\Phi - F]) = 0 \ ; \qquad (3.2.4c)$$

$$(e_{+1} \cdot \Phi \cdot e_{+2}, \ [d_H(e^{-\phi}e_{+1} \cdot \Phi) - \frac{1}{2}e^\phi d^\dagger(e^{-2\phi}e_{+1})\Phi - F]\Gamma^{MN}) = 0 \ . \qquad (3.2.4d)$$

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[3]In general the structure group that one obtains is very complicated but, fortunately, in these thesis we will stay in contexts where it will be much simpler.

((3.2.4a) already appeared in [40], [41] and [42].) In (3.2.4c) and (3.2.4d) (,) means the ten-dimensional Chevalley-Mukai pairing. Equations (3.2.4) are *necessary and sufficient* for supersymmetry to hold [11]. To also solve the equations of motion, one needs to impose the Bianchi identities, which we recall that away from sources (branes and orientifolds) read

$$dH = 0 \; , \qquad d_H F = 0 \; . \tag{3.2.5}$$

It should be noted that equations (3.2.4a) and (3.2.4b) are very elegant: apart from the first equation in (3.2.4a) (expressing that K has to be a Killing vector) they are formulated in terms of differential forms and exterior calculus only and they are much simpler to treat than the original SUSY conditions. Unfortunately, they are necessary to supersymmetry to hold but not sufficient and they must be completed with (3.2.4c) and (3.2.4d) (which we will call *pairing equations*).

Depending on the behaviour of the pairing equations we can consider three different cases: there are some cases where the pairing equations can be recast in an elegant form much like (3.2.4b), this is the case for two-dimensional $N = (2,0)$ vacua studied in [17] and [18]; we will call such a case a *good case*. There are cases where the pairing equations cannot be recast in an elegant form and their geometrical meaning remains difficult to understand, this is the case studied in [43], [44], and [20]; we will call such cases the *ugly* cases.

Finally, there are the *lucky* cases. In such cases the pairing equations are completely redundant and can be dropped; supersymmetry is completely equivalent to equations (3.2.4a) and (3.2.4b) and we can forget about the additional vectors e_{+i}. We will see in the next chapter that AdS_6 and AdS_7 vacua fall in this final set (other examples are given by four-dimensional vacua).

Of course it would be important to obtain some methods to understand from the beginning whether a particular case under investigation is a good, ugly or lucky case. Unfortunately at the moment it is not clear if such a criterion exists[4] and so one has to investigate case by case the behaviour of the pairing equations.

In the next chapters of this thesis we will consider many examples for all the cases just explained.

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[4]Even if it is clear that lucky and good cases are associated to simpler structures on $T \oplus T^*$ the contrary is not true, and there cases where the structure group is quite simple but nevertheless the pairing equations are not redundant and they take a complicated form. An example of such a phenomenon will be given later on this thesis where we will discuss compactification to four-dimensional $N = 2$ supergravity.

Chapter 4

The lucky cases: AdS_7 and AdS_6 solutions

In this chapter we will discuss two example of lucky cases: as already outlined we will study AdS_7 and AdS_6 solutions in type II supergravity. In the AdS_7 case we will numerically classify the solutions of this kind. Such a classification has proven to be related to brane constructions engineering $(1,0)$ six-dimensional SCFTs. In the AdS_6 case our analysis will be unfortunately less complete, but nevertheless we will be able to reduce the problem of finding AdS_6 solutions to a system of two partial differential equations.

We will start by reviewing the results at disposal in the literature without the use of GCG, then we will move to discuss the G-structure approach to AdS_7 solutions and finally we will discuss the AdS_6 case.

4.1 Higher AdS vacua without Generalized Complex Geometry

In this section we will discuss what results were at disposal in the literature without the use of GCG techniques. We will see that only one kind of AdS_7 solutions had been found in the context of eleven-dimensional supergravity (the so-called Freund-Rubin solutions), it was also known that the Freund-Rubin solution are the only AdS_7 vacua that one can find in eleven-dimensional supergravity (and so in massless type IIA too) but nothing was known about massive type IIA and type IIB SUGRA. The situation was similar for AdS_6 solutions, where the only vacuum which had been found was the Brandhuber and Oz solution in massive type IIA [45] (it is shown in [46] that this is the only AdS_6 that can be found in massless type IIA); on the other hand the type IIB analysis and the eleven-dimensional analysis was restricted to some type IIB T-duals of the Brandhuber

and Oz solution. During the discussion of the supergravity solutions we will point out also some aspects of the CFT duals.

4.1.1 The Freund-Rubin AdS$_7$ solution

Let us discuss the Freund-Rubin AdS$_7$ vacuum in eleven-dimensional supergravity and show that this is the only AdS$_7$ vacuum that can be constructed in eleven dimensions. The field content of eleven-dimensional supergravity is very simple: it contains obviously the metric G_{MN}, M, $N = 0, \ldots, 10$; the gravitino Ψ_M, which for each value of M is a 32-component Majorana spinor and, of course, the antisymmetric three-form A_3 that we already introduced in (2.1.20); it is subject to the obvious gauge invariance

$$A_3 \rightarrow A_3 + d\Lambda_2 \ , \tag{4.1.1}$$

where Λ_2 is a two-form.

In analogy with (3.1.1) we take our eleven-dimensional metric of the form[1]

$$ds_{11}^2 = ds_{\text{AdS}_7}^2 + ds_{\mathcal{M}_4}^2 \ . \tag{4.1.2}$$

As usual, supersymmetric configurations are obtained putting to zero the gravitino field and imposing that its supersymmetry variation vanishes

$$\nabla_M \epsilon + \frac{1}{288} \big(\Gamma_M^{NPQR} - 8\delta_M^N \Gamma^{PQR} \big) G_{NPQR} \epsilon = 0 \ , \tag{4.1.3}$$

where ϵ is the eleven-dimensional SUSY parameter (which is a 32-component Majorana spinor) and $G \equiv \frac{1}{4!} G_{NPQR} dx^N \wedge dx^P \wedge dx^Q \wedge dx^R$ is the field-strength of A_3. To preserve the maximal symmetry of the external background we need to impose that the field-strength G is a purely internal four-form

$$G \equiv \frac{1}{4!} G_{npqr} dy^n \wedge dy^p \wedge dy^q \wedge dy^r \ , \tag{4.1.4}$$

and, given that the internal space is four-dimensional, it is straightforward to deduce that it must be proportional to the internal volume form

$$G = g(y) \text{vol}_{\mathcal{M}_4} \ , \tag{4.1.5}$$

where $g(y)$ is a function on $\mathcal{M}_4$.

Exactly as we discussed for four-dimensional vacua (Section 3.1), given that the metric gets factorized as in (4.1.2), we have that the gamma matrices Γ_M take a similar decomposition

$$\Gamma_\mu = \gamma_\mu^{(7)} \otimes \gamma^{(4)} \ , \qquad \Gamma_m = 1^{(7)} \otimes \gamma_m^{(4)} \ , \tag{4.1.6}$$

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[1]For simplicity we have omitted a possible warping factor. We will see in Section 4.2.9 that this does not spoil the generality of the solution.

where $\gamma_\mu^{(7)}$ and $\gamma_m^{(4)}$ are gamma matrices in seven and four dimensions respectively, whereas $\gamma^{(4)}$ is the chiral gamma matrix in four dimensions. Let us discuss now the decomposition of the SUSY parameter ϵ: given the decomposition (4.1.6) we have that ϵ acquires an identical decomposition

$$\epsilon = \zeta \otimes \eta + \text{c.c.} , \tag{4.1.7}$$

where ζ and η are Dirac spinors on the external and internal manifold respectively and we added the complex conjugates to ensure that ϵ is Majorana as required.

As we explained in section 3.1, in order to preserve the maximal symmetry of the external background, we need to impose that ζ satisfies the Killing spinor equation

$$\nabla_\mu \zeta = \frac{\Lambda}{2} \gamma_\mu^{(7)} \zeta , \tag{4.1.8}$$

where Λ is the inverse of the AdS$_7$ radius

$$\Lambda = \frac{1}{R_{\text{AdS}_7}} . \tag{4.1.9}$$

Given the decomposition just explained we can come back to our SUSY equation (4.1.3). It gets decomposed in two equations

$$\left(\nabla_\mu + \frac{1}{6} g \gamma_\mu \otimes 1\right)\epsilon = \gamma_\mu^{(7)} \zeta \otimes \left(\frac{\Lambda}{2} - \frac{g}{6}\right)\eta + \text{c.c.} = 0 ,$$

$$\left(\nabla_m - \frac{1}{3} g 1 \otimes \gamma^{(4)} \gamma_m\right)\epsilon = \zeta \otimes \left(\nabla_m - \frac{g}{3} \gamma^{(4)} \gamma_m\right)\eta + \text{c.c.} = 0 , \tag{4.1.10}$$

that combined tell us $g = 3\Lambda$ and, more important,

$$\left(\nabla_m - \Lambda \gamma^{(4)} \gamma_m\right)\eta = 0 , \tag{4.1.11}$$

which expresses the requirement that η is a *Killing* spinor on $\mathcal{M}_4$. This request implies that the cone $C(\mathcal{M}_4)$ constructed over $\mathcal{M}_4$ admits a covariantly constant spinor; but in five dimensions the only manifold with restricted holonomy is $\mathbb{R}^5$ or one of its orbifolds. This consideration leads us to conclude that the only AdS$_7$ vacuum solution at disposal in eleven-dimensional supergravity is given by AdS$_7 \times S^4$ and its orbifolds AdS$_7 \times S^4/\Gamma$. Of course such solutions are dual to the famous $(2,0)$ theory living on the M5 branes.

4.1.2 The Brandhuber and Oz AdS$_6$ solution

Let us move to discuss the known facts about AdS$_6$ vacua that can be obtained without using GCG. As already remarked the only known AdS$_6$ solution is given by the so-called Brandhuber and Oz solution in massive type IIA supergravity, whose metric takes the form of AdS$_6 \times S^4$; in [46] it has been shown that this solution in massive type IIA supergravity is unique and no other AdS$_6$ vacua can be found in the massive case. We

will review how the Brandhuber and Oz can be deduced starting from some holographic considerations concerning five-dimensional CFTs. Finally we will discuss some further solutions that can be obtained in type IIB by considering T dualities of the Brandhuber and Oz configuration: the standard abelian T-dual obtained along the $U(1)$ fiber of S^4 and, the more mysterious, non abelian T-dual recently considered in [47] and [48].

In [49] it has been argued that a five-dimensional CFT can be found by considering the following brane configuration: consider a Type I' setup on $\mathbb{R}^9 \times I$ where we denoted with I an interval or, equivalently, $S^1/\mathbb{Z}_2$, let us call this coordinate x_9. At the end points of the interval, or at the fixed points of the $\mathbb{Z}_2$ action, we have two orientifolds (two O_8 planes) and so, to cancel the brane charge of them, we need to add to our setup 16 D8 branes located at some points along x_9 and extended along the other directions. Finally, we put in our configuration a bunch of N D_4 branes, extended along x_0, x_1, x_2, x_3, x_4 and located at some points along the other directions. Let us divide the D8 branes in two sets: starting from an O8 and moving along x_9, consider the first N_f D8 branes, with $N_f < 8$,[2] and call the values of their x_9 coordinates x_9^i, $1 \leq i \leq N_f$ (with of course $x_1 \leq x_2 \leq \cdots \leq x_{N_f}$).

Let us describe the five-dimensional $N = 1$ theory living on the N D4 branes: having a bunch of N D4, with an O8, we obtain a $5d$ $N = 1$ gauge theory with gauge group $USp(2N)$, with an antisymmetric hypermultiplet (given by open strings streched between the D4 and the O8), moreover we have N_f foundamental hypermultiplets which are created by open D4-D8 branes. The Coulomb branch of the theory is given by the position along x_9 (that we denote with ϕ) of the bunch of D4s and, instead, the position of the D4 along (x_5, x_6, x_7, x_8) goes in correspondence with the Higgs branch of the theory. The masses of the N_f foundamental hypermultiplets are given by the positions x_9^i of the corresponding D8s. The CFT limit is reached when both the bunch of D4s and the D8s approach the origin $x_9 = 0$ where we have the orientifold plane; from a field theory point of view this correspond to the origin of the Coulomb branch of the $USp(2N)$ theory coupled to $N_f < 8$ *massless* foundamental hypermultiplets. Our aim is now to give a supergravity description of such a limit.

To this end, we start by recalling the metric determined on $\mathbb{R}^9 \times I$ by the D8 branes (before to reach the fixed point). It writes as

$$ds^2 = H_8^{-1/2}(x_9)dx_{0,8}^2 + H_8^{1/2}(x_9)dx_9^2 \ , \tag{4.1.12}$$

with the function $H_8(x_9)$ given by

$$H_8(x_9) = a + 16x_9 - \sum^{N_f} |x_9 - x_9^i| - \sum^{N_f} |x_9 + x_9^i| \ , \tag{4.1.13}$$

where a is a constant that it is related to the inverse of the bare coupling, and so it goes to 0 in the CFT limit. H_8 also determines the dilaton via the relation

$$e^\phi = H_8^{-5/4}(x_9) \ . \tag{4.1.14}$$

Since we are in a configuration containing D8s, it is natural to expect that we have a non-zero Romans mass and indeed the expression for it is given by a piecewise constant function like

$$m(x_9) = \frac{1}{4\pi\sqrt{\alpha'}}(16 - 2i) \qquad x_9^i < x_9 < x_9^{i+1} \qquad i = 0, 1, \ldots, N_f - 1 \ , \tag{4.1.15}$$

where, in order to use a compact formula, we have denoted with x_9^0 the position of the orbifold $x_9 = 0$.

To proceed it is convenient to define a new coordinate

$$z = \left(\frac{2x_9}{3}\sqrt{\frac{8 - N_f}{2\pi}}\right)^{3/2} \ , \tag{4.1.16}$$

in terms of wich the metric (4.1.12) becomes conformally flat when the N_f D8s are on top of the O8

$$ds^2 = \Omega^2(z)\big(dx_{0,8}^2 + dz^2\big) \ , \qquad \Omega(z) \equiv \big(\frac{3}{4\pi}(8 - N_f)z\big)^{-1/6} \ , \tag{4.1.17}$$

the expressions for the dilaton and the Romans mass are given by

$$e^\phi = \Omega^5(z) \ , \qquad m = \frac{8 - N_f}{2\pi\sqrt{\alpha'}} \ . \tag{4.1.18}$$

The metric gets further modified by the N D4 branes that we need to introduce, they give an additional warp factor in the metric and also, as natural, induce a non vanishing six-form flux F_6:

$$\begin{aligned}
ds^2 &= \Omega^2(z)\big[H_4^{-1/2}(r)dx_{0,4}^2 + H_4^{-1/2}(r)(dx_{\mathbb{R}^4}^2 + dz^2)\big] \ , \\
e^\phi &= \Omega^5(z)H_4^{-1/4}(r) \ , \\
F_6 &= d^5x \wedge dH_4^{-1}(r) \ ,
\end{aligned} \tag{4.1.19}$$

where $r^2 = \tilde{r}^2 + z^2$ and $\tilde{r}^2 = x_5^2 + x_6^2 + x_7^2 + x_8^2$, and the additional warping function is given, in the near-horizon limit, by

$$H_4(r) = \frac{Q_4}{r^{10/3}} \ , \qquad Q_4 = \left(\frac{2^{11}\pi^4}{3^4(8 - Nf)}\right)^{1/3} N \ . \tag{4.1.20}$$

It is now convenient to define an angular coordinate α such that $\tilde{r} = r\sin\alpha$ and $z = r\cos\alpha$. In this way one can see that the background (4.1.19) is nothing but a *warped*

product of the form $\text{AdS}_6 \times S_4$

$$ds^2 = \frac{W^2 L^2}{4} \left[9ds^2(\text{AdS}_6) + 4ds^2(S^4) \right] ,$$

$$F_4 = 5L^4 W^{-2} \sin^3 \alpha d\alpha \wedge \text{Vol}(S^3) ,$$

$$e^{-\phi} = \frac{3L}{2W^5} , \qquad W = (\alpha \cos \theta)^{-1/6} , \tag{4.1.21}$$

where we have replaced F_6 with its Hodge dual F_4, L denotes the AdS_6 radius $L = \frac{3}{2} Q_4^{1/4}$ and the metric on S^4 is given by

$$ds^2(S^4) = d\alpha^2 + \sin^2 \alpha ds^2(S^3) . \tag{4.1.22}$$

From (4.1.16) we see that z has to be positive; combining this observation with the definition of α given by $z = r \cos \alpha$ we see that $0 \leq \alpha \leq \pi/2$ and so we conclude that the internal manifold is not an entire S^4 but only half of an S^4. Moreover we can see that this solution diverges for $\alpha \to \pi/2$ since the dilaton diverges for such a value; such a singularity reflects in the supergravity picture the presence of the O8/D8 system. Finally, we see that away the singular point the curvature and the dilaton go like

$$\mathcal{R} \propto -\frac{m^{1/3}}{L^2} , \qquad e^{\phi} \propto \frac{1}{Lm^{5/6}} , \tag{4.1.23}$$

and so we see that the supergravity regime is valid when

$$\frac{m^{1/3}}{L^2} \ll 1 , \qquad Lm^{5/6} \gg 1 , \tag{4.1.24}$$

both these conditions can be satisfied by taking the AdS_6 radius large, $L \gg 1$.

Further developments

As anticipated, in [46] it is shown that the Brandhuber and Oz solution is the unique solution that one can find in *massive* type IIA supergravity. Nevertheless, one can consider some further developments based on this solution. First of all one can consider T-dualities: we know that the internal manifold S^4 can be seen as an S^3 fibration over an interval. Writing S^3 as an S^1-Hopf fibration over S^2

$$ds^2(S^3) = \frac{1}{4}[d\phi_1^2 + \sin^2 \phi_1 d\phi_2^2 + (d\phi_3 + \cos \phi_1 d\phi_2)^2] , \tag{4.1.25}$$

one obtains that the Killing spinors ϵ_1 and ϵ_2 are completely independent of the internal $U(1)$ coordinate ϕ_3. This fact allows us to consider the standard, abelian, T-duality of the Brandhuber and Oz solution without breaking supersymmetry. In this way one obtains

one example of AdS_6 vacuum in Type IIB supergravity, we just quote the final result which is given by

$$ds^2 = \frac{1}{4}W^2L^2\Big[9ds^2(AdS_6) + 4d\alpha^2 + \sin^2\alpha(d\phi_1^2 + \sin^2\phi_1 d\phi_2^2) + \frac{16}{W^4L^4\sin^2\alpha}d\phi_3^2\Big] \ ,$$

$$B = -\cos\phi_1 d\phi_2 \wedge d\phi_3 \ , \qquad e^\phi = \frac{4}{3L^2(m\cos\alpha)^{2/3}\sin\theta} \ ,$$

$$F_3 = \frac{5}{8}L^4(m\cos\alpha)^{1/3}\sin^3\alpha\sin\phi_1 d\alpha \wedge d\phi_1 \wedge d\phi_2 \ , \qquad F_1 = md\phi_3 \ . \qquad (4.1.26)$$

The dilaton shows that we have a new singularity at $\alpha = 0$: this represents the standard singularity that one always gets when T-dualizes along a Hopf direction in a S^3 that shrinks somewhere. It represents an NS5 smeared along the T-dual S^1; one expects worldsheet instantons to modify the metric so that the NS5 singularity gets localized along that direction, as in [50]. As for the singularity at $\alpha = \frac{\pi}{2}$, it now cannot be associated with an O8–D8 system as it was in IIA, since we are in IIB. It probably now represents a smeared O7–D7 system; it is indeed always the case that T-dualizing a brane along a parallel direction in supergravity gives a smeared version of the correct D-brane solution on the other side, as we just saw for the NS5-brane.

A more exotic solution in type IIB is considered in [48], [47]: by performing a *non-abelian* T-duality transformation of the Brandhuber and Oz solution the authors have found a new solution in type IIB whose *local* metric writes

$$ds^2 = \frac{W^2L^2}{4}\Big[9ds^2(\mathrm{AdS}_6) + 4d\alpha^2\Big] + e^{-2\hat{A}}dr^2 + \frac{r^2e^{2\hat{A}}}{r^2 + e^{4\hat{A}}}ds^2(S^2) \ ,$$

$$B_2 = \frac{r^3}{r^2 + e^{4\hat{A}}}\mathrm{Vol}(S^2) \ , \qquad e^{-\phi} = \frac{3L}{2W^5}e^{\hat{A}}\sqrt{r^2 + e^{4\hat{A}}} \ ,$$

$$F_1 = -G_1 - mrdr \ , \qquad F_3 = \frac{r^2}{r^2 + e^{4\hat{A}}}[-rG_1 + me^{4\hat{A}}dr] \wedge \mathrm{Vol}(S^2) \ , \qquad (4.1.27)$$

with

$$e^{\hat{A}} = \frac{WL}{2}\sin\alpha \ , \qquad G_1 = \frac{5}{8W^2}L^4\sin^3\alpha d\theta \ . \qquad (4.1.28)$$

The global properties of such a solution are still mysterious: beyond the singularities at $\alpha = 0$ and $\alpha = \pi/2$ that we just discussed, we see that this solution seems to be non-compact. This feature for sure needs to be investigated better in the future; the authors of [47] have assumed that the coordinate r is stopped at some values $R > 0$, nevertheless the solution looks perfectly well-defined for arbitrary r and so it is not clear how such a cut-off should be interpreted.

Finally, one can consider orbifolds of the Brandhuber and Oz solution: this has been done in [51] where the authors have discussed the various orbifolds that one can do starting from the Brandhuber and Oz solution, the CFT_5 duals are also analyzed and it is shown that all these duals correspond to *linear* quiver gauge theories.

4.1.3 Summary

Let us summarize the results of this section. We have seen that, without using GCG, some partial results were already at disposal in the literature concerning AdS six-dimensional and seven-dimensional vacua. Concerning AdS_7 vacua it was known that in massless type IIA the Freund-Rubin solution represents the only vacuum that one can find. A similar result was at disposal in the six-dimensional case, where it was known that the Brand-huber and Oz solution (and orbifolds) represents that only solution that can be obtained in massive type IIA; via T-dualities one can obtain some examples of IIB solutions. In the next sections we will show how GCG allows us to extends these results.

4.2 AdS$_7$ using generalized complex geometry

In this section we will see how GCG, and in particular the power of the system of equations (3.2.4) allows us to extend the analysis of AdS_7 vacua in type II supergravity. We will see that no solutions can be found in type IIB but the situation is much richer in *massive* type IIA, where we will be able to give a classification (at least at numerical level) of AdS_7 vacua. Such a classification has been very useful in subsequent works to increase our understanding about SCFTs in six dimensions (see [52], [16] and [53]) .

4.2.1 Supersymmetry and pure spinor equations in three dimensions

To start with, following the GCG philosophy, we want to find a system of differential equations on differential forms that is equivalent to preserved supersymmetry for solutions of the type $\text{AdS}_7 \times M_3$. We will derive it by a commonly-used trick: namely, by considering AdS_{d+1} as a warped product of Mink_d and $\mathbb{R}$. We will begin in section 4.2.1 by reviewing a system equivalent to supersymmetry for $\text{Mink}_6 \times M_4$. In section 4.2.1 we will then translate it to a system for $\text{AdS}_7 \times M_3$.

Mink$_6 \times M_4$

For $\text{Mink}_6 \times M_4$ solutions, [54] found a system in terms of a $SU(2) \times SU(2)$ structure on M_4, described by a pair of bispinors $\phi^{1,2}$. Similarly to the $\text{Mink}_4 \times M_6$ case these bispinors are characterized as bilinears of the internal parts $\eta^{1,2}$ of the supersymmetry

parameters in $(A.0.2)$:[3]

$$\phi^1_\mp = e^{-A_4}\eta^1_+ \otimes \eta^{2\,\dagger}_\mp \,, \qquad \phi^2_\mp = e^{-A_4}\eta^1_+ \otimes \eta^{2c\,\dagger}_\mp \,, \qquad (4.2.1)$$

where the warping function A_4 is defined by

$$ds^2_{10} = e^{2A_4}ds^2_{\mathrm{Mink}_6} + ds^2_{M_4} \,. \qquad (4.2.2)$$

The upper index in $(4.2.1)$ is relevant to IIA, the lower index to IIB; so in IIA we have that $\phi^{1,2}$ are both odd forms, and in IIB that they are both even.[4]

The system equivalent to supersymmetry now reads [54] [5]

$$d_H\big(e^{2A_4-\phi}\mathrm{Re}\phi^1_\mp\big) = 0 \,, \qquad (4.2.3a)$$

$$d_H\big(e^{4A_4-\phi}\mathrm{Im}\phi^1_\mp\big) = 0 \,, \qquad (4.2.3b)$$

$$d_H\big(e^{4A_4-\phi}\phi^2_\mp\big) = 0 \,, \qquad (4.2.3c)$$

$$e^\phi F = \mp 16 *_4 \lambda(dA_4 \wedge \mathrm{Re}\phi^1_\mp) \,, \qquad (4.2.3d)$$

$$(\overline{\phi^1_\pm}, \phi^1_\pm) = (\overline{\phi^2_\pm}, \phi^2_\pm) = \frac{1}{4} \,. \qquad (4.2.3e)$$

Here, as usual, ϕ is the dilaton; $d_H = d - H\wedge$ is the twisted exterior derivative; A_4 was defined in $(4.2.2)$; F is the *internal* RR flux, which, using the request of maximal symmetry of the Mink_6 vacuum, determines the external flux via self-duality:

$$F_{(10)} \equiv F + e^{6A_4}\mathrm{vol}_6 \wedge *_4\lambda F \,. \qquad (4.2.4)$$

Actually, $(4.2.3)$ contains an assumption: that the norms of the η^i are equal. For a noncompact M_4, it might be possible to have different norms; $(4.2.3)$ would then have to be slightly changed. (See [55, Sec. A.3] for comments on this in the $\mathrm{Mink}_4 \times M_6$ case.) As shown in appendix A, however, for our purposes such a generalization is not relevant.

With this caveat, the system $(4.2.3)$ is equivalent to supersymmetry for $\mathrm{Mink}_6 \times M_4$. Historically, it has been found by direct computation in [54] but, of course, it can also be found as a consequence of the ten-dimensional system in $(3.2.4)$ and in particular, since we are in a lucky case, of the first two equations $(3.2.4a)$ and $(3.2.4b)$.

---

[3]The index $\mp$ on spinors denotes chirality, and $\eta^c \equiv B_4\eta^*$ denotes Majorana conjugation; for more details see appendix A. The factors e^{-A_4} are included for later convenience.

[4]We could also characterize $\phi^{1,2}$ in terms of a compatibility relation, very similar to the relations $(3.1.17)$ that we discussed for four-dimensional vacua. However for our purposes, the characterization in terms of spinorial bilinears works better

[5]We have massaged a bit the original system in [54], by eliminating $\mathrm{Re}\phi^1_\mp$ from the first equation of their (4.11).

$\text{AdS}_7 \times M_3$

As we anticipated, we will now use the fact that AdS can be used as a warped product of Minkowski space with a line. We would like to classify solutions of the type $\text{AdS}_7 \times M_3$. These in general will have a metric

$$ds_{10}^2 = e^{2A_3} ds_{\text{AdS}_7}^2 + ds_{M_3}^2 \tag{4.2.5}$$

where A_3 is a new warping function (different from the A_4 in (4.2.2)). Since

$$ds_{\text{AdS}_7}^2 = \frac{d\rho^2}{\rho^2} + \rho^2 ds_{\text{Mink}_6}^2 \ , \tag{4.2.6}$$

(4.2.5) can be seen as a subset of (4.2.2) if we take

$$e^{A_4} = \rho e^{A_3} \ , \qquad ds_{M_4}^2 = \frac{e^{2A_3}}{\rho^2} d\rho^2 + ds_{M_3}^2 \ . \tag{4.2.7}$$

Let us now move to discuss the additional assumptions that we have to impose in order to preserve the SO(6,2) invariance of AdS_7: A_3 should be a function of M_3. The fluxes F and H, which for a Mink_6 vacuum were arbitrary forms on M_4, should now be forms on M_3. For IIA, $F = F_0 + F_2 + F_4$: in order not to break $\text{SO}(6,2)$, we impose $F_4 = 0$, since it would necessarily have a leg along AdS_7; for IIB, $F = F_1 + F_3$.

Following this logic, solutions to type II equations of motion of the form $\text{AdS}_7 \times M_3$ are a subclass of solutions of the form $\text{Mink}_6 \times M_4$. In appendix A, we also show how the $\text{AdS}_7 \times M_3$ supercharges get translated in the $\text{Mink}_6 \times M_4$ framework, and that the internal spinors have equal norm, as we anticipated around (4.2.4). Using (A.0.10), we also learn how to express the $\phi^{1,2}$ in (4.2.1) in terms of bilinears of spinors $\chi_{1,2}$ on M_3:

$$\phi_\mp^1 = \frac{1}{2}\left(\psi_\mp^1 + ie^{A_3}\frac{d\rho}{\rho} \wedge \psi_\pm^1 \right) \ , \qquad \phi_\mp^2 = \mp\frac{1}{2}\left(\psi_\mp^2 + ie^{A_3}\frac{d\rho}{\rho} \wedge \psi_\pm^2 \right) \ , \tag{4.2.8}$$

with

$$\psi^1 = \chi_1 \otimes \chi_2^\dagger \ , \qquad \psi^2 = \chi_1 \otimes \chi_2^{c\,\dagger} \ . \tag{4.2.9}$$

As usual, we have implicitly mapped forms to bispinors via the Clifford map, and in (4.3.4) the subscripts $\pm$ refer to taking the even or odd form part. (Recall also that $\phi_-^{1,2}$ is relevant to IIA, and $\phi_+^{1,2}$ to IIB; see (4.2.3).) The spinors $\chi_{1,2}$ have been taken to have unit norm.

As usual, to use the GCG approach we must impose the constraint that $\psi^{1,2}$ are not arbitrary but they can be written as bispinors (4.2.9). We will see in a moment how to solve this constraint in a very elegant way.

We can now use (4.2.8) in (4.2.3). Each of those equations can now be decomposed in a part that contains $d\rho$ and one that does not. Thus, the number of equations would

double. However, for (4.2.3a), (4.2.3b) and (4.2.3c), the part that does not contain $d\rho$ actually follows from the part that does. The "norm" equation, (4.2.3e), simply reduces to a similar equation for a three-dimensional norm. Summing up:

$$d_H \text{Im}(e^{3A_3 - \phi}\psi_\pm^1) = -2e^{2A_3 - \phi}\text{Re}\psi_\mp^1 \ , \tag{4.2.10a}$$

$$d_H \text{Re}(e^{5A_3 - \phi}\psi_\pm^1) = 4e^{4A_3 - \phi}\text{Im}\psi_\mp^1 \ , \tag{4.2.10b}$$

$$d_H(e^{5A_3 - \phi}\psi_\pm^2) = -4ie^{4A_3 - \phi}\psi_\mp^2 \ , \tag{4.2.10c}$$

$$\pm \frac{1}{8}e^\phi *_3 \lambda F = dA_3 \wedge \text{Im}\psi_\pm^1 + e^{-A_3}\text{Re}\psi_\mp^1 \ , \tag{4.2.10d}$$

$$dA_3 \wedge \text{Re}\psi_\mp^1 = 0 \ , \tag{4.2.10e}$$

$$(\psi_+^{1,2}, \overline{\psi_-^{1,2}}) = -\frac{i}{2} \ ; \tag{4.2.10f}$$

again with the upper sign for IIA, and the lower for IIB.

One could ask why we have used this apparently cumbersome procedure of starting from a Mink$_6$ vacuum and then extract the conditions for an AdS$_7$ vacuum, instead of starting directly from the system (3.2.4) written for an AdS$_7$ vacuum. The reason is given by the fact that, for an AdS$_7$ vacuum, it is very difficult to show that the pairing equations are redundant or, in other words, that we are in a lucky case. This represents one of the most unhappy features of the system (3.2.4) that we already described at the end of Chapter 3 (and in the Introduction too): a criterion is not known, at the moment, to determine if a particular class of solutions lives in an ugly, lucky or good case. Finding such a criterion could be very important in order to extend the use of the system (3.2.4) to a broader family of solutions.

We pass now to describe how we can solve the algebraic constraints that follow from the definition of $\psi^{1,2}$ in (4.2.9).

4.2.2 Parameterization of the pure spinors

To solve the constraints contained in (4.2.9) we start by considering first the simpler case $\chi_1 = \chi_2$; the more interesting case $\chi_1 \neq \chi_2$ is very simple but a little bit tedious and so we will just discuss the idea and we quote the final result. We will use the Pauli matrices σ_i as gamma matrices, and use $B_3 = \sigma_2$ as a conjugation matrix (so that $B_3 \sigma_i = -\sigma_i^t B_3 = -\sigma_i^* B_3$). We will define

$$\chi^c \equiv B_3 \chi^* \ , \qquad \overline{\chi} \equiv \chi^t B_3 \ ; \tag{4.2.11}$$

notice that $\chi^{c\dagger} = \chi^t B_3^\dagger = \overline{\chi}$.

We will now evaluate $\psi^{1,2}$ in (4.2.9) when $\chi_1 = \chi_2 \equiv \chi$; as we noted in section 4.2.1, χ is normalized to one. Notice first a general point about the Clifford map $\alpha_k =$

$\frac{1}{k!}\alpha_{i_1\ldots i_k}dx^{i_1}\wedge\ldots\wedge dx^{i_k}\mapsto \not\alpha_k\equiv\frac{1}{k!}\alpha_{i_1\ldots i_k}\gamma^{i_1\ldots i_k}$ in three dimensions (and, more generally, in any odd dimension). After applying the Clifford map on a (poly)form ω we have a very simple way of computing the Hodge dual of ω: one can show that the following relation holds in three euclidean dimensions

$$-i*\lambda\to 1\ ,\tag{4.2.12}$$

and so we see that the Hodge dual of α is related in a very simple way to α when we consider the corresponding bispinors. Since in three dimensions the Hodge dual exchanges odd-forms with even-forms, we see that when evaluating $\psi^{1,2}$, we can give the corresponding forms as an even form, or as an odd form, or as a mix of the two.

Let us first consider $\chi\otimes\chi^\dagger$. We can choose to express it as an odd form. In its Fierz expansion, both its one-form part and its three-form part are a priori non-zero; we can parameterize them as

$$\chi\otimes\chi^\dagger=\frac{1}{2}(\not e_3-i\mathrm{vol}_3)\ .\tag{4.2.13}$$

e_3 is clearly a real vector, whose name has been chosen for later convenience. The fact that the three-form part is simply $-\frac{i}{2}\mathrm{vol}_3$ follows from $||\chi||=1$. Notice also that

$$\not e_3\chi=\sigma_i\chi e_3^i=\sigma_i\chi\chi^\dagger\sigma^i\chi=\frac{1}{2}(-\not e_3-3i\mathrm{vol}_3)\chi\quad\Rightarrow\quad \not e_3\chi=\chi\tag{4.2.14}$$

where we have used (4.2.13), and that $\sigma_i\alpha_k\sigma^i=(-)^k(3-2k)\alpha_k$ on a k-form. (4.2.14) also implies that e_3 has norm one. Coming now to $\chi\otimes\overline{\chi}$, we notice that the three-form part in its Fierz expansion is zero, since $\overline{\chi}\chi=\chi^t B_3\chi=0$. The one-form part is now a priori no longer real; so we write

$$\chi\otimes\overline{\chi}=\frac{1}{2}(\not e_1+i\not e_2)\ .\tag{4.2.15}$$

Similar manipulations as in (4.2.14) show that $(\not e_1+i\not e_2)\chi=0$; using this, we get that

$$e_i\cdot e_j=\delta_{ij}\ .\tag{4.2.16}$$

In other words, $\{e_i\}$ is a vielbein, as notation would suggest.

Two spinors

Moving to consider the more general case with $\chi_1\neq\chi_2$ one could proceed as follow: we have seen that a single three-dimensional spinor defines a vielbein, it is therefore natural (and correct) to assume that χ_1 and χ_2 define a pair of vielbeins $\{e_i^1\}$ and $\{e_i^2\}$ respectively. Even if such a way of proceeding would be perfectly legal, it is not the best way of performing the computation: indeed one has two different vielbeins defined on the same manifold $\mathcal{M}_3$, and in the SUSY equations would appear both. Two vielbeins on the

same manifolds are not independent and they are related by a change of basis. Working in this way one would obtain a system of equations where the variables appearing into the equations are not independent but related in a hidden way.

For this reason we prefer to use a different strategy: we take a third spinor χ which, in a certain sense, is an average between the two χ_1 and χ_2 and we consider the single vielbein $\{e_i\}$ generated by it. In this way one can parametrize the bispinors $\psi^{1,2}$ by using this vielbein and by using three angles $(\theta_1, \theta_2, \psi)$ that, in a sense, expresses all the changes of basis that we need. Proceeding in this way we finally parametrizes our bispinors $\psi^{1,2}$ as

$$\psi_+^1 = \frac{e^{i\theta_1}}{2} \left[\cos(\psi) + e_1 \wedge (-ie_2 + \sin(\psi)e_3)\right] \ , \quad \psi_-^1 = \frac{e^{i\theta_1}}{2} \left[e_3 - i\sin(\psi)e_2 - i\cos(\psi)\text{vol}_3\right] \ ;$$
$$(4.2.17a)$$

$$\psi_+^2 = \frac{e^{i\theta_2}}{2} \left[\sin(\psi) - (ie_2 + \cos(\psi)e_1) \wedge e_3\right] \ , \quad \psi_-^2 = \frac{e^{i\theta_2}}{2} \left[e_1 + i\cos(\psi)e_2 - i\sin(\psi)\text{vol}_3\right] \ .$$
$$(4.2.17b)$$

4.2.3 Analysis of the equations and topology of the solutions

Armed with the parameterization (4.2.17), we are now ready to attack the system (4.2.10) for $\text{AdS}_7 \times M_3$ solutions. In the first subsection we will analyze the equations and obtaining the local form of the solutions and the differential equations that they have to satisfy for being supersymmetric. Then, in the second subsection we will discuss global issues about the topology of the local solutions, we will show in particular that *compact* solutions can be found.

Analysis of the equations

We will start by looking at the equations in (4.2.10) that do not involve any fluxes. These are (4.2.10e), and the lowest-component form part of (4.2.10a), (4.2.10b) and (4.2.10c).

First of all, we can observe quite quickly that the IIB case cannot possibly work. (4.2.10a), (4.2.10b) and (4.2.10c) all have a zero-form part coming from their right-hand side, which, using (4.2.17), read respectively

$$\cos(\psi)\cos(\theta_1) = 0 \ , \qquad \cos(\psi)\sin(\theta_1) = 0 \ , \qquad \sin(\psi)e^{i\theta_2} = 0 \ . \qquad (4.2.18)$$

These cannot be satisfied for any choice of ψ, θ_1 and θ_2. Thus we can already exclude the IIB case. This is one of the places where the power of GCG is more manifest in the entire literature: using GCG one can obtain, performing a trivial computation, the very general result that in type IIB AdS_7 vacua are forbidden.[6]

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[6]In [52] all the 6d SCFTs that can be constructed in F-theory have been classified. However it is well-

Having disposed of IIB so quickly, we will devote the rest of our analysis to IIA. Actually, as we already discussed in section (4.1.1), we can get something new only with non-zero Romans mass, $F_0 \neq 0$. This is because for $F_0 = 0$ we can lift to eleven-dimensional supergravity where we have only the Freund-Rubin solution. We will thus focus on $F_0 \neq 0$, and use the case $F_0 = 0$ as a control.

In IIA, the lowest-degree equations of (4.2.10a), (4.2.10b) and (4.2.10c) are one-forms; they are less dramatic than (4.2.18), but still rather interesting. Using (4.2.17), after some manipulations we get

$$e_1 = -\frac{1}{4}e^A \sin(\psi)d\theta_2 \ , \qquad e_2 = \frac{1}{4}e^A(d\psi + \tan(\psi)d(5A - \phi)) \ ,$$
$$e_3 = \frac{1}{4}e^A \left(-\cos(\psi)d\theta_1 + \frac{\cot(\theta_1)}{\cos(\psi)}d(5A - \phi) \right) \ , \tag{4.2.19}$$

and

$$xdx = (1 + x^2)d\phi - (5 + x^2)dA \ , \tag{4.2.20}$$

where

$$x \equiv \cos(\psi)\sin(\theta_1) \ , \tag{4.2.21}$$

and we have dropped the subscript $_3$ on the warping function: $A \equiv A_3$ from now on. Notice that (4.2.19) determine the vielbein. Usually (i.e. in other dimensions), the geometrical part of the differential system coming from supersymmetry gives the *derivative* of the forms defining the metric. In this case, the forms themselves are determined in terms of derivatives of the angles appearing in our parameterizations. This will allow us to give a more complete and concrete classification than is usually possible. In other words, the AdS$_7$ case can be considered as a particularly lucky case among the lucky cases: the equations for supersymmetry, when are written in terms of G-structures, not only exclude the possibility of having a type IIB supersymmetric vacuum, but also in type IIA give the general form of the metric in a very simple way.

We still have (4.2.10e). Notice that (4.2.10a) allows to write it as $dA \wedge d(e^{3A - \phi}x) = 0$. Using also (4.2.20), we get

$$dA \wedge d\phi = 0 \ . \tag{4.2.22}$$

This means that ϕ is functionally dependent on A:[7]

$$\phi = \phi(A) \ . \tag{4.2.23}$$

(4.2.20) then means that x too is functionally dependent on A: $x = x(A)$.

_____

known that F-theory goes beyond type IIB supergravity, therefore their results are *not* in contradiction with the absence of supersymmetric AdS$_7$ solutions in type IIB supergravity.

[7](4.2.23) excludes the case where A is constant in a region. However, it is easy to see that this case cannot work. Indeed, in this case (4.2.20) can be integrated as $e^\phi \propto \sqrt{1 - x^2}$, which is incompatible with (4.2.24) below.

Let us now move to (4.2.10d), which gives us the RR flux. First we compute F_0 from (4.2.10d):

$$F_0 = 4xe^{-A-\phi}\frac{3 - \partial_A\phi}{5 - 2x^2 - \partial_A\phi} \ . \tag{4.2.24}$$

The Bianchi identity for F_0 says that it should be (piecewise) constant. It will thus be convenient to use (4.2.24) to eliminate $\partial_A\phi$ from our equations.

Before we go on to analyze our equations, let us also introduce the new angle β by

$$\sin^2(\beta) = \frac{\sin^2(\psi)}{1 - x^2} \ . \tag{4.2.25}$$

We can now use x as defined in (4.2.21) to eliminate θ_1, and β to eliminate ψ. This turns out to be very convenient in the following, especially in our analysis of the metric in section 4.2.5 below (which was our original motivation to introduce β).

After these preliminaries, let us give the expression for F_2 as one obtains it from (4.2.10d):

$$F_2 = \frac{1}{16}\sqrt{1 - x^2}e^{A-\phi}(xe^{A+\phi}F_0 - 4)\mathrm{vol}_{S^2} \ , \tag{4.2.26}$$

where

$$\mathrm{vol}_{S^2} = \sin(\beta)d\beta \wedge d\theta_2 \tag{4.2.27}$$

is formally identical to the volume form for a round S^2 with coordinates $\{\beta, \theta_2\}$. We will see later that this is no coincidence.

Finally, let us look at the three-form part of (4.2.10a), (4.2.10b) and (4.2.10c). One of them can be used to determine H:

$$H = \frac{1}{8}e^{2A}\sqrt{1 - x^2}\frac{6 + xF_0e^{A+\phi}}{4 + xF_0e^{A+\phi}}dx \wedge \mathrm{vol}_{S^2} \ , \tag{4.2.28}$$

while the other two turn out to be identically satisfied.

Our analysis is not over: we should of course now impose the equation of motion, and the Bianchi identities for our fluxes. The equation of motion for F_2, $d * F_2 + H * F_0 = 0$, follows automatically from (4.2.10d), much as it happens in the pure spinor system for $\mathrm{AdS}_4 \times M_6$ solutions [10]. We should then impose the Bianchi identity for F_2, which reads $dF_2 - HF_0 = 0$ (away from sources). This does not follow manifestly from (4.2.10d), but in fact it is a consequence of the explicit expressions (4.2.24), (4.2.26) and (4.2.28) above. When $F_0 \neq 0$, it also implies that the B field such that $H = dB$ can be locally written as

$$B_2 = \frac{F_2}{F_0} + b \tag{4.2.29}$$

for a closed two-form b. Using a gauge transformation, it can be assumed to be proportional (by a constant) to vol_{S^2}; we then have that it is a constant, $\partial_A b = 0$.

The equation of motion for H, which reads for us $d(e^{7A-2\phi} *_3 H) = e^{7A} F_0 *_3 F_2$ (again away from sources), is also automatically satisfied, as shown in general in [56]. Finally, since we have checked all the conditions for preserved supersymmetry, the Bianchi identities and the equations of motion for the fluxes, the equations of motion for the dilaton and for the metric will now follow [37].

4.2.4 The system of ODEs

Let us now sum up the results of our analysis of (4.2.10). Most of our equations determine some fields: (4.2.19) give the vielbein, and (4.2.24), (4.2.26), (4.2.28) give the fluxes. The only genuine differential equations we have are (4.2.20), and the condition that F_0 should be constant. Recalling that ϕ is functionally dependent on A, (4.2.23), these two equations can be written as

$$\partial_A \phi = 5 - 2x^2 + \frac{8x(x^2 - 1)}{4x - F_0 e^{A+\phi}} \ , \tag{4.2.30a}$$

$$\partial_A x = 2(x^2 - 1) \frac{x e^{A+\phi} F_0 + 4}{4x - F_0 e^{A+\phi}} \ . \tag{4.2.30b}$$

We thus have reduced the existence of a supersymmetric solution of the form $\text{AdS}_7 \times M_3$ in IIA to solving the system of ODEs (4.2.30). It might look slightly unsettling that we are essentially using at this point A as a coordinate, which might not always be a wise choice (since A might not be monotonic). For that matter, our analysis has so far been completely local; we will start looking at global issues in section 4.2.5, and especially 4.2.6.

Unfortunately we have not been able to find analytic solutions to (4.2.30), other than in the $F_0 = 0$ case (which we will see in section 4.2.9). For the more interesting $F_0 \neq 0$ case, we can gain some intuition by noticing that the system becomes autonomous (i.e. it no longer has explicit dependence on the "time" variable A) if one defines $\tilde{\phi} \equiv \phi + A$. The system for $\{\partial_A \tilde{\phi}, \partial_A x\}$ can now be thought of as a vector field in two dimensions; we plot it in figure 4.1.

We will study the system (4.2.30) numerically in the final part of this Chapter. Before we do that, we should understand what boundary conditions we should impose. We will achieve this by analyzing global issues about our setup, that we have so far ignored.

4.2.5 Metric

The metric

$$ds^2_{M_3} = e_a e_a \tag{4.2.31}$$
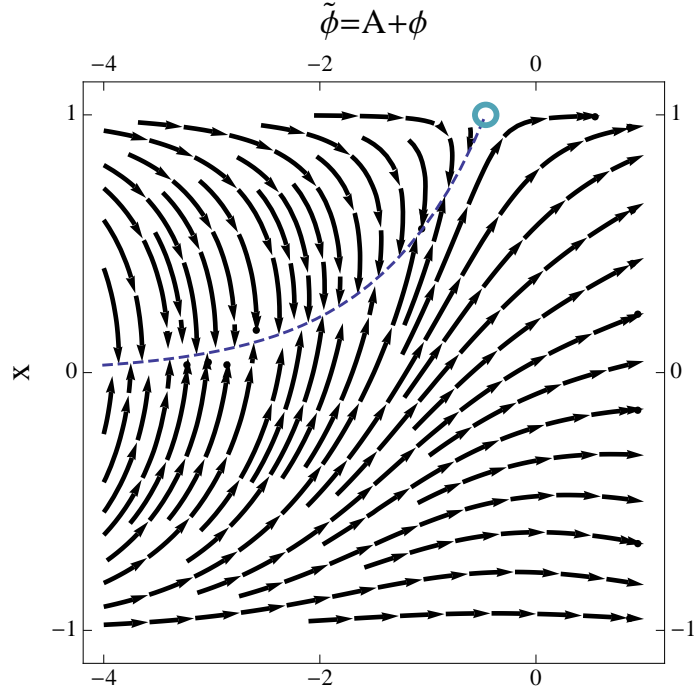
Figure 4.1: A plot of the vector field induced by (4.2.30) on $\{\tilde{\phi} \equiv \phi + A, x\}$, for $F_0 = 40/2\pi$ (in agreement with flux quantization, (4.2.49) below). The green circle represents the point $\{\phi + A = \log(4/F_0), x = 1\}$, whose role will become apparent in section 4.2.7. The dashed line represents the locus along which the denominators in (4.2.30) vanish.

following from (4.2.19) looks quite complicated. However, it simplifies enormously if we rewrite it in terms of β in (4.2.25):[8]

$$ds^2_{M_3} = e^{2A}(1 - x^2)\left[\frac{16}{(4x - e^{A+\phi}F_0)^2}\, dA^2 + \frac{1}{16}\, ds^2_{S^2}\right]\, , \qquad ds^2_{S^2} = d\beta^2 + \sin^2(\beta)d\theta^2_2\, .$$
(4.2.32)

Without any Ansatz, the metric has taken the form of a fibration of a round S^2, with coordinates $\{\beta, \theta_2\}$, over an interval with coordinate A. Notice that none of the scalars appearing in (4.2.32) (and indeed in the fluxes (4.2.24), (4.2.26), (4.2.28)) were originally intended as coordinates, but rather as functions in the parameterization of the pure spinors $\psi^{1,2}$. Usually, one would then need to introduce coordinates independently, and to make an Ansatz about how all functions should depend on those coordinates, sometimes imposing the presence of some particular isometry group in the process.

Here, on the other hand, the functions we have introduced are suggesting themselves as coordinates to us rather automatically. Since so far our expressions for the metric

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[8]In fact, the definition of β was originally found by trying to understand the global properties of the metric (4.2.31). Looking at a slice $x =$const, one finds that the metric in $\{\theta_1, \theta_2\}$ has constant positive curvature; the definition of β becomes then natural. Nontrivially, this definition also gets rid of non-diagonal terms of the type $dAd\theta_1$ that would arise from (4.2.19).

and fluxes were local, we are free to take their suggestion. We will take β to be in the range $[0, \pi]$, and θ_2 to be periodic with period 2π, so that together they describe an S^2 as suggested by (4.2.32), and also by the two-form (4.2.27) that appeared in (4.2.26), (4.2.28).[9]

It is not hard to understand why this S^2 has emerged. The holographic dual of any solutions we might find is a $(1, 0)$ CFT in six dimensions. Such a theory would have $SU(2)$ R-symmetry; an $SU(2)$ isometry group should then appear naturally on the gravity side as well. This is what we are seeing in (4.2.32).

The fact that the S^2 in (4.2.32) is rotated by R-symmetry also helps to explain a possible puzzle about IIB. Often, given a IIA solution, one can produce a IIB one via T-duality along an isometry. We saw examples of this kind when we describe the T-dual of the Brandhuber and Oz solution in (4.1.2). But the IIB case died very quickly in section 4.2.3 and there are no solutions. Here is how this puzzle is resolved. Since the $SU(2)$ isometry group of the S^2 is an R-symmetry, supercharges transform as a doublet under it. Thus even the strange IIB geometry produced by T-duality along a U(1) isometry of S^2 would not be supersymmetric.

Even though we have promoted β and θ_2 to coordinates, it is hard to do the same for A, which actually enters in the seven-dimensional metric (see (4.2.5)). We would like to be able to cover cases where A is non-monotonic. One possibility would be to use A as a coordinate piecewise. We find it clearer, however, to introduce a coordinate r defined by $dr = 4e^A \frac{\sqrt{1-x^2}}{4x - e^{A+\phi} F_0} dA$, so that the metric now reads

$$ds_{M_3}^2 = dr^2 + \frac{1}{16} e^{2A}(1 - x^2) ds_{S^2}^2 \ . \tag{4.2.33}$$

In other words, r measures the distance along the base of the S^2 fibration. Now A, x and ϕ have become functions of r. From (4.2.30) and the definition of r we have

$$\partial_r \phi = \frac{1}{4} \frac{e^{-A}}{\sqrt{1-x^2}} (12x + (2x^2 - 5) F_0 e^{A+\phi}) \ ,$$
$$\partial_r x = -\frac{1}{2} e^{-A} \sqrt{1-x^2} (4 + x F_0 e^{A+\phi}) \ , \tag{4.2.34}$$
$$\partial_r A = \frac{1}{4} \frac{e^{-A}}{\sqrt{1-x^2}} (4x - F_0 e^{A+\phi}) \ .$$

We have introduced a square root in the system, but notice that $-1 \leq x \leq 1$ already follows from requiring that $ds_{M_3}^2$ in (4.2.32) has positive signature. (We choose the positive branch of the square root.)

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[9]A slight variation is to take $\mathbb{RP}^2 = S^2/\mathbb{Z}_2$ instead of S^2; this will not play much of a role in what follows, except for some solutions with O6-planes that we will mention in sections 4.2.9 and 4.2.10.

Let us also record here that the NS three-form also simplifies in the coordinates introduced in this section:

$$H = -(6e^{-A} + xF_0 e^{\phi})\text{vol}_3 \,, \tag{4.2.35}$$

where vol_3 is the volume form of the metric $ds^2_{M_3}$ in (4.2.33) or (4.2.32).

We have obtained so far that the metric is the fibration of an S^2 (with coordinates (β, θ_2)) over a one-dimensional space. The $SU(2)$ isometry group of the S^2 is to be identified holographically with the R-symmetry group of the $(1,0)$-superconformal dual theory. For holographic applications, we would actually like to know whether the total space of the S^2-fibration can be made compact. We now move to discuss this point.

4.2.6 Compact solutions

To make the fibration compact, one possible strategy would be for r to be periodically identified, so that the topology of M_3 would become $S^1 \times S^2$. This is actually impossible: from (4.2.34) we have

$$\partial_r(xe^{3A-\phi}) = -2\sqrt{1-x^2}e^{2A-\phi} \leq 0 \,. \tag{4.2.36}$$

Now, $xe^{3A-\phi}$ is continuous;[10] for r to be periodically identified, $xe^{3A-\phi}$ should be a periodic function. However, thanks to (4.2.36), it is nowhere-increasing. It also cannot be constant, since x would be ± 1 for all r, which makes the metric in (4.2.32) vanish. Thus r cannot be periodically identified.

We then have to look for another way to make M_3 compact. The only other possibility is in fact to shrink the S^2 at two values of r, which we will call r_N and r_S; the topology of M_3 would then be S^3. The subscripts stand for "north" and "south"; we can visualize these two points as the two poles of the S^3, and the other, non-shrunk copies of S^2 over any $r \in (r_N, r_S)$ to be the "parallels" of the S^3. Of course, since (4.2.34) does not depend on r, we can assume without any loss of generality that $r_N = 0$.

We will now analyze this latter possibility in detail.

4.2.7 Local analysis around poles

We have just suggested to make M_3 compact by having the S^2 fiber over an interval $[r_N, r_S]$, and by shrinking it at the two extrema. In this case M_3 would be homeomorphic to S^3.

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[10]This might not be fully obvious in presence of D8-branes, but we will see later that it is true even in that case, basically because ϕ is a physical field, and A and x appear as coefficients in the metric.

To realize this idea, from (4.2.33) we see that x should go to 1 or -1 at the two poles r_N and r_S. To make up for the vanishing of the $\sqrt{1-x^2}$'s in the denominators in (4.2.34), we should also make the numerators vanish. This is accomplished by having $e^{A+\phi} = \pm 4/F_0$ at those two poles (which is obviously only possible when $F_0 \neq 0$). We can now also see that $\partial_r x \sim -4e^{-A}\sqrt{1-x^2} \leq 0$ around the poles. Since, as we noticed earlier, $-1 \leq x \leq 1$, x should actually be 1 at r_N, and -1 at r_S. Summing up:

$$\left\{x = 1,\ e^{A+\phi} = \frac{4}{F_0}\right\} \quad \text{at } r = r_N\ , \qquad \left\{x = -1,\ e^{A+\phi} = -\frac{4}{F_0}\right\} \quad \text{at } r = r_S\ . \quad (4.2.37)$$

Since we made both numerators and denominators in (4.2.34) vanish at the poles, we should be careful about what happens in the vicinity of those points. We want to study the system around the boundary conditions (4.2.37) in a power-series approach. (The same could also be done directly with (4.2.30).) Let us first expand around r_N. As mentioned earlier, thanks to translational invariance in r we can assume $r_N = 0$ without any loss of generality. We get

$$\phi = -A_0^+ + \log\left(\frac{4}{F_0}\right) - 5e^{-2A_0^+}r^2 + \frac{172}{9}e^{-4A_0^+}r^4 + O(r)^6\ ,$$

$$x = 1 - 8e^{-2A_0^+}r^2 + \frac{400}{9}e^{-4A_0^+}r^4 + O(r)^6\ , \qquad\qquad (4.2.38)$$

$$A = A_0^+ - \frac{1}{3}e^{-2A_0^+}r^2 - \frac{4}{27}e^{-4A_0^+}r^4 + O(r)^6\ .$$

A_0^+ here is a free parameter. The way it appears in (4.2.38) is explained by noticing that (4.2.34) is symmetric under

$$A \rightarrow A + \Delta A\ , \qquad \phi \rightarrow \phi - \Delta A\ , \qquad x \rightarrow x\ , \qquad r \rightarrow e^{\Delta A}r\ . \qquad (4.2.39)$$

Applying (4.2.38) to (4.2.33), and setting for a moment $r_N = 0$, we find that the metric has the leading behavior

$$ds^2_{M_3} = dr^2 + r^2 ds^2_{S^2} + O(r)^4 = ds^2_{\mathbb{R}^3} + O(r)^4\ . \qquad (4.2.40)$$

This means that the metric is regular around $r = r_N$. The expansion of the fluxes (4.2.26), (4.2.28) is

$$F_2 = -\frac{10}{3}F_0 e^{-A_0^+}r^3 \text{vol}_{S^2} + O(r)^5\ , \qquad H = -10e^{-A_0^+}r^2 dr \wedge \text{vol}_{S^2} + O(r)^3\ . \quad (4.2.41)$$

As for the B field, recall that it can be written as in (4.2.29). (4.2.41) shows that around $r = r_N = 0$, the term F_2/F_0 is regular as it is, without the addition of b; this suggests that one should set $b = 0$. To make this more precise, consider the limit

$$\lim_{r \to 0} \int_{\Delta_r} H = \lim_{r \to 0} \int_{S^2_r} B_2 \qquad (4.2.42)$$

where Δ_r is a three-dimensional ball such that $\partial\Delta_r = S_r^2$. In (4.2.29), the first term goes to zero because $x \to 1$; so the limit is equal to $\int_{S^2} b$, which is constant. This constant signals a delta in H. So we are forced to conclude that

$$b = 0 \tag{4.2.43}$$

near the pole. (However, we will see in section 4.2.8 that b can become non-zero if one crosses a D8 while going away from the pole.)

To be more precise, (4.2.43) should be understood up to gauge transformations. B is not a two-form, but a 'connection on a gerbe', in the sense that it transforms non-trivially on chart intersections: on $U \cap U'$, $B_U - B_{U'}$ can be a 'small' gauge transformation $d\lambda$, for λ a 1-form, or more generally a 'large' gauge transformation, namely a two-form whose periods are integer multiples of $4\pi^2$. In our case, if we cover S^3 with two patches U_N and U_S, around the equator we can have $B_N - B_S = N\pi\mathrm{vol}_{S^2}$. In this case $\int_{S^3} H = B_N - B_S = N\pi\mathrm{vol}_{S^2} = (4\pi^2)N$, in agreement with flux quantization for H. Thus $b = 0$ is also gauge equivalent to any integer multiple of $\pi\mathrm{vol}_{S^2}$. In practice, however, we will prefer to work with $b = 0$ around the poles, and perform a gauge transformation whenever

$$\hat{b}(r) \equiv \frac{1}{4\pi} \int_{S_r^2} B_2 \tag{4.2.44}$$

gets outside the "fundamental region" $[0, \pi]$. In other words, we will consider $\hat{b}$ to be a variable with values in $[0, \pi]$, and let it begin and end at 0 at the two poles. $\hat{b}$ will then wind an integer number N of times around $[0, \pi]$, and this will make sure that $\int_{S^3} H = (4\pi^2)N$, thus taking care of flux quantization for H.

So far we have discussed the expansion around the north pole; a similar discussion holds for the expansion around the south pole r_S. The expressions that replace (4.2.38), (4.2.40), (4.2.41) can be obtained by using the symmetry of (4.2.34) under

$$x \to -x , \qquad F_0 \to -F_0 , \qquad r \to -r . \tag{4.2.45}$$

The free parameter A_0^+ can now be changed to a possibly different free parameter A_0^-.

We have hence checked that the boundary conditions (4.2.37) are compatible with our system (4.2.34), and that they give rise to a regular metric at the poles.

4.2.8   D8

There is one more ingredient that we will need in section to exhibit compact solutions: brane sources. In presence of branes the metric cannot be called regular: their gravitational backreaction will give rise to a singularity. A random singularity would call into

question the validity of a solution, since the curvature and possibly the dilaton[11] would diverge there, making the supergravity approximation untrustworthy. We are however sure of the existence of D-branes, in spite of the singularities in their geometry, because we have an open string realization for them.

D8-branes in particular are even more benign, in a way, because the singularity manifests itself simply as a discontinuity in the derivatives of the coefficients in the metric. In general relativity, such a discontinuity would be subject to the so-called Israel junction conditions [58], which are a consequence of the Einstein equations. As we mentioned earlier, in our case, however, supersymmetry guarantees that the equations of motion for the dilaton and metric are automatically satisfied [37]. Hence, the conditions on the first derivatives will follow from imposing continuity of the fields and supersymmetry.

Let us be more concrete. We will suppose we have a stack of n_{D8} D8-branes, possibly with a worldvolume gauge field-strength f_2 (not to be confused with the RR field-strength F_2), which induces a D6-brane charge distribution on it. The Bianchi identity for such an object reads

$$d_H F = \frac{1}{2\pi} n_{\mathrm{D8}} e^{\mathcal{F}} \delta \qquad \Rightarrow \qquad d\tilde{F} = \frac{1}{2\pi} n_{\mathrm{D8}} e^{2\pi f_2} \delta \qquad (\delta \equiv dr\delta(r)) \ . \qquad (4.2.46)$$

As usual $\mathcal{F} = B_2 + 2\pi f_2$; recall from section 4.2.1 that $F = F_0 + F_2$; and likewise we have defined

$$\tilde{F} \equiv e^{-B_2} F = F_0 + (F_2 - B_2 F_0) \ . \qquad (4.2.47)$$

In other words, $\tilde{F} = F_0 + \tilde{F}_2$, with $\tilde{F}_2 = F_2 - B_2 F_0$. Since $\tilde{F}_2$ is closed away from sources, it makes sense to define

$$n_2 \equiv \frac{1}{2\pi} \int_{S^2} \tilde{F}_2 \ . \qquad (4.2.48)$$

Flux quantization then requires n_2 to be an integer, and that

$$F_0 = \frac{n_0}{2\pi} \ , \qquad (4.2.49)$$

with n_0 an integer. (We are working in string units where $l_s = 1$.) Integrating now (4.2.46) across the magnetized stack of D8's gives

$$\Delta n_0 = n_{\mathrm{D8}} \ , \qquad \Delta \tilde{F}_2 = f_2 \Delta n_0 \ . \qquad (4.2.50)$$

All physical fields should be continuous across the D8 stack. For example, $\Delta\phi = 0$. Also, the coefficients of the metric should not jump; in particular, from (4.2.5), we see that $\Delta A = 0$. Also, since x appears in front of $ds_{S^2}^2$ in (4.2.33), we should have $\Delta x = 0$.

Imposing that the B field does not jump is trickier. A first caveat is that B would actually be allowed to jump by a gauge transformation, as discussed in section 4.2.7.

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[11]In presence of Romans mass, the string coupling is bounded by the inverse radius of curvature in string units: $e^\phi \propto \frac{l_s}{R_{\mathrm{curv}}}$, and is actually generically of the order of the bound [57].

However, we find it less confusing to put the intersection between the charts U_{N} and U_{S} away from the D8's, and to treat $\int_{S^2} B_2$ as a periodic variable as described in section 4.2.7.

Thus we will simply impose that B does not jump. First, recall that it can be written as in (4.2.29), when $F_0 \neq 0$. The b term was shown in (4.2.43) to be vanishing near the pole, but we will soon see that this conclusion is not valid between D8's. In fact, it is connected to the flux integer n_2 defined in (4.2.48): from (4.2.29) we have

$$\tilde{F}_2 = -F_0 b \; ; \tag{4.2.51}$$

integrating this on S^2, we get $2\pi n_2 = -F_0 \int_{S^2} b$, or in other words

$$b = -\frac{n_2}{2F_0}\mathrm{vol}_{S^2} \; . \tag{4.2.52}$$

We can use our result (4.2.26) for F_2; for this section, it will be convenient to define

$$p \equiv \frac{1}{16}x\sqrt{1 - x^2}e^{2A} \; , \qquad q \equiv \frac{1}{4}\sqrt{1 - x^2}e^{A - \phi} \; , \tag{4.2.53}$$

so that

$$F_2 = (pF_0 - q)\mathrm{vol}_{S^2} \; . \tag{4.2.54}$$

From this and (4.2.52) we now have

$$B_2 = \left(p - \frac{q}{F_0} - \frac{n_2}{2F_0} \right)\mathrm{vol}_{S^2} \; . \tag{4.2.55}$$

Let us call n_0, n_2 the flux integers on one side of the D8 stack, and n_0', n_2' the fluxes on the other side. Let us at first assume that both n_0 and n_0' are non-zero. Then, equating B on the two sides, we see that p cancels out, and we get

$$\frac{1}{n_0} \left(q + \frac{1}{2}n_2 \right) = \frac{1}{n_0'} \left(q + \frac{1}{2}n_2' \right) \; , \tag{4.2.56}$$

or in other words

$$q|_{r=r_{\mathrm{D8}}} = \frac{n_2'n_0 - n_2n_0'}{2(n_0' - n_0)} \; , \tag{4.2.57}$$

with q defined as in (4.2.53). Notice that, in (4.2.29), the term F_2/F_0 and b can both separately jump, while the whole B_2 is staying continuous. For this reason, as we anticipated in section 4.2.7, the conclusion $b = 0$ (which implies $n_2 = 0$ by (4.2.52)) will hold near the poles, but can cease to hold after one crosses a D8. (4.2.57) is also satisfying in that it is symmetric under exchange $\{n_0, n_2\} \leftrightarrow \{n_0', n_2'\}$. Notice also that, under a gauge transformation for the B field, $n_2 \to n_2 + n_0\Delta B$, $n_2' \to n_2' + n_0'\Delta B$, and (4.2.57) remains unchanged.

