INTERTWINING SEMICLASSICAL SOLUTIONS TO A SCHRÖDINGER-NEWTON SYSTEM

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Abstract. We study the problem

$$\begin{cases}(-\varepsilon \mathrm{i}\nabla + A(x))^2 u + V(x)u = \varepsilon^{-2}\left(\frac{1}{|x|} * |u|^2\right)u,\\ u \in L^2(\mathbb{R}^3, \mathbb{C}), \qquad \varepsilon\nabla u + \mathrm{i}Au \in L^2(\mathbb{R}^3, \mathbb{C}^3),\end{cases}$$

where $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ is an exterior magnetic potential, $V\colon \mathbb{R}^3 \to \mathbb{R}$ is an exterior electric potential, and ε is a small positive number. If $A = 0$ and $\varepsilon = \hbar$ is Planck's constant this problem is equivalent to the Schrödinger-Newton equations proposed by Penrose in [23] to describe his view that quantum state reduction occurs due to some gravitational effect. We assume that A and V are compatible with the action of a group G of linear isometries of $\mathbb{R}^3$. Then, for any given homomorphism $\tau : G \to \mathbb{S}^1$ into the unit complex numbers, we show that there is a combined effect of the symmetries and the potential V on the number of semiclassical solutions $u : \mathbb{R}^3 \to \mathbb{C}$ which satisfy $u(gx) = \tau(g)u(x)$ for all $g \in G$, $x \in \mathbb{R}^3$. We also study the concentration behavior of these solutions as $\varepsilon \to 0$.

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1. Introduction

The *Schrödinger-Newton equations* were proposed by Penrose [23] to describe his view that quantum state reduction is a phenomenon that occurs because of some gravitational influence. They consist of a system of equations obtained by coupling together the linear Schrödinger equation of quantum mechanics with the Poisson equation from Newtonian mechanics. For a single particle of mass m this system has the form

$$\begin{cases}-\frac{\hbar^2}{2m}\Delta\psi + V(x)\psi + U\psi = 0,\\ -\Delta U + 4\pi\kappa|\psi|^2 = 0,\end{cases} \tag{1.1}$$

where ψ is the complex wave function, U is the gravitational potential energy, V is a given potential, $\hbar$ is Planck's constant, and $\kappa := \mathrm{G}m^2$, G being Newton's constant. According to Penrose, the solutions ψ of this system are the *basic stationary states* into which a superposition of such states is to decay within a certain timescale, cf. [22, 23, 18, 19, 24].

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After rescaling by

$$\psi(x) = \frac{1}{\hbar}\frac{\hat{\psi}(x)}{\sqrt{2\kappa m}}, \quad V(x) = \frac{1}{2m}\hat{V}(x), \quad U(x) = \frac{1}{2m}\hat{U}(x),$$

system (1.1) can be written as

$$\begin{cases} -\hbar^2\Delta\hat{\psi} + \hat{V}(x)\hat{\psi} + \hat{U}\hat{\psi} = 0, \\ -\hbar^2\Delta\hat{U} + 4\pi|\hat{\psi}|^2 = 0. \end{cases} \tag{1.2}$$

The second equation in (1.2) can be explicitly solved with respect to $\hat{U}$, so this system is equivalent to the single nonlocal equation

$$-\hbar^2\Delta\hat{\psi} + \hat{V}(x)\hat{\psi} = \frac{1}{\hbar^2}\left(\int_{\mathbb{R}^3}\frac{|\hat{\psi}(\xi)|^2}{|x-\xi|}d\xi\right)\hat{\psi} \quad \text{in } \mathbb{R}^3. \tag{1.3}$$

We shall consider a more general equation having a similar structure, namely

$$(-\varepsilon\mathrm{i}\nabla + A(x))^2 u + V(x)u = \frac{1}{\varepsilon^2}\left(\frac{1}{|x|} * |u|^2\right)u \quad \text{in } \mathbb{R}^3, \tag{1.4}$$

where $A\colon \mathbb{R}^3 \to \mathbb{R}^3$ is an exterior magnetic potential, i is the imaginary unit and $*$ denotes the convolution operator. We are interested in semiclassical states, i.e. in solutions of this equation for $\varepsilon \to 0$.

The existence of one solution can be traced back to Lions' paper [15]. In the nonmagnetic case $A = 0$ equation (1.4) and related equations have been investigated by many authors, see e.g. [2, 10, 11, 12, 13, 16, 17, 18, 20, 25, 26, 19] and the references therein. Recently, Wei and Winter [27] showed the existence of positive multibump solutions which concentrate at local minima, local maxima or nondegenerate critical points of the potential V as $\varepsilon \to 0$. The magnetic case $A \neq 0$ was recently studied in [6] where it was shown that equation (1.4) has a family of solutions having multiple concentration regions located around the (possibly degenerate) minima of V.

In this paper we consider the situation where A and V are symmetric and we look for semiclassical solutions of equation (1.4) having specific symmetries. The absolute value of the solutions we obtain concentrates at points which need not be local extrema, nor nondegenerate critical points of V (in fact, we do not even assume that V is differentiable). We state our main results in the following section and give some explicit examples.

2. Statement of results

2.1. **The results.** Let G be a closed subgroup of the group $O(3)$ of linear isometries of $\mathbb{R}^3$, $A : \mathbb{R}^3 \to \mathbb{R}^3$ be a C^1-function, and $V : \mathbb{R}^3 \to \mathbb{R}$ be a bounded continuous function with $\inf_{\mathbb{R}^3} V > 0$, which satisfy

$$A(gx) = gA(x) \quad \text{and} \quad V(gx) = V(x) \quad \text{for all } g \in G,\ x \in \mathbb{R}^3. \tag{2.1}$$

Given a continuous homomorphism of groups $\tau : G \to \mathbb{S}^1$ into the group $\mathbb{S}^1$ of unit complex numbers, we look for solutions to the problem

$$\begin{cases} (-\varepsilon\mathrm{i}\nabla + A)^2 u + V(x)u = \varepsilon^{-2}\left(\frac{1}{|x|} * |u|^2\right)u, \\ u \in L^2(\mathbb{R}^3, \mathbb{C}), \\ \varepsilon\nabla u + \mathrm{i}Au \in L^2(\mathbb{R}^3, \mathbb{C}^3), \end{cases} \tag{2.2}$$

which satisfy

$$u(gx) = \tau(g)u(x) \quad \text{for all } g \in G,\ x \in \mathbb{R}^3, \tag{2.3}$$

This implies that the absolute value $|u|$ of u is G-invariant, i.e.

$$|u(gx)| = |u(x)| \quad \text{for all } g \in G,\, x \in \mathbb{R}^3,$$

whereas the phase of $u(gx)$ is that of $u(x)$ multiplied by $\tau(g)$. A concrete example is given in subsection 2.2 below.

Note that if u satisfies (2.2) and (2.3) then $e^{\mathrm{i}\theta}u$ satisfies (2.2) and (2.3) for every $\theta \in \mathbb{R}$. We shall say that u and v are *geometrically distinct* if $e^{\mathrm{i}\theta}u \neq v$ for all $\theta \in \mathbb{R}$.

We introduce some notation. For $x \in \mathbb{R}^3$, we denote by Gx the G-orbit of x and by G_x the G-isotropy subgroup of x, i.e.

$$Gx := \{gx\colon g \in G\}, \quad G_x := \{g \in G : gx = x\}.$$

A subset X of $\mathbb{R}^3$ is G-invariant if $Gx \subset X$ for every $x \in X$. The G-orbit space of X is the set

$$X/G := \{Gx : x \in X\}$$

of G-orbits of X with the quotient topology.

Let $\#Gx$ denote the cardinality of Gx, and define

$$\ell_{G,V} := \inf_{x\in\mathbb{R}^3} (\#Gx)V^{3/2}(x),$$

$$M_\tau := \left\{ x \in \mathbb{R}^3 : (\#Gx)V^{3/2}(x) = \ell_{G,V},\ G_x \subset \ker \tau \right\}.$$

Assumption (2.1) implies that M_τ is G-invariant. Observe that the points of M_τ need not be neither local minima nor local maxima of V.

Given $\rho > 0$ we set $B_\rho M_\tau := \{x \in \mathbb{R}^3 : \operatorname{dist}(x, M_\tau) \leq \rho\}$, and write

$$\operatorname{cat}_{B_\rho M_\tau/G}(M_\tau/G)$$

for the Lusternik-Schnirelmann category of M_τ/G in $B_\rho M_\tau/G$.

Finally, we denote by E_1 the least energy of a nontrivial solution to problem

$$\begin{cases} -\Delta u + u = (\frac{1}{|x|} * u^2)u, \\ u \in H^1(\mathbb{R}^3, \mathbb{R}). \end{cases} \tag{2.4}$$

We shall prove the following results.

Theorem 2.1. *Assume there exists* $\alpha > 0$ *such that the set*

$$\left\{ x \in \mathbb{R}^3 : (\#Gx)V^{3/2}(x) \leq \ell_{G,V} + \alpha \right\} \tag{2.5}$$

is compact. Then, given $\rho, \delta > 0$, *there exists* $\widehat{\varepsilon} > 0$ *such that, for every* $\varepsilon \in (0, \widehat{\varepsilon})$, *problem* (2.2) *has at least*

$$\operatorname{cat}_{B_\rho M_\tau/G}(M_\tau/G)$$

geometrically distinct solutions u *which satisfy* (2.3) *and*

$$\left| \frac{1}{4} \int_{\mathbb{R}^3} \left(\frac{1}{|x|} * |u|^2 \right) |u|^2 - \varepsilon^5 \ell_{G,V} E_1 \right| < \varepsilon^5 \delta. \tag{2.6}$$

The last inequality says that the energy of the solutions is arbitrarily close to $\varepsilon^3 \ell_{G,V} E_1$ for ε small enough. So considering different groups G and G' for which $\ell_{G,V} \neq \ell_{G',V}$ will lead to solutions with energy in disjoint ranges.

