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Cohomologies of crossed modules

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Contents

Introduction **4**

1 Crossed modules **8**
1.1 Crossed modules . . . 8
1.2 The actor of a crossed module . . . 12
1.3 Actions of crossed modules . . . 13

2 Algebraic models for connected 3-types **15**
2.1 Braided crossed modules . . . 16
2.2 Crossed squares . . . 17
2.3 2-crossed modules . . . 20
2.4 2-crossed modules with trivial Peiffer lifting . . . 22
2.5 Crossed squares and 2-crossed modules . . . 23

3 Categorical groups **24**
3.1 Monoidal categories . . . 24
3.2 Categorical groups . . . 27
3.3 Actions of categorical groups . . . 28
3.4 Categorical crossed modules . . . 30

4 Crossed squares **34**
4.1 Crossed square version of homotopy kernels . . . 35
4.2 $\ker \mathbf{T}$ as a strict categorical $\mathbf{\Gamma}$-crossed module . . . 39
4.3 Crossed square version of homotopy cokernels . . . 41
4.4 $\ker \mathbf{T}$ as a strict categorical $\dfrac{\mathbf{\Gamma}}{<\mathbf{G},\mathbf{T}>}$-crossed module . . . 44
4.5 Images of crossed modules . . . 49

5 Cohomologies **50**
5.1 Cohomology of a group with coefficients in crossed module . . 50
5.1.1 Dedecker Cohomology . . . 50
5.1.2 Borovoi Cohomology . . . 51
5.1.3 Noohi Cohomology . . . 53

5.2 Cohomology with coefficients in categorical crossed modules . . . 56
5.3 Cohomology with coefficients in braided crossed modules . . . 57
5.4 Cohomology in 2-crossed modules . . . 78
5.5 Cohomology in crossed squares . . . 95

A Cohomology of groups **109**
A.1 Group cohomology with abelian coefficients via cochains . . . 109
A.2 Serre cohomology . . . 111

B **112**
B.1 Groupoids [7] . . . 112
B.2 Groups in a category . . . 113
B.3 Simplicial groups . . . 113

Bibliography **118**

Introduction

Since the 60's, in the mathematical research, there have been several theories developed, concerning cohomology with non-abelian coefficients. Despite the efforts made by many researchers in order to make uniform the context, today there isn't a general theory that unifies the various approaches. Indeed, within the same algebraic setting exist small differences between the various theories.

Our aim is to study the low-dimensional cohomology theory in the category of crossed modules.

A lot of people have worked to define the cohomology of a group with coefficients in a crossed module. In general, the approach is to use an explicit cocycle description of the cohomology groups. The original idea goes back to Dedecker [19], [20], [21]. He defined the cohomology of a group Γ with coefficients in a crossed module, considering a trivial Γ-action on crossed module. After many years, Borovoi [1] treated the general case, with a generic action. Another approach is given by Lue [35].

Firstly, we want to recall some known facts concerning the group cohomology in the abelian case. Given a group Γ, let G be a Γ-module. A derivation from Γ to G is a function $\eta : \Gamma \to G$ such that $\eta(\sigma\,\tau) = \eta(\sigma)\,{}^{\sigma}\eta(\tau)$. In this case, the set $\mathrm{Der}(\Gamma, G)$ of all derivations has a natural abelian group structure, with a composition given by punctual composition. Each element g in the abelian group G defines a derivation η_g (called inner derivation), given by $\eta_g(\sigma) = g\,{}^{\sigma}g^{-1}$. The map

$$\begin{array}{rcl} \gamma : G & \longrightarrow & \mathrm{Der}(\Gamma, G) \\ g & \longrightarrow & \eta_g \end{array}$$

is a homomorphism of abelian groups. This homomorphism can be considered as the starting point for the creation of the first abelian cohomology groups. In fact, the kernel of γ is precisely $H^0(\Gamma, G)$ while the cokernel of γ is $H^1(\Gamma, G)$, according to the kernel-cokernel diagram:

$$\begin{array}{ccccccc} H^0(\Gamma,G) & \longrightarrow & G & \xrightarrow{\quad\gamma\quad} & \mathrm{Der}(\Gamma,G) & \longrightarrow & H^1(\Gamma,G) \\ & & & \searrow \qquad \nearrow & & & \\ & & & \gamma(G) & & & \end{array}$$

where, as known:

$$H^0(\Gamma, G) = \ker(\gamma) = G^{\Gamma} \quad \text{and} \quad H^1(\Gamma, G) = \operatorname{coker}(\gamma) = \frac{\operatorname{Der}(\Gamma, G)}{\operatorname{Ider}(\Gamma, G)}$$

with $\operatorname{Ider}(\Gamma, G)$ denoting the group of inner derivations.
In general, if the group Γ acts on a non-abelian group G, the set of derivations $\operatorname{Der}(\Gamma, G)$ is just a pointed set. If G is a Γ-crossed module, in [49] Withehead showed, defining a special product of derivations, that $\operatorname{Der}(\Gamma, G)$ has a natural monoid structure. Furthermore, he characterized the group of units $\operatorname{Der}^*(\Gamma, G)$ in relation to the automorphisms of Γ and G. Finally, Whitehead showed that the set of inner derivations $\operatorname{Ider}(\Gamma, G)$ is a normal subgroup of $\operatorname{Der}^*(\Gamma, G)$. Therefore, in a similar way to the abelian case, given G a Γ-crossed module, Lue in [35] defined the cohomology groups in dimension 0 and 1 of Γ with coefficients in G as follows:

$$H^0_L(\Gamma, G) = \{g \in G \ : \ \forall \sigma \in \Gamma, {}^{\sigma}g = g\}$$

and

$$H^1_L(\Gamma, G) = \operatorname{coker}(T) = \frac{\operatorname{Der}^*(\Gamma, G)}{\operatorname{Ider}(\Gamma, G)}.$$

Serre [45] was the first one to construct a low-dimensional cohomology theory for a group Γ with coefficients in a non-abelian group, considering a Γ-group G. He defined a group $H^0_S(\Gamma, G)$ and a pointed set $H^1_S(\Gamma, G)$ (see A.2). This set satisfies the property of cohomological functors, in particular a short exact sequence gives rise to an exact sequence with six terms in cohomology, but $H^1(\Gamma, G)$ hasn't a group structure. At a later time, Guin [30], considering a group G equipped with a Γ-crossed module structure, defined a notion of 1-cocycle to obtain a cohomology group $H^1_G(\Gamma, G)$. The Guin cohomology is a particular case of the more general Borovoi cohomology.

Given a $\mathbf{\Gamma}$-categorical group $\mathbf{G}$, in [14], the authors have defined the category $\operatorname{Der}(\mathbf{\Gamma}, \mathbf{G})$ of derivations from $\mathbf{\Gamma}$ into $\mathbf{G}$, which is a pointed groupoid. If $(\mathbf{G}, \mathbf{T}, \nu, \chi)$ is a categorical $\mathbf{\Gamma}$-crossed module, then $\operatorname{Der}(\mathbf{\Gamma}, \mathbf{G})$ has a natural monoidal structure, which is inherited from the $\mathbf{\Gamma}$-crossed module structure in $\mathbf{G}$. Then, they have considered a Whitehead categorical group of derivations $\operatorname{Der}^*(\mathbf{\Gamma}, \mathbf{G})$ as the Picard categorical group, $\mathcal{P}(\operatorname{Der}(\mathbf{\Gamma}, \mathbf{G}))$, of the monoidal category $\operatorname{Der}(\mathbf{\Gamma}, \mathbf{G})$.
There is a homomorphism of categorical groups

$$\mathbf{G} \xrightarrow{\overline{\mathbf{T}}} \operatorname{Der}^*(\mathbf{\Gamma}, \mathbf{G})$$

given by inner derivations. There are natural isomorphisms $\overline{\nu}$ and $\overline{\chi}$ such that $(\mathbf{G}, \overline{\mathbf{T}}, \overline{\nu}, \overline{\chi})$ is a categorical $\operatorname{Der}^*(\mathbf{\Gamma}, \mathbf{G})$-crossed module.
Therefore, in a similar way to the abelian case and Lue cohomology, they

have defined the cohomology of categorical groups, in dimension 0 and 1, of $\mathbf{\Gamma}$ with coefficients in categorical $\mathbf{\Gamma}$-crossed module $\mathbf{G}$ as follows:

$$\mathcal{H}^0(\mathbf{\Gamma},\mathbf{G}) = \ker(\overline{\mathbf{T}} : \mathbf{G} \to \mathrm{Der}^*(\mathbf{\Gamma},\mathbf{G}))$$

and

$$\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G}) = \frac{\mathrm{Der}^*(\mathbf{\Gamma},\mathbf{G})}{< \mathbf{G}, \overline{\mathbf{T}} >}.$$

The first one is the kernel of the homomorphism $\overline{\mathbf{T}}$ of categorical groups while the second one is the quotient categorical group for the categorical crossed module $(\mathbf{G}, \overline{\mathbf{T}}, \overline{\nu}, \overline{\chi})$.

There are some well-known particular crossed modules associated to categorical crossed modules, for example:

- braided crossed modules $\partial : G_1 \to G_0$, endowed with an action by a crossed module $\partial' : \Gamma_1 \to \Gamma_0$ and with the braiding equivariant respect this action;
- 2-crossed modules (introduced by Guin-Valery and Loday [30]);
- crossed squares (introduced by Daniel Conduché [16]).

In the Chapter 5, we present the cohomology in these three particular cases.

Before describing this last chapter, we want to recall what we have done in Chapter 4. We show how some known results related to the crossed modules can be extended to the context of crossed squares.
It is well-known that if $\partial : G_1 \to G_0$ is a crossed module, then there exists an action of $\mathrm{coker}(\partial)$ on $\ker(\partial)$ making the composition

$$\ker \partial \hookrightarrow G_1 \xrightarrow{\partial} G_0 \twoheadrightarrow \mathrm{coker}\, \partial \tag{1}$$

a crossed module.
We show that given a crossed square, using its representation as a strict categorical crossed module $\mathbf{T} : \mathbf{G} \to \mathbf{\Gamma}$, we obtain a 2-dimensional version of (1):

$$\ker \mathbf{T} \longrightarrow \mathbf{G} \xrightarrow{\mathbf{T}} \mathbf{\Gamma} \longrightarrow \frac{\mathbf{\Gamma}}{< \mathbf{G}, \mathbf{T} >} \tag{2}$$

for suitable categorical groups $\ker \mathbf{T}$ (kernel categorical group, see [33] and [48]) and $\dfrac{\mathbf{\Gamma}}{< \mathbf{G}, \mathbf{T} >}$ (quotient categorical group, see [14]), where (2) is a strict categorical crossed module.
In Chapter 4, we describe in terms of crossed modules these strict categorical groups, the kernel and cokernels by the homotopical versions. The homotopical kernel is obtained by the construction of a pullback while the homotopical

cokernel by a generalized semi-direct product. Moreover, in Proposition 4.4.1 we describe, in terms of crossed squares, the strict categorical crossed module (2).

Finally, in Chapter 5, in all the three cases before emphasized, the zero-th cohomology categorical group is strict and we present it as a crossed module (under the equivalence between strict categorical groups and crossed modules).
Given a braided crossed module equivariant respect an action of crossed module $\partial' : \Gamma_1 \to \Gamma_0$, in Proposition 5.3.3, we show that the crossed module associated to the zero-th cohomology categorical group is a braided crossed module equivariant respect an action of crossed module $\partial' : \Gamma_1 \to \Gamma_0$.
Given a 2-crossed module, in Proposition 5.4.3, we show that the crossed module associated to the zero-th cohomology categorical group has a 2-crossed module structure.
Given a crossed square, in Proposition 5.5.2, we show that the crossed module associated to the zero-th cohomology categorical group has a crossed square structure.

In the first two cases, we do the same for the first cohomology categorical group, being a strict categorical group.
In the third case, given a crossed square we consider its representation as a strict categorical crossed module $\mathbf{T} : \mathbf{G} \to \mathbf{\Gamma}$ and the categorical crossed module $\overline{\mathbf{T}} : \mathbf{G} \to \mathrm{Der}^*(\mathbf{\Gamma}, \mathbf{G})$. We have that the first cohomology categorical group is just a categorical group (not strict).

Then we define a category $\mathbf{D}$, included in $\mathrm{Der}^*(\mathbf{\Gamma}, \mathbf{G})$, such that we can consider a restriction of the homomorphism $\overline{\mathbf{T}}$:

$$\overline{\mathbf{T}} : \mathbf{G} \to \mathbf{D}$$

and this is a strict categorical $\mathbf{D}$-crossed module (generalization of that happens in the context of the crossed modules to crossed squares). As a consequence of Proposition 4.4.1, in Proposition 5.5.3, we can give a description as crossed square of the strict categorical crossed module $\mathcal{H}^0(\mathbf{\Gamma}, \mathbf{G}) \to \mathcal{H}^1_N(\mathbf{\Gamma}, \mathbf{G})$, where $\mathcal{H}^1_N(\mathbf{\Gamma}, \mathbf{G})$ is the quotient categorical group for the strict categorical crossed module $\overline{\mathbf{T}} : \mathbf{G} \to \mathbf{D}$.

Chapter 1

Crossed modules

The notion of crossed module, that generalizes the notion of a G-module, goes back to Whitehead [49] in the course of his studies on the algebraic structure of the second group of relative homotopy. The relevance of crossed modules to homotopy types follows from the existence of a classifying space functor B (see [34], [9]) assigning to a crossed module $\mathcal{L} = (\partial : G_1 \to G_0)$ a connected pointed CW-space $B\mathcal{L}$ that is defined as the geometrical realization of the nerve of the crossed module (see [9]). The only two non-trivial homotopy groups of $B\mathcal{L}$ are respectively given by $\pi_1 = \mathrm{coker}(\partial)$ and $\pi_2 = \ker(\partial)$. Moreover, for any connected pointed CW-complex X with base point x_0, there is a crossed module $\mathcal{L}X$ and a map $X \to B\mathcal{L}X$ inducing an isomorphism of π_1 and π_2. If X^1 is the 1-skeleton of X, then $\mathcal{L}X$ is the Whitehead crossed module $\pi_2(X, X^1, x_0) \to \pi_1(X^1, x_0)$. These results reveal that crossed modules model all pointed homotopy 2-types (a result due originally to Mac Lane and Whitehead [36] although with the old teminology of 3-types).

In this chapter, we first recall the algebraic definition of crossed module, that is a group endowed with an additional structure related to its group of automorphisms. Then we briefly review some known results on the theory of crossed modules.

1.1 Crossed modules

Definition 1.1.1. *[49] A* crossed module *consists of a group homomorphism $\partial : G_1 \to G_0$, endowed with a left action G_0 on G_1 (denoted by $(g, \alpha) \mapsto {}^{g}\alpha$), satisfying:*

$$\partial({}^{g}\alpha) = g\,\partial(\alpha)\,g^{-1} \qquad \forall \alpha \in G_1, \forall g \in G_0; \tag{1.1}$$

$${}^{\partial\alpha_1}\alpha_2 = \alpha_1\,\alpha_2\,\alpha_1^{-1} \qquad \forall \alpha_1, \alpha_2 \in G_1. \tag{1.2}$$

The first one is called pre-crossed module property and the second one the Peiffer identity.

The two conditions are equivalent to the request of the commutativity of the following diagram:

$$\begin{array}{ccc}
G_1 \times G_1 & \xrightarrow{\chi} & G_1 \\
\Big\downarrow {\scriptstyle (\partial, id_{G_1})} & & \Big\downarrow {\scriptstyle id_{G_1}} \\
G_0 \times G_1 & \xrightarrow{\xi} & G_1 \\
\Big\downarrow {\scriptstyle (id_{G_0}, \partial)} & & \Big\downarrow {\scriptstyle \partial} \\
G_0 \times G_0 & \xrightarrow{\chi} & G_0
\end{array}$$

where χ represents the conjugation action for the group G_1 and G_0 respectively, ξ represents the given action of G_0 on G_1.

Examples. (a) Every group G can be seen as a trivial crossed module $1 \to G$.

(b) Let G be a group, the identity homomorphism of G, sending everything $g \in G$ to the same element g, is a crossed module. In this case, G acts on itself by conjugation.

(c) Let G_1 be a normal subgroup of G_0, the inclusion $\partial : G_1 \hookrightarrow G_0$ is a crossed module. In this case, G_0 acts on the left of G_1 by conjugation.

(d) Any epimorphism $\partial : G_1 \to G_0$ with central kernel is a crossed module. An element $g \in G_0$ acts on $\alpha \in G_1$ by ${}^{g}\alpha = \tilde{g}\,\alpha\,\tilde{g}^{-1}$ where $\tilde{g}$ is any lifting of g to G_1 .

(e) Let $\xi : G_0 \times G_1 \to G_1$ be an action of groups; the pair $(\xi, 1 : G_1 \to G_0)$, where 1 is the trivial map, is a crossed module if and only if G_1 is abelian.

(f) Let G be a group and let $\mathrm{Aut}(G)$ be the automorphism group of G. Conjugation gives a homomorphism

$$\partial : G \to \mathrm{Aut}(G)$$

and the last is a crossed module, with an action of $\mathrm{Aut}(G)$ on G given by ${}^{\varphi}g = \varphi(g)$ for all $\varphi \in \mathrm{Aut}(G)$ and $g \in G$.

(g) The homomorphism $\partial : \mathrm{SL}_2(L) \hookrightarrow \mathrm{GL}_2(L) \twoheadrightarrow \mathrm{PGL}_2(L) = \frac{\mathrm{GL}_2(L)}{Z_2(L)}$, where L is a field, is a crossed module with an action of $\mathrm{PGL}_2(L)$ on $\mathrm{SL}_2(L)$ given by:

$${}^{[C]}B = C\,B\,C^{-1} \qquad \forall C \in \mathrm{GL}_2(L), \forall B \in \mathrm{SL}_2(L).$$

(h) J. H. C. Whitehead [49], who introduced the notion of a crossed module, considered the boundary homomorphism $\partial : \pi_2(X, Y, x_0) \to \pi_1(Y, x_0)$ (where X is a topological space and $Y \subset X$ is a pointed subspace with the base point x_0). There exists an action of $\pi_1(A, x_0)$ on $\pi_2(X, A, x_0)$ which makes the boundary map a crossed module.

(i) [44] Any simplicial group $G_\bullet$ (see B.3 for the definition), yields a crossed module, the crossed 1-cube $M(G_\bullet, 1)$ associated with simplicial group $G_\bullet$, defined by

$$\partial : \frac{NG_1}{d_2(NG_2)} \to G_0$$

where ∂ is induced from d_1 and G_0 acts on $\frac{NG_1}{d_2(NG_2)}$ by conjugation via s_0, i.e. if $g \in G_0$ and $x \in \frac{NG_1}{d_2(NG_2)}$, then ${}^g x = s_0(g)\, x\, s_0(g)^{-1}$.

Definition 1.1.2. *A* morphism between crossed modules $\partial : G_1 \to G_0$ *and* $\partial' : G_1' \to G_0'$ *is a pair* $<\varphi, \psi>$ *where* $\varphi : G_1 \to G_1'$ *and* $\psi : G_0 \to G_0'$ *are homomorphisms such that the diagram*

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi} & G_1' \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \partial'} \\ G_0 & \xrightarrow{\psi} & G_0' \end{array}$$

commutes and $\varphi({}^g\alpha) = {}^{\psi(g)}\varphi(\alpha)$ *for all* $\alpha \in G_1$ *and* $g \in G_0$. *If* $\partial = \partial'$ *and* φ, ψ *are automorphisms then* $<\varphi, \psi>$ *is an automorphism of* $\partial : G_1 \to G_0$. *The group of automorphisms of* $\partial : G_1 \to G_0$ *is denoted by* $\mathrm{Aut}(G_1, G_0, \partial)$.

Crossed modules and their morphisms form a category. That will be denoted by $\mathcal{CM}$.

Definition 1.1.3. *Let* $\partial : G_1 \to G_0$, $\partial' : G_1' \to G_0'$ *be a crossed modules and* $<\varphi, \psi>, <\varphi', \psi'>$ *be a morphism between* $\partial : G_1 \to G_0$ *and* $\partial' : G_1' \to G_0'$. *A transformation between* $<\varphi, \psi>$ *and* $<\varphi', \psi'>$ *is given by a function* $\theta : G_0 \to {G_1}'$ *satisfying:*

$$\begin{aligned} &\varphi'(\alpha)\, \theta(g) = \theta(\partial(\alpha)\, g)\, \varphi(\alpha); \\ &\partial'\, \theta(g)\, \psi(g) = \psi'(g); \\ &\theta(g_1\, g_2) = \theta(g_1)\, {}^{\psi(g_1)}\theta(g_2); \end{aligned}$$

for all $\alpha \in G_1$ *and* $g, g_1, g_2 \in G_0$.

$<\varphi, \psi>$ and $<\varphi', \psi'>$ are homotopy equivalent if there exists a transformation between them.

Now recall some known results arising from the definition of crossed module.

Lemma 1.1.1. *Let* $\partial : G_1 \to G_0$ *be a crossed module. Then*

(i) *the group* $\ker \partial$ *is central in* G_1*;*

(ii) $\ker \partial$ *is* G_0*-invariant;*

(iii) $\operatorname{Im}\partial$ *is normal in* G_0 .

Corollary 1.1.1.1. *The action of* G_0 *on* G_1 *induces an action of* $\operatorname{coker}\partial$ *on the abelian group* $\ker\partial$.

It was known to Verdier in 1965, that groups in the category $\mathcal{GPD}$ of groupoids (see for definition Appendix B.1 and B.2) are equivalent to crossed modules.

From a crossed module $\partial : G_1 \to G_0$ we construct a groupoid $\mathcal{G}$ with a set of objects G_0, as set of arrows the semi-direct product $G_1 \rtimes G_0$, associated with the given action ξ, as follows:

$$G_1 \rtimes G_0 \underset{t}{\overset{s}{\rightrightarrows}} G_0, \qquad i : G_0 \to G_1 \rtimes G_0.$$

The source map s, target map t and the unit map i are defined respectively by

$$\begin{array}{rcl} s : G_1 \rtimes G_0 & \longrightarrow & G_0 \\ (\alpha, g) & \longrightarrow & g \end{array} \quad \begin{array}{rcl} t : G_1 \rtimes G_0 & \longrightarrow & G_0 \\ (\alpha, g) & \longrightarrow & \partial(\alpha)\, g \end{array} \quad \begin{array}{rcl} i : G_0 & \longrightarrow & G_1 \rtimes G_0 \\ g & \longrightarrow & (1, g) \end{array}$$

while the composition of arrows in $\mathcal{G}$ is given by $(\alpha', g') \circ (\alpha, g) = (\alpha' \alpha, g)$. The groupoid $\mathcal{G}$ is a group in the category of groupoids. We can define the functor m on objects and on arrows

$$\begin{array}{rl} m : G_0 \times G_0 \longrightarrow G_0 \\ (g_1, g_2) \longrightarrow g_1\, g_2 \end{array} \qquad \begin{array}{rl} m : (G_1 \rtimes G_0) \times (G_1 \rtimes G_0) \longrightarrow G_1 \rtimes G_0 \\ ((\alpha_1, g_1), (\alpha_2, g_2)) \longrightarrow (\alpha_1\, {}^{g_1}\alpha_2, g_1\, g_2) \end{array}$$

respectively. Therefore, m on the objects is the product on G_0 and m on the arrows is the usual semi-direct product. If $\mathbf{1}$ denotes the terminal category with one object $*$, we can define $e : \mathbf{1} \to \mathcal{G}$ on objects and on arrows

$$\begin{array}{rcl} e : * & \longrightarrow & G_0 \\ * & \longrightarrow & 1_{G_0} \end{array} \qquad \begin{array}{rcl} e : 1_* & \longrightarrow & G_1 \rtimes G_0 \\ 1_* & \longrightarrow & (1_{G_1}, 1_{G_0}) \end{array}$$

respectively. Therefore, e associates with $*$ the neutral element of G_0 and with the arrow 1_* the neutral element of $G_1 \rtimes G_0$. Finally, the functor inv is given by

$$\begin{array}{rcl} inv : G_0 & \longrightarrow & G_0 \\ g & \longrightarrow & g^{-1} \end{array} \qquad \begin{array}{rcl} inv : G_1 \rtimes G_0 & \longrightarrow & G_1 \rtimes G_0 \\ (\alpha, g) & \longrightarrow & ({}^{g^{-1}}\alpha^{-1}, g^{-1}) \end{array}$$

on objects and on arrows, respectively. Therefore, inv associates with any object the inverse in G_0 and with any arrow the inverse in $G_1 \rtimes G_0$.

Conversely, let $\mathcal{G}$ be a group in $\mathcal{GPD}$. $\mathcal{G}$ is a groupoid with set of arrows G, set of objects G_0, source and target maps $s, t : G \to G_0$ and unit map i, that is:

$$\mathbf{G} : \qquad G \times_{\circ} G \xrightarrow{\ \circ\ } G \underset{t}{\overset{s}{\rightrightarrows}} G_0, \qquad i : G_0 \to G.$$

Since $\mathcal{G}$ is a group in $\mathcal{GPD}$, we have three functors $m : \mathcal{G} \times \mathcal{G} \to \mathcal{G}$, $e : 1 \to \mathcal{G}$ and $inv : \mathcal{G} \to \mathcal{G}$ such that the obvious diagrams commute (see for details Appendix B.2). We can show that G and G_0 are groups and s, t, i are homomorphisms of groups. The multiplications, the neutral elements and the inverse ones in G and G_0 are respectively induced by the functors m, e and inv. Then we can define $t_{|\mathrm{ker}s} : \mathrm{ker}s \to G_0$, with an action of G_0 on $\mathrm{ker}s$ given by conjugation, i.e if $g \in G_0$ and $k \in \mathrm{ker}s$, then ${}^g k = i(g)\, k\, i(g)^{-1}$. $t_{|\mathrm{ker}s} : \mathrm{ker}s \to G_0$ turns out to be a crossed module.

Recall that a cat^1-group is a group G with two endomorphisms d_0, $d_1 : G \to G$ such that

$$d_1\, d_0 = d_0 \qquad d_0\, d_1 = d_1 \qquad [\mathrm{ker}d_0, \mathrm{ker}d_1] = 1.$$

A morphism of cat^1-groups $(G, d_0, d_1) \to (G^{'}, d_0^{'}, d_1^{'})$ is a group homomorphism $f : G \to G^{'}$ such that $d_i^{'}\, f = f\, d_i$, i = 0, 1.

The category of crossed modules is equivalent to the category of cat^1-groups (see [34]). Given a crossed module $\partial : G_1 \to G_0$, the corresponding cat^1-group is $(G_1 \rtimes G_0, d_0, d_1)$ where $d_0(\alpha, g) = (1, g)$, $d_1(\alpha, g) = (1, \partial(\alpha)\, g)$ for all $(\alpha, g) \in G_1 \rtimes G_0$.

Another description of the category of crossed modules is given by the equivalent category of simplicial groups (see B.3 for the definition) whose Moore complex with length 1 (see [34]).

1.2 The actor of a crossed module

Norrie, in [42], defines actor crossed modules and shows how they provide an analogue of automorphism groups of groups.

For a crossed module $\partial : G_1 \to G_0$, denote by $\mathrm{Der}\,(G_0, G_1)$ the set of all derivations from G_0 to G_1, i.e. all maps $\eta : G_0 \to G_1$ such that for all g_1, $g_2 \in G_0$,

$$\eta(g_1\, g_2) = \eta(g_1)\, {}^{g_1}\eta(g_2).$$

Each such derivation η defines endomorphisms $\psi (= \psi_\eta)$ and $\varphi (= \varphi_\eta)$ of G_0 and G_1 respectively, given

$$\psi(g) = \partial(\eta(g))\, g \quad \text{and} \quad \varphi(\alpha) = \eta(\partial(\alpha))\, \alpha.$$

Whitehead (see [49]) defined a multiplication in $\mathrm{Der}\,(G_0, G_1)$ by the formula $\eta_1 \cdot \eta_2 = \eta$, where

$$\eta(g) = \eta_1(\psi_{\eta_2}(g))\, \eta_2(g)\; (= \varphi_{\eta_1}(\eta_2(g))\, \eta_1(g)).$$

This turns $\mathrm{Der}\,(G_0, G_1)$ into a monoid, with the identity element the derivation which maps each element of G_0 into the identity element of G_1. The Whitehead group $\mathrm{Der}^*(G_0, G_1)$ is defined to be the group of units of $\mathrm{Der}\,(G_0, G_1)$. The following Proposition combines results from [49] and [35].

Proposition 1.2.1. *The following statements are equivalent:*

(i) $\eta \in \mathrm{Der}^*(G_0, G_1)$;

(ii) $\varphi_\eta \in \mathrm{Aut}\, G_1$;

(iii) $\psi_\eta \in \mathrm{Aut}\, G_0$.