A constraint on the discontinuity should also come from the F_2 Bianchi identity (4.2.46). Using (4.2.54), we see that the only discontinuities are coming from the jump in F_0, so that we get

$$d_H F = \Delta F_0 (1 + p\mathrm{vol}_{S^2})\delta = \Delta F_0 \, e^{p\mathrm{vol}_{S^2}}\delta \ . \tag{4.2.58}$$

Comparing this with (4.2.46) we see that $\mathcal{F} = p\mathrm{vol}_{S^2}$. It also follows that

$$d\tilde{F}_2 = \Delta F_0 (-B_2 + p\mathrm{vol}_{S^2})\delta = \frac{\Delta F_0}{F_0}\left(q + \frac{1}{2}n_2\right)\mathrm{vol}_{S^2} \ . \tag{4.2.59}$$

The expression on the right-hand side is not ambiguous thanks to (4.2.54). Comparing (4.2.59) with (4.2.46) again, we see that $f_2 = \frac{1}{F_0}\left(q + \frac{n_2}{2}\right)$. Going back to (4.2.50), we learn that

$$\frac{\Delta n_2}{\Delta n_0} = \frac{1}{n_0}\left(q + \frac{1}{2}n_2\right) \ . \tag{4.2.60}$$

This is actually nothing but (4.2.57) again.

(4.2.59) shows that our D8 is actually also charged under F_2, and thus that it is actually a D8/D6 bound state.

Finally, in our analysis so far we have left out the case where F_0 is zero on one of the sides of the D8 stack, say the right side, so that $n_0' = 0$. This time we cannot apply (4.2.55) on the right side of the D8. An expression for B in this case will be given in (4.2.68) below. Imposing continuity of B this time does not lead to (4.2.57), but to a different condition in terms of the integration constants appearing in (4.2.68). However, the Bianchi identity for F_2 can still be applied on the left side of the D8, where $F_0 \neq 0$; this still leads to (4.2.57). In other words, in this case we have (4.2.57) plus an extra condition imposing continuity of B. This will be important in our example with two D8's in section 4.2.11.

Let us summarize the results of this section. We have obtained that one can insert D8's in our setup, provided their position r_{D8} is such that the condition (4.2.57) is satisfied. When F_0 is non-zero on both sides of the D8, this ensures that the Bianchi for F_2 is satisfied, and that B is continuous. In the special case where $F_0 = 0$ on one side, continuity of B has to be imposed independently.

4.2.9 Explicit solutions: review of the $F_0 = 0$ solution

We are now able to discuss some explicit AdS$_7$ vacua. To start with we will review the solution one can get for $F_0 = 0$.

As we remarked in section 4.2.3, in the massless case one can always lift to eleven-dimensional supergravity, and there we can only have AdS$_7 \times S^4$ (or an orbifold thereof).

The metric simply reads

$$ds^2_{11} = R^2 \left(ds^2_{\mathrm{AdS_7}} + \frac{1}{4} ds^2_{S^4} \right) , \tag{4.2.61}$$

being R an overall radius. Let us now have a look at how this reduces to IIA. It is not obvious whether the reduction will preserve any supersymmetry; but, as we will now see, this can be arranged.

To reduce, we have to choose an isometry. Since S^4 has Euler characteristic $\chi = 2$, like any even-dimensional sphere, any vector field has at least two zeros, and so our reduction will have at least two points where the dilaton goes to zero; we expect some other strange feature at those two points, and as we will see this expectation is borne out.

How should we choose the isometry? We can think about U(1) isometries on S^d as rotations in $\mathbb{R}^{d+1}$. The infinitesimal generator v is an element of the Lie algebra $\mathfrak{so}(d+1)$, namely an antisymmetric $(d+1) \times (d+1)$ matrix v. Moreover, two such elements v_i that can be related by conjugation, $v_1 = Ov_2O^t$, for $O \in \mathrm{SO}(d+1)$, can be thought of as equivalent. Any antisymmetric matrix can be put in a canonical block-diagonal form where every block is of the form $\begin{pmatrix} 0 & a \\ -a & 0 \end{pmatrix}$, with a an angle. For even d, this implies that there is at least one zero eigenvalue, which corresponds to the fact that there is no vector field without zeros on the sphere. For $d = 4$, we have two angles a_1 and a_2. Our solution can be reduced along any of these vector fields, but we also want the reduction to preserve some supersymmetry. The infinitesimal spinorial action of the vector field we just described is proportional to $a_1\gamma_{12} + a_2\gamma_{34}$. If we demand that this matrix annihilates at least one spinor χ (so that, at the finite level, χ is kept invariant), we get either $a_1 = a_2$ or $a_1 = -a_2$.

To make things more concrete, let us introduce a coordinate system on S^4 adapted to the isometry we just found:

$$ds^2_{S^4} = d\alpha^2 + \sin^2(\alpha)ds^2_{S^3} = d\alpha^2 + \sin^2(\alpha)\left(\frac{1}{4}ds^2_{S^2} + (dy + C_1)^2 \right) , \quad dC_1 = \frac{1}{2}\mathrm{vol}_{S^2} \tag{4.2.62}$$

with $\alpha \in [0, \pi]$. We have written the S^3 metric as a Hopf fibration over S^2; the $1/4$ is introduced so that all spheres have unitary radius. The reduction will now proceed along the vector

$$\partial_y . \tag{4.2.63}$$

We can actually generalize this a bit by considering the orbifold $S^4/\mathbb{Z}_k$, where $\mathbb{Z}_k$ is taken to be a subgroup of the U(1) generated by ∂_y. This is equivalent to multiplying the $(dy + C_1)^2$ term in (4.2.62) by $\frac{1}{k^2}$.

We can now reduce the eleven-dimensional metric (4.2.61), quotiented by the $\mathbb{Z}_k$ we just mentioned, using the string-frame reduction $ds^2_{11} = e^{-\frac{2}{3}\phi}ds^2_{10} + e^{\frac{4}{3}\phi}(dy + C_1)^2$. We

obtain a metric of the form (4.2.5), with

$$e^{2A} = R^2 e^{\frac{2}{3}\phi} = \frac{R^3}{2k}\sin(\alpha) \ , \qquad ds^2_{M_3} = \frac{R^3}{8k}\sin(\alpha)\left(d\alpha^2 + \frac{1}{4}\sin^2(\alpha)ds^2_{S^2}\right) \ . \qquad (4.2.64)$$

We can now compare what we have just obtained with the system of equations (4.2.30), that, putting $F_0 = 0$, can be easily solved explicitly:

$$x = \sqrt{1 - e^{4(A-A_0)}} \ , \qquad \phi = 3A - \phi_0 \qquad (4.2.65)$$

where A_0 and ϕ_0 are two integration constants. This can be seen to be the same as (4.2.64) by taking

$$x = \cos(\alpha) \ , \qquad A_0 = \frac{1}{2}\log\left(\frac{R^3}{2k}\right) \ , \qquad \phi_0 = 3\log R \ . \qquad (4.2.66)$$

The fluxes can now be computed from (4.2.26) and (4.2.28):

$$F_2 = -\frac{1}{2}k\mathrm{vol}_{S^2} \ , \qquad H = -\frac{3}{32}\frac{R^3}{k}\sin^3(\alpha)d\alpha \wedge \mathrm{vol}_{S^2} \ ; \qquad (4.2.67)$$

the B field then can be written as

$$B_2 = \frac{3}{32}\frac{R^3}{k}\left(x - \frac{x^3}{3}\right)\mathrm{vol}_{S^2} + b \qquad (4.2.68)$$

where again b is a closed two-form. The simple result for F_2 in (4.2.67) could be expected from the fact that the metric (4.2.62) is an S^1 fibration over S^2 with Chern class $c_1 = -k$.

However, (4.2.64) might appear problematic for two reasons. First of all, the warping function goes to zero at the two poles $\alpha = 0$, $\alpha = \pi$. Second, $ds^2_{M_3}$ would be singular at the poles even if it were not multiplied by an overall factor $e^{2A} = \frac{R^3}{2k}\sin(\alpha)$, because of the $1/4$ in front of $ds^2_{S^2}$. Indeed, when we expand it around, say, $\alpha = 0$, we find $d\alpha^2 + \frac{\alpha^2}{4}ds^2_{S^2}$; this would be regular without the $1/4$, but as it stands it has a conical singularity.

However, these singularities at the poles have the behavior one expects near a D6. Near the north pole $\alpha = 0$, $ds^2_{M_3}$ in (4.2.64) looks like $ds^2_{M_3} \sim \alpha\left(d\alpha^2 + \frac{1}{4}\alpha^2 ds^2_{S^2}\right)$. In terms of the r variable we used in (4.2.33), this looks like

$$ds^2_{M_3} \sim dr^2 + \left(\frac{3}{4}r\right)^2 ds^2_{S^2} \ . \qquad (4.2.69)$$

Near the ordinary flat-space D6-brane metric, $ds^2_{M_3} \sim \rho^{-1/2}(d\rho^2 + \rho^2 ds^2_{S^2})$, which also looks like (4.2.69) with $r = \frac{4}{3}\rho^{3/4}$.

The presence of D6's could actually be inferred more directly. First of all, we know that D6-branes result from loci where the size of the eleventh dimension goes to zero; this indeed happens at the two poles. Moreover, from the expression of F_2 in (4.2.67),

the integral of F_2 over the S^2 is constant and equal to $-2\pi k$. We can take the S^2 close to the north or the south pole, where it signals the presence of D6-brane charge. More precisely, there are k anti-D6-branes at the north pole and k D6-branes at the south pole.

One crucial difference with the usual D6 behavior, however, is the presence of the NS three-form H. From (4.2.67) we see that it does not vanish near the D6. Rather, it diverges: near the anti-D6 at $r = r_\mathrm{N} = 0$,[12]

$$H \sim r^{-1/3}\mathrm{vol}_3 \ . \tag{4.2.70}$$

We should remember, in any case, that this solution is non-singular in eleven dimensions; the diverging behavior in (4.2.70) is cured by M-theory, just like the divergence of the curvature of (4.2.69) is.

The simultaneous presence of D6's and anti-D6's in a BPS solution might look unsettling at first, since in flat space they cannot be BPS together. It is true that the conditions imposed on the supersymmetry parameters ϵ_i by a D6 and by an anti-D6 brane are incompatible. But in flat space the ϵ_i are constant, while in our present case they are not. The condition changes from the north pole to the south pole; so much so that an anti-D6 is BPS at the north pole, and a D6 is BPS at the south pole. In figure 4.2 we show some parameters for the solution as a function of the r defined in (4.2.33), for uniformity with latter cases. We also show the radius of the transverse sphere, which near the poles has the angular coefficient 3/4 of (4.2.69).

We have obtained this massless IIA solution by reducing the M-theory solution $\mathrm{AdS}_7 \times S^4/\mathbb{Z}_k$, but other orbifolds would be possible as well. One could for example have quotiented by the $\hat{D}_{k-2}$ groups, which would have resulted in IIA in an orientifold by the action of the antipodal map on the S^2. The transverse S^2 would have been replaced by an $\mathbb{RP}^2$; at the poles we would have had O6's together with the k D6's/anti-D6's of the A_k case.

We will see in section 4.2.11 solutions with $F_0 \neq 0$ and without any D6-branes. But we will at first try in the next subsection to introduce F_0 without any D8-branes.

4.2.10 Explicit solutions: massive solution without D8-branes

In section 4.2.9 we reviewed the only solution for $F_0 = 0$, related to $\mathrm{AdS}_7 \times S^4$ by dimensional reduction; it has a D6 and an anti-D6 at the poles of $M_3 \cong S^3$.

We now start looking at what happens in presence of a non-zero Romans mass, $F_0 \neq 0$. We saw in section 4.2.7 that in this case it is possible for the poles to be regular points.

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[12]It is interesting to ask what happens in the Minkowski limit. From (4.2.35) we see that $H = -6e^{-A}\mathrm{vol}_3$; taking $R \to \infty$, e^{-A} tends to zero except than in a region $\alpha \ll R^{-1/3}$, which gets smaller and smaller in the limit.
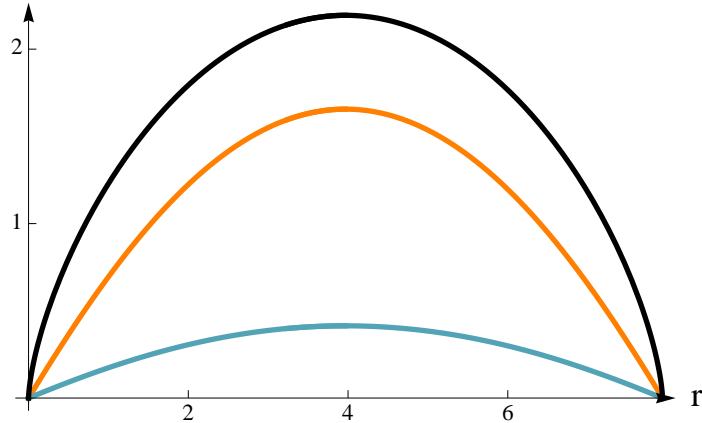
Figure 4.2: Massless solution in IIA. We show here the radius of the S^2 (orange), the warping factor e^{2A} (black; multiplied by a factor 1/20), and the string coupling e^ϕ (green; multiplied by a factor 5). We see that the warping goes to zero at the two poles. The angular coefficient of the orange line can be seen to be 3/4 as in (4.2.69). The two singularities are due to k D6 and k anti-D6 (in this picture, $k = 20$).

It remains to be seen whether those boundary conditions can be joined by a solution of the system (4.2.34).

We can for example impose the boundary condition (4.2.37) at $r = r_{\rm N}$, and evolve numerically towards positive r using (4.2.34). The procedure is standard: we use the approximate power-series solution (4.2.38) from $r = r_{\rm N} = 0$ to a very small r, and then use the values of A, ϕ, x thus found as boundary conditions for a numerical evolution of (4.2.34). One example of solution is shown in figure 4.3(a). It stops at a finite value of r, where it resembles there the south pole behavior of the massless case in figure 4.2; for example, e^A goes to zero at the right extremum.

This is actually easy to understand already from the system, both in (4.2.30) and in (4.2.34). As A and ϕ get negative, they suppress the terms containing F_0, and the system tends to the one for the massless case.

An alternative, and perhaps more intuitive, understanding can be found using the form (4.2.30) of the system, which we drew in figure 4.1 as a vector field flow on the space $\{A+\phi, x\}$. The green circle in that figure represents the point $\{A+\phi = \log(4/F_0), x = 1\}$, which is the appropriate boundary condition for the north pole in (4.2.37). In that figure the 'time' variable is A. From (4.2.38), we see that A has a local maximum at $r = r_{\rm N}$. So the stream in figure 4.1 has to be followed backwards, starting from the green circle at the top. We can see that the integral curve asymptotically approaches $x = -1$, but does not get there in finite 'time'; in other words, $A \to -\infty$. The flow corresponding to the solution in figure 4.3(a) is shown in figure 4.3(b).

(a)   (b)
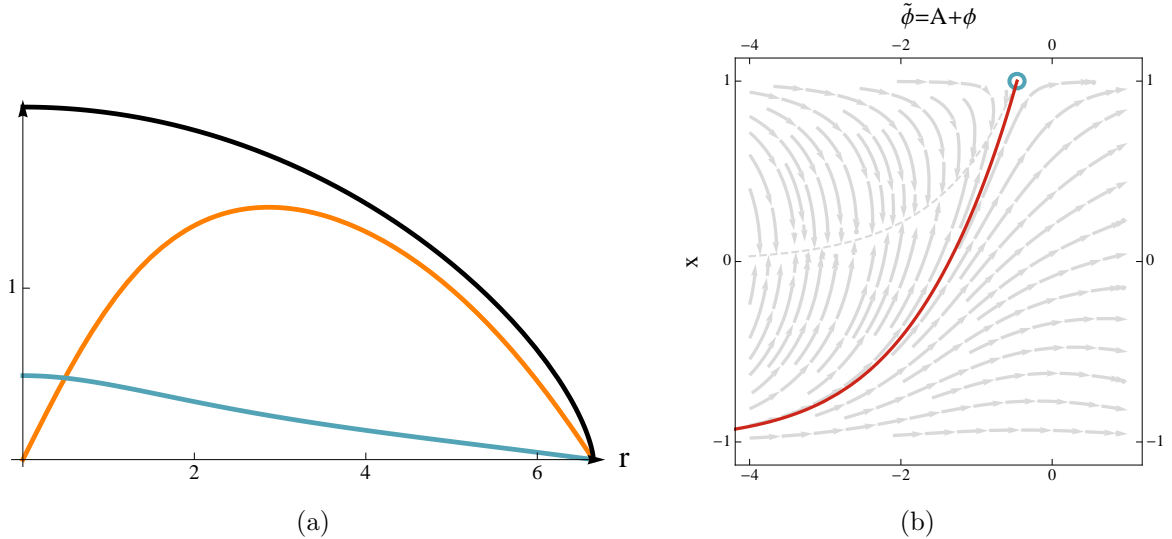
Figure 4.3: Solution for $F_0 = 40/2\pi$. We imposed regularity at the north pole, and evolved towards positive r. In (a) we again plot the radius of the S^2 (orange), the warping factor e^{2A} (black; multiplied by a factor $1/20$), and the string coupling e^ϕ (green; multiplied by a factor 5). With increasing r, the plot gets more and more similar to the one for the massless case in figure 4.2. There is a stack of D6's at the south pole (in this picture, $k = 112$ of them), as in the massless case, although this time it also has a diverging NS three-form H. Notice that the size of the S^2 goes linearly near both poles, but with angular coefficients 1 near the north pole (appropriate for a regular point) and $3/4$ for the south pole (appropriate for a D6, as seen in (4.2.69)). In (b), we see the path described by the solution in the $\{A + \phi, x\}$ plane, overlaid to the vector field shown in figure 4.1.

In the massless case, we saw in section 4.2.9 that the singularities at the poles are actually D6-branes. In this case too we have D6's at the south pole. This is confirmed by considering the integral of F_2 along a sphere S^2 in the limit where it reaches the south pole: it gives a non-zero number. By tuning A_0^+, this can be arranged to be 2π times an integer k, where k is the number of D6-branes at the south pole. The presence of these D6-branes without any anti-D6 is not incompatible with the Bianchi identity $dF_2 - HF_0 = k\delta_{D6}$, because integrating it gives $-F_0 \int H = k$. In other words, the flux lines of the D6's are absorbed by H-flux, as is often the case for flux compactifications. Notice also that these D6's are calibrated; the computation runs along similar lines as the one we presented for the massless solution in section 4.2.9.

To be more precise, the singularity is not the usual D6 singularity, in that there is also a NS three-form H diverging as in (4.2.70). This is consistent with the prediction in [59, Eq. (4.15)] (given there in Einstein frame), and in general with the analysis of [60, 61], which found that it is problematic to have ordinary D6-brane behavior in a massive

$\text{AdS}_7 \times S^3$ setup precisely like the one we are considering here. (In the language of [60], the parameter α of our solution goes to a negative constant; this enables the solution to exist and to evade the global no-go they found, but at the cost of the diverging H in (4.2.70), [59, Eq. (4.15)].) More precisely, the asymptotic behavior we find is the one discovered in [61, Eq. (3.4)].

Thus the singularity at the south pole in figure 4.3 is the same we found in the massless case we saw in section 4.2.9. In that case, the singularity is cured by M-theory. In the present case, the non-vanishing Romans mass prevents us from doing that. However, we still think it should be interpreted as the appropriate response to a D6; for this reason we think it is a physical solution.

So far we have examined what happens when we impose that the north pole is regular. It is also possible to have a D6 and anti-D6 singularity at both poles, as in the previous section, or an O6 at one of the poles (keeping D6's at the other pole). Roughly speaking, this corresponds to a trajectory similar to the one in figure 4.3(b), in which one "misses" the green circle to the left or to the right, respectively. As we have seen, the D6 solution is very similar to the massless one. The O6 solutions also turn out to be very similar to their massless counterpart:[13] near the pole, their asymptotics is $e^A \sim r^{-1/5}$, $e^\phi \sim r^{-3/5}$, $x \sim 1 - r^{4/5}$. This leads to the same asymptotics for the metric as in the massless O6 solution near the critical radius $\rho_0 = g_s l_s$. Once again, however, in the massive case we have a diverging NS three-form; this time $H \sim r^{-3/5}\text{vol}_3$. Finally, in such a case the S^2 is replaced by an $\mathbb{RP}^2$ because of the orientifold action.

4.2.11 Explicit solutions: regular massive solution with D8-branes

We will now examine what happens in presence of D8-branes.

The first possibility that comes to mind is to put all of them together in a single stack. The idea is the following. We once again use the power-series expansion (4.2.38) from $r = r_\text{N} = 0$ to a small r, and use the resulting values of A, ϕ and x as boundary conditions for a numerical evolution of (4.2.34). This time, however, we should stop the evolution at a value of r where (4.2.57) is satisfied. At this point F_0 will change, and (4.2.34) will change as well. Generically, the evolution on the other side of the D8 will lead to a D6 or an O6 singularity, as discussed in section 4.2.10. However, if F_0 is negative, according to (4.2.37), the point $\{x = -1, e^{A+\phi} = -\frac{4}{F_0}\}$ leads to a regular South Pole. Fortunately, our solution still has a free parameter, namely $A_0^+ = A(r_\text{N})$. By fine-tuning this parameter, we can try to reach $\{x = -1, e^{A+\phi} = -\frac{4}{F_0}\}$ and obtain a regular solution.

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[13] In the different setup of [62], an O6 in presence of F_0 gets modified in such a way that its singularity disappears. This does not happen here.

Alternatively, after stopping the evolution from the North Pole to the D8, one can look for a similar solution starting from the South Pole, and then match the two — in the sense that one should make sure that A, ϕ, and x are continuous. One combination of them, namely q, will already match by construction. It is then enough to match two variables, say A and x; this can be done by adjusting A_0^+ and A_0^-.

Naively, however, we face a problem when we try to choose the flux parameters on the two sides of the D8's. We concluded in (4.2.43) that near the poles we should have $b = 0$; this seems to imply, via (4.2.52), that $n_2 = 0$ on both sides of the D8. (4.2.57) would then lead to $q = 0$ on the D8, which can only be true at the poles $x = \pm 1$.

This confusion is easily cleared once we remember that B can undergo a large gauge transformation that shifts it by $k\pi \mathrm{vol}_{S^2}$, as we explained towards the end of section 4.2.7. We saw there that we can keep track of this by introducing the variable $\hat{b}$ in (4.2.44). We now simply have to make sure that $\hat{b}$ winds an integer amount of times N around the fundamental domain $[0, \pi]$; this can be interpreted as the presence of N large gauge transformations, or as the presence of a non-zero quantized flux $N = \frac{1}{4\pi^2} \int H$.

We still face one last apparent problem. It might seem that making sure that $\hat{b}$ winds an integer amount of times requires a further fine-tuning on the solution; this we cannot afford, since we have already used both our free parameters $A_0^\pm$ to make sure all the variables are continuous, and that the poles are regular.

Fortunately, such an extra fine-tuning is in fact not necessary. Let us call (n_0, n_2) the flux parameters before the D8, and (n_0', n_2') after it. For simplicity let us also assume $n_2' = 0$, so that no large gauge transformations are needed on that side. As we remarked at the end of section 4.2.8, $\Delta n_2 = n_2' - n_2 = -n_2$ should be an integer multiple of $\Delta n_0 = n_0' - n_0 = n_{\mathrm{D8}}$: $\Delta n_2 = \mu \Delta n_0$, $\mu \in \mathbb{Z}$. To take care of flux quantization, it is enough to also demand that $n_2 = N n_0$ for N integer. Indeed, from (4.2.49), (4.2.52), (4.2.44), we see that in that case at the North Pole we get $\hat{b} = -\pi N$; since this is an integer multiple of π, it can be brought to zero by using large gauge transformations. Together, the conditions we have imposed determine $n_0' = n_0 \left(1 - \frac{N}{\mu}\right)$.

All this gives a strategy to obtain solutions with one D8 stack. We show one concrete example in figure 4.4. One might find it intuitively strange that the D8-branes are not "slipping" towards the South Pole. The branes back-react on the geometry, bending the S^3, much as a rubber band on a balloon. This by itself, however, would not be enough to prevent them from slipping. Rather, we also have to take into account the Wess–Zumino term in the brane action. This term controls the interaction between the brane and the fluxes and balances the gravitational term which would push the D8s to the pole. At the equilibrium the D8s are stable exactly where they are.

We can also look for a configuration with two stacks of D8-branes, again with regular

(a)   (b)
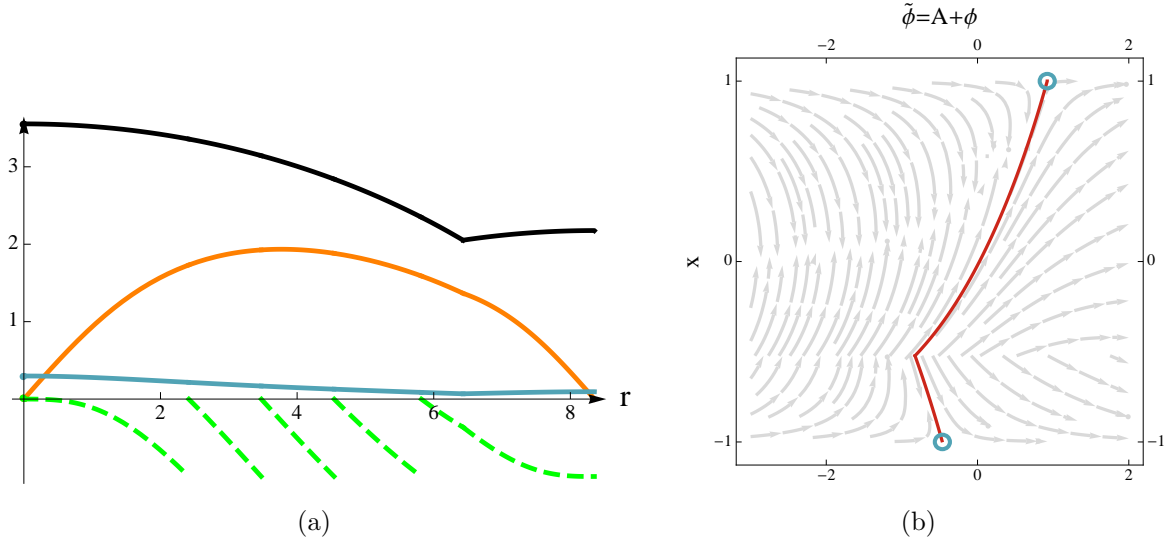
Figure 4.4: Regular solution with one D8 stack. Its position can be seen in the graph as the value of r where the derivatives of the functions jump; it is fixed by (4.2.57). In (a) we again plot the radius of the S^2 (orange), the warping factor e^{2A} (black; rescaled by a factor $1/20$), and the string coupling e^ϕ. We also plot $\frac{1}{\pi}\hat{b}(r) = \frac{1}{4\pi^2}\int_{S_r^2} B_2$ (dashed, light green); to guide the eye, we have periodically identified it as described in section 4.2.7. (The apparent discontinuities are an artifact of the identification.) The fact that it starts and ends at $\hat{b} = 0$ is in compliance with flux quantization for H; we have $\frac{1}{4\pi^2}\int H = -5$. The flux parameters are $\{n_0, n_2\} = \{10, -50\}$ on the left (namely, near the north pole), $\{-40, 0\}$ on the right (near the south pole). In (b), we see the path described by the solution in the $\{A + \phi, x\}$ plane, overlaid to the relevant vector field, that this time changes with n_0.

poles. The easiest thing to attempt is a symmetric configuration where the two stacks have the same number of D8's, with opposite D6 charge. As for the solution with one D8, (4.2.37) implies F_0 at the north pole and negative F_0 at the south pole. For our symmetric configuration, these two values will be opposite, and there will be a central region between the two D8 stacks where $F_0 = 0$.

We show one such solution in figure 4.5. As for the previous solution with one D8, we have started from the North Pole and South Pole; now, however, we did not try to match these two solutions directly, but we inserted a massless region in between. From the northern solutions, again we found at which value of $r = r_{\text{D8}}$ it satisfies (4.2.57). We then stopped the evolution of the system there, evaluated A, ϕ, x at r_{D8}, and used them as a boundary condition for the evolution of (4.2.34), now with $F_0 = 0$. Now we matched this solution to the southern one; namely, we found at which values of $r = r_{\text{D8}'}$ their A, ϕ and x matched. This requires translating the southern solution in r by an appropriate amount, and picking $A_0^- = A_0^+$. Given the symmetry of our configuration, this is not

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surprising: the southern solution is related to the northern one under (4.2.45). Moreover, matching a region with $F_0 \neq 0$ to the massless one means imposing an extra condition, namely the continuity of B in r_{D8}, as we mentioned at the end of 4.2.8.

(a)   (b)
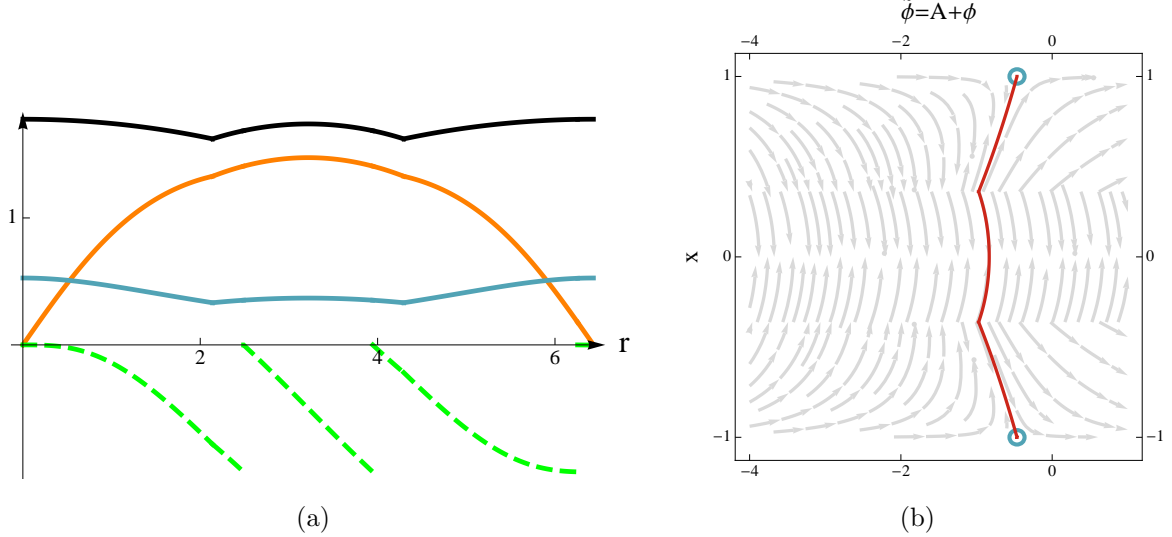
Figure 4.5: Regular solution with two D8 stacks. As in figure 4.4, their positions are the two values of r where the derivatives of the functions jump. In (a) we again plot the radius of the S^2 (orange), the warping factor e^{2A} (black; rescaled by a factor $1/20$), and the string coupling e^{ϕ} (green; rescaled by a factor 5), and $\hat{b}$ (as in figure 4.4; this time $\frac{1}{4\pi^2} \int H = -3$). The flux parameters are: $\{n_0, n_2\} = \{40, 0\}$ on the left (namely, near the north pole); $\{0, -40\}$ in the middle; $\{-40, 0\}$ on the right (near the south pole). The region in the middle thus has $F_0 = 0$; it is indeed very similar to the massless case of figure 4.2. In (b), we see the path described by the solution in the $\{A + \phi, x\}$ plane, overlaid to the relevant vector field, that again changes with n_0.

The parameter $A_0^+ = A_0^- = A_0$ would at this point be still free. However, one still has to impose flux quantization for H. As we recalled above, this is equivalent to requiring that the periodic variable $\hat{b}$ starts and ends at zero. Unlike the case with one D8 above, this time we do need a fine-tuning to achieve this, since the expression for B is not simply controlled by the massive expression (4.2.55). Fortunately we can use the parameter A_0 for this purpose. The solution in the end has no moduli.

As for the solution with one D8 stack we saw earlier, in this case too the D8-branes are not "slipping" towards the North and South Pole because of their interaction with the RR flux: each of the two stacks is calibrated. In this case, intuitively this interaction can be understood as the mutual electric attraction between the two D8 stacks, which indeed have opposite charge under F_2; the balance between this attraction and the "elastic" DBI term is what stabilizes the branes.

Let us also remark that for both solutions (the one with one D8 stack, and the one with two) it is easy to make sure, by taking the flux integers to be large enough, that the curvature and the string coupling e^ϕ are as small as one wishes, so that we remain in the supergravity regime of string theory. In figures 4.4 and 4.5 they are already rather small (moreover, in the figure we use some rescalings for visualization purposes).

Thus we have found regular solutions, with one or two stacks of D8-branes. It is now in principle possible to go on, and to add more D8's. The general analysis concerning the classification of the solutions has been performed in [16].

4.3 AdS$_6$ using generalized complex geometry

In this section we will see how the GCG approach works for AdS$_6$ vacua. We will obtain, starting from the system (3.2.4), the necessary and sufficient conditions for having an AdS$_6$ vacuum solution, focalizing our attention to the type IIB case, since we already know from section (4.1.2) that in massive type IIA the analysis is exhausted by the Brandhuber and Oz solution, whereas in Appendix B we will show that there no supersymmetric AdS$_6$ vacua in eleven dimensional supergravity and hence in massless type IIA too. The analysis for an AdS$_6$ vacuum is much more complicated than the corresponding one for an AdS$_7$ vacuum, and for this reason we will not able to find new explicit example of AdS$_6$ vacua in type IIB. Nevertheless we will show that using GCG one can reduce the problem of finding AdS$_6$ vacua to a system of *two* partal differential equations. A full analysis of this system constitutes an interesting open problem. We will see also that the two known examples of solutions (that as we remarked in section 4.1.2 have been obtained by T-dualities of the Brandhuber and Oz solution) arise as particular cases of solutions of our system of PDEs.

4.3.1 Supersymmetry and pure spinor equations for AdS$_6$

We will start by presenting the system of pure spinor equations that we need to solve. The full derivation of our system from (3.2.4) will be not presented here but it can be found in [13], here we just quote the final system and some particular aspects of it. An important feature is that the spinor decomposition we have to start with is clumsier than the one in other dimensions. We have seen that usually, the ten-dimensional spinors ϵ_a are the sum of two (or sometimes even one) tensor products. For AdS$_4 \times M_6$, for example, we simply have the spinorial Ansatz (3.1.3). The analogue of this for Mink$_6 \times M_4$ is given

in Appendix A, and it writes

$$\epsilon_1 = \zeta_{6+} \otimes \eta^1_{4+} + \zeta^c_{6+} \otimes \eta^{1\,c}_{4+}$$
$$\epsilon_2 = \zeta_{6+} \otimes \eta^2_{4\mp} + \zeta^c_{6+} \otimes \eta^{2\,c}_{4\pm} \qquad (\text{Mink}_6 \times M_4; \text{ IIA/IIB}) , \qquad (4.3.1)$$

where $(\ )^c \equiv C(\ )^*$ denotes Majorana conjugation. For $\text{AdS}_6 \times M_4$, however, such an Ansatz cannot work: as usual, since we are dealing with an AdS_6 vacuum, we need to impose that the ζ_6 obey the Killing spinor equation on AdS_6,

$$\nabla_\mu \zeta_6 = \frac{1}{2} \gamma^{(6)}_\mu \zeta_6 , \qquad (4.3.2)$$

and solutions to this equation cannot be chiral, while the ζ_{6+} in (4.3.1) are chiral. This issue does not arise in AdS_4 because in that case $(\zeta_{4+})^c$ has negative chirality; here $(\zeta_{6+})^c$ has positive chirality. This forces us to add "by hand" to (4.3.1) a second set of spinors with negative chirality, ending up with the unpromising-looking

$$\epsilon_1 = \zeta_+\eta^1_+ + \zeta^c_+\eta^{1\,c}_+ + \zeta_-\eta^1_- + \zeta^c_-\eta^{1\,c}_-$$
$$\epsilon_2 = \zeta_+\eta^2_\mp + \zeta^c_+\eta^{2\,c}_\mp + \zeta_-\eta^2_\pm + \zeta^c_-\eta^{2\,c}_\pm \qquad (\text{AdS}_6 \times M_4; \text{ IIA/IIB}) \qquad (4.3.3)$$

where we have dropped the $_6$ and $_4$ labels (and the $\otimes$ sign), as we will do elsewhere. Attractive or not, (4.3.3) will turn out to be the correct one for our classification.

Having described the spinorial Ansatz that we need, we first describe the forms appearing in the system. If we were interested in the Minkowski case, the system would only contain the bispinors $\eta^1_+ \otimes \eta^{2\,\dagger}_+$ and $\eta^1_+ \otimes (\eta^{2c}_+)^\dagger$. Notice the strong similarity with the AdS_4 already discussed in chapter 3 and indeed one can show that this would describe an $SU(2) \times SU(2)$ structure on $TM_4 \oplus T^*M_4$. Since in (4.3.3) we also have the negative chirality spinors η^1_- and η^{1c}_-, there are many more forms we can build. We have the even forms:[14]

$$\phi^1_\pm = e^{-A}\eta^1_\pm \otimes \eta^{2\,\dagger}_\pm , \qquad \phi^2_\pm = e^{-A}\eta^1_\pm \otimes (\eta^{2c}_\pm)^\dagger \equiv e^{-A}\eta^1_\pm \otimes \overline{\eta^2_\pm} ; \qquad (4.3.4a)$$

and the odd forms:

$$\psi^1_\pm = e^{-A}\eta^1_\pm \otimes \eta^{2\,\dagger}_\mp , \qquad \psi^2_\pm = e^{-A}\eta^1_\pm \otimes (\eta^{2c}_\mp)^\dagger \equiv e^{-A}\eta^1_\pm \otimes \overline{\eta^2_\mp} . \qquad (4.3.4b)$$

The factors e^{-A} are inserted so that the bispinors have unit norm, in a sense to be clarified shortly; A is the warping function, defined as usual by

$$ds^2_{10} = e^{2A}ds^2_{\text{AdS}_6} + ds^2_{M_4} . \qquad (4.3.5)$$

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[14]Notice that the ^{1} or ^{2} on ϕ has nothing to do with the ^{1} or ^{2} on the η's; rather, it has to do with whether the second spinor is Majorana conjugated (^{2}) or not (^{1}). Another caveat is that the $_\pm$ does not indicate the degree of the form, as it is often the case in similar contexts; all the ϕ's in (4.3.4a) are even forms. One can think of the $_\pm$ as indicating whether these forms are self-dual or anti-self-dual.

Already by looking at (4.3.4a), we see that we have *two* $SU(2) \times SU(2)$ structures on $TM_4 \oplus T^*M_4$. It can be shown that both these structures together define an Identity-structure, i.e. a vielbein on M_4. We will see in section 4.3.2 how to parameterize both (4.3.4a) and (4.3.4b) in terms of the vielbein they define. In other words, to stay closer to what we said in Chapter 3, one could express the fact the structure group on the generalized tangent bundle $TM_4 \oplus T^*M_4$ is given by two $SU(2) \times SU(2)$ structures by considering the compatibility relations between the bispinors $(\phi_\pm^{1,2}, \psi_\pm^{1,2})$; however, as we did for AdS$_7$ vacua, it will be much more convenient to explicitly solve the compatibility relations by requiring that $(\phi_\pm^{1,2}, \psi_\pm^{1,2})$ are not arbitrary but that can be written as bispinors.

In the meantime, we can already now notice that the (4.3.4a) and (4.3.4b) can be assembled more conveniently using the $SU(2)$ R-symmetry. This is the group that rotates $\binom{\zeta}{\zeta^c}$ and each of $\binom{\eta_\pm^a}{\eta_\pm^{a\,c}}$ as a doublet. One can check that (4.3.3) is then left invariant, so it is a symmetry; since it acts on the external spinors, we call it an R-symmetry. It is the manifestation of the R-symmetry of a five-dimensional SCFT. Something very similar can be noticed in the AdS$_7$ system, we have not done this discussion here but it can be found in the original work [12]. Since for AdS$_6$ vacua the analysis is considerably more complicated than for AdS$_7$, the $SU(2)$ symmetry will be used from the very beginning to yield more manageable results. Let us define

$$\Phi_\pm \equiv \begin{pmatrix} \eta_\pm^1 \\ \eta_\pm^{1\,c} \end{pmatrix} \otimes \begin{pmatrix} \eta_\pm^{2\,\dagger} & \overline{\eta_\pm^2} \end{pmatrix} = \begin{pmatrix} \phi_\pm^1 & \phi_\pm^2 \\ -(\phi_\pm^2)^* & (\phi_\pm^1)^* \end{pmatrix}$$
$$= \mathrm{Re}\phi_\pm^1 \mathrm{Id}_2 + i(\mathrm{Im}\phi_\pm^2 \sigma_1 + \mathrm{Re}\phi_\pm^2 \sigma_2 + \mathrm{Im}\phi_\pm^1 \sigma_3) \equiv \Phi_\pm^0 \mathrm{Id}_2 + i\Phi_\pm^\alpha \sigma_\alpha \ , \qquad (4.3.6a)$$

$$\Psi_\pm \equiv \begin{pmatrix} \eta_\pm^1 \\ \eta_\pm^{1\,c} \end{pmatrix} \otimes \begin{pmatrix} \eta_\mp^{2\,\dagger} & \overline{\eta_\mp^2} \end{pmatrix} = \begin{pmatrix} \psi_\pm^1 & \psi_\pm^2 \\ -(\psi_\pm^2)^* & (\psi_\pm^1)^* \end{pmatrix}$$
$$= \mathrm{Re}\psi_\pm^1 \mathrm{Id}_2 + i(\mathrm{Im}\psi_\pm^2 \sigma_1 + \mathrm{Re}\psi_\pm^2 \sigma_2 + \mathrm{Im}\psi_\pm^1 \sigma_3) \equiv \Psi_\pm^0 \mathrm{Id}_2 + i\Psi_\pm^\alpha \sigma_\alpha \ . \qquad (4.3.6b)$$

$\sigma_\alpha, \alpha = 1, 2, 3$, are the Pauli matrices. Here and in what follows, the superscript ^{0} denotes an $SU(2)$ singlet, and not the zero-form part; the superscript $^\alpha$ denotes an $SU(2)$ triplet, not a one-form. We hope this will not create confusion.

We can now give the system of equations equivalent to preserved supersymmetry:

$$d_H \left[e^{3A-\phi}(\Psi_- - \Psi_+)^0 \right] - 2e^{2A-\phi}(\Phi_- + \Phi_+)^0 = 0 \ , \qquad (4.3.7a)$$

$$d_H \left[e^{4A-\phi}(\Phi_- - \Phi_+)^\alpha \right] - 3e^{3A-\phi}(\Psi_- + \Psi_+)^\alpha = 0 \ , \qquad (4.3.7b)$$

$$d_H \left[e^{5A-\phi}(\Psi_- - \Psi_+)^\alpha \right] - 4e^{4A-\phi}(\Phi_- + \Phi_+)^\alpha = 0 \ , \qquad (4.3.7c)$$

$$d_H \left[e^{6A-\phi}(\Phi_- - \Phi_+)^0 \right] - 5e^{5A-\phi}(\Psi_- + \Psi_+)^0 = -\frac{1}{4}e^{6A} *_4 \lambda F \ , \qquad (4.3.7d)$$

$$d_H \left[e^{5A-\phi}(\Psi_- + \Psi_+)^0 \right] = 0 \ ; \qquad (4.3.7e)$$

$$||\eta^1||^2 = ||\eta^2||^2 = e^A \ . \qquad (4.3.7f)$$

As usual, ϕ here is the dilaton; $d_H = d - H\wedge$; A was defined in (4.3.5). Since we are in IIB the RR-fluxes are $F = F_1 + F_3$ and they are "totally" internal and, as usual, they determine the external fluxes via the relation

$$F_{(10)} = F + e^{6A}\mathrm{vol}_6 \wedge *_4\lambda F \; . \tag{4.3.8}$$

Again, we remind the reader that the superscript ^{0} denotes a singlet part, and $^\alpha$ a triplet part, as in (4.3.6).

The last equation, (4.3.7f), can be reformulated in terms of Φ and Ψ. Since $\|\eta^a\|^2 \equiv \|\eta^a_+\|^2 + \|\eta^a_-\|^2$, we can define $\|\eta^1_+\| = e^{A/2}\cos(\alpha/2)$, $\|\eta^1_-\| = e^{A/2}\sin(\alpha/2)$, $\|\eta^2_+\| = e^{A/2}\cos(\tilde\alpha/2)$, $\|\eta^2_-\| = e^{A/2}\sin(\tilde\alpha/2)$, where $\alpha, \tilde\alpha \in [0,\pi]$; we then get

$$\begin{aligned}
(\Phi^0_+, \Phi^0_+) &= \frac{1}{8}\cos^2(\alpha/2)\cos^2(\tilde\alpha/2) \; , & (\Phi^0_-, \Phi^0_-) &= -\frac{1}{8}\sin^2(\alpha/2)\sin^2(\tilde\alpha/2) \; ; \\
(\Psi^0_+, \Psi^0_-) &= \frac{1}{8}\cos^2(\alpha/2)\sin^2(\tilde\alpha/2) \; , & (\Psi^0_-, \Psi^0_+) &= -\frac{1}{8}\sin^2(\alpha/2)\cos^2(\tilde\alpha/2) \; .
\end{aligned} \tag{4.3.9}$$

We can check immediately that (4.3.7) imply the equations of motion for the flux, by acting on (4.3.7d) with d_H and using (4.3.7e). The equations of motion for the metric and dilaton are as usual satisfied (as shown in general in [37] for IIA, and in [38] for IIB); the equations of motion for H are also implied, since they are [56] for Minkowski$_4$ compactifications (which include Minkowski$_5$ as a particular case, and hence also AdS$_6$ by a conical construction). We will see later that the Bianchi identities for F and H are also automatically satisfied for this case, as was the case for AdS$_7$ vacua.

In summary, in this section we have presented the system (4.3.7), which is equivalent to preserved supersymmetry for backgrounds of the form AdS$_6 \times M_4$. The forms Φ and Ψ are not arbitrary: they are constructed as spinor bilinears in (4.3.6), (4.3.4). We will now give the general solution to those constraints, and then proceed in section 4.3.3 to analyze the system.

4.3.2 Parameterization of the pure spinors

We have introduced in section 4.3.1 the even forms $\Phi_\pm$ and the odd forms $\Psi_\pm$ (see (4.3.6), (4.3.4a), (4.3.4b)). These are the main characters in the system (4.3.7), which is equivalent to preserved supersymmetry. Before we start using the system, however, we need to characterize what sorts of forms $\Phi_\pm$ and $\Psi_\pm$ can be: this is what we will do in this section.

Even forms

We will first deal with $\Phi_\pm$. We will actually first focus on Φ_+, and then quote the results for Φ_-. The computations in this subsection are actually pretty standard, and we will

be brief.

Let us start with the case $\eta_+^1 = \eta_+^2 \equiv \eta_+$. Assume also for simplicity that $||\eta_+||^2 = 1$. In this case the bilinears define an $SU(2)$ structure:

$$\eta_+\eta_+^\dagger = \frac{1}{4}e^{-ij_+} , \qquad \eta_+\overline{\eta_+} = \frac{1}{4}\omega_+ , \tag{4.3.10}$$

where the two-forms j_+, ω_+ satisfy

$$j_+ \wedge \omega_+ = 0 , \qquad \omega_+^2 = 0 , \qquad \omega_+ \wedge \overline{\omega_+} = 2j_+^2 = -\text{vol}_4 . \tag{4.3.11}$$

We can also compute

$$\eta_+^c\eta_+^{c\dagger} = \frac{1}{4}e^{ij_+} , \qquad \eta_+^c\eta_+^\dagger = -\frac{1}{4}\overline{\omega_+} . \tag{4.3.12}$$

Let us now consider the case with two different spinors, $\eta_+^1 \neq \eta_+^2$; let us again assume that they have unit norm. We can define (in a similar way as in [63])

$$\eta_{0+} = \frac{1}{2}(\eta_+^1 - i\eta_+^2) , \qquad \tilde{\eta}_{0+} = \frac{1}{2}(\eta_+^1 + i\eta_+^2) . \tag{4.3.13}$$

Consider now $a_+ = \eta_+^{2\dagger}\eta_+^1$, $b_+ = \overline{\eta_+^2}\eta_+^1$. $\{\eta_+^2, \eta_+^{2c}\}$ is a basis for spinors on M_4; a_+, b_+ are then the coefficients of η_+^1 along this basis. Since η_+^a have both unit norm, we have $|a_+|^2 + |b_+|^2 = 1$. By multiplying η_+^a by phases, we can assume that a_+ and b_+ are for example purely imaginary, and we can then parameterize them as $a_+ = -i\cos(\theta_+)$, $b_+ = i\sin(\theta_+)$. Going back to (4.3.13), we can now compute their inner products:

$$\eta_{0+}^\dagger\eta_{0+} = \cos^2\left(\frac{\theta_+}{2}\right) , \qquad \eta_{0+}^\dagger\tilde{\eta}_{0+} = 0 , \qquad \overline{\eta_{0+}}\tilde{\eta}_{0+} = \frac{1}{2}\sin(\theta_+) . \tag{4.3.14}$$

From this we can in particular read off the coefficients of the expansion of $\tilde{\eta}_{0+}$ along the basis $\{\eta_{0+}, \eta_{0+}^c\}$. This gives $\tilde{\eta}_{0+} = \frac{1}{||\eta_{0+}||^2}(\eta_{0+}^\dagger\tilde{\eta}_{0+}\eta_{0+} + \overline{\eta_{0+}}\tilde{\eta}_{0+}\eta_{0+}^c) = \tan\left(\frac{\theta_+}{2}\right)\eta_{0+}^c$. Recalling (4.3.13), and defining now $\eta_{0+} = \cos\left(\frac{\theta_+}{2}\right)\eta_+$, we get

$$\eta_+^1 = \cos\left(\frac{\theta_+}{2}\right)\eta_+ + \sin\left(\frac{\theta_+}{2}\right)\eta_+^c , \qquad \eta_+^2 = i\left(\cos\left(\frac{\theta_+}{2}\right)\eta_+ - \sin\left(\frac{\theta_+}{2}\right)\eta_+^c\right) . \tag{4.3.15}$$

From this it is now easy to compute $\eta_+^1\eta_+^{2\dagger}$ and $\eta_+^1\overline{\eta_+^2}$. Recall, however, that in the course of our computation we have first fixed the norms and then the phases of η_+^a. The norms of the spinors we need in this Chapter are not one; they were actually already parameterized before (4.3.9), so as to satisfy (4.3.7f). The factor e^A, however, simplifies with the e^{-A} in the definition (4.3.4a). Let us also restore the phases we earlier fixed, by rescaling

$\eta_\pm^1 \to e^{iu_\pm}\eta_\pm^1$, $\eta_\pm^2 \to e^{it_\pm}\eta_\pm^2$. All in all we get

$$\phi_+^1 = \frac{1}{4}\cos(\alpha/2)\cos(\tilde{\alpha}/2)e^{i(u_+ - t_+)}\cos(\theta_+)\exp\left[-\frac{1}{\cos(\theta_+)}(ij_+ + \sin(\theta_+)\mathrm{Re}\omega_+)\right],$$

(4.3.16a)

$$\phi_+^2 = \frac{1}{4}\cos(\alpha/2)\cos(\tilde{\alpha}/2)e^{i(u_+ + t_+)}\sin(\theta_+)\exp\left[\frac{1}{\sin(\theta_+)}(\cos(\theta_+)\mathrm{Re}\omega_+ + i\mathrm{Im}\omega_+)\right].$$

(4.3.16b)

The formulas for $\phi_-^{1,2}$ can be simply obtained by changing $\cos(\alpha/2) \to \sin(\alpha/2)$, $\cos(\tilde{\alpha}/2) \to \sin(\tilde{\alpha}/2)$, and $_+ \to _-$ everywhere. The only difference to keep in mind is that the last equation in (4.3.11) is now replaced with $\omega_- \wedge \overline{\omega_-} = 2j_-^2 = \mathrm{vol}_4$.

Odd forms

We now turn to the bilinears of "mixed type", i.e. the $\psi_\pm^{1,2}$ we defined in (4.3.4b), which result in odd forms. We will again start from the case where $\eta_\pm^1 = \eta_\pm^2 \equiv \eta_\pm$.

There are two vectors we can define:

$$v_m = \eta_-^{2\dagger}\gamma_m\eta_+^1, \qquad w_m = \overline{\eta_-^2}\gamma_m\eta_+^1. \tag{4.3.17}$$

In bispinor language, we can compute

$$\eta_+\eta_-^\dagger = \frac{1}{4}(1+\gamma)v, \qquad \eta_+^c\eta_-^{c\dagger} = \frac{1}{4}(1+\gamma)\overline{v}, \tag{4.3.18a}$$

$$\eta_-\eta_+^\dagger = \frac{1}{4}(1-\gamma)\overline{v}, \qquad \eta_-^c\eta_+^{c\dagger} = \frac{1}{4}(1-\gamma)v, \tag{4.3.18b}$$

and

$$\eta_+\eta_-^{c\dagger} = \frac{1}{4}(1+\gamma)w, \qquad \eta_+^c\eta_-^\dagger = -\frac{1}{4}(1+\gamma)\overline{w}, \tag{4.3.18c}$$

$$\eta_-\eta_+^{c\dagger} = -\frac{1}{4}(1-\gamma)w, \qquad \eta_-^c\eta_+^\dagger = \frac{1}{4}(1-\gamma)\overline{w}. \tag{4.3.18d}$$

(In four Euclidean dimensions, the chiral $\gamma = *_4\lambda$, so that $(1+\gamma)v = v + *_4v$, and so on. See [11, App. A] for more details.) For the more general case where $\eta_\pm^1 \neq \eta_\pm^2$, we can simply refer back to (4.3.15). For example we get

$$\psi_+^1 = \frac{e^{i(u_+ - t_-)}}{4}\cos(\alpha/2)\sin(\tilde{\alpha}/2)(1+\gamma)\left[\cos\left(\frac{\theta_+ + \theta_-}{2}\right)\mathrm{Re}v + i\cos\left(\frac{\theta_+ - \theta_-}{2}\right)\mathrm{Im}v + \right.$$
$$\left. - \sin\left(\frac{\theta_+ + \theta_-}{2}\right)\mathrm{Re}w + i\sin\left(\frac{\theta_+ - \theta_-}{2}\right)\mathrm{Im}w\right].$$

(4.3.19)

For the time being we do not show the lengthy expressions for the other odd bispinors ψ_+^2 and $\psi_-^{1,2}$, because they will all turn out to simplify quite a bit as soon as we impose the zero-form equations in (4.3.7).

The v and w we just introduced are a complex vielbein; let us see why. First, a standard Fierz computation gives

$$v \cdot \eta_+ = 0 \ , \qquad \overline{v} \cdot \eta_+ = 2\eta_- \ , \tag{4.3.20}$$

where $\cdot$ denotes Clifford product. Multiplying from the left by $\eta_-^\dagger$, we obtain

$$v^2 = 0 \ , \qquad v \llcorner \overline{v} = v^m \overline{v}_m = 2 \ . \tag{4.3.21}$$

Similarly to (4.3.20), we can compute the action of w:

$$w \cdot \eta_\pm = 0 \ , \qquad \overline{w} \cdot \eta_\pm = \pm 2\eta_\mp^c \ . \tag{4.3.22}$$

Multiplying by $\overline{\eta_\mp}$, we get

$$w^2 = 0 \ , \qquad w \llcorner \overline{w} = 2 \ . \tag{4.3.23}$$

From (4.3.20) we can also get $v \cdot \eta_+ \overline{\eta_-} = 0$, $\overline{v} \cdot \eta_+ \overline{\eta_-} = 2\eta_- \overline{\eta_-}$, whose zero-form parts read

$$v \llcorner w = 0 = \overline{v} \llcorner w \ . \tag{4.3.24}$$

Together, (4.3.21), (4.3.23), (4.3.24) say that

$$\{\mathrm{Re}v, \ \mathrm{Re}w, \ \mathrm{Im}v, \ \mathrm{Im}w\} \tag{4.3.25}$$

are a vielbein.

We can also now try to relate the even forms of section 4.3.2 to this vielbein. From (4.3.20) we also see $v \cdot \eta_+ \overline{\eta_+} = 0$, which says $v \wedge \omega_+ = 0$; similarly one gets $\overline{v} \wedge \omega_- = 0$. Also, (4.3.22) implies that $w \cdot \eta_+ \overline{\eta_+} = w \cdot \omega_+ = 0$, and thus that $w \wedge \omega_\pm = 0$. So we have $\omega_+ \propto v \wedge w$, $\omega_- \propto \overline{v} \wedge w$. One can fix the proportionality constant by a little more work:

$$\omega_+ = -v \wedge w \ , \qquad \omega_- = \overline{v} \wedge w \ . \tag{4.3.26a}$$

Similar considerations also determine the real two-forms:

$$j_\pm = \pm \frac{i}{2}(v \wedge \overline{v} \pm w \wedge \overline{w}) \ . \tag{4.3.26b}$$

So far we have managed to parameterize all the pure spinors $\Phi_\pm$, $\Psi_\pm$ in terms of a vielbein given by (4.3.25). The expressions for Φ_+ are given in (4.3.16); Φ_- is given by changing $(\cos(\alpha/2), \cos(\tilde{\alpha}/2)) \to (\sin(\alpha/2), \sin(\tilde{\alpha}/2))$, and $_+ \to {}_-$ everywhere. The forms $j_\pm$, $\omega_\pm$ are given in (4.3.26) in terms of the vielbein. Among the odd forms of $\Psi_\pm$, we have only quoted one example, (4.3.19); similar expressions exist for ψ_+^2 and for $\psi_-^{1,2}$. We will summarize all this again after the simplest supersymmetry equations will allow us to simplify the parameterization quite a bit.

4.3.3 General analysis

We will now use the parameterization obtained for Φ and Ψ in section 4.3.2 in the system (4.3.7). As anticipated in the introduction, we will reduce the system to the two PDEs (4.3.38a), (4.3.39), and we will determine the local form of the metric and of the fluxes in terms of a solution to those equations.

Zero-form equations

The only equations in (4.3.7) that have a zero-form part are (4.3.7a) and (4.3.7c):

$$(\Phi_+ + \Phi_-)^0_0 = 0 \ , \qquad (\Phi_+ + \Phi_-)^\alpha_0 = 0 \ . \tag{4.3.27}$$

The subscript $_0$ here denotes the zero-form part. (Recall that the superscripts ^{0} and $^\alpha$ denote $SU(2)$ singlets and triplets respectively.) To simplify the analysis, it is useful to change variables so as to make the $SU(2)$ R-symmetry more manifest.

In (4.3.16), apart for the overall factor $\cos(\alpha/2)\cos(\tilde\alpha/2)/4$, we have $\phi^1_{+0} \propto e^{i(u_+ - t_+)}\cos(\theta_+)$, $\phi^2_{+0} \propto e^{i(u_+ + t_+)}\sin(\theta_+)$. The singlet is $\mathrm{Re}\phi^1_{+0} \propto \cos(\theta_+)\cos(u_+ - t_+)$, and it is a good idea to give it a name, say x_+. On the other hand, the triplet is $\{\mathrm{Im}\phi^2_+, \mathrm{Re}\phi^2_+, \mathrm{Im}\phi^1_+\} \propto \{\sin(\theta_+)\sin(u_+ + t_+), \sin(\theta_+)\cos(u_+ + t_+), \cos(\theta_+)\sin(u_+ - t_+)\}$. If we sum their squares, we obtain:

$$\sin^2(\theta_+) + \cos(\theta_+)^2 \sin^2(u_+ - t_+) = x^2_+ \tan^2(u_+ - t_+) + \sin^2(\theta_+) = 1 - x^2_+ \ . \tag{4.3.28}$$

This suggests that we parameterize the triplet using the combination $\sqrt{1 - x^2_+}\, y^\alpha$, where y^α should obey $y_\alpha y^\alpha = 1$ and can be chosen to be the $\ell = 1$ spherical harmonics on S^2. What we are doing is essentially changing variables on an S^3, going from coordinates that exhibit it as an $S^1 \times S^1$ fibration over an interval to coordinates that exhibit it as an S^2 fibration over an interval:

$$\left\{ \cos(\theta_+) e^{i(u_+ - t_+)}, \sin(\theta_+) e^{i(u_+ + t_+)} \right\} \to \left\{ x_+, \sqrt{1 - x^2_+}\, y^\alpha \right\} \ . \tag{4.3.29}$$

An identical discussion can of course be given for $\phi^{1,2}_-$. Summing up, we are led to the following definitions:

$$x_\pm \equiv \cos(\theta_\pm)\cos(u_\pm - t_\pm) \ , \quad \sin\beta_\pm \equiv \frac{\sin(\theta_+)}{\sqrt{1 - x^2_+}} \ , \quad \gamma_\pm \equiv \frac{\pi}{2} - u_\pm - t_\pm \ , \tag{4.3.30}$$

and

$$y^\alpha_\pm \equiv \Big(\sin(\beta_\pm)\cos(\gamma_\pm), \ \sin(\beta_\pm)\sin(\gamma_\pm), \ \cos(\beta_\pm) \Big) \ , \tag{4.3.31}$$

in terms of which

$$\Phi_{+0} = \cos(\alpha/2)\cos(\tilde\alpha/2) \left(x_+ + i y^\alpha_+ \sqrt{1 - x^2_+}\, \sigma_\alpha \right) \ ,$$
$$\Phi_{-0} = \sin(\alpha/2)\sin(\tilde\alpha/2) \left(x_- + i y^\alpha_- \sqrt{1 - x^2_-}\, \sigma_\alpha \right) \ . \tag{4.3.32}$$

Going back to (4.3.27), summing the squares of all four equations we get $\cos^2(\alpha/2)\cos^2(\tilde{\alpha}/2) = \sin^2(\alpha/2)\sin^2(\tilde{\alpha}/2)$. Given that α and $\tilde{\alpha} \in [0, \pi]$, this is uniquely solved by

$$\tilde{\alpha} = \pi - \alpha \ . \tag{4.3.33}$$

Now (4.3.27) reduces to

$$-x_- = x_+ \equiv x \ , \qquad -y_-^\alpha = y_+^\alpha \equiv y^\alpha \ . \tag{4.3.34}$$

In terms of the original parameters, this means $\theta_+ = \theta_-$, $u_- = u_+$, $t_- = t_+ + \pi$.

The parameterization obtained in section 4.3.2 now simplifies considerably:

$$\phi_\pm^1 = \pm\frac{1}{8}\sin\alpha\cos\theta\, e^{i(u-t)} \exp\left[-\frac{1}{\cos\theta}(ij_\pm + \sin\theta\mathrm{Re}\omega_\pm)\right] \ , \tag{4.3.35a}$$

$$\phi_\pm^2 = \pm\frac{1}{8}\sin\alpha\sin\theta\, e^{i(u+t)} \exp\left[\frac{1}{\sin\theta}(\cos\theta\mathrm{Re}\omega_+ + i\mathrm{Im}\omega_+)\right] \ ; \tag{4.3.35b}$$

$$\psi_\pm^1 = \mp\frac{1}{8}(1 \pm \cos\alpha)e^{i(u-t)}(1 \pm \gamma)\left[\cos\theta\mathrm{Re}v \pm i\mathrm{Im}v \mp \sin\theta\mathrm{Re}w\right] \ , \tag{4.3.35c}$$

$$\psi_\pm^2 = \mp\frac{1}{8}(1 \pm \cos\alpha)e^{i(u+t)}(1 \pm \gamma)\left[\sin\theta\mathrm{Re}v \pm i\mathrm{Im}w \pm \cos\theta\mathrm{Re}w\right] \ . \tag{4.3.35d}$$

We temporarily reverted here to a formulation where $SU(2)_\mathrm{R}$ is not manifest; however, in what follows we will almost always use the $SU(2)$-covariant variables x and y^α introduced above.

Geometry

We will now describe how we analyzed the higher-form parts of (4.3.7), although not in such detail as in section 4.3.3.

The only equations that have a one-form part are (4.3.7b). From (4.3.35c), (4.3.35d), we see that the second summand $(\Psi_+ + \Psi_-)_1^\alpha$ is a linear combination of the forms in the vielbein (4.3.25). The first summand consists of derivatives of the parameters we have previously introduced. This gives three constraints on the four elements of the vielbein. We used it to express $\mathrm{Im}v$, $\mathrm{Re}w$, $\mathrm{Im}w$ in terms of $\mathrm{Re}v$;[15] the resulting expressions are at this point still not particularly illuminating, and we will not give them here. These expressions are not even manifestly $SU(2)$-covariant at this point; however, once one uses them into $\Phi_\pm$ and $\Psi_\pm$, one does find $SU(2)$-covariant forms. Just by way of example, we have

$$(\Phi_+ + \Phi_-)_2^\alpha = -\frac{1}{3}e^{-3A+\phi}\sin\alpha\,\mathrm{Re}v \wedge d\left(y^\alpha\sin\alpha\, e^{4A-\phi}\sqrt{1-x^2}\right) \ ,$$

$$(\Psi_- - \Psi_+)_1^\alpha = y^\alpha\sqrt{1-x^2}\sin^2(\alpha)\mathrm{Re}v + \frac{1}{3}e^{-3A+\phi}\cos\alpha\, d\left(y^\alpha\sin\alpha\, e^{4A-\phi}\sqrt{1-x^2}\right) \ . \tag{4.3.36}$$

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[15]Doing so requires $x \neq 0$; the case $x = 0$ will be analyzed separately in section 4.3.4.