For $u \in H^1(\mathbb{R}^3, \mathbb{R})$ set

$$\|u\|_\varepsilon^2 := \int_{\mathbb{R}^3} (\varepsilon^2 |\nabla u|^2 + u^2).$$

The following theorem describes the module of the solutions given by Theorem 2.1 as $\varepsilon \to 0$.

Theorem 2.2. *Let u_n be a solution to problem (2.2) which satisfies (2.3) and (2.6) for $\varepsilon = \varepsilon_n > 0$, $\delta = \delta_n > 0$. Assume $\varepsilon_n \to 0$ and $\delta_n \to 0$. Then, after passing to a subsequence, there exists a sequence (ξ_n) in $\mathbb{R}^3$ such that $\xi_n \to \xi \in M_\tau$, $G_{\xi_n} = G_\xi$, and*

$$\varepsilon_n^{-3} \left\| |u_n| - \sum_{g\xi_n \in G\xi_n} \omega_\xi \left(\frac{\cdot - g\xi_n}{\varepsilon_n} \right) \right\|_{\varepsilon_n}^2 \to 0,$$

where ω_ξ is the unique ground state of problem

$$-\Delta u + V(\xi) u = \left(\frac{1}{|x|} * u^2 \right) u, \qquad u \in H^1(\mathbb{R}^3, \mathbb{R}),$$

which is positive and radially symmetric with respect to the origin.

Next, we give an example which illustrates our results.

2.2. **Rotationally invariant potentials.** Let $\mathbb{S}^1$ act on $\mathbb{R}^3 \equiv \mathbb{C} \times \mathbb{R}$ by $e^{\mathrm{i}\theta}(z,t) := (e^{\mathrm{i}\theta} z, t)$, and let A and V satisfy assumption (2.1) for the cyclic group G_m generated by $e^{2\pi \mathrm{i}/m}$, for some $m \in \mathbb{N}$. For example, the standard magnetic potential $A(x_1, x_2, x_3) := (-x_2, x_1, 0)$ associated to the constant magnetic field $B(x) = (0,0,2)$ has this property for every m.

For each $j = 0, 1, \dots, m-1$ we look for solutions to problem (2.2) which satisfy

$$u(e^{2\pi \mathrm{i}/m} z, t) = e^{2\pi \mathrm{i} j/m} u(z,t) \quad \text{for all } (z,t) \in \mathbb{C} \times \mathbb{R}. \tag{2.7}$$

Solutions of this type arise in a natural way in some problems where the magnetic potential is singular and the topology of the domain produces an Aharonov-Bohm type effect, cf. [1, 8]. Taking $\tau_j(g) := g^j$ we see that these are solutions of the type furnished by Theorem 2.1.

If V satisfies

$$V_0 := \inf_{x \in \mathbb{R}^3} V < \liminf_{|x| \to \infty} V(x) \quad \text{and} \quad m V_0^{3/2} < \inf_{t \in \mathbb{R}} V^{3/2}(0,t), \tag{2.8}$$

then assumption (2.5) in Theorem 2.1 is satisfied, $\ell_{G_m,V} = m V_0^{3/2}$ and M_τ is simply the set of minima of V,

$$M = \left\{ x \in \mathbb{R}^3 : V(x) = V_0 \right\}.$$

Thus, for each $j = 0, 1, \dots, m-1$ and ρ, $\delta > 0$, Theorem 2.1 yields at least $\operatorname{cat}_{B_\rho M/G_m}(M/G_m)$ geometrically distinct solutions to problem (2.2) satisfying (2.7) and (2.6), for ε small enough.

For each k dividing m the potentials A and V satify assumption (2.1) for G_k and V satisfies (2.8) with k instead of m. Property (2.6) implies that the solutions obtained for G_k are different from those for G_m if $k \neq m$ and ε is small enough.

This paper is organized as follows. In section 3 we discuss the variational problem related to the existence of solutions to problem (2.2) satisfying (2.3). We also outline the strategy for proving Theorem 2.1. Sections 4 and 5 are devoted to the construction of an *entrance map* and a *local baryorbit map* which will help us estimate the Lusternik-Schnirelmann category of a suitable sublevel set of the variational functional for ε small enough. Finally, in section 6 we prove Theorems 2.1 and 2.2.

3. The variational problem

Set $\nabla_{\varepsilon,A}u := \varepsilon\nabla u + \mathrm{i}Au$ and consider the real Hilbert space

$$H^1_{\varepsilon,A}(\mathbb{R}^3,\mathbb{C}) := \{u \in L^2(\mathbb{R}^3,\mathbb{C}) : \nabla_{\varepsilon,A}u \in L^2(\mathbb{R}^3,\mathbb{C}^3)\}$$

with the scalar product

$$\langle u,v\rangle_{\varepsilon,A,V} := \mathrm{Re}\int_{\mathbb{R}^3}\left(\nabla_{\varepsilon,A}u\cdot\overline{\nabla_{\varepsilon,A}v} + V(x)u\overline{v}\right). \tag{3.1}$$

We write

$$\|u\|_{\varepsilon,A,V} := \left(\int_{\mathbb{R}^3}\left(|\nabla_{\varepsilon,A}u|^2 + V(x)\,|u|^2\right)\right)^{1/2}$$

for the corresponding norm.

If $u \in H^1_{\varepsilon,A}(\mathbb{R}^3,\mathbb{C})$, then $|u| \in H^1(\mathbb{R}^3,\mathbb{R})$ and

$$\varepsilon\,|\nabla|u(x)|| \leq |\varepsilon\nabla u(x) + \mathrm{i}A(x)u(x)| \quad \text{for a.e. } x \in \mathbb{R}^3. \tag{3.2}$$

This is called the diamagnetic inequality [14]. Set

$$\mathbb{D}(u) := \int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{|u(x)|^2|u(y)|^2}{|x-y|}\,dxdy.$$

The standard Hardy–Littlewood–Sobolev inequality [14, Theorem 4.3] yields

$$\left|\int_{\mathbb{R}^3}\int_{\mathbb{R}^3}\frac{f(x)h(y)}{|x-y|}\,dx\,dy\right| \leq C\|f\|_{L^{6/5}(\mathbb{R}^3)}\|h\|_{L^{6/5}(\mathbb{R}^3)} \tag{3.3}$$

for all $f,h \in L^{6/5}(\mathbb{R}^3)$, where C is a positive constant independent of f and h. In particular,

$$\mathbb{D}(u) \leq C\|u\|^4_{L^{12/5}(\mathbb{R}^3)} \tag{3.4}$$

for every $u \in H^1_{\varepsilon,A}(\mathbb{R}^3,\mathbb{C})$.

The energy functional $J_{\varepsilon,A,V} : H^1_{\varepsilon,A}(\mathbb{R}^3,\mathbb{C}) \to \mathbb{R}$ associated to problem (2.2), defined by

$$J_{\varepsilon,A,V}(u) := \frac{1}{2}\|u\|^2_{\varepsilon,A,V} - \frac{1}{4\varepsilon^2}\mathbb{D}(u),$$

is of class C^2, and its derivative is given by

$$J'_{\varepsilon,A,V}(u)v := \langle u,v\rangle_{\varepsilon,A,V} - \frac{1}{\varepsilon^2}\,\mathrm{Re}\int_{\mathbb{R}^3}\left(\frac{1}{|x|} * |u|^2\right)u\overline{v}.$$

Therefore, the solutions to problem (2.2) are the critical points of $J_{\varepsilon,A,V}$. We write $\nabla_\varepsilon J_{\varepsilon,A,V}(u)$ for the gradient of $J_{\varepsilon,A,V}$ at u with respect to the scalar product (3.1).

The action of G on $H^1_{\varepsilon,A}(\mathbb{R}^3,\mathbb{C})$ defined by $(g,u) \mapsto u_g$, where

$$(u_g)(x) := \tau(g)u(g^{-1}x),$$

satisfies

$$\langle u_g, v_g\rangle_{\varepsilon,A,V} = \langle u, v\rangle_{\varepsilon,A,V} \qquad \text{and} \qquad \mathbb{D}(u_g) = \mathbb{D}(u)$$

for all $g \in G$, $u, v \in H^1_{\varepsilon,A}(\mathbb{R}^3, \mathbb{C})$. Hence, $J_{\varepsilon,A,V}$ is G-invariant. By the principle of symmetric criticality [21, 28], the critical points of the restriction of $J_{\varepsilon,A,V}$ to the fixed point space of this G-action, denoted by

$$\begin{aligned} H^1_{\varepsilon,A}(\mathbb{R}^3, \mathbb{C})^\tau &= \left\{u \in H^1_{\varepsilon,A}(\mathbb{R}^3, \mathbb{C}) : u_g = u\right\} \\ &= \left\{u \in H^1_{\varepsilon,A}(\mathbb{R}^3, \mathbb{C}) : u(gx) = \tau(g)u(x) \;\; \forall x \in \mathbb{R}^3,\; g \in G\right\}, \end{aligned}$$

are the solutions to problem (2.2) which satisfy (2.3). Those which are nontrivial lie on the *Nehari manifold*

$$\mathcal{N}^\tau_{\varepsilon,A,V} := \left\{u \in H^1_{\varepsilon,A}(\mathbb{R}^3, \mathbb{C})^\tau : u \neq 0,\; \varepsilon^2 \|u\|^2_{\varepsilon,A,V} = \mathbb{D}(u)\right\},$$

which is a C^2-manifold radially diffeomorphic to the unit sphere in $H^1_{\varepsilon,A}(\mathbb{R}^3, \mathbb{C})^\tau$. The critical points of the restriction of $J_{\varepsilon,A,V}$ to $\mathcal{N}^\tau_{\varepsilon,A,V}$ are precisely the nontrivial solutions to (2.2) which satisfy (2.3).