Moreover, $\triangle : Der^*(G_0, G_1) \to Aut\,(G_1, G_0, \partial)$ *defined by* $\triangle(\eta) = <\varphi_\eta, \psi_\eta>$ *is a homomorphism of groups and there is an action of* $Aut(G_1, G_0, \partial)$ *on* $Der^*(G_0, G_1)$ *given by* $({}^{<\varphi,\psi>}\eta)(g) = \varphi\,\eta\,\psi^{-1}(g)$*, which makes* $\triangle : Der^*(G_0, G_1) \to Aut\,(G_1, G_0, \partial)$ *a crossed module. This crossed module is called the actor crossed module of the crossed module* $\partial : G_1 \to G_0$ *.*

There is a morphism of crossed modules

$$\begin{array}{ccc} G_1 & \xrightarrow{\eta} & \mathrm{Der}^*(G_0, G_1) \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\triangle} \\ G_0 & \xrightarrow{\gamma} & \mathrm{Aut}\,(G_1, G_0, \partial) \end{array}$$

defined as follows. Let $\alpha \in G_1$, then $\eta_\alpha : G_0 \to G_1$ defined by $\eta_\alpha(g) = \alpha\,{}^g\alpha^{-1}$ is an inner derivation associated with α and the map $\alpha \to \eta_\alpha$ defines a homomorphism $\eta : G_1 \to \mathrm{Der}^*(G_0, G_1)$ of groups. Let $\gamma : G_0 \to \mathrm{Aut}\,(G_1, G_0, \partial)$ be the homomorphism $g \mapsto <\varphi_g, \psi_g>$, where $\varphi_g(\alpha) = {}^g\alpha$ and $\psi_g(\overline{g}) = g\,\overline{g}\,g^{-1}$ for all $\alpha \in G_1$ and $g, \overline{g} \in G_0$.

1.3 Actions of crossed modules

Norrie (see [42]) uses this actor to define actions of crossed modules. An action of a crossed module $\partial' : \Gamma_1 \to \Gamma_0$ on a crossed module $\partial : G_1 \to G_0$ is defined to be a morphism of crossed modules from $\partial' : \Gamma_1 \to \Gamma_0$ to the actor of $\partial : G_1 \to G_0$, that is:

$$\begin{array}{ccc} \Gamma_1 & \xrightarrow{\varrho_1} & \mathrm{Der}^*(G_0, G_1) \\ {\scriptstyle\partial'}\downarrow & & \downarrow{\scriptstyle\triangle} \\ \Gamma_0 & \xrightarrow{\varrho_2} & \mathrm{Aut}\,(G_1, G_0, \partial). \end{array}$$

We have seen that this is equivalent to requiring the following conditions:

(i) ϱ_1, ϱ_2 are homomorphisms of groups;

(ii) $\varrho_1(\beta)(g_1\, g_2) = \varrho_1(\beta)(g_1) \cdot {}^{g_1}\varrho_1(\beta)(g_2)$ (since $\varrho_1(\beta) \in \mathrm{Der}(G_0, G_1)$);

(iii) $\varphi_{\varrho_1(\beta)} \in \mathrm{Aut}(G_1)$ where $\varphi_{\varrho_1(\beta)}(\alpha) = \varrho_1(\beta)(\partial(\alpha))\,\alpha$ (since $\varrho_1(\beta) \in \mathrm{Der}^*(G_0, G_1)$);

(iv) $\varrho_2 =< \varrho_2^{'}, \varrho_2^{''} > \in \mathrm{Aut}\,(G_1, G_0, \partial)$

where $\varrho_2^{'} : \Gamma_0 \to \mathrm{Aut}\,(G_1)$ (then Γ_0 acts on G_1, ${}^{\sigma}\alpha := \varrho_2^{'}(\sigma)(\alpha)$)
and $\varrho_2^{''} : \Gamma_0 \to \mathrm{Aut}\,(G_0)$ (then Γ_0 acts on G_0, ${}^{\sigma}g := \varrho_2^{''}(\sigma)(g)$)
such that : $\partial({}^{\sigma}\alpha) = {}^{\sigma}\partial(\alpha)$
${}^{\sigma}({}^{g}\alpha) = {}^{{}^{\sigma}g}\,({}^{\sigma}\alpha)$

(v) the above diagram commutes, then:
$\varrho_1(\beta)(\partial(\alpha)) \cdot \alpha = {}^{\partial'(\beta)}\alpha$;
$\partial\varrho_1(\beta)(g) \cdot g = {}^{\partial'(\beta)}g$;

(vi) $\varrho_1({}^{\sigma}\beta)(g) = {}^{\sigma}(\varrho_1(\beta)({}^{\sigma^{-1}}g))$ (equivariant condition);

for all $\alpha \in G_1$, $\beta \in \Gamma_1$, g, g_1, $g_2 \in G_0$ and $\sigma \in \Gamma_0$.
Therefore, an action of a crossed module $\partial' : \Gamma_1 \to \Gamma_0$ on a crossed module $\partial : G_1 \to G_0$ is equivalent to an action of Γ_0 on G_0 and on G_1 (and hence an action of Γ_1 on G_0 and on G_1 via ∂') and a function $h : \Gamma_1 \times G_0 \to G_1$, defined by $h(\beta, g) = \varrho_1(\beta)(g)$, such that the above axioms become:

(i) $h(\beta_1\,\beta_2, g) = {}^{\beta_1}h(\beta_2, g)\,h(\beta_1, g)$;

(ii) $h(\beta, g_1\,g_2) = h(\beta, g_1)\,{}^{g_1}h(\beta, g_2)$;

(iv) the map ∂ preserve the actions of Γ_0 and ${}^{\sigma}({}^{g}\alpha) = {}^{{}^{\sigma}g}\,({}^{\sigma}\alpha)$;

(v) $h(\beta, \partial(\alpha)) = {}^{\beta}\alpha\,\alpha^{-1}$, $\partial\,h(\beta, g) = {}^{\beta}g\,g^{-1}$;

(vi) $h({}^{\sigma}\beta, {}^{\sigma}g) = {}^{\sigma}h(\beta, g)$;

for all $\alpha \in G_1$, β, β_1, $\beta_2 \in \Gamma_1$, g, g_1, $g_2 \in G_0$ and $\sigma \in \Gamma_0$.

Remark 1.3.1. *In particular, the action of a group Γ, seen as the crossed module $1 \to \Gamma$, on the crossed module $\partial : G_1 \to G_0$ is reduced to having two actions of Γ on G_0 and G_1 (denoted by ${}^{\sigma}g$, ${}^{\sigma}\alpha$ for all $\sigma \in \Gamma$, $\alpha \in G_1$, $g \in G_0$) such that the following relations hold:*

$$\partial({}^{\sigma}\alpha) = {}^{\sigma}(\partial(\alpha)) \qquad \forall\sigma \in \Gamma,\ \forall\alpha \in G_1;$$
$${}^{\sigma}({}^{g}\alpha) = {}^{{}^{\sigma}g}({}^{\sigma}\alpha) \qquad \forall\sigma \in \Gamma,\ \forall\alpha \in G_1,\ \forall g \in G_0.$$

If the crossed module $\partial' : \Gamma_1 \to \Gamma_0$ acts on the crossed module $\partial : G_1 \to G_0$, Norrie, in [42], constructs the following semi-direct product of these crossed modules:

$$(\partial, \partial') : G_1 \rtimes \Gamma_1 \to G_0 \rtimes \Gamma_0$$

where Γ_1 acts on G_1 via ∂' and Γ_0 acts on G_0 with the induced action of $\partial' : \Gamma_1 \to \Gamma_0$ on $\partial : G_1 \to G_0$. The action of $G_1 \rtimes \Gamma_1$ on $G_0 \rtimes \Gamma_0$ is defined by:

$${}^{(g,\sigma)}(\alpha, \beta) = ({}^{g}({}^{\sigma}\alpha)\,h({}^{\sigma}\beta, g)^{-1},\ {}^{\sigma}\beta).$$

Chapter 2

Algebraic models for connected 3-types

Groups are algebraic models of connected 1-types: that is, there is a classifying space functor:

$$B : \mathcal{GP} \to (\text{pointed connected } CW - \text{complexes});$$

such that for any group G, the associated classifying space BG satisfies:

$$\pi_1(BG) \cong G \qquad \text{and} \qquad \pi_i(BG) = 1 \qquad \text{for } i > 1.$$

Furthermore any pointed connected CW-complex X with $\pi_i(X) = 1$ for $i > 1$ is the homotopy type of $B\pi_1(X)$.

Crossed modules, introduced by Whitehead, are algebraic models of connected 2-types. There is a classifying space functor:

$$B : \mathcal{CM} \to (\text{pointed connected } CW - \text{complexes})$$

such that if $\mathcal{L} = (\partial : G_1 \to G_0)$ is a crossed module then $B\mathcal{L}$ has

$$\pi_1(B\mathcal{L}) \cong \text{coker}(\partial) \qquad \pi_2(B\mathcal{L}) \cong \ker(\partial) \qquad \pi_i(B\mathcal{L}) = 1 \qquad \text{for } i > 2$$

Furthermore any pointed connected CW-complex with $\pi_i(X) = 1$ for $i > 2$ is the homotopy type of $B\mathcal{L}$ for some crossed module $\mathcal{L} = (\partial : G_1 \to G_0)$.

Crossed squares arose from a study of excision in algebraic K-theory, introduced by Loday and Guin-Walery in 1981(see [30]). They also form algebraic models of connected 3-type (see [34]).

The use of simplicial groups as algebraic models of homotopy types is of long standing (see [34]). Counduché showed in 1983, in [16], that the category of simplicial groups with Moore complex of lenght 2 is equivalent to that one of 2-crossed modules.

Brown and Gilbert introduced in 1988, in [8], the braided crossed modules for an algebraic models of 3-types. Then they showed that these structure are closely related to simplicial groups; they proved that the category of braided (regular) crossed modules is equivalent to that of simplicial groups with Moore complex of length 2. This gives a composite equivalence between the category of braided crossed modules and that of 2-crossed modules.

The category of braided crossed modules is equivalent to the category of reduced simplicial groups with Moore complex of length 2.

So braided crossed modules of groups, 2-crossed modules and crossed squares are seen to arise from algebraic consideration and are all algebraic models for homotopy 3-types.

2.1 Braided crossed modules

Definition 2.1.1. *[8] A braided crossed module of groups*

$$\partial : G_1 \to G_0$$

is a crossed module with a braiding function $\{-,-\} : G_0 \times G_0 \to G_1$ *satisfying the following axioms:*

(i) $\{g_1, g_2\, g_3\} = \{g_1, g_2\}\, {}^{g_2}\{g_1, g_3\}$;

(ii) $\{g_1\, g_2, g_3\} = {}^{g_1}\{g_2, g_3\}\, \{g_1, g_3\}$;

(iii) $\partial\{g_1, g_2\} = g_1\, g_2\, g_1^{-1}\, g_2^{-1}$;

(iv) $\{\partial(\alpha), g\} = \alpha\, {}^{g}\alpha^{-1}$;

(v) $\{g, \partial(\alpha)\} = {}^{g}\alpha\, \alpha^{-1}$;

for all $\alpha \in G_1$ *and* $g,\ g_1,\ g_2,\ g_3 \in G_0$.
If the braiding is symmetric, we also have:

(vi) $\{g_1, g_2\}\, \{g_2, g_1\} = 1$,

then the crossed module $\partial : G_1 \to G_0$ *is called symmetric crossed module.*

Let $\partial : G_1 \to G_0$ be a braided crossed module, we recall some useful identities that are used in the proofs of many statements:

(a) ${}^{g_1}\{g_1^{-1}, g_2\} = \{g_1, g_2\}^{-1} = {}^{g_2}\{g_1, g_2^{-1}\}$;

(b) ${}^{g_1\, g_2}\{g_1^{-1}, g_2^{-1}\} = \{g_1, g_2\}$;

(c) ${}^{g}\{g, g\} = \{g, g\}$;

(d) $\{g_1\, g_2, g_3\} = \{g_1, g_2\, g_3\, g_2^{-1}\}\, \{g_2, g_3\}$;

(e) $\{g_1, g_2\, g_3\} = \{g_1, g_3\}\, \{g_3\, g_1\, g_3^{-1}, g_2\}$;

(f) ${}^{g_1}\{g_2, g_3\} = \{g_1\, g_2\, g_1^{-1}, g_1\, g_3\, g_1^{-1}\}$;

(g) $\{\partial(\alpha_1)\, g_1, \partial(\alpha_2)\, g_2\}\, \alpha_2\, {}^{g_2}\alpha_1 = \alpha_1\, {}^{g_1}\alpha_2\, \{g_1, g_2\}$;

for all $\alpha_1,\, \alpha_2 \in G_1$ and $g,\, g_1,\, g_2,\, g_3 \in G_0$.

We call Γ-equivariant braided crossed module a braided crossed module $\partial : G_1 \to G_0$ equipped with an action by a group Γ and the braidings are assumed to be Γ-equivariant in the sense that ${}^{\sigma}\{g_1, g_2\} = \{{}^{\sigma}g_1,\, {}^{\sigma}g_2\}$.

A morphism between braided crossed modules is a morphism between crossed modules which is compatible with the braiding map $\{-,-\}$. The category of braided crossed modules will be denoted by $\mathcal{BCM}$.

2.2 Crossed squares

Definition 2.2.1. *A crossed square is a commutative diagram of groups*

$$\begin{array}{ccc} G_1 & \xrightarrow{\bar{p}_1} & \Gamma_1 \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \partial'} \\ G_0 & \xrightarrow{\bar{p}_0} & \Gamma_0 \end{array}$$

together with actions of the group Γ_0 on G_1, Γ_1 and G_0 (and hence actions of Γ_1 on G_1 and G_0 via ∂' and of G_0 on G_1 and Γ_1 via $\bar{p}_0$) and a function $h : \Gamma_1 \times G_0 \to G_1$, such that the following axioms are satisfied:

(i) *the maps $\bar{p}_1$, ∂ preserve the actions of Γ_0. Furthermore, with the given actions the maps ∂', $\bar{p}_0$ and $\partial'\,\bar{p}_1 = \bar{p}_0\partial$ are crossed modules;*

(ii) $\bar{p}_1\, h(\beta, g) = \beta\, {}^{g}\beta^{-1}$, $\partial\, h(\beta, g) = {}^{\beta}g\, g^{-1}$;

(iii) $h(\bar{p}_1(\alpha), g) = \alpha\, {}^{g}\alpha^{-1}$, $h(\beta, \partial(\alpha)) = {}^{\beta}\alpha\, \alpha^{-1}$;

(iv) $h(\beta_1\, \beta_2, g) = {}^{\beta_1}h(\beta_2, g)\, h(\beta_1, g)$, $h(\beta, g_1\, g_2) = h(\beta, g_1)\, {}^{g_1}h(\beta, g_2)$;

(v) $h({}^{\sigma}\beta, {}^{\sigma}g) = {}^{\sigma}h(\beta, g)$;

for all $\alpha \in G_1$, β, β_1, $\beta_2 \in \Gamma_1$, g, g_1, $g_2 \in G_0$ and $\sigma \in \Gamma_0$.

Note that in these axioms a term such as ${}^{\beta}\alpha$ is α acted on by β, and so ${}^{\beta}\alpha = {}^{\partial'(\beta)}\alpha$. It is a consequence of i) that $\partial, \overline{p}_1$ are crossed modules. Further, by (iv), h is normalized and by iii), G_0 acts trivially on $\ker\overline{p}_1$ and Γ_1 acts trivially on $\ker\partial$.

Given a crossed square as above, we have some useful identities (see [34], [6]):

(a) ${}^{\beta}({}^{g}\alpha)\, h(\beta, g) = h(\beta, g)\, {}^{g}({}^{\beta}\alpha)$;

(b) ${}^{\beta_1}({}^{g_1}h(\beta_2, g_2))\, h(\beta_1, g_1) = h(\beta_1, g_1)\, {}^{g_1}({}^{\beta_1}h(\beta_2, g_2))$;

(c) $h(\bar{p}_1\, h(\beta, g_1), g_2) = h(\beta, g_1)\, {}^{g_2}h(\beta, g_1)^{-1}$;

(d) $h(\beta_2, \partial\, h(\beta_1, g)) = {}^{\beta_2}h(\beta_1, g)\, h(\beta_1, g)^{-1}$;

(e) $h(\bar{p}_1(\alpha_1), \partial(\alpha_2)) = \alpha_1\, \alpha_2\, \alpha_1^{-1}\, \alpha_2^{-1}$;

(f) $h(\beta_1\, {}^{g_1}\beta_1^{-1}, {}^{\beta_2}g_2\, g_2^{-1}) = h(\beta_1, g_1)\, h(\beta_2, g_2)\, h(\beta_1, g_1)^{-1}\, h(\beta_2, g_2)^{-1}$;

(g) ${}^{\beta}h(\beta^{-1}, g) = h(\beta, g)^{-1} = {}^{g}h(\beta, g^{-1})$;

(h) ${}^{\beta}({}^{g}h(\beta, g)) = h(\beta, g)$;

(l) $h(\bar{p}_1(\alpha_1)\, \beta_1, \partial(\alpha_2) g_2)\, \alpha_2\, {}^{g_2}\alpha_1 = \alpha_1\, {}^{\beta_1}\alpha_2\, h(\beta_1, g_2)$;

for all α, α_1, $\alpha_2 \in G_1$ and g, g_1, $g_2 \in G_0$. The last three identities do not appear in any text and they are deducted from the axiom (iv).

Definition 2.2.2. *A morphism of crossed squares*

$$\begin{array}{ccccccc} G_1 & \xrightarrow{\bar{p}_1} & \Gamma_1 & & G'_1 & \xrightarrow{\bar{p}'_1} & \Gamma'_1 \\ \partial \big\downarrow & & \big\downarrow \partial' & \xrightarrow{\phi} & \bar{\partial} \big\downarrow & & \big\downarrow \bar{\partial}' \\ G_0 & \xrightarrow{\bar{p}_0} & \Gamma_0 & & G'_0 & \xrightarrow{\bar{p}'_0} & \Gamma'_0 \end{array}$$

consists of four group homomorphisms $\phi_{G_1} : G_1 \to G'_1$ $\phi_{G_0} : G_0 \to G'_0$, $\phi_{\Gamma_1} : \Gamma_1 \to \Gamma'_1$, $\phi_{\Gamma_0} : \Gamma_0 \to \Gamma'_0$ such that the resulting cube of group homomorphisms is commutative; $\phi_{G_1}(h(\beta, g)) = h(\phi_{\Gamma_1}(\beta), \phi_{G_0}(g))$ for every $\beta \in \Gamma_1$, $g \in G_0$; each of the homomorphisms ϕ_{G_1}, ϕ_{G_0}, ϕ_{Γ_1} is ϕ_{Γ_0}-equivariant.

Crossed squares and their morphisms form a category, that will be denoted by $\mathcal{CS}$.

Examples. (a) Given a pair of normal subgroups N_1, N_2 of a group G, we can form the following square:

$$\begin{array}{ccc} N_1 \cap N_2 & \longrightarrow & N_1 \\ \big\downarrow & & \big\downarrow \\ N_2 & \longrightarrow & G \end{array}$$

in which each morphism is an inclusion crossed module and there is a commutator map

$$\begin{array}{rcl} h : N_1 \times N_2 & \longrightarrow & N_1 \cap N_2 \\ (n_1, n_2) & \longrightarrow & [n_1, n_2]. \end{array}$$

This forms a crossed square of groups.

(b) [44] Any simplicial group $G_\bullet$ yields a crossed square, the crossed 2-cube $M(G_\bullet, 2)$ associated with simplicial group $G_\bullet$, defined by:

$$\begin{array}{ccc} \dfrac{NG_2}{d_3(NG_3)} & \longrightarrow & ker\partial_1^1 \\ \downarrow & & \downarrow \\ ker\partial_0^1 & \longrightarrow & G_1 \end{array}$$

for suitable maps. This is part of the construction that shows that all connected 3-types are modelled by crossed squares.

(c) [42] Let

$$\begin{array}{ccc} G_1 & \xrightarrow{\bar{p}_1} & \Gamma_1 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ G_0 & \xrightarrow{\bar{p}_0} & \Gamma_0 \end{array}$$

be a crossed square with a function $h : \Gamma_1 \times G_0 \to G_1$. Then $< \bar{p}_1, \bar{p}_0 >$ is a morphism of crossed modules, and $\partial' : \Gamma_1 \to \Gamma_0$ acts on $\partial : G_1 \to G_0$.

(d) Let

$$\begin{array}{ccc} G_1 & \xrightarrow{\bar{p}_1} & \Gamma_1 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ G_0 & \xrightarrow{\bar{p}_0} & \Gamma_0 \end{array}$$

be a crossed square with a function $h : \Gamma_1 \times G_0 \to G_1$. Then we can construct the semi-direct crossed module (see for the description the section 1.3) and an other one, given by:

$$(\bar{p}_1, \bar{p}_0) : G_1 \rtimes G_0 \to \Gamma_1 \rtimes \Gamma_0.$$

The actions of G_0 on G_1 and of Γ_0 on Γ_1 are the natural actions and the action of $\Gamma_1 \rtimes \Gamma_0$ on $G_1 \rtimes G_0$ is defined by:

$${}^{(\beta,\sigma)}(\alpha, g) = ({}^{\partial'(\beta)\,\sigma}\alpha\, h(\beta, {}^{\sigma}g), {}^{\sigma}g).$$

(e) [42] If $\partial : G_1 \to G_0$ is a crossed module, then we have the following crossed square:

$$\begin{array}{ccc} G_1 & \xrightarrow{\eta} & \mathrm{Der}^*(G_0, G_1) \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\triangle} \\ G_0 & \xrightarrow{\gamma} & \mathrm{Aut}\,(G_1, G_0, \partial) \end{array}$$

with the function $h : \mathrm{Der}^*(G_0, G_1) \times G_0 \to G_1$ given by $(\varepsilon, g) \to \varepsilon(g)$ and where $\mathrm{Aut}\,(G_1, G_0, \partial)$ acts on G_1 and on G_0 via the appropriate projections.

(f) [15] Crossed squares can be seen as crossed modules in the category of crossed modules and they provide algebraic models of connected 3-types.

2.3 2-crossed modules

Definition 2.3.1. *A 2-crossed module is a normal complex of groups* [1]

$$G_1 \xrightarrow{\partial} G_0 \xrightarrow{p_0} \Gamma_0$$

together with actions of Γ_0 *on all three groups and a mapping* $\{-,-\} : G_0 \times G_0 \to G_1$ *satisfying the following axioms:*

(i) *the action on* Γ_0 *on itself is by conjugation,* ∂ *and* p_0 *are* Γ_0*-equivariant;*

(ii) $\partial\{g_1, g_2\} = g_1\, g_2\, g_1^{-1}\, {}^{p_0(g_1)}g_2^{-1}$*;*

(iii) $\{\partial(\alpha_1), \partial(\alpha_2)\} = \alpha_1\, \alpha_2\, \alpha_1^{-1}\, \alpha_2^{-1}$*;*

(iv) $\{\partial(\alpha), g\}\, \{g, \partial(\alpha)\} = \alpha\, {}^{p_0(g)}\alpha^{-1}$*;*

(v) $\{g_1, g_2\, g_3\} = \{g_1, g_2\}\, \{g_1, g_3\}\, \{\partial(\{g_1, g_3\})^{-1}, {}^{p_0(g_1)}\, g_2\}$*;*

(vi) $\{g_1\, g_2, g_3\} = \{g_1, g_2\, g_3\, g_2^{-1}\}\, {}^{p_0(g_1)}\{g_2, g_3\}$*;*

(vii) ${}^{\sigma}\{g_1\, g_2\} = \{{}^{\sigma}g_1, {}^{\sigma}\, g_2\}$*;*

for all α, α_1, $\alpha_2 \in G_1$, g, g_1, g_2, $g_3 \in G_0$ *and* $\sigma \in \Gamma_0$.

The pairing $\{-,-\} : G_0 \times G_0 \to G_1$ is often called the Peiffer lifting of the 2-crossed module. Note that we have not specified that G_0 acts on G_1. We could have done that as follows: if $g \in G_0$ and $\alpha \in G_1$, define:

$${}^{g}\alpha := \alpha\, \{\partial(\alpha)^{-1}, g\}.$$

The homomorphism $\partial : G_1 \to G_0$, endowed with this action, is a crossed module. Now (iv) and (v) simplify to the following expressions:

$$\{g_1, g_2\, g_3\} = \{g_1, g_2\}^{\,p_0(g_1) g_2}\, \{g_1, g_3\};$$

$$\{g_1\, g_2, g_3\} = {}^{g_1}\{g_2, g_3\}\, \{g_1,\, {}^{p_0(g_2)}g_3\}.$$

Let $G_1 \xrightarrow{\partial} G_0 \xrightarrow{p_0} \Gamma_0$ be a 2-crossed module, we recall some useful identities that are used in the proofs of many statements:

---

[1]

Definition 2.3.2. *A chain complex of groups is a sequence (of any length, finite or infinite) of groups and homomorphisms, for instance,*

$$\dots \longrightarrow C_n \xrightarrow{\partial_n} C_{n-1} \xrightarrow{\partial_{n-1}} C_{n-2} \longrightarrow \dots,$$

in which each composite $\partial_{n-1} \circ \partial_n$ *is the trivial homomorphism.*
The chain complex is normal if each image $\partial_n(C_n)$ *is a normal subgroup of the next group* C_{n-1}.

(a) ${}^{g_1}\{g_1^{-1}, g_2\} = \{g_1, {}^{p_0(g_1^{-1})}g_2\}^{-1}$;

(b) ${}^{p_0(g_1)g_2}\{g_1, g_2^{-1}\} = \{g_1, g_2\}^{-1}$;

(c) $\{\partial(\alpha_1)\, g_1, \partial(\alpha_2)\, g_2\}\, {}^{p_0(g_1)}\alpha_2\, {}^{p_0(g_1)g_2}\alpha_1 = \alpha_1\, {}^{g_1}\alpha_2\, \{g_1, g_2\}$;

(d) $\{\partial(\alpha), g\} = \alpha\, {}^{g}\alpha^{-1}$;

(f) $\{g, \partial(\alpha)\} = {}^{g}\alpha\, {}^{p_0(g)}\alpha^{-1}$;

for all $\alpha_1,\, \alpha_2,\, \alpha \in G_1$ and $g_1,\, g_2,\, g \in G_0$.

Definition 2.3.3. *A morphism of 2-crossed modules is given by a diagram*

$$\begin{array}{ccccc} G_1 & \xrightarrow{\partial} & G_0 & \xrightarrow{p_0} & \Gamma_0 \\ \downarrow{\scriptstyle\varphi} & & \downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\chi} \\ G_1' & \xrightarrow{\partial'} & G_0' & \xrightarrow{p_0'} & \Gamma_0' \end{array}$$

such that commutes and

$$\psi({}^{\sigma}g) = {}^{\chi(\sigma)}\psi(g), \qquad \varphi({}^{\sigma}\alpha) = {}^{\chi(\sigma)}\varphi(\alpha), \qquad \{\psi(g_1), \psi(g_2)\} = \varphi(\{g_1, g_2\}),$$

for all $\alpha \in G_1$, g, g_1, $g_2 \in G_0$ *and* $\sigma \in \Gamma_0$.

2-Crossed modules and their morphisms form a category, that will be denoted by $2\!-\!\mathcal{CM}$.

Examples. (a) Any crossed module $\partial : G_1 \to G_0$ gives a 2-crossed module:

$$1 \xrightarrow{1} G_1 \xrightarrow{\partial} G_0$$

with the obvious actions. This construction is functorial and $\mathcal{CM}$ can be considered to be a full subcategory of $2\!-\!\mathcal{CM}$ in this way.

Viceversa, any 2-crossed module having trivial top dimensional group is a crossed module.

(b) $G_1 \xrightarrow{\partial} G_0$ is a braided crossed module if and only if

$$G_1 \xrightarrow{\partial} G_0 \xrightarrow{1} 1$$

is a 2-crossed module. In this way, we can consider the functor from $\mathcal{BCM}$ to $2\!-\!\mathcal{CM}$ and $\mathcal{BCM}$ is a full subcategory of $2\!-\!\mathcal{CM}$.

(c) $G_1 \xrightarrow{\partial} G_0$ is a Γ_0-equivariant braided crossed module if and only if $G_1 \xrightarrow{\partial} G_0 \xrightarrow{1} \Gamma_0$ is a 2-crossed module.

(d) If $G_1 \xrightarrow{\partial} G_0 \xrightarrow{p_0} \Gamma_0$ is a 2-crossed module, then $< 1, p_0 >$ is a morphism of crossed modules from $\partial : G_1 \to G_0$ to $1 \to \Gamma_0$ with an action of $1 \to \Gamma_0$ on $\partial : G_1 \to G_0$.

If

$$G_1 \xrightarrow{\partial} G_0 \xrightarrow{p_0} \Gamma_0$$

is a 2-crossed module, obviously we have that $\mathrm{Im}\,\partial$ is a normal subgroup of G_0. Now we recall a well-known Proposition (with a small abuse of notation).

Proposition 2.3.1. *If* $G_1 \xrightarrow{\partial} G_0 \xrightarrow{p_0} \Gamma_0$ *is a 2-crossed module then there is an induced crossed module structure on*

$$p_0 \,:\, \frac{G_0}{Im\,\partial} \longrightarrow \Gamma_0\,.$$

2.4 2-crossed modules with trivial Peiffer lifting

Suppose we have a 2-crossed module

$$G_1 \xrightarrow{\partial} G_0 \xrightarrow{p_0} \Gamma_0$$

with the extra condition that $\{g_1, g_2\} = 1$ for all g_1, $g_2 \in G_0$. The obvious thing to do is to see what each of the defining properties of a 2-crossed module give in this case.

(i) There is an action of Γ_0 on G_0 and on G_1 and the maps ∂, p_0 are Γ_0-equivariant (this gives nothing new in this special case).

(ii) The Peiffer identity holds for $p_0 : G_0 \to \Gamma_0$, i.e. p_0 is a crossed module.

(iii) G_1 is an abelian group.

(iv) The Peiffer lifting $\{-,-\}$ is trivial, i.e. ${}^{p_0(g)}\alpha = \alpha$, so $p_0(G_0)$ has trivial action on G_1.