We chose these particular 2-form and 1-form triplet combinations because they are involved in the 2-form part of (4.3.7c). The result is a triplet of equations of the form $y^\alpha E_2 + dy^\alpha \wedge E_1 = 0$, where E_i are i-forms and $SU(2)_R$ singlets. If we multiply this by y_α, we obtain $E_2 = 0$ (since $y_\alpha dy^\alpha = 0$); then also $E_1 = 0$ necessarily. The latter gives a simple expression for Rev, the one-form among the vielbein (4.3.25) that we had not determined yet:

$$\mathrm{Re}v = -\frac{e^{-A}}{\sin\alpha} d(e^{2A}\cos\alpha) \ . \tag{4.3.37}$$

Once this is used, the two-form equation $E_2 = 0$ is automatically satisfied.

There are some more two-form equations from (4.3.7). The easiest is (4.3.7e), which gives

$$d\left(\frac{e^{4A-\phi}}{x}\cot\alpha \, d(e^{2A}\cos\alpha) + \frac{1}{3x}e^{2A}\sqrt{1-x^2}d\left(e^{4A-\phi}\sqrt{1-x^2}\sin\alpha\right)\right) = 0 \ . \tag{4.3.38a}$$

Locally, this can be solved by saying

$$\boxed{x\,dz = e^{4A-\phi}\cot\alpha \, d(e^{2A}\cos\alpha) + \frac{1}{3}e^{2A}\sqrt{1-x^2}d\left(e^{4A-\phi}\sqrt{1-x^2}\sin\alpha\right)} \tag{4.3.38b}$$

for some function z. The two-form part of (4.3.7a) reads, on the other hand,

$$\boxed{e^{-8A}d(e^{6A}\cos\alpha)\wedge dz = d(xe^{2A-\phi}\sin\alpha)\wedge d(e^{2A}\cos\alpha)} \ . \tag{4.3.39}$$

If one prefers, dz can be eliminated, giving

$$3\sin(2\alpha)dA\wedge d\phi = d\alpha\wedge\left(6dA+\sin^2(\alpha)\left(-dx^2 - 2(x^2+5)dA + (1+2x^2)d\phi\right)\right) \ . \tag{4.3.40}$$

We will devote the whole section 4.3.5 to analyze the PDEs (4.3.38a), (4.3.39) and we will also exhibit two explicit solutions.

Taking the exterior derivative of (4.3.39) one sees that $d\alpha \wedge dA \wedge dz = 0$. Wedging (4.3.38a) with an appropriate one-form, one also sees $d\alpha \wedge dA \wedge dx = 0$. Taken together, these mean that only two among the remaining variables (α, x, A, ϕ) are really independent. For example we can take α and x to be independent, and

$$A = A(\alpha, x) \ , \qquad \phi = \phi(\alpha, x) \ . \tag{4.3.41}$$

We are not done with the analysis of (4.3.7), but there will be no longer any purely geometrical equations: the remaining content of (4.3.7) determines the fluxes, as we will see in the next subsection. Let us then pause to notice that at this point we have already determined the metric: three of the elements of the vielbein (4.3.25) were determined

already at the beginning of this section in terms of Rev, and the latter was determined in (4.3.37). This gives the metric

$$ds^2 = \frac{\cos\alpha}{\sin^2(\alpha)}\frac{dq^2}{q} + \frac{1}{9}q(1-x^2)\frac{\sin^2(\alpha)}{\cos\alpha}\left(\frac{1}{x^2}\left(\frac{dp}{p} + 3\cot^2(\alpha)\frac{dq}{q}\right)^2 + ds_{S^2}^2\right) , \quad (4.3.42)$$

where the S^2 is spanned by the functions β and γ introduced in (4.3.31) (namely, $ds_{S^2}^2 = d\beta^2 + \sin^2(\beta)d\gamma^2$), and we have eliminated A and ϕ in favor of

$$q \equiv e^{2A}\cos\alpha , \qquad p \equiv e^{4A-\phi}\sin\alpha\sqrt{1-x^2} . \quad (4.3.43)$$

These variables could also be used in the equations (4.3.38a), (4.3.39) above, with marginal simplification. Notice that positivity of (4.3.42) requires $|x| \leq 1$.

Thus we have found in this section that the internal space M_4 is an S^2 fibration over a two-dimensional space Σ, which we can think of as spanned by the coordinates (α, x).

Fluxes

We now turn to the three-form part of (4.3.7b). This is an $SU(2)_{\mathrm{R}}$ triplet. It can be written as $y^\alpha H = \epsilon^{\alpha\beta\gamma}y^\beta dy^\gamma \wedge \tilde{E}_2 + y^\alpha \mathrm{vol}_{S^2} \wedge \tilde{E}_1$, where $\tilde{E}_i$ are i-forms and $SU(2)_{\mathrm{R}}$ singlets. Actually, from (4.3.38a) and (4.3.39) it follows that $\tilde{E}_2 = 0$; we are then left with a single equation setting $H = \mathrm{vol}_{S^2} \wedge \tilde{E}_1$:

$$H = -\frac{1}{9x}e^{2A}\sqrt{1-x^2}\sin\alpha\left[-\frac{6dA}{\sin\alpha} + 2e^{-A}(1+x^2)d(e^A\sin\alpha) + \sin\alpha\, d(\phi+x^2)\right]\wedge\mathrm{vol}_{S^2} . \quad (4.3.44)$$

As expected, H is a singlet under $SU(2)_{\mathrm{R}}$.

All the four-form equations in (4.3.7e), (4.3.7a), (4.3.7c) turn out to be automatically satisfied. We can then finally turn our attention to (4.3.7d), which we have ignored so far. It gives the following expressions for the fluxes:

$$F_1 = \frac{e^{-\phi}}{6x\cos\alpha}\left[\frac{12dA}{\sin\alpha} + 4e^{-A}(x^2-1)d(e^A\sin\alpha) + e^{2\phi}\sin\alpha\, d(e^{-2\phi}(1+2x^2))\right] ; \quad (4.3.45a)$$

$$F_3 = \frac{e^{2A-\phi}}{54}\sqrt{1-x^2}\frac{\sin^2(\alpha)}{\cos\alpha}\left[\frac{36dA}{\sin\alpha} + 4e^{-A}(x^2-7)d(e^A\sin\alpha) + \right.$$
$$\left. + e^{2\phi}\sin\alpha\, d(e^{-2\phi}(1+2x^2))\right]\wedge\mathrm{vol}_{S^2} . \quad (4.3.45b)$$

The Bianchi identities

$$dH = 0 , \quad dF_1 = 0 , \quad dF_3 + H \wedge F_1 = 0 , \quad (4.3.46)$$

are all automatically satisfied, using of course the PDEs (4.3.38a), (4.3.39). As usual, this statement is actually true only if one assumes that the various functions appearing in those equations are smooth. As in [12], one can introduce sources by relaxing this condition.

4.3.4 The case $x = 0$

In section 4.3.3, we used the three-form part of (4.3.7b) to express $\mathrm{Im}v$, $\mathrm{Re}w$, $\mathrm{Im}w$ in terms of $\mathrm{Re}v$. This actually can only be done for $x \neq 0$: the expressions we get contain x in the denominator, as can be seen for example in (4.3.38a). This left out the case $x = 0$; we will analyze it in this section, showing that it leads to a single solution, discussed in [48], which is exactly the abelian T-dual of the Brandhuber and Oz solution that we have recalled in (4.1.26).

Keeping in mind that $-x_- = x_+ = x$ (from (4.3.34)), from (4.3.30) we have $x = \cos(\theta)\cos(u - t)$. Imposing $x = 0$ then means either $\theta = \frac{\pi}{2}$ or $u - t = \frac{\pi}{2}$. Of these two possibilities, the first does not look promising, because on the S^3 parameterized by $(\cos(\theta)e^{i(u-t)}, \sin(\theta)e^{i(u+t)})$ it effectively restricts us to an S^1: only the function $u+t$ is left in the game, and indeed going further in the analysis one finds that the metric becomes degenerate.[16] The second possibility, $u - t = \frac{\pi}{2}$, restricts us instead to an $S^2 \subset S^3$; we will now see that this possibility survives. It gives

$$\beta = \theta \ , \qquad t = -\frac{1}{2}\gamma \ , \qquad u = \frac{\pi}{2} - \frac{1}{2}\gamma \ . \qquad (4.3.47)$$

This leads to a dramatic simplification in the whole system. The one-form equations from (4.3.7b) do not involve $\mathrm{Im}v$ any more; we can now use them to solve for $\mathrm{Re}v$, $\mathrm{Re}w$, $\mathrm{Im}w$ (rather than for $\mathrm{Im}v$, $\mathrm{Re}w$, $\mathrm{Im}w$ as we did in previous subsections, for $x \neq 0$). This strategy would actually have been possible for $x \neq 0$ too, but it would have led to far more involved expressions; for this reason we decided to isolate the $x = 0$ case and to treat it separately in this subsection. We get

$$\mathrm{Re}v = \frac{e^{-3A+\phi}}{3\cos\alpha}d(\sin\alpha\, e^{4A-\phi}) \ , \quad \mathrm{Re}w = \frac{e^A}{3}\sin\alpha\, d\beta \ , \quad \mathrm{Im}w = -\frac{e^A}{3}\sin\alpha\,\sin\beta d\gamma \ . \qquad (4.3.48)$$

We now turn to the 2-form equation in (4.3.7c). As in the previous subsections of this section, this can be separated into a 2-form multiplying y^α and a 1-form multiplying dy^α, which have to vanish separately:

$$d(e^{5A-\phi}\mathrm{Re}v) = 0 \ , \qquad e^{5A-\phi}(3 - 4\sin^2(\alpha))\mathrm{Re}v = d(e^{6A-\phi}\sin\alpha\cos\alpha) \ . \qquad (4.3.49)$$

Hitting the second equation with d and using the first, we find $\sin\alpha\cos\alpha\, d\alpha \wedge \mathrm{Re}v = 0$, and hence, recalling (4.3.48), to $\sin\alpha d\alpha \wedge d(4A - \phi) = 0$. Now, $\sin\alpha$ is not allowed

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[16]At the stage of (4.3.48) below, one would find $\mathrm{Re}w \propto \mathrm{Im}v$.

to vanish because of (4.3.48) (recall that $\mathrm{Re}v$, $\mathrm{Re}w$, $\mathrm{Im}w$ are part of a vielbein); hence $d\alpha \wedge d(4A - \phi) = 0$. This can be interpreted as saying that $4A - \phi$ is a function of α. On the other hand, using (4.3.48) in the first in (4.3.49), we get $d(\frac{e^{2A}}{\cos\alpha}) \wedge d(\sin\alpha\, e^{4A-\phi}) = 0$, which shows that $A = A(\alpha)$, and hence also that $\phi = \phi(\alpha)$. Going back to the second in (4.3.49), it now reads

$$2(\cos^2(\alpha) + 2)\partial_\alpha A + \sin^2(\alpha)\partial_\alpha\phi = \sin(2\alpha) . \tag{4.3.50}$$

Turning to (4.3.7e), its 2-form part reads

$$d(e^{5A-\phi}\mathrm{Im}v) = 0 \qquad \Rightarrow \qquad \mathrm{Im}v = e^{-(5A-\phi)}dz \tag{4.3.51}$$

for some function z. This completes (4.3.48).

Finally, (4.3.7a) gives

$$\left(d(e^{-2A}\cos\alpha) + 2e^{-3A}\sin\alpha\,\mathrm{Re}v\right) \wedge \mathrm{Im}v = 0 . \tag{4.3.52}$$

In view of (4.3.51), the parenthesis has to vanish by itself; this leads to

$$4(7\cos^2(\alpha) - 4)\partial_\alpha A + 4\sin^2(\alpha)\partial_\alpha\phi = -\sin(2\alpha) . \tag{4.3.53}$$

Notice that now (4.3.50) and (4.3.53) are two *ordinary* (as opposed to partial) differential equations, which can be solved explicitly:

$$e^A = \frac{c_1}{\cos^{1/6}(\alpha)} , \qquad e^\phi = \frac{c_2}{\sin\alpha\,\cos^{2/3}(\alpha)} , \tag{4.3.54}$$

where c_i are two integration constants. These are exactly the warping and dilaton presented in (4.1.26), for $c_1 = \frac{3}{2}Lm^{-1/6}$, $c_2 = 4/(3L^2 m^{2/3})$. It is now possible to derive the fluxes, as we did in subsection 4.3.3 for $x \neq 0$, and check that they coincide with those in (4.1.26). The metric can now be computed too, using the vielbein (4.3.49), (4.3.51); it also agrees with (4.1.26) .

Notice finally that, although we have found it convenient to treat the $x = 0$ case separately from the rest, it is in fact a particular case of the general treatment (although a slightly degenerate one). Indeed one can check that (4.3.38b) is satisfied by (4.3.54); in contrast to the general case, this does not determine a function z, but we can use (4.3.40), where z has been eliminated, instead of (4.3.39), which contains z. Thus the solution presented in this subsection is already an example of our general formalism. In section 4.3.5 we will see another, more elaborate example.

4.3.5 The PDEs and the nonabelian T-dual

In section 4.3.3, we reduced the problem of finding $\mathrm{AdS}_6 \times M_4$ solutions to the two PDEs (4.3.38a), (4.3.39). In this section we will recover via a simple Ansatz the known

solution [48], originally obtained via nonabelian T-duality. (As we mentioned, one can also see the $x = 0$ case as a particular solution to the PDEs.)

Many PDEs are reduced to ODEs by a separation of variables Ansatz. For our nonlinear PDEs, this does not work. However, we will now see that a particular case does lead to a solution, namely:

$$\phi = f(\alpha) + \log(x) , \qquad A = A(\alpha) . \tag{4.3.55}$$

Notice that this Ansatz restricts x to be in $(0, 1]$. (We already observed after (4.3.42) that $|x| \leq 1$ in general.)

We begin by considering (4.3.38b). With (4.3.55), after a few manipulations it reduces to

$$\begin{aligned}
dz = d &\left(e^{6A-f} \frac{\sin \alpha}{6x^2} \right) - \frac{1}{3} e^{2A} d(e^{4A-f} \sin \alpha) + \\
&+ \frac{1}{x^2} \left[-\frac{1}{6} e^{4A} d(e^{2A-f} \sin \alpha) + e^{4A-f} \cot \alpha \, d(e^{2A} \cos \alpha) \right] .
\end{aligned} \tag{4.3.56}$$

The first line in (4.3.56) is manifestly exact, since everything is a function of α alone. The second line is of the form $\frac{1}{x^2} d(\text{function}(\alpha))$, and cannot be exact unless it vanishes, which leads to

$$d(e^{2A-f} \sin \alpha) = 6e^{-f} \cot \alpha \, d(e^{2A} \cos \alpha) . \tag{4.3.57}$$

The first line in (4.3.56) then determines dz (and can be integrated to produce z). We can now use this expression for dz in (4.3.39). Most terms in (4.3.39) actually vanish because they involve wedges of forms proportional to $d\alpha$; the only one surviving is of the form $d(e^{6A} \cos \alpha) \wedge dx$. In other words, we are forced to take

$$e^A = c_1 (\cos \alpha)^{-1/6} , \tag{4.3.58}$$

with c_1 an integration constant. Plugging this back into (4.3.57) we get

$$e^f = c_2 \frac{(\cos \alpha)^{-1/3}}{\sin^3 \alpha} \tag{4.3.59}$$

for c_2 another integration constant.

This is actually the solution found in [48]. To see this, one needs to identify

$$x = \frac{e^{2\hat{A}}}{\sqrt{r^2 + e^{4\hat{A}}}} , \tag{4.3.60}$$

where $\hat{A}$ has been introduced in (4.1.27). One can check that indeed the fluxes (4.3.44), (4.3.45) and metric (4.3.42) give the expressions in (4.1.27). Again, we see that the metric looks non-compact; it might be possible to find a suitable analytic continuation, with the help of the PDEs (4.3.38a), (4.3.39) just found.

Chapter 5

The good case: two-dimensional Minkowski vacua

In this chapter we will describe the only known example of good case: Mink_2 $N = (2, 0)$ vacua. We will see that in this case the pairing equations are not redundant but can be written in an elegant form by imposing the additional assumption that the internal manifold enjoys a $SU(4) \times SU(4)$ structure. Some results that we will obtain was already at disposal in literature [18] but we will be able to extend in many respects that analysis. We will also point out the intimate relation between the failure of the supersymmetry-calibrations correspondence and the pairing equations, a relation already discussed in full generality in [64].

5.1 A motivation: the supersymmetry-calibration correspondence

Apart from being an example (the only known example at the moment) of good case, the interest on Mink_2 $N = (2, 0)$ as also another motivation; in this section we will explain such a reason of interest.

After the work [10], which for the first time rewrote SUSY conditions for $N = 1$ four-dimensional vacua in terms of GCG, a very interesting (and perhaps unexpected) observation was done in [65]: it was found that the conditions for a Mink_4 vacuum, when expressed in the pure spinors formulation, are in one-to-one correspondence with the differential conditions satisfied by the calibration forms for *all* the admissible, static, magnetic D-branes in such a background.[1] It is therefore natural to ask whether the correspondence (which we will call the supersymmetry-calibrations correspondence) is

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[1]An analogous story holds also for AdS_4 vacua [40].

also valid in more general situations and can be applied in dimensions different than four.

Motivated by this question, in [54] it was checked that the supersymmetry-calibrations correspondence continues to hold also for Mink$_6$ vacua preserving eight real supercharges, and this led the authors to formulate the following conjecture: the supersymmetry-calibrations correspondence is valid for all Mink$_d$ vacua (with d even) preserving a Weyl spinor on the external manifold.

Specializing the discussion to the case of Mink$_2$, $N = (2, 0)$ vacua, the authors of [54] conjectured that the conditions for unbroken supersymmetry should be

$$
d_H(e^{2A-\phi}\text{Re}\psi_1) = \pm\frac{\alpha}{16}e^{2A} *_8 \lambda(f) \ ,
$$
$$
d_H(e^{2A-\phi}\psi_2) = 0 \ , \tag{5.1.1}
$$

where $d_H \equiv d - H\wedge$ and $\psi_1 = \frac{1}{e^A}\eta^1_+\eta^{2\dagger}_\mp$, $\psi_2 = \frac{1}{e^A}\eta^1_+\eta^{2c\dagger}_\mp$ are the familiar polyforms constructed as bilinears of the internal SUSY parameters η^1_+ and $\eta^2_\pm$ (which are Weyl spinors); the upper (lower) sign is as usual for IIA (IIB). It is worth emphasizing that the correspondence was formulated for η^1_+ and $\eta^2_\pm$ being *pure* spinors on the internal manifold;[2] this assumption implies that the structure group on the generalized tangent bundle $T_8 \oplus T_8^*$ reduces to $SU(4) \times SU(4)$.

In [18] it has been shown, by making the additional assumption that η^1_+ and η^2_+ are *proportional*, that in type IIB the conjecture of [54] fails to be valid: the authors indeed have shown that the equations (5.1.1) are not completely equivalent to supersymmetry and that they must be completed, in this particular case, with the condition

$$
d_H^{\mathcal{J}_2}(e^{-\phi}\text{Im}\psi_1) = -\frac{\alpha}{16}f \ , \tag{5.1.2}
$$

where $d_H^{\mathcal{J}_2} \equiv [d_H, \mathcal{J}_2 \cdot]$ (used for the first time in physical context in [66]), and $\mathcal{J}_2$ is the generalized almost complex structure associated to the pure spinor on the generalized tangent bundle ψ_2 (further details are given in section 5.4.1). They also gave a geometrical interpretation of this equation in terms of calibrations, motivated by the results obtained in dimensions greater than 2.

The authors of [18] conjectured that the final result does not change by removing the assumption of proportionality between η^1_+ and η^2_+, but they did not test this final statement; however they suggested that the ten-dimensional system (3.2.4) could be useful in order to show such a conjecture. In this chapter we will pursue such a program: we

---

[2]Contrary to what happens in lower dimensions, in eight dimensions not every Weyl (not Majorana) spinor is pure: as reviewed in section 5.4.1 an eight-dimensional Weyl spinor is pure if and only it satisfies an additional algebraic condition (5.4.1). From this it follows that the situation considered in [54] is not the most general one for a Mink$_2$, $N = (2, 0)$ vacuum.

will remove the assumption of proportionality between η_+^1 and $\eta_\mp^2$ and show the validity of the results of [18] in the general case with non proportional spinors.

As a further generalization we will also show that the conditions for unbroken supersymmetry can be recast in an elegant form every time $\mathcal{M}_8$ enjoys an $SU(4) \times SU(4)$ structure, no matter whether η_+^1 and $\eta_\mp^2$ are pure or not. In other words, we will see that the conditions for unbroken supersymmetry, and in particular the pairing equations, take an elegant formulation if we assume that it exists a pair of pure spinors $\tilde{\eta}_+^1$ and $\tilde{\eta}_\mp^2$ (which in general will not coincide with the SUSY parameters η_+^1 and $\eta_\mp^2$). In this way we will conclude that Mink$_2$ $N = (2,0)$ vacua, when the internal manifold enjoys a $SU(4) \times SU(4)$ structure, are good cases.

From the point of view of the supersymmetry-calibration correspondence we will also show that the additional equation (5.1.2) (or its generalization (5.5.16)) is exactly in correspondence with the pairing equations, and so we conclude that the pairing equations, when they are not redundant, parametrize the failure of the supersymmetry-calibration correspondence, a relation already proved in generality in [64].

5.2 Spinorial Ansatz and two-dimensional geometry

In this section we will discuss how the ten-dimensional SUSY parameters ϵ_1 and ϵ_2 decompose in order to have an $N = (2,0)$, Mink$_2$ vacuum, namely a configuration of the form Mink$_2 \times \mathcal{M}_8$ (with $\mathcal{M}_8$ compact) enjoying the maximal symmetry of Mink$_2$ and where two real supercharges are preserved. We will also describe what kind of geometrical quantities are defined by a single Weyl (Not Majorana) spinor ζ in two dimensions.

5.2.1 Spinorial Ansatz

Let us start by imposing the request that the ten-dimensional metric takes the form of a Mink$_2$ vacuum and that the spinorial Ansatz gives two-dimensional $N = (2,0)$ supersymmetry. Exactly as we did in the precedent chapter for AdS$_{6,7}$ solutions we require that the metric takes the form

$$ds_{10}^2(x,y) = e^{2A(y)} ds_{\text{Mink}_2}^2(x) + ds_{\mathcal{M}_8}^2(y) \ , \tag{5.2.1}$$

x^μ are the coordinates on Mink$_2$ and y^m are the coordinates on the internal manifold $\mathcal{M}_8$. the warping function is $A(y)$.

As explained in section 5.1, the supersymmetry-calibration correspondence requires that the Mink$_d$ vacuum preserves $2^{d/2}$ supercharges (with d even). For this reason we look for a vacuum which preserves 2 real supercharges in two dimensions and such a number

of supercharges is given by a two-dimensional, complex, Weyl spinor ζ. For this reason the ten-dimensional SUSY parameters ϵ_1 and ϵ_2 take the form

$$\epsilon_1 = \zeta\eta_+^1 + \zeta^c\eta_+^{c\,1} \ ,$$
$$\epsilon_2 = \zeta\eta_\mp^2 + \zeta^c\eta_\mp^{c\,2} \ , \tag{5.2.2}$$

where as usual the upper sign is for IIA, the lower for IIB. ζ denotes a Weyl spinor (of positive chirality) in two dimensions and $\eta_\pm^i$ are two Weyl spinors on $\mathcal{M}_8$.[3] Since we are not imposing also a Majorana condition on ζ (recall that in two Lorentzian dimensions Majorana-Weyl spinors can be defined) we see that ζ defines two real supercharges in two dimensions and (5.2.1) is an $N = (2, 0)$ vacuum. Similarly to (5.2.2), the ten-dimensional gamma matrices Γ_M decompose as

$$\Gamma_\mu = e^A \gamma_\mu \otimes 1 \ , \qquad \Gamma_m = \gamma^{(2)} \otimes \gamma_m \ , \tag{5.2.3}$$

where γ_μ and γ_m are the real two-dimensional and eight-dimensional gamma matrices respectively, and $\gamma^{(2)}$ is the chiral operator in two dimensions. M goes from 0 to 9.

To have a vacuum solution we need that the external spinor ζ satisfies a Killing spinor equation, that for a Minkowski vacuum simply requires that ζ is constant

$$D_\mu \zeta = 0 \ . \tag{5.2.4}$$

5.2.2   Geometry defined by two-dimensional spinors

Given the spinorial Ansatz (5.2.2) we want now to develop what kind of geometrical quantities can be defined using ζ and ζ^c.

Given ζ of positive chirality we can introduce the barred spinor $\bar{\zeta} = \zeta^\dagger \gamma_0$ and a straightforward calculation shows that it has negative chirality. We can now define the bilinears $\zeta \otimes \bar{\zeta}$ and $\zeta \otimes \bar{\zeta}^c$ obtaining a couple of one-forms (or vectors), z_μ and a_μ:

$$\zeta \otimes \bar{\zeta} = \frac{1}{2}\bar{\zeta}\gamma_\mu\zeta\gamma^\mu = z_\mu dx^\mu \ ,$$
$$\zeta \otimes \bar{\zeta}^c = \frac{1}{2}\bar{\zeta}^c\gamma_\mu\zeta\gamma^\mu = a_\mu dx^\mu \ ; \tag{5.2.5}$$

our aim is now to understand the geometrical properties of both.

To start with, z and a are null: a simple Fierz computation gives

$$2z\zeta = \bar{\zeta}\gamma_\mu\zeta\gamma^\mu\zeta = \gamma^\mu\zeta\bar{\zeta}\gamma_\mu\zeta = 0 \ , \tag{5.2.6}$$

---

[3]We will work with real gamma matrices both in Mink$_2$ and in $\mathcal{M}_8$; such a basis in eight dimensions can be defined in terms of octonions [67]. Therefore the Majorana conjugates ζ^c and $\eta_\pm^{c\,i}$ are just the naive conjugates $(\zeta)^*$ and $(\eta_\pm^i)^*$.

where we used the well-known relation $\gamma^\mu C_k \gamma_\mu = (d - 2k)C_k$. From (5.2.6) it follows $z^2 = 0$ and an identical computation shows that also a is null. Moreover z and a are proportional since we have

$$z \zeta \bar{\zeta}^c = 0 \ , \tag{5.2.7}$$

as an obvious consequence of (5.2.6); recalling the formula $\gamma^\mu C_k = (dx^\mu \wedge + g^{\mu\nu}\iota_\nu)C_k$, (5.2.7) can be rephrased as

$$z \wedge a = z \llcorner a = 0 \ , \tag{5.2.8}$$

telling us that a is proportional to z

$$a = g(x)z \ . \tag{5.2.9}$$

Finally, recalling that in two Lorentzian dimensions we have the identification

$$\gamma^{(2)} C_k = *_2 \lambda C_k \ , \tag{5.2.10}$$

relating the action from the left of the chiral operator to the Hodge dual operator, we conclude that both z and a are self-duals

$$*_2 z = z \ , \qquad *_2 a = a \ . \tag{5.2.11}$$

We can also determine the reality properties of these vectors. Evaluating the expression $\gamma^0 (\zeta\bar{\zeta})^\dagger \gamma^0$ (and the analogous one including $\zeta\bar{\zeta}^c$) one deduces that z is real and a is complex.

To conclude this section we note that z and a are d-closed: indeed the external differential acts on a bispinor of odd degree as

$$dz = d(\zeta\bar{\zeta}) = \frac{1}{2} \left[\gamma^\mu, D_\mu \zeta\bar{\zeta} \right] \ , \tag{5.2.12}$$

and using (5.2.4) one obtains

$$dz = da = 0 \ . \tag{5.2.13}$$

5.3 Supersymmetry conditions: general discussion

In this section we will specialize the system (3.2.4) to describe two-dimensional $N = (2,0)$ vacua obtaining a set of conditions for these particular backgrounds. The pairing equations will look a bit scary at first sight but in the next sections we will see that the situation is completely different when $\mathcal{M}_8$ enjoys a $SU(4) \times SU(4)$ structure.

5.3.1 Factorization

As explained in section 5.2 we will consider backgrounds with a metric of the form (5.2.1) and with a spinorial Ansatz like (5.2.2), we will also impose that our configuration is a vacuum, i.e. that the maximal symmetry of Mink_2 is preserved by all the fields.

Given the spinorial Ansatz (5.2.2) we can immediately compute the polyform Φ (equation (3.2.2))

$$
\begin{aligned}
\Phi &= \mp\big((\zeta\bar{\zeta})(\eta_+^1\eta_\mp^{2\dagger}) + (\zeta^c\bar{\zeta})(\eta_+^{1c}\eta_\mp^{2\dagger}) + \text{c.c.}\big) \\
&= \mp 2\text{Re}\left(e^A z \wedge \psi_1 + e^A a \wedge \psi_2\right) ,
\end{aligned}
\tag{5.3.1}
$$

where the decomposition (5.2.3) of the ten-dimensional gamma matrices is used. In (5.3.1) z and a are the two-dimensional vectors defined in (5.2.5), whereas with ψ_1 and ψ_2 we denote the eight-dimensional bilinears

$$
\psi_1 \equiv \eta_+^1\eta_\mp^{2\dagger} , \qquad \psi_2 \equiv \eta_+^1\eta_\mp^{2c\dagger} .
\tag{5.3.2}
$$

Notice that, since not every eight-dimensional Weyl spinor η_+ is pure, ψ_1 and ψ_2 are not in general pure spinors on the generalized eight-dimensional tangent bundle $T_8 \oplus T_8^*$ and so in general they do not induce a reduction of the structure group to $SU(4) \times SU(4)$. Further details about this point will be presented in section 5.4.

We need also the vectors K and $\tilde{K}$ appearing in (3.2.1). To this end we start by computing K_1 obtaining

$$
32K_1 = e^{-A}\big[4z||\eta_+^1||^2 + 2a(\eta_+^1)^2 + \text{c.c}\big] , \qquad ||\eta_+^1||^2 \equiv \eta_+^{1\dagger}\eta_+^1 , \qquad (\eta_+^1)^2 \equiv (\eta_+^{1t}\eta_+^1)
\tag{5.3.3}
$$

notice that $||\eta_+^1||^2$ is real, and $(\eta_+^1)^2$ is complex. If we now impose[4]

$$
||\eta_+^1||^2 = ||\eta_\mp^2||^2 , \qquad (\eta_+^1)^2 = (\eta_\mp^2)^2 ,
\tag{5.3.4}
$$

we see that K_2 takes exactly the same expression of K_1. Therefore we conclude that K and $\tilde{K}$ are

$$
K = \frac{e^{-A}}{8}\big(z||\eta_+^1||^2 + \text{Re}(a(\eta_+^1)^2)\big) , \qquad \tilde{K} = 0 .
\tag{5.3.5}
$$

It remains to consider the factorization of the fluxes and of the NSNS three-form H. Exactly as we did in Chapter 4 the requirement of maximal symmetry in two dimensions imposes that all these fields (and also the dilaton) do not depend on the external

---

[4]When $\mathcal{M}_8$ is compact a famous no-go theorem requires the presence of sources with negative tension like orientifold planes [68], [69]. The requirement that such orientifolds be supersymmetric imposes the conditions (5.3.4) that therefore has to be considered as a necessary condition and not as an assumption [56].

coordinates x^μ. Moreover the indices structure of them must be of the form

$$H = H_0 + H_2 \ ,$$
$$F = F_0 + F_2 = f + e^{2A}\text{vol}_2 \wedge *_8\lambda(f) \ , \tag{5.3.6}$$

where the indices indicate the number of external components, f is an internal polyform and the self-duality of F (equation (2.1.9)) is used. We can now move to discuss the system of equations (3.2.4) for these particular vacua.

5.3.2 Symmetry equations

To begin we consider the symmetry equations, i.e. the equations (3.2.4a). The first equation require that K would be a Killing vector, however K takes the expression (5.3.5) and we already know that z and a are Killing vectors by construction (they are constant), therefore we obtain the constraints

$$||\eta_+^1||^2 = \alpha e^A \ ,$$
$$(\eta_+^1)^2 = (\beta + i\delta)e^A \ , \tag{5.3.7}$$

where α, β and δ are real constants. Moving to the second equation in (3.2.4a) it is straightforward to see (using (5.3.5)) that this equation implies

$$H_2 = 0 \ , \tag{5.3.8}$$

therefore in the following we will write H to simply indicate H_0.

5.3.3 Exterior equation

We turn now to discuss the exterior equation (3.2.4b). We start by evaluating the r.h.s. in (3.2.4b); it reads

$$-(\tilde{K} \wedge + \iota_K)F = \frac{1}{8}(\alpha + \beta\text{Re}(g) - \delta\text{Im}(g))e^{2a}z \wedge *_8\lambda(f) \ , \tag{5.3.9}$$

where we used (5.3.5), (5.3.6), (5.3.7), (5.2.9), the self-duality of z and the relation (valid for any d even)

$$*\lambda(dx^\mu\wedge) = -\iota_\mu * \lambda \ . \tag{5.3.10}$$

Therefore, using the expression (5.3.1) for the polyform Φ, equation (3.2.4b) becomes

$$d_H\big(e^{A-\phi}\text{Re}(z \wedge \psi_1 + a \wedge \psi_2)\big) = \mp\frac{1}{16}(\alpha + \beta\text{Re}(g) - \delta\text{Im}(g))e^{2A}z \wedge *_8\lambda(f) \ , \tag{5.3.11}$$

that can be decomposed in the couple of equations

$$d_H(e^{A-\phi}\text{Re}\psi_1) = \pm\frac{\alpha}{16}e^{2A} *_8 \lambda(f) \ ,$$
$$d_H(e^{A-\phi}\psi_2) = \pm\frac{\beta + i\delta}{16}e^{2A} *_8 \lambda(f) \ . \tag{5.3.12}$$

5.3.4 Pairing equations

It remains to consider the pairing equations (3.2.4c) and (3.2.4d). We will present the computation only for (3.2.4c) since (3.2.4d) is completely parallel. The first part of the analysis will be very similar to the corresponding one presented in [11] for four-dimensional vacua and therefore we will be brief.

To start with, we have to choose the vectors e_{+1} and e_{+2}. Since we have $K_1 = K_2 = K$ we can take $e_{+1} = e_{+2} = e_+$ as well, moreover we take e_+ purely external as K and the action of the gamma matrices $\overrightarrow{\gamma}_+$ and $\overleftarrow{\gamma}_+$ takes the form

$$\overrightarrow{\gamma}_+ = e^A e_+ \wedge + e^{-A} e_{+\llcorner} , \qquad \overleftarrow{\gamma}_+ (-)^{\deg} = e^A e_+ \wedge - e^{-A} e_{+\llcorner} . \tag{5.3.13}$$

Now we can compute the various terms appearing in (3.2.4c): since e_+ is purely external the term containing $d^\dagger(e^{-2\phi} e_+)$ vanishes, moreover the term $d_H(e^{-\phi}\Phi \cdot e_+)$ can be massaged using

$$\left\{ d, \overleftarrow{\gamma}_+ (-)^{\deg} \right\} = e^{-A}\partial_+ + dA \wedge \overrightarrow{\gamma}_+ . \tag{5.3.14}$$

Summarizing, (3.2.4c) becomes

$$\left(\gamma_+ \cdot \Phi \cdot \gamma_+, \Gamma^{MN}\left[dA \wedge \gamma_+ e^{-\phi}\Phi - 2f\right]\right) = 0 , \tag{5.3.15}$$

where we used (2.1.9) and (5.3.6).

We have now to evaluate (5.3.15) for the various possible choices of the indices M and N. It is straightforward to see that for M and N both internal or external the equation reduces to an identity and so it has no content. Therefore the only non trivial equations come when we have $M = m$ and $N = \mu$. We start by computing the factor

$$-2\left(\gamma_+ \cdot \Phi \cdot \gamma_+, \Gamma^m \Gamma^\mu f\right) = \pm\frac{1}{16}\bar\epsilon_1 \gamma_+ \Gamma^m \Gamma^\mu f \gamma_+ \epsilon_2 , \tag{5.3.16}$$

where we used the identity

$$\left(\gamma_{+1} \cdot \Phi \cdot \gamma_{+2}, C\right) = -\frac{(-)^{\deg\Phi}}{32}\bar\epsilon_1 \gamma_{+1} C \gamma_{+2}\epsilon_2 , \tag{5.3.17}$$

that can be found in [11]. Using now the equations (5.2.2) and (5.2.3), we can further massage (5.3.16) obtaining

$$-2\left(\gamma_+ \cdot \Phi \cdot \gamma_+, \Gamma^m \Gamma^\mu f\right) = \frac{1}{16}\left(\bar\zeta\gamma_+\gamma^\mu\gamma_+\zeta\, \eta_+^{1\dagger}\gamma^m f\eta_\mp^2 + \bar\zeta\gamma_+\gamma^\mu\gamma_+\zeta^c\, \eta_+^{1\dagger}\gamma^m f\eta_\mp^{2c} + \text{c.c.}\right) , \tag{5.3.18}$$

where the reality of the gamma matrices γ^m and γ^μ was used.

A similar treatment can be reserved to the other term in (5.3.15) which finally takes the form

$$e^{-\phi}\left(\gamma_+ \cdot \Phi \cdot \gamma_+, \Gamma^m \Gamma^\mu[dA \wedge \gamma_+\Phi]\right) = \pm\frac{e^{-\phi}}{4}\left(\bar\zeta\gamma_+\gamma^\mu\gamma_+\zeta\, \eta_+^{1\dagger}\gamma^m \partial A\eta_\mp^2 + \bar\zeta\gamma_+\gamma^\mu\gamma_+\zeta^c\, \eta_+^{1\dagger}\gamma^m \partial A\eta_\mp^{2c} + \text{c.c.}\right) . \tag{5.3.19}$$

To proceed further we observe that the four-dimensional bilinears take the form

$$\bar{\zeta}\gamma_+\gamma^\mu\gamma_+\zeta \propto e_+^\mu \ ,$$
$$\bar{\zeta}\gamma_+\gamma^\mu\gamma_+\zeta^c \propto \bar{g}e_+^\mu \ , \tag{5.3.20}$$

therefore, requiring that (5.3.15) has a solution which is independent from ζ, we conclude that (5.3.18) and (5.3.19) give rise to the following equations

$$\mathrm{Re}(4\eta_+^{1\dagger}\gamma^m\partial A\eta_+^1 \pm e^\phi\eta_+^{1\dagger}\gamma^m f\eta_\mp^2) = 0 \ ,$$
$$4\eta_+^{1\dagger}\gamma^m\partial A\eta_+^{1c} \pm e^\phi\eta_+^{1\dagger}\gamma^m f\eta_\mp^{2c} = 0 \ , \tag{5.3.21}$$

that can be recast in a more familiar fashion

$$\mathrm{ReTr}\left(\eta_\mp^2\eta_+^{1\dagger}\gamma^m\left(4\partial A\frac{\eta_+^1\eta_\mp^{2\dagger}}{||\eta_\mp^2||^2} \pm e^\phi f\right)\right) = 0 \ ,$$
$$\mathrm{Tr}\left(\eta_\mp^{2c}\eta_+^{1\dagger}\gamma^m\left(4\partial A\frac{\eta_+^{1c}\eta_\mp^{2c\dagger}}{||\eta_\mp^2||^2} \pm e^\phi f\right)\right) = 0 \ , \tag{5.3.22}$$

or, in terms of the six-dimensional Chevalley-Mukai pairing, as

$$\mathrm{Re}\big(\gamma^m\bar{\psi}_1, dA \wedge \psi_1 \mp \frac{\alpha}{8}e^{\phi+A} *_8 \lambda(f)\big) = 0 \ ,$$
$$\big(\gamma^m\bar{\psi}_2, dA \wedge \bar{\psi}_1 \mp \frac{\alpha}{8}e^{\phi+A} *_8 \lambda(f)\big) = 0 \ . \tag{5.3.23}$$

Finally, equation (3.2.4d) can be treated in the same way and the final result is

$$\mathrm{Re}\big(\bar{\psi}_1\gamma^m, dA \wedge \psi_1 \mp \frac{\alpha}{8}e^{\phi+A} *_8 \lambda(f)\big) = 0 \ ,$$
$$\big(\bar{\psi}_2\gamma^m, dA \wedge \psi_1 \mp \frac{\alpha}{8}e^{\phi+A} *_8 \lambda(f)\big) = 0 \ . \tag{5.3.24}$$

5.3.5 Summary

We have rewritten the conditions for unbroken supersymmetry (equations (3.2.4)) for a Mink$_2$, $(2,0)$ vacuum solution. The resulting system of equations is given by (5.3.7), (5.3.8), (5.3.12), (5.3.23) and (5.3.24). Unfortunately equations (5.3.23) and (5.3.24) are not as elegant as (5.3.12) and this is a typical feature of the system (3.2.4). However we will see that, assuming that the structure group of $\mathcal{M}_8$ is $SU(4) \times SU(4)$, the pairing equations can be recast in a concise and elegant form.

5.4 Supersymmetry conditions: the pure case

In this section we will see (motivated by the results found in [18]) how SUSY conditions can be rewritten in a compact form if we make the assumption that the internal spinors

η^1_+ and $\eta^2_\mp$ are *pure*.[5] In this case it is possible to show that the structure group of the generalized tangent bundle $T_8 \oplus T_8^*$ reduces to $SU(4) \times SU(4)$ and this allows a better formulation of the pairing equations. The equations that we will find are already present in [18] but, contrary to that work, we will not assume that the two spinors η^1_+ and $\eta^2_\mp$ are proportional (notice that such an assumption can be done in Type IIB only). Therefore our results in this section can be seen as the extension from the *strict SU(4)* case (treated in [18]) to the $SU(4) \times SU(4)$ case.

5.4.1 Pure spinors and generalized Hodge diamonds

To pursue our goal we need some further technical elements concerning GCG and pure spinors on the generalized tangent bundle. For this reason we will devote this subsection to develop such elements and to recall what the purity condition on eight-dimensional spinors implies and what geometrical structures can be defined on $\mathcal{M}_8$ when η^1_+ and $\eta^2_\mp$ are pure.

Contrary to what happens in lower dimensions, in eight dimensions Weyl spinors are not necessarily pure, as shown by a simple counting argument: in eight dimensions the space of pure spinors has real dimension 14 whereas the space of Weyl spinors has real dimension 16. More explicitly, a given eight-dimensional Weyl spinor of (say) positive chirality η_+ is pure if and only if it satisfies the additional algebraic condition

$$\eta^t_+ \eta_+ = 0 \ . \tag{5.4.1}$$

Notice that a Majorana-Weyl spinor cannot be pure. In this section we will suppose that both η^1_+ and $\eta^2_\mp$ satisfy (5.4.1) and hence that they are pure.

Ley us now review how some basic facts already explained in section 3.1 gets translated in eight dimensions. A pure spinor in eight dimensions implies that the structure group on the tangent bundle gets reduced to $SU(4)$. This is equivalent to saying that on the manifold a real two-form J and a $(4,0)$-form (with respect to the almost complex structure defined by J) called Ω can be defined and they satisfy

$$\frac{1}{16}\Omega \wedge \bar{\Omega} = \frac{1}{24}J^4 = \mathrm{vol}_8 \ , \qquad J \wedge \Omega = 0 \ . \tag{5.4.2}$$

Having introduced J we can reformulate the purity condition of η_+ by saying that it is annihilated by the gamma matrices holomorphic with respect to the almost complex structure defined by J.

When we introduce a second spinor $\eta^2_\mp$ it is useful to consider the generalized tangent bundle $T \oplus T^*$, since in this enlarged space the structure group is always $SU(4) \times SU(4)$.

---

[5]Recall that a spinor is said to be pure if it is annihilated by exactly half of the gamma matrices.

On $T \oplus T^*$ we can define a $Cl(8,8)$ algebra, with the corresponding gamma matrices given by

$$\Gamma_\Lambda = \left\{ \partial_1 \llcorner, \ldots, \partial_8, dx^1 \wedge, \ldots, dx^8 \wedge \right\} \ , \tag{5.4.3}$$

and with spinors simply given by the differential forms of all degrees. On $T \oplus T^*$ it is then convenient to define the bilinears ψ_1 and ψ_2 as in (5.3.2) and, thanks to the purity of η_+^1 and $\eta_\mp^2$, we are sure that they are pure with respects to the $Cl(8,8)$ algebra defined in (5.4.3).

Until now our discussion has been just a rephrasing in eight dimensions of the concepts explained in section 3.1. As anticipated, to rewrite the pairing equations (5.3.23) and (5.3.24) in an elegant form we need new technical tools, that we now turn to describe.

We have seen that, given a pure spinor η_+ on the ordinary tangent bundle, one can associate an almost complex structure $I^m{}_n$, which is simply related to the two-form J by the relation $g = -JI$, and that is an operator $I : T \to T$ such that $I^2 = -1$. A very similar concepts exists for generalized pure spinors: given a pure spinor ψ_i on the generalized tangent bundle one can associate a *generalized almost complex structure* $\mathcal{J}_i$ (GACS), i.e. an operator $\mathcal{J}_i : T \oplus T^* \to T \oplus T^*$ such that $\mathcal{J}_i^2 = -1$; the relation between ψ_i and $\mathcal{J}_i$ is given by the requirement that the i-eigenbundle of $\mathcal{J}_i$ coincides with the annihilator of ψ_i. Notice that, since the pure spinors ψ^i are polyforms, it is well defined the action of $\mathcal{J}_i \cdot \psi^i$ on them.

Finally, it can be shown that the compatibility relations (analogous to (3.1.17) but in eight dimensions) between two bispinors ψ_1 and ψ_2, constructed as bilinears of η_+^1 and $\eta_\mp^2$, can be translated in terms of the corresponding GACSs by requiring that they commute.

To rewrite the pairing equations (5.3.23) and (5.3.24) in an elegant form we need to introduce an appropriate basis for the differential forms on $\mathcal{M}_8$. To this end it is useful to consider the so-called generalized Hodge diamond, which constitutes a basis for the differential forms of any degrees constructed starting from ψ_1 and ψ_2. We can represent this basis as follows:

$$
\begin{array}{ccccccccc}
 & & & & \psi_1 & & & & \\
 & & & \psi_1 \gamma^{i_2} & & \gamma^{\bar{i}_1} \psi_1 & & & \\
 & & \psi_1 \gamma^{i_2 j_2} & & \gamma^{\bar{i}_1} \psi_1 \gamma^{i_2} & & \gamma^{\bar{i}_1 \bar{j}_1} \psi_1 & & \\
 & \psi_2 \gamma^{\bar{i}_2} & & \gamma^{\bar{i}_1} \psi_1 \gamma^{i_2 j_2} & & \gamma^{\bar{i}_1 \bar{j}_1} \psi_1 \gamma^{j_2} & & \gamma^{i_1} \bar{\psi}_2 & \\
\psi_2 & & \gamma^{\bar{i}_1} \psi_2 \gamma^{\bar{i}_2} & & \gamma^{\bar{i}_1 \bar{j}_1} \psi_1 \gamma^{i_2 j_2} & & \gamma^{\bar{i}_1} \bar{\psi}_2 \gamma^{i_2} & & \bar{\psi}_2 \\
 & \gamma^{\bar{i}_1} \psi_2 & & \gamma^{i_1 j_1} \bar{\psi}_1 \gamma^{\bar{i}_2} & & \gamma^{i_1} \bar{\psi}_1 \gamma^{\bar{i}_2 \bar{j}_2} & & \bar{\psi}_2 \gamma^{i_2} & \\
 & & \gamma^{i_1 j_1} \bar{\psi}_1 & & \gamma^{i_1} \bar{\psi}_1 \gamma^{\bar{i}_2} & & \bar{\psi}_1 \gamma^{\bar{i}_2 \bar{j}_2} & & \\
 & & & \gamma^{i_1} \bar{\psi}_1 & & \bar{\psi}_1 \gamma^{\bar{i}_2} & & & \\
 & & & & \bar{\psi}_1 & & & &
\end{array}
\tag{5.4.4}
$$

where the action of the gamma matrices on ψ_i is obviously obtained from the same action

on the spinors η^i.

This basis has the property of being *orthogonal*: every form has vanishing Chevalley-Mukai pairing with every form in the diamond, except with the ones symmetric with respect to the central point. So for example ψ_1 has non vanishing pairing only with $\bar{\psi}_1$, $\psi_1 \gamma^{i_2}$ only with $\bar{\psi}_1 \gamma^{\bar{i}_2}$ and so on. Another important technical property of this basis is that its entries are eigenfunctions for the action of $(\mathcal{J}_1 \cdot, \mathcal{J}_2 \cdot)$ corresponding to ψ_1 and ψ_2, and also for the operator $*_8 \lambda$. More explicitly, the eigenvalues for all these operators are

$$
\begin{array}{ll}
(\mathcal{J}_1 \cdot, \mathcal{J}_2 \cdot) : & \begin{array}{ccccc}
& & (4i, 0) & & \\
& (3i, i) & & (3i, -i) & \\
& (2i, 2i) & (2i, 0) & (2i, -2i) & \\
(i, 3i) & (i, i) & (i, -i) & (i, -3i) & \\
(0, 4i) \quad (0, 2i) & (0, 0) & (0, -2i) \quad (0, -4i) \\
(-i, 3i) & (-i, i) & (-i, -i) & (-i, -3i) \\
& (-2i, 2i) & (-2i, 0) & (-2i, -2i) & \\
& (-3i, i) & & (-3i, -i) & \\
& & (-4i, 0) & &
\end{array}
\end{array}
$$

$$
*_8 \lambda : \quad
\begin{array}{ccccc}
& & & + & & & & \\
& & + & & - & & & \\
& + & & - & & + & & \\
+ & & - & & + & & - & \\
+ & & - & + & & - & & + \\
& - & & + & & - & + & \\
& & + & & - & & + & \\
& & & - & + & & & \\
& & & & + & & &
\end{array}
\tag{5.4.5}
$$

5.4.2 Rewriting SUSY conditions in the pure case

We have now all the instruments necessary to massage the system of equations found in section 5.3 with the assumption that η^1_+ and $\eta^2_\mp$ are pure.

First of all, to stay closer to the results of [18], we perform the following redefinitions:

$$
\psi_1 = \frac{1}{e^A} \eta^1_+ \eta^{2\dagger}_\mp , \qquad \psi_2 = \frac{1}{e^A} \eta^1_+ \eta^{2c\dagger}_\mp .
\tag{5.4.6}
$$

Next we move to the symmetry equations (5.3.7): it is straightforward to see that the second equation implies

$$
\beta = \delta = 0 ,
\tag{5.4.7}
$$

since η^1_+ and $\eta^2_\mp$ are pure. Therefore we can interpret the geometrical role of β and δ as parametrizing the departure from the purity condition. We will discuss this last statement in a more geometrical language in section 5.5.

Moving to the exterior equations (5.3.12), taking into account the redefinition (5.4.6) and the vanishing of β and δ, they become

$$
\begin{aligned}
d_H(e^{2A-\phi} \mathrm{Re}\psi_1) &= \pm \frac{\alpha}{16} e^{2A} *_8 \lambda(f) , \\
d_H(e^{2A-\phi} \psi_2) &= 0 .
\end{aligned}
\tag{5.4.8}
$$

It remains to consider the pairing equations. To start with we see that, using the orthogonality of the generalized Hodge diamond, the second equation in (5.3.23) can be simplified

$$(\gamma^{i_1}\bar{\psi}_2, f) = 0 \ , \tag{5.4.9}$$

and analogously the second equation in (5.3.24) becomes

$$(\bar{\psi}_2\gamma^{i_2}, f) = 0 \ . \tag{5.4.10}$$

Collecting the results we have the following expression for the pairing equations[6]

$$\begin{aligned}
&\left(\gamma^{i_1}\bar{\psi}_1, dA \wedge \psi_1 \mp \frac{\alpha}{8}e^\phi *_8 \lambda(f)\right) = 0 \ , \\
&(\gamma^{i_1}\bar{\psi}_2, f) = 0 \ , \\
&\left(\bar{\psi}_1\gamma^{\bar{i}_2}, dA \wedge \psi_1 \mp \frac{\alpha}{8}e^\phi *_8 \lambda(f)\right) = 0 \ , \\
&(\bar{\psi}_2\gamma^{i_2}, f) = 0 \ .
\end{aligned} \tag{5.4.11}$$

By a direct computation, using the properties contained in (5.4.5), it can be shown that the equations in (5.4.11) are equivalent to the single equation

$$d_H^{\mathcal{J}_2}(e^{-\phi}\mathrm{Im}\psi_1) = \pm\frac{\alpha}{16}f \ , \tag{5.4.12}$$

where $d_H^{\mathcal{J}_2} \equiv [d_H, \mathcal{J}_2\cdot]$. The equivalence between (5.4.11) and (5.4.12) is in appendix C.

5.4.3 Summary

Let us summarize the results of this section. We have shown that, assuming the purity of the spinorial parameters η_+^1 and $\eta_\mp^2$, SUSY equations can be reformulated in terms of three conditions

$$\boxed{\begin{aligned}
&d_H(e^{2A-\phi}\mathrm{Re}\psi_1) = \pm\frac{\alpha}{16}e^{2A} *_8 \lambda(f) \ , \\
&d_H(e^{2A-\phi}\psi_2) = 0 \ , \\
&d_H^{\mathcal{J}_2}(e^{-\phi}\mathrm{Im}\psi_1) = \pm\frac{\alpha}{16}f \ .
\end{aligned}} \tag{5.4.13}$$

These equations were already found in [18] under the simplifying hypothesis of strict $SU(4)$ structure (and so only Type IIB theory was considered in that work). Therefore we have shown in this section that the results of [18] can be extended to the more general situation in which the SUSY parameters are *not* proportional, and this allows to treat Type IIA and Type IIB on the same footing.

---

[6]Notice that we have removed the real part in front of the first equations in (5.3.23) and (5.3.24). This is due to the fact that now the holomorphic (or anti-holomorphic) gamma matrices appear.

Our result is in perfect agreement with the results of [64]: in that work it is shown that the calibrations issues involve only the symmetry equations (3.2.4a) and the exterior equation (3.2.4b). On the other hand the pairing equations (3.2.4c) and (3.2.4d) have no counterpart in the calibrations recipe and indeed we find that the additional equation (5.4.12) is given exactly by the pairing equations.

Of course it would be interesting to look for a generalization of the supersymmetry-calibrations correspondence which takes into account the pairing equations. Obtaining such a correspondence could give a more geometrical understanding of the pairing equations and perhaps a better formulation for them.

5.5 Beyond the pure case

In this section we will remove the hypothesis that η^1_+ and $\eta^2_\mp$ are pure spinors on $\mathcal{M}_8$. Nevertheless we will assume that a pair of pure spinors $\tilde{\eta}^1_+$ and $\tilde{\eta}^2_\mp$ on $\mathcal{M}_8$ exists. In other words we will assume that the structure group of the generalized tangent bundle on $\mathcal{M}_8$ is still $SU(4) \times SU(4)$ but the SUSY parameters η^1_+ and $\eta^2_\mp$ are *not* the spinors realizing the reduction of the structure group. It will become clear in section 5.5.1 that, at least locally, given a Weyl spinor η one can always obtain a corresponding pure spinor $\tilde{\eta}$, by simply taking its real and imaginary parts and by rescaling them; however globally some obstructions can occur. In this section we will assume that such global obstructions do not occur and that we can find a pair of globally defined pure spinors.

5.5.1 Parametrization of non-pure spinors

Given the assumption that a pair of pure spinors on $\mathcal{M}_8$ exists we want to determine a parametrization of η^1_+ and $\eta^2_\mp$ in terms of the pure spinors $\tilde{\eta}^1_+$ and $\tilde{\eta}^2_\mp$.

To this end we start by recalling that a Weyl spinor (not Majorana) η[7] can be written in terms of two Majorana-Weyl spinors χ_1 and χ_2 as follows

$$\eta = \chi_1 + i\chi_2 . \tag{5.5.1}$$

(5.5.1) gives us a simple geometrical interpretation of the purity condition (5.4.1) as an orthonormality property of the spinors χ_1 and χ_2: indeed it is straightforward to see that η is pure if and only if χ_1 and χ_2 satisfy

$$\chi_1^t \chi_1 = \chi_2^t \chi_2 , \qquad \chi_1^t \chi_2 = 0 . \tag{5.5.2}$$

In other words, a Weyl spinor η is pure if and only if its Majorana-Weyl components χ_1 and χ_2 have the same norms (the first condition in (5.5.2)) and they are orthogonal (the

---

[7]We have not written the chirality of η since the discussion does not depend on it.

second condition in (5.5.2)). On the other hand, we see that the obstacles to the purity of η are given by a difference in the norms of χ_1 and χ_2 or if they are not orthogonal.

To proceed, suppose that we have, beyond the non-pure spinor η, a pure spinor $\tilde{\eta}$ with the same chirality and with components $\tilde{\chi}_1$ and $\tilde{\chi}_2$. For future convenience we take the norms of $\tilde{\chi}_1$ and $\tilde{\chi}_2$ to be equal to $e^{A(y)}$ (where $A(y)$ is of course the warping factor appearing in (5.2.1))

$$\tilde{\chi}_1^t \tilde{\chi}_1 = \tilde{\chi}_2^t \tilde{\chi}_2 = e^{A(y)} \quad \Rightarrow \quad ||\tilde{\eta}||^2 = 2e^{A(y)} \ , \tag{5.5.3}$$

we also apply a rotation to $\tilde{\eta}$ in order to put $\tilde{\chi}_1$ along χ_1. A pictorial description of this construction is given in figure 5.1 which shows that η can be parametrized in terms of $\tilde{\eta}$ (and its complex conjugate) via the formula

$$2\eta = \left(A_1 + iB_1 e^{-i\theta_1}\right)\tilde{\eta} + \left(A_1 + iB_1 e^{i\theta_1}\right)\tilde{\eta}^c \ , \tag{5.5.4}$$

where the real quantities A_1 and B_1 are given by

$$A_1 = \sqrt{\frac{\chi_1^t \chi_1}{e^{A(y)}}} \ , \qquad B_1 = \sqrt{\frac{\chi_2^t \chi_2}{e^{A(y)}}} \ , \tag{5.5.5}$$

and θ_1 parametrizes the angle between χ_1 and χ_2.

As a check of the validity of this parametrization notice that $\tilde{\eta}$ is a pure spinor of fixed norm, hence it has 13 real components; on the other hand A_1, B_1 and θ_1 are real coefficients. This gives us a total of 16 real components for η which is correct for a Weyl non-pure spinor on $\mathcal{M}_8$. We note also that in the pure limit we have $A_1 = B_1$ and $\theta_1 = \frac{\pi}{2}$ for a total of 14 real components as it should.

These considerations can be applied to the SUSY parameters η_+^1 and $\eta_\mp^2$ which in terms of the pure spinors $\tilde{\eta}_+^1$ and $\tilde{\eta}_\mp^2$ read

$$\begin{aligned} 2\eta_+^1 &= c_1 \tilde{\eta}_+^1 + c_2 \tilde{\eta}_+^{1c} \ , \\ 2\eta_\mp^2 &= c_3 \tilde{\eta}_\mp^2 + c_4 \tilde{\eta}_\mp^{2c} \ , \end{aligned} \tag{5.5.6}$$

where

$$\begin{aligned} c_1 &\equiv A_1 + iB_1 e^{-i\theta_1} \ , & c_2 &\equiv A_1 + iB_1 e^{i\theta_1} \ , \\ c_3 &\equiv A_2 + iB_2 e^{-i\theta_2} \ , & c_4 &\equiv A_2 + iB_2 e^{i\theta_2} \ , \end{aligned} \tag{5.5.7}$$

we will see in a moment that all these coefficients are *constant* on $\mathcal{M}_8$. Thanks to the parametrization (5.5.6) we can now massage the conditions for unbroken supersymmetry deduced in section 5.3.
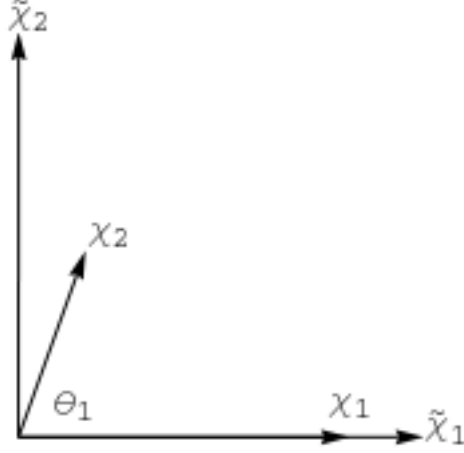
Figure 5.1: A pictorial description of the parametrization (5.5.4). The Majorana-Weyl components of the non-pure spinor η are χ_1 and χ_2; they can be represented as a couple of vectors with different norms and forming an angle θ_1. On the other hand the Majorana-Weyl components of the pure spinor $\tilde{\eta}$ are given by $\tilde{\chi}_1$ and $\tilde{\chi}_2$; they have the same norm $\tilde{\chi}_1^t \tilde{\chi}_1 = \tilde{\chi}_2^t \tilde{\chi}_2 = e^{A(y)}$ and they are orthogonal. A_1 and B_1 appearing in (5.5.4) are given by $A_1 = \sqrt{\frac{\chi_1^t \chi_1}{e^{A(y)}}}$, $B_1 = \sqrt{\frac{\chi_2^t \chi_2}{e^{A(y)}}}$.

5.5.2   Symmetry equations

We start by massaging the symmetry equations that we already wrote in full generality in (5.3.7). Putting (5.5.6) in (5.3.7) and using the assumption that $||\tilde{\eta}_+^1||^2 = ||\tilde{\eta}_\mp^2||^2 = 2e^A$ we obtain, after some manipulations, the equations

$$\beta + i\delta = c_1 c_2 , \qquad \beta + i\delta = c_3 c_4 ,$$
$$2\alpha = |c_1|^2 + |c_2|^2 , \qquad 2\alpha = |c_3|^2 + |c_4|^2 . \tag{5.5.8}$$

If we recall the definitions of the coefficients c_i given in (5.5.7), we see that (5.5.8) leads to

$$\alpha = A_{1,2}^2 + B_{1,2}^2 , \qquad \beta = A_{1,2}^2 - B_{1,2}^2 , \qquad \delta = 2A_{1,2}B_{1,2}\cos\theta_{1,2} , \tag{5.5.9}$$

which clarifies the geometrical interpretation of β and δ: they express the departure from the purity condition, β parametrizes a difference in the norms of the Majorana-Weyl components, δ keeps into account a lacking of orthogonality.

As an immediate consequence of (5.5.9) we see that $c_1 = c_3$, $c_2 = c_4$ and, more important, that they are *constant* as promised. We therefore rewrite (5.5.6) as

$$2\eta_+^1 = c_1 \tilde{\eta}_+^1 + c_2 \tilde{\eta}_+^{1c} ,$$
$$2\eta_\mp^2 = c_1 \tilde{\eta}_\mp^2 + c_2 \tilde{\eta}_\mp^{2c} . \tag{5.5.10}$$

5.5.3 Exterior equations

Let us now consider the exterior equations (5.3.12). Having introduced the pure spinors $\tilde{\eta}^1_+$ and $\tilde{\eta}^2_\mp$ we can use the parametrization (5.5.10) to deduce an analogous parametrization of the bilinears ψ_1 and ψ_2 in terms of the pure spinors $\tilde{\psi}_1$ and $\tilde{\psi}_2$ constructed from $\tilde{\eta}^1_+$ and $\tilde{\eta}^2_\mp$:

$$
\psi_1 = \frac{1}{4}\left[|c_1|^2\tilde{\psi}_1 + |c_2|^2\bar{\tilde{\psi}}_1 + c_1\bar{c}_2\tilde{\psi}_2 + \bar{c}_1 c_2\bar{\tilde{\psi}}_2\right] ,
$$
$$
\psi_2 = \frac{1}{4}\left[c_1 c_2(\tilde{\psi}_1 + \bar{\tilde{\psi}}_1) + c_1^2\tilde{\psi}_2 + c_2^2\bar{\tilde{\psi}}_2\right] , \tag{5.5.11}
$$

in the pure limit we have $\psi_1 = A_1^2\tilde{\psi}_1$ and $\psi_2 = A_1^2\tilde{\psi}_2$ as it should. (5.5.11) can be put into (5.3.12) that becomes

$$
2\alpha\, d_H\big(e^{2A-\phi}\mathrm{Re}\tilde{\psi}_1\big) + c_1\bar{c}_2 d_H\big(e^{2A-\phi}\tilde{\psi}_2\big) + \bar{c}_1 c_2 d_H\big(e^{2A-\phi}\bar{\tilde{\psi}}_2\big) = \pm\frac{\alpha}{4}e^{2A}*_8\lambda(f) ,
$$
$$
2c_1 c_2 d_H\big(e^{2A-\phi}\mathrm{Re}\tilde{\psi}_1\big) + c_1^2 d_H\big(e^{2A-\phi}\tilde{\psi}_2\big) + c_2^2 d_H\big(e^{2A-\phi}\bar{\tilde{\psi}}_2\big) = \pm\frac{c_1 c_2}{4}e^{2A}*_8\lambda(f) , \tag{5.5.12}
$$

where we used $2\alpha = |c_1|^2 + |c_2|^2$. At first sight these equations are not as pleasant as one might wish however, by simply expressing the coefficients c_1, c_2 and α in terms of A_1, B_1 and θ_1 as in (5.5.7) and (5.5.9), and by separating the real and the imaginary part in the second equation in (5.5.12), it can be shown with some simple manipulations that they are equivalent to

$$
d_H(e^{2A-\phi}\mathrm{Re}\tilde{\psi}_1) = \pm\frac{1}{8}e^{2A}*_8\lambda(f) ,
$$
$$
d_H(e^{2A-\phi}\tilde{\psi}_2) = 0 . \tag{5.5.13}
$$

Rewritten in this form the geometrical content of these equations is much more transparent: apart from the trivial redefinition $\tilde{\psi}_1 \to \frac{\alpha}{2}\tilde{\psi}_1$ we see that (5.5.13) take *exactly* the same form of the equations (5.4.8) which are valid in the pure case. In other words, we have deduced that, given the assumption that the structure group on $T_8 \oplus T_8^*$ is $SU(4) \times SU(4)$, the exterior equations, when expressed in terms of pure spinors on the generalized tangent bundle, take *always* the same form, no matter whether the spinorial parameters η^1_+ and $\eta^2_\mp$ are pure or not. It is possible that a better understanding of such a behaviour can be obtained from the calibrations perspective.