The radial projection $\pi_{\varepsilon,A,V} : H^1_{\varepsilon,A}(\mathbb{R}^3, \mathbb{C})^\tau \setminus \{0\} \to \mathcal{N}^\tau_{\varepsilon,A,V}$ is given by

$$\pi_{\varepsilon,A,V}(u) := \frac{\varepsilon \|u\|_{\varepsilon,A,V}}{\sqrt{\mathbb{D}(u)}} u. \tag{3.5}$$

Note that

$$J_{\varepsilon,A,V}(\pi_{\varepsilon,A,V}(u)) = \frac{\varepsilon^2 \|u\|^4_{\varepsilon,A,V}}{4\mathbb{D}(u)} \qquad \text{for all } u \in H^1_{\varepsilon,A}(\mathbb{R}^3, \mathbb{C})^\tau \setminus \{0\}. \tag{3.6}$$

Recall that $J_{\varepsilon,A,V} : \mathcal{N}^\tau_{\varepsilon,A,V} \to \mathbb{R}$ is said to satisfy the *Palais-Smale condition* $(PS)_c$ at the level c if every sequence (u_n) such that

$$u_n \in \mathcal{N}^\tau_{\varepsilon,A,V}, \qquad J_{\varepsilon,A,V}(u_n) \to c, \qquad \nabla_{\mathcal{N}^\tau_{\varepsilon,A,V}} J_{\varepsilon,A,V}(u_n) \to 0,$$

contains a convergent subsequence. Here $\nabla_{\mathcal{N}^\tau_{\varepsilon,A,V}} J_{\varepsilon,A,V}(u)$ denotes the orthogonal projection of $\nabla_\varepsilon J_{\varepsilon,A,V}(u)$ onto the tangent space to $\mathcal{N}^\tau_{\varepsilon,A,V}$ at u. The following holds.

Proposition 3.1. *For every $\varepsilon > 0$, the functional $J_{\varepsilon,A,V} : \mathcal{N}^\tau_{\varepsilon,A,V} \to \mathbb{R}$ satisfies $(PS)_c$ at each level*

$$c < \varepsilon^3 \min_{x \in \mathbb{R}^3 \setminus \{0\}} (\#Gx) V_\infty^{3/2} E_1,$$

where $V_\infty := \liminf_{|x|\to\infty} V(x)$.

Proof. This was proved in [5] for $\varepsilon = 1$. For $\varepsilon > 0$ the assertion follows after performing the change of variable $u_\varepsilon(x) := u(\varepsilon x)$ since a straightforward computation shows that

$$\varepsilon^{-3} J_{\varepsilon,A,V}(u) = J_{1,A_\varepsilon,V_\varepsilon}(u_\varepsilon) \quad \text{and} \quad \varepsilon^{-3/2} \nabla_{\mathcal{N}^\tau_{\varepsilon,A,V}} J_{\varepsilon,A,V}(u) = \nabla_{\mathcal{N}^\tau_{1,A_\varepsilon,V_\varepsilon}} J_{1,A_\varepsilon,V_\varepsilon}(u_\varepsilon),$$

where $A_\varepsilon(x) := A(\varepsilon x)$ and $V_\varepsilon(x) := V(\varepsilon x)$. □

$\mathbb{S}^1$ acts on $H^1_{\varepsilon,A}(\mathbb{R}^3, \mathbb{C})^\tau$ by scalar multiplication: $(e^{\mathrm{i}\theta}, u) \mapsto e^{\mathrm{i}\theta} u$. The Nehari manifold $\mathcal{N}^\tau_{\varepsilon,A,V}$ and the functional $J_{\varepsilon,A,V}$ are invariant under this action. Two solutions of (2.2) are geometrically distinct iff they lie on different $\mathbb{S}^1$-orbits. Equivariant Lusternik-Schnirelmann theory yields the following result, see e.g. [7].

Proposition 3.2. *If $J_{\varepsilon,A,V}:\mathcal{N}^{\tau}_{\varepsilon,A,V}\to\mathbb{R}$ satisfies $(PS)_c$ at each level $c\leq\overline{c}$, then $J_{\varepsilon,A,V}$ has at least*

$$\operatorname{cat}\left[\left(\mathcal{N}^{\tau}_{\varepsilon,A,V}\cap J^{\overline{c}}_{\varepsilon,A,V}\right)/\mathbb{S}^1\right]$$

critical $\mathbb{S}^1$-orbits in $\mathcal{N}^{\tau}_{\varepsilon,A,V}\cap J^{\overline{c}}_{\varepsilon,A,V}$.

Here $(\mathcal{N}^{\tau}_{\varepsilon,A,V}\cap J^{\overline{c}}_{\varepsilon,A,V})/\mathbb{S}^1$ denotes the $\mathbb{S}^1$-orbit space of $\mathcal{N}^{\tau}_{\varepsilon,A,V}\cap J^{\overline{c}}_{\varepsilon,A,V}$, where, as usual, $J^{c}_{\varepsilon,A,V}:=\left\{u\in H^1_{\varepsilon,A}(\mathbb{R}^3,\mathbb{C}):J_{\varepsilon,A,V}(u)\leq c\right\}$.

To prove Theorem 2.1 we will show that

$$\operatorname{cat}_{B_\rho M_\tau/G}M_\tau/G\leq\ \operatorname{cat}\left[\left(\mathcal{N}^{\tau}_{\varepsilon,A,V}\cap J^{d}_{\varepsilon,A,V}\right)/\mathbb{S}^1\right]\tag{3.7}$$

for some $d=d(\varepsilon)\in(c^{\tau}_{\varepsilon,A,V},\varepsilon^3\min_{x\in\mathbb{R}^3\setminus\{0\}}(\#Gx)V_\infty^{3/2}E_1)$, where

$$c^{\tau}_{\varepsilon,A,V}:=\inf_{\mathcal{N}^{\tau}_{\varepsilon,A,V}}J_{\varepsilon,A,V}.\tag{3.8}$$

To obtain inequality (3.7) we shall construct maps

$$M_\tau/G\xrightarrow{\iota_\varepsilon}\mathcal{C}/\mathbb{S}^1\xrightarrow{\beta_\varepsilon}B_\rho M_\tau/G,$$

whose composition is the inclusion $M_\tau/G\hookrightarrow B_\rho M_\tau/G$, where $\mathcal{C}$ is a union of connected components of $\mathcal{N}^{\tau}_{\varepsilon,A,V}\cap J^{d}_{\varepsilon,A,V}$. A standard argument then yields

$$\operatorname{cat}_{B_\rho M_\tau/G}M_\tau/G\leq\operatorname{cat}\left(\mathcal{C}/\mathbb{S}^1\right)\leq\operatorname{cat}\left[\left(\mathcal{N}^{\tau}_{\varepsilon,A,V}\cap J^{d}_{\varepsilon,A,V}\right)/\mathbb{S}^1\right].$$

The main ingredients for defining these maps are contained in the following two sections.

4. The entrance map

For any positive real number λ we consider the problem

$$\begin{cases}-\Delta u+\lambda u=(\frac{1}{|x|}*u^2)u,\\ u\in H^1(\mathbb{R}^3,\mathbb{R}).\end{cases}\tag{4.1}$$

Its associated energy functional $J_\lambda\colon H^1(\mathbb{R}^3,\mathbb{R})\to\mathbb{R}$ is given by

$$J_\lambda(u)=\frac{1}{2}\left\|u\right\|^2_\lambda-\frac{1}{4}\mathbb{D}(u),\qquad\text{with}\quad\left\|u\right\|^2_\lambda:=\int_{\mathbb{R}^3}\left(\left|\nabla u\right|^2+\lambda u^2\right).$$

Its Nehari manifold will be denoted by

$$\mathcal{M}_\lambda:=\left\{u\in H^1(\mathbb{R}^3,\mathbb{R}):\ u\neq0,\quad\left\|u\right\|^2_\lambda=\mathbb{D}(u)\right\}.$$

We set

$$E_\lambda:=\inf_{u\in\mathcal{M}_\lambda}J_\lambda(u).$$

The critical points of J_λ on $\mathcal{M}_\lambda$ are the nontrivial solutions to (4.1). Note that u solves (2.4) if and only if $u_\lambda(x):=\lambda u(\sqrt{\lambda}x)$ solves (4.1). Therefore,

$$E_\lambda=\lambda^{3/2}E_1.$$

Minimizers of J_λ on $\mathcal{M}_\lambda$ are called ground states. Lieb established in [13] the existence and uniqueness of ground states up to sign and translations. Recently Ma and Zhao [17] showed that every positive solution to problem (4.1) is radially symmetric, and they concluded from this fact that the positive solution to this problem is unique up to translations. We denote by ω_λ the positive solution to problem (4.1) which is radially symmetric with respect to the origin.