Axioms (v),(vi) and (vii) vanish and consequently G_0 has trivial action on G_1.

Example. The following diagram

$$\begin{array}{ccc} G_1 & \longrightarrow & 1 \\ {\scriptstyle \partial}\downarrow & & \downarrow \\ G_0 & \xrightarrow{p_0} & \Gamma_0 \end{array}$$

is a crossed square if and only if $G_1 \xrightarrow{\partial} G_0 \xrightarrow{p_0} \Gamma_0$ is a 2-crossed module whit trivial Peiffer lifting.

2.5 Crossed squares and 2-crossed modules

Suppose that

$$\begin{array}{ccc} G_1 & \xrightarrow{\bar{p}_1} & \Gamma_1 \\ {\scriptstyle \partial}\downarrow & & \downarrow{\scriptstyle \partial'} \\ G_0 & \xrightarrow{\bar{p}_0} & \Gamma_0 \end{array}$$

is a crossed square, then its associated 2-crossed module is given by:

$$G_1 \xrightarrow{\overline{\partial}} G_0 \ltimes \Gamma_1 \xrightarrow{p_0} \Gamma_0$$

where $\overline{\partial}(\alpha) = (\partial(\alpha), \bar{p}_1(\alpha^{-1}))$ and $p_0(g, \beta) = \partial'(\beta)\, \bar{p}_0(g)$ (this is a part of the construction to transform morphisms of crossed modules into butterflies [40]). The semi-direct product of Γ_1 on G_0 is formed by making G_0 act on Γ_1 via Γ_0, i.e

$${}^{g}\beta = {}^{\bar{p}_0(g)}\beta \quad \forall \beta \in \Gamma_1, \forall g \in G_0$$

where the Γ_0-action is the given one. Conduché, in [17], defined the Peiffer lifting in this situation by

$$\{(g_1, \beta_1), (g_2, \beta_2)\} = h(\beta_1, g_1\, g_2\, g_1^{-1})^{-1}.$$

We thus have two ways of going from a simplicial group $G_\bullet$ to a 2-crossed module:

(a) directly to get

$$\frac{NG_2}{d_3(NG_3)} \longrightarrow NG_1 \longrightarrow G_0$$

(b) indirectly via $M(G_\bullet, 2)$ and then by the above construction to get

$$\frac{NG_2}{d_3(NG_3)} \longrightarrow ker\partial_1^1 \ltimes ker\partial_0^1 \longrightarrow G_1$$

and they give the same homotopy type. More precisely, G_1 decomposes as $s_0(G_0) \ltimes Ker\partial_0^1$ and the $Ker\partial_0^1$ factor in the middle term of (b) maps down to that in this decomposition by the identity map, thus ∂_0^1 induces a quotient map from (b) to (a) with kernel isomorphic to

$$1 \longrightarrow ker\partial_0^1 \xrightarrow{=} ker\partial_0^1$$

which is acyclic/contractible.

Chapter 3

Categorical groups

In [14], the authors develop a cohomological theory with coefficients in a categorical crossed module. Our purpose (see Chapter 5) is to describe in details the cohomology in some strict cases. This chapter serves to start from recall the concept of categorical crossed module, so the definition of monoidal category (category enriched by a tensor product), categorical group and actions of categorical groups. At the end of this chapter we are giving some examples of categorical crossed modules that they will be the protagonists of the Chapter 5.

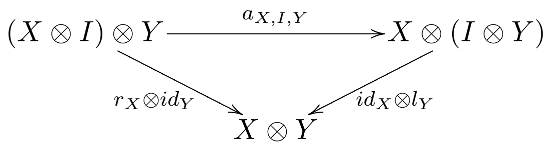
3.1 Monoidal categories

Definition 3.1.1. *A monoidal category* $\mathbf{C} = (\mathbf{C}, \otimes, a, I, l, r)$ *consists of a category* $\mathbf{C}$*, a bifunctor (tensor product)* $\otimes : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$*, an object* I *(unit) and natural isomorphisms called, respectively, the associativity, left unit and right unit constraints:*

$$a = \{a_{X,Y,Z} : (X \otimes Y) \otimes Z \to X \otimes (Y \otimes Z)\}_{X,Y,Z \in Ob(\mathbf{C})};$$
$$l = \{l_X : I \otimes X \to X\}_{X \in Ob(\mathbf{C})};$$
$$r = \{r_X : X \otimes I \to X\}_{X \in Ob(\mathbf{C})};$$

such that for any objects X, Y, Z, W *in* $Ob(\mathbf{C})$ *the following diagrams (associativity coherence and unit coherence) commute:*

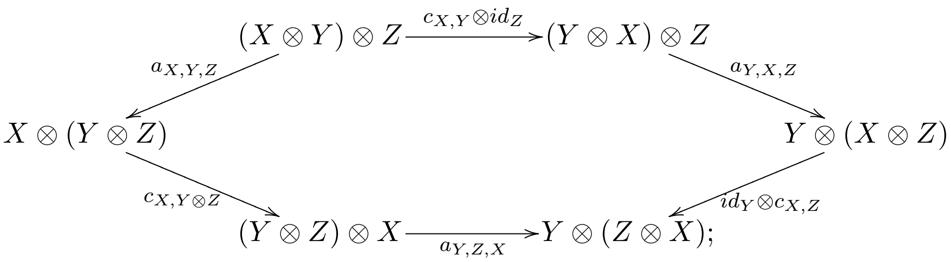
$$\begin{array}{ccc}
((X \otimes Y) \otimes Z) \otimes W & \xrightarrow{a_{X\otimes Y,Z,W}} & (X \otimes Y) \otimes (Z \otimes W) \\
\downarrow{\scriptstyle a_{X,Y,Z}\otimes id_W} & & \\
(X \otimes (Y \otimes Z)) \otimes W & & \downarrow{\scriptstyle a_{X,Y,Z\otimes W}} \\
\downarrow{\scriptstyle a_{X,Y\otimes Z,W}} & & \\
X \otimes ((Y \otimes Z) \otimes W) & \xrightarrow[id_X\otimes a_{Y,Z,W}]{} & X \otimes (Y \otimes (Z \otimes W))
\end{array}$$

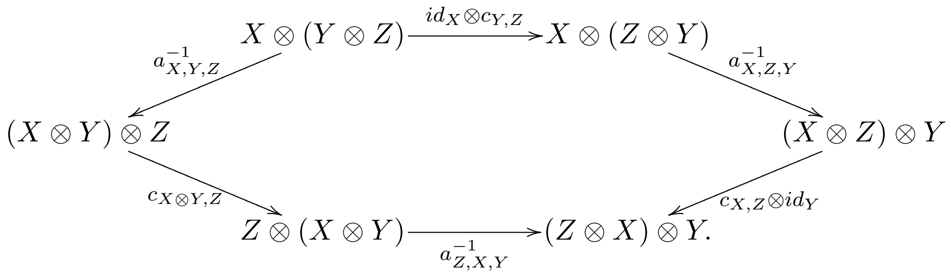


Definition 3.1.2. *A braided monoidal category is a monoidal category* $\mathbf{C}$ *equipped with a family of natural isomorphisms*

$$c = \{c_{X,Y} : X \otimes Y \to Y \otimes Y\}_{X,Y \in Ob(\mathbf{C})},$$

such that for any objects X, Y, Z *in* $Ob(\mathbf{C})$ *the following diagrams (associativity coherence) commute:*





Definition 3.1.3. *A symmetric monoidal category is a braided monoidal category* $\mathbf{C}$ *for which the braiding satisfies* $c_{X,Y} = c^{-1}_{Y,X}$, *for all objects* X *and* Y.

A (braided, symmetric) monoidal category is called strict if $a_{X,Y,Z}$, l_X, r_X are all identity morphisms, for all objects X, Y, Z in $\mathrm{Ob}(\mathbf{C})$. In this case, we have:

$$(X \otimes Y) \otimes Z = X \otimes (Y \otimes Z) \quad \text{and} \quad I \otimes X = X = X \otimes I. \tag{3.1}$$

Definition 3.1.4. *A monoidal functor between monoidal categories,* $\mathbf{C} = (\mathbf{C}, \otimes, a, I_{\mathbf{C}}, l, r)$ *and* $\mathbf{D} = (\mathbf{D}, \otimes, a', I_{\mathbf{D}}, l', r')$, *consists of a functor* $F : \mathbf{C} \to \mathbf{D}$ *equipped with:*

- *a natural isomorphism* $\Phi_{X,Y} : F(X) \otimes F(Y) \to F(X \otimes Y)$;
- *an isomorphism* $\Phi_I : I_{\mathbf{D}} \to F(I_{\mathbf{C}})$;

such that for any objects X, Y, Z *in* $\mathbf{C}$ *the following diagrams commutate:*

$$\begin{array}{ccc}
(F(X)\otimes F(Y))\otimes F(Z) & \xrightarrow{a'_{F(X),F(Y),F(Z)}} & F(X)\otimes(F(Y)\otimes F(Z)) \\
\Big\downarrow{\scriptstyle \Phi_{X,Y}\otimes id_{F(Z)}} & & \Big\downarrow{\scriptstyle id_{F(X)}\otimes\Phi_{Y,Z}} \\
F(X\otimes Y)\otimes F(Z) & & F(X)\otimes F(Y\otimes Z) \\
\Big\downarrow{\scriptstyle \Phi_{X\otimes Y,Z}} & & \Big\downarrow{\scriptstyle \Phi_{X,Y\otimes Z}} \\
F((X\otimes Y)\otimes Z) & \xrightarrow[F(a_{X,Y,Z})]{} & F(X\otimes(Y\otimes Z));
\end{array}$$

$$\begin{array}{ccc}
F(X)\otimes I_{\mathbf{D}} & \xrightarrow{id_{F(X)}\otimes\Phi_I} & F(X)\otimes F(I_{\mathbf{C}}) \\
\Big\downarrow{\scriptstyle r'_{F(X)}} & & \Big\downarrow{\scriptstyle \Phi_{X,I_{\mathbf{C}}}} \\
F(X) & \xleftarrow[F(r_X)]{} & F(X\otimes I_{\mathbf{C}});;
\end{array}$$

$$\begin{array}{ccc}
I_{\mathbf{D}}\otimes F(Y) & \xrightarrow{\Phi_I\otimes id_{F(Y)}} & F(I_{\mathbf{C}})\otimes F(Y) \\
\Big\downarrow{\scriptstyle l'_{F(Y)}} & & \Big\downarrow{\scriptstyle \Phi_{I_{\mathbf{C}},Y}} \\
F(Y) & \xleftarrow[F(l_Y)]{} & F(I_{\mathbf{C}}\otimes Y).
\end{array}$$

Definition 3.1.5. *A braided monoidal functor between braided monoidal categories,* $\mathbf{C} = (\mathbf{C}, \otimes, a, I_{\mathbf{C}}, l, r, c)$ *and* $\mathbf{D} = (\mathbf{D}, \otimes, a', I_{\mathbf{D}}, l', r', c')$, *is a monoidal functor* $(F : \mathbf{C} \to \mathbf{D}, \Phi)$ *such that the following diagram commutes for all* $X, Y \in Ob(\mathbf{C})$*:*

$$\begin{array}{ccc}
F(X)\otimes F(Y) & \xrightarrow{c'_{F(X),F(Y)}} & F(Y)\otimes F(X) \\
\Big\downarrow{\scriptstyle \Phi_{X,Y}} & & \Big\downarrow{\scriptstyle \Phi_{Y,X}} \\
F(X\otimes Y) & \xrightarrow[F(c_{X,Y})]{} & F(Y\otimes Z).
\end{array}$$

A symmetric monoidal functor is simply a braided monoidal functor between symmetric monoidal categories.

A (braided, symmetric) monoidal functor is called strict if $\Phi_{X,Y}$, Φ_I are identity morphisms, for all objects X, Y in $\mathrm{Ob}(\mathbf{C})$. In this case, we have:

$$F(X)\otimes F(Y) = F(X\otimes Y) \quad \text{and} \quad F(I_{\mathbf{C}}) = I_{\mathbf{D}}.$$

Definition 3.1.6. *Suppose that*

$$\mathbf{C} = (\mathbf{C}, \otimes, a, I_{\mathbf{C}}, l, r, c) \quad \textit{and} \quad \mathbf{D} = (\mathbf{D}, \otimes, a', I_{\mathbf{D}}, l', r', c')$$

are two monoidal categories and $(F : \mathbf{C} \to \mathbf{D}, \Phi)$ *and* $(F' : \mathbf{C} \to \mathbf{D}, \Phi')$ *are two monoidal functors between these categories.*

A monoidal natural transformation $\alpha : (F, \Phi) \Rightarrow (F', \Phi')$ *between these functors is a natural transformation* $\alpha : F \Rightarrow F'$ *between the underlying functors such that the diagrams*

$$\begin{array}{ccc} F(X) \otimes F(Y) & \xrightarrow{\alpha_X \otimes \alpha_Y} & F'(X) \otimes F'(Y) \\ {\scriptstyle \Phi_{X,Y}} \downarrow & & \downarrow {\scriptstyle \Phi'_{X,Y}} \\ F(X \otimes Y) & \xrightarrow[\alpha_{X \otimes Y}]{} & F'(X \otimes Y) \end{array} \qquad \begin{array}{ccc} & I_{\mathbf{D}} & \\ {\scriptstyle \Phi_I} \swarrow & & \searrow {\scriptstyle \Phi'_I} \\ F(I_{\mathbf{C}}) & \xrightarrow[\alpha_{I_{\mathbf{C}}}]{} & F'(I_{\mathbf{C}}) \end{array}$$

commute for all objects $X, Y \in Ob(\mathbf{C})$.

3.2 Categorical groups

Definition 3.2.1. *A categorical group* $\mathbf{G}$ *(see [23], [46], [2], [24]) is a monoidal category* $(\mathbf{G}, \otimes, a, I, l, r)$ *such that:*

- $\mathbf{G}$ *is a groupoid (see Appendix B.1);*
- *for each object* X*, there is an object* X^* *(inverse) and an arrow* $\eta_X : I \to X \otimes X^*$.

If $\mathbf{G}$ is a categorical group, then it is possible to choose an arrow $\varepsilon_X : X^* \otimes X \to I$ in such a way that $(X, X^*, \eta_X, \varepsilon_X)$ is a duality, this means that the two following diagrams commute:

$$\begin{array}{ccccc} X^* & \xrightarrow{r^{-1}_{X^*}} & X^* \otimes I & \xrightarrow{id_{X^*} \otimes \eta_X} & X^* \otimes X \otimes X^* \\ & {\scriptstyle id_{X^*}} \searrow & & & \downarrow {\scriptstyle \varepsilon_X \otimes id_{X^*}} \\ & & X^* & \xleftarrow[l_{X^*}]{} & I \otimes X^*; \end{array}$$

$$\begin{array}{ccccc} X & \xrightarrow{l^{-1}_{X}} & I \otimes X & \xrightarrow{\eta_X \otimes id_X} & X \otimes X^* \otimes X \\ & {\scriptstyle id_X} \searrow & & & \downarrow {\scriptstyle id_X \otimes \varepsilon_X} \\ & & X & \xleftarrow[r_X]{} & X \otimes I. \end{array}$$

Moreover, one can choose $I^* = I$.

Categorical groups and their monoidal functors form a category, that will be denoted by $\mathcal{CG}$.

A categorical group is said to be braided (symmetric) (see [32]) if it is braided (symmetric) as a monoidal category.

A (braided, symmetric) categorical group is called strict if it is strict as a monoidal category and η_X is an identity morphism, for all objects X in $\mathrm{Ob}(\mathbf{C})$. In this case, we have the identities (3.1) and furthermore

$$X \otimes X^* = I = X^* \otimes X.$$

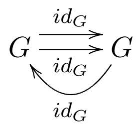
Strict categorical groups and their strict monoidal functors form a category, that will be denoted by $\boldsymbol{Str\mathcal{CG}}$.

Remark 3.2.1. *If we consider a monoidal functor between categorical groups, then each canonical isomorphism Φ_I can be deduced from $\Phi_{X,Y}$.*

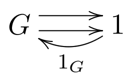
A natural transformation between morphisms of categorical groups is a monoidal natural transformation between the underlying monoidal functors.

It easy to see that the category $\boldsymbol{Str\mathcal{CG}}$ is equivalent to the category of groups in $\mathcal{GPD}$. The last one being equivalent to $\mathcal{CM}$, it follows that $\boldsymbol{Str\mathcal{CG}}$ is equivalent to $\mathcal{CM}$. Given a crossed module $\partial : G_1 \to G_0$ we denote by $\mathbf{G}(\partial)$ the strict categorical group associated with it.

Examples. (a) If G is a group, the associated discrete category $G[0]$



is a strict categorical group where the tensor product is given by the group operation. If G is an abelian group, then the category denoted by $G[1]$:



is also a strict categorical group where the tensor product is given by the group operation.

(b) Let $\mathbf{G}$ be a categorical group. $\mathrm{Eq}(\mathbf{G})$ is the categorical group of the equivalences of $\mathbf{G}$; the set of objects of $\mathrm{Eq}(\mathbf{G})$ are the monoidal functors $(F : \mathbf{G} \to \mathbf{G}, \Phi)$ with F an equivalence of categories and the arrows are the monoidal natural transformations between them. The composition in $\mathrm{Eq}(\mathbf{G})$ is given by the usual vertical composition of natural transformations. It is clear that $\mathrm{Eq}(\mathbf{G})$ is a groupoid. The composition of functors and the horizontal composition of the natural transformations define a tensor functor $\mathrm{Eq}(\mathbf{G}) \otimes \mathrm{Eq}(\mathbf{G}) \to \mathrm{Eq}(\mathbf{G})$. Thus, $\mathrm{Eq}(\mathbf{G})$ is a categorical group in which $I = id_{\mathbf{G}}$ and an inverse for an object (F, Φ) is obtained by taking a quasi-inverse F^{-1} of F.

(c) $\mathrm{Aut}(\mathbf{G})$ is the categorical subgroup of $\mathrm{Eq}(\mathbf{G})$, whose objects, called automorphisms, are strict monoidal functor (F, Φ), where F is an isomorphism.

3.3 Actions of categorical groups

Fix a categorical group $\mathbf{\Gamma}$. A $\mathbf{\Gamma}$-categorical group (see [24]) consists of a categorical group $\mathbf{G}$ together with a morphism of categorical groups (a

G-action) $(F,\mu):\mathbf{\Gamma}\to \mathrm{Eq}(\mathbf{G})$. Equivalently, we have a functor

$$\begin{array}{rcl} ac:\mathbf{\Gamma}\times\mathbf{G} & \longrightarrow & \mathbf{G} \\ (X,A) & \longrightarrow & {}^{X}A \end{array}$$

together with three natural isomorphisms:

$$\begin{array}{l} \psi_{X,A,B}:\ {}^{X}(A\otimes B)\to {}^{X}A\otimes {}^{X}B; \\ \phi_{I,A}:\ {}^{I}A\to A; \\ \phi_{X,Y,A}:\ {}^{(X\otimes Y)}A\to {}^{X}({}^{Y}A); \end{array}$$

such that, for any objects X, Y, Z in $\mathrm{Ob}(\mathbf{\Gamma})$ and A, B, C in $\mathrm{Ob}(\mathbf{G})$, the following diagrams commutate:

$$\begin{array}{ccc}
{}^{(X\otimes(Y\otimes Z))}A & \xrightarrow{\phi_{X,Y\otimes Z,A}} & {}^{X}({}^{(Y\otimes Z)}A) \\
\uparrow{\scriptstyle {}^{a_{X,Y,Z}}A} & & \downarrow{\scriptstyle {}^{X}\phi_{Y,Z,A}} \\
{}^{((X\otimes Y)\otimes Z)}A & & {}^{X}({}^{Y}({}^{Z}A)) \\
{\scriptstyle \phi_{X\otimes Y,Z,A}}\searrow & & \nearrow{\scriptstyle \phi_{X,Y,{}^{Z}A}} \\
& {}^{(X\otimes Y)}({}^{Z}A); &
\end{array}$$

$$\begin{array}{ccc}
{}^{(X\otimes I)}A & \xrightarrow{\phi_{X,I,A}} & {}^{X}({}^{I}A) \\
{\scriptstyle {}^{r_X}A}\searrow & & \swarrow{\scriptstyle {}^{X}\phi_{I,A}} \\
& {}^{X}A; &
\end{array}$$

$$\begin{array}{ccc}
{}^{X}(A\otimes(B\otimes C)) & \xrightarrow{\psi_{X,A,B\otimes C}} & {}^{X}A\otimes {}^{X}(B\otimes C) \\
\uparrow{\scriptstyle {}^{X}a_{A,B,C}} & & \downarrow{\scriptstyle id_{{}^{X}A}\otimes\psi_{X,B,C}} \\
{}^{X}((A\otimes B)\otimes C) & & {}^{X}A\otimes({}^{X}B\otimes{}^{X}C) \\
\downarrow{\scriptstyle \psi_{X,A\otimes B,C}} & & \uparrow{\scriptstyle a_{{}^{X}A,{}^{X}B,{}^{X}C}} \\
{}^{X}(A\otimes B)\otimes{}^{X}C & \xrightarrow[\psi_{X,A,B}\otimes id_{{}^{X}C}]{} & ({}^{X}A\otimes{}^{X}B)\otimes{}^{X}C;
\end{array}$$

$$\begin{array}{ccc}
{}^{(X\otimes Y)}(A\otimes B) & \xrightarrow{\phi_{X\otimes Y,A,B}} & {}^{(X\otimes Y)}A\otimes{}^{(X\otimes Y)}B \\
\downarrow{\scriptstyle \phi_{X,Y,A\otimes B}} & & \downarrow{\scriptstyle \phi_{X,Y,A}\otimes\phi_{X,Y,B}} \\
{}^{X}({}^{Y}(A\otimes B)) & \xrightarrow[\psi_{X,{}^{Y}A,{}^{Y}B}\circ{}^{X}\psi_{Y,A,B}]{} & {}^{X}({}^{Y}A)\otimes{}^{X}({}^{Y}B);
\end{array}$$

Note that a canonical morphism $\phi_{I,A}:\ {}^{I}A\to A$ can be deduced from $\phi_{X,Y,A}$.

Definition 3.3.1. *Let* $\mathbf{G}$ *and* $\mathbf{G}'$ *be* $\mathbf{\Gamma}$*-categorical groups. A morphism* $(\mathbf{T}, \varphi) : \mathbf{G} \to \mathbf{G}'$ *consists of a categorical group morphism* $\mathbf{T} = (T, \mu)$ *and a natural transformation* φ

$$\begin{array}{ccc} \mathbf{\Gamma} \times \mathbf{G} & \xrightarrow{ac} & \mathbf{G} \\ {\scriptstyle Id\times T}\downarrow & \varphi \Downarrow & \downarrow{\scriptstyle T} \\ \mathbf{\Gamma} \times \mathbf{G}' & \xrightarrow[ac]{} & \mathbf{G}' \end{array}$$

compatible with ψ*,* ϕ *and* ϕ_I *in the sense of [24].*

$\mathbf{\Gamma}$-categorical groups and morphisms of $\mathbf{\Gamma}$-categorical groups are the objects and 1-cells of a 2-category, denoted by $\mathbf{\Gamma}$-$\mathcal{CG}$, where a 2-cell $\alpha : (\mathbf{T}, \varphi) \Rightarrow (\mathbf{T}', \varphi')$ is a 2-cell $\alpha : \mathbf{T} \Rightarrow \mathbf{T}'$ in $\mathcal{CG}$ satisfying the corresponding compatibility condition with φ and φ'.

3.4 Categorical crossed modules

Definition 3.4.1. *Fix a categorical group* $\mathbf{\Gamma}$*. A categorical* $\mathbf{\Gamma}$*-precrossed module consists of a triple* $(\mathbf{G}, \mathbf{T}, \nu)$*, where* $\mathbf{G}$ *is a* $\mathbf{\Gamma}$*-categorical group,* $\mathbf{T} = (T, \mu) : \mathbf{G} \to \mathbf{\Gamma}$ *is a morphism of categorical groups and*

$$\nu = (\nu_{X,A} : T({}^{X}A) \otimes X \to X \otimes T(A))_{(X,A)\in Ob(\mathbf{\Gamma})\times Ob(\mathbf{G})}$$

is a family of natural isomorphisms in $\mathbf{\Gamma}$ *such that the following diagrams commute:*

$$\begin{array}{ccc} T({}^{X}({}^{Y}A)) \otimes X \otimes Y & \xrightarrow{T(\phi^{-1}_{X,Y,A})\otimes id_{X\otimes Y}} & T({}^{(X\otimes Y)}A) \otimes X \otimes Y \\ {\scriptstyle \nu_{X,{}^{Y}A}\otimes id_Y}\downarrow & & \downarrow{\scriptstyle \nu_{X\otimes Y,A}} \\ X \otimes T({}^{Y}A) \otimes Y & \xrightarrow[id_X\otimes \nu_{Y,A}]{} & X \otimes Y \otimes T(A); \end{array}$$

$$\begin{array}{ccc} T({}^{X}A) \otimes T({}^{X}B) \otimes X & \xrightarrow{id_{T({}^{X}A)}\otimes \nu_{X,B}} & T({}^{X}A) \otimes X \otimes T(B) \\ {\scriptstyle \mu_{{}^{X}A,{}^{X}B}\otimes id_X}\downarrow & & \downarrow{\scriptstyle \nu_{X,A}\otimes id_{T(B)}} \\ T({}^{X}A \otimes {}^{X}B) \otimes X & & X \otimes T(A) \otimes T(B) \\ {\scriptstyle T(\psi^{-1}_{X,A,B})}\downarrow & & \downarrow{\scriptstyle id_X\otimes \mu_{A,B}} \\ T({}^{X}(A \otimes B)) \otimes X & \xrightarrow[\nu_{X,A\otimes B}]{} & X \otimes T(A \otimes B). \end{array}$$

Now, a morphism of categorical $\mathbf{\Gamma}$*-precrossed modules is a triple*

$$(\mathbf{F}, \eta, \alpha) : (\mathbf{G}, \mathbf{T}, \nu) \to (\mathbf{G}', \mathbf{T}', \nu')$$

with $(\mathbf{F},\eta):\mathbf{G}\to\mathbf{G}'$ *a morphism in* $\mathbf{\Gamma}$-$\mathcal{CG}$ *and* $\alpha : T \Rightarrow T'F$ *a 2-cell in* $\mathcal{CG}$ *such that, for any* X *in* $Ob(\mathbf{\Gamma})$ *and* A *in* $Ob(\mathbf{G})$*, the following diagram is commutative (which corresponds to the coherence condition for* $\alpha : T\Rightarrow T'F$ *being a 2-cell in* $\mathbf{\Gamma}$-$\mathcal{CG}$*):*

$$\begin{array}{ccccc}
T({}^{X}A)\otimes X & & \xrightarrow{\nu_{X,A}} & & X\otimes T(A)\\
\downarrow{\scriptstyle \alpha_{{}^{X}A}\otimes id_X} & & & & \downarrow{\scriptstyle id_X\otimes\alpha_A}\\
T'F({}^{X}A)\otimes X & \xrightarrow{T'(\eta_{X,A})\otimes id_X} & T'({}^{X}F(A))\otimes X. & \xrightarrow{\nu'_{X,F(A)}} & X\otimes T'F(A)
\end{array}$$

Definition 3.4.2. *A categorical* $\mathbf{\Gamma}$*-crossed module consists of a 4-tuple* $(\mathbf{G},\mathbf{T},\nu,\chi)$*, where* $(\mathbf{G},\mathbf{T},\nu)$ *is a categorical* $\mathbf{\Gamma}$*-precrossed module and*

$$\chi=(\chi_{A,B}:{}^{T(A)}B\otimes A\to A\otimes B)_{(A,B)\in Ob(\mathbf{G})\times Ob(\mathbf{G})}$$

is a family of natural isomorphisms in $\mathbf{G}$ *such that the following diagrams commutate:*

$$\begin{array}{ccccc}
{}^{T(A\otimes B)}C\otimes A\otimes B & & \xrightarrow{\chi_{A\otimes B,C}} & & A\otimes B\otimes C\\
\downarrow{\scriptstyle \mu^{-1}_{A,B}\,C\,\otimes\, id_{A\otimes B}} & & & & \uparrow{\scriptstyle id_A\,\otimes\,\chi_{B,C}}\\
{}^{(T(A)\otimes T(B))}C\otimes A\otimes B & \xrightarrow{\phi_{T(A),T(B),C}\,\otimes\, id_{A\otimes B}} & {}^{T(A)}({}^{T(B)}C)\otimes A\otimes B & \xrightarrow{\chi_{A,\,{}^{T(B)}C}\,\otimes\, id_B} & A\otimes{}^{T(B)}C\otimes B;
\end{array}$$

$$\begin{array}{ccc}
{}^{T(A)}(B\otimes C)\otimes A & \xrightarrow{\chi_{A,B\otimes C}} & A\otimes B\otimes C\\
\downarrow{\scriptstyle \psi_{T(A),B,C}\otimes id_A} & & \uparrow{\scriptstyle \chi_{A,B}\otimes id_C}\\
{}^{T(A)}B\otimes{}^{T(A)}C\otimes A & \xrightarrow{id_{{}^{T(A)}B}\otimes\chi_{A,C}} & {}^{T(A)}B\otimes A\otimes C;
\end{array}$$

$$\begin{array}{ccccc}
{}^{X}({}^{T(A)}B\otimes A) & & \xrightarrow{{}^{X}\chi_{A,B}S} & & {}^{X}(A\otimes B)\\
\downarrow{\scriptstyle \psi_{X,\,{}^{T(A)}B,A}} & & & & \downarrow{\scriptstyle \psi_{X,A,B}}\\
{}^{X}({}^{T(A)}B)\otimes{}^{X}A & & & & {}^{X}A\otimes{}^{X}B\\
\downarrow{\scriptstyle \phi^{-1}_{X,T(A),B}\,\otimes\, id_{{}^{X}A}} & & & & \uparrow{\scriptstyle \chi_{{}^{X}A,\,{}^{X}B}}\\
{}^{(X\otimes T(A))}B\otimes{}^{X}A & \xrightarrow{\nu^{-1}_{X,A}\,B\,\otimes\, id_{{}^{X}A}} & {}^{(T({}^{X}A)\otimes X)}B\otimes{}^{X}A & \xrightarrow{\phi_{T({}^{X}A),X,B}\,\otimes\, id_{{}^{X}A}} & {}^{T({}^{X}A)}({}^{X}B)\otimes{}^{X}A;
\end{array}$$

$$\begin{array}{ccc}
T({}^{T(A)}B\otimes A) & \xrightarrow{T(\chi_{A,B})} & T(A\otimes B)\\
\downarrow{\scriptstyle \mu^{-1}_{{}^{T(A)}B,A}} & & \downarrow{\scriptstyle \mu^{-1}_{A,B}}\\
T({}^{T(A)}B)\otimes T(A) & \xrightarrow{\nu_{T(A),B}} & T(A)\otimes T(B).
\end{array}$$

Now, a morphism of categorical $\mathbf{\Gamma}$*-crossed modules*

$$(\mathbf{F}, \eta, \alpha) : (\mathbf{G}, \mathbf{T}, \nu, \chi) \to (\mathbf{G}', \mathbf{T}', \nu', \chi')$$

is a morphism between the underlying categorical $\mathbf{\Gamma}$*-precrossed modules such that, for any* A, B *in* $Ob(\mathbf{G})$*, the following diagram (expressing a compatibility condition between the natural isomorphisms* χ *and* χ'*) commutates:*

$$\begin{array}{ccc}
F({}^{T(A)}B \otimes A) & \xrightarrow{\quad F(\chi_{A,B}) \quad} & F(A \otimes B) \\
{\scriptstyle can}\downarrow & & \downarrow{\scriptstyle can} \\
F({}^{T(A)}B) \otimes F(A) & & F(A) \otimes F(B) \\
{\scriptstyle \eta_{T(A),B} \otimes id_{F(A)}}\downarrow & & \uparrow{\scriptstyle \chi'_{F(A),F(B)}} \\
{}^{T(A)}F(B) \otimes F(A) & \xrightarrow[\alpha_A F(B) \otimes id_{F(A)}]{} & {}^{T'F(A)}F(B) \otimes F(A).
\end{array}$$

A categorical $\mathbf{\Gamma}$-crossed module $(\mathbf{G}, \mathbf{T}, \nu, \chi)$ is called:

- *semistrict* if $\mathbf{G}$ and $\mathbf{\Gamma}$ are strict categorical groups, the action of $\mathbf{\Gamma}$ on $\mathbf{G}$ is strict and T is strictly equivariant (i.e., ν is an identity);
- *special semistrict* if $\mathbf{G}$ is a strict categorical group and $\mathbf{\Gamma}$ is a discrete categorical group acting strictly on $\mathbf{G}$;
- *strict* if it is semistrict and χ is an identity.