5.5.4 Pairing equations

It remains to massage the pairing equations that, as usual, are much more intricate than the others. The strategy can be easily described: as we have seen in section 5.4 and in appendix C, generalized complex geometry (and in particular the generalized

Hodge diamond (5.4.4) and its properties (5.4.5)) gives us a way to rewrite the pairing equations in a fancy form when η_+^1 and $\eta_\mp^2$ are pure. It is therefore conceivable that a similar simplification arises also in the non-pure case, thanks to the generalized Hodge diamond constructed from $\tilde{\psi}_1$ and $\tilde{\psi}_2$.

Given the strategy just described we show in appendix D that the pairing equations (5.3.23), (5.3.24) can be rewritten in terms of the pure spinors $\tilde{\psi}_1$ and $\tilde{\psi}_2$ as[8]

$$\left(\gamma^{i_1}\bar{\tilde{\psi}}_1, dA \wedge \tilde{\psi}_1 \mp \frac{1}{4}e^\phi *_8 \lambda(f)\right) = 0 ,$$
$$\left(\bar{\tilde{\psi}}_1\gamma^{\bar{i}_2}, dA \wedge \tilde{\psi}_1 \mp \frac{1}{4}e^\phi *_8 \lambda(f)\right) = 0 , \tag{5.5.14a}$$

and

$$\left(\gamma^{i_1}\bar{\tilde{\psi}}_2, *_8\lambda(f)\right) \mp \frac{8\bar{c}e^\phi}{\alpha}\left(\gamma^{i_1}\bar{\tilde{\psi}}_2, \frac{dA}{4} \wedge \tilde{\psi}_2\right) = \pm\omega^{i_1} ,$$
$$\left(\bar{\tilde{\psi}}_2\gamma^{i_2}, *_8\lambda(f)\right) \mp \frac{8\bar{c}e^\phi}{\alpha}\left(\bar{\tilde{\psi}}_2\gamma^{i_2}, \frac{dA}{4} \wedge \tilde{\psi}_2\right) = \pm\sigma^{i_2} , \tag{5.5.14b}$$

where we defined

$$\omega^{i_1} \equiv \mp\frac{(b+2\alpha)\bar{d}e^{2\phi}}{2\alpha\bar{e}}\left(\gamma^{i_1}\bar{\tilde{\psi}}_1, *_8\lambda(f)\right) ,$$
$$\sigma^{i_2} \equiv \mp\frac{(b+2\alpha)\bar{d}e^{2\phi}}{2\alpha\bar{e}}\left(\tilde{\psi}_1\gamma^{i_2}, *_8\lambda(f)\right) , \tag{5.5.15}$$

and the quantities b, c, d, e are defined in (D.0.2). We note that, apart from the trivial redefinition $\tilde{\psi}_1 \to \frac{\alpha}{2}\tilde{\psi}_1$ already noted after (5.5.13), (5.5.14a) again reproduces the corresponding ones valid in the pure case (first and third equations in (5.4.11)), on the other hand (5.5.14b) are similar to the pure case (second and fourth equations in (5.4.11)) but contain additional deformation pieces (that of course vanish in the pure limit). It is therefore natural to look for a formulation of (5.5.14) which is similar to (5.4.12), and indeed, using the same techniques of appendix C, we can recast (5.5.14) as

$$d_H^{\tilde{\mathcal{J}}_2}\left(e^{-\phi}\mathrm{Im}\tilde{\psi}_1\right) = \pm\frac{1}{8}f - \mathrm{Re}\left(\frac{2\bar{c}e^\phi}{\alpha} *_8 \lambda(dA\wedge\tilde{\psi}_2)\right) + \mathrm{Re}\left(\gamma^{\bar{i}_1}\tilde{\psi}_2\hat{\omega}_{\bar{i}_1}\right) - \mathrm{Re}\left(\psi_2\gamma^{\bar{i}_2}\hat{\sigma}_{\bar{i}_2}\right) , \tag{5.5.16}$$

where we introduced

$$\hat{\omega}_{\bar{i}_1} \equiv \delta_{\bar{i}_1j_1}\left(\gamma^{j_1}\bar{\tilde{\psi}}_2, \gamma^{\bar{l}_1}\tilde{\psi}_2\right)^{-1}\delta_{\bar{l}_1i_1}\omega^{i_1} , \qquad \hat{\sigma}_{\bar{i}_2} \equiv \delta_{\bar{i}_2j_2}\left(\bar{\tilde{\psi}}_2\gamma^{j_2}, \tilde{\psi}_2\gamma^{\bar{l}_2}\right)^{-1}\delta_{\bar{l}_2i_2}\sigma^{i_2} . \tag{5.5.17}$$

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[8]The indices i_1 and i_2 should be intended as $\tilde{i}_1$, $\tilde{i}_2$, meaning that we are taking holomorphic indices with respect to the almost complex structures defined by the pure spinors $\tilde{\eta}_+^1$ and $\tilde{\eta}_\mp^2$ respectively. However we will use the notations i_1 and i_2 just for simplicity.

5.5.5 Summary

Summarizing the results of this section, we have removed the purity condition (5.4.1) on the spinorial parameters. Nevertheless we have assumed that a couple of pure spinors $\tilde{\eta}^1_+$ and $\tilde{\eta}^2_\mp$ on $\mathcal{M}_8$ exists and in this way we have obtained the parametrization (5.5.6). The conditions for unbroken supersymmetry enforce the coefficients of this parametrization to be *constant* on $\mathcal{M}_8$. Moreover we have shown that the exterior equations (5.3.12), when rewritten in terms of the pure spinors $\tilde{\psi}_1$ and $\tilde{\psi}_2$, take exactly the same form of the pure case (5.4.8). On the other hand the pairing equations (3.2.4c), (3.2.4d) are different but nevertheless can be recast in an elegant form (5.5.16) which can be interpreted as a deformation of (5.4.12) valid in the non-pure case. The final system of equations is

$$
\begin{aligned}
&d_H(e^{2A-\phi}\mathrm{Re}\tilde{\psi}_1) = \pm\tfrac{1}{8}e^{2A} *_8 \lambda(f) \ , \\
&d_H(e^{2A-\phi}\tilde{\psi}_2) = 0 \ , \\
&d_H^{\tilde{\mathcal{J}}_2}\big(e^{-\phi}\mathrm{Im}\tilde{\psi}_1\big) = \pm\tfrac{1}{8}f - \mathrm{Re}\big(\tfrac{2\bar{c}e^\phi}{\alpha} *_8 \lambda(dA \wedge \tilde{\psi}_2)\big) + \mathrm{Re}\big(\gamma^{\bar{i}_1}\tilde{\psi}_2\hat{\omega}_{\bar{i}_1}\big) - \mathrm{Re}\big(\psi_2\gamma^{\bar{i}_2}\hat{\sigma}_{\bar{i}_2}\big) \ ,
\end{aligned}
$$

$$(5.5.18)$$

where the quantities $\hat{\omega}_{\bar{i}_1}$ and $\hat{\sigma}_{\bar{i}_2}$ are defined in (5.5.17).

Chapter 6

The ugly case: four-dimensional $N = 2$ backgrounds

In this chapter we will describe an example of ugly case by considering the conditions for unbroken supersymmetry for a four-dimensional reduction of type II supergravity with a spinorial ansatz including *two* external, four-dimensional, spinors. In other words, we will consider backgrounds that can be seen as solutions of $N = 2$ four-dimensional supergravity. By considering an analogous system of equations for four-dimensional $N = 2$ supergravity we will map the ten-dimensional equations to the corresponding four dimensional ones. We will see that the ten-dimensional system has more equations than the four-dimensional one and such equations can be interpreted as obstructions to the possibility of lifting a four-dimensional solution to ten-dimension. As just explained this is an example of ugly case: the pairing equations in this case are not redundant, they carry additional information but unfortunately they cannot be written in a fancy way. On the other hand we will discover that they suggest a G-structure system for four-dimensional $N = 2$ solutions without making any Ansatz on the form of the solution.

6.1 A motivation: lifting of four-dimensional solutions

It is well known that there is no guarantee that supergravity theories in dimensions lower than ten make sense as quantum theories. Indeed usually supersymmetry is not sufficient to eliminate the ultraviolet divergences that typically arise in gravity theories.[1] However in many cases, for AdS/CFT correspondence applications for example, it would be very important to know if a certain supersymmetric solution of a lower dimensional super-

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[1] Perhaps a notably exception is given by four-dimensional $N = 8$ supergravity

gravity theory can be extended to a solution of a "quantum" supergravity theory. On the other hand, ten-dimensional type II supergravities, and eleven dimensional supergravity too, are usually considered as "good" quantum supergravity theories since they can be embedded in string theory (or M-theory).

Many lower-dimensional supergravity actions can be seen as the product of a reduction procedure from ten-dimensional supergravity action. However, to understand whether a particular solution, found in four-dimensional $N = 2$ supergravity, can be embedded in ten-dimensions or not, the reduction procedure is not the best way to proceed: sometimes the reduced action misses important subtleties and the truncation of modes that defines it is often "non-consistent", in that it misses some equations of motion of the ten-dimensional action. This means that lifting a solution to ten dimensions is not guaranteed to work. For "vacuum" solutions (namely those of the type Mink$_4 \times M_6$ or AdS$_4 \times M_6$) experience shows that it is sometimes faster to look for solutions directly in ten dimensions.

Let us be more precise by recalling a typical example of reduction from ten to four dimensions: as is well known, reducing type II on a Calabi–Yau yields ungauged $N = 2$ supergravity; internal fluxes then correspond to gauging the theory (see for example [70–72]). The G-structure approach suggests that this might be true more generally for $SU(3) \times SU(3)$ structure manifolds [73]. This has been argued for by proceeding in two steps [74,75]: first, the ten-dimensional theory is formally rewritten as a four-dimensional action; second, one truncates to a finite set of internal forms. Already in the first step, one needs to set to zero certain modes that would in principle lead to additional gravitino fields, beyond the two one expects for an $N = 2$ theory; these are best avoided because they would lead to null states without a gauge invariance to gauge them away. This in turn leads to setting to zero also some internal RR fluxes, associated to the "edge of the Hodge diamond" (in the Calabi–Yau case they would correspond to cohomologies like $h^{1,0}$, $h^{2,0}$). In the second step, finding an appropriate set of internal forms is in general challenging [76], although it can be done on coset manifolds [77,78].

In this Chapter, we present an alternative approach to lifting four-dimensional BPS solutions to ten dimensions. We consider ten-dimensional type II theories on fibrations

$$ds^2_{10} = ds^2_4(x) + ds^2_6(x,y) \ : \tag{6.1.1}$$

the metric on the internal six-dimensional space M_6 (with coordinates y^m) is allowed to depend on the coordinates x^μ of the spacetime M_4 (corresponding to varying scalars in four dimensions);[2] a natural Ansatz is made for the supersymmetry parameters. We

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[2]The fibration is assumed to be topologically trivial; for this reason, we need not introduce connection terms in (6.1.1), and we can work in the gauge $g_{\mu m} = 0$ (for which there is no obstruction, since the fibration is trivial). In many applications M_4 is homeomorphic to $\mathbb{R}^4$, and the fibration is automatically

organize the *ten-dimensional* supersymmetry equations in such a way as to resemble the supersymmetry equations one gets in a four-dimensional $N = 2$ supergravity. The system includes some equations corresponding to multiplets one usually throws away in reductions to gauged supergravity. These additional equations which are not present in the four-dimensional system can be seen as an obstruction to the possibility of lifting solutions from four dimensions to ten.

In our approach, we avoid completely the truncation problem, since we are just rewriting the supersymmetry equations in four-dimensional language. We also avoid the gravitino problem: we are not attempting to write an action, but simply rewriting the supersymmetry equations. And indeed we get some equations that appear to be formally associated to the extra gravitinos, and some associated to the "edge of the diamond" vector multiplets which are usually set to zero (sections 6.5.6 and 6.5.7 below).

In the next sections we will describe how such an alternative approach to the problem of lifting solutions from four dimensions to ten dimensions can be applied in concrete.

6.2 Geometry of four-dimensional spinors

We begin by reviewing some facts about the geometry defined by four-dimensional spinors. In particular, in section 6.2.4 we show which exterior differentials ((6.2.32) below) are equivalent to the covariant derivatives of two spinors in the timelike case (to be defined in section 6.2.2). This result will be useful both for section 6.3, where we consider four-dimensional $N = 2$ supergravity, and in section 6.5 where we consider type II supergravity. The general case, beyond the timelike assumption, will be considered in section 6.6.

6.2.1 One spinor

Let us consider a single four-dimensional spinor ζ_+, of positive chirality. Most of this material was reviewed in [11] and [79].

It will be convenient to work with real gamma matrices. In this basis, the Majorana conjugate of ζ_+ is simply the naive conjugate $(\zeta_+)^* \equiv \zeta_-$. If we also introduce barred spinors $\overline{\zeta_\pm} \equiv \zeta_\pm^\dagger \gamma^0$, we can also define form bilinears; in particular, a one-form (or vector)

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topologically trivial. On the other hand, in other cases (such as in presence of black holes, as we will see shortly) spacetime does have nontrivial topological features, and the fibration may be non-trivial. It is outlined in [20] how our results would be changed in presence of such a non-trivial fibration.

(6.1.1) also sets to zero the so-called warping function A, an overall function of the internal coordinates y^m which in this context is not particularly natural; this complication could be easily added to our formalism. For similar reasons, the dilaton ϕ will be taken to depend on the spacetime coordinates, but not on the internal directions.

k and a two-form ω. This can be summarized by saying[3]

$$\zeta_+ \otimes \overline{\zeta}_+ = k + i * k \ , \qquad \zeta_+ \otimes \overline{\zeta}_- \equiv \omega \ . \tag{6.2.1}$$

It also follows that $*\omega = i\omega$.

By Fierzing we can see

$$k\,\zeta = -k\,\zeta_+ \qquad \Rightarrow \qquad k\,\zeta_+ = 0 \ , \tag{6.2.2}$$

which implies that $k^2 = 0$. (6.2.2) also implies that $[k, \zeta_+ \otimes \overline{\zeta}_-] = 0$, which translated into forms reads $k \wedge \omega = 0$. This means in turn that there exists a w such that

$$\omega = k \wedge w \ , \tag{6.2.3}$$

where w is a *complex* one-form, which also annihilates ζ_+. Since we now have

$$k\zeta_+ = w\zeta_+ = 0 \ , \tag{6.2.4}$$

ζ_+ is annihilated by two combinations of gamma matrices; in other words, it is a pure spinor.

One can now also show that

$$k \cdot w = w^2 = 0 = k^2 \ , \qquad w \cdot \bar{w} = 2 \ . \tag{6.2.5}$$

We can think of k and w as elements of a local frame: $k = e_+$, $w = e_2 - ie_3$. We have now exhausted the list of one-forms we can define from ζ_+ alone; we see that a single spinor is not enough to define a vielbein (similarly to the discussion for 10d spinors). In group theory terms, this is because ζ_+ has a stabilizer isomorphic to the group of two-dimensional translations $\mathbb{R}^2$,[4] and thus defines an $\mathbb{R}^2$ structure, rather than an identity structure which would be necessary to define a vielbein. It is then often convenient to complete the vielbein by introducing an additional null real one-form e_+ such that

$$(e_+)^2 = 0 \ , \qquad e_+ \cdot k = 1 \ , \qquad e_+ \cdot w = 0 \ , \tag{6.2.6}$$

as was done in [79] (and, of course, in [11] in ten dimensions with the construction that we reviewed in Chapter 3).

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[3]In this chapter $*$ will be the four-dimensional Hodge star operator unless otherwise noted.

[4]The stabilizer of the light-like vector k is $SO(2) \ltimes \mathbb{R}^2$; w breaks the $SO(2)$ to the identity. For more details see [11]. Alternatively, one can compute the stabilizer of ζ directly. In a vielbein where $k = e^+$, the stabilizer is spanned by γ^{+i}, where $i \neq -$. These generate the abelian Lie algebra $\mathbb{R}^2$.

6.2.2 Two spinors: the timelike and null cases

Since we will deal with $N = 2$ supergravity, we will also need to study the structure defined by two spinors ζ_{1+}, ζ_{2+}. To make our equations more readable, we will drop the subscript $_+$: it will be understood from now on that ζ_i, $i = 1, 2$ are Weyl spinors of positive chirality. Their Majorana conjugates will then be Weyl spinors of negative chirality and, as usual in $N = 2$ supergravity, they will be denoted with an upper index ζ^i:

$$\gamma\zeta_i = \zeta_i \ , \qquad \gamma\zeta^i = -\zeta^i \ . \tag{6.2.7}$$

The barred versions of the ζ_i will be denoted by $\overline{\zeta^i}$, since they have opposite chirality; and likewise for their complex conjugates $\overline{\zeta_i}$:

$$\overline{\zeta^i}\gamma = -\overline{\zeta^i} \ , \qquad \overline{\zeta_i}\gamma = \overline{\zeta_i} \ . \tag{6.2.8}$$

Each of the two spinors ζ_i will now define its own one-forms k_i, w_i, and two-forms $\omega_i = k_i \wedge w_i$, following section 6.2.1. However, we are now also able to define mixed bilinears:

$$\zeta_1 \otimes \overline{\zeta^2} \equiv v + i * v \ , \qquad \zeta_1 \otimes \overline{\zeta_2} \equiv \mu(1 + i\mathrm{vol}) + \omega \ . \tag{6.2.9}$$

This new ω satisfies $*\omega = i\omega$, just like the ω_i associated to the individual ζ_i.

The new vector v is almost entirely fixed by the k_i associated to the individual ζ_i as in (6.2.1). Indeed, one can show, in a similar way as (6.2.2),

$$v\zeta_1 = 0 \ , \qquad v\zeta_2 = -k_2\zeta_1 \ , \qquad \bar{v}\zeta_2 = 0 \ . \tag{6.2.10}$$

This in turn implies

$$v \cdot k_i = 0 \ , \qquad v^2 = 0 \ , \qquad v \cdot \bar{v} = -k_1 \cdot k_2 \ . \tag{6.2.11}$$

When the k_i are not proportional, these relations determine v up to a phase (as we will see later more explicitly). When the k_i are proportional, however, the ambiguity is greater. In other words, there is no formula for v in terms of the k_i alone. We could only find 'asymmetrical' formulas such as

$$v = \frac{1}{2\mu}(k_2 \cdot k_1 w_1 - k_2 \cdot w_1 k_1) = -\frac{1}{2\bar{\mu}}(k_1 \cdot k_2 w_2 - k_1 \cdot w_2 k_2) \ . \tag{6.2.12}$$

We saw in section 6.2.1 that a single spinor ζ defines an $\mathbb{R}^2$ structure. We now see that the structure defined by two spinors ζ_i depends again on whether they are parallel or not. When they are not parallel, the one-forms

$$k_1 \ , \qquad k_2 \ , \qquad \mathrm{Re}v \ , \qquad \mathrm{Im}v \tag{6.2.13}$$

together constitute a vielbein (up to an overall rescaling), thanks to $k_i^2 = 0$ and to (6.2.11). Thus the ζ_i define an identity structure. On the other hand, when the ζ_i are parallel their common stabilizer is just the stabilizer of one of them, namely $\mathbb{R}^2$.

In what follows, the vector

$$k \equiv \frac{1}{2}(k_1 + k_2) \tag{6.2.14}$$

will play a special role. When the two ζ_i are not proportional (and thus define an identity structure, and a vielbein), k is the sum of two different lightlike vectors, and thus must be timelike; from now on, we will call this *the timelike case*. On the other hand, when the ζ_i are proportional, the k_i are also proportional, and k is null; we will then call this *the null case* from now on.

Another useful indicator of which case we are dealing with is the complex quantity μ in (6.2.9). By comparing with (6.2.1), we see that μ should vanish in the null case (when the ζ_i are parallel). In fact one can be more precise:

$$-16|\mu|^2 = (\overline{\zeta^1}\zeta^2)(\overline{\zeta_2}\zeta_1) = \mathrm{Tr}(\zeta_1\overline{\zeta^1}\zeta^2\overline{\zeta_2}) = \mathrm{Tr}((1+\gamma)k_1(1-\gamma)k_2) = \mathrm{Tr}(2(1+\gamma)k_1k_2) = 8k_1 \cdot k_2 \;. \tag{6.2.15}$$

As we have seen, in the timelike case there is a natural vielbein, while in the null case there is none. Again to retain full generality, we will find it useful to introduce two null vectors

$$e_{+1} \;, \qquad e_{+2} \;, \tag{6.2.16}$$

satisfying

$$(e_{+i})^2 = 0 \;, \qquad e_{+i} \cdot k_i = 1 \;, \qquad e_{+i} \cdot w_i = 0 \;. \tag{6.2.17}$$

In the timelike case, k_1 and k_2 do not coincide and are both null; so we can just take e_{+1} proportional to k_2, and e_{+2} proportional to k_1. The proportionality constants can be fixed using (6.2.17):

$$e_{+1} = -\frac{k_2}{2|\mu|^2} \;, \qquad e_{+2} = -\frac{k_1}{2|\mu|^2} \qquad \text{(timelike case)}. \tag{6.2.18}$$

From (6.2.12) and (6.2.15) we then also get

$$v = -\bar{\mu}w_1 = \mu\bar{w}_2 \qquad \text{(timelike case)}. \tag{6.2.19}$$

In the null case, on the other hand, k_1 and k_2 are proportional: we have

$$\zeta_2 = \alpha(x)\,\zeta_1 \tag{6.2.20}$$

which leads us to

$$k_2 = |\alpha(x)|^2\,k_1 \;. \tag{6.2.21}$$

Therefore we conclude that there is no natural candidate for the e_{+i}. On the other hand, there is no need to pick two of them, and we can just as well say

$$e_{+2} = \frac{e_{+1}}{|\alpha|^2} \qquad \text{(null case)} \tag{6.2.22}$$

where the proportionality between e_{+1} and e_{+2} is fixed by requiring

$$e_{+i} \cdot k_i = 1 \qquad i = 1, 2 \ . \tag{6.2.23}$$

6.2.3 $SU(2)$-covariant formalism

In the timelike case, it will often be useful to collect the bilinears we introduced in section 6.2.2 in an $SU(2)$-covariant fashion.

We define[5]

$$\begin{aligned}
\zeta_i \otimes \overline{\zeta^j} &= (1 + i*) v_i{}^j = (1 + i*)(k\delta_i{}^j + v^x \sigma_{x\,i}{}^j) \ , \\
\zeta_i \otimes \overline{\zeta_j} &= \mu\epsilon_{ij}(1 + i\mathrm{vol}) + o_{ij} = \mu\epsilon_{ij}(1 + i\mathrm{vol}) + o^x \epsilon_{ik}\sigma^k_{xj} \ ,
\end{aligned} \tag{6.2.24}$$

which summarize (6.2.1), (6.2.9); notice that

$$v^1 = \mathrm{Re}\, v \ , \qquad v^2 = \mathrm{Im}\, v \ , \qquad v^3 = \frac{1}{2}(k_1 - k_2) \ , \tag{6.2.25}$$

while the vector k is precisely the same vector which we defined in (6.2.14).

Many of the properties we saw earlier can be now summarized more quickly. For example, one can find

$$v_{(i}{}^j \zeta_{k)} = 0 \quad \Rightarrow \quad v_{(i}{}^j v_{k)}{}^l = 0 \ , \tag{6.2.26}$$

which summarizes (6.2.2), (6.2.10) and (6.2.11). This tells us that $k \cdot v_x = 0$, $v_x \cdot v_y = \frac{1}{3}\delta_{xy} v_z \cdot v_z$, $k^2 = v_z \cdot v_z$. To get the overall normalization, one can instead perform a computation similar to (6.2.15), and obtain

$$v_i{}^k \bar{v}^l{}_j = 2|\mu|^2 \epsilon_{ij}\epsilon^{kl} \ . \tag{6.2.27}$$

This gives us

$$k^2 = -|\mu|^2 \ , \qquad v_x \cdot v_y = \delta_{xy}|\mu|^2 \ , \qquad k \cdot v_x = 0 \ . \tag{6.2.28}$$

This means that

$$\left\{ e^0 \equiv \frac{1}{|\mu|}k, \ e^x \equiv \frac{1}{|\mu|}v^x \right\} \tag{6.2.29}$$

is a vielbein.

---

[5]In our conventions, $\sigma^i_{x\,j}$ are the conventional Pauli matrices, while $\sigma_{x\,j}{}^i$ are their transposes; notice that the position of the index x does not play any role. Moreover, $\epsilon_{ij} = \epsilon^{ij} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$. We lower (raise) indices acting from the left (right) with ϵ: so for example $\sigma_{x\,ij} = \epsilon_{ik}\sigma_{x\,j}{}^k$, $\sigma^{ij}_x = \sigma^i_{x\,k}\epsilon^{kj}$.

6.2.4 Spinor derivatives in the timelike case, and spin connection

In supergravity we also need to discuss spinorial covariant derivatives $\nabla_\mu \zeta_i$. As usual, such covariant derivatives can be conveniently parameterized in terms of the so-called intrinsic torsions of a G-structure. As we saw in section 6.2.2, our two spinors define an identity structure in the timelike case, and an $\mathbb{R}^2$ structure in the null case. The timelike case is thus significantly simpler, since the intrinsic torsion is in this case nothing but the spin connection itself. We will discuss this case here, and leave the general case (where one might have null loci somewhere) to section 6.6.1.

In the timelike case, one might then think that the information about $\nabla_\mu \zeta_i$ is completely captured by the covariant derivatives of the vielbein (6.2.29). One might go to a frame where the ζ_i are constant, and reconstruct then $\nabla_\mu \zeta_i = \frac{1}{4} \omega_\mu^{ab} \gamma_{ab} \zeta_i$ from $\nabla_\mu e^a$ or de^a. This, however, forgets the information about the inner product $\mu = \frac{1}{4} \overline{\zeta_2} \zeta_1$. To see this more clearly, define

$$\nabla_\mu \zeta_i = p_{\mu\,i}{}^j \zeta_j \ . \tag{6.2.30}$$

The $4 \times 2 \times 2$ complex components of these p are more than the 4×6 real components of the spin connection ω^{ab}. The mismatch is due to the derivatives of μ. Indeed, we can compute in terms of these $p_i{}^j$ the covariant derivatives of the bilinears μ, k and v^x, and hence of the vielbein $\{e^0, e^x\}$ from (6.2.29); from the latter we can get the spin connection. Decomposing $p_i{}^j = p^0 \delta_i{}^j + p^x \sigma_{x\,i}{}^j$, we get

$$p^0 = \frac{d\mu}{2\mu} \ , \qquad p^x = \frac{1}{2} \left(\omega^{0x} + \frac{i}{2} \epsilon^{xyz} \omega_{yz} \right) \ . \tag{6.2.31}$$

Hence the components of the p^x correspond to the spin connection, while p^0 corresponds to the derivatives of μ. Thus the information in $\nabla_\mu \zeta_i$ is contained in the spin connection and in $d\mu$. The spin connection can be extracted from de^a, but since we need $d\mu$ anyway, we might as well use directly dk, dv^x.

We conclude then that the information in $\nabla_\mu \zeta_i$ is contained in

$$d\mu \ , \qquad dk_i \ , \qquad dv \ . \tag{6.2.32}$$

6.3 $N = 2$ four-dimensional supergravity: timelike solutions

In this section, we will reformulate the supersymmetry equations for four-dimensional $N = 2$ supergravity in terms of differential forms, using what we have learned in section 6.2. We will assume that the solution is timelike, in the sense specified in section 6.2.2:

namely, the Killing vector $k = \frac{1}{2}(k_1 + k_2)$ is taken to be timelike. This case is 'generic': even if, in a given solution, there happen to be subsets where $k^2 = 0$, these have measure zero. There are, however, also (non-generic) null solutions, where $k^2 = 0$ everywhere, an important example being $N = 1$ vacua. For this reason, in section 6.6 we will also take a look at the general case, giving a set of equations which will be inspired by the corresponding set that we will write in ten dimensions.

The timelike case is also notable in that the differential equations one obtains are much nicer than in the general case, in that they can be formulated only in terms of exterior differentials of spinors bilinears and nothing more. The equations that we will write in this section were already derived in [80], but we will be able to show that only a subset of their system is actually needed for supersymmetry.

After some general comments about $N = 2$ gauged supergravity in section 6.3.1, we will reformulate the conditions for supersymmetry (6.3.2) in sections 6.3.2, 6.3.3, 6.3.4, and briefly summarize in section 6.3.5 for the reader's convenience.

6.3.1 Supersymmetry equations

We will start by quickly recalling here some features of four-dimensional gauged $N = 2$ theories. This section is not meant to be a review of the general formalism, for which the reader may consult for example [81]. We will follow a notation similar to [80], which recently applied G-structures to the general theory.

A general $N = 2$ theory consists of:

- a graviton multiplet, which contains the metric $g_{\mu\nu}$, two gravitinos $\psi_{i\mu}$, $i = 1, 2$, and a vector A^0_μ (with field-strength $T_{\mu\nu}$);

- n_v vector multiplets, which contain vectors A^a_μ (with field-strength $G^a_{\mu\nu}$), gaugini λ^{ia} and complex scalars t^a ($a = 1, \ldots, n_v$), parametrizing a special Kähler manifold $\mathcal{SK}$;

- n_h hypermultiplets, which contain $4n_h$ scalars q^u ($u = 1, \ldots, 4n_h$) and $2n_h$ hyperini κ_α ($\alpha = 1, \ldots, 2n_h$); in this case the q^u span a quaternionic manifold $\mathcal{Q}$, whose vielbein is denoted by $U^{i\alpha}_u$.

The $n_v + 1$ vectors are then usually grouped with the notation A^Λ_μ ($\Lambda = 0, \ldots, n_v$). In gauged supergravity, some of these vectors will gauge some symmetries of the scalar manifolds $\mathcal{SK}$ and $\mathcal{Q}$, whose generators will be denoted by k^a_Λ and k^u_Λ respectively. This means that the vectors will appear in covariant derivatives

$$Dt^a = dt^a + g\, A^\Lambda k^a_\Lambda \,, \qquad Dq^u = dq^u + g\, A^\Lambda k^u_\Lambda \,. \tag{6.3.1}$$

Supersymmetry dictates that the Killing vectors k_Λ^a should be generated by momentum maps P_Λ, and that the k_Λ^u by hyper-momentum maps P_Λ^u.

We will look for solutions where the fermions are set to zero, so that supersymmetry will be unbroken if and only if the variations of the fermions $\psi_{i\mu}$, λ^{ia}, κ_α are zero. These read

$$\delta_\zeta \psi_{i\mu} = D_\mu \zeta_i + \left(T_{\mu\nu}^+ \gamma^\nu \epsilon_{ij} - \frac{1}{2} \gamma_\mu S_x \sigma_{ij}^x \right) \zeta^j = 0 \ , \tag{6.3.2a}$$

$$\delta_\zeta \kappa_\alpha = i U_{\alpha\, iu} D q^u \zeta^i + N_\alpha^i \zeta_i = 0 \ , \tag{6.3.2b}$$

$$\delta_\zeta \lambda^{ia} = i D t^a \zeta^i + \left((G^{a+} + W^a) \epsilon^{ij} + \frac{i}{2} W^{a\, x} \sigma_x^{ij} \right) \zeta_j = 0 \ . \tag{6.3.2c}$$

Here, one- and two-forms act as bispinors as we explained in Chapter 2 ; for Pauli matrix conventions, see footnote 5. The quantities S_x, W^a, W_x^a and N_α^i are related to the gauging data (the Killing vectors on $\mathcal{SK}$ and $\mathcal{Q}$ and their (hyper)-momentum maps). The covariant derivatives act on spinors as

$$D_\mu \zeta_i = \left(\nabla_\mu + \frac{i}{2} \hat{Q}_\mu \right) \zeta_i + \frac{i}{2} \hat{A}_\mu^x \sigma_i^{x\, j} \zeta_j \ , \tag{6.3.3}$$

where the connection $\hat{A}_\mu^x$ is defined from the $SU(2)$ connection A^x on the quaternionic manifold

$$\hat{A}_\mu^x \equiv \partial_\mu q^u A_u^x + g A_\mu^\Lambda P_\Lambda^x \ . \tag{6.3.4}$$

We will now analyze the geometrical content of (6.3.2). Unlike what happens in ten dimensions, each of these variations can be analyzed separately.

6.3.2 Gravitino equations

We will deal first with the gravitino equation $\delta \psi_{i\mu} = 0$ from (6.3.2a).

In general, the gravitino equation is the hardest to analyze in supergravity, since it involves derivatives of the spinors. However, as we saw in section 6.2.4, in the timelike case the information contained in $\nabla_\mu \zeta_i$ is equivalent to the information contained in the exterior derivatives of μ, k and v_x. Hence, the gravitino equation (6.3.2a) is equivalent to the equations for these quantities that can be computed from it; in the $SU(2)$-covariant formalism of section 6.2.3, they read

$$\boxed{\begin{aligned} D\mu &= S_x v_x - 2\iota_k T^+ \ , \\ dk &= -2\mathrm{Re}(S_x \bar{o}_x + 2\bar{\mu} T^+) \ , \\ Dv_x &= 2\epsilon_{xyz} \mathrm{Im}(\bar{S}_y o_z) \ . \end{aligned}} \tag{6.3.5}$$

The twisted external differential D acts as

$$D\mu = d\mu + i\hat{Q}\mu \,, \qquad Dv_x = dv_x + \epsilon_{xyz}\,\hat{A}_y \wedge v_z \,. \tag{6.3.6}$$

Apart from a few redefinitions, these are the same as (3.1), (3.3), (3.5) in [80]. Their (3.2) and (3.4) can be safely dropped: the system (6.3.5) is equivalent to the four-dimensional gravitino equation (6.3.2a) in the timelike case, already as it is. It is particularly pleasing that these equations only involve exterior differentials.

6.3.3   Hyperino equations

We now analyze the content of the hyperino equations (6.3.2b).

Since they do not involve any derivatives of the ζ_i, they are easier to understand. Their full geometrical content can be obtained by expanding along an appropriate basis of spinors. Since we are in the timelike case, this basis can be taken to be the ζ_i themselves. To project the (6.3.2b) on this basis, we can simply multiply them from the left by $\overline{\zeta_k}$. As we mentioned, the N^i_α in (6.3.2b) can be derived from the gauging data; the precise formula is

$$N^i_\alpha = g\,U_{\alpha j u}\bar{\mathcal{L}}^\Lambda k_\Lambda{}^u \epsilon^{ji} \,. \tag{6.3.7}$$

We get the equation

$$U_{\alpha i u}\big(i[k\,\delta^i_k + v_x\,\sigma^{x\,i}_k]\cdot Dq^u - g\,\bar{\mathcal{L}}^\Lambda k_\Lambda{}^u \mu\,\delta^i_k\big) = 0 \,. \tag{6.3.8}$$

Now, since $U_{\alpha i u}$ is a vielbein on the quaternionic manifold $\mathcal{Q}$, we have $U^u_{\alpha i}U^{\alpha i}_v = \delta^u_v$. Moreover, the tensors

$$\Omega^{xv}_u = i\sigma^{xk}_i U^{\alpha i}_u U^v_{\alpha k} \tag{6.3.9}$$

are a triplet of complex structures defined on $\mathcal{Q}$. Using this, one obtains the single equation

$$\boxed{\,i\,k \cdot Dq^v + \Omega^{xv}{}_u v_x \cdot Dq^u - g\,\bar{\mathcal{L}}^\Lambda k_\Lambda^v\,\mu = 0\,} \,. \tag{6.3.10}$$

This equation already appeared in [80, Eq. (3.24)].

6.3.4   Gaugino equations

It remains to consider the gaugino equations (6.3.2c). Just like for the hyperino equations, these do not involve any spinor derivatives; hence, again we can extract their full geometrical meaning by multiplying them from the left by $\overline{\zeta_k}$. This gives

$$i\,(v)_k{}^i \cdot Dt^a + \epsilon^{ij}o_{kj}\llcorner G^{a+} - \mu W^a\delta^i_k - \frac{i}{2}\,\mu W^{a\,x}\sigma^{x\,i}{}_k = 0 \,. \tag{6.3.11}$$

This appeared in [80, Eq. (3.7)]. It is a set of four scalar equations; we can also recast them as a single equation for a 1-form:

$$\boxed{2i\bar{\mu}Dt^a - 4\iota_k G^{a+} + 2W^a k - iW^{ax}v^x = 0}\ .\qquad(6.3.12)$$

This expression will be particularly useful for our comparison with ten-dimensional supersymmetry in section 6.5.

6.3.5 Summary: four-dimensional timelike case

We have found in this section that preserved supersymmetry is equivalent to the system given by the boxed equations (6.3.5), (6.3.10), (6.3.12). These come respectively from the variations of the gravitino, of the hyperinos and of the gauginos. The system is formulated in terms of exterior calculus only, and it does not have any redundancy.

6.4 Ten dimensions

We will now consider supersymmetry in ten-dimensional type II supergravity. As anticipated in the introduction, we will specialize the system of equations (3.2.4) to a topologically trivial fibration:

$$ds_{10}^2 = ds_4^2(x) + ds_6^2(x,y)\ .\qquad(6.4.1)$$

M_4 is a four-dimensional spacetime with coordinates x^μ and Lorentzian metric g_4, and M_6 is a compact space with coordinates y^m and Riemannian metric g_6, admitting an $SU(3) \times SU(3)$ structure. We will not introduce any a priori constraint on either g_4 or g_6. The extension of our results to fibrations which are topologically non-trivial is outlined in [20] and we will not review it here.

In section 6.5 we will rewrite this system assuming the timelike hypothesis and we will compare it to the four-dimensional system presented in section 6.3. Our results in this section will also help us in section 6.6, where we will present a system equivalent to supersymmetry in four-dimensional $N = 2$ supergravity without the timelike assumption.

We will start by discussing in sections 6.4.1 and 6.4.2 how to specialize it to the factorized geometry (6.4.1). We will then apply those considerations to each of the equations in (3.2.4), in sections 6.4.3, 6.4.4, 6.4.5; section 6.4.6 is a brief summary.

6.4.1 Factorization

As anticipated, we will consider a metric of the form (6.4.1), where x^μ are the coordinates on the four-dimensional space-time M_4 and y^m are the coordinates on the internal

manifold M_6.

As usual we decompose gamma matrices as

$$\Gamma^{(10)}_\mu = \gamma^{(4)}_\mu \otimes 1^{(6)} \,, \qquad \Gamma^{(10)}_m = \gamma^{(4)}_5 \otimes \gamma^{(6)}_m \,. \tag{6.4.2}$$

($\gamma^{(4)}_5$ was called simply γ in section 6.2). Passing to discuss the decomposition of the SUSY parameters ϵ_i, we recall that we want to look for solutions which can be understood as solutions of four-dimensional $N = 2$ supergravity, and so we impose a spinorial Ansatz of the form

$$\begin{aligned}
\epsilon_1 &= \zeta_1(x)\, \eta^1_+(x,y) \,+\, \zeta^1(x)\, \eta^1_-(x,y) \\
\epsilon_2 &= \zeta_2(x)\, \eta^2_\mp(x,y) \,+\, \zeta^2(x)\, \eta^2_\pm(x,y) \,.
\end{aligned} \tag{6.4.3}$$

Here ζ_i are spinors on M_4 of positive chirality (they are the same spinors the we introduced in section 6.2), and η^i_+ are spinors on M_6 of positive chirality, while $\eta^i_- = (\eta^i_+)^*$ are their Majorana conjugates, so that ϵ_i are Majorana.[6] Notice that $N = 1$ flux vacua (namely, solutions where M_4 is a maximally symmetric space, Minkowski$_4$ or AdS$_4$), can be obtained from (6.4.3) by setting $\zeta_1 = \zeta_2$. However, for solutions with four supercharges which are not vacua, one could use a more general Ansatz involving four ζ_i obeying some constraints.

The spinor Ansatz (6.4.3) immediately lets us compute some of the ingredients of (3.2.4): namely, $\Phi, K, \tilde{K}$. First we evaluate $\Phi = \epsilon_1 \otimes \overline{\epsilon_2}$:

$$\begin{aligned}
\Phi &= 2\mathrm{Re}[\mp\zeta_1\overline{\zeta^2} \wedge \phi_\mp + (\zeta_1\overline{\zeta_2}) \wedge \phi_\pm] \\
&= 2\mathrm{Re}\left[\mp(v + i*v) \wedge \phi_\mp + \left(\mu\left(1 + i\mathrm{vol}_4\right) + \omega\right) \wedge \phi_\pm\right] \,,
\end{aligned} \tag{6.4.4}$$

where we have used (6.2.9), and, as in [10],

$$\phi_\pm = \eta^1_+ \eta^{2\,\dagger}_\pm \tag{6.4.5}$$

are the six-dimensional pure spinors, which together define an $SU(3) \times SU(3)$ structure. The origin of the signs in (6.4.4) is explained in [20, App. A].

Let us now compute K_1 and K_2. As in the case of $N = 1$ vacua [11, Sec. 4.1.2], the six-dimensional components of these vectors vanish ($K^m_i = 0$, $m = 1, \ldots, 6$) because $\eta^\dagger_- \gamma^m \eta_+ = 0$. For external indices we have

$$K^\mu_1 = \frac{1}{4}k^\mu_1 ||\eta^1_+||^2 \,, \qquad K^\mu_2 = \frac{1}{4}k^\mu_2 ||\eta^2_+||^2 \,. \tag{6.4.6}$$

We now assume for simplicity that the norms of the η^a are equal:

$$||\eta^1_+||^2 = ||\eta^2_2||^2 \,. \tag{6.4.7}$$

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[6]We work in a basis where gamma matrices are real in four dimensions and purely imaginary in six, so that Majorana conjugation is just naive complex conjugation.

We do not expect that allowing unequal norms would lead to a substantial change in our discussion. For $N = 1$ vacua, one can actually even show that (6.4.7) is necessary [55, App. A.3]. K and $\tilde{K}$ take then the form

$$K^\mu = c \left(k_1^\mu + k_2^\mu \right) , \qquad \tilde{K}^\mu = c \left(k_1^\mu - k_2^\mu \right) , \qquad c \equiv \frac{||\eta||^2}{8} . \tag{6.4.8}$$

The name c anticipates that we will soon find that it has to be a constant.

6.4.2 Fluxes

The next ingredient of (3.2.4) we need to consider is the RR polyform F and the NSNS flux H too. Contrary to what we did in the precedent Chapters, in the following it will be useful to decompose F as

$$F = F_0 + F_1 + F_2 + F_3 + F_4 \tag{6.4.9}$$

where F_i is a polyform with exactly i *external* indices (and *not* the RR form with i overall indices). In particular the self duality conditions write

$$F_4 = *\lambda F_0 , \qquad F_3 = *\lambda F_1 , \qquad F_2 = *\lambda F_2 . \tag{6.4.10}$$

An analogous decomposition is valid also for the three-form H

$$H = H_0 + H_1 + H_2 + H_3 \tag{6.4.11}$$

and for the B field: $B = B_0 + B_1 + B_2$. Locally we have $H_0 = d_6 B_0$, $H_1 = d_4 B_0 + d_6 B_1$, $H_2 = d_4 B_1 + d_6 B_2$, $H_3 = d_4 B_2$.

We will now make a few assumptions about these fluxes. Since we have already assumed $\partial_m g_{\mu\nu} = 0$ (see footnote 2), it is natural to also assume $\partial_m B_{\mu\nu} = 0$, or in other words $d_6 B_2 = 0$. We have also assumed that the fibration is trivial; the analogue of this is to assume that $B_1 = 0$. So locally we now have

$$H_2 = 0 , \qquad H_1 = d_4 B_0 . \tag{6.4.12}$$

In this situation, $d_H = d_4 + d_4 B_0 + d_6 + d_6 B_1 + H_3 \wedge = e^{-B_0 \wedge} d e^{B_0 \wedge} + H_3 \wedge$.

This suggests that it is convenient to use the so-called b-transformation with $b = B_0$:

$$\Phi \to e^{b\wedge}\Phi , \tag{6.4.13}$$

such a transformation is a symmetry of (3.2.4b) if we also perform the transformations

$$H = H - db , \qquad K \to K , \qquad \tilde{K} \to \tilde{K} + \iota_K b , \qquad F \to e^{b\wedge}F , \tag{6.4.14}$$

although now F should satisfy $F = *_b \lambda F$ rather than (2.1.9), where $*_b = e^{b\wedge} * \lambda e^{-b\wedge}\lambda$. This is also a symmetry of (3.2.4a), provided

$$L_K b = 0 . \tag{6.4.15}$$

Making this symmetry work for (3.2.4c) and (3.2.4d) is more complex in general. However, one can show that it does indeed work when b is purely internal as in our Ansatz.

Having introduced such a transformation, it is convenient to work with a system where the differential is $d + H_3\wedge$, and Φ is replaced by $e^{B_0} \wedge \Phi$. This means that (6.4.4) is now modified by modifying the internal pure spinors

$$\phi_\pm \to \phi_\pm^{B_0} \equiv e^{B_0} \wedge \phi_\pm . \tag{6.4.16}$$

Actually, however, since from now on we will only work with the $\phi_\pm^{B_0}$, we will drop the ^{B_0} superscript and write simply $\phi_\pm$ in all our equations.

6.4.3 Symmetry equations: (3.2.4a)

Having specified in section 6.4.1 our Ansatz (6.4.1), (6.4.3), and what it implies on the various ingredients in the supersymmetry equations (3.2.4), we can now start seeing what those equations become.

We will start by (3.2.4a). The condition that K is a Killing vector for the ten-dimensional metric (6.4.1) splits according to whether the indices are both external, both internal, or one of each. Recalling (6.4.8), the last case gives

$$\nabla_m ||\eta_+||^2 k_\mu = 0 . \tag{6.4.17a}$$

Thus $||\eta||^2$ does not depend on the internal coordinates. The residual dependence of $||\eta||^2$ from the external coordinates can be reabsorbed in the definition of the four-dimensional spinors; hence we can simply assume it is a constant (as anticipated in (6.4.8)). In the following we will fix systematically $||\eta_+||^2 = 2$ in order to have $\bar{\epsilon}_i e_{+i}\epsilon_i = 16$ as in [11]. We will also make use of names such as k_i, k and v^3 in order to stay closer to section 6.3, so that (6.4.8) now reads $K = \frac{1}{2}k$, $\tilde{K} = \frac{1}{2}v^3$. To be precise, however, these vectors are not exactly the same as those we encountered in four dimensions, because in four dimensions the natural metric would be the one in the Einstein frame, which differs from the ds_4^2 in (6.4.1) by a function of the dilaton. Since our aim is not to reduce the ten-dimensional theory to four dimensions, however, we will not perform any rescaling and will work with the string frame metric.

Of course in (3.2.4c) and (3.2.4d) we also have the two extra vectors e_{+1}, e_{+2}, which must satisfy (3.2.3). In this section we will just assume that they will be purely external.

The resulting pairing equations involving them will be very complicated but still useful in order to suggest a G-structure system in four dimensions which is valid without the timelike hypothesis. On the other hand we will see in the next section that imposing the timelike hypothesis gives the possibility of making the choices (6.2.18). We will see that such a possibility strongly simplifies the resulting system of equations.

Coming back to (3.2.4a), the purely external and purely internal case give

$$\nabla_{(\mu}k_{\nu)} = (L_k g_4)_{\mu\nu} = 0 \ , \tag{6.4.17b}$$

$$\nabla_{(m}k_{n)} = (L_k g_6)_{mn} = 0 \ : \tag{6.4.17c}$$

namely, k is a Killing vector for the four-dimensional metric $g_{\mu\nu}$, and it is a symmetry for the internal metric g_{mn} as well.

Using our assumption (6.4.12), the equation $d\tilde{K} = i_K H$ yields

$$d_4\tilde{k} = \iota_k H_3 \ , \tag{6.4.18a}$$

$$0 = \iota_k H_1 \ . \tag{6.4.18b}$$

(6.4.18b) can also be written as $L_k B_0 = 0$, which we promised at the end of section 6.4.2. We will also see later that, in the timelike case, (6.4.18b) and (6.4.17c) both follow from other equations (namely, from invariance under k of the internal pure spinors).

6.4.4 Exterior equation: (3.2.4b)

From now on, in this section and in the next, for simplicity of notation we will specialize to IIA; the IIB case is very similar, and differs by some signs only.

In a similar manner as we did for equations (3.2.4a), equation (3.2.4b) splits in five pieces according to the number of external components involved:

$$d_6\mathrm{Re}\big(e^{-\phi}\mu\phi_+\big) = -\frac{1}{4}i_k F_1 \ , \tag{6.4.19a}$$

$$d_4\mathrm{Re}\big(e^{-\phi}\mu\phi_+\big) - d_6\mathrm{Re}\big(e^{-\phi}v \wedge \phi_-\big) = -\frac{1}{4}\big(i_k F_2 + v^3 \wedge F_0\big) \ , \tag{6.4.19b}$$

$$- d_4\mathrm{Re}\big(e^{-\phi}v \wedge \phi_-\big) + d_6\mathrm{Re}\big(e^{-\phi}\omega \wedge \phi_+\big) = -\frac{1}{4}\big(i_k F_3 + v^3 \wedge F_1\big) \ , \tag{6.4.19c}$$

$$d_4\mathrm{Re}\big(e^{-\phi}\omega \wedge \phi_+\big) - d_6\mathrm{Re}\big(e^{-\phi}i * v \wedge \phi_-\big)$$
$$+ H_3 \wedge \mathrm{Re}\big(e^{-\phi}\mu \wedge \phi_+\big) = -\frac{1}{4}\big(i_k F_4 + v^3 \wedge F_2\big) \ , \tag{6.4.19d}$$

$$- d_4\mathrm{Re}\big(e^{-\phi}i * v \wedge \phi_-\big) + d_6\mathrm{Re}\big(e^{-\phi}i\mu\mathrm{vol}_4 \wedge \phi_+\big)$$
$$- H_3 \wedge \mathrm{Re}\big(e^{-\phi}v \wedge \phi_-\big) = -\frac{1}{4}v^3 \wedge F_3 \ . \tag{6.4.19e}$$

Recall that the $\phi_\pm$ here include the internal B_0 field as in (6.4.16).

For the particular case of four-dimensional vacua, where the dependence on M_4 is trivial, (6.4.19) reduce to the pure spinor equations of [10]. In general, they contain information both about the geometry of M_4 (via d_4 of the external forms) and about the way the metric of M_6 changes as a function of the spacetime coordinates x^μ (via $d_4\phi_\pm$). In particular, in the timelike case we will see that (6.4.19b) and (6.4.19c) give first-order equations for the dependence of the scalars in the vector multiplets and hypermultiplets respectively. These first-order equations would give rise to attractor-like equations in black hole applications.

6.4.5 Pairing equations: (3.2.4c), (3.2.4d)

We now turn to (3.2.4c), (3.2.4d). As emphasized in many places, this is an ugly case and indeed the resulting pairing equations are very complicated to obtain. In [20] they have been massaged to give the final form:

$$e_{+2} \cdot (-4q_1 + S_3 v + \iota_v T^+) + i(\mu e_{+2} - \iota_{e_{+2}}\omega) \cdot f_1 = 0 \ , \tag{6.4.20a}$$

$$2e_{+2} \cdot \left(p_1 - i(\bar\phi_+, d_4\phi_+) - i(\bar\phi_-, d_4\phi_-) \right) - \iota_{e_{+1}}(\bar\mu e_{+2} - \iota_{e_{+2}}\bar\omega)_\llcorner T^+ =$$

$$= -\bar\nu S_3 - i(2e_{+1} \cdot \bar v e_{+2} - \bar\nu w_2) \cdot f_1 \ , \tag{6.4.20b}$$

$$e_{+2} \cdot \left(v(\gamma^{\bar i_1}\phi_-, F_0) + \iota_v(\gamma^{\bar i_1}\phi_-, F_2) \right) + (\mu e_{+2} - \iota_{e_{+2}}\omega) \cdot (\gamma^{\bar i_1}\phi_+, F_1) = 0 \ , \tag{6.4.20c}$$

$$- \bar\nu(\gamma^{\bar i_1}\phi_-, F_0) + \iota_{e_{+1}}(\bar\mu e_{+2} - \iota_{e_{+2}}\bar\omega)_\llcorner(\gamma^{\bar i_1}\phi_-, F_2) = (2e_{+1} \cdot \bar v e_{+2} - \bar\nu w_2) \cdot (\gamma^{\bar i_1}\phi_+, F_1) \ , \tag{6.4.20d}$$

where

$$\nu \equiv \mu e_{+1} \cdot e_{+2} - \iota_{e_{+1}}\iota_{e_{+2}}\omega \ , \tag{6.4.21}$$

and we defined the quantities

$$S_3 \equiv i(\bar\phi_+, F_0) \ , \qquad f_1 = dx^\mu(\bar\phi_-, F_{1\,\mu}) \ , \qquad T^+_{\mu\nu} \equiv i(\bar\phi_+, F_{2\,\mu\nu}) \ . \tag{6.4.22}$$

Together, (6.4.20a) are all the components of (3.2.4c) relevant to the Ansatz (6.4.3) introduced in this section.

One can deal with (3.2.4d) in a similar way. We get

$$e_{+1} \cdot (-4\bar q_2 + \bar S_3 v + \iota_v T^-) + i(\bar\mu e_{+1} + \iota_{e_{+1}}\bar\omega) \cdot f_1 = 0 \ , \tag{6.4.23a}$$

$$e_{+1} \cdot \left(-2\bar p_2 + 2i(\phi_+, d_4\bar\phi_+) + 2i(\bar\phi_-, d_4\phi_-) - (\mu e_{+2} - \iota_{e_{+2}}\omega)_\llcorner T^- \right) =$$

$$= -\nu \bar S_3 - i(2e_{+2} \cdot \bar v e_{+1} + \nu \bar w_1) \cdot f_1 \ , \tag{6.4.23b}$$

$$e_{+1} \cdot \left(v(\phi_-\gamma^{\bar j_2}, F_0) - \iota_v(\phi_-\gamma^{\bar j_2}, F_2) \right) + (\bar\mu e_{+1} + \iota_{e_{+1}}\bar\omega) \cdot (\bar\phi_+\gamma^{\bar j_2}, F_1) = 0 \ , \tag{6.4.23c}$$

$$\nu(\phi_-\gamma^{\bar j_2}, F_0) - \iota_{e_{+1}}(\mu e_{+2} - \iota_{e_{+2}}\omega) \cdot (\phi_-\gamma^{\bar j_2}, F_2) = -(2e_{+2} \cdot \bar v e_{+1} + \nu \bar w_1) \cdot (\bar\phi_+\gamma^{\bar j_2}, F_1) \ , \tag{6.4.23d}$$

Let us comment the equations just written. Equations (6.4.20a), (6.4.20b), (6.4.23a) and (6.4.23b) involve the "intrinsic torsions" q_i and p_i: they reflect the fact that in general the exterior equations (6.4.19) and the symmetry equation (6.4.18a) are not sufficient to fully reconstruct the external gravitino equation (6.3.2a); indeed such equations will suggest a G-structure system in four-dimensions which is valid in the general case. On the other hand, we have seen in section 6.3.2 that the gravitino equations (6.3.5) in the timelike case can be obtained by considering only the derivatives of the vielbein defined by k, $\tilde{k}$ and v (together with the proportionality factor μ); therefore we will see in the next section that equations (6.4.20a), (6.4.20b), (6.4.23a) and (6.4.23b) can be dropped by imposing the timelike hypothesis. The other equations appering in (6.4.20) and (6.4.23) instead have nothing to do with the four-dimensional intrinsic torsions and indeed they are not redundant even in the timelike case. We will discuss in the next section the interpretation of such equations.

6.4.6   Summary: ten-dimensional system

In this section, we have applied the ten-dimensional system (3.2.4) to a (topologically trivial) fibration (6.4.1), with a spinor Ansatz (6.4.3), (6.4.7), and a few assumptions summarized in footnote 2 and in (6.4.12). The conditions of preserved supersymmetry are equivalent to equations (6.4.17), (6.4.18), (6.4.19), (6.4.20), (6.4.23). (These last two were given with the additional assumption $\phi = H_3 = 0$.) These equations are not as pleasant as one might wish, but fortunately we will be able to do much better in the next section. There, we will apply the system in the timelike case, and reduce it to a much more pleasant-looking form, which will closely parallel the "boxed" system seen in section 6.3.

6.5   Ten-dimensional system in the timelike case

In this section, we will rewrite the equations we found in section 6.4 in the timelike case. The conditions for supersymmetry are expected to be much simpler in this case and the reason why we expect such a behaviour is easy to understand: as we discussed in section 3.2, in order to rewrite the SUSY conditions using G-structures we need to introduce the additional vectors e_{+i}. Such vectors are on a large extent arbitrary, indeed they have just to satisfy the constraints (3.2.3). The fact that e_{+i} are not directly defined by the SUSY parameters ϵ_i can be seen as the origin of the very complicated form that equations (3.2.4c) and (3.2.4c) assume. On the other hand, equations (6.2.18) tell us that in the timelike case a natural choice for e_{+i} exists. In this section we will use such a choice and we will see what kind of equations can be obtained.

Most of the equations we will obtain organize themselves in a way that closely parallels the boxed system (6.3.5), (6.3.10), (6.3.11) in section 6.3. As we will see, however, there will also be equations formally associated with "gravitino multiplets".

We will start in section 6.5.1 with a discussion of how ten-dimensional fields organize themselves on a spacetime $M_4 \times M_6$, where M_6 is a $SU(3) \times SU(3)$ structure manifold. Many fields come from forms: the RR fields, but also the internal metric and B field, through the pure spinors $\phi_\pm$. A useful basis for internal forms is given by the "generalized Hodge diamond", that we introduced in Chapter 5 in eight dimensions and that we will write in six dimensions in (6.5.2) below. The most substantial part of the multiplets will correspond to the interior of that diamond. The edges are usually discarded in $N = 2$ reductions for reasons we will review below; however, we are not performing a reduction, and we will need to keep the corresponding representations from the edges.

Section 6.5.1 will then dictate the way we organize our ten-dimensional equations in later subsections. We will describe those corresponding to the four-dimensional gravitino equations in section 6.5.2, and to the universal and non-universal hypermultiplets in sections 6.5.3 and 6.5.4. We will then have vector multiplets from the bulk of the diamond (section 6.5.5) and our new vector multiplets from the edge (section 6.5.6). Finally, we will have in section 6.5.7 some new equations associated with gravitino multiplets, in a sense we will clarify.

6.5.1 Organizing the fields

We will first review how the ten-dimensional fields produce the various four-dimensional fields in a reduction, and then how these get organized in multiplets for $N = 2$ compactifications. Most of this material is by now standard. One purpose in reviewing it here is to introduce a few definitions that will be useful later. Another purpose is that some of our equations will be in "vector" representations associated to the edge of the diamond in (6.5.2) below; these will organize themselves in multiplets which are not commonly considered in the literature, as we will see.

Scalars

We will start by considering spacetime scalars that come from deforming the internal NSNS fields, g_{mn} and $B_{0\,mn}$. These degrees of freedom are determined [9] (see [55, Sec. 3] for a review) by the internal pure spinors $\phi_\pm$, along with the internal dilaton and spinors:

$$\{g_{mn}, B_{0\,mn}, \phi, \eta^{1,2}\} \leftrightarrow \phi_\pm \tag{6.5.1}$$

Hence the deformations δg_{mn}, $\delta B_{0\,mn}$ come from the deformations $\delta\phi_\pm$ of the pure spinors. We hence need to expand these latter deformations in an appropriate basis for internal

forms.

The most natural basis is given by the generalized Hodge diamond, that in six dimensions writes:

$$
\begin{array}{ccccccc}
& & & \phi_+ & & & \\
& & \phi_+\gamma^{i_2} & & \gamma^{\bar{i}_1}\phi_+ & & \\
& \phi_-\gamma^{\bar{i}_2} & & \gamma^{\bar{i}_1}\phi_+\gamma^{i_2} & & \gamma^{i_1}\bar{\phi}_- & \\
\phi_- & & \gamma^{\bar{i}_1}\phi_-\gamma^{\bar{j}_2} & & \gamma^{i_1}\bar{\phi}_-\gamma^{j_2} & & \bar{\phi}_- \\
& \gamma^{\bar{i}_1}\phi_- & & \gamma^{i_1}\bar{\phi}_+\gamma^{\bar{j}_2} & & \bar{\phi}_-\gamma^{i_2} & \\
& & \gamma^{i_1}\bar{\phi}_+ & & \bar{\phi}_+\gamma^{\bar{i}_2} & & \\
& & & \bar{\phi}_+ & & &
\end{array}
\qquad (6.5.2)
$$

It should be emphasized that the basis (6.5.2) is a basis at every point: if we expand a form in M_6 in terms of (6.5.2), we will get functions, not numbers. Nevertheless, the basis provides a way to neatly organize our equations. As mentioned in Section 6.1, this is similar in spirit to the pre-truncation computations in [74] (corresponding to their section 2).

We can now expand $\delta\phi_\pm$ in the basis (6.5.2). As we already explain in Appendix C each variation can only produce forms that are "not too far" from the original pure spinor. This means that it can only contain forms in the zeroth and second row of (6.5.2). In the same way, $\delta\phi_-$ can only contain forms in the zeroth and second column. To shorten our notation, let us introduce indices counting the (infinitely many) forms in these entries of (6.5.2). First of all,

$$
\delta\phi_+^a = \{\gamma^{i_1}\bar{\phi}_+\gamma^{\bar{j}_2}\} \ . \tag{6.5.3}
$$

Let us then define[7]

$$
-D\bar{t}^a = (\delta\phi_+^a, d_4\phi_+) \qquad (\mathbf{3},\bar{\mathbf{3}}) \ . \tag{6.5.4}
$$

(Remember again that ϕ_+ here actually includes a e^{B_0}, as in (6.4.16); hence these t^a are complex. The way these scalars are defined is reminiscent of how the vector multiplet scalars in a Calabi–Yau compactification are integrals over two-cycles of the form B_0+iJ.) We have stressed the $SU(3)\times SU(3)$ representation in which these scalars transform. These $D\bar{t}^a$ can be morally thought of as suitable covariant derivatives of scalars $\bar{t}^a$ defined by expanding the variation of ϕ_+ along the forms $\delta\phi_+^a$, with a connection piece coming from the fact that the $\delta\phi_+^a$ are themselves not closed. This issue potentially comes up even in Calabi–Yau compactifications, where one expands along harmonic forms ω_i, which a priori should vary when one varies the metric. However, in that case the connection is flat and can be gauged away; one would want this to happen for a more general reduction to $N=2$ supergravity [76]. Since we are not trying to reduce to an $N=2$ effective

---

[7]In this section the pairing $(\,,\,)$ denotes the six-dimensional one; we also define $(a_6,\beta_4\wedge b_6)=\beta_4\wedge(a_6,b_6)$, where a_6,b_6 are internal forms and β_4 is an external form.

theory, but merely to reorganize the supersymmetry equations in ten dimensions in a way inspired from $N = 2$ in four dimensions, we can afford to leave this issue unresolved. The definition (6.5.4) should be thought of as a bookkeeping device more than a detailed attempt at writing a four-dimensional effective theory. Similar considerations will apply to the symbols D we will introduce from now on. For example, one can similarly define

$$\delta\phi_-^\alpha = \{\gamma^{i_1}\bar{\phi}_-\gamma^{j_2}\} \tag{6.5.5}$$

and

$$D(z^\alpha + i\tilde{z}^\alpha) = \text{Re}(\delta\phi_-^\alpha, d_4\phi_-) + i\text{Im}(\delta\phi_-^\alpha, d_4\phi_-) \qquad (\mathbf{3},\mathbf{3}) . \tag{6.5.6}$$

The expansion of $d_4\phi_\pm$ along the forms on the boundary of the diamond (6.5.2) (namely, $\phi_+\gamma^{i_2}$ and the others with one gamma acting on $\phi_\pm$) does not directly correspond to deformations of g_{mn} and B_{mn}, but rather to changes in the spinors $\eta^{1,2}$ determined by $\phi_\pm$ (see for example [11, Sec. 2.3]).

Finally, the expansion of $d_4\phi_\pm^a$ along the corners of (6.5.2), $(\bar{\phi}_+, d_4\phi_+)$, will appear as a connection in some of our equations.

Other scalars come from the RR sector. As we did earlier, it is convenient to consider the decomposition (6.4.9) of F as $\sum F_i$, where F_i has i external indices. From F_1 we have

$$D(\xi^\alpha + i\tilde{\xi}^\alpha) \equiv -\frac{1}{2}e^\phi(\delta\phi_-^\alpha, F_1) \qquad (\mathbf{3},\mathbf{3}) ; \tag{6.5.7}$$

$$D(\xi + i\tilde{\xi}) \equiv 2e^\phi(\bar{\phi}_-, F_1) \qquad (\mathbf{1},\mathbf{1}) . \tag{6.5.8}$$

Notice that, for the scalars (6.5.7), (6.5.8), the symbol D is now hiding something more than the (perhaps flat) connection we mentioned for (6.5.4). Here, on top of the fact that d_4 can act on the forms $\delta\phi_-^\alpha$, $\bar{\phi}_-$, we also have the fact that (locally) $F_1 = d_4C_0 + d_6C_1$. If we defined the scalar $(\xi + i\tilde{\xi})$ by $2e^\phi(\bar{\phi}_-, C_0)$, we would see that (6.5.8) contains a term proportional to C_1, which signals that the scalar is gauged under one of the spacetime vectors originated by RR fields, which we will see in section 6.5.1, which is in line with expectations from actual reductions in presence of internal flux. Again, since we are not actually performing a reduction, we will be content with the definition (6.5.8) and will not try to resolve the symbol D down to its constituents.

We also have D of scalars in "vector" representations:

$$(\phi_+\gamma^{i_2}, F_1) \qquad (\mathbf{1},\mathbf{3}) ; \qquad (\gamma^{\bar{i}_1}\phi_+, F_1) \qquad (\bar{\mathbf{3}},\mathbf{1}) . \tag{6.5.9}$$

These shall remain nameless, for reasons to become clear later. Finally we have the dilaton ϕ, and a scalar a which can be defined by dualizing the spacetime NSNS three-form:

$$H_3 = *d_4a . \tag{6.5.10}$$

Actually this dualization procedure is only possible when $d_4 * H_3 = 0$. This is not guaranteed in general, since the equation of motion for H reads in general $d(e^{4A-2\phi} * H) = -e^{4A} \sum_n F_n \wedge *F_{n+2}$, and the right hand side might sometimes not vanish. This corresponds roughly to a case where one wants to include both magnetic and electric gaugings at the same time. When H_3 cannot be dualized, one cannot define the scalar a, and one would have to work with multiplets involving tensors. Although supersymmetric actions for such multiplets have been studied (see for example [82]), we will gloss over this subtlety, and assume (6.5.10).