Fix a radial function $\varrho \in C^\infty(\mathbb{R}^3, \mathbb{R})$ such that $\varrho(x) = 1$ if $|x| \leq \frac{1}{2}$ and $\varrho(x) = 0$ if $|x| \geq 1$. For $\varepsilon > 0$ set $\varrho_\varepsilon(x) := \varrho(\sqrt{\varepsilon}x)$, $\omega_{\lambda,\varepsilon} := \varrho_\varepsilon \omega_\lambda$ and

$$\upsilon_{\lambda,\varepsilon} = \frac{\|\omega_{\lambda,\varepsilon}\|_\lambda}{\sqrt{\mathbb{D}(\omega_{\lambda,\varepsilon})}}\,\omega_{\lambda,\varepsilon}. \tag{4.2}$$

Note that $\operatorname{supp}(\upsilon_{\lambda,\varepsilon}) \subset B(0, 1/\sqrt{\varepsilon}) := \{x \in \mathbb{R}^3 : |x| \leq 1/\sqrt{\varepsilon}\}$ and $\upsilon_{\lambda,\varepsilon} \in \mathcal{M}_\lambda$. An easy computation shows that

$$\lim_{\varepsilon \to 0} J_\lambda(\upsilon_{\lambda,\varepsilon}) = \lambda^{3/2} E_1. \tag{4.3}$$

Observe that

$$\ell_{G,V} := \inf_{x \in \mathbb{R}^3} (\#Gx) V^{3/2}(x) < V^{3/2}(0) < \infty.$$

We assume from now on that there exists $\alpha > 0$ such that the set

$$\left\{ y \in \mathbb{R}^3 : (\#Gy) V^{3/2}(y) \leq \ell_{G,V} + \alpha \right\}$$

is compact. Then

$$M_{G,V} := \left\{ y \in \mathbb{R}^3 : (\#Gy) V^{3/2}(y) = \ell_{G,V} \right\}$$

is a compact G-invariant set and all G-orbits in $M_{G,V}$ are finite. We split $M_{G,V}$ according to the orbit type of its elements as follows: we choose subgroups $G_1, \ldots, G_m$ of G such that the isotropy subgroup G_x of every point $x \in M_{G,V}$ is conjugate to precisely one of the G_i's, and we set

$$M_i := \left\{ y \in M_{G,V} : G_y = gG_i g^{-1} \text{ for some } g \in G \right\}.$$

Since isotropy subgroups satisfy $G_{gx} = gG_x g^{-1}$, the sets M_i are G-invariant and, since V is continuous, they are closed and pairwise disjoint, and

$$M_{G,V} = M_1 \cup \cdots \cup M_m.$$

Moreover, since

$$|G/G_i|\, V^{3/2}(y) = (\#Gy) V^{3/2}(y) = \ell_{G,V} \quad \text{for all } \; y \in M_i,$$

the potential V is constant on each M_i. Here $|G/G_i|$ denotes the index of G_i in G. We denote by V_i the value of V on M_i.

Let $\upsilon_{i,\varepsilon} := \upsilon_{V_i,\varepsilon}$ be defined as in (4.2) with $\lambda := V_i$. For $\xi \in M_i$ set

$$\phi_{\varepsilon,\xi}(x) := \upsilon_{i,\varepsilon}\left(\frac{x - \xi}{\varepsilon}\right) \exp\left(-\mathrm{i}A(\xi) \cdot \left(\frac{x-\xi}{\varepsilon}\right)\right).$$

The proofs of the following two lemmas are similar to those of Lemmas 1 and 2 in [3], so we shall omit them.

Lemma 4.1. *Uniformly in $\xi \in M_i$, we have that*

$$\lim_{\varepsilon \to 0} \varepsilon^{-3} J_{\varepsilon,A,V}\left[\pi_{\varepsilon,A,V}(\phi_{\varepsilon,\xi})\right] = V_i^{3/2} E_1,$$

where $\pi_{\varepsilon,A,V}$ is as in (3.5).

It is well known that the map $G/G_\xi \to G\xi$ given by $gG_\xi \mapsto g\xi$ is a homeomorphism, see e.g. [9]. So, if $G_i \subset \ker \tau$ and $\xi \in M_i$, then the map

$$G\xi \to \mathbb{S}^1, \quad g\xi \mapsto \tau(g),$$

is well defined and continuous. Set

$$\psi_{\varepsilon,\xi}(x) := \sum_{g\xi \in G\xi} \tau(g) v_{i,\varepsilon}\left(\frac{x - g\xi}{\varepsilon}\right) e^{-\mathrm{i}A(g\xi)\cdot\left(\frac{x-g\xi}{\varepsilon}\right)}. \tag{4.4}$$

Lemma 4.2. *Assume that $G_i \subset \ker\tau$. Then, the following hold:*
(a) For every $\xi \in M_i$ and $\varepsilon > 0$, one has that

$$\psi_{\varepsilon,\xi}(gx) = \tau(g)\psi_{\varepsilon,\xi}(x) \qquad \forall g \in G,\ x \in \mathbb{R}^3.$$

(b) For every $\xi \in M_i$ and $\varepsilon > 0$, one has that

$$\tau(g)\psi_{\varepsilon,g\xi}(x) = \psi_{\varepsilon,\xi}(x) \qquad \forall g \in G,\ x \in \mathbb{R}^3.$$

(c) One has that

$$\lim_{\varepsilon\to 0} \varepsilon^{-3} J_{\varepsilon,A,V}\left[\pi_{\varepsilon,A,V}(\psi_{\varepsilon,\xi})\right] = \ell_{G,V} E_1.$$

uniformly in $\xi \in M_i$.

Let

$$M_\tau := \{y \in M_{G,V} : G_y \subset \ker\tau\} = \bigcup_{G_i \subset \ker\tau} M_i.$$

Proposition 4.3. *The map $\widehat{\iota}_\varepsilon : M_\tau \to \mathcal{N}^\tau_{\varepsilon,A,V}$ given by*

$$\widehat{\iota}_\varepsilon(\xi) := \pi_{\varepsilon,A,V}(\psi_{\varepsilon,\xi})$$

is well defined and continuous, and satisfies

$$\tau(g)\widehat{\iota}_\varepsilon(g\xi) = \widehat{\iota}_\varepsilon(\xi) \qquad \forall \xi \in M_\tau,\ g \in G.$$

Moreover, given $d > \ell_G E_1$, there exists $\varepsilon_d > 0$ such that

$$\varepsilon^{-3}\ J_{\varepsilon,A,V}(\widehat{\iota}_\varepsilon(\xi)) \le d \qquad \forall \xi \in M_\tau,\ \varepsilon \in (0,\varepsilon_d).$$

Proof. This follows immediately from Lemma 4.2. □

5. A local baryorbit map

Let $W : \mathbb{R}^3 \to \mathbb{R}$ be a bounded, uniformly continuous function with $\inf_{\mathbb{R}^3} W > 0$ and such that $W(gx) = W(x)$ for all $g \in G$, $x \in \mathbb{R}^3$. We assume that the set

$$\left\{ y \in \mathbb{R}^3 : (\#Gy) W^{3/2}(y) \le \ell_{G,W} + \alpha \right\} \tag{5.1}$$

is compact, where $\ell_{G,W} := \inf_{x\in\mathbb{R}^3}(\#Gx)W^{3/2}(x)$, and consider the real-valued problem

$$\begin{cases} -\varepsilon^2\Delta v + W(x)v = \frac{1}{\varepsilon^2}\left(\frac{1}{|x|} * u^2\right) u, \\ v \in H^1(\mathbb{R}^3,\mathbb{R}), \\ v(gx) = v(x) \ \ \forall x \in \mathbb{R}^3,\ g \in G. \end{cases} \tag{5.2}$$

We write

$$\langle v, w\rangle_{\varepsilon,W} := \int_{\mathbb{R}^3} \left(\varepsilon^2 \nabla v \cdot \nabla w + W(x) v w\right), \quad \|v\|^2_{\varepsilon,W} := \int_{\mathbb{R}^3} \left(|\varepsilon\nabla v|^2 + W(x)v^2\right),$$

and set

$$H^1(\mathbb{R}^3,\mathbb{R})^G := \{v \in H^1(\mathbb{R}^3,\mathbb{R}) : v(gx) = v(x)\ \forall x \in \mathbb{R}^3,\ g \in G\}.$$

The nontrivial solutions of (5.2) are the critical points of the energy functional

$$J_{\varepsilon,W}(v) = \frac{1}{2}\|v\|^2_{\varepsilon,W} - \frac{1}{4\varepsilon^2}\mathbb{D}(v)$$

on the Nehari manifold

$$\mathcal{M}^G_{\varepsilon,W} := \{v \in H^1(\mathbb{R}^3,\mathbb{R})^G : v \neq 0,\ \ \|v\|^2_{\varepsilon,W} = \varepsilon^{-2}\mathbb{D}(v)\}.$$

Set

$$c^G_{\varepsilon,W} := \inf_{\mathcal{M}^G_{\varepsilon,W}} J_{\varepsilon,W} = \inf_{\substack{v\in H^1(\mathbb{R}^3,\mathbb{R})^G \\ v\neq 0}} \frac{\varepsilon^2 \|v\|^4_{\varepsilon,W}}{4\mathbb{D}(v)}. \tag{5.3}$$

We wish to study the behavior of "minimizing sequences" for the family of problems (5.2), parametrized by ε, as $\varepsilon \to 0$. This is described in Proposition 5.4 below. We start with some lemmas.