Examples. (a) Any crossed module of groups $\partial : G_1 \to G_0$ is a categorical crossed module when both G_1 and G_0 are seen as discrete categorical groups. This is a trivial example of strict categorical crossed module.

(b) Let $(G_1 \xrightarrow{\partial} G_0 \xrightarrow{p_0} \Gamma_0, \{,\})$ be a 2-crossed module. Then, following [13], it has an associated categorical $\Gamma_0[0]$-crossed module $(\mathbf{G}(\partial), \mathbf{T}, id, \chi)$ where $\mathbf{G}(\partial)$ is the strict categorical group associated with crossed module $\partial : G_1 \to G_0$, the morphism of categorical groups $\mathbf{T} = (T, \mu)$ with T defined as p_0 on the objects and as trivial map on the arrows of $\mathbf{G}(\partial)$ and μ = identity. ν is the identity and $\chi_{g_1,g_2} = (\{g_1, g_2\}, {}^{p_0(g_1)}g_2\, g_1)$, for all $g_1, g_2 \in G_0$, where $\{-,-\}$ is the Peiffer lifting. So 2-crossed modules are examples of special semistrict categorical crossed modules.

(c) In [12], a categorical $\mathbf{\Gamma}$-module is defined as a braided categorical group $(\mathbf{G}, c)$ provided with a $\mathbf{\Gamma}$-action such that

$$c_{{}^X\!A,\, {}^X\!B}\, \psi_{X,A,B} = \psi_{X,B,A}\, {}^X c_{A,B}$$

for any $X \in \mathrm{Ob}(\mathbf{G})$ and $A,\ B \in \mathrm{Ob}(\mathbf{\Gamma})$. If $\mathbf{G}$ is a categorical $\mathbf{\Gamma}$-module, the trivial morphism $\mathbf{1} : \mathbf{G} \to \mathbf{\Gamma}$ is a categorical $\mathbf{\Gamma}$-crossed module where, for any $A,\ B \in \mathrm{Ob}(\mathbf{G})$, $\chi_{A,B} : {}^I B \otimes A \to A \otimes B$ is given by the braiding $c_{B,A}$, up to composition with the obvious canonical isomorphism.

This example contains the following two special cases.

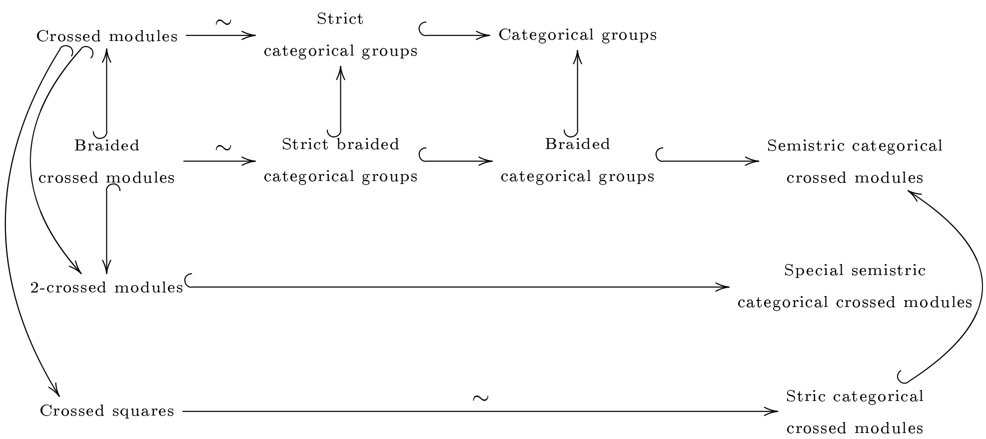
1. Let $\partial : G_1 \to G_0$ be a braided crossed module equipped with an action by a crossed module $\partial' : \Gamma_1 \to \Gamma_0$. The braiding are assumed to be equivariant respect the action, that is ${}^{\sigma}\{g_1, g_2\} = \{{}^{\sigma}g_1, {}^{\sigma}g_2\}$, for all σ in Γ_0 and g_1, g_2 in G_0. Then $(\mathbf{G}(\partial), c)$ is a categorical $\mathbf{G}(\partial')$-module where $c_{g_2,g_1} = (\{g_1, g_2\}, g_2\, g_1)$, for all $g_1, g_2 \in G_0$. So we have an example of semistrict categorical crossed module.

2. Let $\partial : G_1 \to G_0$ be a Γ_0-equivariant braided crossed module (see 2.1 for the definition). Then $(\mathbf{G}(\partial), c)$ is a categorical $\Gamma_0[0]$-module where $c_{g_2,g_1} = (\{g_1, g_2\}, g_2\, g_1)$, for all g_1, $g_2 \in G_0$. So we have another example of special semistrict categorical crossed module.

Notice that this last case is a special case both of (b) and (c)1..

(d) [15] Crossed squares correspond, up to isomorphisms, to strict categorical crossed modules.

Finally, we want to summarize with the following diagram the inclusions and the equivalences between the categories presented.



Chapter 4

Crossed squares

First of all, let us recall a few well-known facts about crossed modules. Let $\partial : G_1 \to G_0$ be a crossed module, then:

(a) $\ker \partial$ is G_0-invariant;

(b) $\mathrm{Im}\, \partial$ is normal in G_0;

(c) there is an action of $\mathrm{coker}\, \partial$ on the abelian group $\ker \partial$ such that

$$\ker \partial \hookrightarrow G_1 \xrightarrow{\partial} G_0 \twoheadrightarrow \mathrm{coker}\, \partial \tag{4.1}$$

is a crossed module.

We are going to show that these properties hold, in a 2-dimensional form, provided we change the notions of kernels and cokernels by the homotopical versions. Using the representation of strict categorical crossed modules given by crossed squares, we show that, starting from a strict categorical crossed module $\mathbf{T} : \mathbf{G} \to \mathbf{\Gamma}$, we obtain a 2-dimensional version of (4.1):

$$\ker \mathbf{T} \longrightarrow \mathbf{G} \xrightarrow{\mathbf{T}} \mathbf{\Gamma} \longrightarrow \frac{\mathbf{\Gamma}}{< \mathbf{G}, \mathbf{T} >}$$

for suitable categorical groups $\ker \mathbf{T}$ and $\dfrac{\mathbf{\Gamma}}{< \mathbf{G}, \mathbf{T} >}$ (introduced in [33], [48] and [14]), where $\ker \mathbf{T} \longrightarrow \dfrac{\mathbf{\Gamma}}{< \mathbf{G}, \mathbf{T} >}$ is shown to be a strict categorical crossed module.

4.1 Crossed square version of homotopy kernels

In literature, there are two versions of the kernel of a morphism of crossed module

$$\begin{array}{ccc} G_1 & \xrightarrow{\varphi} & G_1' \\ \partial \downarrow & & \downarrow \partial' \\ G_0 & \xrightarrow{\psi} & G_0'. \end{array}$$

The strict version is introduced by Norrie in [42]. In this approach, she considers crossed modules as the objects of a category $\mathcal{CM}$ and the kernel of the morphism $< \varphi, \psi >$ is $\partial_{|\ker_\varphi} : \ker\varphi \to \ker\psi$.

The homotopical version is analyzed in [14] where the authors consider crossed modules as the objects of a 2-category (thanks the equivalence between strict categorical groups and crossed modules). The kernel is given by the homotopy fibre over the unit object of the morphism of categorical groups $\mathbf{G}(\partial) \to \mathbf{G}(\partial')$. In this case the objects of the kernel are the elements of the pullback $G_0 \times_{G_0'} G_1'$ (see 6.7. REMARK in [14]).

In this section we use this last version.

Let us consider the crossed square

$$\begin{array}{ccc} G_1 & \xrightarrow{\bar{p}_1} & \Gamma_1 \\ \partial \downarrow & & \downarrow \partial' \\ G_0 & \xrightarrow{\bar{p}_0} & \Gamma_0. \end{array} \tag{4.2}$$

If we call $\mathbf{G}$ the strict categorical group associated with $\partial : G_1 \to G_0$ and $\mathbf{\Gamma}$ the strict categorical group associated with $\partial' : \Gamma_1 \to \Gamma_0$, then there is a strict categorical crossed module $\mathbf{T} : \mathbf{G} \to \mathbf{\Gamma}$ (see the example (d) in 3.4). The kernel $\ker \mathbf{T}$ of $\mathbf{T} : \mathbf{G} \to \mathbf{\Gamma}$ is a strict categorical group with $\mathrm{Ob}(\ker \mathbf{T}) = G_0 \times_{\Gamma_0} \Gamma_1$.

An arrow in $\ker \mathbf{T}$ from (g_1, β_1) to (g_2, β_2) is an arrow $g_1 \xrightarrow{(\alpha, g_2)} g_2$, with $g_1 = \partial(\alpha)\, g_2$, such that the triangle

$$\begin{array}{ccc} \bar{p}_0(g_1) & \xrightarrow{(\bar{p}_1(\alpha), \bar{p}_0(g_2))} & \bar{p}_0(g_2) \\ & {\scriptstyle (\beta_1, 1)} \searrow \quad \swarrow {\scriptstyle (\beta_2, 1)} & \\ & 1 & \end{array}$$

commutes, that is $\bar{p}_1(\alpha)\, \beta_2 = \beta_1$. Therefore, an arrow in the categorical group $\ker \mathbf{T}$ is uniquely specified by triple (α, g_2, β_2) with $(g_2, \beta_2) \in G_0 \times_{\Gamma_0} \Gamma_1$ and an element $\alpha \in G_1$. The target of (α, g_2, β_2) is given by (g_2, β_2); the source of (α, g_2, β_2) is given by (g_1, β_1) where $g_1 = \partial(\alpha)\, g_2$ and $\beta_1 = \bar{p}_1(\alpha)\, \beta_2$.

It is easy to check that the tensor product on objects is given by the direct product.

Let $(\overline{g}_1, \overline{\beta}_1) \xrightarrow{(\alpha_1, g_1, \beta_1)} (g_1, \beta_1)$ and $(\overline{g}_2, \overline{\beta}_2) \xrightarrow{(\alpha_2, g_2, \beta_2)} (g_2, \beta_2)$ be two arrows in $\ker \mathbf{T}$, where $(\overline{g}_i, \overline{\beta}_i)$ are determined by (α_i, g_i, β_i) for $i = 1, 2$, the tensor product of these two arrows is given by:

$$(\alpha_1, g_1, \beta_1)\,(\alpha_2, g_2, \beta_2) = (\alpha_1\, {}^{g_1}\alpha_2, g_1\, g_2, \beta_1\, \beta_2).$$

Because $\ker \mathbf{T}$ is a strict categorical group, under the equivalence between strict categorical groups and crossed modules, it is equivalent to the crossed module constructed as follows:

$$\overline{\partial} : \mathrm{Ker}t \;\; \rightarrow \;\; G_0 \times_{\Gamma_0} \Gamma_1$$

with $\overline{\partial} = s_{|\mathrm{Ker}t}$, where s and t are the source and target maps, respectively, of the underlying groupoid $\ker \mathbf{T}$. We denote with $\ker \mathbf{T}_1$ the set of arrows in $\ker \mathbf{T}$ and we recall the target map:

$$\begin{array}{rcl} t : \ker \mathbf{T}_1 & \longrightarrow & G_0 \times_{\Gamma_0} \Gamma_1 \\ (\alpha_1, g_1, \theta_1) & \longrightarrow & (g_1, \theta_1) \end{array}$$

while the source map is given by:

$$\begin{array}{rcl} s : \ker \mathbf{T}_1 & \longrightarrow & G_0 \times_{\Gamma_0} \Gamma_1 \\ (\alpha_1, g_1, \theta_1) & \longrightarrow & (g_2, \theta_2) \end{array}$$

where (g_2, θ_2) are given by $g_2 = \partial(\alpha_1)\, g_1$ and $\beta_2 = \bar{p}_1(\alpha_1)\, \beta_1$.
Thus we have

$$\begin{array}{rcl} \overline{\partial} : \mathrm{Ker}t & \rightarrow & G_0 \times_{\Gamma_0} \Gamma_1 \\ (\alpha, 1, 1) & \rightarrow & (\partial(\alpha), \bar{p}_1(\alpha)). \end{array}$$

The product of two arrows $(\alpha_1, 1, 1)$ and $(\alpha_2, 1, 1)$ in $\ker \mathbf{T}$ is $(\alpha_1\, \alpha_2, 1, 1)$ and the product in $G_0 \times_{\Gamma_0} \Gamma_1$ is the direct product, checked above. The action of the group $G_0 \times_{\Gamma_0} \Gamma_1$ on $\mathrm{Ker}t$ is given by:

$${}^{(g,\beta)}(\alpha, 1, 1) = i(g, \beta)(\alpha, 1, 1)(i(g, \beta))^{-1}.$$

We recall that the map i for the groupoid $\ker \mathbf{T}$ is given by:

$$\begin{array}{rcl} i : G_0 \times_{\Gamma_0} \Gamma_1 & \longrightarrow & \ker \mathbf{T}_1 \\ (g, \beta) & \longrightarrow & (1, g, \beta). \end{array}$$

Therefore, using the multiplication defined above on $\ker \mathbf{T}_1$, we have:

$$\begin{array}{rcl} {}^{(g,\beta)}(\alpha, 1, 1) & = & (1, g, \beta)\,(\alpha, 1, 1)\,(1, g, \beta)^{-1} = (1, g, \beta)\,(\alpha, 1, 1) \\ & & (1, g^{-1}, \beta^{-1}) = \\ & = & ({}^{g}\alpha, g, \beta)\,(1, g^{-1}, \beta^{-1}) = ({}^{g}\alpha, 1, 1). \end{array}$$

Because Ker$t$ is isomorphic to G_1, it is clear the isomorphism between $\overline{\partial}$ and a homomorphism

$$\overline{\partial} : G_1 \quad \to \quad G_0 \times_{\Gamma_0} \Gamma_1$$

which, by abuse of notation, we have denoted again by $\overline{\partial}$.

In the category of categorical groups $\mathcal{CG}$ we have a morphism $e_{\mathbf{T}}$: Ker$\mathbf{T} \to \mathbf{G}$ defined on objects and on arrows

$$\begin{array}{rcl} e_{\mathbf{T}0} : G_0 \times_{\Gamma_0} \Gamma_1 & \longrightarrow & G_0 \\ (g,\beta) & \longrightarrow & g \end{array} \quad \begin{array}{rcl} e_{\mathbf{T}1} : \ker \mathbf{T}_1 & \longrightarrow & G_1 \rtimes G_0 \\ (\alpha, g, \beta) & \longrightarrow & (\alpha, g) \end{array}$$

and it is a categorical $\mathbf{G}$-crossed module.

If we interpret these facts in the context of crossed modules, we can prove the following Proposition.

Proposition 4.1.1. *The diagram*

$$\begin{array}{ccc} G_1 & = & G_1 \\ {\scriptstyle \overline{\partial}}\downarrow & & \downarrow{\scriptstyle \partial} \\ G_0 \times_{\Gamma_0} \Gamma_1 & \xrightarrow{p_{G_0}} & G_0 \end{array} \tag{4.3}$$

gives rise to a crossed square (that is a crossed module of crossed modules) with actions, group homomorphism p_{G_0} and function $\widehat{h} : G_1 \times (G_0 \times_{\Gamma_0} \Gamma_1) \to G_1$ defined as following:

- *the action of G_0 on G_1 is the action of the crossed module $\partial : G_1 \to G_0$;*
- *the action of G_0 on $G_0 \times_{\Gamma_0} \Gamma_1$ is defined by ${}^g(g_2, \beta_2) = (g\, g_2\, g^{-1},\, {}^g\beta_2)$;*
- *$p_{G_0} : G_0 \times_{\Gamma_0} \Gamma_1 \to G_0$ is the canonical projection on G_0.*
- *$\widehat{h}(\alpha, (g_2, \beta_2)) := \alpha\, {}^{g_2}\alpha^{-1}$ (notice that $\widehat{h}(\alpha, (g_2, \beta_2)) = h(\bar{p}_1(\alpha), g_2)$ where the function h is given by the crossed square structure of (4.2));*

Proof. The action of G_0 on $G_0 \times_{\Gamma_0} \Gamma_1$ is well defined. We now want to check the five properties making this diagram a crossed square (see definition 2.2.1).

(i) The map $\mathrm{id}_{G_1} : G_1 \to G_1$ obviously preserves the actions of G_0.

The map $\overline{\partial}$ preserves the actions of G_0:

$$\begin{aligned} \overline{\partial}({}^g\alpha) &= (\partial({}^g\alpha), \bar{p}_1({}^g\alpha)) = (g\,\partial(\alpha)\,g^{-1}, \bar{p}_1({}^{\bar{p}_0(g)}\alpha)) = \\ &= (g\,\partial(\alpha)\,g^{-1},\, {}^{\bar{p}_0(g)}\bar{p}_1(\alpha)) = (g\,\partial(\alpha)\,g^{-1},\, {}^g\bar{p}_1(\alpha)) = \\ &= {}^g(\partial(\alpha), \bar{p}_1(\alpha)) = {}^g\overline{\partial}(\alpha). \end{aligned}$$

∂ is a crossed module because (4.2) is a crossed square and we want to prove that p_{G_0} is a crossed module. In fact, we have:

$$\begin{aligned}
p_{G_0}({}^{g}(g_2,\beta_2)) &= p_{G_0}(g\,g_2\,g^{-1},\,{}^{g}\beta_2) = g\,g_2\,g^{-1} = \\
&= g\,p_{G_0}(g_2,\beta_2)\,g^{-1}; \\
{}^{p_{G_0}(g_2,\beta_2)}({g_2}',{\beta_2}') &= {}^{g_2}(g_2,\beta_2) = (g_2\,{g_2}'\,g_2^{-1},\,{}^{g_2}{\beta_2}') = \\
&= (g_2\,{g_2}'\,g_2^{-1},\,{}^{\bar{p}_0(g_2)}{\beta_2}') = \\
&= (g_2\,{g_2}'\,g_2^{-1},\,{}^{\partial'(\beta_2)}{\beta_2}') = \\
&= (g_2\,{g_2}'\,g_2^{-1},\beta_2\,{\beta_2}'\beta_2^{-1}) = \\
&= (g_2,\beta_2)\,({g_2}',{\beta_2}')\,(g_2,\beta_2)^{-1}.
\end{aligned}$$

$p_{G_0}\,\overline{\partial} = \partial\,\mathrm{id}_{G_1}$ is a crossed module because $\partial : G_1 \to G_0$ is a crossed module.

(ii) $\mathrm{id}_{G_1}(\widehat{h}(\alpha,(g_2,\beta_2))) = \alpha\,{}^{g_2}\alpha^{-1} = \alpha\,{}^{(g_2,\beta_2)}\alpha^{-1}$.
Now we want to prove that $\overline{\partial}\,\widehat{h}(\alpha,(g_2,\beta_2)) = {}^{\alpha}(g_2,\beta_2)\,(g_2,\beta_2)^{-1}$ and we develop the two members separately:

$$\begin{aligned}
\overline{\partial}\,\widehat{h}(\alpha,(g_2,\beta_2)) &= (\partial(\alpha\,{}^{g_2}\alpha^{-1}),\bar{p}_1(\alpha\,{}^{g_2}\alpha^{-1})) = \\
&= (\partial(\alpha)\,g_2\,\partial(\alpha)^{-1}\,g_2^{-1},\bar{p}_1(\alpha)\;{}^{g_2}\bar{p}_1(\alpha)^{-1}) = \\
&= (\partial(\alpha)\,g_2\,\partial(\alpha)^{-1}\,g_2^{-1},\bar{p}_1(\alpha)\;{}^{\bar{p}_0(g_2)}\bar{p}_1(\alpha)^{-1}) = \\
&= (\partial(\alpha)\,g_2\,\partial(\alpha)^{-1}\,g_2^{-1},\bar{p}_1(\alpha)\;{}^{\partial'(\beta_2)}\bar{p}_1(\alpha)^{-1}) \\
&= (\partial(\alpha)\,g_2\,\partial(\alpha)^{-1}\,g_2^{-1},\bar{p}_1(\alpha)\,\beta_2\,\bar{p}_1(\alpha)^{-1}\,\beta_2^{-1}); \\
{}^{\alpha}(g_2,\beta_2)\,(g_2,\beta_2)^{-1} &= {}^{\partial(\alpha)}(g_2,\beta_2)\,(g_2,\beta_2)^{-1} = \\
&= (\partial(\alpha)\,g_2\,\partial(\alpha)^{-1},\,{}^{\partial(\alpha)}\beta_2)\,(g_2^{-1},\beta_2^{-1}) = \\
&= (\partial(\alpha)\,g_2\,\partial(\alpha)^{-1}\,g_2^{-1},\,{}^{\bar{p}_0(\partial(\alpha))}\beta_2\,\beta_2^{-1}) = \\
&= (\partial(\alpha)\,g_2\,\partial(\alpha)^{-1}\,g_2^{-1},\,{}^{\partial'(\bar{p}_1(\alpha))}\beta_2\,\beta_2^{-1}) = \\
&= (\partial(\alpha)\,g_2\,\partial(\alpha)^{-1}\,g_2^{-1},\bar{p}_1(\alpha)\,\beta_2\,\bar{p}_1(\alpha)^{-1}\,\beta_2^{-1}).
\end{aligned}$$

In the first development, the next to last passage is given by the fact that (g_2,β_2) belongs to the pullback $G_0 \times_{\Gamma_0} \Gamma_1$.

(iii) $\widehat{h}(\mathrm{id}_{G_1}(\alpha),(g_2,\beta_2)) = \widehat{h}(\alpha,(g_2,\beta_2)) = \alpha\,{}^{g_2}\alpha^{-1} = \alpha\,{}^{(g_2,\beta_2)}\alpha^{-1}$;
$\widehat{h}(\alpha,\overline{\partial}(\alpha')) = \widehat{h}(\alpha,(\partial(\alpha'),\bar{p}_1(\alpha'))) = \alpha\,{}^{\partial(\alpha')}\alpha^{-1} = \alpha\alpha'\alpha^{-1}\alpha'^{-1} =$
$= {}^{\partial(\alpha)}\alpha'\alpha'^{-1} = {}^{\alpha}\alpha'\alpha'^{-1}$.

(iv)

$$\begin{aligned}
\widehat{h}(\alpha\,\alpha',(g_2,\beta_2)) &= \alpha\,\alpha'\,{}^{g_2}(\alpha\,\alpha')^{-1} = \alpha\,\alpha'\,{}^{g_2}\,\alpha'^{-1}\,{}^{g_2}\alpha^{-1} = \\
&= \alpha\,\alpha'\,{}^{g_2}\,\alpha'^{-1}\,\alpha^{-1}\,\alpha\,{}^{g_2}\alpha^{-1} = \\
&= \alpha\,\widehat{h}(\alpha',(g_2,\beta_2))\,\alpha^{-1}\,\widehat{h}(\alpha,(g_2,\beta_2)) =
\end{aligned}$$

$$
\begin{aligned}
&= {}^{\alpha}\widehat{h}(\alpha', (g_2, \beta_2))\, \widehat{h}(\alpha, (g_2, \beta_2));\\
\widehat{h}(\alpha, (g_2, \beta_2)\,(g_2{}', \beta_2{}')) &= \widehat{h}(\alpha, (g_2\, g_2{}', \beta_2\, \beta_2{}')) = \alpha\ {}^{g_2\, g_2{}'}\alpha^{-1} =\\
&= \alpha\, {}^{g_2}\alpha^{-1}\, {}^{g_2}\alpha\, {}^{g_2\, g_2{}'}\alpha^{-1} =\\
&= \alpha\, {}^{g_2}\alpha^{-1}\, {}^{g_2}(\alpha\, {}^{g_2{}'}\alpha^{-1}) =\\
&= \widehat{h}(\alpha, (g_2, \beta_2))\, {}^{g_2}\widehat{h}(\alpha, (g_2{}', \beta_2{}')) =\\
&= \widehat{h}(\alpha, (g_2, \beta_2))\, {}^{(g_2, \beta_2)}\widehat{h}(\alpha, (g_2{}', \beta_2{}')).
\end{aligned}
$$

(v)

$$
\begin{aligned}
\widehat{h}({}^{g}\alpha, {}^{g}(g_2, \beta_2)) &= \widehat{h}({}^{g}\alpha, (g\, g_2\, g^{-1}, {}^{g}\beta_2)) = {}^{g}\alpha\, {}^{g\, g_2\, g^{-1}}({}^{g}\alpha^{-1}) =\\
&= {}^{g}\alpha\, {}^{g\, g_2}\alpha^{-1} = {}^{g}(\alpha\, {}^{g_2}\alpha^{-1}) = {}^{g}\widehat{h}(\alpha, (g_2, \beta_2)).
\end{aligned}
$$

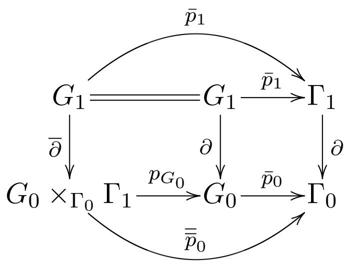
□

Remark 4.1.1. *If* $<\bar{p}_1, \bar{p}_0>$ *is just a morphism of crossed modules then (4.3) is still a crossed square. This is a generalization of the well-known fact in the category of groups that if* $\partial : G_1 \to G_0$ *is a morphism of groups then* $ker\partial \hookrightarrow G_1$ *is a crossed module (of groups).*

4.2 ker T as a strict categorical Γ-crossed module

It is well-known that given a crossed module (of groups) $\partial : G_1 \to G_0$ then $\ker\partial \hookrightarrow G_1 \to G_0$ is a crossed module (of groups). In the context of crossed squares, we prove the following Proposition.

Proposition 4.2.1. *The outer diagram*



(4.4)

gives rise to a crossed square with actions and function $\overline{h} : \Gamma_1 \times (G_0 \times_{\Gamma_0} \Gamma_1) \to G_1$ *defined as following:*

- *the action of* Γ_0 *on* G_1 *is induced by the action of* $\partial' : \Gamma_1 \to \Gamma_0$ *on* $\partial : G_1 \to G_0$ *;*
- *the action of* Γ_0 *on* Γ_1 *is the action of the crossed module* $\partial' : \Gamma_1 \to \Gamma_0$*;*
- *the action of* Γ_0 *on* $G_0 \times_{\Gamma_0} \Gamma_1$ *is defined by* ${}^{\sigma}(g_2, \beta_2) = ({}^{\sigma}g_2, {}^{\sigma}\beta_2)$*;*

- $\overline{h}(\beta, (g_2, \beta_2)) := h(\beta, g_2)$ *where the function* h *is given by the crossed square structure of (4.2);*

Proof. The action of Γ_0 on $G_0 \times_{\Gamma_0} \Gamma_1$ is well defined. $\overline{\bar{p}}_0$ is a group homomorphism because $\bar{p}_0$ is and the diagram (4.4) commutes. Now we want to check the five properties making this diagram a crossed square.