Vectors

The NSNS sector gives rise to four-dimensional vectors via the mixed components $g_{\mu m}$, $B_{\mu m}$. Notice that these components will be set to zero when we give our equations; however, for the time being we find it useful to consider them.

The vectors $g_{\mu m}$, $B_{\mu m}$ both have a single internal vector index. This does not make their $SU(3) \times SU(3)$ representation manifest. However, writing them as $E_{\mu m} = g_{\mu m} + B_{\mu m}$, $E_{m\mu} = g_{m\mu} + B_{m\mu} = g_{\mu m} - B_{\mu m}$ and remembering the stringy origins of these fields make one guess [75] that they belong to the representations

$$g_{\mu m} , \qquad B_{\mu m} : \qquad (\mathbf{3}, \mathbf{1}) \oplus (\bar{\mathbf{3}}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \bar{\mathbf{3}}) . \qquad (6.5.11)$$

A way to confirm this conclusion is to study explicitly how E transforms under internal $O(6, 6)$ transformations $O = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$: one obtains

$$E_{\mu m} \to E_{\mu n}((cE + d)^{-1})^n{}_m , \qquad E_{m\mu} \to (a - (aE + b)(cE + d)^{-1}c)_m{}^n E_{n\mu} \qquad (6.5.12)$$

(where E is the internal $g + B$). Using the expression for the generalized almost complex structures

$$\mathcal{J}_{\pm} = \mathcal{E} \begin{pmatrix} I_1 & 0 \\ 0 & I_2 \end{pmatrix} \mathcal{E}^{-1} , \qquad \mathcal{E} = \begin{pmatrix} 1 & 1 \\ E & -E^t \end{pmatrix} \qquad (6.5.13)$$

(where I_i are two almost complex structures), the $SU(3) \times SU(3)$ subgroup of $O(6, 6)$ can be characterized as

$$O = (\mathcal{E}^{-1})^t \begin{pmatrix} U_1 & 0 \\ 0 & U_2 \end{pmatrix} \mathcal{E}^t \qquad (6.5.14)$$

where U_i satisfy $[U_i, I_i] = 0$ and $U_i^t g U_i = g$ — namely, they are unitary with respect to the internal metric g. Specializing (6.5.12) to this particular O leads to

$$E_{\mu m} \to (U_1^{-1})_m{}^n E_{\mu n} , \qquad E_{m\mu} \to (U_2)_m{}^n E_{n\mu} , \qquad (6.5.15)$$

which confirms (6.5.11).

We also have vectors from the RR sector. The expansion of F_2 (recall that the $_2$ denotes the number of four-dimensional indices) gives the field-strengths

$$T^+ \equiv -\frac{i}{2} e^\phi \left(\bar{\phi}_+, F_2 \right) \qquad (\mathbf{1}, \mathbf{1}) \tag{6.5.16}$$

$$G^{a-} \equiv -\frac{i}{4} e^\phi \left(\delta\phi_+^a, F_2 \right) \qquad (\mathbf{3}, \bar{\mathbf{3}}) \tag{6.5.17}$$

as well as

$$\left(\phi_- \gamma^{\bar{i}_2}, F_2 \right) \qquad (\mathbf{1}, \bar{\mathbf{3}}) \; ; \qquad \left(\gamma^{i_1} \bar{\phi}_-, F_2 \right) \qquad (\mathbf{3}, \mathbf{1}) \; . \tag{6.5.18}$$

Similarly to our comments about (6.5.4), all these field strengths are not simply the exterior derivative of a potential, because of the non-constancy of $\bar{\phi}_+$ and $\delta\phi_+^a$, and because F_2 has two terms: locally, $F_2 = d_6 C_2 + d_4 C_1$ (see our comment after (6.5.8)). As the notation implies, these are actually the self-dual (or anti-self dual) parts of the field-strengths: for example, T^+ satisfies $*T^+ = iT^+$.

Fermions

We also take a quick look at fermions. The spin $3/2$ fields in four dimensions originate from the ten-dimensional gravitinos, with their index taken along the four dimensions. To understand how these transform under $SU(3) \times SU(3)$, recall that the two SU(3) factors come from the stabilizers of the two supersymmetry parameters η^1 and η^2 respectively. That suggests that the ψ^1 transforms under the first SU(3) and is a singlet under the second, and that ψ^2 transforms under the second SU(3) and is a singlet under the first. Taking also four-dimensional chirality into account we get

$$\begin{aligned} \psi^1_{+\mu} & \qquad (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{1}) \; ; \\ \psi^2_{+\mu} & \qquad (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \bar{\mathbf{3}}) \; . \end{aligned} \tag{6.5.19}$$

Spin $1/2$ fields arise both from $\psi^{1,2}_m$ (the internal components of the gravitinos) and from the dilatinos $\lambda^{1,2}$. The latter transform as in (6.5.19):

$$\begin{aligned} \lambda^1_+ & \qquad (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{1}) \; ; \\ \lambda^2_+ & \qquad (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \bar{\mathbf{3}}) \; . \end{aligned} \tag{6.5.20}$$

The $\psi^{1,2}_m$ are subtler because we also have to work out the transformation law under $SU(3) \times SU(3)$ of the internal index m (much as we had to do for $g_{m\mu}$ and $B_{m\mu}$ in section 6.5.1). As it was noticed in [83, Sec. 5.1], the correct transformation law is obtained by assuming that for ψ^1_m the spinorial index transforms under the first $SU(3)$, while the $_m$ index transforms under the *second* $SU(3)$; and likewise for ψ^2_m:

$$\begin{aligned} \psi^1_{+m} & \qquad ((\mathbf{1}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{1})) \otimes ((\mathbf{1}, \mathbf{3}) \oplus (\mathbf{1}, \bar{\mathbf{3}})) \; ; \\ \psi^2_{+m} & \qquad ((\mathbf{1}, \mathbf{1}) \oplus (\mathbf{1}, \bar{\mathbf{3}})) \otimes ((\mathbf{3}, \mathbf{1}) \oplus (\bar{\mathbf{3}}, \mathbf{1})) \; . \end{aligned} \tag{6.5.21}$$

This can be determined by using the $O(d, d)$ transformation laws for fermions, and specializing them to $SU(3) \times SU(3)$ as in (6.5.14). We will not do so explicitly here, but see for example [84, Sec. 3].

Multiplets

We will now collect the vectors and scalars in four-dimensional $N = 2$ multiplets. We will not deal with the fermions, since there are non-trivial mixings between gravitinos and dilatinos [75].

Most multiplets are natural extensions of the ones which are familiar from Calabi–Yau compactifications. There is a vector multiplet transforming in the $(\mathbf{3}, \bar{\mathbf{3}})$, which collects the scalars from (6.5.4), the vectors from (6.5.17), and part of the spinors in (6.5.21). There is a hypermultiplet in the $(\mathbf{3}, \mathbf{3})$, which collects the scalars in (6.5.6), (6.5.7), and again part of the spinors in (6.5.21). Finally, there is a "universal" hypermultiplet in the $(\mathbf{1}, \mathbf{1})$, whose scalars are (6.5.8), the dilaton ϕ, and the axion a defined as usual by (6.5.10). In the Calabi–Yau case, these would result in the usual $h^{1,1}$ vector multiplets and $1 + h^{2,1}$ hypermultiplets.

All this is standard; these multiplets were included in [75]. The situation is a bit more problematic in the "vector" representations, $(\mathbf{1}, \mathbf{3})$, $(\bar{\mathbf{3}}, \mathbf{1})$ and their complex conjugates. These have not been included in reductions to $N = 2$ supergravity — for good reasons, as we will now see. Looking at sections 6.5.1 and 6.5.1, the first thing we notice is that we have more vectors than could be possibly accommodated in vector multiplets: the scalars (6.5.9) will sit in a vector multiplet in the same representation, but their partners could be among (6.5.11) or perhaps (6.5.18).

The reason of this apparent mismatch becomes clear if we consider the case $M_6 = T^6$. This produces an $N = 8$ theory. If we decompose its field content in $N = 2$ multiplets, we find 15 vector multiplets, 10 hypermultiplets, and 6 "gravitino multiplets" which contain a spin 3/2 field, two vectors and a spin 1/2 field. This suggests that we should include a gravitino multiplet in the $(\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3})$. (At this point we are not actually able to tell the difference between a $\mathbf{3}$ and $\bar{\mathbf{3}}$, which are complex conjugates of each other.)

This does not mean we are advocating including gravitino multiplets in $N = 2$ effective theories. These multiplets are allowed classically by supersymmetry, but in general they run into trouble quantum mechanically: their spin 3/2 fields contain zero-norm states that need to be gauged out by a spinorial gauge transformation. These gauge transformations are the supersymmetry parameters; so these multiplets are only allowed when supersymmetry is actually higher, $N > 2$. Even massless gravitinos will probably arise in the context of a Higgs effect, where the spin 1/2 gauge transformations are still present. Thus, they were not included in [75] for good reasons.

In this Chapter, however, we are not actually reducing any theory. We are simply organizing the ten-dimensional equations for supersymmetry from a four-dimensional perspective. The gravitino multiplets will be for us a bookkeeping device; some of our equations will be in the "vector representations", and we now know that some will resemble those in a vector multiplet, while others will resemble the supersymmetry equations for a gravitino multiplet.

We finally want to understand whether the partners of the scalars (6.5.9) come from (6.5.11) or from (6.5.18). To see this, it is again useful to think about the $N = 8$ theory. This theory has $\binom{8}{2} = 28$ vectors, whose field-strengths we will denote by T_{AB}, antisymmetric in AB, and $\binom{8}{4} = 70$ scalars parameterizing a coset space, whose vielbein we will denote by a totally antisymmetric P_{ABCD}. In $N = 2$ terms, the index A should be split in $SU(3) \times SU(3)$ representations. The first four ζ_A come from ϵ_1, while the second four come from ϵ_2. Taking also chirality into account, we see that

$$A \to (\mathbf{1}, \mathbf{1}) \oplus (\mathbf{3}, \mathbf{1}) \oplus (\mathbf{1}, \bar{\mathbf{3}}) \oplus (\mathbf{1}, \mathbf{1}) \ . \tag{6.5.22}$$

As in the generalized Hodge diamond (6.5.2), we can introduce an index i_1 for the $(\mathbf{3}, \mathbf{1})$ and an index $\bar{j}_2$ for the $(\mathbf{1}, \bar{\mathbf{3}})$. The first and second singlet will be denoted by indices 1 and 2. In this language, the RR vectors should be associated to field-strengths which mix indices coming from the first copy of $SU(3)$ (namely, 1 and i_1) with indices from the second copy (2 and $\bar{j}_2$):

$$\text{RR}: \quad T_{1\bar{j}_2} \ (\mathbf{1}, \bar{\mathbf{3}}) \ , T_{12} \ (\mathbf{1}, \mathbf{1}) \ , T_{i_1\bar{j}_2} \ (\mathbf{3}, \bar{\mathbf{3}}) \ , T_{i_12} \ (\mathbf{3}, \mathbf{1}) \ . \tag{6.5.23}$$

Clearly, T_{12} is the graviphoton, $T_{i_1\bar{j}_2}$ are the vectors in (6.5.17), and $T_{1\bar{j}_2}$, T_{i_12} are the vectors from (6.5.18). On the other hand, the following vectors should come from the NSNS sector:

$$\text{NSNS}: \quad T_{1i_1} \ (\mathbf{3}, \mathbf{1}) \ , T_{i_1j_1} \ (\bar{\mathbf{3}}, \mathbf{1}) \ , T_{\bar{j}_1\bar{j}_2} \ (\mathbf{1}, \mathbf{3}) \ , T_{\bar{j}_22} \ (\mathbf{1}, \bar{\mathbf{3}}) \ . \tag{6.5.24}$$

Turning now to the scalars P_{ABCD} of the $N = 8$ theory, in $N = 2$ terms we see that the $2 \times \binom{6}{3} = 40$ scalars which have one "singlet" index (1 or 2) sit in hypermultiplets, while the $\binom{6}{2} + \binom{6}{4} = 30$ which have both 1 and 2, or neither, sit in vector multiplets. The supersymmetry transformations of the spin 1/2 fields λ_{IJK} in $N = 8$ supergravity [85, Sec. 7] schematically read, in absence of gauging, $\delta\lambda_{ABC} \sim T_{[AB}\zeta_{C]+} + P_{ABCD}\zeta_-^D$. In a $N = 2$ truncation, we only have the supersymmetry parameters ζ_1 and ζ_2, and we set to zero ζ_{i_1}, $\zeta_{\bar{j}_2}$. Under $N = 2$ supersymmetry, then, $T_{i_1j_1}$ is mixed with $P_{1i_1j_12}$, while $T_{1\bar{j}_2}$ is not related to any P (since $P_{11\bar{j}_22}$ vanishes by antisymmetry). In other words, only the T which do not have an index 1 or 2 can sit in a vector multiplet:

$$T_{i_1\bar{j}_2} \ , \qquad T_{i_1j_1} \ , \qquad T_{\bar{j}_1\bar{j}_2} \qquad \text{(in vector multiplets)} \ . \tag{6.5.25}$$

This means that the vector which is a partner of (6.5.9) is among (6.5.11), rather than (6.5.18). The multiplet structure is summarized in table 6.1.

scalars		vectors	
$\left.\begin{array}{l}\delta g_{mn}\\ \delta B_{mn}\end{array}\right\} \sim \delta\phi_{\pm} \to \underbrace{(\mathbf{3},\mathbf{3})}_{\text{hm}} \oplus \underbrace{(\mathbf{3},\bar{\mathbf{3}})}_{\text{vm}}$		$\left.\begin{array}{l}\delta g_{\mu m}\\ \delta B_{\mu m}\end{array}\right\} \to \underbrace{(\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\bar{\mathbf{3}})}_{\text{gravitino mult.}} \oplus \underbrace{(\bar{\mathbf{3}},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})}_{\text{vm}}$	
$F \to \underbrace{(\mathbf{1},\mathbf{1})}_{\text{univ. hm}} \oplus \underbrace{(\mathbf{3},\mathbf{3})}_{\text{hm}} \oplus \underbrace{(\bar{\mathbf{3}},\mathbf{1}) \oplus (\mathbf{1},\mathbf{3})}_{\text{vm}}$		$F \to \underbrace{(\mathbf{1},\mathbf{1})}_{\text{graviphoton}} \oplus \underbrace{(\mathbf{3},\bar{\mathbf{3}})}_{\text{vm}} \oplus \underbrace{(\mathbf{3},\mathbf{1}) \oplus (\mathbf{1},\bar{\mathbf{3}})}_{\text{gravitino mult.}}$	

Table 6.1: Summary of the multiplet structure.

6.5.2   External gravitino equations

We will now start collecting the ten-dimensional supersymmetry equations. We will start in this section by collecting those that constrain the four-dimensional geometry. As argued in section 6.2.4, the information contained in the covariant derivatives $\nabla_\mu \zeta_i$ can be completely extracted from the exterior derivatives $d\mu$, dk_i, dv.

The exterior derivatives $d\mu$, dv^3, dv can be extracted from section 6.4, while dk will have to be rederived. (Recall that $k = \frac{1}{2}(k_1 + k_2)$, $v^3 = \frac{1}{2}(k_1 - k_2)$.) The equation for dv^3 was given in (6.4.18a), and we repeat it here for convenience:

$$d_4 v^3 = \iota_k H_3 \ . \tag{6.5.26a}$$

$d\mu$ and dv can be obtained by taking the pairing of (6.4.19b) with $\bar{\phi}_+$ and of (6.4.19c) with ϕ_- respectively:

$$d_4\mu - i\mu \left(\bar{\phi}_+, e^\phi d_4(e^{-\phi}\phi_+)\right) = s^x v^x - 2\iota_k T^+ \ , \tag{6.5.26b}$$

$$d_4 v + 2iv \wedge \left(\bar{\phi}_-, e^\phi d_4(e^{-\phi}\phi_-)\right) = -\frac{1}{2}\omega \bar{s}^- + \frac{1}{2}\bar{\omega}s^+ - ie^\phi \iota_k(\bar{\phi}_-, F_3) - ie^\phi v^3 \wedge (\bar{\phi}_-, F_1) \tag{6.5.26c}$$

where T^+ was defined in (6.5.16), and

$$s^+ = 4i\,(\bar{\phi}_+, d_6\bar{\phi}_-) \ , \qquad s^- = 4i\,(\bar{\phi}_+, d_6\phi_-) \ , \qquad s^3 = ie^\phi\,(\bar{\phi}_+, F_0) \ . \tag{6.5.27}$$

These s_i are morally related to the Killing prepotentials $\mathcal{P}_x$ of $N = 2$ supergravity; this identification agrees with [74, Eq. (2.139),(2.140)] (up to a change in conventions). Notice that (6.5.26b) takes exactly the same form of the four-dimensional counterpart (6.3.5) by simply putting

$$D\mu \equiv d_4\mu - i\mu \left(\bar{\phi}_+, e^\phi d_4(e^{-\phi}\phi_+)\right) \ , \tag{6.5.28}$$

this result suggests the identification

$$\hat{Q} = -(\bar{\phi}_+, e^\phi d_4(e^{-\phi}\phi_+)) \ . \tag{6.5.29}$$

Finally, we have to compute an expression for dk. Such an equation is not explicitly present in the original system of equations (3.2.4). In [20] it is shown that this equation would originate from the pairing equations (3.2.4c), (3.2.4d); however, we found it easier to compute it from scratch. We start from the equation

$$D_{[M}K_{N]} - \frac{1}{2}H_{MNQ}\tilde{K}^Q = -\frac{e^\phi}{256}\bar{\epsilon}_1\Gamma_{[M|}F\,\Gamma_{|N]}\epsilon_2 \ , \tag{6.5.30}$$

which is valid in ten dimensions without making any assumption about compactifications. To specialize (6.5.30) to compactifications, recall that k and v^3 have only external components (and that they only depend on the external coordinates). Using the decompositions of spinors and fluxes in sections 6.4.1 and 6.4.2, one finally obtains

$$d_4 k - \iota_{v^3}H = 2\mathrm{Re}\left[-\bar{\omega}\,s_3 + e^\phi(\bar{\phi}_-, F_1)\llcorner(*\bar{v}) - 2\bar{\mu}\,T^+\right] \ . \tag{6.5.31}$$

Together, (6.5.26a), (6.5.26b), (6.5.26c) and (6.5.31) exhaust the constraints on the geometry of the external spacetime M_4. They are the analogues of (6.3.5) for four-dimensional $N = 2$ supergravity.

6.5.3 Universal hypermultiplet

As we saw in section 6.5.1, the scalars in the universal hypermultiplet are the dilaton ϕ, the axion a defined in (6.5.10), and the complex scalar ξ defined by (6.5.8).

The equations for these scalars are not easy to find in the system we gave in section 6.4. They are hidden inside some equations that would seem to constrain four-dimensional geometry. There are several of these equations: the equation for $d_4\omega$ in (6.4.19d) and for $d_4 * v$ in (6.4.19e); (6.4.17b), saying that K_μ is a Killing vector for M_4; and the 'pairing' equations (6.4.20a), (6.4.20b), (6.4.23a), (6.4.23b). One can eliminate from these equations all the four-dimensional intrinsic torsions by using (6.5.26a), (6.5.26b), (6.5.26c) and (6.5.31). Some of the equations become trivial; four stay non-trivial, and can be interpreted as the equations for the scalars in the universal hypermultiplet.

This derivation is laborious, however, and in this case it might be preferable to present an alternative logic, stemming directly from the supersymmetry equations.

The idea is to start from the dilatino equations in ten dimensions. These read $(-\frac{1}{2}H + \partial\phi)\epsilon_1 + \frac{e^\phi}{16}\Gamma^M F\Gamma_M\epsilon_2 = 0$, $(\frac{1}{2}H + \partial\phi)\epsilon_2 + \frac{e^\phi}{16}\Gamma^M\lambda(F)\Gamma_M\epsilon_1 = 0$. They do not contain any intrinsic torsions, either internal or external (unless one defines the intrinsic torsions by including the NSNS flux, as was done in [11, App. B]). One way to simplify it is to use $\gamma^\mu C_k\gamma_\mu = (-)^k(4 - 2k)C_k$, where C_k is k-form in four dimensions, and write

$$\Gamma^M F\Gamma_M = 2(4F_0 + 2F_1) + \Gamma^m F\Gamma_m \ . \tag{6.5.32}$$

The last term can be eliminated using the internal gravitino, which in turn produces a term involving the internal Dirac operator $\gamma^m D_m$. In this way one can produce several equations. Some have an internal free index, and naturally belong to the "edge of the diamond" equations that we will present in sections 6.5.6 and 6.5.7. The ones without an internal index are

$$L_K \phi = 0 \ , \qquad L_K a = -4\mathrm{Im}(\mu \bar{s}^3) \ , \qquad L_K(\xi + i\tilde{\xi}) = i(\mu \bar{s}^- - \bar{\mu} s^+) \ ; \qquad (6.5.33\mathrm{a})$$

$$v^3 \cdot da + \mathrm{Re}(v \cdot D(\xi - i\tilde{\xi})) = 0 \ , \quad -2v^3 \cdot d\phi + \mathrm{Im}(v \cdot D(\xi - i\tilde{\xi})) = 4\mathrm{Re}(\mu \bar{s}^3) \ , \quad (6.5.33\mathrm{b})$$

$$v^3 \cdot D(\xi + i\tilde{\xi}) - v \cdot d(a + 2i\phi) = i(\mu \bar{s}^- + \bar{\mu} s^+) \ . \qquad (6.5.33\mathrm{c})$$

These can be written exactly as (6.3.10): it is enough to define

$$q^1 = a \ , \qquad q^2 = \xi \ , \qquad q^3 = \tilde{\xi} \ , \qquad q^4 = 2\phi \ , \qquad (6.5.34)$$

to take the matrices Ω^x to be

$$\Omega^1 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \ , \qquad \Omega^2 = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix} \ , \qquad \Omega^3 = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{pmatrix} \ .$$

$$(6.5.35)$$

and the gauging vectors

$$g \, \bar{\mathcal{L}}^\Lambda k_\Lambda^u = \begin{pmatrix} -4\bar{s}^3 \\ -(\bar{s}^+ + \bar{s}^-) \\ i(\bar{s}^- - \bar{s}^+) \\ 0 \end{pmatrix} \ . \qquad (6.5.36)$$

The Ω^x in (6.5.35) can be thought of as a block of dimension 4 of the triplet of complex structures Ω^x (which appeared in section 6.3.3) characterizing the quaternionic structure of the space of fields of a four-dimensional $N = 2$ supergravity. Notice that indeed (6.5.35) satisfy $\Omega^x \Omega^y = -\delta^{xy} 1 + \epsilon^{xyz} \Omega^z$.

6.5.4 Non-universal hypermultiplets

We will now present the equations corresponding to the other, non-universal, hypermultiplets. As we saw in section 6.5.1, the scalars are the z^α, $\tilde{z}^\alpha$ defined in (6.5.6), which come from deformations of the internal metric along the forms (6.5.5), and the ξ^α, $\tilde{\xi}^\alpha$ defined in (6.5.8), which come from fluxes.

It is natural to look for these equations in (6.4.19a), (6.4.19c), (6.4.19e), by taking their pairing with the forms $\delta\phi_-^\alpha$ in (6.5.5). However, it turns out that the pairings $(\delta\phi_-^\alpha, (6.4.19a))$ and $(\delta\phi_-^\alpha, (6.4.19e))$ are redundant: they give equations that can also be

obtained from $(\delta\phi_-^\alpha, (6.4.19c))$. We hence need to consider only the latter pairing. Using the self-duality property $(2.1.9)$, one can show that

$$(\delta\phi_-^\alpha, F_3) = -i * (\delta\phi_-^\alpha, F_1) \ . \tag{6.5.37}$$

This allows to rewrite $(\delta\phi_-^\alpha, (6.4.19c))$ as

$$v \wedge (\delta\phi_-^\alpha, d_4\phi_-) - (v^3 - i\iota_k*)D(\xi + i\tilde{\xi}) = -\omega(\delta\phi_-^\alpha, d_6\phi_+) - \bar{\omega}(\delta\phi_-^\alpha, d_6\bar{\phi}_+) \ . \tag{6.5.38}$$

This equation becomes more familiar once we decompose it along the basis of two-forms $\{k \wedge v^x, v^x \wedge v^y\}$. The components along $k \wedge v^x$ give the equations

$$L_k z^\alpha = L_k \tilde{z}^\alpha = 0 \ , \tag{6.5.39a}$$

$$L_k D(\xi + i\tilde{\xi}) = \mu(\delta\phi_-^\alpha, d_6\phi_+) + \bar{\mu}(\delta\phi_-^\alpha, d_6\bar{\phi}_+) \ . \tag{6.5.39b}$$

$(6.5.39b)$ comes from $k \wedge v^3$, and is exactly the same as $(\delta\phi_-^\alpha, (6.4.19a))$; $(6.5.39a)$ come from $k \wedge v^1$, or equivalently $k \wedge v^2$. Notice that $(6.5.39a)$ also follow from the statement $(6.4.17c)$, that the action of k preserves the internal metric g_{mn}. The components of $(6.5.38)$ along $v^x \wedge v^y$ give

$$v \cdot D(\xi^\alpha + i\tilde{\xi}^\alpha) + v^3 \cdot D(z^\alpha + i\tilde{z}^\alpha) = 0 \ , \tag{6.5.39c}$$

$$v \cdot D(z^\alpha + i\tilde{z}^\alpha) - v^3 \cdot D(\xi^\alpha + i\tilde{\xi}^\alpha) = -\mu(\delta\phi_-^\alpha, d_6\phi_+) + \bar{\mu}(\delta\phi_-^\alpha, d_6\bar{\phi}_+) \ . \tag{6.5.39d}$$

$(6.5.39d)$ comes from $v^1 \wedge v^2$, and is exactly the same as $(\delta\phi_-^\alpha, (6.4.19e))$. On the other hand, $(6.5.39c)$ comes from $v^1 \wedge v^3$, or equivalently from $v^2 \wedge v^3$. In black hole applications, these equations would often give that hypermultiplet scalars do not flow, but sometimes that they do, as in [86].

All the equations in $(6.5.39)$ can again be rewritten as in $(6.3.10)$ by taking

$$q^{1\alpha} = z^\alpha \qquad q^{2\alpha} = \xi^\alpha \qquad q^{3\alpha} = \tilde{\xi}^\alpha \qquad q^4 = \tilde{z}^\alpha \tag{6.5.40}$$

and the Ω^x, in each dimension 4 block corresponding to a hypermultiplet, are again given by $(6.5.35)$. The gauging vectors are given by

$$g\bar{\mathcal{L}}^\Lambda k_\Lambda^u = \begin{pmatrix} 0 \\ i(\delta\phi_-^\alpha, d_6\phi_+) + i(\overline{\delta\phi_-^\alpha}, d_6\phi_+) \\ (\delta\phi_-^\alpha, d_6\phi_+) - (\overline{\delta\phi_-^\alpha}, d_6\phi_+) \\ 0 \end{pmatrix} \ . \tag{6.5.41}$$

6.5.5 Vector multiplets

Now we turn to the equations corresponding to the vector multiplets. As we saw in section $6.5.1$, the scalars in the vector multiplets sit in the components along $\delta\phi_+^a$ in $(6.5.3)$ and its conjugate. The pairing $(\delta\phi_+^a, (6.4.19b))$ gives

$$-i\mu \, D\bar{t}^a + \frac{i}{2}\bar{W}^{ax}v^x = 2\iota_k G^{a-} \ , \tag{6.5.42}$$

where the $D\bar{t}^a$ and G^{a-} were defined in (6.5.4), (6.5.17), and the W's are defined as

$$\bar{W}^{a+} \equiv 4\left(\delta\phi_+^a, d_6\phi_-\right), \quad \bar{W}^{a-} \equiv 4\left(\delta\phi_+^a, d_6\bar{\phi}_-\right), \quad \bar{W}^{a3} \equiv e^\phi\left(\delta\phi_+^a, F_0\right). \tag{6.5.43}$$

(6.5.42) is exactly the complex conjugate of (6.3.12). The W^{ax} correspond to the four-dimensional shifts appearing in (6.3.2c) and (6.3.12). They are related to the derivatives of the shifts s^x with respects to the geometrical moduli. This feature was already remarked in [87, Eq. (3.77)].

It is also interesting to note that there does not seem to exist an equivalent of the shift W^a. In fact, thanks to this, we see that

$$L_k t^a = 0 \ . \tag{6.5.44}$$

Another notable consequence of (6.5.42) would be, in black hole applications, the attractor equation for vector multiplet scalars.

It remains to consider the parings of (6.4.19d) with $\delta\phi_+^a$. However one can show that the resultant equations are exactly equivalent to (6.5.42), therefore they do not give any additional information. This result is not a surprise since we already know that (6.3.11) is sufficient to fully reconstruct the gaugino equations in the four-dimensional timelike case.

6.5.6 New vector multiplets: edge of the diamond

As we saw in section 6.5.1 (see the summary in table 6.1), we also expect equations associated to the "edge of the diamond". They come from different places: to start with we have the equations coming from (6.4.19) and expanded along the edge of diamond. These are similar to the counterparts just discussed in the interior of the diamond; however, in this case we have fewer redundacies than in sections 6.5.4 and 6.5.5, and thus the full amount of equations is bigger. We have also the pairing equations (6.4.20c), (6.4.20d), (6.4.23c) and (6.4.23d).[8] All these equations will be shown in this section and in the next.

In this section, we will give the ones corresponding to the new vector multiplet in the

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[8]Recall that the other pairing equations (6.4.20a), (6.4.20b), (6.4.23a) and (6.4.23b) are redundant in the timelike case since they determines external torsions which are determined by the equations in section 6.5.2.

$(\bar{\mathbf{3}}, \mathbf{1}) \oplus (\mathbf{1}, \mathbf{3})$ (see table 6.1). They take the form

$$\mu(\gamma^{\bar{i}_1}\phi_+, F_1) = 2e^{-\phi}(\gamma^{\bar{i}_1}\phi_+, d_6\bar{\phi}_+)(k - v^3) + (\gamma^{\bar{i}_1}\phi_-, F_0)v - 2\left(\frac{e^{-\phi}}{\bar{\mu}}v \cdot (\gamma^{\bar{i}_1}\phi_-, d_4\phi_+)\right)v^3 ,$$
(6.5.45a)

$$\bar{\mu}(\bar{\phi}_+\gamma^{\bar{i}_2}, F_1) = 2e^{-\phi}(\bar{\phi}_+\gamma^{\bar{i}_2}, d_6\phi_+)(k + v^3) + (\phi_-\gamma^{\bar{i}_2}, F_0)v + 2\left(\frac{e^{-\phi}}{\mu}v \cdot (\bar{\phi}_+\gamma^{\bar{i}_2}, d_4\phi_-)\right)v^3 .$$
(6.5.45b)

They come from (6.4.19c) expanded along the edge of the diamond, combined with (6.4.20d) and (6.4.23d). (Equations (6.4.19a) and (6.4.19e) are redundant like it happened for the non universal hypermultiplet in section 6.5.4). The reason we do not see any field-strengths in the vector multiplet equations (6.5.45) is that their gauge potentials would be $g_{\mu m}$ and $B_{\mu m}$ (see again table 6.1), which we have set to zero.

6.5.7 Gravitino multiplet: edge of the diamond

We finally show the equations associated to the gravitino multiplet of table 6.1. This was not discussed in section 6.3, since it is usually not considered in four-dimensional theories, for reasons explained in section 6.5.1. As discussed there, we are not advocating including gravitino multiplets in a reduction; here, we are not reducing, but rather organizing the ten-dimensional supersymmetry equations in a way which makes it natural to compare with four dimensions.

The equations read :

$$\iota_k(\gamma^{i_1}\bar{\phi}_-, F_2) = -2\bar{\mu}e^{-\phi}(\gamma^{i_1}\bar{\phi}_-, d_4\bar{\phi}_+) - 2Q_L^x v_x \qquad (6.5.46a)$$

$$\iota_k(\phi_-\gamma^{\bar{i}_2}, F_2) = -2\bar{\mu}e^{-\phi}(\phi_-\gamma^{\bar{i}_2}, d_4\bar{\phi}_+) - 2Q_R^x v_x \qquad (6.5.46b)$$

$$\iota_{v^3}(\gamma^{\bar{i}_1}\phi_-, d_4\phi_+) = \iota_{v^3}(\phi_-\gamma^{\bar{i}_2}, d_4\bar{\phi}_+) = 0 = \iota_k(\gamma^{\bar{i}_1}\phi_-, d_4\phi_+) = \iota_k(\phi_-\gamma^{\bar{i}_2}, d_4\bar{\phi}_+) , \quad (6.5.46c)$$

$$(\gamma^{\bar{i}_1}\phi_+, d_6\bar{\phi}_+) = (\gamma^{\bar{i}_1}\phi_-, d_6\bar{\phi}_-) , \qquad (\bar{\phi}_+\gamma^{\bar{i}_2}, d_6\phi_+) = -(\phi_-\gamma^{\bar{i}_2}, d_6\bar{\phi}_-) , \qquad (6.5.46d)$$

where

$$Q_L^x = \begin{pmatrix} e^{-\phi}(\gamma^{i_1}\bar{\phi}_-, d_6\phi_-) \\ ie^{-\phi}(\gamma^{i_1}\bar{\phi}_-, d_6\phi_-) \\ \frac{1}{2}(\gamma^{i_1}\bar{\phi}_-, F_0) \end{pmatrix} , \qquad Q_R^x = \begin{pmatrix} e^{-\phi}(\phi_-\gamma^{\bar{i}_2}, d_6\bar{\phi}_-) \\ -ie^{-\phi}(\phi_-\gamma^{\bar{i}_2}, d_6\bar{\phi}_-) \\ \frac{1}{2}(\phi_-\gamma^{\bar{i}_2}, F_0) \end{pmatrix} . \qquad (6.5.47)$$

(6.5.46a) and (6.5.46b) are obtained by taking $(\gamma^{\bar{i}_1}\phi_-, (6.4.19b))$ and $(\phi_-\gamma^{\bar{i}_2}, (6.4.19b))$, while (6.5.46c) are obtained from $(\gamma^{\bar{i}_1}\phi_-, (6.4.19c))$ and $(\phi_-\gamma^{\bar{i}_2}, (6.4.19c))$. (6.5.46d) are a consequence of (6.4.20c) and of (6.4.23c).

From (6.5.46c), together with (6.5.39a) and (6.5.44), we see that the internal pure spinors $\phi_\pm$ are left invariant by the action of k. Since, as we recalled at the beginning

of section 6.5.1, the internal metric and B_0 field are determined by them, we recover (6.4.17c), (6.4.18b).

With all the caveats given above, we can use our rough discussion in section 6.5.1 to get a sense of where the new equations (6.5.46) come from. In $N = 8$ supergravity, in absence of gaugings the supersymmetry transformations of gravitinos look like $\delta\psi_{\mu\, A+} \sim D_\mu\zeta_{A+} + T_{AB}^+\gamma_\mu\zeta_-^B$. Recall that for us the index A is split in $1, i_1, \bar{j}_2, 2$ (see (6.5.22)). Moreover, only ζ_1 and ζ_2 are kept, while the remaining ζ_{i_1} and $\zeta_{\bar{j}_2}$ are set to zero. The gravitino equations for $A = 1, 2$ now become the usual $N = 2$ gravitino equations (6.3.2a) (corrected there by gaugings); for $A = i_1, \bar{j}_2$ they become the new equations

$$T_{i_11+}^{\mu\nu}\gamma_\nu\zeta^1 + T_{i_12+}^{\mu\nu}\gamma_\nu\zeta^2 = 0 = T_{\bar{j}_21+}^{\mu\nu}\gamma_\nu\zeta^1 + T_{\bar{j}_22+}^{\mu\nu}\gamma_\nu\zeta^2 \ . \tag{6.5.48}$$

In $N = 8$ we also have the supersymmetry variations of the spin $1/2$ fields, which again read $\delta\lambda_{ABC} \sim T_{[AB}\zeta_{C]+} + P_{ABCD}\zeta_-^D$. Now, $\delta\lambda_{i_1\bar{j}_21}$ and $\delta\lambda_{i_1\bar{j}_22}$ give rise to the vector multiplet equations in 6.5.5; $\delta\lambda_{i_1j_11}$ and $\delta\lambda_{i_1j_12}$ give rise to the "edge" vector multiplets discussed in section 6.5.6; the $\delta\lambda$'s with no 1 or 2 index give rise to the hypermultiplets, discussed in section 6.5.3 and 6.5.4. We still have $\delta\lambda_{i_112}$ and $\delta\lambda_{\bar{j}_212}$; these give

$$T_{i_1[1}\zeta_{2]} = 0 = T_{\bar{j}_2[1}\zeta_{2]} \ . \tag{6.5.49}$$

Both (6.5.48) and (6.5.49), suitably corrected by gaugings, are the origin of (6.5.46).

6.5.8 Summary: ten-dimensional timelike case

In this section, we refined our results of section 6.4 for the timelike case. Namely, we still work with a fibration (6.4.1), with spinor Ansatz (6.4.3), (6.4.7) and assumptions summarized in footnote 2 and (6.4.12), but now we assume that the spinors ζ_i do not coincide. We have found a system that is equivalent to preserved supersymmetry in this case. It consists of equations (6.5.26a), (6.5.26b), (6.5.26c), (6.5.31); (6.5.33), (6.5.39), (6.5.42), (6.5.45), (6.5.46). Most of the equations correspond to the "boxed" system in section 6.3; the ones which do not, (6.5.46), capture extra equations which it would be challenging to obtain in a four-dimensional effective approach, as anticipated in the introduction and discussed in detail in section 6.5.1.

6.6 An aside: $N = 2$ four-dimensional supergravity, general case

We found in section 6.3 a system of form equations which is equivalent to preserved supersymmetry in the timelike case. This was based on the fact (see section 6.2.4)

that all the intrinsic torsions can be reconstructed from $d\mu$, dv, dk_i. This had no clear counterpart in the general case, and at that point it was not clear how to proceed.

In this section, we will see that the general ten-dimensional system we presented in section 6.4 suggests a system of equations which are valid in the general case. Unfortunately, this system will be much more complicated than the one in section 6.3. Of course, it is possible that a better alternative exists; finding one might also suggest how to improve the general ten-dimensional system (3.2.4).

We will start in section 6.6.1 by presenting a set of derivatives of forms from which the covariant derivatives of spinors can be fully reconstructed, in the general case. We will not present the full derivation of the equivalence between such set of derivatives of forms and the covariant derivatives of the SUSY parameters (it can be found in [20]), but we will just quote the final result and we give some qualitative explanation.

We will then describe our equations for the gravitinos (section 6.6.2), the hyperinos (section 6.6.3), and the gauginos (section 6.6.4).

6.6.1 Intrinsic torsions in the general case

We begin by some mathematical preliminaries about how to reexpress the covariant derivatives $\nabla\zeta_i$ in terms of exterior algebra of forms. In the timelike case, this was done in 6.2.4. In the general case, we have to use a different procedure; we will take inspiration from the intrinsic torsion for an $\mathbb{R}^2$ structure defined by a single spinor [79], which we now review.

One spinor

A possible definition of intrinsic torsions for one Weyl spinor ζ was introduced in [79]; we will briefly review it here.

As usual, the idea is to expand the derivative of the object defining the structure in terms of a basis; given this program, we observe that, once that the auxiliary vector e_+ has been introduced, the pair of spinors

$$\zeta_+ , \qquad e_+ \cdot \zeta_- , \tag{6.6.1}$$

constitute a basis for the four-dimensional spinors of positive chirality. Similarly, we can construct a basis for the four-dimensional spinors of negative chirality by considering the pair ζ_- and $e_+ \cdot \zeta_+$. Having defined the appropriate basis for the spinors, it is natural to expand the derivative of the four-dimensional SUSY parameter ζ_+ on it. This leads to the intrinsic torsions via the formula

$$\nabla_\mu \zeta_+ = p_\mu \zeta_+ + q_\mu e_+ \cdot \zeta_- . \tag{6.6.2}$$

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where p_μ, q_μ are complex one-forms.

From (6.6.2) we see that *all* the information contained in the covariant derivative of the spinor $\nabla_\mu \zeta_+$ is contained into the one-forms p_μ and q_μ. The goal is then to obtain a set of differential conditions, on differential forms and not on spinors, which allows to fully reconstruct the intrinsic torsions.

This problem has been solved in [79], where it is shown that the intrinsic torsions for a *single* spinor of positive chirality are fully reconstruct by considering the following external differential

$$dk \ , \qquad dw \ , \qquad de_+ \ . \tag{6.6.3}$$

Two spinors

We will now consider the case where we have two spinors ζ_i of positive chirality.

As in section 6.6.1, to define intrinsic torsions we need a basis of spinors. Recall from section 6.2.2 that the two ζ_i are independent in the timelike case (and so they can be used to define a basis in this case), but that they are proportional in the null case. Between these two extremes one has a "general" case in which the two spinors ζ_i are parallel in some points and independent in others. The timelike case has already been dealt with in section 6.2.4, with the result that one needs to compute the exterior derivatives of μ, k, v^x.

In the general case, however, we have to proceed differently; we will follow the procedure of section 6.6.1. The e_{+i} introduced in section 6.2.2 now give us two possible bases for spinors, along the lines of (6.6.1). These bases are related, of course; for example we compute[9]

$$\zeta_2 = \frac{1}{2} e_{+1} \cdot \bar{v} \, \zeta_1 - \frac{1}{2} \mu \, e_{+1} \cdot \zeta^1 \tag{6.6.4}$$

and analogously

$$\zeta_1 = \frac{1}{2} e_{+2} \cdot v \, \zeta_2 + \frac{1}{2} \mu \, e_{+2} \cdot \zeta^2 \ . \tag{6.6.5}$$

However, we will most often keep the two bases separate. We can now simply introduce two sets of p_μ and q_μ:

$$\nabla_\mu \zeta_i = p_\mu^i \, \zeta_i + q_\mu^i \, e_{+i} \cdot \zeta^i \tag{6.6.6}$$

where the indices are not summed.

We should now try to identify a system of PDE in terms of exterior algebra, from which one can reconstruct both the p^i and q^i in (6.6.6), similarly to how (6.6.3) reconstructs p and q in section 6.6.1. We saw in section 6.6.1 that the system (6.6.3) is necessary and sufficient to fully reconstruct intrinsic torsions for the case with one spinor. With

---

[9]Notice that the coefficient $\frac{1}{2} e_{+1} \cdot \bar{v}$ is precisely the same coefficient $\alpha(x)$ that we encountered in (6.2.20).

two spinors, the most obvious procedure would be to "double" that computation, and consider the equations for dk_i, dw_i, de_{+i}.

This is obviously sufficient to determine p^i and q^i. However, it is not suited to the applications to supergravity we will pursue later in this section. The reason is that dw_i, de_{+i} would contain two 'spurious' one-forms ρ^i, which would not be determined by supersymmetry, since they do not appear in the covariant derivatives $\nabla_\mu \zeta_i$. One might try to improve the system (6.6.3) so that ρ never appears (for example, by taking appropriate projections of dw and de_+). This would certainly deserve further investigation but here we will however follow a different approach: we will take inspiration from the ten dimensional system (3.2.4), specialized as in section 6.4, to deduce a necessary and sufficient system of PDEs to reconstruct p^i and q^i.

First of all, (6.4.19) and (3.2.4a) suggest that we have to consider the exterior derivative of $\zeta_1 \overline{\zeta_2}$ and $\zeta_1 \overline{\zeta_2^c}$

$$d(\zeta_1 \overline{\zeta_2}) , \qquad d(\zeta_1 \overline{\zeta_2^c}) , \tag{6.6.7}$$

together with the tensor

$$\nabla_{(\mu}(k_1 + k_2)_{\nu)} , \qquad \nabla_{[\mu}(k_1 - k_2)_{\nu]} . \tag{6.6.8}$$

Together, (6.6.7) and (6.6.8) contain almost all of the components of p^i and q^i. The only ones left are[10]

$$e_{+1} \cdot p_2 , \qquad e_{+2} \cdot p_1 , \qquad e_{+1} \cdot q_2 , \qquad e_{+2} \cdot q_1 . \tag{6.6.9}$$

To determine these remaining components using differential forms, we finally take inspiration from (3.2.4c), (3.2.4d): indeed we see that (6.6.9) are exactly the components of the intrinsic torsions appearing in (6.4.20a), (6.4.20b), (6.4.23a) and (6.4.23b); it is possible to verify that such components can be computed by considering the following expressions involving the four-dimensional pairings

$$
\begin{aligned}
(d(e_{+1}\zeta_1\overline{\zeta^2}), e_{+1}\zeta^1\overline{\zeta^2}) &= -16i e_{+1} \cdot \bar{q}_2 , \qquad (d(\zeta_1\overline{\zeta^2}e_{+2}), \zeta_1\overline{\zeta_2}e_{+2}) = 16i e_{+2} \cdot q_1 , \\
(d(e_{+1}\zeta_1\overline{\zeta_2}), e_{+1}\zeta^1\overline{\zeta^2}e_{+2}) &+ (d(e_{+1}\zeta^1\overline{\zeta_2}), e_{+1}\zeta_1\overline{\zeta^2}e_{+2}) - 16i d^\dagger e_{+1} = 32i e_{+1} \cdot p_2 , \quad (6.6.10) \\
(d(\zeta_1\overline{\zeta^2}e_{+2}), e_{+1}\zeta^1\overline{\zeta_2}e_{+2}) &+ (d(\zeta_1\overline{\zeta_2}e_{+2}), e_{+1}\zeta^1\overline{\zeta^2}e_{+2}) + 16i d^\dagger e_{+2} = -32i e_{+2} \cdot p_1 .
\end{aligned}
$$

6.6.2 Gravitino

Our strategy will be to evaluate using supersymmetry the various intrinsic torsions we identified in section 6.6.1. There are three classes of equations: those coming from $d(\zeta_1\overline{\zeta^2})$, $d(\zeta_1\overline{\zeta_2})$, those from $\nabla_\mu k_{1\nu} + \nabla_\nu k_{2\mu}$ and the torsions "along k_i" (6.6.10).

---

[10]This precisely parallels the fact that the intrinsic torsions Q^1_{+2NP} and Q^2_{+1NP} were the ones not determined by (3.2.4b) and (3.2.4a); see [11, (B.20)].

We can evaluate $d(\zeta_1\bar{\zeta}^2)$, $d(\zeta_1\bar{\zeta}_2)$ in the language of (6.2.9). That means computing $d\mu$, dv, $d\omega$, $d*v$. Actually, before we give the expressions for these exterior derivatives, notice that one more exterior derivative is the antisymmetrization of $\nabla_\mu k_{1\nu} + \nabla_\nu k_{2\mu}$, namely $d(k_1 - k_2)$. In (6.3.5), we already gave formulas for $d(k_1 - k_2)$ and dv, collected together in a triplet dv_x (see (6.3.6)). This time, however, we do not need to use dk. The exterior derivatives we need are then

$$d\mu = S_x v_x - \iota_K T^+ \,, \tag{6.6.11a}$$

$$dv_x = 2\epsilon_{xyz}\text{Im}(\bar{S}_y\omega_z) \,, \tag{6.6.11b}$$

$$d\omega = i\hat{Q} \wedge \omega + \frac{i}{2}(A^+ \wedge \omega_1 - A^- \wedge \omega_2) + \frac{3}{2}i*(-S_+\bar{v} + S_-v + S_3(k_1 + k_2)) - (k_1 - k_2) \wedge T^+ \,, \tag{6.6.11c}$$

$$d*v = -2(\bar{\mu}S^+ + \mu\overline{S^-})\text{vol}_4 \,. \tag{6.6.11d}$$

We also have to compute the symmetrization of $\nabla_\mu k_{1\nu} + \nabla_\nu k_{2\mu}$, which simply gives

$$\nabla_{(\mu}k_{\nu)} = 0 \,. \tag{6.6.12}$$

This is simply the statement that k is a Killing vector.

Finally, we have to compute the remaining torsions (6.6.10). These give

$$e_{+1} \cdot (4q_2 + S^- k_2 + S^3 v - 2\iota_v T^+ - i\mu A^-) = 0 \,, \tag{6.6.13a}$$

$$e_{+1}^\mu \left(2p_{2\,\mu} - S^-\bar{w}_2^\mu - 2(\bar{\lambda}e_{+2} - \iota_{e_{+2}}\bar{\omega})^\nu \left(T_{\mu\nu}^+ - \frac{1}{2}S^3 g_{\mu\nu}\right) - 2i\hat{Q}_\mu - 2ie_{+2}\cdot v A_\mu^-\right) = 0 \,, \tag{6.6.13b}$$

$$e_{+2} \cdot (4q_1 - S^+ k_1 + S^3 \bar{v} + 2\iota_{\bar{v}} T^+ + i\mu A^+) = 0 \,, \tag{6.6.13c}$$

$$e_{+2}^\mu \left(2p_{1\,\mu} + S^+\bar{w}_1^\mu - 2(\bar{\lambda}e_{+1} + \iota_{e_{+1}}\bar{\omega})^\nu \left(T_{\mu\nu}^+ + \frac{1}{2}S^3 g_{\mu\nu}\right) - 2i\hat{Q}_\mu - 2ie_{+1}\cdot \bar{v} A_\mu^-\right) = 0 \,. \tag{6.6.13d}$$

6.6.3 Hyperino equations

Now we turn to the hyperino equations (6.3.2b). As we remarked in section 6.3.3 to extract all the information contained in these equations it is sufficient to expand them in basis of spinors. Moreover, as explained in section 6.6.1 two possible bases of spinors of positive chirality are given by

$$\zeta_i \,, \quad e_{+i}\zeta^i \,, \qquad i = 1,\, 2 \,. \tag{6.6.14}$$

In principle a minimal set of equations which is equivalent to the original hyperino equations can be obtained by choosing a specific basis (such as the bases with index

$i = 1$) and expanding the (6.3.2b) along this basis. However, it turns out that it is more convenient to expand (6.3.2b) along *all* the spinors appearing in (6.6.14). Of course the resultant system of equations will be redundant but it can be written in a neat form.

Obviously, when we expand (6.3.2b) along ζ_i we simply re-obtain equations (6.3.10). However, in this case these equations are not sufficient, and they must be completed by the equations obtained by expanding (6.3.2b) along $e_{+i}\zeta^i$. To this end, it is useful to introduce

$$E^{ki}_\mu \equiv \bar{\zeta}^k e_{+k} \gamma_\mu \zeta^i = -4 \begin{pmatrix} \bar{w}_1 & \bar{\mu} e_{+1} + \iota_{e_{+1}}\bar{\omega} \\ -\bar{\mu} e_{+2} + \iota_{e_{+2}}\bar{\omega} & \bar{w}_2 \end{pmatrix}_\mu = E_\mu \epsilon^{ki} + E_{\mu\,x}\sigma^{xik} \ , \quad (6.6.15)$$

where k is not summed. This matrix does not actually transform well under $SU(2)$; it is a bookkeeping device. It would be possible to define a three-index tensor $\bar{\zeta}^k e_{+j}\gamma^\mu\zeta^i$ that does transform well, but for our current aim this would be an overkill. We also introduce

$$C^k{}_i \equiv \bar{\zeta}^k e_{+k}\zeta_j = 4 \begin{pmatrix} 1 & e_{+1}\cdot\bar{v} \\ e_{+2}\cdot v & 1 \end{pmatrix} = C\delta^k{}_i + C^x\sigma^{x\,k}{}_i \ . \quad (6.6.16)$$

Returning now to (6.3.2b) we can expand them along $\bar{\zeta}^k e_{+k}$ and, after some manipulations very similar to those that lead to (6.3.10), we obtain the equation

$$iE \cdot Dq^v - \Omega^{x\,v}{}_u E^x \cdot Dq^u - Cg\bar{\mathcal{L}}^\Lambda k^v_\Lambda - ig\Omega^{x\,v}{}_u C^x \bar{\mathcal{L}}^\Lambda k^u_\Lambda = 0 \ . \quad (6.6.17)$$

In the general case, the hyperino equations are equivalent, in a slightly redundant manner, to (6.6.17) and (6.3.10).

6.6.4 Gaugino equations

Finally we move to the gaugino equations (6.3.2c). Contrary to the hyperino equations just discussed, in this case it is not convenient to use the $SU(2)$ formalism; therefore we will rewrite (6.3.2c) as

$$i\,Dt^a\zeta^i + G^{a+}\zeta_2 + W^a\zeta_2 - \frac{i}{2}W^{a-}\zeta_1 + \frac{i}{2}W^{a3}\zeta_2 = 0 \ , \quad (6.6.18a)$$

$$i\,Dt^a\zeta^2 - G^{a+}\zeta_1 - W^a\zeta_1 + \frac{i}{2}W^{a3}\zeta_1 + \frac{i}{2}W^{a+}\zeta_2 = 0 \ . \quad (6.6.18b)$$

To proceed we expand equations (6.6.18a) along $\bar{\zeta}_2$ and $\bar{\zeta}^2 e_{+2}$, and (6.6.18b) along $\bar{\zeta}_1$ and $\bar{\zeta}^1 e_{+1}$ obtaining the system

$$i\,\bar{v} \cdot Dt^a + G^{a+} \llcorner \omega_2 - \frac{i}{2}\mu W^{a-} = 0 \ , \tag{6.6.19a}$$

$$-\,i\big((\iota_{e_{+2}}\bar{\omega}) - \bar{\mu}e_{+2}\big)\llcorner Dt^a - 2\iota_{e_{+2}}\iota_{k_2}G^{a+} + W^a - \frac{i}{2}(e_{+2} \cdot v)W^{a-} + \frac{i}{2}W^{a3} = 0 \ , \tag{6.6.19b}$$

$$i\,v \cdot Dt^a - G^{a+} \llcorner \omega_1 - \frac{i}{2}\mu W^{a+} = 0 \ , \tag{6.6.19c}$$

$$-\,i\big((\iota_{e_{+1}}\bar{\omega}) + \bar{\mu}e_{+1}\big)\llcorner Dt^a + 2\iota_{e_{+1}}\iota_{k_1}G^{a+} - W^a + \frac{i}{2}(e_{+1} \cdot \bar{v})W^{a+} + \frac{i}{2}W^{a3} = 0 \ . \tag{6.6.19d}$$

The system (6.6.19) is therefore completely equivalent to the original equation (6.3.2c) without any redundancy.

Chapter 7

Holomorphic Chern-Simons theory coupled to off-shell Kodaira-Spencer gravity

This Chapter is devoted to Holomorphic Chern-Simons theory (HCS), a theory introduced by Witten in [24] as the target space field theory describing the dynamics of a stack of N 5-branes of topological string theory of the B type living on a Calabi-Yau complex three-fold X. In section 7.1 we will present an introduction to the theory, we will discover its main features and we will describe our goal. Then we will move to discuss how this theory can be formulated consistently also when the geometrical backgrounds are *off shell*. Finally, we will see how the BV formalism allows us to uncover an $N = 2$ twisted SUSY algebra which is responsible for the semi-topological character of the model.

7.1   Statement of the problem

The action of HCS

$$\Gamma = \int_X \Omega \wedge \text{Tr}\big(\frac{1}{2}\, A\, \bar{\partial}_Z\, A + \frac{1}{3}\, A^3\big) \, , \tag{7.1.1}$$

is a six-dimensional analogue of the topological three-dimensional Chern-Simons action [88]. The gauge field A, encoding the open string degrees of freedom, is a one-form with values in the Lie algebra of $SU(N)$ of type $(0, 1)$ with respect to the chosen complex structure on X

$$A = dZ^{\bar{\imath}}\, A_{\bar{\imath}}(Z, \bar{Z}) = dZ^{\bar{\imath}}\, A_{\bar{\imath}}^a(Z, \bar{Z})\, T^a. \tag{7.1.2}$$

In the formula above T^a are the $SU(N)$ generators and Tr is the trace in its fundamental representation. $\bar{\partial}_Z$ is the Dolbeault operator relative to complex coordinates $(Z^i, Z^{\bar{\imath}})$

compatible with the chosen complex structure

$$\bar{\partial}_Z = dZ^{\bar{\imath}} \frac{\partial}{\partial Z^{\bar{\imath}}} \,. \tag{7.1.3}$$

The HCS action (7.1.1) depends therefore on *two* different classical geometrical data. One of them is the complex structure that one picks on X. The other is Ω, the globally defined holomorphic (3,0)-form on X that we already encountered, in the supergravity discussion, in Chapter 3

$$\Omega = \Omega_{ijk}(Z, \bar{Z}) \, dZ^i \wedge dZ^j \wedge dZ^k \; = \rho(Z, \bar{Z}) \, \epsilon_{ijk} \, dZ^i \wedge dZ^j \wedge dZ^k \,, \tag{7.1.4}$$

which, for Calabi-Yau three-folds, is unique up to a rescaling. Ω and the complex structure on X are in correspondence with the *closed* moduli parametrizing the closed string vacuum in which the 5-branes live. Since the (3,0)-form Ω depends on the complex structure on X, the moduli space of closed strings is the total space of a complex line bundle whose base is the moduli space of complex structures on X and whose fiber is the holomorphic (3,0)-form.

To exhibit explicitly the dependence of the theory on the complex structure of X it is convenient to introduce the Beltrami parametrization of the differentials dZ^i

$$dZ^i = \Lambda^i_j \left(dz^j + \mu^j_{\bar{\jmath}} \, dz^{\bar{\jmath}} \right), \tag{7.1.5}$$

where $(z^i, z^{\bar{\imath}})$ is a *fixed* system of complex coordinates. The Beltrami differential

$$\mu \equiv \mu^i \frac{\partial}{\partial z^i} \equiv \mu^i_{\bar{\jmath}} \, dz^{\bar{\jmath}} \frac{\partial}{\partial z^i} \tag{7.1.6}$$

is a (0,1)-form with values in the holomorphic tangent $T^{(1,0)} X$. The action (7.1.1) rewrites in the system of coordinates $(z^i, z^{\bar{\imath}})$ as follows

$$\Gamma_0(\Omega, \mu) = \int_X \Omega \wedge \left(\frac{1}{2} A \, \overline{\nabla} \, A + \frac{1}{3} \, A^3 \right), \tag{7.1.7}$$

where

$$\overline{\nabla} \equiv \bar{\partial} - \mu^i \, \partial_i \,, \qquad \bar{\partial} \equiv dz^i \frac{\partial}{\partial z^i} \,. \tag{7.1.8}$$

In this formulation, the dependence of the theory on the closed moduli is captured by the two *classical* backgrounds fields — Ω and μ.

The original action (7.1.1) is invariant under Ω-preserving holomorphic reparametrizations. The coupling of A to the classical background μ promotes this *global* invariance into a *local* symmetry under which μ transforms as a gauge field

$$s_{\text{diff}} \, \mu^i = -\bar{\partial} \, \xi^i + \xi^j \partial_j \, \mu^i - \partial_j \, \xi^i \, \mu^j \,. \tag{7.1.9}$$

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In (7.1.9), ξ^i is the ghost of Ω-preserving local diffeomorphisms

$$\partial\, i_\xi(\Omega) = 0\,, \tag{7.1.10}$$

where i_ξ is the contraction of a form with the vector field $\xi^i\,\partial_i$.

The backgrounds Ω and μ in (7.1.7) must satisfy the classical equations of motion of the closed topological string theory:

$$\mathcal{F}^i \equiv \bar{\partial}\mu^i - \mu^j\partial_j\mu^i = 0\,, \tag{7.1.11}$$

$$\hat{\overline{\nabla}}\,\Omega \equiv \overline{\nabla}\Omega + \partial_i\,\mu^i\,\Omega = 0\,. \tag{7.1.12}$$

The first of such equations is the celebrated Kodaira-Spencer equation [26] which expresses the integrability of the Beltrami differential; the second equation expresses the holomorphicity of Ω in the complex structure associated to μ^i. Indeed the action (7.1.7) is invariant under the gauge BRST symmetry[1]

$$s\,A = -\overline{\nabla}c - [A, c]_+\,,$$
$$s\,c = -c^2\,, \tag{7.1.13}$$

where $c = c^a\,T^a$ is the anti-commuting ghost associated to $SU(N)$ gauge transformations, *only if* the closed string equation of motions (7.1.11) and (7.1.12) are satisfied. It should be kept in mind that A and c are the dynamical variables of HCS while μ^i, Ω and ξ^i are classical non-dynamical fields.

For the purpose of investigating the *quantum* properties of HCS field theory, like its renormalization and its anomalies, it is useful to extend both gravitational backgrounds μ and Ω to be generic *off-shell* functions. Hence in the next sections we will write down the appropriate generalization of the action (7.1.7) valid also when μ and Ω do not satisfy their equations of motion (7.1.11) and (7.1.12). Nevertheless, as mentioned above, the closed string fields will still be treated as non-dynamical backgrounds. In the context of string theory our result could help understanding the back-reaction of the 5-branes on the closed string vacuum, since the presence of branes modifies the equation of motions (7.1.11) and (7.1.12) and puts the backgrounds off-shell.

The standard method to go "off-shell" is to introduce new fields acting as Lagrange multipliers whose equations of motions are precisely the closed string equations (7.1.11) and (7.1.12) and whose gauge transformation properties are such that the action is gauge invariant even for off-shell backgrounds. This strategy has been adopted by the authors

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[1]In this second part of the thesis, we will adopt the convention that fields and operators carrying odd ghost number *anti-commute* with fields and operators carrying odd form degree. In particular, the BRST operator s and the Dolbeault differential $\overline{\nabla}$ anti-commute.

of [89],[2] who were able to solve, so-to-say, half of the problem: they introduced a Lagrange multiplier whose equation of motion is the Kodaira-Spencer equation (7.1.11), but they did not reformulate the second equation (7.1.12) in the same way. We achieve this task in the present article.

The reason why the authors of [89], whose main focus is the closed target space field theory, have restricted Ω to be holomorphic, has to do with the different status that equations (7.1.11) and (7.1.12) enjoy in the Kodaira-Spencer field theory: Eqs. (7.1.11), which are the classical equations of motion derived from the Kodaira-Spencer action [26], are equivalent to the BRST-invariance of the closed vertex operators associated to the complex structure moduli. This is the standard relation between the second quantized classical equations of motion and first-quantized vertex operators.

Eq. (7.1.12), instead, is not an equation of motion of Kodaira-Spencer field theory. The Ω which enters the Kodaira-Spencer action must be holomorphic and hence it is a parameter and not a dynamical field of Kodaira-Spencer theory. From this point of view, Kodaira-Spencer theory does not provide the second quantized formulation for the first-quantized vertex operator (of non-standard world-sheet ghost number) associated to Ω.

On the other hand, in the open string field theory it seems to be perfectly sensible to treat Ω and μ on the same footing: we will therefore introduce Lagrange multipliers whose equations of motion coincide with both (7.1.11) and (7.1.12) and will determine their gauge transformation properties. We will find it necessary to enlarge the $SU(N)$ gauge symmetry to include a number of new ghost (and ghost-for-ghost) fields which can be thought of as "descendants" of the Lagrange multipliers and which ensure the nilpotency of the full BRST transformations.

Since Ω becomes, in our construction, an off-shell background, the HCS action that we will derive enjoys a larger reparametrization invariance than the original action (7.1.7). This invariance include reparametrizations which are not Ω preserving:

$$
\begin{aligned}
s_{\text{diff}}\, \mu^i &= -\bar{\partial}\, \xi^i + \xi^j \partial_j\, \mu^i - \partial_j\, \xi^i\, \mu^j \,, \\
s_{\text{diff}}\, \Omega &= \partial\, i_\xi(\Omega) \,, \\
s_{\text{diff}}\, A &= \xi^i \partial_i\, A \,, \qquad s_{\text{diff}}\, \xi^i = \xi^j\, \partial_j\, \xi^i \,,
\end{aligned}
\tag{7.1.14}
$$

together with analogous transformations for all the other dynamical fields that we will

---

[2] A different method to couple HCS to off-shell gravitational backgrounds has been put forward in [90]. Contrary to our approach, the (3,0)-form Ω is not treated in [90] as a background independent of the complex structure μ and, correspondingly, the Ω equation (7.1.12) is still implicitly assumed, much like in the treatment of [89]. Moreover, the strategy employed to lift the Kodaira-Spencer constraint (7.1.11) entails the inclusion among the dynamical fields of the (1,0) component of the gauge field, together with a series of satellite fields, thus introducing a large gauge redundancy and making the dependence on the complex structure μ fairly implicit.

introduce. We will refer to the reparametrization invariance (7.1.14) acting on off-shell μ^i and Ω as *chiral* diffeomorphism invariance.

7.2 Chiral reparametrization invariance

The coupling of HCS to the holomorphic Beltrami differentials (7.1.5) is determined by requiring invariance under chiral reparametrizations. Chiral reparametrizations act on the Beltrami differentials as follows

$$s_{\text{diff}} \, \mu^i = -\bar{\partial} \, \xi^i + \xi^j \partial_j \, \mu^i - \partial_j \, \xi^i \, \mu^j \,, \tag{7.2.1}$$

where ξ^i is the anti-commuting ghost field of chiral diffeomorphisms:

$$s_{\text{diff}} \, \xi^i = \xi^j \, \partial_j \, \xi^i \,. \tag{7.2.2}$$

It is important to keep in mind that s_{diff} is nilpotent for *generic* μ^i, independently of the validity of the Kodaira-Spencer equation (7.1.11). On the space of Beltrami differentials μ^i which do satisfy Eq. (7.1.11) there exists a natural action of non-chiral (standard) reparametrizations which follows from the definition (7.1.5): one can show [91] that the actions of chiral and non-chiral reparametrizations coincide on such space if one identifies the chiral ghost ξ^i with the following combinations of the ghosts $(c^i, c^{\bar{\imath}})$ of standard diffeomorphisms

$$\xi^i = c^i + \mu^i_{\bar{\jmath}} \, c^{\bar{\jmath}} \,. \tag{7.2.3}$$

There is no notion of standard reparametrizations of "off-shell" Beltrami differentials, i.e. of μ^i's which do not satisfy the Kodaira-Spencer equation: invariance under chiral diffeomorphisms (7.2.1) represents the extention of reparametrization invariance appropriate for off-shell μ^i.