Lemma 5.1. $0 < (\inf_{\mathbb{R}^3} W)^{3/2} E_1 \leq \varepsilon^{-3} c^G_{\varepsilon,W}$ *for every* $\varepsilon > 0,$ *and*

$$\limsup_{\varepsilon\to 0} \varepsilon^{-3} c^G_{\varepsilon,W} \leq \ell_{G,W} E_1,$$

Proof. Set $W_0 := \inf_{\mathbb{R}^3} W$ and write $v_\varepsilon(x) := v(\varepsilon x)$. Then $\|v_\varepsilon\|^2_{W_0} = \varepsilon^{-3}\|v\|^2_{\varepsilon,W_0}$ and $\mathbb{D}(v_\varepsilon) = \varepsilon^{-5}\mathbb{D}(v)$. If follows immediately from (5.3) that

$$W_0^{3/2} E_1 \leq c^G_{1,W_0} = \varepsilon^{-3} c^G_{\varepsilon,W_0} \leq \varepsilon^{-3} c^G_{\varepsilon,W}.$$

To prove the second inequality, take $\xi \in \mathbb{R}^3$ such that $(\#G\xi)W^{3/2}(\xi) = \ell_{G,W}E_1$. Write $G\xi := \{\xi_1, ..., \xi_m\}$. Fix $0 < \rho < \frac{1}{2}\min_{i\neq j}|\xi_i - \xi_j|$, and let $W_\rho := \sup_{B(\xi_1,\rho)} W$. Let $\upsilon_{\rho,\varepsilon} := \upsilon_{W_\rho,\varepsilon}$ be defined as in (4.2) with $\lambda := W_\rho$. Set

$$w_{\rho,\varepsilon}(x) := \sum_{i=1}^m \upsilon_{\rho,\varepsilon}\left(\frac{x-\xi_i}{\varepsilon}\right).$$

If $\sqrt{\varepsilon} \leq \rho$, then $\operatorname{supp}(w_{\rho,\varepsilon}) \subset \cup_{i=1}^m B(\xi_i,\rho)$. Therefore $w_{\rho,\varepsilon} \in \mathcal{M}^G_{\varepsilon,W_\rho}$ and

$$\varepsilon^{-3} c^G_{\varepsilon,W} \leq \varepsilon^{-3} J_{\varepsilon,W}(w_{\rho,\varepsilon}) \leq \varepsilon^{-3} J_{\varepsilon,W\rho}(w_{\rho,\varepsilon}) = mJ_{W_\rho}(\upsilon_{\rho,\varepsilon}).$$

It follows from (4.3) that

$$\limsup_{\varepsilon\to 0} \varepsilon^{-3} c^G_{\varepsilon,W} \leq mW_\rho^{3/2} E_1.$$

Letting $\rho \to 0$, we conclude that

$$\limsup_{\varepsilon\to 0} \varepsilon^{-3} c^G_{\varepsilon,W} \leq (\#G\xi)W^{3/2}(\xi)E_1 = \ell_{G,W}E_1,$$

as claimed. □

Lemma 5.2. *Let* $\varepsilon_n > 0$ *and* $\xi_n \in \mathbb{R}^3$ *such that* $\varepsilon_n \to 0$ *and* $(W(\xi_n))$ *converges. Set* $\widehat{W}_n(x) := W(\varepsilon_n x + \xi_n)$ *and* $\widehat{W} := \lim_{n\to\infty} W(\xi_n)$. *Then, for every sequence* (u_n) *in* $H^1(\mathbb{R}^3,\mathbb{R})$ *such that* $u_n \rightharpoonup u$ *weakly in* $H^1(\mathbb{R}^3,\mathbb{R})$ *and every* $w \in H^1(\mathbb{R}^3,\mathbb{R})$, *the following hold:*

$$\lim_{n\to\infty}\left(\langle u_n, w\rangle_{1,\widehat{W}_n} - \langle u_n - u, w\rangle_{1,\widehat{W}_n}\right) = \langle u, w\rangle_{1,\widehat{W}}$$

and

$$\lim_{n\to\infty}\left(\|u_n\|^2_{1,\widehat{W}_n} - \|u_n - u\|^2_{1,\widehat{W}_n}\right) = \|u\|^2_{1,\widehat{W}}.$$

Proof. The argument is similar for both equalities. We prove the second one. Since (u_n) is bounded in $L^2(\mathbb{R}^3)$ there exists $C > 2\|u\|_{L^2(\mathbb{R}^3)}$ such that

$$\begin{aligned}&\left|\|u_n\|^2_{1,\widehat{W}_n} - \|u_n - u\|^2_{1,\widehat{W}_n} - \|u\|^2_{1,\widehat{W}}\right| \\ &\le \left|\|u_n\|^2_{1,\widehat{W}} - \|u_n - u\|^2_{1,\widehat{W}} - \|u\|^2_{1,\widehat{W}}\right| + \int_{\mathbb{R}^3}\left|(\widehat{W}_n - \widehat{W})(2u_n u - u^2)\right| \\ &\le o(1) + C\left\|(\widehat{W}_n - \widehat{W})u\right\|_{L^2(\mathbb{R}^3)}.\end{aligned}$$

Given $\varepsilon > 0$ we fix $R > 0$ such that

$$\int_{|x|\ge R}(\widehat{W}_n - \widehat{W})^2 u^2 \le (2\sup_{x\in\mathbb{R}^3} W)^2\int_{|x|\ge R} u^2 < \varepsilon^2.$$

Since W is uniformly continuous, there exists $\delta > 0$ such that

$$|W(\varepsilon_n x + \xi_n) - W(\xi_n)| < \frac{\varepsilon}{C} \quad \text{if } |x| < \frac{\delta}{\varepsilon_n}.$$

Fix $n_0 \in \mathbb{N}$ such that $|W(\xi_n) - \widehat{W}| < \frac{\varepsilon}{C}$ and $\frac{\delta}{\varepsilon_n} > R$ if $n \ge n_0$. Then,

$$\int_{|x|\le R}(\widehat{W}_n - \widehat{W})^2 u^2 < \varepsilon^2 \quad \text{for all } n \ge n_0.$$

Therefore,

$$\lim_{n\to\infty}\left\|(\widehat{W}_n - \widehat{W})u\right\|_{L^2(\mathbb{R}^3)} = 0.$$

This concludes the proof. □

Lemma 5.3. *Let (z_n) be a sequence in $\mathbb{R}^N$. Then, after passing to a subsequence, there exist a closed subgroup Γ of G and a sequence (ζ_n) in $\mathbb{R}^N$ such that*
(a) $(\mathrm{dist}(Gz_n, \zeta_n))$ *is bounded,*
(b) $G_{\zeta_n} = \Gamma$,
(c) if $|G/\Gamma| < \infty$ then $|g\zeta_n - \tilde{g}\zeta_n| \to \infty$ for all $g, \tilde{g} \in G$ with $\tilde{g}g^{-1} \notin \Gamma$,
(d) if $|G/\Gamma| = \infty$, there exists a closed subgroup Γ' of G such that $\Gamma \subset \Gamma'$, $|G/\Gamma'| = \infty$ and $|g\zeta_n - \tilde{g}\zeta_n| \to \infty$ for all $g, \tilde{g} \in G$ with $\tilde{g}g^{-1} \notin \Gamma'$.

Proof. See Lemma 3.2 in [5]. □

Set

$$M_{G,W} := \left\{y \in \mathbb{R}^3 : (\#Gy)W^{3/2}(y) = \ell_{G,W}\right\}$$

Abusing notation we write again G_i and M_i for the groups and the sets defined as in Section 4 but now for W instead of V. So the value of W on M_i is constant and we denote it by W_i. We fix $\widehat{\rho} > 0$ such that

$$\begin{aligned}&|y - gy| > 2\widehat{\rho} && \text{if } gy \ne y \in M_{G,W},\\ &\mathrm{dist}(M_i, M_j) > 2\widehat{\rho} && \text{if } i \ne j,\end{aligned} \tag{5.4}$$

For $\rho \in (0, \widehat{\rho})$, let

$$M_i^\rho := \{y \in \mathbb{R}^3 : \mathrm{dist}(y, M_i) \le \rho, \ \ G_y = gG_i g^{-1} \text{ for some } g \in G\},$$

and for each $\xi \in M_i^\rho$ and $\varepsilon > 0$, define

$$\theta_{\varepsilon,\xi}(x) := \sum_{g\xi\in G\xi}\omega_i\left(\frac{x - g\xi}{\varepsilon}\right),$$

where ω_i is unique positive ground state of problem (4.1) with $\lambda := W_i$ which is radially symmetric with respect to the origin. Set

$$\Theta_{\rho,\varepsilon} := \{\theta_{\varepsilon,\xi} : \xi \in M_1^\rho \cup \cdots \cup M_m^\rho\}.$$

The following holds.