(i) The map $\bar{p}_1$ preserves the actions of Γ_0 because (4.2) is a crossed square.
The map $\overline{\partial}$ preserves the actions of Γ_0:
$\overline{\partial}({}^{\sigma}\alpha) = (\partial({}^{\sigma}\alpha), \bar{p}_1({}^{\sigma}\alpha)) = ({}^{\sigma}\partial(\alpha), {}^{\sigma}\bar{p}_1(\alpha)) = {}^{\sigma}(\partial(\alpha), \bar{p}_1(\alpha)) =$
$= {}^{\sigma}\overline{\partial}(\alpha)$.
∂' is a crossed module because (4.2) is a crossed square and we want to prove that $\overline{\bar{p}}_0$ is a crossed module. The pre-crossed module property holds because $\bar{p}_0$ satisfies the pre-crossed module property. It also holds the Peiffer condition:

$$\begin{aligned}
{}^{\overline{\bar{p}}_0(g_2,\beta_2)}(g_2{}', \beta_2{}') &= {}^{\bar{p}_0(g_2)}(g_2{}', \beta_2{}') = \\
&= ({}^{\bar{p}_0(g_2)}g_2{}', {}^{\bar{p}_0(g_2)}\beta_2{}') = \\
&= (g_2\, g_2{}'\, g_2^{-1}, {}^{\partial'(\beta_2)}\beta_2{}') = \\
&= (g_2\, g_2{}'\, g_2^{-1}, \beta_2\, \beta_2{}'\beta_2^{-1}); \\
(g_2, \beta_2)\,(g_2{}', \beta_2{}')\,(g_2, \beta_2)^{-1} &= (g_2, \beta_2)\,(g_2{}', \beta_2{}')\,(g_2^{-1}, \beta_2^{-1}) = \\
&= (g_2\, g_2{}'\, g_2^{-1}, \beta_2\, \beta_2{}'\beta_2^{-1}).
\end{aligned}$$

$\overline{\bar{p}}_0\, \overline{\partial} = \partial'\, \bar{p}_1$ is a crossed module because (4.2) is a crossed square.

(ii) $\bar{p}_1(\overline{h}(\beta, (g_2, \beta_2))) = \bar{p}_1(h(\beta, g_2)) = \beta\, {}^{g_2}\beta^{-1} = \beta\, {}^{\bar{p}_0(g_2)}\beta^{-1} =$
$= \beta\, {}^{\overline{\bar{p}}_0(g_2,\beta_2)}\beta^{-1} = \beta\, {}^{(g_2,\beta_2)}\beta^{-1}$.
Now we want to show that $\overline{\partial}\, \overline{h}(\beta, (g_2, \beta_2)) = {}^{\beta}(g_2, \beta_2)\,(g_2, \beta_2)^{-1}$. We develop the two members separately:

$$\begin{aligned}
\overline{\partial}\, \overline{h}(\beta, (g_2, \beta_2)) &= (\partial\, \overline{h}(\beta, (g_2, \beta_2)), \bar{p}_1\, \overline{h}(\beta, (g_2, \beta_2))) = \\
&= (\partial\, h(\beta, g_2), \bar{p}_1\, h(\beta, g_2)) = ({}^{\beta}g_2\, g_2^{-1}, \beta\, {}^{g_2}\beta^{-1}); \\
{}^{\beta}(g_2, \beta_2)\,(g_2, \beta_2)^{-1} &= {}^{\partial'(\beta)}(g_2, \beta_2)\,(g_2, \beta_2)^{-1} = \\
&= ({}^{\partial'(\beta)}g_2, {}^{\partial'(\beta)}\beta_2)\,(g_2^{-1}, \beta_2^{-1}) = \\
&= ({}^{\partial'(\beta)}g_2\, g_2^{-1}, {}^{\partial'(\beta)}\beta_2\, \beta_2^{-1}) = \\
&= ({}^{\beta}g_2\, g_2^{-1}, \beta\, {}^{\partial'(\beta_2)}\, \beta^{-1}) = \\
&= ({}^{\beta}g_2\, g_2^{-1}, \beta\, {}^{\bar{p}_0(g_2)}\, \beta^{-1}) = ({}^{\beta}g_2\, g_2^{-1}, \beta\, {}^{g_2}\, \beta^{-1}).
\end{aligned}$$

In the development of the first member, the last passage is allowed since (4.2) is a crossed square. In the second, the next to last passage is given by the fact that (g_2, β_2) belongs to the pullback $G_0 \times_{\Gamma_0} \Gamma_1$.

(iii) $\overline{h}(\bar{p}_1(\alpha), (g_2, \beta_2)) = h(\bar{p}_1(\alpha), g_2) = \alpha\, {}^{g_2}\alpha^{-1} = \alpha\, {}^{(g_2,\beta_2)}\alpha^{-1}$;
$\overline{h}(\beta, \overline{\partial}(\alpha)) = \overline{h}(\beta, (\partial(\alpha), \bar{p}_1(\alpha))) = h(\beta, \partial(\alpha)) = {}^{\beta}\alpha\, \alpha^{-1}$.

(iv)

$$\begin{aligned}\overline{h}(\beta\,\beta', (g_2,\beta_2)) &= h(\beta\,\beta', g_2) = {}^{\beta}h(\beta', g_2)\, h(\beta, g_2) = \\ &= {}^{\beta}\overline{h}(\beta', (g_2,\beta_2))\, \overline{h}(\beta, (g_2,\beta_2)); \\ \overline{h}(\beta, (g_2,\beta_2)\,(g_2{}',\beta_2{}')) &= \overline{h}(\beta, (g_2\, g_2{}', \beta_2\,\beta_2{}')) = h(\beta, g_2\, g_2{}') = \\ &= h(\beta, g_2)\, {}^{g_2}h(\beta, g_2{}') = \\ &= \overline{h}(\beta, (g_2,\beta_2))\ {}^{(g_2,\beta_2)}\overline{h}(\beta, (g_2{}',\beta_2{}')).\end{aligned}$$

(v)

$$\begin{aligned}\overline{h}({}^{\sigma}\beta, {}^{\sigma}(g_2,\beta_2)) &= \overline{h}({}^{\sigma}\beta, ({}^{\sigma}g_2, {}^{\sigma}\beta_2)) = h({}^{\sigma}\beta, {}^{\sigma}g_2) = {}^{\sigma}h(\beta, g_2) = \\ &= {}^{\sigma}\overline{h}(\beta, (g_2,\beta_2)).\end{aligned}$$

□

Remark 4.2.1. *In the category of groups, it is obvious that the following composition*

$$ker\,\partial \hookrightarrow G_1 \xrightarrow{\partial} G_0$$

is the trivial homomorphism.

If we interpret this fact in the context of crossed modules, we can prove that the morphism of crossed modules (4.4) is homotopy equivalent to the trivial morphism. In fact, there exists a transformation between them given by a function $\theta : G_0 \times_{\Gamma_0} \Gamma_1 \to \Gamma_1$*, defined by* $\theta(g,\beta) = \beta^{-1}$*.*

4.3 Crossed square version of homotopy cokernels

We can consider the quotient categorical group $\dfrac{\mathbf{\Gamma}}{<\mathbf{G},\mathbf{T}>}$ as defined in [14]. We have $\mathrm{Ob}(\dfrac{\mathbf{\Gamma}}{<\mathbf{G},\mathbf{T}>}) = \Gamma_0$ and the tensor product on objects in $\dfrac{\mathbf{\Gamma}}{<\mathbf{G},\mathbf{T}>}$ is the same as the product in Γ_0. Then $\dfrac{\mathbf{\Gamma}}{<\mathbf{G},\mathbf{T}>}$ is a strict categorical group because Γ_0 is a group.

We are going to describe morphisms in this category specifying the general construction given in [14].

Definition 4.3.1. *A premorphism in* $\dfrac{\mathbf{\Gamma}}{<\mathbf{G},\mathbf{T}>}$ *is uniquely specified by* (g,β,σ_2) *with* $(\beta, \bar{p}_0(g)\,\sigma_2) \in \Gamma_1 \rtimes \Gamma_0$ *(the set of arrows of* $\mathbf{\Gamma}$*),* $g \in G_0$*. The target of* (g,β,σ_2) *is* $\bar{p}_0(g)\,\sigma_2$ *and the source of* (g,β,σ_2) *is given by* σ_1 *where*

$$\sigma_1 = \partial'(\beta)\,\bar{p}_0(g)\,\sigma_2. \tag{4.5}$$

Definition 4.3.2. *A morphism in* $\dfrac{\mathbf{\Gamma}}{<\mathbf{G},\mathbf{T}>}$ *from* σ_1 *to* σ_2 *is a class of premorphisms* $[g,\beta,\sigma_2]$ *where* (g,β,σ_2) *and* $(g\,',\beta\,',\sigma_2)$ *are equivalent if there is an arrow in* $\mathbf{G}$ *from* g *to* $g\,'$*, that is an* $\alpha\in G_1$ *such that* $g=\partial(\alpha)\,g\,'$ *and the diagram*

$$\begin{array}{ccc} \sigma_1 & \xrightarrow{(\beta,\bar{p}_0(g)\,\sigma_2)} & \bar{p}_0(g)\,\sigma_2 \\ & {\scriptstyle (\beta\,',\bar{p}_0(g\,')\,\sigma_2)}\searrow \qquad \swarrow{\scriptstyle (\bar{p}_1(\alpha),\bar{p}_0(g\,')\,\sigma_2)} & \\ & \bar{p}_0(g\,')\,\sigma_2 & \end{array}$$

commutes in $\mathbf{G}$*. Therefore*

$$\beta\,\bar{p}_1(\alpha)=\beta\,'.$$

Given two morphisms $\sigma_1\xrightarrow{[g,\beta,\sigma_2]}\sigma_2\xrightarrow{[g\,',\beta\,',\sigma_3]}\sigma_3$, we define their composition by

$$\sigma_1\xrightarrow{[g\,g\,',\beta\,{}^{g}\beta\,',\sigma_3]}\sigma_3\ .$$

Given two morphisms $\sigma_1\xrightarrow{[g_1,\beta_1,\sigma_2]}\sigma_2$ and $\sigma_1\,'\xrightarrow{[g_2,\beta_2,\sigma_2\,']}\sigma_2\,'$, their tensor product is given by

$$[g\,{}^{\sigma_2}g_2,\overline{\beta},\sigma_2\,\sigma_2\,'].$$

$\overline{\beta}$ is given by the composition of the following three morphisms:

$$\begin{array}{c} \partial\,'(\beta_1)\,\bar{p}_0(g_1)\,\sigma_2\,\partial\,'(\beta_2)\,\bar{p}_0(g_2)\,\sigma_2\,' \\ \big\downarrow{\scriptstyle (\beta_1\,{}^{\bar{p}_0(g_1)\,\sigma_2}\beta_2,\bar{p}_0(g_1)\,\sigma_2\,\bar{p}_0(g_2)\,\sigma_2\,')} \\ \bar{p}_0(g_1)\,\sigma_2\,\bar{p}_0(g_2)\,\sigma_2\,' \\ \Big\| \\ \bar{p}_0(g_1)\,\bar{p}_0({}^{\sigma_2}g_2)\,\sigma_2\,\sigma_2\,' \\ \Big\| \\ \bar{p}_0(g_1\,{}^{\sigma_2}g_2)\,\sigma_2\,\sigma_2\,' \end{array}$$

hence

$$\overline{\beta}=\beta_1\,{}^{\bar{p}_0(g_1)\,\sigma_2}\beta_2.$$

Because $\dfrac{\mathbf{\Gamma}}{<\mathbf{G},\mathbf{T}>}$ is a strict categorical group it is equivalent to the crossed module constructed as follows:

$$d:\mathrm{Ker}t\ \ \to\ \ \mathrm{Ob}(\frac{\mathbf{\Gamma}}{<\mathbf{G},\mathbf{T}>})=\Gamma_0$$

with $d = s_{|\mathrm{Ker}t}$, where s and t are the source and target maps, respectively, of the underlying groupoid $\dfrac{\mathbf{\Gamma}}{<\mathbf{G},\mathbf{T}>}$. We denote with $\dfrac{\mathbf{\Gamma}}{<\mathbf{G},\mathbf{T}>_1}$ the set of arrows in $\dfrac{\mathbf{\Gamma}}{<\mathbf{G},\mathbf{T}>}$ and we consider the target map:

$$\begin{array}{rcl} t : \dfrac{\mathbf{\Gamma}}{<\mathbf{G},\mathbf{T}>_1} & \longrightarrow & \Gamma_0 \\ (g, \beta, \sigma_2) & \longrightarrow & \sigma_2 \end{array}$$

while the source map:

$$\begin{array}{rcl} s : \dfrac{\mathbf{\Gamma}}{<\mathbf{G},\mathbf{T}>_1} & \longrightarrow & \Gamma_0 \\ (g, \beta, \sigma_2) & \longrightarrow & \partial'(\beta)\, \bar{p}_0(g)\, \sigma_2. \end{array}$$

Thus we have

$$\begin{array}{rcl} d : \mathrm{Ker}t & \to & \Gamma_0 \\ (g, \beta, 1) & \to & \partial'(\beta)\, \bar{p}_0(g). \end{array}$$

The product of two arrows $[g, \beta, 1]$ and $[g', \beta', 1]$ in $\mathrm{Ker}t$ is

$$[g\, g', \beta\, {}^{\bar{p}_0(g)}\beta', 1] = [g\, g', \beta\, {}^{g}\beta', 1].$$

The action of the group Γ_0 on $\mathrm{Ker}t$ is given by:

$${}^{\sigma}[g, \beta, 1] = i(\sigma)\, [g, \beta, 1]\, i(\sigma)^{-1}.$$

We recall that the map i for the groupoid $\dfrac{\mathbf{\Gamma}}{<\mathbf{G},\mathbf{T}>}$ is given by:

$$\begin{array}{rcl} i : \Gamma_0 & \longrightarrow & \dfrac{\mathbf{\Gamma}}{<\mathbf{G},\mathbf{T}>_1} \\ \sigma & \longrightarrow & (1, 1, \sigma) \end{array}$$

Therefore, using the multiplication defined above on $\dfrac{\mathbf{\Gamma}}{<\mathbf{G},\mathbf{T}>_1}$, we have:

$$\begin{array}{rcl} {}^{\sigma}[g, \beta, 1] & = & [1, 1, \sigma]\, [g, \beta, 1]\, [1, 1, \sigma]^{-1} = \\ & = & [1, 1, \sigma]\, [g, \beta, 1]\, [1, 1, \sigma^{-1}] = \\ & = & [{}^{\sigma}g,\, {}^{\sigma}\beta, \sigma]\, [1, 1, \sigma^{-1}] = \\ & = & [{}^{\sigma}g,\, {}^{\sigma}\beta, 1]. \end{array}$$

It is easy to observe that: $\mathrm{Ker}t$ is isomorphic to $\dfrac{G_0 \ltimes \Gamma_1}{\sim}$ where $(g_1, \beta_1) \sim (g_2, \beta_2)$ if there is an $\alpha \in G_1$ such that

$$g_1 = \partial(\alpha)\, g_2, \tag{4.6}$$

$$\beta_1 = \beta_2\, \bar{p}_1(\alpha)^{-1}. \tag{4.7}$$

We can also show that the group $\dfrac{G_0 \ltimes \Gamma_1}{\sim}$ is isomorphic to the generalized semi-direct product $G_0 \ltimes^{G_1} \Gamma_1$ of G_0 and Γ_1 along G_1, introduced by Noohi in [40]. By definition, $G_0 \ltimes^{G_1} \Gamma_1$ is equal to $\dfrac{G_0 \ltimes \Gamma_1}{N}$, where $N = \{(\partial(\alpha), \bar{p}_1(\alpha)^{-1}), \quad \alpha \in G_1\}$.
In fact, $(g_1, \beta_1) \sim (g_2, \beta_2)$ if there is an $\alpha \in G_1$ such that the identities (4.6) and (4.7) hold. Then we have:

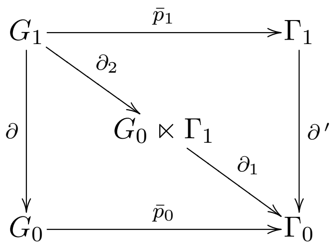
$$\begin{aligned} (g_1, \beta_1) &= (\partial(\alpha)\, g_2, \beta_2\, \bar{p}_1(\alpha)^{-1}) = (\partial(\alpha)\, g_2, \bar{p}_1(\alpha)^{-1\; \partial'(\bar{p}_1(\alpha))}\beta_2) = \\ &= (\partial(\alpha)\, g_2, \bar{p}_1(\alpha)^{-1\; \bar{p}_0(\partial(\alpha))}\beta_2) = (\partial(\alpha)\, g_2, \bar{p}_1(\alpha)^{-1\; \partial(\alpha)}\beta_2) = \\ &= (\partial(\alpha), \bar{p}_1(\alpha)^{-1})\,(g_2, \beta_2) \end{aligned}$$

So we have a homomorphism

$$d : G_0 \ltimes^{G_1} \Gamma_1 \;\; \to \;\; \Gamma_0$$

which, by abuse of notation, we have denoted again by d.

Remark 4.3.1. *Starting from the crossed square (4.2), Conduché in [17] introduced a construction called the mapping cone complex, given by:*



where $\partial_2(\alpha) = (\partial(\alpha), \bar{p}_1(\alpha^{-1}))$ *and* $\partial_1(g, \beta) = \partial'(\beta)\, \bar{p}_0(g)$. *It is immediate to observe that the generalized semi-direct product* $G_0 \ltimes^{G_1} \Gamma_1$ *is obtained from the mapping cone complex as* $\mathrm{coker}\, \partial_2$.

From the previous remark, we obtain the following Proposition.

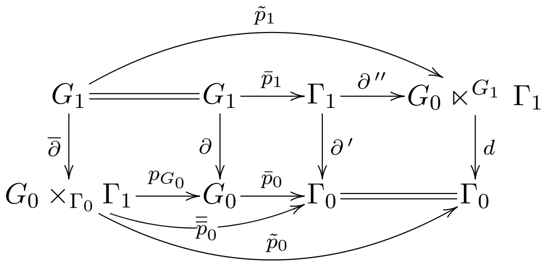
Proposition 4.3.1. $d : G_0 \ltimes^{G_1} \Gamma_1 \to \Gamma_0$ *is a crossed module*

Proof. Proposition 2.3.1.

4.4 $\ker \mathbf{T}$ as a strict categorical $\dfrac{\mathbf{\Gamma}}{<\mathbf{G}, \mathbf{T}>}$-crossed module

It is well-known that given a crossed module (of groups) $\partial : G_1 \to G_0$ then $\ker\partial \hookrightarrow G_1 \to G_0 \twoheadrightarrow \mathrm{coker}\partial$ is a crossed module (of groups). In the context of a crossed squares, we prove the following Proposition.

Proposition 4.4.1. *The outer diagram*



(4.8)

gives rise to a crossed square with actions, group homomorphism ∂'' *and function* $\overline{\overline{h}} : (G_0 \ltimes^{G_1} \Gamma_1) \times (G_0 \times_{\Gamma_0} \Gamma_1) \to G_1$ *defined as following:*

- *the action of* Γ_0 *on* G_1 *is induced by the action of* $\partial' : \Gamma_1 \to \Gamma_0$ *on* $\partial : G_1 \to G_0$ *;*
- *the action of* Γ_0 *on* $G_0 \ltimes^{G_1} \Gamma_1$ *is the action of a crossed module* $d : G_0 \ltimes^{G_1} \Gamma_1 \to \Gamma_0$*;*
- *the action of* Γ_0 *on* $G_0 \times_{\Gamma_0} \Gamma_1$ *is defined by* ${}^{\sigma}(g_2, \beta_2) = ({}^{\sigma}g_2, {}^{\sigma}\beta_2)$ *(the same action seen in the crossed square (4.4));*
- $\partial'' : \Gamma_1 \to G_0 \ltimes^{G_1} \Gamma_1$ *is the canonical inclusion map of* Γ_1 *in* $G_0 \ltimes^{G_1} \Gamma_1$*;*
- $\overline{\overline{h}}((g_1, \beta_1), (g_2, \beta_2)) := h(\beta_1, g_1\, g_2\, g_1^{-1})\, h(\beta_2, g_1)^{-1}$ *where the function* h *is given by the crossed square structure of (4.2).*

Proof. $\tilde{p}_0 = \overline{\overline{p}}_0$ is a group homomorphism, where $\overline{\overline{p}}_0$ is defined in (4.4). $\tilde{p}_1(\alpha) = (1, \bar{p}_1(\alpha))$ is a group homomorphism because $\bar{p}_1$ is and it is easy to check that $d\tilde{p}_1 = \partial' \bar{p}_1 = \bar{p}_0\, \partial = \tilde{p}_0\, \overline{\partial}$ (so the diagram (4.8) commutes and the last map is a crossed module). $\overline{\overline{h}}$ is well defined, in fact we have:

$$
\begin{aligned}
&\overline{\overline{h}}((\partial(\alpha)\, g_1, \beta_1\, \bar{p}_1(\alpha)^{-1}), (g_2, \beta_2)) = \\
&= h(\beta_1\, \bar{p}_1(\alpha)^{-1}, \partial(\alpha)\, g_1\, g_2\, g_1^{-1}\, \partial(\alpha)^{-1})\, h(\beta_2, \partial(\alpha)\, g_1)^{-1} = \\
&= h(\beta_1\, \bar{p}_1(\alpha)^{-1}, {}^{\bar{p}_0\,\partial(\alpha)}(g_1\, g_2\, g_1^{-1}))\, {}^{\partial(\alpha)}h(\beta_2, g_1)^{-1}\, h(\beta_2, \partial(\alpha))^{-1} = \\
&= {}^{\bar{p}_0\,\partial(\alpha)}h({}^{\bar{p}_0\,\partial(\alpha)^{-1}}(\beta_1\, \bar{p}_1(\alpha)^{-1}), g_1\, g_2\, g_1^{-1})\, \alpha\, h(\beta_2, g_1)^{-1}\, \alpha^{-1}\, \alpha\, {}^{\beta_2}\alpha^{-1} = \\
&= \alpha\, h({}^{\partial'\bar{p}_1(\alpha)^{-1}}(\beta_1\, \bar{p}_1(\alpha)^{-1}), g_1\, g_2\, g_1^{-1})\, \alpha^{-1}\, \alpha\, h(\beta_2, g_1)^{-1}\, {}^{\beta_2}\alpha^{-1} = \\
&= \alpha\, h(\bar{p}_1(\alpha)^{-1}\, \beta_1, g_1\, g_2\, g_1^{-1})\, h(\beta_2, g_1)^{-1}\, {}^{\beta_2}\alpha^{-1} = \\
&= \alpha\, {}^{\partial'\bar{p}_1(\alpha)^{-1}}h(\beta_1, g_1\, g_2\, g_1^{-1})\, h(\bar{p}_1(\alpha)^{-1}, g_1\, g_2\, g_1^{-1})\, h(\beta_2, g_1)^{-1}\, {}^{\beta_2}\alpha^{-1} = \\
&= \alpha\, \alpha^{-1}\, h(\beta_1, g_1\, g_2\, g_1^{-1})\, \alpha\, \alpha^{-1}\, {}^{g_1\, g_2\, g_1^{-1}}\alpha\, h(\beta_2, g_1)^{-1}\, {}^{\beta_2}\alpha^{-1} = \\
&= h(\beta_1, g_1\, g_2\, g_1^{-1})\, {}^{g_1\,\beta_2}({}^{g_1^{-1}}\alpha)\, h(\beta_2, g_1)^{-1}\, {}^{\beta_2}\alpha^{-1} = \\
&= h(\beta_1, g_1\, g_2\, g_1^{-1})\, h(\beta_2, g_1)^{-1}\, {}^{\beta_2\, g_1}({}^{g_1^{-1}}\alpha)\, {}^{\beta_2}\alpha^{-1} = \\
&= h(\beta_1, g_1\, g_2\, g_1^{-1})\, h(\beta_2, g_1)^{-1}\, {}^{\beta_2}\alpha\, {}^{\beta_2}\alpha^{-1} = \\
&= h(\beta_1, g_1\, g_2\, g_1^{-1})\, h(\beta_2, g_1)^{-1}.
\end{aligned}
$$

The equalities above are consequences of the axioms of the crossed square (4.2). We also want to emphasize that in the eighth passage we have used the fact that (g_2, β_2) belongs to the pullback $G_0 \times_{\Gamma_0} \Gamma_1$ and we have $\bar{p}_0(g_2) = \partial'(\beta_2)$ and in general for any $\alpha_1 \in G_1$, we have:

$$^{g_2}\alpha_1 = {}^{\bar{p}_0(g_2)}\alpha_1 = {}^{\partial'(\beta_2)}\alpha_1 = {}^{\beta_2}\alpha_1.$$

Instead, in the ninth passage, we used the property (a) of the crossed square (4.2) (see section 2.2).

Now we want to check the five properties making the diagram (4.8) a crossed square.

(i) The map $\tilde{p}_1$ preserves the actions of Γ_0; in fact:

$$\tilde{p}_1({}^{\sigma}\alpha) = (1, \bar{p}_1({}^{\sigma}\alpha)) = (1, {}^{\sigma}\bar{p}_1(\alpha)) = {}^{\sigma}(1, \bar{p}_1(\alpha)) = {}^{\sigma}\tilde{p}_1(\alpha).$$

We have already seen that the map $\overline{\partial}$ preserves the actions of Γ_0.

d is a crossed module and $\tilde{p}_0$ is a crossed module because $\overline{\overline{p}}_0$ is.

(ii) We want to prove that $\tilde{p}_1(\overline{\overline{h}}((g_1,\beta_1),(g_2,\beta_2))) = (g_1,\beta_1)\,{}^{(g_2,\beta_2)}(g_1,\beta_1)^{-1}$ and we develop the two members separately:

$$\begin{aligned}
&\tilde{p}_1(\overline{\overline{h}}((g_1,\beta_1),(g_2,\beta_2))) = \tilde{p}_1(h(\beta_1, g_1\, g_2\, g_1^{-1})\, h(\beta_2, g_1)^{-1}) = \\
&= (1, \bar{p}_1(h(\beta_1, g_1\, g_2\, g_1^{-1})\, h(\beta_2, g_1)^{-1})) = \\
&= (1, \beta_1\,{}^{g_1\, g_2\, g_1^{-1}}\beta_1^{-1}\,{}^{g_1}\beta_2\, \beta_2^{-1}); \\
&(g_1,\beta_1)\,{}^{(g_2,\beta_2)}(g_1,\beta_1)^{-1} = (g_1,\beta_1)\,{}^{\tilde{p}_0(g_2,\beta_2)}(g_1,\beta_1)^{-1} = \\
&= (g_1,\beta_1)\,{}^{\bar{p}_0(g_2)}(g_1^{-1}, {}^{g_1^{-1}}\beta_1^{-1}) = (g_1,\beta_1)\,(g_2\, g_1^{-1}\, g_2^{-1}, {}^{g_2\, g_1^{-1}}\beta_1^{-1}) = \\
&= (g_1\, g_2\, g_1^{-1}\, g_2^{-1}, \beta_1\,{}^{g_1\, g_2\, g_1^{-1}}\beta_1^{-1}) = \\
&= (\partial\, h(\beta_2, g_1)^{-1}\ \cdot 1, \beta_1\,{}^{g_1\, g_2\, g_1^{-1}}\beta_1^{-1}\,{}^{g_1}\beta_2\, \beta_2^{-1}\ \cdot \bar{p}_1\, h(\beta_2, g_1)).
\end{aligned}$$

So $\tilde{p}_1(\overline{\overline{h}}((g_1,\beta_1),(g_2,\beta_2))) \sim (g_1,\beta_1)\,{}^{(g_2,\beta_2)}(g_1,\beta_1)^{-1}$ in $\dfrac{G_0 \ltimes \Gamma_1}{\sim} \cong G_0 \ltimes^{G_1} \Gamma_1$.