Matter fields with only anti-holomorphic indices transform under chiral diffeomorphisms as scalars

$$s_{\text{diff}} \, \phi_{\bar{\imath}\bar{\jmath}\ldots} = \xi^i \, \partial_i \, \phi_{\bar{\imath}\bar{\jmath}\ldots} \,. \tag{7.2.4}$$

For example, the transformation law under chiral reparametrizations of the gauge field $A = A_{\bar{\imath}} \, dx^{\bar{\imath}}$ is

$$s_{\text{diff}} \, A_{\bar{\imath}} = \xi^i \, \partial_i \, A_{\bar{\imath}} \,. \tag{7.2.5}$$

The action of chiral diffeomorphisms on tensors with holomorphic indices is instead

$$s_{\text{diff}} \, \phi^{i\cdots}_{\bar{\imath}\bar{\jmath}\ldots;k\ldots} = \xi^j \, \partial_j \, \phi^{i\cdots}_{\bar{\imath}\bar{\jmath}\ldots;k\ldots} - \partial_j \, \xi^i \, \phi^{j\cdots}_{\bar{\imath}\bar{\jmath}\ldots;k\ \ldots} + \partial_k \, \xi^j \, \phi^{i\cdots}_{\bar{\imath}\bar{\jmath}\ldots;j\ \ldots} + \cdots \,. \tag{7.2.6}$$

For example, in the following Section we will introduce the Lagrange multiplier $C_i = C_{i\bar{\imath}}\, dx^{\bar{\imath}}$ which transforms under chiral reparametrizations as follows

$$s_{\text{diff}}\, C_{i\bar{\imath}} = \xi^j\, \partial_j\, C_{i\bar{\imath}} + \partial_i\, \xi^j\, C_{j\bar{\imath}}\,. \tag{7.2.7}$$

For chiral reparametrizations there is a natural definition of *covariant anti-holomorphic* derivative

$$\hat{\nabla}_{\bar{k}}\, \phi^{i\ldots}_{\bar{\imath}\bar{\jmath}\ldots;k\ldots} = \nabla_{\bar{k}}\, \phi^{i\ldots}_{\bar{\imath}\bar{\jmath}\ldots;k\ldots} + \partial_j\, \mu^i_{\bar{k}}\, \phi^{j\ldots}_{\bar{\imath}\bar{\jmath}\ldots;k\ldots} - \partial_k\, \mu^j_{\bar{k}}\, \phi^{i\ldots}_{\bar{\imath}\bar{\jmath}\ldots;j\ldots} + \cdots\,. \tag{7.2.8}$$

There is instead no natural notion of covariant *holomorphic* derivative. However the holomorphic derivative of a tensor with no holomorphic indices is a tensor with one holomorphic lower index.[3]

We will use the notation

$$\hat{\overline{\nabla}} \equiv dx^{\bar{k}}\, \hat{\nabla}_{\bar{k}} \equiv \overline{\nabla} + \hat{\Gamma}\,, \tag{7.2.9}$$

where the connection $\hat{\Gamma}$ denotes the appropriate tensor product of matrices with holomorphic indices

$$(\hat{\Gamma})^i_j = dx^{\bar{k}}\, \partial_j \mu^i_{\bar{k}} \tag{7.2.10}$$

acting on holomorphic tensors in the usual way. For example

$$\hat{\overline{\nabla}}\, V_i \equiv \overline{\nabla} V_i - \partial_i\, \mu^j\, V_j\,. \tag{7.2.11}$$

7.3   Gauge invariance

The variation of the HCS action

$$\Gamma_0 = \frac{1}{2} \int_X \Omega\, \mathrm{Tr}\big(A\, \overline{\nabla}\, A + \frac{2}{3}\, A^3\big) \tag{7.3.1}$$

under the BRST gauge transformations

$$\begin{aligned}
s\, A &= -\overline{\nabla}\, c - [A, c]_+\,, \\
s\, c &= -c^2
\end{aligned} \tag{7.3.2}$$

is:

$$\begin{aligned}
s\, \Gamma_0 &= \frac{1}{2} \int_X \Omega\, \mathrm{Tr}\big(\overline{\nabla}\, c\, \overline{\nabla}\, A + A\, \overline{\nabla}^2 c\big) = \\
&= \frac{1}{2} \int_X \Omega\, \overline{\nabla}\, \mathrm{Tr}\big(c\, \overline{\nabla}\, A\big) + \Omega\, \mathrm{Tr}\big(c\, \overline{\nabla}^2 A + A\, \overline{\nabla}^2 c\big) \\
&= \frac{1}{2} \int_X \hat{\overline{\nabla}}(\Omega)\, \mathrm{Tr}\big(c\, \overline{\nabla}\, A\big) + \Omega\, \mathrm{Tr}\big(c\, \overline{\nabla}^2 A + A\, \overline{\nabla}^2 c\big)\,, \tag{7.3.3}
\end{aligned}$$

----

[3] "Natural" in this context means that the connection in Eq. (7.2.8) depends only on μ^i and not on the choice of a metric.

where

$$\hat{\overline{\nabla}}\,\Omega \equiv \overline{\nabla}\Omega + \partial_i\,\mu^i\,\Omega\,,$$
$$\overline{\nabla} \equiv \bar{\partial} - \mu^i\,\partial_i \equiv dx^{\bar{\imath}}\,\nabla_{\bar{\imath}} \equiv dx^{\bar{\imath}}\left(\partial_{\bar{\imath}} - \mu^i_{\bar{\imath}}\,\partial_i\right). \tag{7.3.4}$$

The curvature of the $\overline{\nabla}$-differential is

$$\overline{\nabla}^2 = -dx^{\bar{\imath}}\,dx^{\bar{\jmath}}\left(\partial_{\bar{\imath}}\,\mu^j_{\bar{\jmath}} - \mu^i_{\bar{\imath}}\,\partial_i\,\mu^j_{\bar{\jmath}}\right)\partial_j = -\mathcal{F}^i\,\partial_i\,, \tag{7.3.5}$$

where

$$\mathcal{F}^j \equiv \bar{\partial}\,\mu^j - \mu^i\,\partial_i\,\mu^j \tag{7.3.6}$$

is the Kodaira-Spencer (0,2)-form with values in the holomorphic tangent.

Eq. (7.3.3) shows that Γ_0 is gauge-invariant only if both Ω and μ^i are "on-shell", i.e. if they satisfy the equations

$$\mathcal{F}^i \equiv \bar{\partial}\mu^i - \mu^j\partial_j\mu^i = 0\,, \qquad \hat{\overline{\nabla}}\,\Omega \equiv \overline{\nabla}\Omega + \partial_i\,\mu^i\,\Omega = 0\,. \tag{7.3.7}$$

The first equation is equivalent to the nilpotency of $\overline{\nabla}$ while the second expresses the holomorphicity of Ω in the complex structure defined by μ. Let us introduce the Lagrange multipliers

$$C_i \equiv C_{i\bar{\imath}}\,dx^{\bar{\imath}}, \tag{7.3.8}$$

a (0,1)-form with values in the holomorphic cotangent, in corrispondence with the first of (7.3.7), and

$$B \equiv dx^{\bar{\imath}}\,dx^{\bar{\jmath}}\,B_{\bar{\imath}\bar{\jmath}}\,, \tag{7.3.9}$$

a (0,2)-form, in correspondence with the second equation.

If their BRST variations are taken to be

$$s\,B = -\mathrm{Tr}\left(c\,\overline{\nabla}\,A\right),$$
$$s\,C_i = \mathrm{Tr}\left(-c\,\partial_i\,A + \partial_i\,c\,A\right), \tag{7.3.10}$$

the action

$$\Gamma = \frac{1}{2}\int_X\left[\Omega\,\mathrm{Tr}\left(A\,\overline{\nabla}\,A + \frac{2}{3}A^3\right) + \Omega\left(\overline{\nabla}\,B + \mathcal{F}^i\,C_i\right)\right] \tag{7.3.11}$$

is then BRST invariant for generic, "off-shell" backgrounds Ω and μ^i.[4]

---

[4]The gauge transformation laws of C_i in Eq. (7.3.10) differ from those given in [89] but they are equivalent to them when Ω is on-shell.

The BRST transformations (7.3.2) and (7.3.10) are not nilpotent when acting on the Lagrange multipliers

$$s^2 B = \mathrm{Tr}(c \, \overline{\nabla}^2 c) - \overline{\nabla} \, \mathrm{Tr}\,(A \, c^2) = -\mathcal{F}^i \, \mathrm{Tr}(c \, \partial_i c) - \overline{\nabla} \, \mathrm{Tr}\,(A \, c^2)\,,$$

$$s^2 C_i = \mathrm{Tr}\left(-c \, \partial_i \overline{\nabla} \, c + \partial_i c \overline{\nabla} c\right) - \partial_i \, \mathrm{Tr}\,(A \, c^2) =$$
$$= \overline{\nabla} \, \mathrm{Tr}\,(c \, \partial_i c) + \mathrm{Tr}\, c \, [\overline{\nabla}, \partial_i]\, c - \partial_i \,(\mathrm{Tr}\, A \, c^2) =$$
$$= \hat{\overline{\nabla}} \, \mathrm{Tr}\,(c \, \partial_i c) - \partial_i \, \mathrm{Tr}(A \, c^2)\,, \tag{7.3.12}$$

where we made use of the relation

$$\left[\partial_i, \overline{\nabla}_{\bar{\imath}}\right] = -\partial_i \, \mu^j_{\bar{\imath}} \, \partial_j \,. \tag{7.3.13}$$

The lack of nilpotency of (7.3.10) is due to the existence of new local symmetries of the action (7.3.11)

$$B \to B' = B + \mathcal{F}^i \, f_i + \overline{\nabla} \, d\,, \qquad C_i \to C'_i = C_i - \hat{\overline{\nabla}} \, f_i + \partial_i \, d\,, \tag{7.3.14}$$

with parameters $d \equiv d_{\bar{\imath}} \, dx^{\bar{\imath}}$ and f_i which are, respectively, a (0,1)-form and a section of the holomorphic cotangent. The transformations (7.3.14) are symmetries of the action (7.3.11) since they leave invariant the combination

$$\overline{\nabla} \, B + \mathcal{F}^i \, C_i \to \overline{\nabla} \, B + \mathcal{F}^i \, C_i + \left(\overline{\nabla}(\mathcal{F}^i \, f_i) - \mathcal{F}^i \, \hat{\overline{\nabla}} \, f_i\right) + \left(\overline{\nabla}^2 \, d + \mathcal{F}^i \, \partial_i \, d\right) =$$
$$= \overline{\nabla} \, B + \mathcal{F}^i \, C_i\,. \tag{7.3.15}$$

In the equation above we made use of (7.3.5) and of the Bianchi identity for $\mathcal{F}^i$:

$$0 = \epsilon^{\bar{\imath}\bar{\jmath}\bar{k}} \left[\overline{\nabla}_{\bar{\imath}}, \left[\overline{\nabla}_{\bar{\jmath}}, \overline{\nabla}_{\bar{k}}\right]\right] = -\epsilon^{\bar{\imath}\bar{\jmath}\bar{k}} \, \overline{\nabla}_{\bar{\imath}} \left(\mathcal{F}^i_{\bar{\jmath}\bar{k}} \, \partial_i\right) + \epsilon^{\bar{\imath}\bar{\jmath}\bar{k}} \, \mathcal{F}^i_{\bar{\jmath}\bar{k}} \, \partial_i \left(\overline{\nabla}_{\bar{\imath}}\right) =$$
$$= -\epsilon^{\bar{\imath}\bar{\jmath}\bar{k}} \, \overline{\nabla}_{\bar{\imath}} \mathcal{F}^i_{\bar{\jmath}\bar{k}} \, \partial_i + \epsilon^{\bar{\imath}\bar{\jmath}\bar{k}} \, \mathcal{F}^i_{\bar{\jmath}\bar{k}} \, [\partial_i, \overline{\nabla}_{\bar{\imath}}] = -\epsilon^{\bar{\imath}\bar{\jmath}\bar{k}} \left[\overline{\nabla}_{\bar{\imath}} \mathcal{F}^i_{\bar{\jmath}\bar{k}} + \mathcal{F}^j_{\bar{\jmath}\bar{k}} \, \partial_j \, \mu^i_{\bar{\imath}}\right] \partial_i\,, \tag{7.3.16}$$

which can equivalently be written as

$$\hat{\overline{\nabla}} \, \mathcal{F}^i = 0\,. \tag{7.3.17}$$

Henceforth the BRST transformations

$$s \, A = -\overline{\nabla} \, c - [A, c]_+\,,$$
$$s \, c = -c^2\,,$$
$$s \, B = -\mathrm{Tr}\,(c \, \overline{\nabla} \, A) - \mathcal{F}^i \, f_i - \overline{\nabla} \, d\,,$$
$$s \, C_i = \mathrm{Tr}\,(-c \, \partial_i \, A + \partial_i \, c \, A) - \hat{\overline{\nabla}} \, f_i + \partial_i \, d\,,$$
$$s \, d = \mathrm{Tr}\,(A \, c^2)\,,$$
$$s \, f_i = -\mathrm{Tr}\,(c \, \partial_i c)\,, \tag{7.3.18}$$

where f_i and d are *anti-commuting* fields with ghost number $+1$, are nilpotent when acting on A, c, B and C_i. The transformations (7.3.18) are however *not* nilpotent when acting on d and f_i:

$$s^2 d = -\frac{1}{3} \overline{\nabla} \operatorname{Tr} c^3 ,$$

$$s^2 f_i = -\frac{1}{3} \partial_i \operatorname{Tr} c^3 . \tag{7.3.19}$$

The reason why the BRST rules (7.3.18) are not nilpotent on d and f_i can be traced back to the fact that the replacements

$$d \to d' = d + \overline{\nabla} e , \qquad f_i \to f'_i = f_i + \partial_i e \tag{7.3.20}$$

leave unchanged the transformations of B and C_i in (7.3.18). Therefore, by introducing a scalar *commuting* ghost-for-ghost field e of ghost number $+2$, we obtain at last the fully nilpotent BRST transformations of the action (7.3.11)

$$s A = -\overline{\nabla} c - [A, c]_+ ,$$

$$s c = -c^2 ,$$

$$s B = -\operatorname{Tr} (c \overline{\nabla} A) - \mathcal{F}^i f_i - \overline{\nabla} d ,$$

$$s C_i = \operatorname{Tr} (-c \partial_i A + \partial_i c A) - \hat{\overline{\nabla}} f_i + \partial_i d ,$$

$$s d = \operatorname{Tr} (A c^2) - \overline{\nabla} e ,$$

$$s f_i = -\operatorname{Tr} (c \partial_i c) + \partial_i e ,$$

$$s e = \frac{1}{3} \operatorname{Tr} c^3 . \tag{7.3.21}$$

The structure of these BRST transformations is possibly made more transparent by the remark that the c-dependent terms in the BRST variations of B, d and e are precisely the forms which appear in the BRST descent equations that are generated by the holomorphic Chern-Simons (0,3)-form and the on-shell $\overline{\nabla}$:

$$s \, \Gamma^{(0,3)} = -\overline{\nabla} \, \Gamma^{(0,2)} , \quad s \, \Gamma^{(0,2)} = -\overline{\nabla} \, \Gamma^{(0,1)} ,$$

$$s \, \Gamma^{(0,1)} = -\overline{\nabla} \, \Gamma^{(0,0)} , \quad s \, \Gamma^{(0,0)} = 0 \quad \text{if } \overline{\nabla}^2 = 0 , \tag{7.3.22}$$

where

$$\Gamma^{(0,3)} = \operatorname{Tr} \left(A \overline{\nabla} A + \frac{2}{3} A^3 \right) ,$$

$$\Gamma^{(0,2)} = \operatorname{Tr} (c \overline{\nabla} A) ,$$

$$\Gamma^{(0,1)} = -\operatorname{Tr} (A c^2) ,$$

$$\Gamma^{(0,0)} = -\frac{1}{3} \operatorname{Tr} c^3 . \tag{7.3.23}$$

Therefore, when $\mathcal{F}^i = 0$, the cocycle

$$
\begin{aligned}
\tilde{\Gamma}^{(0,3)} &= \Gamma^{(0,3)} + \overline{\nabla} B \,, \\
\tilde{\Gamma}^{(0,2)} &= \Gamma^{(0,2)} + s\, B + \overline{\nabla} d = 0 \,, \\
\tilde{\Gamma}^{(0,1)} &= \Gamma^{(0,1)} + s\, d + \overline{\nabla} e = 0 \,, \\
\tilde{\Gamma}^{(0,0)} &= \Gamma^{(0,0)} + s\, e = 0
\end{aligned}
\tag{7.3.24}
$$

is a solution of the descent equations (7.3.22) which is BRST equivalent to the Chern-Simons cocycle (7.3.23) and whose $(0,3)$ component is precisely the form which appears in the off-shell action (7.3.11).

Summarizing, the (0,3)-form which appears in the off-shell Chern-Simons action is the representative of the solution of the cohomology problem (7.3.22) which is characterized by the vanishing of the components of lower form-degree: its top-form component is, when $\overline{\nabla}^2 = 0$, s-invariant — not just s-invariant modulo $\overline{\nabla}$. The terms in (7.3.21) involving f_i and C_i are necessary to make $\tilde{\Gamma}^{(0,3)} + \mathcal{F}^i\, C_i$ s-invariant even when $\overline{\nabla}^2 \neq 0$.

The action (7.3.11) contains only covariant anti-holomorphic derivatives and therefore is manifestly invariant under chiral diffeomorphisms of both fields and backgrounds

$$
\begin{aligned}
s_{\text{diff}} A &= \xi^i\, \partial_i\, A \,, & s_{\text{diff}} c &= \xi^i\, \partial_i\, c \,, \\
s_{\text{diff}} B &= \xi^i\, \partial_i\, B \,, & s_{\text{diff}} C_i &= \xi^j\, \partial_j\, C_i + \partial_i\, \xi^j\, C_j \,, \\
s_{\text{diff}} d &= \xi^i \partial_i\, d \,, & s_{\text{diff}} f &= \xi^i \partial_i\, f \,, & s_{\text{diff}} e &= \xi^i \partial_i\, e \,, \\
s_{\text{diff}} \mu^i &= -\hat{\overline{\nabla}}\, \xi^i \,, & s_{\text{diff}} \xi^i &= \xi^j\, \partial_j\, \xi^i \,, & s_{\text{diff}} \Omega &= \partial\, i_\xi(\Omega) \,.
\end{aligned}
\tag{7.3.25}
$$

Moreover the gauge BRST transformations (7.3.21) are also manifestly covariant, since they are expressed in terms of anti-holomorphic covariant derivatives and holomorphic derivative of chiral reparametrizations scalars. Therefore the off-shell action (7.3.11) is invariant under the nilpotent total BRST operator s_{tot}

$$
s_{\text{tot}} \equiv s_{\text{diff}} + s \,,
\tag{7.3.26}
$$

which encodes both the $SU(N)$ gauge symmetry and the global Ω-preserving holomorphic reparametrization symmetry of the original action (7.1.1).

7.4 Anti-fields and the chiral N=2 structure of the BRST transformations

It is known [92] that the structure of the BRST symmetry of 3-dimensional (real) CS theory becomes considerably more transparent when one considers, together with the

gauge connection A and the ghost field c, also their corresponding anti-fields A^* and c^*, which are, respectively, a 2-form of ghost number -1 and a 3-form of ghost number -2. All these fields can be collected in one single superfield, a polyform:

$$\mathcal{A} = c + A + A^* + c^*, \tag{7.4.1}$$

whose total grassmannian degree $f = n_{\text{ghost}} + n_{\text{form}}$, the sum of ghost number n_{ghost} and form degree n_{form}, is $f = +1$. The BRST transformations of both fields and anti-fields of the 3-dimensional CS theory write nicely in terms of $\mathcal{A}$ as follows

$$(s + d)\,\mathcal{A} + \mathcal{A}^2 = 0. \tag{7.4.2}$$

In this Section we will see that a similar strategy of collecting fields in polyforms of given Grassmann parity f also elucidates the geometrical content of the BRST transformations of the HCS theory coupled to off-shell gravitational backgrounds. For HCS theory, of course, the grassmann parity f is given by the sum of the ghost number and of the *anti-holomorphic* form degree.

Let us first write down the BRST transformations of the anti-fields of the dynamical fields.[5] The anti-field of a (0,1)-form $A = A_{\bar{\imath}}\,dx^{\bar{\imath}}$ is naturally a (3,2)-form, A^*, whose BRST variation is

$$s\,A^* = -\Omega\,\overline{\nabla}\,A - \frac{1}{2}\hat{\overline{\nabla}}\Omega\,A + \cdots, \tag{7.4.3}$$

where the dots denote the contribution from fields other than A. In order to obtain an anti-field which sits in the same superfield (7.4.1) as c and A, it is convenient to introduce the holomorphic density ρ

$$\Omega = \rho\,\epsilon_{ijk}\,dz^i\,dz^j\,dz^k \tag{7.4.4}$$

and to pull out a factor of ρ from the definition of the anti-field A^*:

$$A^* \to \rho\,A^* \qquad c^* \to \rho\,c^*. \tag{7.4.5}$$

The redefined A^* becomes a (0,2)-form and (7.4.3) gets replaced by

$$s\,A^* = -\overline{\nabla}\,A - \frac{1}{2}\frac{\hat{\overline{\nabla}}\,\rho}{\rho}\,A + \cdots. \tag{7.4.6}$$

We will use the notation

$$\frac{\hat{\overline{\nabla}}\,\rho}{\rho} = \frac{\overline{\nabla}\,\rho - \partial_i\mu^i\,\rho}{\rho} \equiv \hat{\overline{\nabla}}\log\rho. \tag{7.4.7}$$

---

[5]For a condensed introduction to anti-fields and the Batalin-Vilkovisky (BV) formalism see [93].

Redefining both A^* and c^* in this way, we obtain for their BRST transformations the expressions

$$
\begin{aligned}
s\,A^* = {}& -\overline{\nabla}\,A - A^2 - [c, A^*]_+ + 2\,C^{*\,i}\,\partial_i\,c + \\
& -\frac{1}{2}\left(\overline{\nabla}\log\rho\right)A - B^*\,\overline{\nabla}\,c - \left(\overline{\nabla}\,B^* + \left(\overline{\nabla}\log\rho\right)B^*\right)c + \\
& +\left(\partial_i\,C^{*\,i} + \left(\partial_i\log\rho\right)C^{*\,i}\right)c + c^2\,d^*\,, \\
s\,c^* = {}& -[c, c^*]_+ - \overline{\nabla}\,A^* - [A, A^*]_+ + 2\,C^{*\,i}\,\partial_i\,A + 2\,f^{*\,i}\,\partial_i\,c + \\
& -\left(\overline{\nabla}\log\rho\right)A^* - B^*\,\overline{\nabla}A + \left(\partial_i\,C^{*\,i} + \left(\partial_i\log\rho\right)C^{*\,i}\right)A + \\
& +\left(\partial_i\,f^{*\,i} + \left(\partial_i\log\rho\right)f^{*\,i}\right)c + [A, c]_+\,d^* + c^2\,e^*\,.
\end{aligned}
\tag{7.4.8}
$$

The Lagrange multipliers B, d, e sit in a single superfield with $f = 2$. Therefore the corresponding anti-fields B^*, d^*, e^* are holomorphic densities with $f = 0$. The multipliers C_i and f_i have $f = +1$, and thus $C^{*\,i}$ and $f^{*\,i}$ are holomorphic densities with $f = +1$. We will find it convenient to redefine $C^{*\,i}$ and $f^{*\,i}$ by pulling out a factor of ρ, as we did with A^* and c^*,

$$
C^{*\,i} \to \rho\,C^{*\,i}\,, \qquad f^{*\,i} \to \rho\,f^{*\,i}\,,
\tag{7.4.9}
$$

so that the new anti-fields $C^{*\,i}$ and $f^{*\,i}$ are forms of anti-holomorphic degree 2 and 3 respectively. We obtain therefore for the BRST transformation laws of the anti-fields of the Lagrange multipliers:

$$
\begin{aligned}
s\,B^* &= -\frac{1}{2}\,\hat{\overline{\nabla}}\rho\,, \\
s\,d^* &= \partial_i\left(\rho\,C^{*\,i}\right) - \hat{\overline{\nabla}}\,B^*\,, \\
s\,e^* &= \partial_i\left(\rho\,f^{*\,i}\right) - \overline{\nabla}\,d^*\,, \\
s\,C^{*\,i} &= -\frac{1}{2}\,\mathcal{F}^i\,, \\
s\,f^{*\,i} &= -\hat{\overline{\nabla}}\,C^{*\,i} - \frac{B^*\,\mathcal{F}^i + \left(\hat{\overline{\nabla}}\,\rho\right)C^{*\,i}}{\rho}\,.
\end{aligned}
\tag{7.4.10}
$$

The BRST transformation laws (7.4.10) make clear that the three anti-fields B^*, d^*, e^* can be put together with the holomorphic density ρ to form a "complete" BRST multiplet

$$
\mathcal{B}^* \equiv \rho + 2\,B^* + 2\,d^* + 2\,e^*
\tag{7.4.11}
$$

containing components of all degrees $n_{\text{form}} = 0, 1, 2, 3$ which transform as follows:

$$
\begin{aligned}
s\,\rho &= 0\,, \\
s\,(2\,B^*) &= -\hat{\overline{\nabla}}\rho\,, \\
s\,(2\,d^*) &= -\hat{\overline{\nabla}}\,(2\,B^*) + \partial_i\left(2\,\rho\,C^{*\,i}\right), \\
s\,(2\,e^*) &= -\overline{\nabla}\,(2\,d^*) + \partial_i\left(2\,\rho\,f^{*\,i}\right).
\end{aligned}
\tag{7.4.12}
$$

To form a complete multiplet out of $C^{*\,i}$ and $f^{*\,i}$ we need a (0,0)-form of ghost number 1 and a (0,1)-form of ghost number 0 with values in the holomorphic tangent: The natural candidates are ξ^i, the chiral reparametrizations ghost, and μ^i, the Beltrami differentials. This motivates considering the total BRST operator

$$s_{\text{tot}} = s_{\text{diff}} + s\,, \qquad (7.4.13)$$

which encodes both the chiral reparametrizations invariance and the gauge symmetry of HCS theory. Indeed, one can check that by defining

$$\mathbb{M}^i \equiv \xi^i + \mu^i + 2\,C^{*i} + 2\left(f^{*i} - \frac{2}{\rho}\,B^*\,C^{*i}\right) \equiv \xi^i + \mu^i + 2\,C^{*i} + 2\,f_n^{*i}\,, \qquad (7.4.14)$$

the transformation rules for $C^{*\,i}$ and $f^{*\,i}$ in (7.4.10) assume the form

$$s_{\text{tot}}\,\mathbb{M}^i = -\left(\bar{\partial}\,\mathbb{M}^i - \mathbb{M}^j\,\partial_j\,\mathbb{M}^i\right)\,. \qquad (7.4.15)$$

This equation also reproduces the correct BRST transformations for ξ^i and μ^i. From the same equation it also follows that the anti-holomorphic derivative acting on super-fields $\Phi^{i\,\cdots}_{\bar{i}\bar{j}\dots;k\dots}$

$$\hat{\nabla}_{\bar{k}}(\mathbb{M})\,\Phi^{i\,\cdots}_{\bar{i}\bar{j}\dots;k\dots} \equiv$$
$$\equiv \bar{\partial}_{\bar{k}}\,\Phi^{i\,\cdots}_{\bar{i}\bar{j}\dots;k\dots} - \mathbb{M}^j\,\partial_j\,\Phi^{i\,\cdots}_{\bar{i}\bar{j}\dots;k\dots} + \partial_j\,\mathbb{M}^i_{\bar{k}}\,\Phi^{j\,\cdots}_{\bar{i}\bar{j}\dots;k\dots} - \partial_k\,\mathbb{M}^j_{\bar{k}}\,\Phi^{i\,\cdots}_{\bar{i}\bar{j}\dots;j\dots} + \cdots \qquad (7.4.16)$$

is covariant under the transformations (7.4.15). Moreover the covariant differential

$$\hat{\bar{\nabla}}(\mathbb{M}) \equiv dx^{\bar{k}}\,\hat{\nabla}_{\bar{k}}(\mathbb{M}) \qquad (7.4.17)$$

satisfies

$$\{s_{\text{tot}}, \hat{\bar{\nabla}}(\mathbb{M})\} + \hat{\bar{\nabla}}(\mathbb{M})^2 = 0\,. \qquad (7.4.18)$$

This means that the operator

$$\delta \equiv s_{\text{tot}} + \hat{\bar{\nabla}}(\mathbb{M}) \qquad (7.4.19)$$

is nilpotent:

$$\delta^2 = 0\,. \qquad (7.4.20)$$

It is easily seen that the transformations (7.4.12) rewrite in terms of this super-covariant anti-holomorphic derivative as

$$s_{\text{tot}}\,\mathcal{B}^* = -\hat{\bar{\nabla}}(\mathbb{M})\,\mathcal{B}^*\,. \qquad (7.4.21)$$

The introduction of the flat super-Beltrami $\mathbb{M}^i$ allows one to recast the BRST transformations of the gauge supermultiplet c, A, A^*, c^* in a form which is analogous to the transformations (7.4.3) of the three-dimensional theory. Defining the modified anti-fields

$$A_n^* = A^* - \frac{B^*}{\rho}\, A - \frac{d^*}{\rho}\, c\,, \quad c_n^* = c^* - 2\,\frac{B^*}{\rho}\, A_n^* - \frac{d^*}{\rho}\, A - \frac{e^*}{\rho}\, c \tag{7.4.22}$$

and the superfield

$$\mathcal{A} \equiv c + A + A_n^* + c_n^*\,, \tag{7.4.23}$$

the transformations of the gauge multiplet in (7.3.21) and (7.4.8) write as

$$s_{\text{tot}}\, \mathcal{A} = -\overline{\nabla}(\mathbb{M})\, \mathcal{A} - \mathcal{A}^2\,. \tag{7.4.24}$$

Let us turn to the BRST transformations of the Lagrange multipliers. To form a complete BRST multiplet $\mathcal{B}$ out of B, d, e we need to introduce the anti-field ρ^*, with ghost number -1 and anti-holomorphic form degree 3, corresponding to the background ρ.

Let us comment on the significance of BRST transformations of the backgrounds and of their anti-fields. Backgrounds (or coupling constants) can appear both in the classical action and in the gauge-fixing term. Backgrounds which appear only in the gauge-fixing term are of course *unphysical*. It is convenient in various contexts to extend the action of the BRST operator on the unphysical backgrounds by introducing corresponding fermionic super-partners to form trivial BRST doublets (see [94] and references therein). The BRST variation of physical backgrounds (or coupling constants) must instead be put to zero since varying a physical coupling constant is, by definition, not a symmetry. Indeed in HCS theory the gauge BRST transformations of the (physical) backgrounds ρ and μ^i vanish, as indicated in (7.4.12) and (7.4.21). However in the BV formalism it is natural to consider also the anti-fields corresponding to physical backgrounds. Anti-fields of backgrounds do not appear in the BV action since the BRST variation of the physical backgrounds vanish. Their BRST variations are naturally defined in the BV formalism by the derivatives of the BV action with respect to the backgrounds. For HCS theory the BRST variations of the anti-fields of ρ and μ^i can be defined to be

$$s\,\rho^* = -\frac{\partial \Gamma_{BV}}{\partial \rho} = -\text{Tr}\,\Big(\frac{1}{2}\,A\,\overline{\nabla}\,A + \frac{1}{3}\,A^3\Big) - \frac{1}{2}\,\overline{\nabla}\,B - \frac{1}{2}\,\mathcal{F}^i\,C_i\,,$$

$$\frac{1}{\rho}\,s\,\mu_i^* = -\frac{1}{\rho}\,\frac{\partial \Gamma_{BV}}{\partial \mu^i} = \frac{1}{2}\,\Big(-\text{Tr}\,(A\,\partial_i\,A) + \partial_i\,B - \hat{\overline{\nabla}}\,C_i - (\hat{\overline{\nabla}}\log\rho)\,C_i\Big) +$$

$$-\text{Tr}\,(A^*\,\partial_i\,c) - \frac{B^*}{\rho}\,\big(\text{Tr}\,(c\,\partial_i\,A) - \partial_i\,d + \hat{\overline{\nabla}} f_i\big) - \overline{\nabla}\big(\frac{B^*}{\rho}\big)\,f_i +$$

$$-C^{*j}\,\partial_{[i}\,f_{j]} + \big(\partial_j\,C^{*\,j} + (\partial_j\log\rho)\,C^{*\,j}\big)\,f_i + \frac{d^*}{\rho}\,\partial_i\,e\,, \tag{7.4.25}$$

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where Γ_{BV} is the BV action.[6]

The content of the relation (7.4.25) is that the variations of the action with respect to the physical backgrounds are BRST-closed: since the BRST transformations do depend on the backgrounds this is not self-evident but it is ensured by the general BV formula. In the enlarged field space which includes anti-fields of backgrounds such variations are BRST-trivial.

The superfield which collects together B, d, e and ρ^* and has nice BRST transformation laws turns out to be

$$\mathcal{B} = e + d + B_n + 2\,\rho_n^*\,, \tag{7.4.26}$$

where

$$B_n \equiv B - 2\,C^{*i}\,f_i - \text{Tr}\,(A_n^*\,c)\,,$$
$$2\,\rho_n^* \equiv 2\,\rho^* - 2\,C^{*i}\,C_i - 2\,f_n^{*i}\,f_i - \text{Tr}\,(A_n^*\,A + c_n^*\,c)\,. \tag{7.4.27}$$

One can check that the BRST transformation laws for B, d, e rewrite in terms of $\mathcal{B}$ as follows

$$s_{\text{tot}}\,\mathcal{B} = -\overline{\nabla}(\mathbb{M})\,\mathcal{B} + \frac{1}{3}\,\text{Tr}\,\mathcal{A}^3\,. \tag{7.4.28}$$

The Lagrange multipliers C_i and f_i sit in a superfield which contains also a 2-form of ghost number -1 and a 3-form of ghost number -2 with values in the holomorphic cotangent. Looking at (7.4.14) one sees that these should be identified with the anti-fields μ_i^* and ξ_i^* of the backgrounds μ^i and ξ^i. Since $\mathbb{M}^i$ is valued in the holomorphic tangent, $\mathbb{M}_i^*$ is naturally a holomorphic density. Choosing its components to be

$$\mathbb{M}_i^* = \rho\,f_i + (\rho\,C_i + 2\,B^*\,f_i) + 2\,\mu_i^* + 2\,\xi_i^*\,, \tag{7.4.29}$$

its BRST transformation writes

$$s_{\text{tot}}\,\mathbb{M}_i^* = -\hat{\overline{\nabla}}(\mathbb{M})\,\mathbb{M}_i^* + \mathcal{B}^*\,\partial_i\,\mathcal{B} - \mathcal{B}^*\,\text{Tr}\,\mathcal{A}\,\partial_i\,\mathcal{A}\,. \tag{7.4.30}$$

The BRST transformations of all fields and backgrounds and their anti-fields write in a nice compact form in terms of the coboundary operator δ:

$$\delta\,\mathbb{M}^i + \mathbb{M}^j\,\partial_j\,\mathbb{M}^i = 0\,,$$
$$\delta\mathcal{A} + \mathcal{A}^2 = 0\,,$$
$$\delta\mathcal{B} = \frac{1}{3}\,\text{Tr}\,\mathcal{A}^3\,,$$
$$\delta\,\mathbb{M}_i^* = \mathcal{B}^*\,\partial_i\,\mathcal{B} - \mathcal{B}^*\,\text{Tr}\,\mathcal{A}\,\partial_i\,\mathcal{A}\,,$$
$$\delta\,\mathcal{B}^* = 0\,. \tag{7.4.31}$$

---

[6]In Eq. (7.4.25) we defined the functional derivative of Γ_{BV} with respect to ρ by keeping constant the *true* anti-fields A^*, c^*, C^{*i} and f^{*i}, and not the redefined ones in (7.4.5),(7.4.9).

Let us comment on the geometrical interpretation of the BRST transformations (7.4.31). The first of (7.4.31) tells us that the super-Beltrami field $\mathbb{M}^i$ has flat Kodaira-Spencer curvature with respect to the differential δ. The second equation expresses the flatness of the gauge super-connection $\mathcal{A}$. Since $\mathcal{A}$ is flat, the Chern-Simons polyform

$$\Gamma_{CS} = \mathrm{Tr}\big(\mathcal{A}\,\delta\,\mathcal{A} + \frac{2}{3}\,\mathcal{A}^3\big) = -\frac{1}{3}\,\mathrm{Tr}\,\mathcal{A}^3 \tag{7.4.32}$$

is a δ-cocycle. The third equation in (7.4.31) says that such cocycle is δ-exact, being the δ-variation of $\mathcal{B}$. Taking the ∂_i derivative of this equation one obtains

$$\delta\,\partial_i\,\mathcal{B} = \mathrm{Tr}\,\partial_i\,\mathcal{A}\,\mathcal{A}^2 = \delta\,\mathrm{Tr}\,\mathcal{A}\,\partial_i\,\mathcal{A}\,. \tag{7.4.33}$$

This means that $\Omega_i \equiv \partial_i\,\mathcal{B} - \mathrm{Tr}\,\mathcal{A}\,\partial_i\,\mathcal{A}$ is a δ-cocycle

$$\delta\,\big(\partial_i\,\mathcal{B} - \mathrm{Tr}\,\mathcal{A}\,\partial_i\,\mathcal{A}\big) = \delta\,\Omega_i = 0\,. \tag{7.4.34}$$

The fourth equation in (7.4.31)

$$\mathcal{B}^*\,\Omega_i = \delta\,\mathbb{M}^*_i \tag{7.4.35}$$

implies therefore

$$\delta\,\mathcal{B}^*\,\Omega_i = 0\,. \tag{7.4.36}$$

This is consistent with the fifth equation in (7.4.31) and implies that Ω_i is also δ-trivial

$$\Omega_i = \delta\,\mathcal{C}_i\,, \qquad \mathbb{M}^*_i \equiv \mathcal{B}^*\,\mathcal{C}_i\,. \tag{7.4.37}$$

7.4.1 The action

Not only the BRST transformations but also the action rewrites in a neat form in terms of superfields. The BV action corresponding to the gauge invariant action (7.3.11) is

$$2\,\Gamma_{BV} = \rho\,\mathrm{Tr}\,\big(A\,\overline{\nabla}\,A + \frac{2}{3}\,A^3\big) + \rho\,\overline{\nabla}\,B + \rho\,\mathcal{F}^i\,C_i - 2\,\rho\,A^*\,s\,A - 2\,\rho\,c^*\,s\,c + $$
$$-2\,B^*\,s\,B - 2\,d^*\,s\,d - 2\,e^*\,s\,e - 2\,\rho\,C^{*i}\,s\,C_i - 2\,\rho\,f^{*i}\,s\,f_i\,, \tag{7.4.38}$$

where we chose to think of Γ_{BV} as a $(0,3)$-form with values in the holomorphic densities rather than a $(3,3)$-form as in Eq. (7.3.11).

We have seen that when working with the superfields it is natural to promote the gauge BRST operator to the total s_{tot} which includes the chiral diffeomorphisms, by introducing the chiral reparametrization ghost ξ^i which should be thought of as a background, in the same way as ρ and μ^i. The corresponding BV action has extra terms with respect to the gauge BV action (7.4.38) which are proportional to the background ξ^i. It is this extended

action which writes most simply in terms of superfields. Of course one can always recover the gauge action (7.4.38) by putting ξ^i to zero.

A direct computation shows that (the extended) Γ_{BV} is the $(0,3)$-component of the following polyform with values in the holomorphic densities

$$
\begin{aligned}
2\,\Gamma_{BV} &= -\mathcal{B}^* \, s_{\mathrm{tot}}\, \mathcal{B} - \mathbb{M}_i^* \, s_{\mathrm{tot}}\, \mathbb{M}^i - \mathcal{B}^* \, \mathrm{Tr}\left(\mathcal{A}\, s_{\mathrm{tot}}\, \mathcal{A}\right) = \\
&= \mathcal{B}^* \, \mathrm{Tr}\left(\mathcal{A}\, \hat{\bar{\nabla}}(\mathbb{M})\, \mathcal{A} + \frac{2}{3}\, \mathcal{A}^3\right) + \mathcal{B}^* \, \hat{\bar{\nabla}}(\mathbb{M})\, \mathcal{B} + \mathbb{M}_i^*\left(\bar{\partial}\,\mathbb{M}^i - \mathbb{M}^j\,\partial_j\,\mathbb{M}^i\right).
\end{aligned} \tag{7.4.39}
$$

We see therefore that, in much the same way as it happens for 3d CS theory [92], the BV action is obtained from the classical action (7.3.11) by replacing every field and background with the superfield to which it belongs

$$
\begin{aligned}
A \to \mathcal{A}\,, \qquad B \to \mathcal{B}\,, \qquad \rho\, C_i \to \mathbb{M}_i^*\,, \\
\mu^i \to \mathbb{M}^i\,, \qquad \rho \to \mathcal{B}^*\,.
\end{aligned} \tag{7.4.40}
$$

7.5 Anti-holomorphic dependence of physical correlators

The stress-energy tensor of a topological quantum field theory is a BRST anti-commutator

$$
T_{\mu\nu} = \{s, G_{\mu\nu}\}\,, \tag{7.5.1}
$$

where $G_{\mu\nu}$ is the supercurrent. If both $T_{\mu\nu}$ and $G_{\mu\nu}$ are conserved one obtains a corresponding relation for the charges

$$
P_\mu = \{s, G_\mu\}\,, \tag{7.5.2}
$$

where P_μ is the generator of translations and G_μ is a vector supersymmetry. Since P_μ is implemented on local fields by space-time derivatives

$$
\partial_\mu = \{s, G_\mu\}\,, \tag{7.5.3}
$$

the relation (7.5.2) proves that correlators of local observables of topological field theories are space-time independent.

HCS theory is, in a sense, semi-topological: it does not depend on the full space-time metric but only on the Beltrami differential μ^i. Consequently we expect that a holomorphic version of the relation (7.5.3) holds for HCS:

$$
\hat{\nabla}_{\bar{\imath}} = \{s, G_{\bar{\imath}}\}\,. \tag{7.5.4}
$$

In this section we want to explore the validity of such a relation. We will find that a suitable $G_{\bar{\imath}}$ does indeed exist if we enlarge the functional space upon which $G_{\bar{\imath}}$ acts to include the anti-fields of both the dynamical fields and the backgrounds μ^i and Ω.

It is convenient to introduce a field $\gamma^{\bar{\imath}}(\bar{z})$, which depends only on the anti-holomorphic coordinates $z^{\bar{\imath}}$ and define the scalar operator

$$G_{\bar{\gamma}} = \gamma^{\bar{\imath}} G_{\bar{\imath}} , \qquad (7.5.5)$$

which carries ghost number -1. It turns out that a suitable $G_{\bar{\gamma}}$ which satisfies (7.5.4) is defined by the following simple action on the superfields that we introduced in Section 7.4

$$G_{\bar{\gamma}} \mathcal{A} = i_{\bar{\gamma}}(\mathcal{A}) , \qquad G_{\bar{\gamma}} \mathcal{B} = i_{\bar{\gamma}}(\mathcal{B}) , \qquad G_{\bar{\gamma}} \mathcal{B}^* = i_{\bar{\gamma}}(\mathcal{B}^*) ,$$
$$G_{\bar{\gamma}} \mathbb{M}^i = i_{\bar{\gamma}}(\mathbb{M}^i) , \qquad G_{\bar{\gamma}} \mathbb{M}^*_i = i_{\bar{\gamma}}(\mathbb{M}^*_i) , \qquad (7.5.6)$$

where $i_{\bar{\gamma}}$ is the contraction of a form with the antiholomorphic vector field $\gamma^{\bar{\imath}} \partial_{\bar{\imath}}$. $G_{\bar{\gamma}}$ so defined is easily seen to satisfy the relation

$$\{s_{\text{tot}}, G_{\bar{\gamma}}\} = \{i_{\bar{\gamma}}, \bar{\partial}\} , \qquad (7.5.7)$$

where s_{tot} is the BRST operator which includes both gauge transformations and chiral diffeomorphisms:

$$s_{\text{tot}} = s_{\text{diff}} + s . \qquad (7.5.8)$$

Note that the gauge BRST operator s acts trivially on the gravitational backgrounds (μ^i, ρ, ξ^i). Let us show that (7.5.7) implies (7.5.4) for the dynamical fields. Indeed, let Φ be a field which is neither μ^i nor ξ^i. We have

$$G_{\bar{\gamma}} s_{\text{diff}} (\Phi) = G_{\bar{\gamma}}(\mathcal{L}_\xi \Phi) = \mathcal{L}_{i_{\bar{\gamma}}(\mu)} \Phi - \mathcal{L}_\xi G_{\bar{\gamma}}(\Phi) ,$$
$$s_{\text{diff}} G_{\bar{\gamma}} (\Phi) = \mathcal{L}_\xi G_{\bar{\gamma}} (\Phi) ,$$
$$\{s_{\text{diff}}, G_{\bar{\gamma}}\} = \mathcal{L}_{i_{\bar{\gamma}}(\mu)} \Phi , \qquad (7.5.9)$$

where $\mathcal{L}_\xi$ denotes the action of chiral diffeomorphisms with parameter ξ^i. Hence

$$\{s, G_{\bar{\gamma}}\} \Phi = \{i_{\bar{\gamma}}, \bar{\partial}\} \Phi - \{s_{\text{diff}}, G_{\bar{\gamma}}\} \Phi = \{i_{\bar{\gamma}}, \hat{\bar{\nabla}}\} \Phi , \qquad (7.5.10)$$

which is equivalent to (7.5.4). Note that on the backgrounds, we have instead

$$\{s, G_{\bar{\gamma}}\} \xi^i = 0 , \qquad \{s, G_{\bar{\gamma}}\} \mu^i = i_{\bar{\gamma}}(\mathcal{F}^i) . \qquad (7.5.11)$$

When writing down explicitly $G_{\bar{\gamma}}$ on the component fields one verifies that its action on the sector which does *not* include the Lagrange multipliers B and C_i does not involve the

antifields of μ_i^* and ρ^*:

$$G_{\bar{\gamma}}\, c = i_{\bar{\gamma}}(A)\,,$$

$$G_{\bar{\gamma}} A = i_{\bar{\gamma}}(A^*) - i_{\bar{\gamma}}\Big(\frac{B^*}{\rho} A\Big) - i_{\bar{\gamma}}\Big(\frac{d^*}{\rho}\Big)\, c\,,$$

$$G_{\bar{\gamma}} A^* = i_{\bar{\gamma}}(c^*) - 2\, i_{\bar{\gamma}}\Big(\frac{B^*}{\rho}\Big)\, A^* - \frac{B^*}{\rho}\, i_{\bar{\gamma}}(A^*) +$$

$$\quad + i_{\bar{\gamma}}\Big(\frac{B^*}{\rho}\Big)\, \frac{B^*}{\rho}\, A + \frac{B^*}{\rho}\, i_{\bar{\gamma}}\Big(\frac{d^*}{\rho}\Big)\, c + \Big(\frac{B^*}{\rho}\Big)^2 i_{\bar{\gamma}}(A)\,,$$

$$G_{\bar{\gamma}} c^* = -2\, i_{\bar{\gamma}}\Big(\frac{B^*}{\rho}\Big)\, c^* - \frac{d^*}{\rho}\, i_{\bar{\gamma}}(A^*) - i_{\bar{\gamma}}\Big(\frac{d^*}{\rho}\Big)\, \frac{B^*}{\rho}\, A\,,$$

$$G_{\bar{\gamma}}\,(\rho) = 2\, i_{\bar{\gamma}}(B^*)\,,$$

$$G_{\bar{\gamma}}\, B^* = i_{\bar{\gamma}}(d^*)\,,$$

$$G_{\bar{\gamma}}\, d^* = i_{\bar{\gamma}}(e^*)\,,$$

$$G_{\bar{\gamma}}\, e^* = 0\,,$$

$$G_{\bar{\gamma}}\, \mu^i = 2\, i_{\bar{\gamma}}(C^{*i})\,,$$

$$G_{\bar{\gamma}}\, C^{*i} = i_{\bar{\gamma}}(f^{*i}) - 2\, i_{\bar{\gamma}}\Big(\frac{B^*}{\rho}\Big)\, C^{*i} - 2\, \frac{B^*}{\rho}\, i_{\bar{\gamma}}(C^{*i})\,,$$

$$G_{\bar{\gamma}}\, f^{*i} = -2\, i_{\bar{\gamma}}\Big(\frac{B^*}{\rho}\Big)\, f^{*i} - 2\, \frac{d^*}{\rho}\, i_{\bar{\gamma}}(C^{*i})\,,$$

$$G_{\bar{\gamma}}\, e = i_{\bar{\gamma}}(d)\,,$$

$$G_{\bar{\gamma}}\, d = i_{\bar{\gamma}}(B) - 2\, i_{\bar{\gamma}}(C^{*i})\, f_i - \mathrm{Tr}\,\big(i_{\bar{\gamma}}(A^*)\, c\big) + i_{\bar{\gamma}}\Big(\frac{B^*}{\rho}\,\mathrm{Tr}\,(A\, c)\Big)\,,$$

$$G_{\bar{\gamma}}\, f_i = i_{\bar{\gamma}}(C_i)\,. \tag{7.5.12}$$

The action of $G_{\bar{\gamma}}$ on B and C_i involves instead the anti-fields μ_i^* and ρ^* whose BRST transformations we introduced in (7.4.25):

$$G_{\bar{\gamma}}\, B = 2\, i_{\bar{\gamma}}(\rho^*) - 2\, i_{\bar{\gamma}}(C^{*i})\, C_i - i_{\bar{\gamma}}\Big(\frac{d^*}{\rho}\Big)\, \mathrm{Tr}\,(A\, c) + \frac{B^*}{\rho}\, \mathrm{Tr}\,\big(A\, i_{\bar{\gamma}}(A)\big) +$$

$$\quad - \mathrm{Tr}\,\big(i_{\bar{\gamma}}(A^*)\, A\big)\,,$$

$$G_{\bar{\gamma}}\, C_i = 2\, i_{\bar{\gamma}}\Big(\frac{\mu_i^*}{\rho}\Big) - 2\, i_{\bar{\gamma}}\Big(\frac{B^*}{\rho}\, C_i\Big) - 2\, i_{\bar{\gamma}}\Big(\frac{d^*}{\rho}\Big)\, f_i\,. \tag{7.5.13}$$

The existence of $G_{\bar{\gamma}}$ therefore reflects the semi-topological character of the theory. Since the relation (7.5.3) valid for topological theories is replaced in HCS by (7.5.4), the correlators of physical local observables $O(z, \bar{z})$

$$F(z, \bar{z}) = \langle O(z, \bar{z}) \cdots \rangle \quad \text{with} \quad s\, O(z, \bar{z}) = 0\,, \tag{7.5.14}$$

where the dots denote insertions of physical observables at space-time points other than $(z, \bar{z})$, satisfy the identity

$$\hat{\nabla}_{\bar{\imath}}\, F(z, \bar{z}) = \big\langle s\, \big(G_{\bar{\imath}}\, O(z, \bar{z})\big) \cdots \big\rangle\,. \tag{7.5.15}$$

One cannot immediately conclude, from this Ward identity (and BRST invariance) that $F(z, \bar{z})$ is a holomorphic function (tensor) on X. This for two reasons.

First of all we have seen that $G_{\bar{\imath}}$ when acting on B and C_i produces the μ_i^* and ρ^*: since ρ and μ^i are not dynamical (one does not integrate over them) the Ward identity (7.5.15) says that the $\bar{z}$-dependence of physical correlators involving B and C_i can be expressed in terms of derivative of correlators with respect to the moduli ρ and μ^i.

Secondly, even if restricted to observables which do not involve the Lagrange multipliers B and C_i, the Ward identity (7.5.15) "almost" implies the holomorphicity of $F(z, \bar{z})$, but not quite. Indeed, the $G_{\bar{\gamma}}$ variations (7.5.12) of fields other than B and C_i contain the dynamical anti-fields, and the functional averages of the BRST variation of operators which depend on the anti-fields are, in general, zero only up to contact terms.

At any rate it is clear that the Ward identity (7.5.15) strongly constrains the anti-holomorphic dependence of physical correlators. This equation should therefore play for the Green functions of physical observables of HCS field theory the role that the holomorphic anomaly equation plays for the open-closed topological string amplitudes [95]. For example, it is conceivable that one could determine, to a large extent, the space-time dependence of physical correlators of HCS using the identity (7.5.15) together with assumptions about the behavior of correlators at infinity. An analogous approach to compute topological open and closed string amplitudes by integrating the holomorphic anomaly equation has been quite successful [26], [96].

It would be very interesting to understand the full *quantum* properties of HCS field theory. Here we will limit ourselves to few brief comments. First of all there is the issue of anomalies: the chiral diffeomorphism symmetry (7.3.25) can, in principle, suffer from anomalies, and, indeed, it does [97]. Chiral diffeomorphism invariance can be restored at the price of introducing a dependence on the anti-holomorphic Beltrami differentials and, possibly, on the Kähler metric. The chiral diffeomorphism invariant theory should display an anomalous Ward identity which controls the anti-holomorphic dependence on the backgrounds very much like (7.5.15) does for the space-time anti-holomorphic dependence.

But, of course, the real question which remains to be addressed is the ultraviolet completeness of the HCS quantum field theory. Being a 6-dimensional gauge theory, HCS theory is superficially not renormalizable. On the other hand its string interpretation suggests the opposite. We believe that the extended supersymmetry structure (7.5.4) capturing the semi-topological character of the theory and the identity (7.5.15) restricting the space-time dependence of quantum correlators should be instrumental in ensuring that the physical sector of the theory is indeed free of ultra-violet divergences.

Chapter 8

Coupling 2-dimensional YM to topological backgrounds

In this Chapter we will move to discuss 2-dimensional topological YM. Even if naïvely it does not possess any topological supersymmetry we will see that such a topological supersymmetry can be obtained, very similarly to what we just did for HCS, by suitably coupling the theory to topological backgrounds. Also in this case we will need to consider the BV formalism even if in a somewhat exotic formulation. We will see in the next Chapter that such a new approach to BV formalism will be useful also for 3-dimensional Chern-Simons theory.

8.1   Introduction to the theory

It is commonly said that 2-dimensional Yang-Mills theory is a topological theory. It is indeed well-known that gauge invariance in two dimensions is sufficient to remove all the propagating local degrees of freedom from the theory. However, this statement can be considered as too naïve in two respects. First of all, as pointed out in [98], the 2-dimensional YM action is not topological, i.e. it is not completely independent from the 2-dimensional metric. To convince oneself about this fact it is convenient to write the 2-dimensional action in a slightly unusual way [98] by introducing, beyond the gauge field $A = A^a T^a$, an additional adjoint scalar $\phi = \phi^a T^a$

$$\Gamma_{YM} = \int_\Sigma \text{Tr}\phi F + \frac{\epsilon}{2} \int_\Sigma d^2x \sqrt{g}\, \text{Tr}\phi^2 \; , \tag{8.1.1}$$

where ϵ is a constant proportional to the square of the standard YM coupling constant and Σ is a 2-dimensional Riemann surface provided with a metric g. The correspondence

with the standard YM action is then recovered by simply integrating out ϕ

$$\Gamma'_{YM} = -\frac{1}{2\epsilon} \int_\Sigma \mathrm{Tr} F^2 \ . \tag{8.1.2}$$

On the other hand the formulation (8.1.1) makes explicit the dependence of the theory from the 2-dimensional metric, via the volume form $d^2x\sqrt{g}$ which appears in the term

$$\frac{\epsilon}{2} \int_\Sigma d^2x \sqrt{g} \, \mathrm{Tr} \phi^2 \ . \tag{8.1.3}$$

The action (8.1.1) also shows that, at least classically, the dependence from the metric gets removed by considering the $\epsilon \to 0$ limit: in this way one obtains the topological action

$$\Gamma_{YM}|_{\epsilon=0} = \int_\Sigma \mathrm{Tr} \phi F \ , \tag{8.1.4}$$

which is manifestly invariant under 2-dimensional reparametrizations.

The second aspect that makes 2-dimensional YM a "bad" topological theory is that both the physical action (8.1.1) and the topological action (8.1.4), do *not* possess any topological supersymmetry: they are only invariant under gauge BRST transformations

$$\begin{aligned} s_{gauge} c &= -c^2 \ , \\ s_{gauge} A &= -Dc \ , \\ s_{gauge} \phi &= -[c, \phi] \ , \end{aligned} \tag{8.1.5}$$

where, as usual, $c = c^a T^a$ is the ghost field associated to the gauge invariance and the gauge covariant derivative is

$$Dc \equiv dc + [A, c]_+ \ , \tag{8.1.6}$$

whereas it is not present any topological supersymmetry like the one of four-dimensional topological YM

$$sA = \psi \ , \tag{8.1.7}$$

where ψ is the *gaugino* field connected with topological supersymmetry.

Usually the topological supersymmetry is introduced in the theory "by hand": in [99] it has been observed that, by adding a decoupled quadratic term in the gravitino field ψ

$$\int_\Sigma \mathrm{Tr} \, \psi \wedge \psi \tag{8.1.8}$$

to the topological action (8.1.4), the resulting theory becomes invariant under topological Yang-Mills BRST transformations, see (8.2.7).

In this Chapter we will discuss how the topological supersymmetry (8.2.7) and the supersymmetric term (8.1.8) can be obtained in a different, and perhaps more natural, way.

Let us briefly explain our strategy: contrary to the topological action (8.1.4), the *global* symmetry of the "physical" action (8.1.1) is not given by generic 2-dimensional diffeomorphisms, but by 2-dimensional *volume-preserving* diffeomorphisms. The standard method to study global symmetries is to promote them to local symmetries, by introducing additional background fields. What we will do in practice is to introduce a specific 2-dimensional background field $f^{(2)}$ (a 2-form on Σ) and to rewrite the non-topological term (8.1.3) as

$$-\frac{1}{2}\int_\Sigma f^{(2)}\mathrm{Tr}\phi^2 \ . \tag{8.1.9}$$

By requiring that the resulting action has the same physical content of the original one, we will be forced to complete the 2-form field $f^{(2)}$ to a *topological $U(1)$* multiplet of total ghost number (form-degree + ghost number) $+2$

$$f^{(2)} \ , \qquad \psi^{(1)} \ , \qquad \gamma^{(0)} \ , \tag{8.1.10}$$

where $\psi^{(1)} = \psi^{(1)}_\mu dx^\mu$ is a 1-form of ghost number $+1$ and $\gamma^{(0)}$ is a 0-form of ghost number $+2$.

Moreover, the introduction of the topological multiplet (8.1.10) will force us to deform the BRST transformations of the physical fields A, ϕ and c. However we will see that the deformed BRST algebra closes only up to the equations of motion of the gauge field A. To face this problem we will make again use of the BV formalism, even if in a new and somewhat exotic formulation: the standard BV recipe requires to double the field space by introducing, for each field Φ of the theory, a corresponding antifield $\tilde{\Phi}$. The action gets deformed by additional terms that schematically can be written as

$$\Gamma_{BV} = \Gamma_{\mathrm{can}} + \cdots = \int_\Sigma \tilde{\Phi}(s\Phi) + \dots \ , \tag{8.1.11}$$

where the dots denote additional terms at least quadratic in the antifields and that, for the specific case at hand, write

$$-\frac{1}{2}\int_\Sigma \gamma^{(0)}\,\mathrm{Tr}\,\tilde{A}\wedge\tilde{A} \ , \tag{8.1.12}$$

where $\tilde{A}$ is the antifield for the gauge field A.

The standard BV interpretation, of course, does not consider the antifields $\tilde{\Phi}$ as independent fields, but rather they are fixed to particular functionals of the physical fields during the gauge-fixing procedure. We will take a different approach: we will just

introduce the antifield $\tilde{A}$ of the gauge field A and we will consider it as an *independent* field. We will also drop the canonical term Γ_{can} and we will obtain the final action (8.2.29) which will be invariant under the BRST transformations (8.2.31); such transformations can be seen as describing the coupling between topological YM and the additional $U(1)$ topological background multiplet (8.1.10).

The connection with the approach of [99], equations (8.2.7) and (8.2.8), is now obtained by looking for *bosonic* backgrounds $(f^{(2)}, \psi^{(1)} = 0, \gamma^{(0)})$ which are invariant under the BRST transformations (8.2.31). Such backgrounds are obtained by taking $\gamma^{(0)}$ constant on Σ and it is easy to show that, for such backgrounds, our action and BRST transformations (8.2.29), (8.2.31) are equivalent to the corresponding obtained by Witten, (8.2.7) and (8.2.8), that therefore can be thought as the *rigid* limit of our approach.

Our approach gives also an understanding, in terms of our backgrounds fields $(f^{(2)}, \psi^{(1)} = 0, \gamma^{(0)})$, of the "non-supersymmetric" action (8.1.1): it is simply obtained by taking the bosonic, supersymmetric, background $\gamma^{(0)} = 0$; in this point the topological supersymmetry breaks down to the gauge symmetry (8.1.5) where the topological supersymmetry is hidden.

8.2   Coupling 2d Yang-Mills to topological backgrounds

The formulation of 2-dimensional Yang-Mills theory which is most directly related to the topological gauge theory [99] involves, beyond the gauge field $A = A^a T^a$, an adjoint scalar field $\phi = \phi^a T^a$. The 2-dimensional action writes

$$\Gamma_{YM} = \int_{\Sigma} \text{Tr} \, \phi \, F + \frac{\epsilon}{2} \int_{\Sigma} d^2 x \, \sqrt{g} \, \text{Tr} \, \phi^2 \ , \tag{8.2.1}$$

Σ is the 2-dimensional world-sheet, F is the field strength

$$F = d \, A + A^2 \ , \qquad A = A_{\mu} \, dx^{\mu} = A^a \, T^a \ , \tag{8.2.2}$$

and T^a, with $a = 1, \dots \dim G$, are generators of the Lie algebra of the gauge group G. The BRST gauge transformations

$$s_{gauge} \, c = -c^2 \ ,$$
$$s_{gauge} \, A = -D \, c \ ,$$
$$s_{gauge} \, \phi = -[c, \phi] \ , \tag{8.2.3}$$

leave the action invariant. By integrating out the scalar field ϕ one recovers the traditional form of the YM action

$$\Gamma'_{YM} = -\frac{1}{2 \, \epsilon} \int_{\Sigma} \text{Tr} \, F^2 \ . \tag{8.2.4}$$

The parameter ϵ is therefore (proportional to) the square of the standard YM coupling constant.

The theory (8.2.1) does not describe propagating degrees of freedom, and yet it does not possess any rigid topological supersymmetry; in this sense it can be thought as the non-topological formulation of a topological theory. Its $\epsilon \to 0$ limit

$$\Gamma_{YM}\big|_{\epsilon=0} = \int_\Sigma \mathrm{Tr}\,\phi\,F\ , \tag{8.2.5}$$

gives rise to an action which is invariant under 2-dimensional reparametrizations and it is therefore often referred to as the "topological limit".

As a matter of fact, even the $\Gamma_{YM}\big|_{\epsilon=0}$ action is *not* really invariant under any topological supersymmetry. In [99] it was observed, however, that action $\Gamma_{YM}\big|_{\epsilon=0}$ can be easily supersymmetrized by adding to it a decoupled quadratic fermionic term

$$\Gamma_{top} = \Gamma_{YM}\big|_{\epsilon=0} - \frac{1}{2}\int_\Sigma \mathrm{Tr}\,\psi \wedge \psi = \int_\Sigma \mathrm{Tr}\,\phi\,F - \frac{1}{2}\int_\Sigma \mathrm{Tr}\,\psi \wedge \psi\ . \tag{8.2.6}$$

This action is indeed invariant under topological Yang-Mills BRST transformations:

$$\begin{aligned}
s\,c &= -c^2 + \phi\ , \\
s\,A &= -D\,c + \psi\ , \\
s\,\psi &= -[c,\psi] - D\,\phi\ , \\
s\,\phi &= -[c,\phi]\ .
\end{aligned} \tag{8.2.7}$$

By switching on ϵ one obtains the action considered in [99]

$$\Gamma_W = \int_\Sigma \mathrm{Tr}\,\phi\,F - \frac{1}{2}\int_\Sigma \mathrm{Tr}\,\psi \wedge \psi + \frac{\epsilon}{2}\int_\Sigma d^2x\,\sqrt{g}\,\mathrm{Tr}\,\phi^2\ . \tag{8.2.8}$$

Γ_W is also invariant under (8.2.7); nevertheless it is not invariant under 2-dimensional diffeomorphisms, since it explicitly depends on a 2-dimensional background metric via the volume form $d^2x\,\sqrt{g}$. The *global* symmetry of Γ_W is just given by *volume preserving* 2-dimensional diffeomorphisms.

The standard and convenient way to study global symmetries is to make them local by introducing suitable backgrounds. Accordingly, in this section, the volume preserving reparametrization symmetry will be treated by replacing both the metric and the coupling constant ϵ with a topological background. This will produce automatically the quadratic fermionic term in (8.2.6), which was introduced by hand in [99].