Proposition 5.4. *Let $\varepsilon_n > 0$ and $v_n \in H^1(\mathbb{R}^3,\mathbb{R})^G$ be such that*

$$\varepsilon_n \to 0, \quad \varepsilon_n^{-3} J_{\varepsilon_n,W}(v_n) \to \widehat{c}, \quad \varepsilon_n^{-3} \left\| \nabla_{\varepsilon_n} J_{\varepsilon_n,W}(v_n) \right\|^2_{\varepsilon_n,W} \to 0, \tag{5.5}$$

where $\widehat{c} := \liminf_{\varepsilon\to 0} \varepsilon^{-3} c^G_{\varepsilon,W}$ and $\nabla_{\varepsilon_n} J_{\varepsilon_n,W}$ is the gradient of $J_{\varepsilon_n,W}$ with respect to the scalar product $\langle\cdot,\cdot\rangle_{\varepsilon_n,W}$. Then, passing to a subsequence, there exist an $i \in \{1,...,m\}$ and a sequence (ξ_n) in $\mathbb{R}^3$ such that
(i) $G_{\xi_n} = G_i$,
(ii) $\xi_n \to \xi \in M_i$,
(iii) $\varepsilon_n^{-3} \left\| |v_n| - \theta_{\varepsilon_n,\xi_n} \right\|^2_{\varepsilon_n,W} \to 0$,
(iv) $\widehat{c} = \lim_{\varepsilon\to 0} \varepsilon^{-3} c^G_{\varepsilon,W} = \ell_{G,W} E_1$.

Proof. A standard argument shows that the sequence $(\varepsilon_n^{-3}\|v_n\|^2_{\varepsilon_n,W})$ is bounded and that

$$\lim_{n\to\infty} \varepsilon_n^{-3}\|v_n\|^2_{\varepsilon_n,W} = \lim_{n\to\infty} \varepsilon_n^{-5}\mathbb{D}(v_n) = 4\widehat{c} =: c > 0.$$

Let $\widetilde{v}_n \in H^1(\mathbb{R}^3,\mathbb{R})^G$ be given by $\widetilde{v}_n(z) := v_n(\varepsilon_n z)$. Then,

$$\|\widetilde{v}_n\|^2_{1,W_n} = \varepsilon^{-3}\|v_n\|^2_{\varepsilon_n,W} \quad \text{and} \quad \mathbb{D}(\widetilde{v}_n) = \varepsilon_n^{-5}\mathbb{D}(v_n),$$

where $W_n(z) := W(\varepsilon_n z)$. Set

$$\delta := \limsup_{n\to\infty} \sup_{y\in\mathbb{R}^3} \int_{B(y,1)} |\widetilde{v}_n|^2 .$$

Since $c > 0$, Lions' lemma [28, Lemma 1.21], together with inequality (3.4), yields that $\delta > 0$. Choose $z_n \in \mathbb{R}^3$ such that

$$\int_{B(z_n,1)} |\widetilde{v}_n|^2 \geq \frac{\delta}{2}$$

and replace (z_n) by a sequence (ζ_n) having the properties stated in Lemma 5.3. Set $\widehat{v}_n(z) := \widetilde{v}_n(z+\zeta_n)$. After passing to a subsequence, we may assume that $\widehat{v}_n \rightharpoonup \widehat{v}$ weakly in $H^1(\mathbb{R}^3,\mathbb{R})$, $\widehat{v}_n(x) \to \widehat{v}(x)$ a.e. on $\mathbb{R}^3$ and $\widehat{v}_n \to \widehat{v}$ in $L^2_{loc}(\mathbb{R}^3,\mathbb{R})$. Choosing $C \geq \operatorname{dist}(\zeta_n, Gz_n)$ for all n, we obtain

$$\int_{B(0,C+1)} |\widehat{v}_n|^2 = \int_{B(\zeta_n,C+1)} |\widetilde{v}_n|^2 \geq \int_{B(z_n,1)} |\widetilde{v}_n|^2 \geq \frac{\delta}{2}.$$

Therefore, $\widehat{v} \neq 0$.
Set $\xi_n := \varepsilon_n\zeta_n$ and $\widehat{W}_n(x) := W(\varepsilon_n x + \xi_n)$. Since W is bounded, a subsequence of $W(\xi_n)$ converges. We set $\widehat{W} := \lim_{n\to\infty} W(\xi_n)$. The weak continuity of $\mathbb{D}'$ [2, Lemma 3.5], together with Lemma 5.2 and assumption (5.5) imply that $\widehat{v}$ is a solution to problem (4.1) with $\lambda := \widehat{W}$.
Since v_n and W are G-invariant we have that $\widehat{v}_n(g^{-1}x) = v_n(\varepsilon_n x + g\xi_n)$, $\widehat{W}_n(g^{-1}x) = W(\varepsilon_n x + g\xi_n)$, and $\widehat{W} := \lim_{n\to\infty} W(g\xi_n)$ for each $g \in G$. Fix $g_1, ..., g_k \in G$ such that $|g_i\zeta_n - g_j\zeta_n| \to \infty$ if $i \neq j$. Then,

$$\widehat{v}_n g_j^{-1} - \sum_{i=j+1}^{k} \widehat{v} g_i^{-1}(\cdot - g_i\zeta_n + g_j\zeta_n) \rightharpoonup \widehat{v} g_j^{-1} \tag{5.6}$$

weakly in $H^1(\mathbb{R}^3,\mathbb{R})$. Applying Lemma 5.2 we obtain

$$\left\| \widehat{v}_n g_j^{-1} - \sum_{i=j+1}^{k} \widehat{v} g_i^{-1} \left(\cdot - g_i\zeta_n + g_j\zeta_n\right) \right\|^2_{1,\widehat{W}_n g_j^{-1}}$$
$$= \left\| \widehat{v}_n g_j^{-1} - \widehat{v} g_j^{-1} - \sum_{i=j+1}^{k} \widehat{v} g_i^{-1} \left(\cdot - g_i\zeta_n + g_j\zeta_n\right) \right\|^2_{1,\widehat{W}_n g_j^{-1}} + \left\| \widehat{v} g_j^{-1} \right\|^2_{1,\widehat{W}} + o(1),$$

and performing the change of variable $y = \varepsilon_n x + g_j\xi_n$ we conclude that

$$\varepsilon_n^{-3} \left\| v_n - \sum_{i=j+1}^{k} \widehat{v} g_i^{-1} \left(\frac{\cdot - g_i\xi_n}{\varepsilon_n} \right) \right\|^2_{\varepsilon_n,W}$$
$$= \varepsilon_n^{-3} \left\| v_n - \sum_{i=j}^{k} \widehat{v} g_i^{-1} \left(\frac{\cdot - g_i\xi_n}{\varepsilon_n} \right) \right\|^2_{\varepsilon_n,W} + \|\widehat{v}\|^2_{1,\widehat{W}} + o(1).$$

Iterating these equalities we conclude that

$$4\widehat{c} = \lim_{n\to\infty} \varepsilon_n^{-3} \|v_n\|^2_{\varepsilon_n,W} = \lim_{n\to\infty} \varepsilon_n^{-3} \left\| v_n - \sum_{i=1}^{k} \widehat{v} g_i^{-1} \left(\frac{\cdot - g_i\xi_n}{\varepsilon_n} \right) \right\|^2_{\varepsilon_n,W} + k \|\widehat{v}\|^2_{1,\widehat{W}}.$$

This implies that $4\widehat{c} \geq k \|\widehat{v}\|^2_{1,\widehat{W}}$ which, together with property *(d)* in Lemma 5.3, implies $|G/\Gamma| < \infty$. Property *(c)* allows us to take $k := |G/\Gamma|$. Then, property *(b)* and Lemma 5.1 yield

$$\ell_{G,W} E_1 \leq \lim_{n\to\infty} (\#G\xi_n) W^{3/2}(\xi_n) E_1 = |G/\Gamma| \widehat{W}^{3/2} E_1$$
$$\leq |G/\Gamma| \frac{1}{4} \|\widehat{v}\|^2_{1,\widehat{W}} \leq \widehat{c} \leq \limsup_{\varepsilon\to 0} \varepsilon^{-3} c^G_{\varepsilon,W} \leq \ell_{G,W} E_1.$$

This proves *(iv)* and gives also

$$\lim_{n\to\infty} \varepsilon_n^{-3} \left\| v_n - \sum_{i=1}^{k} \widehat{v} g_i^{-1} \left(\frac{\cdot - g_i\xi_n}{\varepsilon_n} \right) \right\|^2_{\varepsilon_n,W} = 0. \tag{5.7}$$

Moreover, $(\#G\xi_n)W^{3/2}(\xi_n) \leq \ell_{G,W} + \alpha$ for n large enough. Thus, assumption (2.5) implies, after passing to a subsequence, that $\xi_n \to \xi$. Hence, $W(\xi) = \widehat{W}$ and

$$\ell_{G,W} E_1 \leq (\#G\xi) W(\xi) E_1 \leq |G/\Gamma| \widehat{W}^{3/2} E_1 \leq |G/\Gamma| \frac{1}{4} \|\widehat{v}\|^2_{1,\widehat{W}} \leq \ell_{G,W} E_1.$$

We conclude that $\xi \in M_i$ for some $i = 1, ..., m$, as claimed in *(ii)*. Then, $\widehat{W} = W_i$, $\Gamma = G_\xi = gG_ig^{-1}$ for some $g \in G$, and $\widehat{v}$ is a ground state of problem (4.1) with $\lambda = W_i$.