Now we want to prove that

$$\overline{\partial}\,\overline{\overline{h}}((g_1,\beta_1),(g_2,\beta_2)) = {}^{(g_1,\beta_1)}(g_2,\beta_2)\,(g_2,\beta_2)^{-1}$$

and we develop the two members separately:

$$\begin{aligned}
&\overline{\partial}\,\overline{\overline{h}}((g_1,\beta_1),(g_2,\beta_2)) = \overline{\partial}(h(\beta_1, g_1\, g_2\, g_1^{-1})\, h(\beta_2, g_1)^{-1}) = \\
&= (\partial\, h(\beta_1, g_1\, g_2\, g_1^{-1})\, \partial\, h(\beta_2, g_1)^{-1}, \bar{p}_1\, h(\beta_1, g_1\, g_2\, g_1^{-1}) \\
&\quad \bar{p}_1\, h(\beta_2, g_1)^{-1}) = \\
&= ({}^{\beta_1}(g_1\, g_2\, g_1^{-1})\, g_1\, g_2^{-1}\, g_1^{-1}\, g_1\,{}^{\beta_2}g_1^{-1}, \beta_1\,{}^{g_1\, g_2\, g_1^{-1}}\beta_1^{-1}\,{}^{g_1}\beta_2\, \beta_2^{-1}) =
\end{aligned}$$

$$\begin{aligned}
&= ({}^{\beta_1}(g_1\, g_2\, g_1^{-1})\, g_1\, g_2^{-1\;\bar{p}_0(g_2)} g_1^{-1}, \beta_1\, {}^{g_1}({}^{\partial'(\beta_2)}({}^{g_1^{-1}}\beta_1^{-1}))\, {}^{g_1}\beta_2\, \beta_2^{-1}) = \\
&= ({}^{\beta_1}(g_1\, g_2\, g_1^{-1})\, g_1\, g_2^{-1}\, g_2\, g_1^{-1}\, g_2^{-1}, \beta_1\, {}^{g_1}\beta_2\, \beta_1^{-1}\, {}^{g_1}\beta_2^{-1}\, {}^{g_1}\beta_2\, \beta_2^{-1}) = \\
&= ({}^{\beta_1}(g_1\, g_2\, g_1^{-1})\, g_2^{-1}, \beta_1\, {}^{g_1}\beta_2\, \beta_1^{-1}\, \beta_2^{-1}); \\
&{}^{(g_1,\beta_1)}(g_2,\beta_2)\,(g_2,\beta_2)^{-1} = {}^{\partial'(\beta_1)\,\bar{p}_0(g_1)}(g_2,\beta_2)\,(g_2^{-1},\beta_2^{-1}) = \\
&= ({}^{\beta_1}(g_1\, g_2\, g_1^{-1}), \beta_1\, {}^{g_1}\beta_2\, \beta_1^{-1})\,(g_2^{-1},\beta_2^{-1}) = \\
&= ({}^{\beta_1}(g_1\, g_2\, g_1^{-1})\, g_2^{-1}, \beta_1\, {}^{g_1}\beta_2\, \beta_1^{-1}\, \beta_2^{-1}).
\end{aligned}$$

(iii)

$$\begin{aligned}
&\overline{\overline{h}}(\tilde{p}_1(\alpha),(g_2,\beta_2)) = \overline{\overline{h}}((1,\bar{p}_1(\alpha)),(g_2,\beta_2)) = \\
&= h(\bar{p}_1(\alpha), g_2)\, h(\beta_2, 1)^{-1} = \alpha^{\,g_2}\alpha^{-1} = \alpha^{\,\bar{p}_0(g_2)}\alpha^{-1} = \\
&= \alpha^{\,\bar{p}_0(g_2,\beta_2)}\alpha^{-1} = \alpha^{\,(g_2,\beta_2)}\alpha^{-1}; \\
&\overline{\overline{h}}((g_1,\beta_1),\overline{\partial}(\alpha)) = \overline{\overline{h}}((g_1,\beta_1),(\partial(\alpha),\bar{p}_1(\alpha))) = \\
&= h(\beta_1, g_1\,\partial(\alpha)\, g_1^{-1})\, h(\bar{p}_1(\alpha), g_1)^{-1} = \\
&= h(\beta_1, \partial({}^{g_1}\alpha))\, h(\bar{p}_1(\alpha), g_1)^{-1} = {}^{\beta_1}({}^{g_1}\alpha)\, {}^{g_1}\alpha^{-1}\, {}^{g_1}\alpha\, \alpha^{-1} = \\
&= {}^{\beta_1}({}^{g_1}\alpha)\,\alpha^{-1} = {}^{\partial'(\beta_1)\,\bar{p}_0(g_1)}\alpha\,\alpha^{-1} = {}^{d(g_1,\beta_1)}\alpha\,\alpha^{-1} = {}^{(g_1,\beta_1)}\alpha\,\alpha^{-1}.
\end{aligned}$$

(iv) We want to prove that:

$$\overline{\overline{h}}((g_1,\beta_1)\,(g_1{}',\beta_1{}'),(g_2,\beta_2)) = {}^{(g_1,\beta_1)}\overline{\overline{h}}((g_1{}',\beta_1{}'),(g_2,\beta_2))\,\overline{\overline{h}}((g_1,\beta_1),(g_2,\beta_2))$$

and we develop the two members separately:

$$\begin{aligned}
&\overline{\overline{h}}((g_1,\beta_1)\,(g_1{}',\beta_1{}'),(g_2,\beta_2)) = \overline{\overline{h}}((g_1\, g_1{}', \beta_1\, {}^{g_1}\beta_1{}'),(g_2,\beta_2)) = \\
&= h(\beta_1\, {}^{g_1}\beta_1{}', g_1\, g_1{}'\, g_2\, g_1{}'^{-1}\, g_1^{-1})\, h(\beta_2, g_1\, g_1{}')^{-1} = \\
&= {}^{\beta_1}h({}^{g_1}\beta_1{}', g_1\, g_1{}'\, g_2\, g_1{}'^{-1}\, g_1^{-1})\, h(\beta_1, g_1\, g_1{}'\, g_2\, g_1{}'^{-1}\, g_1^{-1}) \\
&\quad {}^{g_1}h(\beta_2, g_1{}')^{-1}\, h(\beta_2, g_1)^{-1} = \\
&= {}^{\beta_1\, g_1}h(\beta_1{}', g_1{}'\, g_2\, g_1{}'^{-1})\, h(\beta_1, g_1\, g_1{}'^{\,\bar{p}_0(g_2)}(g_1\, g_1{}')^{-1}\, g_2) \\
&\quad {}^{g_1}h(\beta_2, g_1{}')^{-1}\, h(\beta_2, g_1)^{-1} = \\
&= {}^{\beta_1\, g_1}h(\beta_1{}', g_1{}'\, g_2\, g_1{}'^{-1})\, h(\beta_1, g_1\, g_1{}'^{\,\partial'(\beta_2)}(g_1\, g_1{}')^{-1}\, g_2) \\
&\quad {}^{g_1}h(\beta_2, g_1{}')^{-1}\, h(\beta_2, g_1)^{-1} = \\
&= {}^{\beta_1\, g_1}h(\beta_1{}', g_1{}'\, g_2\, g_1{}'^{-1})\, h(\beta_1, \partial\, h(\beta_2, g_1\, g_1{}')^{-1}\, g_2)\, {}^{g_1}h(\beta_2, g_1{}')^{-1} \\
&\quad h(\beta_2, g_1)^{-1} = \\
&= {}^{\beta_1\, g_1}h(\beta_1{}', g_1{}'\, g_2\, g_1{}'^{-1})\, {}^{\beta_1}h(\beta_2, g_1\, g_1{}')^{-1}\, h(\beta_1, g_2)\, h(\beta_2, g_1\, g_1{}') \\
&\quad {}^{g_1}h(\beta_2, g_1{}')^{-1}\, h(\beta_2, g_1)^{-1} = \\
&= {}^{\beta_1\, g_1}h(\beta_1{}', g_1{}'\, g_2\, g_1{}'^{-1})\, {}^{\beta_1}h(\beta_2, g_1\, g_1{}')^{-1}\, h(\beta_1, g_2)\, h(\beta_2, g_1) \\
&\quad {}^{g_1}h(\beta_2, g_1{}')\, {}^{g_1}h(\beta_2, g_1{}')^{-1}\, h(\beta_2, g_1)^{-1} = \\
&= {}^{\beta_1\, g_1}h(\beta_1{}', g_1{}'\, g_2\, g_1{}'^{-1})\, {}^{\beta_1}h(\beta_2, g_1\, g_1{}')^{-1}\, h(\beta_1, g_2);
\end{aligned}$$

$$
\begin{aligned}
&{}^{(g_1,\beta_1)}\overline{\overline{h}}(({g_1}',{\beta_1}'),(g_2,\beta_2))\,\overline{\overline{h}}((g_1,\beta_1),(g_2,\beta_2)) = \\
&= {}^{\beta_1\,g_1}[h({\beta_1}',{g_1}'\,g_2\,{g_1}'^{-1})\,h(\beta_2,{g_1}')^{-1}]\,h(\beta_1,g_1\,g_2\,g_1^{-1}) \\
&\quad h(\beta_2,g_1)^{-1} = \\
&= {}^{\beta_1\,g_1}h({\beta_1}',{g_1}'\,g_2\,{g_1}'^{-1})\,{}^{\beta_1\,g_1}h(\beta_2,{g_1}')^{-1}\;h(\beta_1,{g_1}^{\bar{p}_0(g_2)}\,g_1^{-1}\,g_2) \\
&\quad h(\beta_2,g_1)^{-1} = \\
&= {}^{\beta_1\,g_1}h({\beta_1}',{g_1}'\,g_2\,{g_1}'^{-1})\,{}^{\beta_1\,g_1}h(\beta_2,{g_1}')^{-1}\;h(\beta_1,{g_1}^{\partial'(\beta_2)}\,g_1^{-1}\,g_2) \\
&\quad h(\beta_2,g_1)^{-1} = \\
&= {}^{\beta_1\,g_1}h({\beta_1}',{g_1}'\,g_2\,{g_1}'^{-1})\,{}^{\beta_1\,g_1}h(\beta_2,{g_1}')^{-1}\;h(\beta_1,\partial\,h(\beta_2,g_1)^{-1}\,g_2) \\
&\quad h(\beta_2,g_1)^{-1} = \\
&= {}^{\beta_1\,g_1}h({\beta_1}',{g_1}'\,g_2\,{g_1}'^{-1})\,{}^{\beta_1\,g_1}h(\beta_2,{g_1}')^{-1}\;{}^{\beta_1}h(\beta_2,g_1)^{-1}\,h(\beta_1,g_2) \\
&\quad h(\beta_2,g_1)\,h(\beta_2,g_1)^{-1} = \\
&= {}^{\beta_1\,g_1}h({\beta_1}',{g_1}'\,g_2\,{g_1}'^{-1})\,{}^{\beta_1}[\,{}^{g_1}h(\beta_2,{g_1}')^{-1}\,h(\beta_2,g_1)^{-1}]\,h(\beta_1,g_2) = \\
&= {}^{\beta_1\,g_1}h({\beta_1}',{g_1}'\,g_2\,{g_1}'^{-1})\,{}^{\beta_1}h(\beta_2,g_1\,{g_1}')^{-1}\,h(\beta_1,g_2).
\end{aligned}
$$

In the development of both members, we used the axioms relating to crossed square (4.2). We want to emphasize that, in the development of the first member (second member) in the fourth (third) step, we used the fact that (g_2,β_2) belongs to the pullback $G_0\times_{\Gamma_0}\Gamma_1$ and in the sixth (fifth) step we used the identity (l) for the crossed square (4.2) (see section 2.2).

(v)

$$
\begin{aligned}
&\overline{\overline{h}}({}^{\sigma}(g_1,\beta_1),\,{}^{\sigma}(g_2,\beta_2)) = \overline{\overline{h}}(({}^{\sigma}g_1,\,{}^{\sigma}\beta_1),({}^{\sigma}g_2,\,{}^{\sigma}\beta_2)) = \\
&= h({}^{\sigma}\beta_1,\,{}^{\sigma}g_1\,{}^{\sigma}g_2\,{}^{\sigma}g_1^{-1})\,h({}^{\sigma}\beta_2,\,{}^{\sigma}g_1)^{-1} = \\
&= h({}^{\sigma}\beta_1,\,{}^{\sigma}(g_1\,g_2\,g_1^{-1}))\,h({}^{\sigma}\beta_2,\,{}^{\sigma}g_1)^{-1} = \\
&= {}^{\sigma}h(\beta_1,g_1\,g_2\,g_1^{-1})\,{}^{\sigma}h(\beta_2,g_1)^{-1} = \\
&= {}^{\sigma}(h(\beta_1,g_1\,g_2\,g_1^{-1})\,h(\beta_2,g_1)^{-1}) = \\
&= {}^{\sigma}\overline{\overline{h}}((g_1,\beta_1),(g_2,\beta_2)).
\end{aligned}
$$

□

Transferring in a categorical crossed module language, we can summarize the previous results by saying

$$\ker\mathbf{T}\longrightarrow\mathbf{G}\xrightarrow{\mathbf{T}}\mathbf{\Gamma}\longrightarrow\frac{\mathbf{\Gamma}}{<\mathbf{G},\mathbf{T}>}$$

is a strict categorical crossed module.

Remark 4.4.1. *In the category of groups, it is obvious that the following composition*

$$ker\,\partial\hookrightarrow G_1\xrightarrow{\partial}G_0\twoheadrightarrow coker\,\partial$$

is the trivial homomorphism.

If we interpret this fact in the context of crossed modules, we can prove that the morphism of crossed modules (4.8) is homotopy equivalent to the trivial morphism. In fact, there exists a transformation between them given by a function $\theta : G_0 \times_{\Gamma_0} \Gamma_1 \to G_0 \ltimes^{G_1} \Gamma_1$*, defined by* $\theta(g,\beta) = (1_{G_0}, \beta^{-1})$.

4.5 Images of crossed modules

In this last section, we consider the Norrie's approach (see [42]) of the image of a crossed module morphism and we well prove another result showing the analogy between crossed modules and crossed squares.
It is well-known that given a crossed module (of groups) $\partial : G_1 \to G_0$ then $\mathrm{Im}\partial$ is normal in G_0.

If we interpret these facts in the context of crossed squares, we can prove the following proposition.

Proposition 4.5.1. *Let*

$$\begin{array}{ccc} G_1 & \xrightarrow{\bar{p}_1} & \Gamma_1 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ G_0 & \xrightarrow{\bar{p}_0} & \Gamma_0, \end{array}$$

be a crossed square, the subcrossed module ${\partial'}_{|Im\,\bar{p}_1} : Im\,\bar{p}_1 \to Im\,\bar{p}_0$ *of* $\partial' : \Gamma_1 \to \Gamma_0$ *is normal.*

Proof.

- $\mathrm{Im}\,\bar{p}_0$ is a normal subgroup of Γ_0 (because $\bar{p}_0 : G_0 \to \Gamma_0$ is a crossed module);

- for all $\sigma \in \Gamma_0$ and $\overline{\beta} \in \mathrm{Im}\,\bar{p}_1$ (that is there exists $\overline{\alpha} \in G_1$ such that $\bar{p}_1(\overline{\alpha}) = \overline{\beta}$), we have:

$$^{\sigma}\overline{\beta} = {}^{\sigma}\bar{p}_1(\overline{\alpha}) = \bar{p}_1({}^{\sigma}\overline{\alpha}),$$

  so $^{\sigma}\overline{\beta} \in \mathrm{Im}\,\bar{p}_1$.

- for all $\overline{\sigma} \in \mathrm{Im}\,\bar{p}_0$ (that is there exists $\overline{g} \in G_0$ such that $\bar{p}_0(\overline{g}) = \overline{\sigma}$) and $\beta \in \Gamma_1$, we have:

$$^{\overline{\sigma}}\beta\,\beta^{-1} = {}^{\bar{p}_0(\overline{g})}\beta\,\beta^{-1} = {}^{\overline{g}}\beta\,\beta^{-1} = \bar{p}_1\,h(\beta,\overline{g}),$$

  so $^{\overline{\sigma}}\beta\,\beta^{-1} \in \mathrm{Im}\,\bar{p}_1$.

□

Chapter 5

Cohomologies

In this chapter, we recall and revisit some results of Dedecker [20]-[21], Borovoi [1] and Noohi [41] relative to the cohomology of a group with coefficients in crossed modules.

In the sections 5.3, 5.4 and 5.5, we present (thanks to the article [14] on the cohomology for categorical groups) a low-dimensional cohomology for crossed modules with coefficients in braided crossed modules, 2-crossed modules and crossed squares, respectively.

5.1 Cohomology of a group with coefficients in crossed module

5.1.1 Dedecker Cohomology

The category of Γ-groups (with objects Γ-groups and arrows group homomorphisms respecting the action of Γ) is suitable as a category of coefficients to describe a good cohomological theory of Γ only in dimension 0 and 1 (see A.2). This is not true in dimension 2 and therefore Dedecker replaced the category of Γ-groups with the category of crossed modules. In 1964 Dedecker [20]-[21] defined the cohomology in dimension 2 of a group Γ with coefficients in a crossed module $\partial : G_1 \to G_0$, considering the trivial action of Γ on $\partial : G_1 \to G_0$. The usefulness of this cohomological theory is that:

1. it is functorial;
2. it produces a cohomological exact sequence associated with short exact sequence in the coefficients category, a notion respected by the forgetful functors from $\mathcal{CM}$ to the category of Γ-groups.

For an arbitrary group Γ, Dedecker denotes by:

$$C^0_D(\Gamma, \partial : G_1 \to G_0) = G_1 \qquad \text{0-cochains,}$$
$$C^1_D(\Gamma, \partial : G_1 \to G_0) = \mathrm{App}(\Gamma, G_1) \qquad \text{1-cochains,}$$

$$C^2_D(\Gamma, \partial : G_1 \to G_0) = \mathrm{App}(\Gamma, G_0) \times \mathrm{App}(\Gamma \times \Gamma, G_1) \qquad \text{2-cochains,}$$

where App represents the set of all maps between the underlying sets. Dedecker defines the set of 2-cocycles in the following way:

$$\begin{aligned} Z^2_D(\Gamma, \partial : G_1 \to G_0) = \{ \quad & (p, \varepsilon) \in C^2_D(\Gamma, \partial : G_1 \to G_0) \; / \\ & p(\sigma)\, p(\tau) = \partial\varepsilon(\sigma, \tau)\, p(\sigma\,\tau) \\ & {}^{p(\sigma)}\varepsilon(\tau, \upsilon)\, \varepsilon(\sigma, \tau\,\upsilon) = \varepsilon(\sigma, \tau)\, \varepsilon(\sigma\,\tau, \upsilon) \quad \}. \end{aligned}$$

$Z^2_D(\Gamma, \partial : G_1 \to G_0)$ is a pointed set with as base point the pair of constant functions $(1_{G_0}, 1_{G_1})$, where $1_{G_0}(g) = 1_{G_0}$ and $1_{G_1}(\alpha) = 1_{G_1}$ for all g in G_0 and all α in G_1. An action of the group $C^1_D(\Gamma, \partial : G_1 \to G_0) = App(\Gamma, G_1)$ (with the product given by $(\theta_1 \cdot \theta_2)(\sigma) = \theta_1(\sigma)\, \theta_2(\sigma)$) on the set $Z^2_D(\Gamma, \partial : G_1 \to G_0)$ is given by the following function:

$$* : C^1_D(\Gamma, \partial : G_1 \to G_0) \times Z^2_D(\Gamma, \partial : G_1 \to G_0) \to Z^2_D(\Gamma, \partial : G_1 \to G_0)$$

where we have $\theta * (p_1, \varepsilon_1) = (p_2, \varepsilon_2)$ and:

$$\begin{aligned} (i) \qquad & p_2(\sigma) = \partial\theta(\sigma)\, p_1(\sigma); \\ (ii) \qquad & \varepsilon_2(\sigma, \tau) = \theta(\sigma)\, {}^{p_1(\sigma)}\theta(\tau)\, \varepsilon_1(\sigma, \tau)\, \theta(\sigma\,\tau)^{-1}. \end{aligned}$$

Dedecker considers the orbits of this action that form a set $H^2_D(\Gamma, \partial : G_1 \to G_0)$.

5.1.2 Borovoi Cohomology

Actually Dedecker cohomology does not fully generalizes the abelian case, presented in A.1, since it represents the cohomology for trivial actions of Γ. After more than 20 years, Borovoi [1] gave a full generalization of cohomology of a group Γ with coefficients in a Γ-module in dimension 0, 1, 2. In this theory, the main tool is the notion of action of a group on a crossed module (see section 1.3).

Given an action of a group Γ on the crossed module $\partial : G_1 \to G_0$, Borovoi denotes by:

$$\begin{aligned} & C^0_B(\Gamma, \partial : G_1 \to G_0) = G_1 && \text{0-cochains,} \\ & C^1_B(\Gamma, \partial : G_1 \to G_0) = G_0 \times \mathrm{App}(\Gamma, G_1) && \text{1-cochains,} \\ & C^2_B(\Gamma, \partial : G_1 \to G_0) = \mathrm{App}(\Gamma, G_0) \times \mathrm{App}(\Gamma \times \Gamma, G_1) && \text{2-cochains.} \end{aligned}$$

Borovoi defines

$$H^0_B(\Gamma, \partial : G_1 \to G_0) = (\ker\partial)^\Gamma$$

and this is an abelian group.

$C^1_B(\Gamma, \partial : G_1 \to G_0)$ is a group with the product given by:

$$(g_1, \theta_1)(g_2, \theta_2) = (g_1\, g_2,\ {}^{g_1}\theta_2\, \theta_1)$$

where $({}^{g_1}\theta_2\,\theta_1)(\sigma) = {}^{g_1}\theta_2(\sigma)\,\theta_1(\sigma)$. The inverse of (g,θ) is the pair (g^{-1},θ^*), where $\theta^*(\sigma) = {}^{g^{-1}}\theta(\sigma)^{-1}$.
The 1-cocycles form a subgroup of $C^1_B(\Gamma,\partial : G_1 \to G_0)$, defined as

$$Z^1_B(\Gamma,\partial : G_1 \to G_0) = \{(g,\theta) \in C^1_B \ / \ \theta(\sigma\,\tau) = \theta(\sigma)\,{}^{\sigma}\theta(\tau)\ ,\ {}^{\sigma}g = \partial\theta(\sigma)^{-1}\,g\}.$$

The 1-coboundaries form a normal subgroup of $Z^1_B(\Gamma,\partial : G_1 \to G_0)$, defined as

$$B^1_B(\Gamma,\partial : G_1 \to G_0) = \{(g,\theta) \in Z^1_B \ / \ \exists\alpha \in G_1\ :\ g = \partial(\alpha)\ ,\ \theta(\sigma) = \alpha\,{}^{\sigma}\alpha^{-1}\}.$$

Then Borovoi introduces the following group (in general not abelian):

$$H^1_B(\Gamma,\partial : G_1 \to G_0) = \frac{Z^1_B(\Gamma,\partial : G_1 \to G_0)}{B^1_B(\Gamma,\partial : G_1 \to G_0)}.$$

He defines the following pointed set:

$$\begin{aligned} Z^2_B(\Gamma,\partial : G_1 \to G_0) = \{\ & (p,\varepsilon) \in C^2_B(\Gamma,\partial : G_1 \to G_0)\ / \\ & p(\sigma)\,{}^{\sigma}p(\tau) = \partial(\varepsilon(\sigma,\tau))\,p(\sigma\,\tau) \\ & {}^{p(\sigma)}({}^{\sigma}\varepsilon(\tau,\upsilon))\,\varepsilon(\sigma,\tau\,\upsilon) = \varepsilon(\sigma,\tau)\,\varepsilon(\sigma\,\tau,\upsilon) \quad \} \end{aligned}$$

with as base point the pair of constant functions $(1_{G_0},1_{G_1})$. There is an action of the group $C^1_B(\Gamma,\partial : G_1 \to G_0)$ on the set $Z^2_B(\Gamma,\partial : G_1 \to G_0)$:

$$* : C^1_B(\Gamma,\partial : G_1 \to G_0) \times Z^2_B(\Gamma,\partial : G_1 \to G_0) \to Z^2_B(\Gamma,\partial : G_1 \to G_0).$$

This is defined by $(g_1,\theta_1) * (p_1,\varepsilon_1) = (p_2,\varepsilon_2)$, where:

$$\begin{aligned} (i) \qquad & p_2(\sigma) = g_1^{-1}\partial\theta_1(\sigma)\,p_1(\sigma)\,{}^{\sigma}g_1; \\ (ii) \qquad & \varepsilon_2(\sigma,\tau) = {}^{g_1^{-1}}[\theta_1(\sigma)\,{}^{p_1(\sigma)}({}^{\sigma}\theta_1(\tau))\,\varepsilon_1(\sigma,\tau)\,\theta_1(\sigma\,\tau)^{-1}]. \end{aligned}$$

Borovoi considers the orbits of this action that form a set $H^2_B(\Gamma,\partial : G_1 \to G_0)$.

Remark 5.1.1. *If we regard crossed modules as 2-dimensional forms of groups, this is also a generalization of the Serre cohomology A.2 because there is just the action of the group* Γ *on the crossed module* $\partial : G_1 \to G_0$.

Examples. (a) Let G_1 be an abelian group, then $G_1 \to 1$ is a crossed module and an action of Γ on $G_1 \to 1$ corresponds to an action of Γ on G_1. In this standard example, the Borovoi cohomology recovers the abelian cohomology introduced in A.1 and we have:

$$\begin{aligned} H^0_B(\Gamma,G_1 \to 1) &= G_1^{\Gamma} = H^0(\Gamma,G_1); \\ H^1_B(\Gamma,G_1 \to 1) &= H^1(\Gamma,G_1); \\ H^2_B(\Gamma,G_1 \to 1) &= H^2(\Gamma,G_1). \end{aligned}$$

(b) Let G_0 be a group, then $1 \to G_0$ is a crossed module and an action of Γ on $1 \to G_0$ corresponds to an action of Γ on G_0. Furthermore, in this case, the Borovoi cohomology recovers the Serre cohomology introduced in A.2 and we have

$$\begin{aligned} &H^0_B(\Gamma, 1 \to G_0) = 1; \\ &H^1_B(\Gamma, 1 \to G_0) = G_1^\Gamma = H^0_S(\Gamma, G_0); \\ &H^2_B(\Gamma, 1 \to G_0) = H^1_S(\Gamma, G_0). \end{aligned}$$

(c) Let $\partial : G_1 \to G_0$ be a crossed module endowed with a trivial action of a group Γ. In an analogous way of the previous examples, we have:

$$H^2_B(\Gamma, G_1 \to G_0) = H^2_D(\Gamma, G_1 \to G_0).$$

(d) Finally, we are going to describe the Borovoi cohomology for a specific example. Let us consider the crossed module $\partial : \mathrm{SL}_2(L) \to \mathrm{PGL}_2(L)$ (see example (g) in 1.1), we have:

$$\begin{aligned} &H^0_B(\mathrm{Gal}(L \setminus K), \partial : \mathrm{SL}_2(L) \to \mathrm{PGL}_2(L)) \cong \mathbb{Z}_2 \qquad \text{if } \mathrm{car}\, L \neq 2; \\ &H^0_B(\mathrm{Gal}(L \setminus K), \partial : \mathrm{SL}_2(L) \to \mathrm{PGL}_2(L)) = 1 \qquad \text{if } \mathrm{car}\, L = 2; \\ &H^1_B(\mathrm{Gal}(L \setminus K), \partial : \mathrm{SL}_2(L) \to \mathrm{PGL}_2(L)) = 1. \end{aligned}$$

5.1.3 Noohi Cohomology

In this section, we are going to recall and revisit some well-known facts about the cohomology presented by Noohi in [41].
First of all, Noohi notes that $H^0_B(\Gamma, G_1 \to G_0)$ and $H^1_B(\Gamma, G_1 \to G_0)$ are the kernel and cokernel, respectively, of the crossed module:

$$\begin{aligned} \overline{\overline{\partial}} : G_1 &\longrightarrow Z^1_B(\Gamma, \partial : G_1 \to G_0) \\ \alpha &\longrightarrow (\partial(\alpha), \theta_\alpha) \end{aligned}$$

where $\theta_\alpha(\sigma) = \alpha\, {}^\sigma\alpha^{-1}$ with the action of $Z^1_B(\Gamma, \partial : G_1 \to G_0)$ on G_1 given by

$${}^{(g,\theta)}\alpha := {}^g\alpha.$$

He analyzes also the Borovoi cohomology of a group Γ with coefficients in a Γ-equivariant braided crossed module $\partial : G_1 \to G_0$ (see for the definition the section 2.1). In this case, Noohi observes that $H^1_B(\Gamma, \partial : G_1 \to G_0)$ is abelian. This is true thanks to the following Lemma.

Lemma 5.1.1. *[41] The commutator of the two 1-cocycles (g_1, θ_1) and (g_2, θ_2) in $Z^1_B(\Gamma, \partial : G_1 \to G_0)$ is equal to the 1-coboundary $(\partial(\alpha), \theta_\alpha)$, where $\alpha = \{g_1, g_2\}$.*

It follows, from the above Lemma, that the bracket $\{(g_1,\theta_1),(g_2,\theta_2)\} = \{g_1, g_2\}$ makes the crossed module $\overline{\overline{\partial}} : G_1 \to Z^1_B(\Gamma, G_1 \to G_0)$ (defined above) a braided crossed module.

In the presence of a braiding on $\partial : G_1 \to G_0$, Noohi introduces a second product on $C^1_B(\Gamma, \partial : G_1 \to G_0)$ which makes it into a group as well. Given two 1-cochains (g_1, θ_1) and (g_2, θ_2), their product is the 1-cochain $(g_1\, g_2, \theta)$, where θ is defined by the formula

$$\theta(\sigma) = {}^{g_1}\{g_2\, {}^{\sigma}g_2^{-1},\ {}^{\sigma}g_1^{-1}\}\, {}^{g_1\, {}^{\sigma}g_1^{-1}}\theta_2(\sigma)\, \theta_1(\sigma).$$

When restricted to $Z^1_B(\Gamma, \partial : G_1 \to G_0)$, the above product coincides with the Borovoi product.

Noohi defines the following group homomorphism:

$$\begin{aligned} d : C^1_B(\Gamma, \partial : G_1 \to G_0) &\to Z^2_B(\Gamma, \partial : G_1 \to G_0) \\ (g, \theta) &\to (p, \varepsilon) \end{aligned}$$

- $p(\sigma) = \partial\theta(\sigma)^{-1}\, g\, {}^{\sigma}g^{-1}$;
- $\varepsilon(\sigma, \tau) = \theta(\sigma)^{-1}\, {}^{g\, {}^{\sigma}g^{-1}}({}^{\sigma}\theta(\tau)^{-1})\, \theta(\sigma\,\tau)$.