Let then $f^{(2)}$ be a 2-form field and let us replace the action (8.2.1) with

$$\Gamma_1 = \int_\Sigma \mathrm{Tr}\,\phi\,F - \frac{1}{2}\int_\Sigma f^{(2)}\mathrm{Tr}\,\phi^2\ . \tag{8.2.9}$$

This action of course has not the same physical content as the original action. A generic $f^{(2)}$ admits a Hodge decomposition of the following form

$$f^{(2)} = \Omega^{(2)} + d\,\Omega^{(1)} \ , \tag{8.2.10}$$

where

$$\Omega^{(2)} = \epsilon\,d^2x\,\sqrt{g} \ , \tag{8.2.11}$$

is a representative of $H^2(\Sigma)$ and $\Omega^{(1)}$ a 1-form. For Γ_1 to be equivalent to Γ_ϵ we must remove the degrees of freedom associated to $\Omega^{(1)}$. We do this by introducing a BRST symmetry for the background $f^{(2)}$

$$s\,f^{(2)} = -d\,\psi^{(1)} \ , \tag{8.2.12}$$

where $\psi^{(1)}$ is a fermionic background field of ghost number $+1$. The BRST transformation (8.2.12) is degenerate: we need therefore introduce also a ghost-for-ghost background field $\gamma^{(0)}$ of ghost number $+2$

$$s\,\psi^{(1)} = -d\,\gamma^{(0)} \ , \tag{8.2.13}$$

with

$$s\,\gamma^{(0)} = 0 \ . \tag{8.2.14}$$

The action (8.2.4), however, is not BRST invariant, since

$$s\,\Gamma_1 = -s\left(\frac{1}{2}\int_\Sigma f^{(2)}\mathrm{Tr}\,\phi^2\right) = \frac{1}{2}\int_\Sigma d\,\psi^{(1)}\,\mathrm{Tr}\,\phi^2 = \int_\Sigma \psi^{(1)}\,\mathrm{Tr}\,D\,\phi\,\phi \ . \tag{8.2.15}$$

To cure for this we modify the BRST transformation law for A

$$s\,A = -D\,c - \psi^{(1)}\,\phi + \cdots \ , \tag{8.2.16}$$

so that the BRST variation of the first term in Γ_1 cancels the lack of invariance of the second term:

$$s\,\Gamma_1 = s\int_\Sigma \mathrm{Tr}\,\phi\,F + \int_\Sigma \psi^{(1)}\,\mathrm{Tr}\,D\,\phi\,\phi = 0 \ . \tag{8.2.17}$$

The problem with (8.2.16) is that it is not nilpotent:

$$s^2\,A = -d\,\gamma^{(0)}\,\phi + \cdots \ , \tag{8.2.18}$$

to fix this it is necessary to deform the BRST transformation rule for the ghost c

$$s\,c = -c^2 + \gamma^{(0)}\,\phi \ . \tag{8.2.19}$$

With this modification one has

$$s^2 c = 0 \ , \tag{8.2.20}$$

and also this induces an extra term in $s^2 A$ which cancels the term proportional to $d \gamma_0$:

$$s^2 A = D \left(\gamma^{(0)} \phi \right) - d \gamma^{(0)} \phi + \cdots = \gamma^{(0)} D \phi + \cdots \ . \tag{8.2.21}$$

Although this is still not zero, the lack of nilpotency is now reduced to a term proportional to the equations of motion of A:

$$\frac{\delta \Gamma_1}{\delta A} = D \phi \ , \tag{8.2.22}$$

and

$$s^2 A = 0 \qquad \text{on shell} \ . \tag{8.2.23}$$

The BV formalism provides a systematic way to go off-shell. One introduces the anti-field corresponding to A

$$\tilde{A} \equiv \tilde{A}^a_\mu \, T^a \, dx^\mu \ , \tag{8.2.24}$$

which is a 1-form in the adjoint of the gauge group of ghost number -1, and an anti-field dependent term in the BRST transformation of A

$$s A = -D c + \gamma^{(0)} \tilde{A} - \psi^{(1)} \phi \ . \tag{8.2.25}$$

This makes s nilpotent off-shell on all fields

$$s^2 c = s^2 A = s^2 \phi = s^2 \tilde{A} = 0 \qquad \text{off shell} \ , \tag{8.2.26}$$

as long as the $\tilde{A}$ transforms according to

$$s \tilde{A} = -[c, \tilde{A}] - D \phi \ . \tag{8.2.27}$$

The new term proportional to γ_0 in (8.2.25) spoils the invariance of the action

$$s \, \Gamma_1 = \int_\Sigma \text{Tr} \, \phi \, D \left(\gamma^{(0)} \, \tilde{A} \right) = - \int_\Sigma \gamma^{(0)} \, \text{Tr} \, D \, \phi \, \tilde{A} \ , \tag{8.2.28}$$

and this is anticipated in the BV framework: once an anti-field dependent term is introduced in the BRST transformation of a field, terms quadratic in the anti-fields must be added to the action. Indeed the action

$$\Gamma = \int_\Sigma \text{Tr} \, \phi \, F + \frac{1}{2} \int_\Sigma f^{(2)} \text{Tr} \, \phi^2 - \frac{1}{2} \int_\Sigma \gamma^{(0)} \, \text{Tr} \, \tilde{A} \wedge \tilde{A} \ , \tag{8.2.29}$$

is invariant:

$$s \, \Gamma = 0 \ , \tag{8.2.30}$$

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under BRST transformations of both fields and backgrounds

$$s\,c = -c^2 + \gamma^{(0)}\,\phi\ ,$$
$$s\,A = -D\,c + \gamma^{(0)}\,\tilde{A} - \psi^{(1)}\,\phi\ ,$$
$$s\,\phi = -[c,\phi]\ ,$$
$$s\,\tilde{A} = -[c,\tilde{A}] - D\,\phi\ ,$$
$$s\,f^{(2)} = -d\,\psi^{(1)}\ ,$$
$$s\,\psi^{(1)} = -d\,\gamma^{(0)}\ ,$$
$$s\,\gamma^{(0)} = 0\ . \tag{8.2.31}$$

8.3 The topological supersymmetry

We have seen that the field $\tilde{A}$ emerges naturally in the context of the BV formalism. In the BV framework, the action (8.2.29) would not however be the full action. The standard BV action is given by adding to (8.2.29) a canonical piece linear in the antifields corresponding to all the fields and backgrounds:

$$\Gamma_{\text{can}} = \int_\Sigma s\,c\,\tilde{c} + s\,A\,\tilde{A} + s\,\phi\,\tilde{\phi} + s\,f^{(2)}\,\tilde{f} + s\,\psi^{(1)}\,\tilde{\psi} + s\,\gamma^{(0)}\,\tilde{\gamma}\ , \tag{8.3.1}$$

where

$$s\,\tilde{c} = -[c,,\tilde{c}] - [\phi,\tilde{\phi}] - D\,\tilde{A}\ ,$$
$$s\,\tilde{\phi} = -[c,\tilde{\phi}] - F - \gamma_0\,\tilde{c} + \psi^{(1)}\,\tilde{A} - f^{(2)}\,\phi\ . \tag{8.3.2}$$

The full BV action

$$\Gamma_{BV} = \Gamma + \Gamma_{\text{can}} = \int_\Sigma \text{Tr}\,\phi\,F + \frac{1}{2}\int_\Sigma f^{(2)}\text{Tr}\,\phi^2 +$$
$$+ \int_\Sigma \left[(-c^2 + \gamma^{(0)}\,\phi)\,\tilde{c} + (D\,c - \psi^{(1)}\,\phi)\,\tilde{A} - [c,\phi]\,\tilde{\phi} - d\psi^{(1)}\,\tilde{f} - d\,\gamma^{(0)}\,\tilde{\psi} \right] +$$
$$+ \frac{1}{2}\int_\Sigma \gamma^{(0)}\,\text{Tr}\,\tilde{A} \wedge \tilde{A}\ , \tag{8.3.3}$$

generates the BRST transformations of both fields and anti-fields via the familiar BV formulas.

However the interpretation of $\tilde{A}$ as the anti-field of A is not mandatory. We will argue that an alternative — although exotic — point of view is available in our context and this point of view leads in a natural way to topological symmetry and localization. In the approach that we propose the $\tilde{A}$ is seen as an *independent* auxiliary field whose function is to close the BRST transformations off-shell: at the same time, the action is taken to be Γ, disregarding the canonical piece Γ_{can}.

This approach is consistent since the BRST transformations close on the fields $(\phi, A, c, \tilde{A})$ and leave Γ invariant. The only local gauge symmetry of Γ is the non-abelian gauge symmetry: Γ possesses also a global vector supersymmetry which, together with the gauge symmetry, gives rise to the BRST symmetry in (8.2.31). With this reinterpretation the "ghost field" associated to the topological supersymmetry is the ghost number $+1$ combination $\gamma^{(0)} \tilde{A}$.

The background dependent action Γ in (8.2.29) is invariant under simultaneous transformations of fields and backgrounds. To obtain the action invariant under rigid topological supersymmetry we consider the backgrounds which are left invariant under (8.2.31)

$$d\,\psi^{(1)} = 0 \ , \qquad d\,\gamma^{(0)} = 0 \Leftrightarrow \gamma^{(0)} = \text{constant} \equiv \gamma_0 \ . \tag{8.3.4}$$

One usually restricts oneself to bosonic backgrounds. In this case

$$\psi^{(1)} = 0 \ \ , \tag{8.3.5}$$

and the BRST transformations reduce to

$$\begin{aligned}
s\,c &= -c^2 + \gamma_0\,\phi \ , \\
s\,A &= D\,c + \gamma_0\,\tilde{A} \ , \\
s\,\phi &= -[c, \phi] \ , \\
s\,\tilde{A} &= -[c, \tilde{A}] - D\,\phi \ .
\end{aligned} \tag{8.3.6}$$

By introducing the rescaled fields

$$\hat{\phi} \equiv \gamma_0\,\phi \qquad \hat{\psi} \equiv \gamma_0\,\tilde{A} \ , \tag{8.3.7}$$

with $\hat{\psi}$ and $\hat{\phi}$ of ghost number 1 and 2 respectively, the BRST transformations (8.3.6) become identical to the topological Yang-Mills BRST transformations (8.2.7) and the BRST invariant action Γ

$$\Gamma = \frac{1}{\gamma_0} \left[\int_\Sigma \text{Tr}\,\hat{\phi}\,F - \frac{1}{2} \int_\Sigma \text{Tr}\,\hat{\psi} \wedge \hat{\psi} + \frac{1}{2} \int_\Sigma \frac{f^{(2)}}{\gamma_0} \text{Tr}\,\hat{\phi}^2 \right] \ , \tag{8.3.8}$$

coincides up to a multiplicative factor, with the Witten topological action Γ_W in (8.2.8) with the identification

$$\frac{f^{(2)}}{\gamma_0} = \sqrt{g}\,d^2x \ . \tag{8.3.9}$$

Let us summarize our logic: we started from the 2d YM theory. In order to study its global symmetry — volume preserving reparametrizations — we replaced the 2-dimensional metric and the coupling constant ϵ with a 2-form background field $f^{(2)}$,

at the same time asking that the physics only depend on the cohomology class of $f^{(2)}$. We have seen that this entails both extending the BRST gauge transformations to the background and to deform the BRST transformations of the gauge multiplet. Since the deformed BRST transformations close only up to the equations of motion of the gauge field, it was necessary to introduce $\tilde{A}$ — which in the conventional BV treating would be the anti-field of A. We managed to obtain in this way a BRST invariant theory coupled to topological backgrounds

$$Z[\{f^{(2)}, \psi^{(1)}, \gamma^{(0)}\}] = \int [dA \, d\phi \, d\tilde{A}] \mathrm{e}^{-\Gamma[A, \phi, \tilde{A}; f^{(2)}, \psi^{(1)}, \gamma^{(0)}]} \ . \tag{8.3.10}$$

The Ward identity associated to this symmetry

$$s \, Z[\{f^{(2)}, \psi^{(1)}, \gamma^{(0)}\}] = 0 \ , \tag{8.3.11}$$

encodes the fact that the partition function only depends on the cohomology class of $f^{(2)}$, i.e. the volume-preserving 2-dimensional reparametrization global symmetry of the original theory.

Theories invariant under rigid supersymmetry are now obtained by considering the backgrounds which are *bosonic* fixed points of the deformed BRST operator, i.e. $\gamma^{(0)} = \gamma_0$ constant and $\psi^{(1)} = 0$. For $\gamma_0 \neq 0$ one recovers the topological YM Witten theory and identifies the somewhat mysterious topological gaugino ψ of [99] as $\gamma_0 \, \tilde{A}$.

By choosing the point $\gamma_0 = 0$ in the space of BRST-invariant backgrounds one of course recovers the original YM action (8.2.1): in this limit the topological supersymmetry collapses and the BRST symmetry reduces to the pure gauge one, (8.2.3). The fact that the $\gamma_0 = 0$ point is degenerate in the space of backgrounds, gives a conceptual understanding of why the topological supersymmetry of the standard YM action is "hidden". On generic points $\gamma_0 \neq 0$ of the space of backgrounds the topological supersymmetry is manifest.

8.4 Summary

Summarizing our results we have explained, from a new point of view, how the topological supersymmetry of 2-dimensional YM theory can be achieved without the necessity of introducing any auxiliary fields by hand. We have seen that, by coupling the 2-dimensional field theory to the background fields, one is forced to complete the gauge system with the fermionic field $\tilde{A}$. Such an approach has required an alternative way of seeing the BV formalism too: in this new point of view the antifield A is not seen as an antifield but rather as an independent field. Our final system is expressed by the action (8.2.29), which is invariant under the transformations (8.2.31) involving *both* the physical fields and the

background fields. Therefore, our final system can be seen as given by the coupling of 2-dimensional YM with a topological $U(1)$ multiplet. The connection with the traditional approach of [28] is obtained by looking for bosonic, supersymmetric backgrounds, i.e. backgrounds which are invariant under the BRST transformations (8.2.31).

Let us finally discuss some open points that would be worth to investigate further in the future. Two-dimensional (topological) YM can be seen as a particular example of topological field theory of Schwarz-type: the theory is supposed to be topological since the classical action is, at least on a large extent, independent from the metric on the manifold where the theory is defined. On the other hand it is well-known that another way of obtaining topological field theories is given by topological twisting: one starts with a physical supersymmetric system and by topological twisting extracts a topological field theory where a particular supercharge plays the role of the BRST operator. We have seen that the topological supersymmetry of two-dimensional YM theory can be obtained by coupling the system to the background. We have seen in the precedent Chapter that a very similar result can be obtained for HCS too, by coupling the theory to suitable background fields one obtains a (twisted) $N = 2$ SUSY algebra, which is responsible for the topological properties of the model. It is therefore reasonable that, given a topological field theory of Schwarz type, it is possible to uncover a twisted supersymmetry algebra, by coupling the theory to topological backgrounds. We will discuss in the next Chapter that this is the case also for three-dimensional Chern-Simons theory.

The system (8.2.29), which describes the coupling between two-dimensional YM and a topological $U(1)$ multiplet, can be consistently coupled to two-dimensional topological gravity. One obtains that, to couple consistently two-dimensional YM theory to topological gravity, the presence of the topological $U(1)$ multiplet is crucial. It would be interesting to study further the resulting system coupled to two-dimensional topological gravity and to explore what kind of topological invariants it computes.

Chapter 9

Coupling 3-dimensional vector multiplets to topological backgrounds

In this Chapter we will apply our idea of coupling topological field theories to topological background fields to the case of three-dimensional supersymmetric theories with vector supermultiplets; by considering Chern-Simons theories and topological YM theories. We will see that such an approach, that can be thought as an alternative to current paradigma of coupling supersymmetric field theories to supergravity, has many relevant benefits with respect to the standard one. It allows a straightforward characterization of the backgrounds allowing rigid (twisted) supersymmetry, it allows a clear identification of the geometrical moduli on which the partition function can depend and it allows to compute the functional dependence of the partition on the geometrical backgrounds without performing any regularization procedure and in a completely gauge independent way: we will see indeed that the functional dependence of the partition function from the geometrical moduli can be computed starting from a topological anomaly. In the first section of this Chapter we will introduce the problem and this new point of view. In subsequent sections we will see how such a program can be worked out.

9.1   Statement of the problem

In the last few years there has been considerable progress in the analytical evaluation of partition functions and observables of supersymmetric gauge theories in different dimensions on certain compact manifolds equipped with suitable metrics. The common theme of these computations is localization. Localization is a long-known property of supersymmetric and topological theories, by virtue of which semi-classical approximation becomes,

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in certain cases, exact [28]. In more recent times this property has been exploited with considerable success in the work by Pestun [29] and in many following papers. In Pestun's approach no twisting of supersymmetry is performed. One rather seeks for manifolds and metrics supporting (generalized) covariantly constant spinors which ensure that certain supersymmetry global charges are unbroken. The global supersymmetry charges, even if spinorial in character, function essentially as topological BRST charges. Under favourable conditions one can choose a Lagrangian for which the semi-classical computation in the supersymmetric background is exact.

In three dimensions, a host of results is available. Explicit exact computations have been performed for 3-spheres, both with round and "squashed" metrics, and for Lens spaces. The best understood case is the one for which the complex conjugate of the (generalized) covariantly constant spinor is also covariantly constant. This is referred to as the "real" case in [100]. In all these cases the existence of (generalized) convariantly constant spinors implies in turn the existence of a *Seifert structure* on the 3-manifold. This refers to 3-manifolds with an almost contact metric structure and associated Reeb Killing vector field.

As a matter of fact Seifert 3-manifolds has already made their appearance earlier in the study of (non-supersymmetric) pure Chern-Simons (CS) gauge theories. It was discovered first "experimentally" [101] and then explained using various approaches by different authors [102], [103] that the semiclassical approximation for CS theories becomes exact precisely for Seifert 3-manifolds. Later, starting from [31], this result was rederived by considering the supersymmetric extension of CS: indeed this model is equivalent, after integrating out non-dynamical auxiliary fields, to the bosonic theory. In this way, computability of CS on Seifert manifolds was brought within the more general paradigm of convariantly constant spinors and localization.

In some cases, it is possible to perform localization computations not just for a single isolated Seifert structure, but for families of Seifert metrics depending on some continuous parameters. A significant example is provided by the the squashed metric considered in [35].[1] In those instances the partition function (and the observables) turns out to depend non-trivially on (some of those) parameters. Take for example the case of (supersymmetric) Chern-Simons theory on the squashed spheres, with the squashing considered in [35]:

$$ds^2 = \bar{g}_{\mu\nu}(x; b) \, dx^\mu \otimes dx^\nu = (\sin^2\theta + b^4 \cos^2\theta) \, d\theta^2 + \cos^2\theta \, d\phi_1^2 + b^4 \sin^2\theta \, d\phi_2^2 \; . \quad (9.1.1)$$

At first sight, the fact that the partition function is a non-trivial function of the squashing

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[1] In this Chapter, we will use the word squashed sphere to indicate the case that in [35] is indicated as the "less familiar" one. Namely it means a squashing that preserve just a $U(1) \times U(1)$ symmetry of the sphere.

parameter b^2 is not, *per se*, surprising. Indeed, even if CS theory is topological at classical level, topological invariance is anomalous at quantum level [34]. This means that the quantum CS action does depend on the background metric in a way which is controlled by the anomaly functional:

$$\delta\Gamma_{CS}[g_{\mu\nu}] = \int_{M_3} A_1^{(3)}[g_{\mu\nu}, \psi_{\mu\nu}] = \frac{c}{6} \int_{M_3} \epsilon^{\mu\nu\rho} R_\mu^\alpha D_\nu \psi_{\rho\alpha} d^3x \ , \qquad (9.1.2)$$

where $\psi_{\mu\nu} = \delta g_{\mu\nu}$ is the variation of the metric and c is a computable anomaly coefficient. However, if one plugs the squashed metric (9.1.1) and the variation $\psi_{\mu\nu} = b \, \partial_b \, g_{\mu\nu}$ into the anomaly form (9.1.2), one finds that the anomaly vanishes identically

$$A_1^{(3)}[g_{\mu\nu}(x;b), b \, \partial_b \, g_{\mu\nu}] = 0 \ . \qquad (9.1.3)$$

In this Chapter we will solve this conundrum: we will see that the vanishing of the topological anomaly for the squashed spheres is compatible with the non-trivial dependence of the partition function on the squashing parameter. As a matter of fact, we will show that the topological anomaly captures the precise dependence of the partition function on b. We will extend this results to generic three-dimensional supersymmetric theories, with both Yang-Mills and Chern-Simons terms in the action (YM + CS), involving vector multiplets only.

The resolution of our puzzle will require understanding the appropriate renormalization prescription for quantum effective actions on Seifert manifolds. The time-honored method to identify the renormalization prescriptions associated to certain symmetries is to introduce backgrounds fields which act as sources for the currents associated to those symmetries. This approach has been forcefully advocated more recently in the specific context of supersymmetric gauge theories in [30] and in several following papers.

We also will introduce backgrounds, but our treatment will differ from the one which has become common in the literature on localization of the last few years. Instead of coupling the supersymmetric gauge theory to supergravity, we will *first* consider its topological version and *then* couple it to topological gravity.

This will have the advantage of obtaining the Seifert condition for global supersymmetry in the most straightforward way by avoiding all the complications of spinors. Moreover and most importantly the topological gravity formulation will make immediate to identify the subsets of the topological transformations which preserve the Seifert structure. In the Seifert context Chern-Simons topological (framing) anomaly is BRST trivial [102]. We compute explicitly the corresponding local Wess-Zumino functional. We will use it to derive the dependence on the Seifert moduli of the quantum action directly from the anomalous Ward identity associated to the topological anomaly. Our computation will be manifestly regularization and gauge independent: We will do it without the need of computing explicitly any functional determinant.

9.2 Coupling 3d Chern-Simons to topological gravity

The classical Chern-Simons action [88] is

$$\Gamma_{CS} = \int_{M_3} \text{Tr} \left[\frac{1}{2} A \, d \, A + \frac{1}{3} A^3 \right] , \qquad (9.2.1)$$

where

$$A = A_\mu^a \, T^a \, dx^\mu \qquad (9.2.2)$$

is a 1-form gauge field on a closed 3-manifold M_3. T^a, with $a = 1, \ldots, \dim G$, are generators of the Lie algebra of the simple, connected and simply connected gauge group G. Gauge invariance leads to the nilpotent BRST transformation rules

$$S_0 \, A = -D \, c , \qquad S_0 \, c = -c^2 , \qquad (9.2.3)$$

where $c = c^a \, T^a$ is the ghost field carrying ghost number $+1$ and $D \, c \equiv d \, c + [A, c]_+$ is the covariant differential.

The classical action (9.2.1) is both invariant under diffeomorphisms and independent of the 3-dimensional background metric $g_{\mu\nu}$. In order to study the fate in the quantum theory of this *global* topological symmetry one must couple the theory to suitable backgrounds. This has been done in [94] where it is explained that the backgrounds appropriate for the topological symmetry in question are those of equivariant *topological gravity* [23]

$$s \, g_{\mu\nu} = \psi_{\mu\nu} - \mathcal{L}_\xi \, g_{\mu\nu} , \qquad s \, \psi_{\mu\nu} = \mathcal{L}_\gamma \, g_{\mu\nu} - \mathcal{L}_\xi \, \psi_{\mu\nu} ,$$
$$s \, \xi^\mu = \gamma^\mu - \frac{1}{2} \, \mathcal{L}_\xi \, \xi^\mu , \qquad s \, \gamma^\mu = -\mathcal{L}_\xi \, \gamma^\mu , \qquad (9.2.4)$$

where ξ^μ is the ghost of reparametrizations of ghost number $+1$, $\psi_{\mu\nu}$ is the topological gravitino of ghost number $+1$ and γ^μ is the ghost-for-ghost of ghost number $+2$.

The coupling to background topological gravity induces both deformations in the BRST transformations of the matter fields and extra terms in the action. It also requires introducing new matter fields $\tilde{A}$ and $\tilde{c}$ which sit in the same BRST multiplet as c and A. $\tilde{A}$ and $\tilde{c}$ are Lie algebra-valued 2 and 3-forms

$$\tilde{A} = (\tilde{A})_{\mu\nu}^a \, T^a \, dx^\mu \, dx^\nu , \qquad \tilde{c} = (\tilde{c})_{\mu\nu\rho}^a \, T^a \, dx^\mu \, dx^\nu \, dx^\rho , \qquad (9.2.5)$$

of ghost number -1 and -2 respectively. All the matter fields, as we already recalled in Chapter 7, fit niceley into a single super-field, or polyform, $\mathcal{A}$,

$$\mathcal{A} \equiv c + A + \tilde{A} + \tilde{c} \qquad (9.2.6)$$

whose total fermionic number, given by the form degree plus ghost number, is $+1$. The action of the BRST transformation on the supermultiplet (9.2.6) *before* coupling to topological gravity writes in the compact form

$$\delta_0 \mathcal{A} + \mathcal{A}^2 = 0 \ , \tag{9.2.7}$$

where

$$\delta_0 = S_0 + d \ , \qquad \delta_0^2 = 0 \tag{9.2.8}$$

is the "rigid" coboundary operator of total fermion number $+1$. It is immediate to check that (9.2.7) reduces to (9.2.4) when restricted to the fields c and A.

To describe the coupling to topological gravity it is convenient to consider the operator

$$S \equiv s + \mathcal{L}_\xi \ , \tag{9.2.9}$$

rather than the nilpotent BRST operator s. On the functionals of the backgrounds independent of ξ^μ, S satisfies

$$S^2 = \mathcal{L}_\gamma \tag{9.2.10}$$

and it is therefore nilpotent on the space of equivariant functionals of the topological gravity multiplet.

After these preliminaries, one discovers that the coboundary operator appropriate to describe the BRST transformation rules of the system *after* coupling to topological gravity is simply

$$\delta \equiv S + d - i_\gamma \ , \qquad \delta^2 = 0 \ , \tag{9.2.11}$$

where δ writes in the same form as the rigid ones

$$\delta \mathcal{A} + \mathcal{A}^2 = 0 \ . \tag{9.2.12}$$

When written for the component fields, these transformations become

$$\begin{aligned}
s\, c &= -c^2 - \mathcal{L}_\xi\, c + i_\gamma(A) \ , \\
s\, A &= -D\, c - \mathcal{L}_\xi\, A + i_\gamma(\tilde{A}) \ , \\
s\, \tilde{A} &= -[\tilde{A}, c] - \mathcal{L}_\xi\, \tilde{A} - F + i_\gamma(\tilde{c}) \ , \\
s\, \tilde{c} &= -[\tilde{c}, c] - \mathcal{L}_\xi\, \tilde{c} - D\, \tilde{A} \ .
\end{aligned} \tag{9.2.13}$$

The action also gets modified after coupling to topological gravity: one can check that

$$\Gamma_{CS+t.g.} = \Gamma_{CS} + \frac{1}{2} \int_{M_3} \mathrm{Tr}\, i_\gamma(\tilde{A})\, \tilde{A} \ , \tag{9.2.14}$$

is invariant by transforming both the fields according to (9.2.13) and the backgrounds according to (9.2.4).

The fields $\tilde{A}$ and $\tilde{c}$ have a natural interpretation, in the Batalin-Vilkovisky formalism, as the anti-fields of A and c. However in this Chapter we will make use of the alternative — somewhat exotic — point of view that we developed in Chapter 8. $\tilde{A}$ and $\tilde{c}$ are considere as *independent* auxiliary fields whose function is to close the BRST transformations *off-shell*: at the same time, our action we will be just $\Gamma_{CS+t.g.}$, not the full Γ_{BV}.

Of course, in the formulation in which $(\tilde{A}, \tilde{c})$ are auxiliary fields, $\Gamma_{CS+t.g.}$ maintains the original non-abelian gauge symmetry, which, eventually, will have to be fixed: both the gauge non-abelian symmetry and the the global vector supersymmetry associated to the topological invariance of the matter theory are captured by the BRST symmetry in (9.2.13). In other words, the anti-fields of the BV formulation can be reinterpreted as the auxiliary fields which are necessary to close the global supersymmetry algebra of the supersymmetric CS, *after* twisting that supersymmetry to obtain a topological model.

As a matter of fact, the 3d action (9.2.14) has *more* local gauge invariance than just the standard non-abelian gauge invariance: it is invariant also under the fermionic local symmetry

$$\tilde{A} \to \tilde{A} + i_\gamma(\chi) \ , \tag{9.2.15}$$

where χ is a fermionic scalar gauge parameter in the adjoint of the gauge group. Thus, the commuting field $\tilde{c}$ can be viewed as the ghost associated to this additional local symmetry. This extra gauge invariance is fixed by replacing $\Gamma_{CS+t.g.}$ with

$$\Gamma'_{CS+t.g.} = \Gamma_{CS+t.g.} + s \int_{M_3} \mathrm{Tr}\big(b * i_\gamma(*\tilde{A})\big) =$$
$$= \Gamma_{CS} + \frac{1}{2} \int_{M_3} \mathrm{Tr}\, i_\gamma(\tilde{A})\, \tilde{A} + \int_{M_3} \mathrm{Tr}\big[\Lambda * i_\gamma(*\tilde{A}) - b * i_\gamma(* i_\gamma(\tilde{c})) + b * i_\gamma(* F)\big] +$$
$$+ \int_{M_3} d^3 x\, \psi_{\mu\nu}\, \epsilon^{\alpha\beta\mu}\, \gamma^\nu\, \mathrm{Tr}\, b\, \tilde{A}_{\alpha\beta} \ , \tag{9.2.16}$$

where b is a 0-form anti-ghost of ghost number -2,

$$s\, b = -\mathcal{L}_\xi\, b - [c, b] + \Lambda \ , \qquad s\, \Lambda = -\mathcal{L}_\xi\, \Lambda - [c, \Lambda] + i_\gamma(D\, b) \ , \tag{9.2.17}$$

Λ is a Lagrange multiplier of ghost number -1 and $*$ is the Hodge dual with respect to the background metric $g_{\mu\nu}$.

Summarizing: The action $\Gamma'_{CS+t.g.}$ has the background topological supersymmetry captured by (9.2.4) and (9.2.13) and no other local gauge-invariance beyond the standard non-abelian gauge invariance: $\tilde{A}$ and $\tilde{c}$ can now be consistently thought of as auxiliary, non propagating, fields rather than anti-fields.

9.3 Coupling 3d topological YM with CS term to topological gravity

The 3d topological YM theory is characterized by the BRST transformations

$$S_0\, c = -c^2 + \sigma\ ,$$
$$S_0\, A = -D\, c + \Psi\ ,$$
$$S_0\, \Psi = -[c, \Psi] - D\, \sigma\ ,$$
$$S_0\, \sigma = -[c, \sigma]\ , \tag{9.3.1}$$

Ψ is a fermionic 1-form of ghost number 1 and σ a bosonic 0-form of ghost number 2. Both of them are in the adjoint of the gauge group. It is convenient to introduce a super-field or polyform of total fermionic number (ghost number + form degree) equal to 2:

$$\mathcal{F} = F + \Psi + \sigma\ . \tag{9.3.2}$$

The transformations (9.3.1) write in a nice compact form in terms of the "rigid" coboundary operator $\delta_0 = S_0 + d$

$$\mathcal{F} = \delta_0\, \mathcal{A}_{YM} + \mathcal{A}_{YM}^2\ , \tag{9.3.3}$$

where

$$\mathcal{A}_{YM} = c + A\ . \tag{9.3.4}$$

It is important to observe that the super-field or polyform containing the gauge connection which is appropriate for the YM theory is not the same as the connection polyform $\mathcal{A}$ (9.2.6) of CS.

Let us again denote by s the nilpotent BRST operator after coupling to topological gravity. As seen in the previous section, it is convenient to introduce the operator

$$S_{YM} \equiv s + \mathcal{L}_\xi\ , \qquad S^2 = \mathcal{L}_\xi\ , \tag{9.3.5}$$

where ξ is the ghost associated to reparametrizations. The coboundary operator for topological YM coupled to topological gravity is again given by a formula identical to (9.2.11)

$$\delta_{YM} \equiv S_{YM} + d - i_\gamma\ , \qquad \delta^2 = 0\ , \tag{9.3.6}$$

with δ satisfying

$$\mathcal{F} = \delta_{YM}\, \mathcal{A}_{YM} + \mathcal{A}_{YM}^2\ . \tag{9.3.7}$$

These transformations write in components:

$$S_{YM}\,c = -c^2 + i_\gamma(A) + \sigma \ ,$$
$$S_{YM}\,A = -D\,c + \Psi \ ,$$
$$S_{YM}\,\Psi = -[c, \Psi] + i_\gamma(F) - D\,\sigma \ ,$$
$$S_{YM}\,\sigma = -[c, \sigma] + i_\gamma(\Psi) \ . \tag{9.3.8}$$

Finally, the action of pure topological YM can be taken to be a s-trivial term:

$$\Gamma_{YM+t.g.} = S_{YM}\,\chi \ . \tag{9.3.9}$$

As just remarked, the superfield (9.3.4) appropriate for YM theory is quite different from the corresponding polyform relevant for Chern-Simons theory. However, we will now show that it is possibile to recast the topological YM BRST transformations purely in terms of the Chern-Simons superfield $\mathcal{A}$. To this end, let us pick a contact structure k on the 3-manifold, M_3 which is dual to the background vector field γ^μ:

$$i_\gamma(k) = 1 \ , \qquad \mathcal{L}_\gamma\,k = 0 \ . \tag{9.3.10}$$

1-forms ω on M_3 are naturally decomposed along the horizontal and vertical directions as follows

$$\omega = k\,\omega_V + \omega_H \ , \tag{9.3.11}$$

where

$$\omega_V \equiv i_\gamma(\omega) \ , \qquad i_\gamma(\omega_H) = 0 \ . \tag{9.3.12}$$

Let us therefore decompose the fermionic 1-form Ψ of the topological YM multiplet according to

$$\Psi \equiv k\,\zeta + \Psi_H \ , \tag{9.3.13}$$

with $i_\gamma(\Psi) = \zeta$. Since any horizontal form is i_γ-exact

$$\Psi_H = i_\gamma(\tilde{A}) \ , \tag{9.3.14}$$

we have

$$\Psi \equiv k\,\zeta + i_\gamma(\tilde{A}) \ , \tag{9.3.15}$$

where $\tilde{A}$ is a 2-form of ghost number -1. Note that the decomposition (9.3.15) is not unique and it has the gauge invariance

$$\tilde{A} \to \tilde{A} + i_\gamma(\tilde{c}) \ . \tag{9.3.16}$$

When written in terms of ζ and $\tilde{A}$ the YM topological transformations (9.3.8) rewrite as

$$
\begin{aligned}
S_{YM}\, c &= -c^2 + i_\gamma(A) + \sigma \ , \\
S_{YM}\, A &= -Dc + i_\gamma(\tilde{A}) + k\,\zeta \ , \\
S_{YM}\, \tilde{A} &= -[c, \tilde{A}] - F + i_\gamma(\tilde{c}) + k\,D\,\sigma \ , \\
S_{YM}\, \tilde{c} &= -[c, \tilde{c}] - D\,\tilde{A} - [\sigma, \tilde{A}] + k\,dk\,\zeta \ , \\
S_{YM}\, \sigma &= -[c, \sigma] + \zeta \ , \\
S_{YM}\, \zeta &= -[c, \zeta] + i_\gamma(D\,\sigma) \ ,
\end{aligned}
\tag{9.3.17}
$$

where we introduced the ghost-for-ghost $\tilde{c}$ to take into account the gauge-invariance (9.3.16). When written in this form, the emergence of the CS superfield

$$
\mathcal{A} = \mathcal{A}_{YM} + \tilde{A} + \tilde{c}
\tag{9.3.18}
$$

becomes apparent. Indeed the transformations (9.3.17) can be expressed entirely in terms of $\mathcal{A}$:

$$
\delta_{YM}\,\mathcal{A} + \mathcal{A}^2 = \mathbf{\Phi} \ ,
\tag{9.3.19}
$$

where $\mathbf{\Phi}$ is the polyform of total ghost number $+2$

$$
\mathbf{\Phi} = \sigma + k\,\zeta + k\,D\,\sigma + k\left(dk\,\zeta - [\sigma, \tilde{A}]\right) \ .
\tag{9.3.20}
$$

Eq. (9.3.19) is perfectly equivalent to the original (9.3.8): by means of the decomposition associated to the contact form k, it was possible to reformulate the topological YM transformations in terms of the full CS multiplet.

The important observation is that it is possible to recast (9.3.19) in the CS-form

$$
\delta_{YM}\,\mathcal{A}'(\mathcal{A}) + \mathcal{A}'(\mathcal{A})^2 = 0 \ ,
\tag{9.3.21}
$$

where

$$
\begin{aligned}
\mathcal{A}'(\mathcal{A}) &\equiv \mathcal{A} + \mathbf{\Theta} \ , \\
\mathbf{\Theta} &\equiv k\,\sigma + k\,dk\,\sigma \ .
\end{aligned}
\tag{9.3.22}
$$

The relation (9.3.21) shows that the YM BRST operator δ_{YM} has the same algebraic content as the CS BRST operator δ. As a matter of fact one sees from (9.3.17) that δ_{YM} differs from the CS δ only because it also includes, on top of the CS transformations, the shift symmetry

$$
A \to A + k\,\zeta
\tag{9.3.23}
$$

together with σ, the BRST partner of ζ. This shift symmetry was originally introduced in [102], to explain localization of CS theory on Seifert manifolds.

Since ζ, σ make a trivial BRST doublet, the physical content of δ_{YM} and δ is the same. Indeed, from (9.3.22) one derives the identity

$$S_{YM}\,\Gamma[\mathcal{A}'(\mathcal{A})] = S_{CS}\,\Gamma[\mathcal{A}]\Big|_{\mathcal{A}\to\mathcal{A}'(\mathcal{A})} . \qquad (9.3.24)$$

Thus, given any action $\Gamma[\mathcal{A}]$ invariant under the CS BRST operator S_{CS}, we can obtain an action invariant under S_{YM} by performing the substitutions

$$c' \;=\; c\,, \qquad A' = A + k\,\sigma\,, \qquad \tilde{A}' = \tilde{A}\,, \qquad \tilde{c}' = \tilde{c} + k\,dk\,\sigma\,. \qquad (9.3.25)$$

In particular from the S_{CS}-invariant action (9.2.14), one obtains the S_{YM} invariant action

$$\tilde{\Gamma}_{CS}[A, \tilde{A}, \sigma] = \Gamma_{CS}[A + k\,\sigma] + \frac{1}{2}\int_{M_3} \mathrm{Tr}\, i_\gamma(\tilde{A})\,\tilde{A}\,, \qquad (9.3.26)$$

which is equivalent to the CS action coupled to topological gravity backgrounds. This action is essentially the same as the Beasley-Witten's action [102]. In the symplectic formalism of [102] the term quadratic in $\tilde{A}$ is interpreted as the symplectic 2-form on the space of connections living on the base of the Seifert fibration. In our approach this term emerges naturally from the coupling to the topological backgrounds.

Summarizing it is possible to include a CS-term in the action for the topological YM gauge theory coupled to topological gravity: this is given in (9.3.26). The total action of the topological YM system with a CS term coupled to the topological gravity background will have therefore the form

$$\Gamma_{YM+CS+t.g.} = S_{YM}\,\chi + \Gamma_{CS}[A + k\,\sigma] + \frac{1}{2}\int_{M_3}\mathrm{Tr}\, i_\gamma(\tilde{A})\,\tilde{A}\,. \qquad (9.3.27)$$

9.4 The supersymmetric point

The quantum partition function of the topological YM + CS system in presence of topological gravity backgrounds[2]

$$Z[g_{\mu\nu}, \psi_{\mu\nu}, \gamma^\mu] = \int [dA\,d\tilde{A}\,d\tilde{c}\,d\sigma\,d\zeta]\,\mathrm{e}^{-\Gamma_{YM+CS+t.g.}}\,, \qquad (9.4.1)$$

is an *equivariant* functional of the topological gravity multiplet. This means that it is both independent of ξ^μ and invariant under reparametrizations. At classical level it satisfies the Ward identity

$$s\,Z[g_{\mu\nu}, \psi_{\mu\nu}, \gamma^\mu] = S\,Z[g_{\mu\nu}, \psi_{\mu\nu}, \gamma^\mu] = 0\,, \qquad (9.4.2)$$

---

[2]The notation in (9.4.1) is schematic: we did not include explicitly the ghost sector which fixes the standard YM gauge invariance.

which can be — and actually is — broken by quantum anomalies. We postpone the discussion regarding quantum topological anomalies to Section 9.6.

We can now look for bosonic backgrounds which are left invariant by S:

$$\bar{\psi}_{\mu\nu} = 0 \ , \qquad \mathcal{L}_{\bar{\gamma}}\, \bar{g}^{\mu\nu} \equiv \bar{D}^\mu\, \bar{\gamma}^\nu + \bar{D}^\nu\, \bar{\gamma}^\mu = 0 \ . \tag{9.4.3}$$

The second equation above says that the superghost background $\bar{\gamma}^\mu(x)$ is a Killing vector of the 3-dimensional metric $\bar{g}_{\mu\nu}(x)$. These geometrical data define a so-called *Seifert* structure on a 3-dimensional manifold [102]. We see therefore that supersymmetric YM+CS theories admitting a rigid topological supersymmetry are precisely those defined on 3-dimensional Seifert manifolds. Hence one anticipates that supersymmetric YM+CS theories on Seifert manifolds enjoy localization properties [29]. This fact, originally discovered in a "phenomenological" way in [101], has been subsequently explained using various approaches by different authors [102], [31]. In our approach this follows straighforwardly from the BRST trasformations of topological gravity.

The topological YM+CS action on a fixed Seifert manifold

$$\bar{\Gamma} \;=\; S_{YM}\,\chi + \Gamma_{CS}[A + k\,\sigma] + \int_{M_3} \mathrm{Tr}\,\Big[\frac{1}{2}\,i_{\bar{\gamma}}(\tilde{A})\,\tilde{A} + \Lambda * i_{\bar{\gamma}}(*\tilde{A}) + b * i_{\bar{\gamma}}(* i_{\bar{\gamma}}(\tilde{c} + k\,dk\,\sigma)) +$$
$$+ b * i_{\bar{\gamma}}(*\,(F + D\,(k\,\sigma)))\Big] \ , \tag{9.4.4}$$

is therefore invariant under the following BRST transformations which encode both gauge-invariance and global topological supersymmetry

$$\begin{aligned}
S_{YM}\,c &= -c^2 + i_{\bar{\gamma}}(A) + \sigma \ , \\
S_{YM}\,A &= -Dc + i_{\bar{\gamma}}(\tilde{A}) + k\,\zeta \ , \\
S_{YM}\,\tilde{A} &= -[c, \tilde{A}] - F + i_{\bar{\gamma}}(\tilde{c}) + k\,D\,\sigma \ , \\
S_{YM}\,\tilde{c} &= -[c, \tilde{c}] - D\,\tilde{A} - [\sigma, \tilde{A}] + k\,dk\,\zeta \ , \\
S_{YM}\,\sigma &= -[c, \sigma] + \zeta \ , \\
S_{YM}\,\zeta &= -[c, \zeta] + i_{\bar{\gamma}}(D\,\sigma) \ .
\end{aligned} \tag{9.4.5}$$

9.5 The relation between topological and physical supersymmetry

The rigid topological theory that we obtained by coupling topological YM+CS to topological gravity computes certain (semi)-topological observables of the "physical" globally supersymmetric YM+CS theory living on the same manifold. In particular the topological partition function, which is the object that we consider in this Chapter, is the

same as the superpartition function of the "physical" theory, i.e. the partition function with supersymmetric boundary conditions on both bosons and fermions. Indeed, as we mentioned in Section 9.1, the almost totality of the computations of those (semi)-topological observables performed in recent years, were developed directly in the context of the "physical" theory with spinorial supercharges. We argued that the topological gravity viewpoint provides some benefits, both conceptual and practical. For starters, we just saw in the previous section that the Seifert condition emerges from topological gravity directly — without the necessity to go through covariantly constant spinors [31], add extra symmetries [102], or pick up ingenious gauges [103]. But, above all, the coupling to topological gravity will allow us, in the next sections, to compute the moduli dependence of the partition function of supersymmetric YM+CS theory (involving only vector multiplets) by solving the anomalous topological Ward identities, in a completely regularization and gauge independent way.

In this section we will describe more precisely the relation between the topological YM+CS obtained via the coupling to topological gravity and "physical", spinorial, supersymmetric theory. We will also elucidate how the action that emerges from topological gravity encompasses the topological actions which were introduced in either [102] or [106].

Supersymmetric CS (SCS) theory on curved space has been studied starting from [31], who considered the special example of S^3. The supersymmetric extension of the CS action in flat space [104] writes

$$\Gamma_{SCS} = \Gamma_{CS} + \int d^3x \, \mathrm{Tr} \left(D \, \sigma - \frac{1}{2} \lambda^\dagger \lambda \right) , \qquad (9.5.1)$$

where the scalars D, σ and the Dirac spinor λ are in the adjoint of the gauge group. Since the D, σ and λ are auxiliary non-dynamical fields supersymmetric CS theory (9.5.1) is physically equivalent to pure CS theory. The action (9.5.1) is invariant under the *global* supersymmetry transformations which have the structure

$$\delta \equiv \delta_\epsilon + \delta_{\bar\epsilon} , \qquad (9.5.2)$$

where

$$\delta_\epsilon A_\mu = -\frac{i}{2} \lambda^\dagger \gamma_\mu \epsilon ,$$
$$\delta_\epsilon \sigma = -\frac{1}{2} \lambda^\dagger \epsilon ,$$
$$\delta_\epsilon D = -\frac{i}{2} D_\mu \lambda^\dagger \gamma^\mu \epsilon + \frac{i}{2} [\lambda^\dagger, \sigma] \epsilon ,$$
$$\delta_\epsilon \lambda = -i \sqrt{g} \, \epsilon_{\mu\nu\rho} \gamma^\rho F^{\mu\nu} - D \epsilon + i \gamma^\mu D_\mu \sigma \epsilon ,$$
$$\delta_\epsilon \lambda^\dagger = 0 , \qquad (9.5.3)$$

and $\delta_{\bar\epsilon} \Phi = (\delta_\epsilon \Phi)^\dagger$, for any field $\Phi \in \{A, \sigma, D, \lambda, \lambda^\dagger\}$.

The classical action of ordinary, non supersymmetric, CS action has the peculiarity of being invariant under local diffeomorphisms without the need of introducing a space-time metric. This means that one can study quantum CS theory on any *fixed* curved manifold: topological invariance of CS theory must be thought of as a global symmetry, in the sense that one does not need to integrate over space-time metrics to make sense of the quantum theory. This global symmetry is actually broken by anomalies: but, precisely because one is dealing with a global symmetry, this does not spoil the consistency of the quantum theory.

One might imagine that, by analogy, SCS theory might be made invariant under local supersymmetry transformations without the need of explicitly introducing supergravity backgrounds. If this were so, SCS could be formulated consistently on any manifolds. Let us discuss why this is not the case.

The standard recipe for putting a generic supersymmetric theory on a curved manifold is to first couple it to supergravity, by promoting the global supersymmetry transformations (9.5.3) to local ones. For the SCS theory, this would mean in principle to couple (9.5.1) to the Noether supercurrents

$$S^\mu = \frac{i}{2}\,\lambda^\dagger\,\gamma^\mu\,\sigma\ , \qquad \bar{S}^\mu = -\frac{i}{2}\,\gamma^\mu\,\lambda\,\sigma\ , \tag{9.5.4}$$

by changing the action

$$\Gamma_{SCS}^{\text{curved}} = \Gamma_{SCS} + \psi_\mu^\dagger\,\bar{S}^\mu + S^\mu\,\psi_\mu + \cdots\ , \tag{9.5.5}$$

where ψ_μ is the gravitino field and the dots denote the higher order terms of the Noether procedure. The coupled action $\Gamma_{SCS}^{\text{curved}}$ is invariant — at linearized level — under local supersymmetry transformations (9.5.3) of the fields if the gravitino background also transforms as

$$\delta_\epsilon\,\psi_\mu = D_\mu\epsilon + \cdots\ , \qquad \delta_{\bar\epsilon}\,\psi_\mu^\dagger = D_\mu\epsilon^\dagger + \cdots\ . \tag{9.5.6}$$

However SCS theory is "almost" topological. This is reflected by the fact that the supercurrents (9.5.4) vanish on shell

$$S^\mu = i\,\alpha\,\frac{\delta\,\Gamma_{SCS}}{\delta\lambda}\gamma^\mu\,\sigma + \frac{i}{2}\,(1-\alpha)\lambda^\dagger\,\gamma^\mu\,\frac{\delta\,\Gamma_{SCS}}{\delta D}\ , \tag{9.5.7}$$

where α is an arbitrary parameter. Since, when ϵ is space-dependent, the supersymmetry variation of the flat space action writes in terms of the supercurrents as follows

$$
\begin{aligned}
\left(\delta_\epsilon + \delta_{\bar\epsilon}\right)\Gamma_{SCS} &= \int d^3x\,\left(S^\mu\,D_\mu\,\epsilon + D_\mu\,\epsilon^\dagger\,\bar{S}^\mu\right) = \\
&= \int d^3x\,\left(i\,\alpha\,\frac{\delta\,\Gamma_{SCS}}{\delta\lambda}\gamma^\mu\,\sigma + \frac{i}{2}\,(1-\alpha)\lambda^\dagger\,\gamma^\mu\,\frac{\delta\,\Gamma_{SCS}}{\delta D}\right)D_\mu\,\epsilon + c.c.\ ,
\end{aligned}
\tag{9.5.8}
$$

one sees there is an alternative way to make (the diffeomorphism invariant extension of) Γ_{SCS} *locally* supersymmetric. This alternative method does not require introducing the gravitino: thanks to on-shell vanishing of the supercurrents one can simply modify the supersymmetry variations of $\lambda, \lambda^\dagger$ and D

$$\tilde{\delta}_\epsilon \, \lambda = \delta_\epsilon \, \lambda + i \, \alpha \, \gamma^\mu \, D_\mu \, \epsilon \, \sigma \; ,$$
$$\tilde{\delta}_\epsilon \, D = \delta_\epsilon \, D - \frac{i}{2} \, (1 - \alpha) \, \lambda^\dagger \, \gamma^\mu \, D_\mu \, \epsilon \; , \tag{9.5.9}$$

where the covariant derivatives are those appropriate to the chosen curved manifold. Then, (9.5.8) is obviously equivalent to

$$\tilde{\delta}_\epsilon \, \Gamma_{SCS} = \tilde{\delta}_{\bar{\epsilon}} \, \Gamma_{SCS} = 0 \; , \tag{9.5.10}$$

for space-time dependent ϵ's.

The trouble with this "alternative" way to make the supersymmetry local is that, for α arbitrary and for generic manifolds, the local supersymmetry algebra does not close. By analyzing the supersymmetry commutation relations one discovers [107] that closure of the algebra requires both that the condition

$$\alpha = \frac{2}{3} \tag{9.5.11}$$

is met *and* that the space-time dependence of ϵ be restricted by the differential equation

$$\gamma^\mu \, \gamma^\nu \, D_\mu \, D_\nu \, \epsilon = h \, \epsilon \; . \tag{9.5.12}$$

One concludes that CS theory with rigid supersymmetry can be constructed only on manifolds for which solutions of Eq. (9.5.12) exist. For those special manifolds one can obtain the corresponding rigid supersymmetry transformations by replacing in (9.5.9) the spinors which solve Eq. (9.5.12).

The lesson of this discussion is that, even for the "almost topological" supersymmetric CS theories one cannot neglect the coupling of the (classically vanishing) supercurrents to the supergravity backgrounds. Indeed it has since been understood [105], [100] that the conditions (9.5.11) and (9.5.12) are to be interpreted, in a model independent way, as the equations for the vanishing of the supersymmetry variation of the gravitino background

$$\delta_\epsilon \, \psi_\mu = \delta_{\bar{\epsilon}} \, \psi_\mu^\dagger = 0 \; . \tag{9.5.13}$$

The nice feature of Eqs. (9.5.13) is their universality: they do not depend on the specific theory one is considering and they characterize manifolds on which field theories with global supersymmetry may be constructed. The specific form of supersymmetry does instead depend on both the solution of (9.5.13) and the form of the coupling of the supergravity multiplet to the theory at hand.

It can be shown that in the *real* case when a solution ϵ of $\delta_\epsilon \psi_\mu = 0$ defines by conjugation a solution of $\delta_{\bar\epsilon} \psi_\mu^\dagger = 0$, the vector

$$\bar\gamma^\mu = \epsilon^\dagger \, \Gamma^\mu \, \epsilon \tag{9.5.14}$$

is a (real) Killing vector of the underlying 3-manifold

$$D_\mu \, \bar\gamma_\nu + D_\nu \, \bar\gamma_\mu = 0 \ . \tag{9.5.15}$$

This explains in particular why CS theories on 3-manifolds admitting a U(1) action — known as Seifert 3-manifolds — enjoy the localization property which was originally discovered in [101] in an experimental way. We have seen that in our topological approach, the Seifert condition (9.5.15) emerged directly from the topological gravity BRST transformation laws, with no reference to (generalized) covariantly constant spinors.

We can consider also the supersymmetric YM action in the SCS theory:

$$\Gamma_{SCS+SYM} = \Gamma_{SCS} + \Gamma_{SYM} \ , \tag{9.5.16}$$

where[3]

$$\Gamma_{SYM} = \mathrm{Tr}\Big[\frac{1}{4}\, F_{\mu\nu}^2 + \frac{1}{2}\, D_\mu \sigma \, D^\mu \sigma + \frac{1}{2}\, (D + \frac{\sigma}{r})^2 + \frac{i}{2}\, \lambda^\dagger \gamma^\mu D_\mu \lambda + \frac{i}{2}\, \lambda^\dagger\, [\sigma, \lambda] - \frac{1}{4\,r}\, \lambda^\dagger \lambda \Big] \ . \tag{9.5.17}$$

This latter action is not only invariant under the global supersymmetry transformations (9.5.3), (9.5.9), (9.5.13), but also supersymmetric exact

$$\epsilon^\dagger \epsilon \, \mathcal{L}_{YM} = \tilde\delta_\epsilon \, \tilde\delta_{\bar\epsilon} \, \mathrm{Tr}\, \frac{1}{4}\, \big(\lambda^\dagger \lambda - 2\, D\, \sigma\big) \ . \tag{9.5.18}$$

Therefore, cohomologically, the SCS+SYM system is equivalent to SCS theory.

The supersymmetric SCS+SYM on a *fixed* Seifert manifold can be twisted to give a model with a topological rigid symmetry. This was done in [106]. The physical supersymmetric vector multiplet (9.5.3) includes a Dirac fermion λ which has 4 real components. After the twist of [106], three of those form the topological gaugino Ψ. This, together with the scalar σ and the gauge connection A form the multiplet of topological YM. The twisted supersymmetry transformations of this supermultiplet turn out to have the form (9.3.8), in which the topological gravity field γ^μ is replaced by the Reeb vector field $\bar\gamma^\mu$. The remaining fermion gives rise to a scalar α of ghost number $+1$ which form, together with the auxiliary scalar field D of ghost number $+2$, an additional BRST *trivial* doublet,

$$\begin{aligned} S_{YM}\, \alpha &= -[c, \alpha] + \mathrm{D} + X(A, \sigma) \ , \\ S_{YM}\, \mathrm{D} &= -[c, \mathrm{D}] + i_{\bar\gamma}\, D\, \alpha + [\sigma, \alpha] - \tilde S\, X(A, \sigma) \ . \end{aligned} \tag{9.5.19}$$

---

[3]The parameter r which appears in this formula is the radius of the S^1 of the Seifert fibration.

Note that S_{YM} is nilpotent for any choice of the scalar function $X(A, \sigma)$ of ghost number 2. The twist of the physical supersymmetric theory gives rise to a specific choice for $X(A, \sigma)$. We keep it arbitrary for the moment, to better explain the connections with other approches.

The additional BRST trivial doublet (α, D) can function as an antighost-Lagrange multiplier pair, by adding to the S_{YM} invariant action (9.3.26) an S_{YM}-trivial term

$$\Gamma'_{CS} = \Gamma_{CS}[A + k\,\sigma] + \frac{1}{2} \int_{M_3} k \operatorname{Tr} \Psi\,\Psi + S_{YM} \int_{M_3} k\,dk \operatorname{Tr} \alpha\,\sigma \ . \tag{9.5.20}$$

One can pick

$$X(A, \sigma) = 0 \ . \tag{9.5.21}$$

With this choice the gauge-fixing term in (9.5.20) fixes the Beasley-Witten shift-symmetry: integrating out D puts σ to zero and integrating out α sets $\zeta = 0$ and thus $\Psi = i_\gamma(\tilde{A})$. We recover in this way our original CS action (9.2.14).

The twist of the physical SCS action (9.5.1) discussed in [106] gives instead the

$$X(A, \sigma) = \frac{k\,F}{k\,dk} + \sigma = i_\gamma(*F) + \sigma \ . \tag{9.5.22}$$

Note that this choice of $X(A, \sigma)$ introduces a *spurious* dependence of the BRST operator on the metric compatible with the vector field $\bar{\gamma}^\mu$ which defines the Seifert structure. This dependence should of course drop out in physical observables, but this is not explicit in the framework of [106]. The reason of course is that twisting a physical supersymmetric action corresponds to make a specific choice for the gauge-fixing term of the topological action. This, although sometimes convenient to perform explicit computations, leads to gauge-dependent BRST transformations, somehow obscuring the geometric content of the topological symmetry. One appealing feature of our treatment is that it makes manifest that the theory only depends on the Seifert structure.

9.6 Topological Anomaly for Seifert manifolds

The classical Ward identity (9.4.2) can be broken by quantum anomalies

$$S \log Z[g_{\mu\nu}, \psi_{\mu\nu}, \gamma^\mu] \equiv S \int_{M_3} \Gamma[g_{\mu\nu}] = \int_{M_3} A_1^{(3)}[g_{\mu\nu}, \psi_{\mu\nu}] \ . \tag{9.6.1}$$

The topological anomaly describes therefore the response of the *quantum action density* $\Gamma[g_{\mu\nu}]$ of the YM+CS topological system under a *generic* variation of the metric $\delta g_{\mu\nu} \equiv \psi_{\mu\nu}$

$$\delta \int_{M_3} \Gamma[g_{\mu\nu}] = \int_{M_3} A_1^{(3)}[g_{\mu\nu}, \delta g_{\mu\nu}] \ . \tag{9.6.2}$$

The topological anomaly 3-form $A_1^{(3)}$ is a local cohomology class of ghost number $+1$ of the BRST operator of topological gravity, which must satisfy the Wess-Zumino consistency condition

$$S\, A_1^{(3)}[g_{\mu\nu}, \psi_{\mu\nu}] = -d\, A_2^{(2)}[g_{\mu\nu}, \psi_{\mu\nu}, \gamma^\mu] \ . \qquad (9.6.3)$$

Topological anomalies were classified in [94]. In 3-dimensions we have a single representative of ghost number $+1$

$$A_1^{(3)}[g_{\mu\nu}, \psi_{\mu\nu}] = \frac{c}{6}\, \epsilon^{\mu\nu\rho}\, R_\mu^\alpha\, D_\nu\, \psi_{\rho\alpha}\, d^3x \ , \qquad (9.6.4)$$

c is the anomaly coefficient. From the structure of the anomaly, it is clear that the parity invariant topological YM part of the action cannot contribute to c. A non-trivial c can only come from the CS part $\Gamma_{CS+t.g.}$ of the action. Since this theory is equivalent to bosonic CS, we conclude that c is nothing but the coefficient of the framing anomaly of pure bosonic CS. For $SU(N)$ gauge theories this has been computed in [34]:

$$c_{SU(N)} = \frac{i}{4\,\pi}\frac{N^2 - 1}{k} \qquad (9.6.5)$$

3-dimensional diffeomorphisms are not anomalous. Hence, there exists a renormalization prescription which gives rise to an effective (non-local) action $\Gamma[g_{\mu\nu}]$ which transforms as a 3-form under 3-dimensional *generic* diffeomorphisms. To express this condition, it is useful to introduce the Bardeen-Zumino BRST operator S_{diff} [108] associated to 3-dimensional diffeomorphisms:

$$S_{\text{diff}} = \mathcal{L}_\xi - \{i_\xi, d\} \ , \qquad S_{\text{diff}}^2 = 0 \ , \qquad (9.6.6)$$

where $\xi = \xi^\mu\, \partial_\mu$ is the reparametrization ghost in 3-dimensions, and $\mathcal{L}_\xi$ denotes the action of the Lie derivative along ξ on the metric $g_{\mu\nu}$. The equation

$$S_{\text{diff}}\, \Gamma[g_{\mu\nu}] = 0 \ , \qquad (9.6.7)$$

precisely expresses the fact that the quantum action density $\Gamma[g_{\mu\nu}]$ transforms as 3-form under reparametrizations.

After these preliminaries, let us now make our main observation: When considering YM+CS topological theories on Seifert manifolds, one relaxes the request (9.6.7) of full 3-dimensional reparametrization invariance. One is satisfied with invariance under reparametrizations which preserve the Seifert structure: these are reparametrizations whose ghost fields ξ^μ commute with the Reeb vector $\bar\gamma^\mu$. Let us denote by

$$S_{\text{diff}}^{Seif} = \mathcal{L}_{\xi_{Seif}} - \{i_{\xi_{Seif}}, d\} \ , \qquad \mathcal{L}_{\xi_{Seif}} \bar\gamma = 0 \ , \qquad (9.6.8)$$

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the Bardeen-Zumino BRST operator associated to diffeomorphisms preserving $\bar{\gamma}^\mu$. One also restricts the topological gravity background fields to those left invariant under $\mathcal{L}_{\bar{\gamma}}$

$$\mathcal{L}_{\bar{\gamma}}\, g_{\mu\nu} = \mathcal{L}_{\bar{\gamma}}\, \psi_{\mu\nu} = \mathcal{L}_{\bar{\gamma}}\, \gamma^\mu = 0 \ . \tag{9.6.9}$$

To parametrize solutions of (9.6.9) it is useful to introduce systems of coordinates adapted to the Seifert structure associated to $\bar{\gamma}^\mu$:

$$
\begin{aligned}
(ds)^2_M &= \mathrm{e}^\sigma\, k \otimes k + g_{ij}\, dx^i \otimes dx^j = \\
&= \mathrm{e}^\sigma\, dy \otimes dy + 2\, \mathrm{e}^\sigma\, a_i\, dx^i \otimes dy + (g_{ij} + \mathrm{e}^\sigma\, a_i\, a_j)\, dx^i \otimes dx^j \ ,
\end{aligned}
\tag{9.6.10}
$$

where k is the contact 1-form

$$k \equiv dy + a_i\, dx^i \ , \tag{9.6.11}$$

dual to the Reeb vector field

$$i_{\bar{\gamma}}(k) = 1 \ , \tag{9.6.12}$$

σ, g_{ij} and a_i are fields on the two-dimensional surface Σ_2, associated to the Seifert fibration $\pi : M \to \Sigma_2$. The invariant $\psi_{\mu\nu}$ are analogously parametrized by fermions ζ, ψ_{ij} and ψ_i living on Σ_2, defined as follows:

$$\psi_{\mu\nu} = \begin{pmatrix} \mathrm{e}^\sigma\, \zeta & \mathrm{e}^\sigma\, \psi_i + \mathrm{e}^\sigma\, \zeta\, a_i \\ \mathrm{e}^\sigma\, \psi_i + \mathrm{e}^\sigma\, \zeta\, a_i & \psi_{ij} + \mathrm{e}^\sigma\, (\psi_i\, a_j + \psi_j\, a_i + \zeta\, a_i\, a_j) \end{pmatrix} \ . \tag{9.6.13}$$

Finally, the invariant ξ^μ and γ^μ can be written in terms of fields living on Σ_2 as

$$\xi^\mu = (\xi^0, \xi^i) \equiv (\xi^0, \vec{\xi}) \ , \qquad \gamma^\mu = (\gamma^0, \gamma^i) \equiv (\gamma^0, \vec{\gamma}) \ . \tag{9.6.14}$$

Therefore, in the Seifert case, the effective action $\Gamma^{Seif}[g_{ij}, \sigma, a_i]$ is a functional of the fields σ, g_{ij} and a_i, and the appropriate renormalization prescription writes

$$S_{\mathrm{diff}}^{Seif}\, \Gamma^{Seif}[g_{ij}, \sigma, a_i] = 0 \ . \tag{9.6.15}$$

The action $\Gamma[g_{\mu\nu}]$ which satisfies the (strong) prescription (9.6.7) defines, of course, once written in Seifert adapted coordinates, also an action $\Gamma^{Seif}[g_{ij}, \sigma, a_i]$ satisfiying the (weaker) Seifert renormalization condition (9.6.15):

$$\Gamma^{Seif}[g_{ij}, \sigma, a_i] = \Gamma[g_{\mu\nu}]\big|_{\mathcal{L}_{\bar{\gamma}}g_{\mu\nu}=0} \ . \tag{9.6.16}$$

This effective action satisfies the topological anomaly equation

$$s_{Seif} \int_{M_3} \Gamma^{Seif}[g_{ij}, \sigma, a_i] = \int_{M_3} A_1^{(3)}[g_{ij}, \sigma, a_i; \psi_{ij}, \zeta, \psi_i] \ , \tag{9.6.17}$$

where the r.h.s. is obtained from $A_1^{(3)}[g_{\mu\nu}, \psi_{\mu\nu}]$ by evaluating it for $\bar{\gamma}$-invariant fields (9.6.9). The BRST operator s_{Seif} in the l.h.s. of the equation above encodes topological gravity transformations which preserves the Seifert structure

$$s_{Seif}\, \xi^i = -\tfrac{1}{2}\mathcal{L}_{\vec{\xi}}\xi^i + \gamma^i \,, \qquad s_{Seif}\, \gamma^i = -\mathcal{L}_{\vec{\xi}}\gamma^i \,,$$
$$s_{Seif}\, g_{ij} = -\mathcal{L}_{\vec{\xi}}g_{ij} + \psi_{ij} \,, \qquad s_{Seif}\, \psi_{ij} = -\mathcal{L}_{\vec{\xi}}\psi_{ij} + \mathcal{L}_{\vec{\gamma}}\, g_{ij} \,,$$
$$s_{Seif}\, \xi^0 = -\mathcal{L}_{\vec{\xi}}\xi^0 + \gamma^0 \,, \qquad s_{Seif}\, \gamma^0 = -\mathcal{L}_{\vec{\xi}}\gamma^0 + \mathcal{L}_{\vec{\gamma}}\, \xi^0 \,,$$
$$s_{Seif}\, a_i = -\mathcal{L}_{\vec{\xi}}a_i - \partial_i\xi^0 + \psi_i \,, \quad s_{Seif}\, \psi_i = -\mathcal{L}_{\vec{\xi}}\psi_i + \partial_i\gamma^0 + \mathcal{L}_{\vec{\gamma}}\, a_i \,,$$
$$s_{Seif}\, \sigma = -\mathcal{L}_{\vec{\xi}}\sigma + \zeta \,, \qquad s_{Seif}\, \zeta = -\mathcal{L}_{\vec{\xi}}\zeta + \mathcal{L}_{\vec{\gamma}}\, \sigma \,. \qquad (9.6.18)$$

The invariant gravitational background fields split into three multiplets: one is the 2-dimensional topological gravity multiplet $(\xi^i, \gamma^i, g_{ij}, \psi_{ij})$. Then there is an abelian topological gauge multiplet $(\xi^0, \gamma^0, a_i\, \psi_i)$: their BRST properties are not just the "flat" ones, but they are modified by the coupling to 2-dimensional gravity. Finally there is also an uncharged scalar topological multiplet (σ, ζ): this too is coupled to 2-dimensional topological gravity.