Since the ground state is unique up to sign and translation we must have that $\widehat{v}(z) = \pm\omega_i(z - z_0)$ for some $z_0 \in \mathbb{R}^3$. Observe that $\widehat{v}$ is Γ-invariant. So, if Γ is nontrivial, then $z_0 = 0$ and, since ω_i is radial, equation (5.7) becomes *(iii)*. If, on the other hand, Γ is the trivial group, we replace ξ_n by $\xi_n' := \xi_n + \varepsilon_n z_0$. Since $G\xi_n \cong G$ and $\varepsilon_n \to 0$, ξ_n' has the same properties as ξ_n for n large enough. Moreover, since ω_i is radially symmetric,

$$\widehat{v}\left(\frac{g^{-1}z - \xi_n}{\varepsilon_n}\right) = \pm\omega_i\left(\frac{g^{-1}z - \xi_n'}{\varepsilon_n}\right) = \pm\omega_i\left(\frac{z - g\xi_n'}{\varepsilon_n}\right)$$

and, again, equation (5.7) yields *(iii)*. This completes the proof. □

Proposition 5.5. *Given $\rho \in (0, \widehat{\rho})$ there exist $d_\rho > \ell_{G,W} E_1$ and $\varepsilon_\rho > 0$ with the following property: For every $\varepsilon \in (0, \varepsilon_\rho)$ and every $v \in \mathcal{M}^G_{\varepsilon,W}$ with $J_{\varepsilon,W}(v) \leq \varepsilon^3 d_\rho$ there exists precisely one G-orbit $G\xi_{\varepsilon,v}$ with $\xi_{\varepsilon,v} \in M^\rho_1 \cup \cdots \cup M^\rho_m$ such that*

$$\varepsilon^{-3} \left\| |v| - \theta_{\varepsilon,\xi_{\varepsilon,v}} \right\|^2_{\varepsilon,W} = \min_{\theta \in \Theta_{\rho,\varepsilon}} \| |v| - \theta \|^2_{\varepsilon,W}.$$

Proof. The proof is analogous to that of Proposition 5.3 in [4]. We omit the details. □

Fix $\rho \in (0, \widehat{\rho})$ and $\varepsilon \in (0, \varepsilon_\rho)$. Proposition 5.5 allows us to define a map

$$\widehat{\beta}_{\rho,\varepsilon,0} : \mathcal{M}^G_{\varepsilon,W} \cap J^{\varepsilon^3 d_\rho}_{\varepsilon,W} \longrightarrow \left(M^\rho_1 \cup \cdots \cup M^\rho_m\right)/G \tag{5.8}$$

by taking

$$\widehat{\beta}_{\rho,\varepsilon,0}(v) := G\xi_{\varepsilon,v}.$$

Here, as usual, $J^c_{\varepsilon,W} := \{v \in H^1(\mathbb{R}^3, \mathbb{R}) : J_{\varepsilon,W}(v) \leq c\}$. The map $\widehat{\beta}_{\rho,\varepsilon,0}$ is the G-equivariant analogon to the usual baricenter map. It is only defined for functions in $\mathcal{M}^G_{\varepsilon,W}$ with small enough energy. We call it the *local baryorbit map.* It reflects the fact that such functions concentrate at a unique G-orbit with minimal cardinality as $\varepsilon \to 0$.

6. Proofs of the main results

Let $V_\infty := \liminf_{|x|\to\infty} V(x)$. Assumption (2.5) implies that

$$\ell_{G,V} < \min_{x \in \mathbb{R}^3 \setminus \{0\}} (\#Gx) V^{3/2}_\infty.$$

We fix $\delta_0 > 0$ and $\lambda \in (0, V_\infty)$ such that

$$\ell_{G,V} E_1 + \delta_0 < \min_{x \in \mathbb{R}^3 \setminus \{0\}} (\#Gx) \lambda^{3/2} E_1 < \min_{x \in \mathbb{R}^3 \setminus \{0\}} (\#Gx) V^{3/2}_\infty E_1, \tag{6.1}$$

and define $W(x) := \min\{V(x), \lambda\}$. This W has all properties stated in section 5, in particular, it is uniformly continuous. Moreover, $\ell_{G,W} = \ell_{G,V}$ and $M_{G,W} = M_{G,V}$.

Let $\pi_{\varepsilon,W} : H^1(\mathbb{R}^3, \mathbb{R})^G \setminus \{0\} \to \mathcal{M}^G_{\varepsilon,W}$ denote the radial projection onto the Nehari manifold, which is given by

$$\pi_{\varepsilon,W}(v) := \frac{\varepsilon \|u\|_{\varepsilon,W}}{\sqrt{\mathbb{D}(u)}} v. \tag{6.2}$$

Observe that

$$J_{\varepsilon,W}(\pi_{\varepsilon,W}(v)) = \frac{\varepsilon^2 \|v\|^4_{\varepsilon,W}}{4\mathbb{D}(v)} \qquad \text{for all } v \in H^1(\mathbb{R}^3, \mathbb{R})^G \setminus \{0\}. \tag{6.3}$$

Let $\widehat{\iota}_\varepsilon$ be the map defined in Proposition 4.3 and $\widehat{\beta}_{\rho,\varepsilon,0}$ be as in (5.8). Then, for $d_\rho > \ell_{G,V} E_1$ and $\varepsilon_\rho > 0$ as in Proposition 5.5 the following holds.

Proposition 6.1. *For each $\rho \in (0, \widehat{\rho})$ and $\varepsilon \in (0, \varepsilon_\rho)$, the map*

$$\widehat{\beta}_{\rho,\varepsilon} : \mathcal{N}^\tau_{\varepsilon,A,V} \cap J^{\varepsilon^3 d_\rho}_{\varepsilon,A,V} \to \left(M^\rho_1 \cup \cdots \cup M^\rho_m\right)/G, \qquad \widehat{\beta}_{\rho,\varepsilon}(u) := \widehat{\beta}_{\rho,\varepsilon,0}(\pi_{\varepsilon,W}(|u|)),$$

is well defined and continuous and satisfies

(i) $\widehat{\beta}_{\rho,\varepsilon}(\gamma u) = \widehat{\beta}_{\rho,\varepsilon}(u)$ *for all* $\gamma \in \mathbb{S}^1$,

(ii) $\widehat{\beta}_{\rho,\varepsilon}(\widehat{\iota}_\varepsilon(\xi)) = G\xi$ *for all* $\xi \in M_\tau$ *with* $J_{\varepsilon,A,V}(\widehat{\iota}_\varepsilon(\xi)) \leq \varepsilon^3 d_\rho$.

Proof. If $u \in \mathcal{N}^\tau_{\varepsilon,A,V}$ then $|u| \in H^1(\mathbb{R}^3,\mathbb{R})^G \setminus \{0\}$ and, since $W \leq V$, formulas (6.3) and (3.6), together with the diamagnetic inequality (3.2) yield

$$J_{\varepsilon,W}(\pi_{\varepsilon,W}(|u|)) \leq J_{\varepsilon,V}(\pi_{\varepsilon,V}(|u|)) \leq J_{\varepsilon,A,V}(u). \tag{6.4}$$

So $J_{\varepsilon,W}(\pi_{\varepsilon,W}(|u|)) \leq \varepsilon^3 d_\rho$ if $J_{\varepsilon,A,V}(u) \leq \varepsilon^3 d_\rho$. Therefore, $\widehat{\beta}_{\rho,\varepsilon}$ is well defined. It is straightforward to verify that it has the desired properties. □

Let

$$M^\rho_\tau := \bigcup_{G_i \subset \ker \tau} M^\rho_i.$$

Propositions 4.3 and 6.1 allow us to estimate the Lusternik-Schnirelmann category of low energy sublevel sets as follows.

Corollary 6.2. *For every $\rho \in (0,\widehat{\rho})$ and $d \in (\ell_{G,V}E_1, d_\rho)$ there exists $\varepsilon_{\rho,d} > 0$ such that*

$$\operatorname{cat}_{M^\rho_\tau/G} M_\tau/G \leq \operatorname{cat}\left((\mathcal{N}^\tau_{\varepsilon,A,V} \cap J^{\varepsilon^3 d}_{\varepsilon,A,V})/\mathbb{S}^1\right)$$

for every $\varepsilon \in (0,\varepsilon_{\rho,d})$.