The product in $C^1_B(\Gamma, \partial : G_1 \to G_0)$ is the one defined above and the product in $Z^2_B(\Gamma, \partial : G_1 \to G_0)$ is defined as follows. Let (p_1, ε_1) and (p_2, ε_2) be in $Z^2_B(\Gamma, \partial : G_1 \to G_0)$, the product $(p_1, \varepsilon_1)\,(p_2, \varepsilon_2)$ is the pair (p, ε) where

$$\begin{aligned} &p(\sigma) = p_1(\sigma)\, p_2(\sigma); \\ &\varepsilon(\sigma, \tau) = {}^{p_1(\sigma)}\{p_2(\sigma),\ {}^{\sigma}p_1(\tau)\}\, \varepsilon_1(\sigma, \tau)\, {}^{p_1(\sigma\,\tau)}\varepsilon_2(\sigma, \tau). \end{aligned}$$

The inverse of the element (p, ε) in $Z^2_B(\Gamma, \partial : G_1 \to G_0)$ is the pair (p^*, ε^*) where

$$\begin{aligned} &p^*(\sigma) = p(\sigma)^{-1}; \\ &\varepsilon^*(\sigma, \tau) = \{p(\sigma)^{-1},\ {}^{\sigma}p(\tau)^{-1}\}\, {}^{p(\sigma\,\tau)^{-1}}\varepsilon(\sigma, \tau)^{-1}. \end{aligned}$$

Next he constructs the crossed module:

$$d : \frac{C^1_B(\Gamma, \partial : G_1 \to G_0)}{B^1_B(\Gamma, \partial : G_1 \to G_0)} \to Z^2_B(\Gamma, \partial : G_1 \to G_0)$$

because the group homomorphism d vanishes on the subgroup $B^1_B(\Gamma, \partial : G_1 \to G_0) \subseteq C^1_B(\Gamma, \partial : G_1 \to G_0)$ of 1-coboundaries. Therefore, the group homomorphism d factors through the new homomorphism d, by abuse of notation. The action of $(p, \varepsilon) \in Z^2_B(\Gamma, \partial : G_1 \to G_0)$ on $[g, \theta] \in \frac{C^1_B(\Gamma,\partial:G_1\to G_0)}{B^1_B(\Gamma,\partial:G_1\to G_0)}$ is given by:

$${}^{(p,\varepsilon)}[g, \theta] = [g, \widetilde{\theta}]$$

where $\widetilde{\theta}(\sigma) = {}^{g}\{p(\sigma), {}^{\sigma}g^{-1}\}^{-1}\, \{g, p(\sigma)\}\, {}^{p(\sigma)}\theta(\sigma)$.

It is easy to observe that the kernel of d coincides with $H^1_B(\Gamma, \partial : G_1 \to G_0)$. Noohi shows that the cokernel of d coincides with $H^2_B(\Gamma, \partial : G_1 \to G_0)$. He does so by comparing the action of $C^1_B(\Gamma, \partial : G_1 \to G_0)$ on $Z^2_B(\Gamma, \partial : G_1 \to G_0)$, introduced in 5.1.2, with the multiplication action of $C^1_B(\Gamma, \partial : G_1 \to G_0)$ on $Z^2_B(\Gamma, \partial : G_1 \to G_0)$ via d. More precisely, we have the following Lemma.

Lemma 5.1.2. *[41] Let (g, θ) be in $C^1_B(\Gamma, \partial : G_1 \to G_0)$ and (p, ε) in $Z^2_B(\Gamma, \partial : G_1 \to G_0)$. Let ${}^{(g,\theta)}(p, \varepsilon)$ be the action of $C^1_B(\Gamma, \partial : G_1 \to G_0)$ on $Z^2_B(\Gamma, \partial : G_1 \to G_0)$ introduced in the section 5.1.2. Then,*

$$ {}^{(g,\theta)}(p, \varepsilon) = d(g^{-1}, \widehat{\theta})\, (p, \varepsilon) $$

where $\widehat{\theta} : \Gamma \to G_1$ is defined by $\widehat{\theta}(\sigma) = {}^{g^{-1}}\{p(\sigma), {}^{\sigma}g\}^{-1}\, {}^{g^{-1}}\theta(\sigma)^{-1}$.

Proof.
$d(g^{-1}, \widehat{\theta})\, (p, \varepsilon) = (\widehat{p}, \widehat{\varepsilon})\, (p, \varepsilon) = (\overline{p}, \overline{\varepsilon})$

$$
\begin{aligned}
\widehat{p}(\sigma) &= \partial\widehat{\theta}(\sigma)^{-1}\, g^{-1}\, {}^{\sigma}g = \partial({}^{g^{-1}}\{p(\sigma), {}^{\sigma}g\}^{-1}\, {}^{g^{-1}}\theta(\sigma)^{-1})^{-1}\, g^{-1}\, {}^{\sigma}g = \\
&= g^{-1}\, \partial\, \theta(\sigma)\, p(\sigma)\, {}^{\sigma}g\, p(\sigma)^{-1}\, {}^{\sigma}g^{-1}\, g\; g^{-1}\, {}^{\sigma}g \\
&= g^{-1}\, \partial\, \theta(\sigma)\, p(\sigma)\, {}^{\sigma}g\, p(\sigma)^{-1} \\
\widehat{\varepsilon}(\sigma, \tau) &= \widehat{\theta}(\sigma)^{-1}\, {}^{g^{-1}\,{}^{\sigma}g}(\, {}^{\sigma}\widehat{\theta}(\tau)^{-1})\, \widehat{\theta}(\sigma\,\tau) \\
\overline{p}(\sigma) &= \widehat{p}(\sigma)\, p(\sigma) = g^{-1}\, \partial\, \theta(\sigma)\, p(\sigma)\, {}^{\sigma}g \\
\overline{\varepsilon}(\sigma, \tau) &= {}^{\widehat{p}(\sigma)}\{p(\sigma), {}^{\sigma}\widehat{p}(\tau)\}\, \widehat{\varepsilon}(\sigma, \tau)\, {}^{\widehat{p}(\sigma\,\tau)}\varepsilon(\sigma, \tau) = \\
&= {}^{\partial\widehat{\theta}(\sigma)^{-1}\, g^{-1}\, {}^{\sigma}g}\{p(\sigma), {}^{\sigma}\partial\widehat{\theta}(\tau)^{-1}\, {}^{\sigma}g^{-1}\, {}^{\sigma\,\tau}g\}\, \widehat{\theta}(\sigma)^{-1}\, {}^{g^{-1}\,{}^{\sigma}g}(\, {}^{\sigma}\widehat{\theta}(\tau)^{-1}) \\
&\quad \widehat{\theta}(\sigma\,\tau)\, {}^{\partial\widehat{\theta}(\sigma\,\tau)^{-1}\, g^{-1}\, {}^{\sigma\,\tau}g}\varepsilon(\sigma, \tau) = \\
&= \widehat{\theta}(\sigma)^{-1}\, {}^{g^{-1}\,{}^{\sigma}g}[\, {}^{p(\sigma)}(\, {}^{\sigma}\widehat{\theta}(\tau)^{-1})\, \{p(\sigma), {}^{\sigma}g^{-1}\, {}^{\sigma\,\tau}g\}\, {}^{\sigma}\widehat{\theta}(\tau)]\, \widehat{\theta}(\sigma)\, \widehat{\theta}(\sigma)^{-1} \\
&\quad {}^{g^{-1}\,{}^{\sigma}g}(\, {}^{\sigma}\widehat{\theta}(\tau)^{-1})\, \widehat{\theta}(\sigma\,\tau)\, \widehat{\theta}(\sigma\,\tau)^{-1}\, {}^{g^{-1}\,{}^{\sigma\,\tau}g}\varepsilon(\sigma, \tau)\, \widehat{\theta}(\sigma\,\tau) = \\
&= \widehat{\theta}(\sigma)^{-1}\, {}^{g^{-1}\,{}^{\sigma}g}[\, {}^{p(\sigma)}(\, {}^{\sigma}\widehat{\theta}(\tau)^{-1})\, \{p(\sigma), {}^{\sigma}g^{-1}\, {}^{\sigma\,\tau}g\}]\, {}^{g^{-1}\,{}^{\sigma\,\tau}g}\varepsilon(\sigma, \tau) \\
&\quad \widehat{\theta}(\sigma\,\tau) = \\
&= {}^{g^{-1}}\theta(\sigma)\, {}^{g^{-1}}\{p(\sigma), {}^{\sigma}g\}\, {}^{g^{-1}\,{}^{\sigma}g}[\, {}^{p(\sigma)}(\, {}^{\sigma}(\, {}^{g^{-1}}\theta(\tau)\, {}^{g^{-1}}\{p(\tau), {}^{\tau}g\})) \\
&\quad \{p(\sigma), {}^{\sigma}g^{-1}\, {}^{\sigma\,\tau}g\}]\, {}^{g^{-1}\,{}^{\sigma\,\tau}g}\varepsilon(\sigma, \tau)\, {}^{g^{-1}}\{p(\sigma\,\tau), {}^{\sigma\,\tau}g\}^{-1} \\
&\quad {}^{g^{-1}}\theta(\sigma\,\tau)^{-1} = \\
&= {}^{g^{-1}}\theta(\sigma)\, {}^{g^{-1}}\{p(\sigma), {}^{\sigma}g\}\, {}^{\partial(\, {}^{g^{-1}}\{p(\sigma), {}^{\sigma}g\}^{-1})\, g^{-1}\, p(\sigma)}[\, {}^{\sigma}\theta(\tau)\{\, {}^{\sigma}p(\tau), \\
&\quad {}^{\sigma\,\tau}g\}]\, {}^{g^{-1}\,{}^{\sigma}g}\{p(\sigma), {}^{\sigma}g^{-1}\, {}^{\sigma\,\tau}g\}\, {}^{g^{-1}\,{}^{\sigma\,\tau}g}\varepsilon(\sigma, \tau) \\
&\quad {}^{g^{-1}}\{\partial\, \varepsilon(\sigma, \tau)^{-1}\, p(\sigma)\, {}^{\sigma}p(\tau), {}^{\sigma\,\tau}g\}^{-1}\, {}^{g^{-1}}\theta(\sigma\,\tau)^{-1} = \\
&= {}^{g^{-1}}\theta(\sigma)\, {}^{g^{-1}\, p(\sigma)}(\, {}^{\sigma}\theta(\tau))\, {}^{g^{-1}\; p(\sigma)}\{\, {}^{\sigma}p(\tau), {}^{\sigma\,\tau}g\}\, {}^{g^{-1}}\{p(\sigma), {}^{\sigma}g\} \\
&\quad {}^{g^{-1}\,{}^{\sigma}g}\{p(\sigma), {}^{\sigma}g^{-1}\, {}^{\sigma\,\tau}g\}\, {}^{g^{-1}\,{}^{\sigma\,\tau}g}\varepsilon(\sigma, \tau)\, {}^{g^{-1}}[\, {}^{\sigma\,\tau}g\varepsilon(\sigma, \tau)^{-1}
\end{aligned}
$$

$$
\begin{aligned}
& \{p(\sigma)\, {}^{\sigma}p(\tau),\, {}^{\sigma\,\tau}g\}^{-1}\, \varepsilon(\sigma,\tau)]\, {}^{g^{-1}}\theta(\sigma\,\tau)^{-1} = \\
= \quad & {}^{g^{-1}}\theta(\sigma)\, {}^{g^{-1}p(\sigma)}({}^{\sigma}\theta(\tau))\, {}^{g^{-1}p(\sigma)}\{{}^{\sigma}p(\tau),\, {}^{\sigma\,\tau}g\}\, {}^{g^{-1}}\{p(\sigma),\, {}^{\sigma\,\tau}g\} \\
& {}^{g^{-1}\,\sigma\,\tau g}\varepsilon(\sigma,\tau)\, {}^{g^{-1}\,\sigma\,\tau g}\varepsilon(\sigma,\tau)^{-1}\, {}^{g^{-1}}\{p(\sigma)\, {}^{\sigma}p(\tau),\, {}^{\sigma\,\tau}g\}^{-1}\, {}^{g^{-1}}\varepsilon(\sigma,\tau) \\
& {}^{g^{-1}}\theta(\sigma\,\tau)^{-1} = \\
= \quad & {}^{g^{-1}}[\theta(\sigma)\, {}^{p(\sigma)}({}^{\sigma}\theta(\tau))\, \{p(\sigma){}^{\sigma}p(\tau),\, {}^{\sigma\,\tau}g\}\, \{p(\sigma)\, {}^{\sigma}p(\tau),\, {}^{\sigma\,\tau}g\}^{-1} \\
& \varepsilon(\sigma,\tau)\, \theta(\sigma\,\tau)^{-1}] = \\
= \quad & {}^{g^{-1}}[\theta(\sigma)\, {}^{p(\sigma)}({}^{\sigma}\theta(\tau))\, \varepsilon(\sigma,\tau)\, \theta(\sigma\,\tau)^{-1}].
\end{aligned}
$$

□

Corollary 5.1.2.1. *[41] When $\partial : G_1 \to G_0$ has a Γ-equivariant braiding, the second cohomology set $H^2_B(\Gamma, \partial : G_1 \to G_0)$ inherits a natural group structure, $H^1_B(\Gamma, \partial : G_1 \to G_0)$ is abelian, and there is a natural action of $H^2_B(\Gamma, \partial : G_1 \to G_0)$ on $H^1_B(\Gamma, \partial : G_1 \to G_0)$.*

In the case where the braiding is symmetric, we can do even better.

Lemma 5.1.3. *[41] Suppose that the braiding on $\partial : G_1 \to G_0$ is symmetric. Then, the crossed-module $d : \frac{C^1_B(\Gamma,\partial:G_1\to G_0)}{B^1_B(\Gamma,\partial:G_1\to G_0)} \to Z^2_B(\Gamma, \partial : G_1 \to G_0)$ is braided and symmetric. The braiding is given by*

$$\{(p_1,\varepsilon_1),(p_2,\varepsilon_2)\} = [1, \{p_1,p_2\}]$$

where $\{p_1,p_2\} : \Gamma_0 \to G_1$ is the pointwise bracket of the maps $p_1, p_2 : \Gamma_0 \to G_0$.

Corollary 5.1.3.1. *[41] When the braiding on $\partial : G_1 \to G_0$ is symmetric, the structure on $H^2_B(\Gamma, \partial : G_1 \to G_0)$ is abelian.*

5.2 Cohomology with coefficients in categorical crossed modules

Categorical groups are regarded as 2-dimensional sorts of groups. From this point of view, the cohomology for categorical groups with coefficients in categorical crossed modules [14] can be considered a generalization of the Lue cohomology, with coefficients in crossed modules [35].

We are going to recall the Lue cohomology and we briefly introduce the one with coefficients in categorical crossed modules.

Let G_1 be a G_0-group, $\mathrm{Der}(G_0, G_1)$ is the set of all derivations from G_0 to G_1. This is a pointed set with as base point the function $1_{G_1} : G_0 \to G_1$ where $1_{G_1}(g) = 1_{G_1}$ for all g in G_0.

If we consider a crossed module $\partial : G_1 \to G_0$, $\mathrm{Der}(G_0, G_1)$ becomes a monoid, with the Whitehead product. Then we can take the group $\mathrm{Der}^*(G_0, G_1)$ of the units of $\mathrm{Der}(G_0, G_1)$.

There is a homomorphism γ of groups

$$G_1 \xrightarrow{\gamma} \mathrm{Der}^*(G_0, G_1) \tag{5.1}$$

sending an element α in G_1 to the associated inner derivation. The group homomorphism $\gamma : G_1 \to \mathrm{Der}^*(G_0, G_1)$ is a crossed module with the action of $\mathrm{Der}^*(G_0, G_1)$ on G_1 given by:

$$^{\eta}\alpha = \eta(\partial(\alpha))\,\alpha.$$

So Lue, in [35], defines a cohomology in dimension 0 and 1 as follows:

$$H^0_L(G_0, G_1) = \ker(\gamma);$$
$$H^1_L(G_0, G_1) = \mathrm{coker}(\gamma).$$

Now we want to give an idea of the construction of a low-dimensional cohomology of a categorical group $\mathbf{\Gamma}$ with coefficients in a categorical $\mathbf{\Gamma}$-crossed module (see [14] for details).

Let $\mathbf{G}$ be a $\mathbf{\Gamma}$-categorical group, in [14], the authors define a pointed groupoid $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$ of derivations of categorical groups from $\mathbf{\Gamma}$ to $\mathbf{G}$.

They show that starting with a categorical $\mathbf{\Gamma}$-crossed module $(\mathbf{G}, \mathbf{T}, \nu, \chi)$, the groupoid $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$ becomes a monoidal category. Then they introduce a Whitehead categorical group of derivations $\mathrm{Der}^*(\mathbf{\Gamma}, \mathbf{G})$. This is the Picard categorical group of the monoidal category $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$, that is the subcategory of $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$ given by invertible objects and isomorphisms between them.

There is a homomorphism of categorical groups

$$\mathbf{G} \xrightarrow{\overline{\mathbf{T}}} \mathrm{Der}^*(\mathbf{\Gamma}, \mathbf{G})$$

given by inner derivations. There are natural isomorphisms $\overline{\nu}$ and $\overline{\chi}$ such that $(\mathbf{G}, \overline{\mathbf{T}}, \overline{\nu}, \overline{\chi})$ is a categorical $\mathrm{Der}^*(\mathbf{\Gamma}, \mathbf{G})$-crossed module.

So they define a cohomology in dimension 0 and 1 as follows:

$$\mathcal{H}^0(\mathbf{\Gamma}, \mathbf{G}) = \ker(\overline{\mathbf{T}} : \mathbf{G} \to \mathrm{Der}^*(\mathbf{\Gamma}, \mathbf{G}))$$
$$\mathcal{H}^1(\mathbf{\Gamma}, \mathbf{G}) = \frac{\mathrm{Der}^*(\mathbf{\Gamma}, \mathbf{G})}{< \mathbf{G}, \overline{\mathbf{T}} >}$$

where the first one is the kernel (see [48]) of the categorical group homomorphism $\overline{\mathbf{T}}$ while the second one is the quotient categorical group (see [14]) for the categorical crossed module $(\mathbf{G}, \overline{\mathbf{T}}, \overline{\nu}, \overline{\chi})$.

5.3 Cohomology with coefficients in braided crossed modules

Our work arises from the observation that the Noohi cohomology, of a group Γ with coefficients in a Γ-equivariant braided crossed module $\partial : G_1 \to G_0$,

falls within the setting of the cohomology with coefficients in categorical crossed modules.
We have already observed (see the example (c)2. of section 3.4) that a Γ-equivariant braided crossed module can be seen as a special semistrict categorical $\Gamma[0]$-crossed module, where $\Gamma[0]$ is the discrete category associated with Γ. Thanks to this remark, we notice that

$$\overline{\overline{\partial}} : G_1 \to Z^1_B(\Gamma, \partial : G_1 \to G_0)$$

defined in 5.1.3 is a model for the strict categorical group $\mathcal{H}^0(\Gamma[0], \mathbf{G}(\partial))$, where $\mathbf{G}(\partial)$ is the strict categorical group associated with $\partial : G_1 \to G_0$, and

$$d : \frac{C^1_B(\Gamma, \partial : G_1 \to G_0)}{B^1_B(\Gamma, \partial : G_1 \to G_0)} \to Z^2_B(\Gamma, \partial : G_1 \to G_0)$$

defined in 5.1.3 is a model for the strict categorical group $\mathcal{H}^1(\Gamma[0], \mathbf{G}(\partial))$.

In this section, we want to revisit the cohomology with coefficients in categorical crossed modules for another particular case. We have just seen if $\partial : G_1 \to G_0$ is a braided crossed module, endowed with an action by a crossed module $\partial' : \Gamma_1 \to \Gamma_0$ and the braiding is equivariant respect this action, we have an example of semistrict $\mathbf{G}(\partial')$-categorical crossed module (see the example (c)1. section 3.4). We use $\mathbf{\Gamma}$ to denote $\mathbf{G}(\partial')$ and $\mathbf{G}$ for $\mathbf{G}(\partial)$.

In this case (see [27]), $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$ is a categorical group, so $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G}) = \mathrm{Der}^*(\mathbf{\Gamma}, \mathbf{G})$. The associativity a, left unit l and right unit r of the monoidal structure of $\mathrm{Der}\,(\mathbf{\Gamma}, \mathbf{G})$ are defined by using the canonical isomorphisms of $\mathbf{\Gamma}$, $\mathbf{G}$ and the trivial morphism $\mathbf{1} : \mathbf{\Gamma} \to \mathbf{G}$ (see [14]). In this case, they are all identity maps. Furthermore, for any object in $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$ exists an strict inverse (so that $\eta = \varepsilon =$ identity). Thus $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$ is a strict categorical group.

We are going to describe the objects and arrows of $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$.

Lemma 5.3.1. *A derivation from $\mathbf{\Gamma}$ into $\mathbf{G}$ is uniquely specified by a triple of functions $p : \Gamma_0 \to G_0$, $f : \Gamma_1 \rtimes \Gamma_0 \to G_1$ and $\varepsilon : \Gamma_0 \times \Gamma_0 \to G_1$ satisfying*

$$p(\partial'(\beta)\,\sigma) = \partial f(\beta, \sigma)\, p(\sigma); \tag{5.2}$$

$$f(\beta_1\,\beta_2, \sigma) = f(\beta_1, \partial'(\beta_2)\,\sigma)\, f(\beta_2, \sigma); \tag{5.3}$$

$$f(\beta, \sigma)\,{}^{p(\sigma)}h(\beta, {}^{\sigma}p(\partial'(\beta')\,\sigma'))\,{}^{p(\sigma)}({}^{\sigma}f(\beta', \sigma'))\,\varepsilon(\sigma, \sigma') = $$
$$= \varepsilon(\partial'(\beta)\,\sigma, \partial'(\beta')\,\sigma')\, f(\beta\,{}^{\sigma}\beta', \sigma\,\sigma'); \tag{5.4}$$

$$p(\sigma)\,{}^{\sigma}p(\tau) = \partial(\varepsilon(\sigma, \tau))\, p(\sigma\,\tau); \tag{5.5}$$

$${}^{p(\sigma)}({}^{\sigma}\varepsilon(\tau, \upsilon))\,\varepsilon(\sigma, \tau\,\upsilon) = \varepsilon(\sigma, \tau)\,\varepsilon(\sigma\,\tau, \upsilon). \tag{5.6}$$

Proof.
An object in $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$ is a functor $D : \mathbf{\Gamma} \to \mathbf{G}$ together with a family

of natural isomorphisms $\varphi = \varphi_{\sigma,\tau} : D(\sigma\,\tau) \to D(\sigma)\,{}^{\sigma}D(\tau)$, $\sigma, \tau \in \mathrm{Ob}(\mathbf{\Gamma})$, verifying a coherence condition with respect to the canonical isomorphisms of the action. The functor $D : \mathbf{\Gamma} \to \mathbf{G}$ is defined on objects

$$\begin{array}{rcl} D_0 : \Gamma_0 & \longrightarrow & G_0 \\ \sigma & \longrightarrow & p(\sigma) \end{array}$$

by a map $p : \Gamma_0 \to G_0$ and on arrows

$$\begin{array}{rcl} D_1 : \Gamma_1 \rtimes \Gamma_0 & \longrightarrow & G_1 \rtimes G_0 \\ (\beta, \sigma) & \longrightarrow & (f(\beta, \sigma), f_0(\beta, \sigma)) \end{array}$$

by a pair of functions $f : \Gamma_1 \rtimes \Gamma_0 \to G_1$ and $f_0 : \Gamma_1 \rtimes \Gamma_0 \to G_0$.
Because $D : \mathbf{\Gamma} \to \mathbf{G}$ is a functor, the following diagrams have to commute:

$$\begin{array}{ccc} \Gamma_1 \rtimes \Gamma_0 & \xrightarrow{D_1} & G_1 \rtimes G_0 \\ \downarrow{\scriptstyle s} & & \downarrow{\scriptstyle s} \\ \Gamma_0 & \xrightarrow[D_0]{} & G_0 \end{array} \qquad \begin{array}{ccc} \Gamma_1 \rtimes \Gamma_0 & \xrightarrow{D_1} & G_1 \rtimes G_0 \\ \downarrow{\scriptstyle t} & & \downarrow{\scriptstyle t} \\ \Gamma_0 & \xrightarrow[D_0]{} & G_0 \end{array}$$

- $D_0(s(\beta, \sigma)) = s(D_1(\beta, \sigma)) \quad \Rightarrow \quad p(\sigma) = f_0(\beta, \sigma)$;
- $D_0(t(\beta, \sigma)) = t(D_1(\beta, \sigma)) \quad \Rightarrow \quad p(\partial'(\beta)\,\sigma) = \partial f(\beta, \sigma)\, f_0(\beta, \sigma) \quad \Rightarrow$ $p(\partial'(\beta)\,\sigma) = \partial f(\beta, \sigma)\, p(\sigma)$.

Furthermore, the following conditions of functoriality have to be satisfied:

-
$$\begin{array}{l} D_1((\beta_1, \partial'(\beta_2)\,\sigma) \circ (\beta_2, \sigma)) = D_1(\beta_1, \partial'(\beta_2)\,\sigma) \circ D_1(\beta_2, \sigma) \quad \Rightarrow \\ D_1(\beta_1\,\beta_2, \sigma) = (f(\beta_1, \partial'(\beta_2)\,\sigma), p(\partial'(\beta_2)\,\sigma)) \circ (f(\beta_2, \sigma), p(\sigma)) \quad \Rightarrow \\ (f(\beta_1\,\beta_2, \sigma), p(\sigma)) = (f(\beta_1, \partial'(\beta_2)\,\sigma)\, f(\beta_2, \sigma), p(\sigma)) \quad \Rightarrow \\ f(\beta_1\,\beta_2, \sigma) = f(\beta_1, \partial'(\beta_2)\,\sigma)\, f(\beta_2, \sigma); \end{array} \tag{5.7}$$

- $\forall \sigma \in \Gamma_0$
  $D_1(i(\sigma)) = i(D_0(\sigma)) \quad \Rightarrow \quad D_1(1, \sigma) = (1, D_0(\sigma)) \quad \Rightarrow$
  $(f(1, \sigma), p(\sigma)) = (1, p(\sigma)) \quad \Rightarrow \quad f(1, \sigma) = 1$.

The last request is equivalent to the commutativity of the following diagram:

$$\begin{array}{ccc} \Gamma_1 \rtimes \Gamma_0 & \xrightarrow{D_1} & G_1 \rtimes G_0 \\ \uparrow{\scriptstyle i} & & \uparrow{\scriptstyle i} \\ \Gamma_0 & \xrightarrow[D_0]{} & G_0. \end{array}$$

We can observe that the relation $f(1, \sigma) = 1$ is a consequence of the identity (5.7).

A natural isomorphism φ is defined by the map p and a new function $\varepsilon : \Gamma_0 \times \Gamma_0 \to G_1$

$$D_0(\sigma\,\tau) \xrightarrow{\varphi_{\sigma,\tau}} D_0(\sigma)\,{}^{\sigma}D_0(\tau)$$

$$p(\sigma\,\tau) \xrightarrow{(\varepsilon(\sigma,\tau),p(\sigma\,\tau))} p(\sigma)\,{}^{\sigma}p(\tau)$$

and the codomain of this arrow is defined by the following condition:

- $\partial\,\varepsilon(\sigma,\tau)\,p(\sigma\,\tau) = p(\sigma)\,{}^{\sigma}p(\tau)$.

Let $\sigma \xrightarrow{(\beta,\,\sigma)} \partial'(\beta)\,\sigma$ and $\sigma' \xrightarrow{(\beta',\,\sigma')} \partial'(\beta')\,\sigma'$ be two arrows of $\mathbf{\Gamma}$, the naturalness of φ is equivalent to requiring the commutativity of the following diagram:

$$\begin{array}{ccc}
p(\sigma\,\sigma') & \xrightarrow{(\varepsilon(\sigma,\,\sigma'),\,p(\sigma\,\sigma'))} & p(\sigma)\,{}^{\sigma}p(\sigma') \\
\Big\downarrow{\scriptstyle (f(\beta\,{}^{\sigma}\beta',\,\sigma\,\sigma'),\,p(\sigma\,\sigma'))} & & \Big\downarrow{\scriptstyle (f(\beta,\sigma)\,{}^{p(\sigma)}h(\beta,\,{}^{\sigma}p(\partial'(\beta')\,\sigma'))\,{}^{p(\sigma)}({}^{\sigma}f(\beta',\,\sigma')),\,p(\sigma)\,{}^{\sigma}p(\sigma'))} \\
p(\partial'(\beta)\,\sigma\,\partial'(\beta')\,\sigma') & \xrightarrow[(\varepsilon(\partial'(\beta)\,\sigma,\,\partial'(\beta')\,\sigma'),\,p(\partial'(\beta)\,\sigma\,\partial'(\beta')\,\sigma'))]{} & p(\partial'(\beta)\,\sigma)\,{}^{\partial'(\beta)\,\sigma}p(\partial'(\beta')\,\sigma'),
\end{array}$$

therefore

- $f(\beta,\sigma)\,{}^{p(\sigma)}h(\beta,\,{}^{\sigma}p(\partial'(\beta')\,\sigma'))\,{}^{p(\sigma)}({}^{\sigma}f(\beta',\sigma'))\,\varepsilon(\sigma,\sigma') =$
  $= \varepsilon(\partial'(\beta)\,\sigma,\partial'(\beta')\,\sigma')\,f(\beta\,{}^{\sigma}\beta',\sigma\,\sigma')$.