Writing $A_1^{(3)}[g_{ij}, \sigma, a_i; \psi_{ij}, \zeta, \psi_i]$ in adapted coordinates

$$A_1^{(3)}[g_{ij}, \sigma, a_i; \psi_{ij}, \zeta, \psi_i] = \frac{1}{2}\mathcal{A}\, dy\, \epsilon_{ij}dx^i\, dx^j \,, \qquad (9.6.19)$$

one finds the following expression for $\mathcal{A}$:

$$\mathcal{A} = -\frac{1}{2}\sqrt{g}\, \psi_{ij}\, \mathrm{e}^{\sigma}\left[D^i\, D^j\, f + 3\, D^i f\, D^j\, \sigma + 2\, f\, D^i\sigma\, D^j\sigma + f\, D^i\, D^j\sigma\right] +$$
$$+\frac{1}{2}\sqrt{g}\, \mathrm{e}^{\sigma}\, \psi_i^i\left[D^2\, f + f\,(\frac{1}{2}\, R - \mathrm{e}^{\sigma}\, f^2) + \right.$$
$$\left.+\frac{5}{2}\, D_j f\, D^j\sigma + \frac{3}{2}\, f\, D_j\sigma\, D^j\sigma + \frac{3}{2}\, f\, D^2\, \sigma\right] +$$
$$+\frac{1}{2}\, \epsilon^{ij}\, \mathrm{e}^{\sigma}\, \psi_i\left[6\, \mathrm{e}^{\sigma}\, f\, D_j f - D_j R + \right.$$
$$+6\, \mathrm{e}^{\sigma}\, f^2\, D_j\sigma - R\, D_j\sigma - D_j\, D^2\, \sigma - D_j\sigma\, D^2\, \sigma\Big] +$$
$$+\sqrt{g}\, \mathrm{e}^{\sigma}\, \zeta\left[\mathrm{e}^{\sigma}\, f^3 - \frac{1}{2}\, f\, R - D_i\, f\, D^i\sigma + \right.$$
$$\left.-\frac{1}{2}\, f\, D_i\sigma\, D^i\sigma - f\, D^2\sigma - \frac{1}{2}\, D^2\, f\right] \,. \qquad (9.6.20)$$

In (9.6.20) we have introduced

$$f = \frac{\epsilon^{ij}}{\sqrt{g}}\, f_{ij} = \frac{\epsilon^{ij}}{\sqrt{g}}\,(\partial_i\, a_j - \partial_j\, a_i) \,, \qquad (9.6.21)$$

which is the scalar field dual to the $U(1)$ field strength $f^{(2)} \equiv da$.

The important fact, now, is that $A_1^{(3)}[g_{ij}, \sigma, a_i; \psi_{ij}, \zeta, \psi_i]$ is s_{Seif}-trivial[4]

$$\mathcal{A} = s_{Seif} \, \Gamma_{WZ}^{Seif}[g_{ij}, \sigma, a_i] \; , \tag{9.6.22}$$

where the Wess-Zumino action Γ_{WZ}^{Seif} is the following *local* functional

$$\Gamma_{WZ}^{Seif}[g_{ij}, \sigma, a_i] = \frac{1}{2} \sqrt{g} \, e^{2\sigma} \, f^3 - \sqrt{g} \, \frac{1}{2} \, e^{\sigma} \, f \, \hat{R} - \frac{1}{2} \sqrt{g} \, e^{\sigma} \, f \, D^2 \, \sigma \; . \tag{9.6.23}$$

$\Gamma_{WZ}^{Seif}[g_{ij}, \sigma, a_i]$ is a legitimate Wess-Zumino action since it is both *local* and invariant under reparametrizations which preserve the Seifert structure

$$S_{\text{diff}}^{Seif} \, \Gamma_{WZ}^{Seif}[g_{ij}, \sigma, a_i] = 0 \; . \tag{9.6.24}$$

It should be kept in mind, however, that $\Gamma_{WZ}^{Seif}[g_{ij}, \sigma, a_i]$ — unlike the non-local $\Gamma[g_{\mu\nu}]$ in eq. (9.6.16) — is *not* invariant under the full 3-dimensional S_{diff}.

Hence one can define the effective action

$$\tilde{\Gamma}^{Seif}[g_{ij}, \sigma, a_i] \equiv \Gamma^{Seif}[g_{ij}, \sigma, a_i] - \Gamma_{WZ}^{Seif}[g_{ij}, \sigma, a_i] \; , \tag{9.6.25}$$

which is both s_{Seif}-invariant — i.e. topological in the Seifert sense —

$$s_{Seif} \, \tilde{\Gamma}^{Seif}[g_{ij}, \sigma, a_i] = 0 \; , \tag{9.6.26}$$

and invariant under reparametrizations which preserve $\bar{\gamma}$

$$S_{\text{diff}}^{Seif} \, \tilde{\Gamma}^{Seif}[g_{ij}, \sigma, a_i] = 0 \; , \tag{9.6.27}$$

Summarizing, we have shown that it is always possible to define through Eq. (9.6.25) a quantum action density $\tilde{\Gamma}^{Seif}[g_{ij}, \sigma, a_i]$ which depends on the moduli parametrizing the Seifert structures (which we will characterize in Section 9.7) but not on the specific adapted metric which one picks to quantize the theory.

9.7 Moduli

We have seen that supersymmetric topological backgrounds correspond to solutions of the Killing equations

$$\mathcal{L}_{\bar{\gamma}} \, \bar{g}_{\mu\nu} \equiv \bar{D}_\mu \, \bar{\gamma}_\nu + \bar{D}_\nu \, \bar{\gamma}_\mu = 0 \; . \tag{9.7.1}$$

---

[4]The topological anomaly $A_1^{(3)}$ satisfies also: $A_1^{(3)} = S \, \Gamma_{GCS}^{(3)}[g]$, where $\Gamma_{GCS}[g]$ is the gravitational Chern-Simons action. Since $\Gamma_{GCS}^{(3)}[g]$ is *not* a globally defined 3-form, the anomaly is indeed non-trivial in the 3-dimensional sense. The precise relation between $\Gamma_{GCS}^{(3)}[g]$ and $\Gamma_{WZ}^{Seif}[g_{ij}, \sigma, a_i]$ is described in Appendix F.

Given a solution $\{\bar{g}_{\mu\nu}, \bar{\gamma}^{\mu}\}$ of (9.7.1) we want explore nearby supersymmetric backgrounds $\{\bar{g}_{\mu\nu} + \delta g_{\mu\nu}, \gamma^{\mu} + \delta\gamma^{\mu}\}$. The deformations $\{\delta g_{\mu\nu}, \delta\gamma^{\mu}\}$ must satisfy the linear equation

$$\mathcal{L}_{\bar{\gamma}} \, \delta g_{\mu\nu} + \mathcal{L}_{\delta\gamma} \, \bar{g}_{\mu\nu} = 0 \; . \tag{9.7.2}$$

Let us introduce the vector space

$$V_0 = \Gamma(TM_3) \oplus \mathrm{Sym}_2(TM_3) \; , \tag{9.7.3}$$

where $\Gamma(TM_3)$ is the space of vector fields on M_3 and $\mathrm{Sym}_2(TM_3)$ the space of 2-index symmetric tensors on M_3. The deformation equation (9.7.2) describes therefore the kernel of the linear operator

$$\begin{aligned} Q_0 &: V_0 \to V_1 \; , \\ Q_0 &: (\delta\gamma^{\mu}, \delta g_{\mu\nu}) \to \mathcal{L}_{\bar{\gamma}} \, \delta g_{\mu\nu} + \mathcal{L}_{\delta\gamma} \, \bar{g}_{\mu\nu} \; , \end{aligned} \tag{9.7.4}$$

where

$$V_1 = \mathrm{Sym}_2(TM_3) \; . \tag{9.7.5}$$

We are interested in characterizing physical deformations, i.e. solutions of this equation *modulo gauge equivalences*. Gauge-invariance includes infinitesimal diffeomorphisms

$$(\delta\gamma^{\mu}, \delta g_{\mu\nu}) \sim (\delta\gamma^{\mu}, \delta g_{\mu\nu}) + (\mathcal{L}_{\xi}\delta\gamma^{\mu}, \mathcal{L}_{\xi}\delta g_{\mu\nu}) \; , \tag{9.7.6}$$

where ξ^{μ} is a vector field on M_3. But $\mathcal{L}_{\bar{\gamma}}$-invariant topological deformations of the metric should also be treated as a gauge invariances

$$(\delta\gamma^{\mu}, \delta g_{\mu\nu}) \sim (\delta\gamma^{\mu}, \delta g_{\mu\nu} + \psi_{\mu\nu}) \; , \tag{9.7.7}$$

for any $\mathcal{L}_{\bar{\gamma}}$-invariant $\psi_{\mu\nu}$[5]

$$\psi_{\mu\nu} \in \mathrm{Sym}_2^{inv}(TM_3) \equiv \{\psi_{\mu\nu} \in \mathrm{Sym}_2(TM_3) \big| \, \mathcal{L}_{\bar{\gamma}}\psi_{\mu\nu} = 0\} \; . \tag{9.7.8}$$

We can therefore define a linear operator Q_{-1} which captures both gauge equivalences (9.7.6) and (9.7.7)

$$\begin{aligned} Q_{-1} &: V_{-1} \to V_0 \; , \\ Q_{-1} &: \{\xi^{\mu}, \psi_{\mu\nu}\} \to \{\mathcal{L}_{\xi}\bar{g}_{\mu\nu} + \psi_{\mu\nu}, \mathcal{L}_{\xi}\bar{\gamma}^{\mu}\} \; , \end{aligned} \tag{9.7.9}$$

where

$$V_{-1} = \Gamma(TM_3) \oplus \mathrm{Sym}_2^{inv}(TM_3) \; . \tag{9.7.10}$$

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[5]The explicit form for invariant $\psi_{\mu\nu}$, in adapted coordinates, is given in Eq. (9.6.13).

We have

$$Q_0 \, Q_{-1} = 0 \; . \qquad (9.7.11)$$

One can consider therefore the short exact sequence

$$0 \to V_{-1} \xrightarrow{Q_0} V_0 \xrightarrow{Q_0} V_1 \to 0 \; . \qquad (9.7.12)$$

The associated cohomology space

$$H_0(Q) = \frac{\ker Q_0}{\operatorname{Im} Q_{-1}} \; , \qquad (9.7.13)$$

describes therefore inequivalent deformations around the Seifert structure $\{\bar{g}_{\mu\nu}, \bar{\gamma}^\mu\}$. The kernel of Q_{-1}

$$\ker Q_{-1} = H_{-1}(Q) = \{(\xi^\mu, \psi_{\mu\nu}) : [\xi, \bar{\gamma}] = 0, \psi_{\mu\nu} = -\mathcal{L}_\xi \, \bar{g}_{\mu\nu}\} \; , \qquad (9.7.14)$$

is isomorphic to the commutant $C_{\bar{\gamma}}$ of $\bar{\gamma}^\mu$ in the Lie algebra of vectors fields on M_3:

$$C_{\bar{\gamma}} = \{\gamma^\mu \in \Gamma(TM_3) : [\gamma, \bar{\gamma}] = 0\} \simeq H_{-1}(Q) \; . \qquad (9.7.15)$$

Consider now the map between $C_{\bar{\gamma}}$ and $\operatorname{Sym}_2^{inv}(TM_3)$:

$$\varphi : C_{\bar{\gamma}} \to \operatorname{Sym}_2^{inv}(TM_3) \; ,$$
$$\varphi : \gamma^\mu \to \mathcal{L}_\gamma \bar{g}_{\mu\nu} \; . \qquad (9.7.16)$$

The kernel of φ is made of the isometries of $\bar{g}_{\mu\nu}$ which commute with $\bar{\gamma}^\mu$

$$\ker \varphi = \{\gamma : [\gamma, \bar{\gamma}] = 0, \; \mathcal{L}_\gamma \bar{g}_{\mu\nu} = 0\} \subset C_{\bar{\gamma}} \; . \qquad (9.7.17)$$

The cokernel of φ is, on the other hand, characterized by the $\mathcal{L}_{\bar{\gamma}}$-invariant $\psi_{\mu\nu}$'s which are orthogonal to $\operatorname{Img} \varphi$

$$0 = \int_{M_3} \gamma^\nu \bar{D}^\mu \, \psi_{\mu\nu} = \int_{M_3} \gamma^\nu g_{\nu\mu} \, v^\mu \equiv \langle \gamma, v \rangle \qquad \forall \gamma \in C_{\bar{\gamma}} \; , \qquad (9.7.18)$$

where

$$v^\mu \equiv \bar{g}^{\mu\lambda} \bar{D}^\nu \, \psi_{\lambda\nu} \; . \qquad (9.7.19)$$

The vector v^μ is $\mathcal{L}_{\bar{\gamma}}$-invariant, since $\bar{g}_{\mu\nu}$ and $\psi_{\mu\nu}$ are:

$$\mathcal{L}_{\bar{\gamma}} v^\mu = 0 = [\bar{\gamma}, v] \; . \qquad (9.7.20)$$

Hence $v^\mu \in C_{\bar{\gamma}}$. But since, according to (9.7.18) v^μ is orthogonal to whole $C_{\bar{\gamma}}$, it vanishes

$$v^\mu = \bar{g}^{\mu\lambda} \bar{D}^\nu \, \psi_{\lambda\nu} = 0 \; . \qquad (9.7.21)$$

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We conclude that

$$\text{coker}\,\varphi = \{\psi_{\mu\nu} : \mathcal{L}_{\bar{\gamma}}\,\psi_{\mu\nu} = 0, \bar{D}^{\mu}\,\psi_{\mu\nu} = 0\} , \qquad (9.7.22)$$

and therefore

$$\text{Sym}_2^{inv}(TM_3) = \text{Im}\,\varphi \oplus \text{coker}\,\varphi \simeq \frac{C_{\bar{\gamma}}}{\ker\,\varphi} \oplus \text{coker}\,\varphi . \qquad (9.7.23)$$

Let us now consider the cokernel of Q_0: it is characterized by the equation

$$\int_{M_3} \psi^{\mu\nu} \left(\mathcal{L}_{\bar{\gamma}}\,\delta g_{\mu\nu} + \mathcal{L}_{\delta\gamma}\,\bar{g}_{\mu\nu} \right) = 0 \qquad \forall\, \delta\gamma^{\mu} \in \Gamma(TM_3) \text{ and } \forall\, \delta g_{\mu\nu} \in \text{Sym}_2(TM_3) \quad (9.7.24)$$

This implies

$$\mathcal{L}_{\bar{\gamma}}\,\psi^{\mu\nu} = \bar{D}^{\mu}\,\psi_{\mu\nu} = 0 . \qquad (9.7.25)$$

In other words

$$\text{coker}\,Q_0 = H_1(Q) = \text{coker}\,\varphi . \qquad (9.7.26)$$

The exactness of the sequence (9.7.12) implies therefore

$$T_{\{\bar{g},\bar{\gamma}\}}\mathcal{M} \simeq H_0(Q) = \frac{H_{-1}(Q) \oplus H_1(Q)}{\text{Sym}_2^{inv}(TM_3)} \simeq \ker\,\varphi , \qquad (9.7.27)$$

where $T_{(\bar{\gamma},\bar{g})}\mathcal{M}$ is the tangent to the space of physical moduli of the theory at a point $\{\bar{g}_{\mu\nu}, \bar{\gamma}^{\mu}\}$.

Hence $\bar{g}_{\mu\nu}$-isometries which commute with $\bar{\gamma}^{\mu}$ are in one-to-one correspondence with non-trivial deformations of a given Seifert structure $(\bar{\gamma}^{\mu}, \bar{g}_{\mu\nu})$. $\bar{\gamma}^{\mu}$ itself, of course, is always one of such isometries. The corresponding deformation is a rescaling of $\bar{\gamma}^{\mu}$. Since a rescaling of $\bar{\gamma}^{\mu}$ in the YM + CS action (9.3.27) can be reasorbed in a rescaling of the field $\tilde{A}$, the YM + CS partition function does not depend on this kind of deformation. In conclusion the parameter space which the YM + CS partition function on a Seifert manifold depends on is the quotient space

$$\ker\,\varphi/\sim , \qquad (9.7.28)$$

where the equivalence relation is

$$\gamma' \sim \gamma + \alpha\bar{\gamma}, \qquad \gamma, \gamma' \in \ker\,\varphi , \qquad (9.7.29)$$

with α constant.

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9.8 The topological anomaly for the squashed spheres

Let us now consider the squashed metric on S_3[6]:

$$ds^2 = (l^2 \sin^2 \theta + \tilde{l}^2 \cos^2 \theta) \, d\theta^2 + l^2 \cos^2 \theta \, d\phi_1^2 + \tilde{l}^2 \sin^2 \theta \, d\phi_2^2 \,, \qquad (9.8.1)$$

where $\phi_{1,2} \in [0, 2\pi]$ and $0 \leq \theta \leq \frac{\pi}{2}$, are the Hopf coordinates on S_3.

The vector field

$$\bar{\gamma}^\mu \partial_\mu = \frac{1}{l} \, \partial_{\phi_1} + \frac{1}{\tilde{l}} \, \partial_{\phi_2} = \frac{\partial}{\partial y} \,, \qquad (9.8.2)$$

is, for each value of the squashing parameters $(l, \tilde{l})$, an isometry of $\bar{g}_{\mu\nu}(x; l, \tilde{l})$. A system of coordinates (y, α, β) adapted to the Seifert structure corresponding to $(l, \tilde{l})$ is defined by the relations

$$\theta = \frac{\alpha}{2} \,, \qquad \phi_1 = \frac{y}{l} + \frac{\beta}{2l} + \frac{\epsilon(\alpha, \beta)}{l} \,, \qquad \phi_2 = \frac{y}{\tilde{l}} - \frac{\beta}{2\tilde{l}} + \frac{\epsilon(\alpha, \beta)}{\tilde{l}} \,, \qquad (9.8.3)$$

and their inverse

$$y = \frac{l \, \phi_1 + \tilde{l} \, \phi_2}{2} - \epsilon(\alpha, \beta) \,, \qquad \beta = l \, \phi_1 - \tilde{l} \, \phi_2 \,, \qquad \alpha = 2\theta \,. \qquad (9.8.4)$$

$\epsilon(\alpha, \beta)$ is an arbitrary local function which corresponds to abelian gauge transformations along the fiber of the fibration. The squashed metric (9.8.1) writes in these adapted coordinates as follows

$$ds^2 = (dy + a)^2 + \frac{1}{4} \left[\frac{1}{2} (l^2 + \tilde{l}^2 + (\tilde{l}^2 - l^2) \cos\alpha) \, d\alpha^2 + \sin^2\alpha \, d\beta^2 \right] \,, \qquad (9.8.5)$$

where the abelian gauge connection a and its field strength $f^{(2)}$ are given by

$$a = \frac{1}{2} \cos\alpha \, d\beta + d\epsilon \,,$$

$$f^{(2)} = da = -\frac{1}{2} \sin\alpha \, d\alpha \, d\beta \,. \qquad (9.8.6)$$

The curvature R_2 of the 2-dimensional metric $(g_2)_{ij}$ on the S_2 base of the Seifert fibration is

$$R_2 = \frac{8}{l^2} \frac{(b^4 - 1) \cos\alpha + 2(b^4 + 1)}{((b^4 - 1) \cos\alpha + b^4 + 1)^2} \,,$$

$$\sqrt{g_2} = l \frac{1}{4} \sin\alpha \sqrt{\frac{b^4 - 1}{2} \cos\alpha + \frac{b^4 + 1}{2}} \,, \qquad (9.8.7)$$

---

[6]Our definitions and convetions for the Hopf coordinates for the squashed sphere are reviewed in Appendix E.

where we introduced the ratio

$$b^2 \equiv \frac{\tilde{l}}{l} \ . \tag{9.8.8}$$

The scalar field which is dual to the abelian field strength is therefore

$$f = \frac{\epsilon^{ij} \, \partial_i \, a_j}{\sqrt{g_2}} = \frac{2\sqrt{2}}{l \, \sqrt{(b^4 - 1) \cos \alpha + b^4 + 1}} \ , \qquad \sqrt{g_2} \, f = \frac{1}{2} \sin \alpha \ . \tag{9.8.9}$$

We learnt in the previous Section that, given $\bar{g}_{\mu\nu}(x; l, \tilde{l})$ and $\bar{\gamma} = \frac{1}{l} \partial_{\phi_1} + \frac{1}{\tilde{l}} \partial_{\phi_2}$, the deformations of the Seifert structure are associated to the isometries which commute with $\bar{\gamma}$ modulo $\bar{\gamma}$. For generic $(l, \tilde{l})$ the isometries which commute with $\bar{\gamma}$ are ∂_{ϕ_1} and ∂_{ϕ_2}. Hence we see that b^2 parametrizes precisely the inequivalent deformations of the Seifert structure around a generic point $b \neq 1$. The point $b = 1$ corresponds to the "round" sphere, which has an enhanced symmetry $SU(2)_L \times SU(2)_R$. Around this point more general deformations are possible, since the isometries which commute with, let us say, J_3^R form the full $SU(2)_L$.

Let us compute the topological anomaly for the squashed sphere metric $\bar{g}_{\mu\nu}(x; l, \tilde{l})$. Since $A_1^{(3)}[g_{\mu\nu}, \psi_{\mu\nu}]$ depends only on the conformal class of the metric, we can take, without loss of generality

$$l = 1 \ , \qquad \tilde{l} = b^2 \tag{9.8.10}$$

and put $\bar{g}_{\mu\nu}(x; 1, b^2) \equiv \bar{g}_{\mu\nu}(x; b)$. Then

$$\bar{\psi}_{\mu\nu}(x; b) = b \, \partial_b \, \bar{g}_{\mu\nu}(x; b) = \begin{pmatrix} 4 \, b^4 \cos^2 \theta & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 4 \, b^4 \sin^2 \theta \ , \end{pmatrix} \tag{9.8.11}$$

where the b-derivative is taken by keeping the Hopf coordinates constant. It is easy to verify that the topological anomaly for these backgrounds vanishes for all b's:

$$A_1^{(3)}[\bar{g}, \bar{\psi}] = 0 \ . \tag{9.8.12}$$

This implies that the effective action $\Gamma[g_{\mu\nu}]$ evalutated for the squashed sphere metric $\bar{g}_{\mu\nu}(x; b)$ is independent of b:

$$b \, \partial_b \Gamma[\bar{g}_{\mu\nu}(x; b)] = b \, \partial_b \, \Gamma^{Seif}[\bar{g}_{ij}(X; b), \bar{\sigma}(X; b), \bar{a}_i(X; b)] = 0 \ , \tag{9.8.13}$$

where $X(x; b) \equiv (y, \alpha, \beta)$ are the coordinates adapted to the Seifert structure parametrized by b^2.

However, we explained in Section 9.6 that $\Gamma[g_{\mu\nu}]$ is *not* the action renormalized with the correct Seifert prescription (9.6.15). The action $\tilde{\Gamma}^{Seif}[g_{ij}, \sigma, a_i]$ renormalized according

to the Seifert prescription is given by (9.6.25). When this latter action is evaluated on $\bar{g}_{\mu\nu}(x;b)$, one obtains

$$
\tilde{\Gamma}^{Seif}[\bar{g}_{ij},\bar{\sigma},\bar{a}_i] =
$$
$$
= \Gamma^{Seif}[\bar{g}_{ij}(X;b),\bar{\sigma}(X;b),\bar{a}_i(X;b)] - \Gamma_{WZ}^{Seif}[\bar{g}_{ij}(X;b),\bar{\sigma}(X;b),\bar{a}_i(X;b)] \,.\text{(9.8.14)}
$$

We have just shown that, due to (9.8.12), the first (non-local) term in the r.h.s. of the equation above is b-independent. But the second (local) one is not. Indeed, by plugging (9.8.7) inside(9.6.23) one computes

$$
\Gamma_{WZ}^{Seif}[\bar{g}_{ij}(X;b),\bar{\sigma}(X;b),\bar{a}_i(X;b)] = \frac{c}{6}\int\frac{1}{2}\sqrt{g}\,f\,(f^2 - R_2) =
$$
$$
= -\frac{c}{6}\,(2\,\pi)^2\,\left(b^2 + \frac{1}{b^2}\right)\,, \tag{9.8.15}
$$

which therefore encodes the whole dependence of the Seifert partition function on b^2:

$$
Z^{\text{squashed}}(b) = \mathrm{e}^{\tilde{\Gamma}^{Seif}[\bar{g}_{ij},\bar{\sigma},\bar{a}_i]} = \mathrm{e}^{-\frac{4\,\pi^2\,c}{3}\left[\frac{1}{2}\left(b^2+\frac{1}{b^2}\right)-1\right]}\,Z^{\text{squashed}}(1)\,, \tag{9.8.16}
$$

where $Z(1)$ is the partition function on the "round" sphere, corresponding to $b=1$.

Let us compare the anomaly equation (9.8.16) with the YM+CS partition function on the squashed sphere computed directly by means of localization in [107], taking for simplicity the case of $SU(2)$ gauge group[7]

$$
Z^{\text{squashed}}(b) = \int_{-\infty}^{\infty}\frac{dx}{2\,\pi\,i}\,\mathrm{e}^{-\frac{i\,k\,x^2}{8\,\pi}}\,\sinh\frac{i\,x\,b}{2}\,\sinh\frac{i\,x}{2\,b}\,. \tag{9.8.17}
$$

Here the exponential in the integrand comes from the value of the action on the saddle point, while the hyperbolic sine factors are the results of the 1-loop determinants. Expressing the sinh factors in terms of exponentials, the integration reduces to the sum of Gaussian integrals:

$$
Z^{\text{squashed}}(b) = \mathrm{e}^{\frac{i\,\pi}{2\,k}\left(b^2+\frac{1}{b^2}\right)-\frac{i\,\pi}{4}}\,Z_{CS}^{CFT}\,, \tag{9.8.18}
$$

where

$$
Z_{CS}^{CFT} = \sqrt{\frac{2}{k}}\,\sin\frac{\pi}{k} \tag{9.8.19}
$$

is the round sphere $SU(2)$ CS partition function, obtained by means of surgery from the CFT modular transformations of the WZW 2d conformal model. The result (9.8.18) obtained by direct computation agrees with our prediction (9.8.16) obtained from the

---

[7]The parameter k which appears in this and in the following formulas should be taken as the "shifted" $l + 2$, with respect to the level l of the Kac-Moody current algebra whose conformal blocks map to the states obtained from canonically quantizing the theory.

topological anomaly and the Wess-Zumino Seifert action, once we insert the topological anomaly coefficient for $SU(2)$[8]

$$c_{SU(2)} = \frac{i}{4\,\pi}\frac{2^2 - 1}{k} = \frac{i}{4\,\pi}\frac{3}{k}\ .\tag{9.8.20}$$

9.9 Discussion

The current paradigm for localization relies on spinorial global supercharges. Since the fate and properties of quantum global symmetries are best studied by introducing background fields coupled to currents, the same paradigm has lead to studying the coupling of "physical" supersymmetric theories to off-shell supergravity. In particular the search for globally supersymmetric models has been reduced to the study of generalized covariantly constant spinors.

In this Chapter we proposed an alternative route. Localization is naturally understood in terms of topological scalar supercharges — i.e. in terms of topological theories and BRST symmetries. In this framework it is the coupling of topological field theories to *topological gravity*, not supergravity, which is relevant. In particular we worked out the coupling of both CS and topological Yang Mills theory — i.e. of a generic vector twisted supermultiplet — to topological gravity. The BRST structure of the Chern-Simons supermultiplet looks very different from that of the topological YM theory, when the latter is presented in its familiar formulation valid in arbitrary dimension. We exhibited, however, a new formulation of the BRST transformations of topological YM in 3d purely in terms of the CS supermultiplet. This allowed us to provide a unique (anomalous) Ward identity which characterizes the coupling of a 3d generic twisted vector supermultiplet to topological gravity.

One first advantage of the topological gravity viewpoint is that the structure of topological gravity is the same in all dimensions, a fact which makes the characterization of supersymmetric bosonic backgrounds straightforward. For example, in the context of 3d gauge theories, we have seen that the Seifert condition emerges quite immediately without the need to go through generalized covariantly constant spinors or similar indirect routes peculiar to other approaches. Moreover we have found that, in the 3d context, the off-shell coupling of topological (YM+CS) gauge theories to topological gravity is easily achieved by suitably covariantizing the "rigid" coboundary BRST operator with a universal term which is the form-contraction with the super-ghost field of topological gravity. We also discovered that the anti-fields of the BV formulations of CS theory

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[8]The $\frac{i}{4\,\pi}$ factor which multiplies the anomaly comes from the normalization of the Chern-Simons action, which we took to be $\frac{i}{4\,\pi\,k}$.

are nothing but the auxiliary fields which are required to close off-shell the topological supersymmetry algebra.

But the real payoff of the topological approach was that it made straightforward to identify the subset of local topological transformations which preserves the Seifert backgrounds. These turned out to be 2d topological gravity transformations coupled to topological abelian gauge transformations. This allowed us to give a cohomological characterization of the Seifert background moduli. Moreover we were able to explicitly solve the anomalous Ward identity associated to topological transformations of the gravitational background. The solution involved a Wess-Zumino local action, invariant under the reparametrizations which preserve the Seifert structure.

The triviality of the topological (framing) 3d anomaly when restricted on Seifert backgrounds shows, rigorously and in a completely regularization independent way, that the quantum effective action of the gauge theory on Seifert manifold depends on the Seifert moduli but not on the specific metric adapted to the Seifert structure which one picks to quantize the theory. The Wess-Zumino Seifert action also completely determines the dependence of the partition function on the Seifert moduli. We explicitly showed this in the case of the squashed sphere, for which we recovered the dependence of the partition function on the squashing modulus without computing any functional determinant.

Our discussion in this Chapter was restricted to (twisted) vector supermultiplets in 3d. It would be interesting to extend our results to (twisted) chiral matter. To do this it would be necessary to work out the coupling of topological chiral matter to topological gravitational backgrounds: something which, to our knowledge, has not be done yet[9]. The dependence on the Seifert moduli of the quantum effective action of chiral theories is considerably more complicated than that of vector supermultiplets. We expect therefore that the coupling of topological chiral theories to topological gravitational backgrounds involves some new ingredients. It would be equally interesting to apply our methods to higher dimensions. Realizing this program might reduce the computation of the dependence on the moduli of quantum effective actions of localizable theories to the solution of appropriate anomalous Ward identities.

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[9]The BRST structure of *rigid* topological chiral matter in 3d has been described in [109].

Appendix A

Supercharges for AdS$_7$ solutions

At the beginning of section 4.2.1 we reviewed an old argument that shows how a solution of the form AdS$_7 \times M_3$ can also be viewed as a solution of the type Mink$_6 \times M_4$. In this appendix we show how the AdS$_7 \times M_3$ supercharges get translated in the Mink$_6 \times M_4$ framework.

A decomposition of gamma matrices appropriate to six-dimensional compactifications reads

$$\gamma_\mu^{(6+4)} = e^{A_4}\gamma_\mu^{(6)} \otimes 1 \ , \qquad \gamma_{m+5}^{(6+4)} = \gamma^{(6)} \otimes \gamma_m^{(4)} \ . \tag{A.0.1}$$

Here $\gamma_\mu^{(6)}$, $\mu = 0, \ldots, 5$, are a basis of six-dimensional gamma matrices, while $\gamma_m^{(4)}$, $m = 1, \ldots, 4$ are a basis of four-dimensional gamma matrices. For a supersymmetric Mink$_6 \times M_4$ solution, the supersymmetry parameters can be taken to be

$$\begin{aligned}
\epsilon_1^{(6+4)} &= \zeta_+^0 \otimes \eta_+^1 + \zeta_+^{0\,c} \otimes \eta_+^{1\,c} \ , \\
\epsilon_2^{(6+4)} &= \zeta_+^0 \otimes \eta_\mp^2 + \zeta_+^{0\,c} \otimes \eta_\mp^{2\,c} \ ,
\end{aligned} \tag{A.0.2}$$

where ζ_+ is a constant spinor; $_\mp$ denotes the chirality, and ^{c} Majorana conjugation both in six and four dimensions. Supersymmetry implies that the norms of the internal spinors satisfy $||\eta^1||^2 \pm ||\eta^2||^2 = c_\pm e^{\pm A_4}$, where $c_\pm$ are constant.

On the other hand, for seven-dimensional compactifications a possible gamma matrix decomposition reads

$$\begin{aligned}
\gamma_\mu^{(7+3)} &= e^{A_3}\gamma_\mu^{(7)} \otimes 1 \otimes \sigma_2 \ , \\
\gamma_{i+6}^{(7+3)} &= 1 \otimes \sigma_i \otimes \sigma_1 \ .
\end{aligned} \tag{A.0.3}$$

This time $\gamma_\mu^{(7)}$, $\mu = 0, \ldots, 6$, are a basis of seven-dimensional gamma matrices, and σ_i, $i = 1, 2, 3$, are a basis of gamma matrices in three dimensions (which in flat indices can be taken to be the Pauli matrices). For a supersymmetric solution of the form AdS$_7 \times M_3$,

the supersymmetry parameters are now of the form

$$\epsilon_1^{(7+3)} = (\zeta \otimes \chi_1 + \zeta^c \otimes \chi_1^c) \otimes v_+ \; ,$$
$$\epsilon_2^{(7+3)} = (\zeta \otimes \chi_2 \mp \zeta^c \otimes \chi_2^c) \otimes v_{\mp} \; . \tag{A.0.4}$$

Here, $\chi_{1,2}$ are spinors on M_3, with $\chi_{1,2}^c \equiv B_3 \chi_{1,2}^*$ their Majorana conjugates; a possible choice of B_3 is $B_3 = \sigma_2$. ζ is a spinor on AdS$_7$, and $\zeta^c \equiv B_7 \zeta^*$ is its Majorana conjugate; there exists a choice of B_7 which is real and satisfies $B_7 \gamma_\mu = \gamma_\mu^* B_7$. (It also obeys $B_7 B_7^* = -1$, which is the famous statement that one cannot impose the Majorana condition in seven Lorentzian dimensions.) The ten-dimensional conjugation matrix can then be taken to be $B_{10} = B_7 \otimes B_3 \otimes \sigma_3$; the last factor in (A.0.4), $v_\pm$, are then spinors chosen in such a way as to give the $\epsilon_i^{(7+3)}$ the correct chirality, and to make them Majorana; with the above choice of B_{10}, $v_+ = \frac{1}{\sqrt{2}} \binom{1}{-1}$, $v_- = \frac{1}{\sqrt{2}} \binom{1}{1}$. The minus sign (for the IIA case) in front of the term $\zeta^c \otimes \chi_2^c$ in (A.0.4) is due to the fact that, both in seven Lorentzian and three Euclidean dimensions, conjugation does not square to one: $(\zeta^c)^c = -\zeta$, $(\chi^c)^c = -\chi$.

The presence of the cosmological constant in seven dimensions means that ζ is not constant, but rather that it satisfies the so-called Killing spinor equation, which for $R_{\text{AdS}} = 1$ reads

$$\nabla_\mu \zeta = \frac{1}{2} \gamma_\mu^{(7)} \zeta \; . \tag{A.0.5}$$

One class of solutions to this equation [110, 111] is simply of the form

$$\zeta_+ = \rho^{1/2} \zeta_+^0 \; . \tag{A.0.6}$$

The coordinate ρ appears in (4.2.6), which expresses AdS$_7$ as a warped product of Mink$_6$ and $\mathbb{R}$. ζ_+^0 is a spinor constant along Mink$_6$ and such that $\gamma_{\hat\rho} \zeta_+^0 = \zeta_+^0$ (the hat denoting a flat index).

Just like for Mink$_6 \times M_4$, supersymmetry again implies that the norms of the internal spinors $\chi^{1,2}$ should be related to the warping function: $||\chi_1||^2 \pm ||\chi_2||^2 = c_\pm e^{\pm A_3}$, where $c_\pm$ are constant. We will now see, however, that for AdS$_7 \times M_3$ actually $c_- = 0$. To this end, we will use the equation

$$d\tilde{K} = \iota_K H \tag{A.0.7}$$

of the ten-dimensional system (3.2.4). Recall that K and $\tilde{K}$ are the ten-dimensional vector and one-form defined by $K = \frac{1}{64}(\bar\epsilon_1 \gamma_M^{(10)} \epsilon_1 + \bar\epsilon_2 \gamma_M^{(10)} \epsilon_2) dx^M$ and $\tilde{K} = \frac{1}{64}(\bar\epsilon_1 \gamma_M^{(10)} \epsilon_1 - \bar\epsilon_2 \gamma_M^{(10)} \epsilon_2) dx^M$. Plugging the decomposed spinors (A.0.4) in these definitions and calling $\beta_1 = e^{A_3}(\frac{1}{8}\bar\zeta \gamma_\mu^{(7)} \zeta) dx^\mu$, the part of (A.0.7) along AdS$_7$ leads to $e^{A_3} d_7 \beta_1(||\chi_1||^2 - ||\chi_2||^2) = (d_7\beta_1)c_- = 0$, where d_7 is the exterior derivative along AdS$_7$. (The right hand side does not contribute, because H has only internal components.) On the other hand, using the Killing spinor equation (A.0.5) in AdS$_7$, we have that $d_7\beta_1 = e^{2A_3}(\bar\zeta \gamma_{\mu\nu}^{(7)} \zeta) dx^{\mu\nu} \equiv \beta_2$. A spinor in seven dimensions can be in different orbits (defining an $SU(3)$ or an $SU(2) \ltimes \mathbb{R}^5$

structure [112,113]), but for none of them the bilinear β_2 is identically zero. Consequently, the norms of the two Killing spinors have to be equal, namely $c_- = 0$.

Let us now see how to translate the spinors ϵ_i for an AdS$_7 \times M_3$ solution into a language relevant for Mink$_6 \times M_4$. First, we split the seven-dimensional gamma matrices $\gamma_\mu^{(7)}$; the first six give a basis of gamma matrices in six dimensions, $\tilde{\gamma}_\mu^{(6)} = \rho\gamma_\mu^{(7)}$, $\mu = 0, \dots, 5$, while the radial direction, $\gamma_{\hat\rho}^{(7)} = \gamma^{(6)}$ becomes the chiral gamma in six dimensions. (The hat denotes a flat index.) This split is by itself not enough to turn (A.0.3) into (A.0.1), because the three-dimensional gamma's in (A.0.3) have no $\gamma^{(6)}$ in front. This can be cured by applying a change of basis:

$$\gamma_M^{(6+4)} = O\gamma_M^{(7+3)}O^{-1} \,, \qquad O = \frac{1}{\sqrt{2}}(1 - i\gamma_{\hat\rho}^{(7+3)}) \,, \qquad (A.0.8)$$

with, however, a change of basis in six dimensions: $\gamma_\mu^{(6)} \rightarrow -i\gamma^{(6)}\gamma_\mu^{(6)}$. Likewise, the spinors (A.0.4) are related to (A.0.2) by

$$\epsilon_i^{(6+4)} = O\epsilon_i^{(7+3)} \,, \qquad (A.0.9)$$

if we take

$$\eta_1 = \rho^{1/2}\,\chi_1 \otimes v_+ = \frac{1}{\sqrt{2}}\rho^{1/2}\,\chi_1 \otimes \begin{pmatrix} 1 \\ -1 \end{pmatrix} \,, \qquad \eta_2 = \rho^{1/2}\,\chi_2 \otimes v_\mp = \frac{1}{\sqrt{2}}\rho^{1/2}\,\chi_2 \otimes \begin{pmatrix} 1 \\ \pm 1 \end{pmatrix} \,.$$
$$(A.0.10)$$

Notice that the two η^i have equal norm, because the χ^i have equal norm, as shown earlier. Moreover, since the norm of the χ^i is $e^{A_3/2}$, and because of the factor $\rho^{1/2}$ in (A.0.10), the η^i have norm equal to $\rho^{1/2}e^{A_3/2}$; recalling (4.2.7), this is equal to $e^{A_4/2}$, as it should.

Besides (A.0.6), there is also a second class of solution to the Killing spinor equation $\nabla_\mu\zeta = \frac{1}{2}\gamma_\mu^{(7)}\zeta$ on AdS$_7$: it reads $\zeta = (\rho^{-1/2} + \rho^{1/2}x^\mu\gamma_\mu^{(7)})\zeta_-^0$, where now $\gamma_{\hat\rho}\zeta_-^0 = -\zeta_-^0$. If we plug this into (A.0.4) and use the above procedure (A.0.9) to translate it in the Mink$_6 \times M_4$ language, we find a generalization of (A.0.2) where both a positive and negative chirality six-dimensional spinor appear (namely, $x^\mu\gamma_\mu\zeta_-^0$ and ζ_-^0) instead of just a positive chirality spinor ζ_+^0. Because of the $x^\mu\gamma_\mu$ factor, this spinor Ansatz would break Poincaré invariance if used by itself; if four supercharges of the form (A.0.2) are preserved, Poincaré invariance is present, and these additional supercharges simply signal that an AdS$_7 \times M_3$ solution is $\mathcal{N} = 2$ in terms of Mink$_6 \times M_4$.

Appendix B

AdS$_6$ solutions in eleven-dimensional supergravity

We will show here that there are no AdS$_6 \times M_5$ solutions in eleven-dimensional super-gravity.[1] This case is easy enough that we will deal with it by using the original fermionic form of the supersymmetry equations, without trying to reformulate them in terms of bilinears as we did in the main text for IIB.

We take the eleven-dimensional metric to have the warped product form

$$ds_{11}^2 = e^{2A}ds_{\text{AdS}_6}^2 + ds_{M_5}^2 \ . \tag{B.0.1}$$

In order to preserve the $SO(2,5)$ invariance of AdS$_6$ we take the warping factor to be a function of M_5, and G to be a four-form on M_5. Preserved supersymmetry is equivalent to the existence of a Majorana spinor ϵ satisfying the equation

$$\nabla_M \epsilon + \frac{1}{288}\left(\gamma_M^{(11)NPQR} - 8\delta_M^N \gamma_{(11)}^{PQR}\right)G_{NPQR}\, \epsilon = 0 \ . \tag{B.0.2}$$

We may decompose the eleven-dimensional gamma matrices via

$$\gamma_\mu^{(6+5)} = e^A \gamma_\mu^{(6)} \otimes 1 \ , \qquad \gamma_{m+5}^{(6+5)} = \gamma^{(6)} \otimes \gamma_m^{(5)} \ . \tag{B.0.3}$$

Here $\gamma_\mu^{(6)}$, $\mu = 0, \ldots, 5$ are a basis of six-dimensional gamma matrices ($\gamma^{(6)}$ is the chiral gamma), while $\gamma_m^{(5)}$, $m = 1, \ldots, 5$ are a basis of five-dimensional gamma matrices. The spinor Anzatz preserving $\mathcal{N} = 1$ supersymmetry in AdS$_6$ is

$$\epsilon = \zeta_+ \eta_+ + \zeta_- \eta_- + \text{c.c.} \tag{B.0.4}$$

where $\zeta_\pm$ are the chiral components of a Killing spinor on AdS$_6$ satisfying

$$\nabla_\mu \zeta_\pm = \frac{1}{2}\gamma_\mu^{(6)}\zeta_\mp \ , \tag{B.0.5}$$

---

[1]This conclusion was also reached independently by F. Canoura and D. Martelli.

while $\eta_\pm$ are Dirac spinors on M_5.

Substituting (B.0.4) in (B.0.2) leads to the following equations for the spinors $\eta_\pm$:

$$\frac{1}{2}e^{-A}\eta_\mp \pm \frac{1}{2}\gamma_{(5)}^m \partial_m A\,\eta_\pm + \frac{1}{12}*_5 G_m\gamma_{(5)}^m\eta_\pm = 0\ , \tag{B.0.6a}$$

$$\nabla_m\eta_\pm \pm \frac{1}{4}*_5 G_m\eta_\pm \mp \frac{1}{6}*_5 G_n\gamma_m^{(5)}\gamma_{(5)}^n\eta_\pm = 0\ . \tag{B.0.6b}$$

Using (B.0.6) it is possible to derive the following differential conditions on the norms $\eta_\pm^\dagger\eta_\pm \equiv e^{B_\pm}$ of the internal spinors:

$$*_5 G = \mp 6\,d_5 B_\pm\ , \tag{B.0.7}$$

$$B_+ = -B_- + \text{const.}\ . \tag{B.0.8}$$

We can absorb the constant in a redefinition of η_- so that $B_+ = -B_- \equiv B$; thus

$$*_5\,G = -6\,d_5 B\ . \tag{B.0.9}$$

The equation of motion for G is then automatically satisfied; in absence of sources, the Bianchi identity reads $d_5 G = 0$, resulting in $*_5 G$ being harmonic. This is in contradiction with $*_5 G$ being exact. This still leaves open the possibility of adding M5-branes extended along AdS_6, which would modify the Bianchi identity to $d_5 G = \delta_{\mathrm{M5}}$. However, we will now show that even that possibility is not realized.

Defining $\tilde\eta_\pm \equiv e^{-B/2}\eta_\pm$ we can rewrite (B.0.6b) as

$$\nabla_m\tilde\eta_\pm \pm \partial_n B\,\gamma_m^n\tilde\eta_\pm = 0\ . \tag{B.0.10}$$

Upon rescaling the metric $ds_{M_5}^2 \to e^{-4B}ds_{M_5'}^2$ the equation for $\tilde\eta_+$ becomes

$$\nabla_m'\tilde\eta_+ = 0\ . \tag{B.0.11}$$

In five dimensions the only compact manifold admitting parallel spinors is the torus T^5, so we are forced to set $ds_{M_5'}^2 = ds_{T^5}^2$. Similarly if we rescale the metric $ds_{M_5}^2 \to e^{4B}ds_{M_5''}^2$ the equation for $\tilde\eta_-$ becomes

$$\nabla_m''\tilde\eta_- = 0\ , \tag{B.0.12}$$

so that $ds_{M_5''}^2 = ds_{T^5}^2$.[2] We are thus led to the relation

$$e^{-4B}ds_{M_5'}^2 = e^{4B}ds_{M_5''}^2\ . \tag{B.0.13}$$

Since $ds_{M_5'}^2 = ds_{M_5''}^2 = ds_{T^5}^2$, this implies $B = 0$, and hence $G = 0$ (from (B.0.9)). This makes the whole system collapse to flat space.

---

[2]One might try to avoid this conclusion by setting $\tilde\eta_-$ to zero. However, (B.0.6a) would then also set $\tilde\eta_+$ to zero.

Appendix C

Massaging the pairings: the pure case

In this Appendix we will show the equivalence between the equations (5.4.11) and (5.4.12); we will restrict to the IIB case since the story for IIA is identical.

First of all we need to record some further properties about the generalized Hodge diamond (5.4.4) and about how the deformations of the pure spinors can be arranged into the diamond. Recalling that in type IIB ψ_1 and ψ_2 are even forms on $\mathcal{M}_8$, we see that each row in the diamond has definite parity: the first row, the third and so on contain even forms, whereas the second, the fourth and so on contain odd forms. It is also straightforward to verify that $\gamma^{\bar{i}_1}$ (on the left) and γ^{i_2} (on the right) act as *descending* operators, whereas γ^{i_1} (on the left) and $\gamma^{\bar{i}_2}$ (on the right) act as *raising* operators: so for example by acting with $\gamma^{\bar{i}_1}$ and γ^{i_2} on ψ_1 it descends to the second row, whereas by acting with γ^{i_1} and $\gamma^{\bar{i}_2}$ on $\bar{\psi}_1$ it jumps to the eighth row.

We move to discuss the deformation issues and the recipe is very simple: $\delta\psi_i$ contains only terms of the form $\gamma^{mn}\psi_i$. Concretely this means that $\delta\psi_1$ sits in the zeroth and third row of (5.4.4), and $\delta\psi_2$ sits in the zeroth and third column of (5.4.4) (and of course an identical statement is true for complex conjugates). By combining a deformation with the action of the gamma matrices we conclude that $d_H\psi_1$ sits in the second and fourth row in the diamond, whereas $d_H\psi_2$ sits in the second and fourth column.

We can now show the equivalence between (5.4.11) and (5.4.12). Our strategy is simple: we consider (5.4.12) and the first equation in (5.4.8) and, by expanding both on each position of the diamond, we will see that they are completely equivalent to (5.4.11) plus the first equation in (5.4.8).

Let us start for example with the expansion in the $\psi_1\gamma^{i_2}$ position: the first equation

in (5.4.8) rewrites

$$\left(\bar\psi_1\gamma^{\bar i_2}, 2dA\wedge\psi_1 e^{-\phi}+d_H(e^{-\phi}\psi_1)\right)=-\left(\bar\psi_1\gamma^{\bar i_2},\frac{\alpha}{8}f\right)\,, \tag{C.0.1}$$

whereas (5.4.12), using the properties summarized in (5.4.5), reads

$$\left(\bar\psi_1\gamma^{\bar i_2}, d_H(e^{-\phi}\psi_1)\right)=\left(\bar\psi_1\gamma^{\bar i_2},\frac{\alpha}{8}f\right)\,, \tag{C.0.2}$$

and, simply by subtracting the two we obtain

$$\left(\bar\psi_1\gamma^{\bar i_2}, dA\wedge\psi_1+\frac{\alpha}{8}e^{\phi}f\right)=0\,, \tag{C.0.3}$$

which is precisely the third equation in (5.4.11). An identical consideration shows that by considering the expansion in the $\gamma^{\bar i_1}\psi_1$ position we simply reproduce the first equation in (5.4.11). Next we consider the expansion along $\psi_2\gamma^{\bar i_2}$: the first equation in (5.4.8) gives

$$\left(\bar\psi_2\gamma^{i_2}, d_H(e^{-\phi}\psi_1)\right)=-\left(\bar\psi_2\gamma^{i_2},\frac{\alpha}{8}f\right)\,, \tag{C.0.4}$$

whereas (5.4.12) gives

$$3\left(\bar\psi_2\gamma^{i_2}, d_H(e^{-\phi}\psi_1)\right)=\left(\bar\psi_2\gamma^{i_2},\frac{\alpha}{8}f\right)\,, \tag{C.0.5}$$

and the two equations imply $\left(\bar\psi_2\gamma^{i_2}, f\right)=0$ which is equivalent to the fourth equation in (5.4.11). We move to the expansion along $\gamma^{\bar i_1}\psi_1\gamma^{i_2 j_2}$: the pairing equations (5.4.11) say nothing about this position and indeed both the first equation in (5.4.8) and (5.4.12) say

$$\left(\gamma^{i_1}\bar\psi_1\gamma^{\bar i_2\bar j_2}, d_H(e^{-\phi}\psi_1)-\frac{\alpha}{8}f\right)=0\,, \tag{C.0.6}$$

therefore we conclude that (5.4.12) is redundant in this position.

Identical computations can be repeated for the other positions of the diamond and so we conclude that (5.4.11) and (5.4.12) are equivalent as we claimed.

Appendix D

Massaging the pairings: the non-pure case

In this Appendix we will describe how the pairing equations (3.2.4c), (3.2.4d) can be massaged in the non-pure case in order to obtain the equations (5.5.14). We will discuss the equations (3.2.4c) only, since the discussion for (3.2.4d) is almost identical.

To start with, we recall that for a general Mink$_2$, $N = (2,0)$ vacuum configuration the pairing equations take the form (5.3.23) and (5.3.24). Let us go to consider the first equation in (5.3.23); by putting the parametrizations (5.5.11) into the equation we obtain

$$\left(\gamma^m[a\bar{\tilde{\psi}}_1 + b\bar{\psi}_1 + c\bar{\tilde{\psi}}_2 + \bar{c}\tilde{\psi}_2], \frac{1}{4}dA \wedge [a\tilde{\psi}_1 + b\bar{\psi}_1 + c\bar{\tilde{\psi}}_2 + \bar{c}\tilde{\psi}_2] \mp \frac{\alpha}{8}e^\phi *_8 \lambda(f)\right)+$$

$$+\left(\gamma^m[b\bar{\tilde{\psi}}_1 + a\bar{\psi}_1 + c\bar{\tilde{\psi}}_2 + \bar{c}\tilde{\psi}_2], \frac{1}{4}dA \wedge [b\tilde{\psi}_1 + a\bar{\psi}_1 + c\bar{\tilde{\psi}}_2 + \bar{c}\tilde{\psi}_2] \mp \frac{\alpha}{8}e^\phi *_8 \lambda(f)\right) = 0 ,$$

$$\text{(D.0.1)}$$

where the gamma matrix γ^m has to be intended real and we have introduced the shortcuts

$$a \equiv |c_1|^2 = A_1^2 + B_1^2 + 2A_1B_1\sin\theta_1 , \qquad b \equiv |c_2|^2 = A_1^2 + B_1^2 - 2A_1B_1\sin\theta_1 ,$$

$$c \equiv \bar{c}_1c_2 = A_1^2 + B_1^2\cos(2\theta_1) + iB^2\sin(2\theta_1) , \qquad d \equiv c_1c_2 = A_1^2 - B_1^2 + 2iA_1B_1\cos\theta_1 ,$$

$$e \equiv c_1^2 = (A_1^2 - B_1^2\cos(2\theta_1) + 2A_1B_1\sin\theta_1) + i(2A_1B_1\cos\theta_1 + 2B^2\sin\theta_1\cos\theta_1) ,$$

$$h \equiv c_2^2 = (A_1^2 - B_1^2\cos(2\theta_1) - 2A_1B_1\sin\theta_1) + i(2A_1B_1\cos\theta_1 - 2B^2\sin\theta_1\cos\theta_1) .$$

$$\text{(D.0.2)}$$

Now by taking $\gamma^m \equiv \gamma^{i_1}$ (see footnote 8 for the meaning of the index i_1), (D.0.1) simplifies to

$$\left(\gamma^{i_1}[a\bar{\tilde{\psi}}_1 + c\bar{\tilde{\psi}}_2], \frac{1}{4}dA \wedge [a\tilde{\psi}_1 + \bar{c}\tilde{\psi}_2] \mp \frac{\alpha}{8}e^\phi *_8 \lambda(f)\right)+$$

$$+\left(\gamma^{i_1}[b\bar{\tilde{\psi}}_1 + c\bar{\tilde{\psi}}_2], \frac{1}{4}dA \wedge [b\tilde{\psi}_1 + \bar{c}\tilde{\psi}_2] \mp \frac{\alpha}{8}e^\phi *_8 \lambda(f)\right) = 0 , \qquad \text{(D.0.3)}$$

that is

$$\left(\gamma^{i_1}\bar{\tilde{\psi}}_1, \frac{a^2+b^2}{4}dA \wedge \tilde{\psi}_1 \mp \frac{\alpha}{8}(a+b)e^\phi *_8 \lambda(f)\right) + 2c\left(\gamma^{i_1}\bar{\tilde{\psi}}_2, \frac{\bar{c}}{4}dA \wedge \tilde{\psi}_2 \mp \frac{\alpha}{8}e^\phi *_8 \lambda(f)\right) = 0 \ . \tag{D.0.4}$$

Moving to the second equation in (5.3.23), we can perform the same procedure but in this case we obtain a pair of equations: the first one is obtained by taking $m = i_1$ and the second one is obtained when $m = \bar{i}_1$

$$\bar{d}\left(\gamma^{i_1}\bar{\tilde{\psi}}_1, \frac{b}{4}dA \wedge \tilde{\psi}_1 \mp \frac{\alpha}{8}e^\phi *_8 \lambda(f)\right) + \bar{e}\left(\gamma^{i_1}\bar{\tilde{\psi}}_2, \frac{\bar{c}}{4}dA \wedge \tilde{\psi}_2 \mp \frac{\alpha}{8}e^\phi *_8 \lambda(f)\right) = 0 \ ,$$
$$\bar{d}\left(\gamma^{\bar{i}_1}\tilde{\psi}_1, \frac{a}{4}dA \wedge \bar{\tilde{\psi}}_1 \mp \frac{\alpha}{8}e^\phi *_8 \lambda(f)\right) + \bar{h}\left(\gamma^{\bar{i}_1}\tilde{\psi}_2, \frac{c}{4}dA \wedge \bar{\tilde{\psi}}_2 \mp \frac{\alpha}{8}e^\phi *_8 \lambda(f)\right) = 0 \ . \tag{D.0.5}$$

Before to proceed we note that in the pure limit we have $a = e \neq 0$ whereas $b = c = d = h = 0$; therefore in this case the equations (D.0.4), (D.0.5) collapse to the first two equations in (5.4.11) that are valid in the pure case. To proceed we rewrite the first equation in (D.0.5) as

$$\left(\gamma^{i_1}\bar{\tilde{\psi}}_2, \frac{\bar{c}}{4}dA \wedge \tilde{\psi}_2 \mp \frac{\alpha}{8}e^\phi *_8 \lambda(f)\right) = -\frac{\bar{d}}{\bar{e}}\left(\gamma^{i_1}\bar{\tilde{\psi}}_1, \frac{b}{4}dA \wedge \tilde{\psi}_1 \mp \frac{\alpha}{8}e^\phi *_8 \lambda(f)\right) \ , \tag{D.0.6}$$

that we can put in (D.0.4) and in the complex conjugate of the second equation in (D.0.5) obtaining the algebraic system

$$\left(a^2 + b^2 - \frac{2cd\bar{b}}{\bar{e}}\right)x + \left(a + b - \frac{2c\bar{d}}{\bar{e}}\right)y = 0 \ ,$$
$$\left(da - \frac{h\bar{d}b}{\bar{e}}\right)x + \left(d - \frac{h\bar{d}}{\bar{e}}\right)y = 0 \ ,$$
$$x \equiv \left(\gamma^{i_1}\bar{\tilde{\psi}}_1, \frac{1}{4}dA \wedge \tilde{\psi}_1\right) \ , \qquad y \equiv \left(\gamma^{i_1}\bar{\tilde{\psi}}_1, \mp\frac{\alpha}{8}e^\phi *_8 \lambda(f)\right) \ . \tag{D.0.7}$$

It can be verified that the determinant of this algebraic system vanishes and so we remain with the single equation

$$x = -\frac{\bar{e}(a+b) - 2c\bar{d}}{\bar{e}(a^2+b^2) - 2cd\bar{b}}\, y = -\frac{1}{2\alpha}y \ , \tag{D.0.8}$$

where the last equivalence can be verified using the explicit expressions (D.0.2). Summarizing the pairing equations (5.3.23) rewrite as

$$\left(\gamma^{i_1}\bar{\tilde{\psi}}_1, dA \wedge \tilde{\psi}_1 \mp \frac{1}{4}e^\phi *_8 \lambda(f)\right) = 0 \ ,$$
$$\left(\gamma^{i_1}\bar{\tilde{\psi}}_2, \mp\frac{\alpha}{8}e^\phi *_8 \lambda(f)\right) = -\bar{c}\left(\gamma^{i_1}\bar{\tilde{\psi}}_2, \frac{dA}{4} \wedge \tilde{\psi}_2\right) - \frac{\bar{d}}{\bar{e}}\left(\gamma^{i_1}\bar{\tilde{\psi}}_1, \frac{b}{4}dA \wedge \tilde{\psi}_1 \mp \frac{\alpha}{8}e^\phi *_8 \lambda(f)\right) \ . \tag{D.0.9}$$

The same strategy can be applied of course for the equations (5.3.24).

Appendix E

Squashed 3-spheres in the Hopf coordinates

Let us start from the beginning and construct the squashed metric from C^2

$$ds^2 = |dz_1|^2 + |dz_2|^2 \tag{E.0.1}$$

where

$$z_1 = \rho_1\, e^{i\,\phi_1} \qquad z_2 = \rho_2\, e^{i\,\phi_2} \tag{E.0.2}$$

The S_3 is obtained by embedding the hypersurface

$$\rho^2 \equiv \frac{\rho_1^2}{l^2} + \frac{\rho_2^2}{\tilde{l}^2} = 1 \tag{E.0.3}$$

in C^2 , which gives

$$\rho_1 = l\,\cos\theta \qquad \rho_2 = \tilde{l}\,\sin\theta \qquad 0 \leq \theta \leq \frac{\pi}{2} \tag{E.0.4}$$

Hence

$$ds^2 = \sum_{i=1,2} |d\rho_i + i\,\rho_i\,d\phi_i|^2 = d\rho_1^2 + d\rho_2^2 + \rho_1^2\,d\phi_1^2 + \rho_2^2\,d\phi_2^2 =$$
$$= \left(l^2\,\sin^2\theta + \tilde{l}^2\,\cos^2\theta \right) d\theta^2 + l^2\,\cos^2\theta\,d\phi_1^2 + \tilde{l}^2\,\sin^2\theta\,d\phi_2^2 \tag{E.0.5}$$

and the Killing vector which gives the Seifert structure is

$$\bar{\gamma}^\mu \partial_\mu = \frac{1}{l}\,\partial_{\phi_1} + \frac{1}{\tilde{l}}\,\partial_{\phi_2} \tag{E.0.6}$$

Appendix F

Gravitational CS action and the Seifert WZ action

The topological anomaly is, by definition, non-trivial: there is no local functional of the 3-dimensional metric, trasforming as a 3-form, whose BRST variation gives the anomaly. As a matter of fact, one has

$$A_1^{(3)} = s\,\Gamma_{GCS}^{(3)}(g) \tag{F.0.1}$$

where $\Gamma_{GCS}^{(3)}$ is the 3-dimensional gravitational Chern-Simons action

$$\Gamma_{GCS}^{(3)}(g) = \frac{c}{12}\,\mathrm{Tr}\left(\frac{1}{2}\Gamma\,d\,\Gamma + \frac{1}{3}\,\Gamma^3\right) \tag{F.0.2}$$

Since $\Gamma_{GCS}^{(3)}(g)$ is not a globally defined 3-form on M_3, $A_1^{(3)}$ is indeed non-trivial.

Let us discuss the relation between the Seifert Wess-Zumino action (9.6.23) and the gravitational CS action (F.0.2). Let us introduce the 3-form

$$\Gamma_{WZ}^{(3)\,Seif} = \Gamma_{WZ}^{Seif}\ dy\,\frac{1}{2}\,\epsilon_{ij}dx^i\,dx^j \tag{F.0.3}$$

Then of course

$$s\int_{M_3}\left(\Gamma_{WZ}^{(3)\,Seif} - \Gamma_{GCS}^{(3)}[g_{ij},\sigma,a_i]\right) = 0 \tag{F.0.4}$$

where

$$\Gamma_{GCS}^{(3)}[g_{ij},\sigma,a_i] \equiv \Gamma_{GCS}^{(3)}[g]\big|_{\mathcal{L}_{\bar\gamma}\,g} \tag{F.0.5}$$

is the gravitational CS action evaluated for the metric adapted to the Seifert structure (9.6.10).

Therefore, there exists a local 2-form $\Omega^{(2)}$ such that

$$\Gamma_{WZ}^{(3)\,Seif} - \Gamma_{GCS}^{(3)}[g_{ij},\sigma,a_i] = d\,\Omega^{(2)} \tag{F.0.6}$$

It turns out that

$$\Omega^{(2)} = \frac{1}{4} \sqrt{g} \, f \, \epsilon_{ij} \, g^{jk} \, \Gamma^i_{lk} \, dx^l \, dy \tag{F.0.7}$$

$\Gamma^{(3)}_{GCS}$ is not a globally defined 3-form: This fact is expressed by

$$S_{\text{diff}} \, \Gamma^{(3)}_{GCS} = d \, Q^{(2)}_1 \tag{F.0.8}$$

where S_{diff} is the BRST operator associated to 3-dimensional diffeomorphisms:

$$S_{\text{diff}} = \mathcal{L}_\xi - \{i_\xi, d\} \tag{F.0.9}$$

with $\xi = \xi^\mu \, \partial_\mu$ the reparametrization ghost in 3-dimensions. $Q^{(2)}_1$ is the reparametrization anomaly in 2 dimensions:

$$Q^{(2)}_1 = \frac{1}{2} \operatorname{Tr} \operatorname{M} d\, \Gamma = \frac{1}{2} \, \partial_\mu \, \xi^\nu d \, \Gamma^\mu_\nu \tag{F.0.10}$$

$\Gamma^{(3)\,Seif}_{WZ}$ also is not invariant under S_{diff}. However we can consider reparametrization ghosts ξ^μ which are invariant under the Reeb vector $\bar{\gamma}^\mu$. Let us denote by S^{Seif}_{diff} the reparametrization BRST operator associated to such invariant reparametrization ghosts. The symmetry associated to S^{Seif}_{diff} is the one corresponding to 2-dimensional diffeomorphisms and abelian gauge transformations.

Although $\Gamma^{(3)}_{GCS}[g_{ij}, \sigma, a_i]$ is not invariant even under the restricted $\bar{S}_{\text{diff}}$, $\Gamma^{(3)\,Seif}_{WZ}$ is invariant:

$$S^{Seif}_{\text{diff}} \, \Gamma^{(3)\,Seif}_{WZ} = 0 \tag{F.0.11}$$

as it is manifest from (9.6.23). This equations expresses the fact that $\Gamma^{(3)\,Seif}_{WZ}$ is a form under reparametrizations which do not change the Seifert structure. Hence

$$S^{Seif}_{\text{diff}} \left(\Gamma^{(3)}_{GCS}[g_{ij}, \sigma, a_i] + d\, \Omega^{(2)} \right) = d \left(Q^{(2)}_1 + S^{Seif}_{\text{diff}} \, \Omega^{(2)} \right) \tag{F.0.12}$$

We conclude that

$$Q^{(2)}_1 = S^{Seif}_{\text{diff}} \, \Omega^{(2)} + d\, \Omega^{(1)}_1 \tag{F.0.13}$$

In other words the functional $\Omega^{(2)}$ trivializes the 2-dimensional gravitational anomaly: this is possibile since $\Omega^{(2)}$ is a functional not only of the 2-dimensional metric g_{ij} but also of the abelian field strength f.

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