Proof. Set $\varepsilon_{\rho,d} := \min\{\varepsilon_d, \varepsilon_\rho\}$ where ε_d is as in Proposition 4.3. Fix $\varepsilon \in (0,\varepsilon_{\rho,d})$. Then,

$$J_{\varepsilon,A,V}(\widehat{\iota}_\varepsilon(\xi)) \leq \varepsilon^3 d \quad \text{and} \quad \widehat{\beta}_{\rho,\varepsilon}(\widehat{\iota}_\varepsilon(\xi)) = \xi \quad \text{for all } \xi \in M_\tau.$$

Since $M^\rho_1, ..., M^\rho_m$ are G-invariant and pairwise disjoint, the set

$$\mathcal{C} := \{u \in \mathcal{N}^\tau_{\varepsilon,A,V} \cap J^{\varepsilon^3 d}_{\varepsilon,A,V} : \widehat{\beta}_{\rho,\varepsilon}(u) \in M^\rho_\tau/G\}$$

is a union of connected components of $\mathcal{N}^\tau_{\varepsilon,A,V} \cap J^{\varepsilon^3 d}_{\varepsilon,A,V}$. Therefore,

$$\operatorname{cat}\left(\mathcal{C}/\mathbb{S}^1\right) \leq \operatorname{cat}\left((\mathcal{N}^\tau_{\varepsilon,A,V} \cap J^{\varepsilon^3 d}_{\varepsilon,A,V})/\mathbb{S}^1\right).$$

By Propositions 4.3 and 6.1, the maps

$$M_\tau/G \xrightarrow{\iota_\varepsilon} \mathcal{C}/\mathbb{S}^1 \xrightarrow{\beta_{\rho,\varepsilon}} M^\rho_\tau/G,$$

given by $\iota_\varepsilon(G\xi) := \widehat{\iota}_\varepsilon(\xi)$ and $\beta_{\rho,\varepsilon}(\mathbb{S}^1 u) := \widehat{\beta}_{\rho,\varepsilon}(u)$, are well defined and satisfy $\beta_{\rho,\varepsilon}(\iota_\varepsilon(\xi)) = \xi$ for all $\xi \in M_\tau$. Therefore,

$$\operatorname{cat}_{M^\rho_\tau/G} M_\tau/G \leq \operatorname{cat}\left(\mathcal{C}/\mathbb{S}^1\right).$$

This finishes the proof. □

Another consequence of our previous results is the following.

Corollary 6.3. *If there exists $\xi \in \mathbb{R}^3$ such that $(\#G\xi)V^{3/2}(\xi) = \ell_{G,V}$ and $G_\xi \subset \ker\tau$, then*

$$\lim_{\varepsilon\to\infty} \varepsilon^{-3} c^\tau_{\varepsilon,A,V} = \ell_{G,V}E_1,$$

where $c^\tau_{\varepsilon,A,V} := \inf_{\mathcal{N}^\tau_{\varepsilon,A,V}} J_{\varepsilon,A,V}$.

Proof. Inequality (6.4) yields $c^G_{\varepsilon,W} := \inf_{\mathcal{M}^G_{\varepsilon,W}} J_{\varepsilon,W} \leq \inf_{\mathcal{N}^\tau_{\varepsilon,A,V}} J_{\varepsilon,A,V} =: c^\tau_{\varepsilon,A,V}$. By Proposition 5.4 and Lemma 4.2*(c)*,

$$\ell_{G,W}E_1 = \lim_{\varepsilon\to\infty} \varepsilon^{-3} c^G_{\varepsilon,W} \leq \liminf_{\varepsilon\to 0} \varepsilon^{-3} c^\tau_{\varepsilon,A,V} \leq \limsup_{\varepsilon\to\infty} \varepsilon^{-3} c^\tau_{\varepsilon,A,V} \leq \ell_{G,V}E_1.$$

Since $\ell_{G,W}E_1 = \ell_{G,V}E_1$, our claim follows. □

Proof of Theorem 2.1. Let $\rho, \delta > 0$ be given. We may assume that $\rho \in (0, \widehat{\rho})$ with $\widehat{\rho}$ as in (5.4) and that $\delta \in (0, \delta_0)$ with δ_0 as in (6.1). By Corollary 6.3 there exists $\varepsilon_\delta > 0$ such that

$$\ell_{G,V} E_1 - \delta < \varepsilon^{-3} c^{\tau}_{\varepsilon,A,V} \qquad \text{for all } \varepsilon \in (0, \varepsilon_\delta).$$

Fix $d \in (\ell_{G,V}E_1, \min\{d_\rho, \ell_{G,V}E_1 + \delta\})$ and set $\widehat{\varepsilon} := \min\{\varepsilon_\delta, \varepsilon_{\rho,d}\}$ with $\varepsilon_{\rho,d}$ as in Corollary 6.2. Since (6.1) holds, Proposition 3.1 asserts that $J_{\varepsilon,A,V} : \mathcal{N}^{\tau}_{\varepsilon,A,V} \to \mathbb{R}$ satisfies $(PS)_c$ for every $c \le \varepsilon^3 d$. Applying Proposition 3.2 and Corollary 6.2 we conclude that $J_{\varepsilon,A,V}$ has at least

$$\operatorname{cat}_{M^{\rho}_{\tau}/G} M_\tau / G$$

geometrically distinct solutions $u \in \mathcal{N}^{\tau}_{\varepsilon,A,V}$ satisfying

$$\varepsilon^3 \ell_{G,V} E_1 - \varepsilon^3 \delta < J_{\varepsilon,A,V}(u) = \frac{1}{4\varepsilon^2}\mathbb{D}(u) \le \varepsilon^3 d < \varepsilon^3 \ell_{G,V} E_1 + \varepsilon^3 \delta,$$

for each $\varepsilon \in (0, \widehat{\varepsilon})$, as claimed. □

Proof of Theorem 2.2. After passing to a subsequence, we may assume that $\ell_{G,V}E_1 - \frac{1}{2n} \le \varepsilon_n^{-3} c^{G}_{\varepsilon_n,W}$ and $\delta_n \le \frac{1}{2n}$. Then, inequality (6.4) yields

$$c^{G}_{\varepsilon_n,W} \le J_{\varepsilon_n,W}(\pi_{\varepsilon_n,W}(|u_n|)) \le J_{\varepsilon_n,A,V}(u_n) \le \varepsilon_n^3(\ell_G E_1 + \delta_n) \le c^{G}_{\varepsilon_n,W} + \varepsilon_n^3/n. \tag{6.5}$$

By Ekeland's variational principle [28, Theorem 8.5] we may choose $v_n \in \mathcal{M}^{G}_{\varepsilon_n,W}$ such that

$$\varepsilon_n^{-3} \left\| \pi_{\varepsilon_n,W}(|u_n|) - v_n \right\|^2_{\varepsilon_n,W} \to 0, \tag{6.6}$$

$$\varepsilon_n^{-3} J_{\varepsilon_n,W}(v_n) \to \ell_G E_1 \quad \text{and} \quad \varepsilon_n^{-3} \|\nabla_{\varepsilon_n} J_{\varepsilon_n,W}(v_n)\|^2_{\varepsilon_n,W} \to 0.$$

After passing again to a subsequence, Proposition 5.4 gives a sequence (ξ_n) in $\mathbb{R}^3$ such that $\xi_n \to \xi \in M_\tau$, $G_{\xi_n} = G_\xi$, and

$$\varepsilon_n^{-3} \left\| v_n - \sum_{g\xi_n \in G\xi_n} \omega_\xi \left(\frac{\cdot - g\xi_n}{\varepsilon_n} \right) \right\|^2_{\varepsilon_n,W} \to 0. \tag{6.7}$$

Since $u_n \in \mathcal{N}^{\tau}_{\varepsilon_n,A,V}$, multiplying inequality (6.5) by $4\varepsilon_n^2 \left[\mathbb{D}(|u_n|)\right]^{-1} = 4\varepsilon_n^2 \left[\mathbb{D}(u_n)\right]^{-1} = \left[J_{\varepsilon_n,A,V}(u_n)\right]^{-1}$, using (6.3) and (3.6), and observing that $\varepsilon_n^{-3} J_{\varepsilon_n,A,V}(u_n) \to \ell_G E_1$, we get

$$\begin{aligned}
\left| 1 - \left(\frac{\varepsilon_n \| \, |u_n| \, \|_{\varepsilon_n,W}}{\sqrt{\mathbb{D}(|u_n|)}} \right)^4 \right| &= \left| \left(\frac{\varepsilon_n^2 \|u_n\|^2_{\varepsilon_n,A,V}}{\mathbb{D}(u_n)} \right)^2 - \left(\frac{\varepsilon_n^2 \| \, |u_n| \, \|^2_{\varepsilon_n,W}}{\mathbb{D}(|u_n|)} \right)^2 \right| \\
&\le \left(\varepsilon_n^{-3} J_{\varepsilon_n,A,V}(u_n) \right)^{-1} \frac{1}{n} \to 0.
\end{aligned}$$

Recalling (6.2) and using the diamagnetic inequality (3.2), we then obtain

$$\begin{aligned}
\varepsilon_n^{-3} \| \, |u_n| - \pi_{\varepsilon_n,W}(|u_n|) \, \|^2_{\varepsilon_n,W} &= \left| 1 - \frac{\varepsilon_n \| \, |u_n| \, \|_{\varepsilon_n,W}}{\sqrt{\mathbb{D}(|u_n|)}} \right|^2 \varepsilon_n^{-3} \| \, |u_n| \, \|^2_{\varepsilon_n,W} \\
&\le \left| 1 - \frac{\varepsilon_n \| \, |u_n| \, \|_{\varepsilon_n,W}}{\sqrt{\mathbb{D}(|u_n|)}} \right|^2 4\varepsilon_n^{-3} J_{\varepsilon_n,A,V}(u_n) \to 0. \qquad (6.8)
\end{aligned}$$

Finally, combining (6.6), (6.7), and (6.8) we conclude that

$$\varepsilon_n^{-3} \left\| |u_n| - \sum_{g\xi_n \in G\xi_n} \omega_\xi \left(\frac{\cdot - g\xi_n}{\varepsilon_n} \right) \right\|^2_{\varepsilon_n, W} \to 0.$$

Since $\|v\|^2_\varepsilon \leq C \|v\|^2_{\varepsilon,W}$ for some constant independent of ε, our claim follows. □

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