The coherence condition is equivalent to requiring the commutativity of the following diagram:

$$\begin{array}{ccc}
p(\sigma\,\tau\,\upsilon) & \xrightarrow{(\varepsilon(\sigma,\tau\,\upsilon),p(\sigma\,\tau\,\upsilon))} & p(\sigma)\,{}^{\sigma}p(\tau\,\upsilon) \\
\Big\downarrow{\scriptstyle (\varepsilon(\sigma\,\tau,\upsilon),p(\sigma\,\tau\,\upsilon))} & & \Big\downarrow{\scriptstyle ({}^{p(\sigma)}({}^{\sigma}\varepsilon(\tau,\upsilon)),p(\sigma)\,{}^{\sigma}p(\tau\,\upsilon))} \\
p(\sigma\,\tau)\,{}^{\sigma\,\tau}p(\upsilon) & \xrightarrow[(\varepsilon(\sigma,\tau),p(\sigma\,\tau)\,{}^{\sigma\,\tau}p(\upsilon))]{} & p(\sigma)\,{}^{\sigma}p(\tau)\,{}^{\sigma\,\tau}p(\upsilon),
\end{array}$$

therefore

- ${}^{p(\sigma)}({}^{\sigma}\varepsilon(\tau,\upsilon))\,\varepsilon(\sigma,\tau\,\upsilon) = \varepsilon(\sigma,\tau)\,\varepsilon(\sigma\,\tau,\upsilon)$.

□

Proposition 5.3.1. *An arrow in the categorical group Der*$(\mathbf{\Gamma},\mathbf{G})$ *is uniquely specified by a quadruple* $(\theta,p_1,f_1,\varepsilon_1)$ *with* (p_1,f_1,ε_1) *as in Lemma 5.3.1 and an arbitrary function* $\theta : \Gamma_0 \to G_1$*. The source of* $(\theta,p_1,f_1,\varepsilon_1)$ *is the derivation from* $\mathbf{\Gamma}$ *into* $\mathbf{G}$ *given by* (p_1,f_1,ε_1)*; the target of* $(\theta,p_1,f_1,\varepsilon_1)$ *is the derivation from* $\mathbf{\Gamma}$ *into* $\mathbf{G}$ *given by the triple of functions:*

$$\begin{aligned}
&p_2(\sigma) = \partial\theta(\sigma)\,p_1(\sigma);\\
&f_2(\beta,\sigma) = \theta(\partial'(\beta)\,\sigma)\,f_1(\beta,\sigma)\,\theta(\sigma)^{-1};\\
&\varepsilon_2(\sigma,\tau) = \theta(\sigma)\,{}^{p_1(\sigma)}({}^{\sigma}\theta(\tau))\,\varepsilon_1(\sigma,\tau)\,\theta(\sigma\,\tau)^{-1}.
\end{aligned}$$

Proof.
An arrow in $\mathrm{Der}(\boldsymbol{\Gamma},\mathbf{G})$ from (p_1,f_1,ε_1) to (p_2,f_2,ε_2) is a natural transformation, that is a function $(\theta,\varphi):\Gamma_0\to G_1\rtimes G_0$ such that the square

$$\begin{array}{ccc} p_1(\sigma) & \xrightarrow{(\theta(\sigma),\varphi(\sigma))} & p_2(\sigma) \\ {\scriptstyle (f_1(\beta,\sigma),p_1(\sigma))}\downarrow & & \downarrow{\scriptstyle (f_2(\beta,\sigma),p_2(\sigma))} \\ p_1(\partial'(\beta)\,\sigma) & \xrightarrow[(\theta(\partial'(\beta)\,\sigma),\varphi(\partial'(\beta)\,\sigma))]{} & p_2(\partial'(\beta)\,\sigma) \end{array}$$

commutes in $\mathbf{G}$. Therefore

$$\varphi = p_1, \tag{5.8}$$
$$p_2(\sigma) = \partial\theta(\sigma)\,\varphi(\sigma), \tag{5.9}$$
$$f_2(\beta,\sigma)\,\theta(\sigma) = \theta(\partial'(\beta)\,\sigma)\,f_1(\beta,\sigma). \tag{5.10}$$

This natural transformation has to be compatible with ε_1 and ε_2, that is the following square:

$$\begin{array}{ccc} p_1(\sigma\,\tau) & \xrightarrow{(\varepsilon_1(\sigma,\tau),p_1(\sigma\,\tau))} & p_1(\sigma)\,{}^{\sigma}p_1(\tau) \\ {\scriptstyle (\theta(\sigma\,\tau),p_1(\sigma\,\tau))}\downarrow & & \downarrow{\scriptstyle (\theta(\sigma),p_1(\sigma))\otimes{}^{\sigma}(\theta(\tau),p_1(\tau)) = \atop = (\theta(\sigma)\,{}^{p_1(\sigma)}({}^{\sigma}\theta(\tau)),\,p_1(\sigma)\,{}^{\sigma}p_1(\tau))} \\ p_2(\sigma\,\tau) & \xrightarrow[(\varepsilon_2(\sigma,\tau),p_2(\sigma\,\tau))]{} & p_2(\sigma)\,{}^{\sigma}p_2(\tau) \end{array}$$

has to commute. Therefore

$$\theta(\sigma)\,{}^{p_1(\sigma)}({}^{\sigma}\theta(\tau))\,\varepsilon_1(\sigma,\tau) = \varepsilon_2(\sigma,\tau)\,\theta(\sigma\,\tau).$$

Thus $(\theta,p_1,f_1,\varepsilon_1)$ determines (p_2,f_2,ε_2), and it is simple to check that if $(p_1,f_1,\varepsilon_1)\in\mathrm{Ob}(\mathrm{Der}(\boldsymbol{\Gamma},\mathbf{G}))$ and if (p_2,f_2,ε_2) is defined as above, then $(p_2,f_2,\varepsilon_2)\in\mathrm{Ob}(\mathrm{Der}(\boldsymbol{\Gamma},\mathbf{G}))$.

□

$\mathrm{Der}(\boldsymbol{\Gamma},\mathbf{G})$ is a strict categorical group and the tensor product on objects (see Theorem 5.2 in [14]) is given by:

$$(p_1,f_1,\varepsilon_1)(p_2,f_2,\varepsilon_2) = (p,f,\varepsilon)$$
$$p(\sigma) = p_1(\sigma)\,p_2(\sigma) \tag{5.11}$$
$$f(\beta,\sigma) = f_1(\beta,\sigma)\,{}^{p_1(\sigma)}f_2(\beta,\sigma) \tag{5.12}$$

and ε is defined by the composition of the following sequence of arrows in

G:

$$\begin{array}{c} p_1(\sigma\,\tau)\,p_2(\sigma\,\tau) \\ \Big\downarrow{\scriptstyle (\varepsilon_1(\sigma,\tau),p_1(\sigma\,\tau))\otimes(\varepsilon_2(\sigma,\tau),p_2(\sigma\,\tau))=(\varepsilon_1(\sigma,\tau)\,{}^{p_1(\sigma\,\tau)}\varepsilon_2(\sigma,\tau),p_1(\sigma\,\tau)\,p_2(\sigma\,\tau))} \\ p_1(\sigma)\,{}^{\sigma}p_1(\tau)\,p_2(\sigma)\,{}^{\sigma}p_2(\tau) \\ \Big\downarrow{\scriptstyle (1,p_1(\sigma))\otimes(\{p_2(\sigma),\,{}^{\sigma}p_1(\tau)\},\,{}^{\sigma}p_1(\tau)\,p_2(\sigma))\otimes(1,\,{}^{\sigma}p_2(\tau)) = (\,{}^{p_1(\sigma)}\{p_2(\sigma),\,{}^{\sigma}p_1(\tau)\},p_1(\sigma)\,{}^{\sigma}p_1(\tau)\,p_2(\sigma)\,{}^{\sigma}p_2(\tau))} \\ p_1(\sigma)\,p_2(\sigma)\,{}^{\sigma}p_1(\tau)\,{}^{\sigma}p_2(\tau) \\ \Big\| \\ p_1(\sigma)\,p_2(\sigma)\,{}^{\sigma}(p_1(\tau)\,p_2(\tau)). \end{array}$$

Therefore, we have:

$$\varepsilon(\sigma,\tau) = {}^{p_1(\sigma)}\{p_2(\sigma),\,{}^{\sigma}p_1(\tau)\}\,\varepsilon_1(\sigma,\tau)\,{}^{p_1(\sigma\,\tau)}\varepsilon_2(\sigma,\tau). \tag{5.13}$$

Because $\mathrm{Der}(\mathbf{\Gamma},\mathbf{G})$ is a strict categorical group the set of objects of $\mathrm{Der}(\mathbf{\Gamma},\mathbf{G})$ is a group, thus this product is a group product.

Let

$$(p_1,f_1,\varepsilon_1) \xrightarrow{(\theta_1,p_1,f_1,\varepsilon_1)} (\overline{p}_1,\overline{f}_1,\overline{\varepsilon}_1) \quad \text{and} \quad (p_2,f_2,\varepsilon_2) \xrightarrow{(\theta_2,p_2,f_2,\varepsilon_2)} (\overline{p}_2,\overline{f}_2,\overline{\varepsilon}_2)$$

be two arrows in $\mathrm{Der}(\mathbf{\Gamma},\mathbf{G})$, where $(\overline{p}_i,\overline{f}_i,\overline{\varepsilon}_i)$ are determined by $(\theta_i,p_i,f_i,\varepsilon_i)$ under the Proposition 5.3.1 for $i=1,2$, the tensor product of these two arrows (see Theorem 5.2 in [14]) is given by:

$$\begin{array}{c} p_1(\sigma)\,p_2(\sigma) \\ \Big\downarrow{\scriptstyle (\theta_1(\sigma),p_1(\sigma))\otimes(1,p_2(\sigma))=(\theta_1(\sigma),p_1(\sigma)\,p_2(\sigma))} \\ \overline{p}_1(\sigma)\,p_2(\sigma) \\ \Big\downarrow{\scriptstyle (1,\overline{p}_1(\sigma))\otimes(\theta_2(\sigma),p_2(\sigma))=(\,{}^{\overline{p}_1(\sigma)}\theta_2(\sigma),\overline{p}_1(\sigma)\,p_2(\sigma))} \\ \overline{p}_1(\sigma)\,\overline{p}_2(\sigma). \end{array}$$

Therefore, we obtain:

$$\begin{aligned} &(\theta_1,p_1,f_1,\varepsilon_1)(\theta_2,p_2,f_2,\varepsilon_2) = (\theta,p,f,\varepsilon) \\ &\theta(\sigma) = {}^{\overline{p}_1(\sigma)}\theta_2(\sigma)\,\theta_1(\sigma) = {}^{\partial\theta_1(\sigma)\,p_1(\sigma)}\theta_2(\sigma)\,\theta_1(\sigma) = \theta_1(\sigma)\,{}^{p_1(\sigma)}\theta_2(\sigma) \end{aligned}$$

and $(p,f,\varepsilon) = (p_1,f_1,\varepsilon_1)(p_2,f_2,\varepsilon_2)$, as in (5.11), (5.12) and (5.13).

Because $\mathrm{Der}(\mathbf{\Gamma},\mathbf{G})$ is a strict categorical group it corresponds to the crossed module constructed as follows:

$$\overline{\partial} : \mathrm{Ker}s \;\; \to \;\; \mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma},\mathbf{G}))$$

with $\overline{\partial} = t_{|\mathrm{Ker}s}$, where s and t are the source and target maps, respectively, of the underlying groupoid $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$. We denote with $\mathrm{Der}_1(\mathbf{\Gamma}, \mathbf{G})$ the set of arrows in $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$ and we recall the source map:

$$\begin{array}{rcl} s : \mathrm{Der}_1(\mathbf{\Gamma}, \mathbf{G}) & \longrightarrow & \mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})) \\ (\theta_1, p_1, f_1, \varepsilon_1) & \longrightarrow & (p_1, f_1, \varepsilon_1) \end{array}$$

while the target map:

$$\begin{array}{rcl} t : \mathrm{Der}_1(\mathbf{\Gamma}, \mathbf{G}) & \longrightarrow & \mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})) \\ (\theta_1, p_1, f_1, \varepsilon_1) & \longrightarrow & (p_2, f_2, \varepsilon_2) \end{array}$$

where $(p_2, f_2, \varepsilon_2)$ as in Proposition 5.3.1.
Thus we have

$$\begin{array}{rcl} \overline{\partial} : \mathrm{Ker}s & \rightarrow & \mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})) \\ (\theta, 1, 1, 1) & \rightarrow & (p, f, \varepsilon) \end{array}$$

where

- $p(\sigma) = \partial\, \theta(\sigma)$;
- $f(\beta, \sigma) = \theta(\partial'(\beta)\, \sigma)\, \theta(\sigma)^{-1}$;
- $\varepsilon(\sigma, \tau) = \theta(\sigma)\, {}^{\sigma}\theta(\tau)\, \theta(\sigma\, \tau)^{-1}$.

The product of two arrows $(\theta_1, 1, 1, 1)$ and $(\theta_2, 1, 1, 1)$ in $\mathrm{Ker}s$ is $(\theta, 1, 1, 1)$ where $\theta(\sigma) = \theta_1(\sigma)\, \theta_2(\sigma)$ and the product in $\mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G}))$ as defined above. The inverse of the element $(p, f, \varepsilon) \in \mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G}))$ is the triple $(p^*, f^*, \varepsilon^*)$ where

$$\begin{array}{l} p^*(\sigma) = p(\sigma)^{-1}; \\ f^*(\beta, \sigma) = {}^{p(\sigma)^{-1}} f(\beta, \sigma)^{-1}; \\ \varepsilon^*(\sigma, \tau) = \{p(\sigma)^{-1},\, {}^{\sigma}p(\tau)^{-1}\}\, {}^{p(\sigma\,\tau)^{-1}}\varepsilon(\sigma, \tau)^{-1}. \end{array}$$

The action of the group $\mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G}))$ on $\mathrm{Ker}s$ is given by:

$$^{(p,f,\varepsilon)}(\theta, 1, 1, 1) = i(p, f, \varepsilon)(\theta, 1, 1, 1)(i(p, f, \varepsilon))^{-1}.$$

We recall that the map i for the groupoid $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$ is defined by:

$$\begin{array}{rcl} i : \mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})) & \longrightarrow & \mathrm{Der}_1(\mathbf{\Gamma}, \mathbf{G}) \\ (p, f, \varepsilon) & \longrightarrow & (1, p, f, \varepsilon). \end{array}$$

Therefore, using the multiplication defined above on $\mathrm{Der}_1(\mathbf{\Gamma},\mathbf{G})$ and the inverse in $\mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma},\mathbf{G}))$, we have:

$$\begin{aligned} {}^{(p,f,\varepsilon)}(\theta,1,1,1) &= (1,p,f,\varepsilon)\,(\theta,1,1,1)\,(1,p,f,\varepsilon)^{-1} \\ &= (1,p,f,\varepsilon)\,(\theta,1,1,1)\,(1,p^*,f^*,\varepsilon^*) \\ &= (\widehat{\theta},p,f,\varepsilon)\,(1,p^*,f^*,\varepsilon^*) \\ &= (\widehat{\widehat{\theta}},1,1,1) \end{aligned}$$

$$\text{where} \quad \begin{aligned} \widehat{\theta}(\sigma) &= 1(\sigma)\,{}^{p(\sigma)}\theta(\sigma) = {}^{p(\sigma)}\theta(\sigma); \\ \widehat{\widehat{\theta}}(\sigma) &= \widehat{\theta}(\sigma)\,{}^{p(\sigma)}1(\sigma) = \widehat{\theta}(\sigma) = {}^{p(\sigma)}\theta(\sigma). \end{aligned}$$

Because $\mathrm{Ker}s$ is isomorphic to $\mathrm{App}(\Gamma_0,G_1)$, it is clear the isomorphism between $\overline{\partial}$ and a homomorphism

$$\overline{\partial}:\mathrm{App}(\Gamma_0,G_1) \;\to\; \mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma},\mathbf{G}))$$

which, by abuse of notation, we have denoted again by $\overline{\partial}$.

Remark 5.3.1. *[27] If $\partial:G_1\to G_0$ is a symmetric crossed module, then $\overline{\partial}:App(\Gamma_0,G_1)\to Ob(Der(\mathbf{\Gamma},\mathbf{G}))$ is a symmetric crossed module where the braiding*

$$\{-,-\}:Ob(Der(\mathbf{\Gamma},\mathbf{G}))\times Ob(Der(\mathbf{\Gamma},\mathbf{G}))\to App(\Gamma_0,G_1)$$

is determined by $\{(p_1,f_1,\varepsilon_1),(p_2,f_2,\varepsilon_2)\}(\sigma)=\{p_1(\sigma),p_2(\sigma)\}$. However, if $\{-,-\}$ is just a braiding in $\partial:G_1\to G_0$ (but not a symmetry), then $\overline{\partial}:App(\Gamma_0,G_1)\to Ob(Der(\mathbf{\Gamma},\mathbf{G}))$ is not a braided crossed module.

Now we are going to describe the structure of $\mathcal{H}^0(\mathbf{\Gamma},\mathbf{G})$, presented in [14]. $\mathcal{H}^0(\mathbf{\Gamma},\mathbf{G})$ is well-known to be equivalent to the categorical group of $\mathbf{\Gamma}$-invariant objects $\mathbf{G}^{\mathbf{\Gamma}}$ (see [26]). The associativity a, left unit l and right unit r of the monoidal structure of $\mathbf{G}^{\mathbf{\Gamma}}$ are given by the respective constraints a, l and r of $\mathbf{G}$ and in this case they are all identity maps. Furthermore, for any object in $\mathbf{G}^{\mathbf{\Gamma}}$ there exists a strict inverse. Thus $\mathbf{G}^{\mathbf{\Gamma}}$ is a strict categorical group.

Lemma 5.3.2. *A $\mathbf{\Gamma}$-invariant object of $\mathbf{G}$ is uniquely specified by a pair (g,θ), with $g\in G_0$ and a function $\theta:\Gamma_0\to G_1$ satisfying*

$$\partial\,\theta(\sigma)=g\,{}^{\sigma}g^{-1} \tag{5.14}$$

$$\theta(\sigma\,\tau)=\theta(\sigma)\,{}^{\sigma}\theta(\tau) \tag{5.15}$$

$$\theta(\sigma)=\theta(\partial'(\beta)\,\sigma)\,h(\beta,{}^{\sigma}g) \tag{5.16}$$

Proof.
An object in $\mathbf{G}^{\mathbf{\Gamma}}$ (see [14]) is a pair consisting of an object $g \in G_0$ and a natural transformation, that is a function $(\theta, \varphi) : \Gamma_0 \to G_1 \rtimes G_0$ such that the square

$$\begin{array}{ccc} {}^{\sigma}g & \xrightarrow{(\theta(\sigma),\varphi(\sigma))} & g \\ {\scriptstyle (h(\beta,{}^{\sigma}g),{}^{\sigma}g)}\downarrow & & \| {\scriptstyle (1,g)} \\ {}^{\partial'(\beta)\,\sigma}g & \xrightarrow[(\theta(\partial'(\beta)\,\sigma),\varphi(\partial'(\beta)\,\sigma))]{} & g \end{array}$$

commutes in $\mathbf{G}$. Therefore we have:

$$\begin{aligned} &\varphi(\sigma) = {}^{\sigma}g; \\ &g = \partial\,\theta(\sigma)\,\varphi(\sigma) \quad \Rightarrow \quad g = \partial\,\theta(\sigma)\,{}^{\sigma}g; \\ &\theta(\sigma) = \theta(\partial'(\beta)\,\sigma)\,h(\beta, {}^{\sigma}g). \end{aligned}$$

Furthermore, the following square has to commute:

$$\begin{array}{ccc} {}^{\sigma\,\tau}g & \xrightarrow{(\theta(\sigma\,\tau),\,{}^{\sigma\,\tau}g)} & g \\ \| & & \uparrow {\scriptstyle (\theta(\sigma),\,{}^{\sigma}g)} \\ {}^{\sigma}({}^{\tau}g) & \xrightarrow[{}^{\sigma}(\theta(\tau),\,{}^{\tau}g)=({}^{\sigma}\theta(\tau),\,{}^{\sigma}({}^{\tau}g))]{} & {}^{\sigma}g, \end{array}$$

therefore we obtain:

$$\theta(\sigma\,\tau) = \theta(\sigma)\,{}^{\sigma}\theta(\tau).$$

□

Proposition 5.3.2. *An arrow in the categorical group* $\mathbf{G}^{\mathbf{\Gamma}}$ *is uniquely specified by a triple* (α, g_1, θ_1) *with* (g_1, θ_1) *as in Lemma 5.3.2 and an element* $\alpha \in G_1$*. The source of* (α, g_1, θ_1) *is the* $\mathbf{\Gamma}$*-invariant object of* $\mathbf{G}$ *given by* (g_1, θ_1)*; the target of* (α, g_1, θ_1) *is the* $\mathbf{\Gamma}$*-invariant object of* $\mathbf{G}$ *given by* (g_2, θ_2) *where* $g_2 = \partial(\alpha)\,g_1$ *and* $\theta_2(\sigma) = \alpha\,\theta_1(\sigma)\,{}^{\sigma}\alpha^{-1}$*.*

Proof.
An arrow in $\mathbf{G}^{\mathbf{\Gamma}}$ from (g_1, θ_1) to (g_2, θ_2) is an arrow $g_1 \xrightarrow{(\alpha,g_1)} g_2$ in $\mathbf{G}$ such that the square

$$\begin{array}{ccc} {}^{\sigma}g_1 & \xrightarrow{(\theta_1(\sigma),\,{}^{\sigma}g_1)} & g_1 \\ {\scriptstyle {}^{\sigma}(\alpha,g_1)=({}^{\sigma}\alpha,\,{}^{\sigma}g_1)}\downarrow & & \downarrow {\scriptstyle (\alpha,g_1)} \\ {}^{\sigma}g_2 & \xrightarrow[(\theta_2(\sigma),\,{}^{\sigma}g_2)]{} & g_2 \end{array}$$

commutes in $\mathbf{G}$. Therefore we have:

$$g_2 = \partial(\alpha)\, g_1;$$
$$\alpha\, \theta_1(\sigma) = \theta_2(\sigma)\, {}^{\sigma}\alpha.$$

Thus (α, g_1, θ_1) determines (g_2, θ_2), and it is simple to check that if $(g_1, \theta_1) \in \mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}})$ and if (g_2, θ_2) is defined as above, then $(g_2, \theta_2) \in \mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}})$.

□

$\mathbf{G}^{\mathbf{\Gamma}}$ is a strict categorical group and the tensor product on objects (see [26]) is given by:

$$(g_1, \theta_1)\,(g_2, \theta_2) = (g_1\, g_2, \theta)$$

$$\begin{array}{ccc} {}^{\sigma}g_1 & {}^{\sigma}g_2 & {}^{\sigma}g_1\, {}^{\sigma}g_2 \\ \Big\downarrow{\scriptstyle (\theta_1(\sigma),\, {}^{\sigma}g_1)} & \Big\downarrow{\scriptstyle (\theta_2(\sigma),\, {}^{\sigma}g_2)} & \Big\downarrow{\scriptstyle (\theta_1(\sigma)\, {}^{{}^{\sigma}g_1}\theta_2(\sigma),\, {}^{\sigma}g_1\, {}^{\sigma}g_2)} \\ g_1 & g_2 & g_1\, g_2 \end{array}$$

where

$$\begin{aligned} \theta(\sigma) &= \theta_1(\sigma)\, {}^{{}^{\sigma}g_1}\theta_2(\sigma) = {}^{\partial\theta_1(\sigma)\, {}^{\sigma}g_1}\theta_2(\sigma)\, \theta_1(\sigma) = {}^{g_1\, {}^{\sigma}g_1^{-1}\, {}^{\sigma}g_1}\theta_2(\sigma)\, \theta_1(\sigma) = \\ &= {}^{g_1}\theta_2(\sigma)\, \theta_1(\sigma). \end{aligned}$$

Because $\mathbf{G}^{\mathbf{\Gamma}}$ is a strict categorical group the set of objects of $\mathbf{G}^{\mathbf{\Gamma}}$ is a group, thus this product is a group product.

Let

$$(g_1, \theta_1) \xrightarrow{(\alpha_1, g_1, \theta_1)} (\overline{g}_1, \overline{\theta}_1) \quad \text{and} \quad (g_2, \theta_2) \xrightarrow{(\alpha_2, g_2, \theta_2)} (\overline{g}_2, \overline{\theta}_2)$$

be two arrows in $\mathbf{G}^{\mathbf{\Gamma}}$, where $(\overline{g}_i, \overline{\theta}_i)$ are determined by $(\alpha_i, g_i, \theta_i)$ under the Proposition 5.3.2 for $i = 1, 2$, the tensor product of these two arrows is given by:

$$(\alpha_1, g_1, \theta_1)\,(\alpha_2, g_2, \theta_2) = (\alpha_1\, {}^{g_1}\alpha_2, g_1\, g_2, \theta)$$

where θ is defined as above.

Because $\mathbf{G}^{\mathbf{\Gamma}}$ is a strict categorical group it corresponds to the crossed module constructed as follows:

$$\overline{\overline{\partial}} : \mathrm{Ker}s \;\; \to \;\; \mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}})$$

with $\overline{\overline{\partial}} = t_{|\mathrm{Ker}s}$, where s and t are the source and target maps, respectively, of the underlying groupoid $\mathbf{G}^{\mathbf{\Gamma}}$. We denote with $\mathbf{G}_1^{\mathbf{\Gamma}}$ the set of arrows in $\mathbf{G}^{\mathbf{\Gamma}}$ and we recall the source map:

$$\begin{aligned} s : \mathbf{G}_1^{\mathbf{\Gamma}} &\longrightarrow \mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}}) \\ (\alpha_1, g_1, \theta_1) &\longrightarrow (g_1, \theta_1) \end{aligned}$$

while the target map:

$$\begin{array}{rcl} t:\mathbf{G}_1^{\mathbf{\Gamma}} & \longrightarrow & \mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}}) \\ (\alpha_1, g_1, \theta_1) & \longrightarrow & (g_2, \theta_2) \end{array}$$

where (g_2,θ_2) as in Proposition 5.3.2.
Thus we have

$$\begin{array}{rcl} \overline{\overline{\partial}}:\mathrm{Ker}s & \rightarrow & \mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}}) \\ (\alpha, 1, 1) & \rightarrow & (\partial(\alpha), \theta_\alpha) \end{array}$$

where $\theta_\alpha(\sigma) = \alpha\, {}^{\sigma}\alpha^{-1}$. The product of two arrows $(\alpha_1, 1, 1)$ and $(\alpha_2, 1, 1)$ in $\mathrm{Ker}s$ is $(\alpha_1\,\alpha_2, 1, 1)$ and the product in $\mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}})$ is defined as above. The inverse of the element $(g,\theta) \in \mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}})$ is the pair (g^{-1}, θ^*) where

$$\theta^*(\sigma) = {}^{g^{-1}}\theta(\sigma)^{-1}.$$

The action of the group $\mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}})$ on $\mathrm{Ker}s$ is given by:

$${}^{(g,\theta)}(\alpha, 1, 1) = i(g,\theta)(\alpha,1,1)(i(g,\theta))^{-1}.$$

We recall that the map i for the groupoid $\mathbf{G}^{\mathbf{\Gamma}}$ is given by:

$$\begin{array}{rcl} i:\mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}}) & \longrightarrow & \mathbf{G}_1^{\mathbf{\Gamma}} \\ (g,\theta) & \longrightarrow & (1, g, \theta). \end{array}$$

Therefore, using the multiplication defined above on $\mathbf{G}_1^{\mathbf{\Gamma}}$ and the inverse in $\mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}})$, we have:

$$\begin{array}{rcl} {}^{(g,\theta)}(\alpha,1,1) & = & (1,g,\theta)\,(\alpha,1,1)\,(1,g,\theta)^{-1} = (1,g,\theta)\,(\alpha,1,1)\,(1,g^{-1},\theta^*) = \\ & = & ({}^{g}\alpha, g, \theta)\,(1, g^{-1}, \theta^*) = ({}^{g}\alpha, 1, 1). \end{array}$$

Because $\mathrm{Ker}s$ is isomorphic to G_1, it is clear the isomorphism between $\overline{\overline{\partial}}$ and a homomorphism

$$\overline{\overline{\partial}}: G_1 \rightarrow \mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}})$$

which, by abuse of notation, we have denoted again by $\overline{\overline{\partial}}$.

$\mathcal{H}^0(\mathbf{\Gamma}, \mathbf{G})$ is defined by the kernel (see [14]) as follows:

$$\mathcal{H}^0(\mathbf{\Gamma}, \mathbf{G}) = \ker(\overline{\mathbf{T}}: \mathbf{G} \rightarrow \mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})).$$

In this case, the functor $\overline{\mathbf{T}}: \mathbf{G} \rightarrow \mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$ is defined on objects and on arrows by

$$\begin{array}{rcl} \overline{T}_0: G_0 & \longrightarrow & \mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})) \\ g & \longrightarrow & (p_g, f_g, 1) \end{array} \quad \begin{array}{rcl} \overline{T}_1: G_1 \rtimes G_0 & \longrightarrow & \mathrm{Der}_1(\mathbf{\Gamma}, \mathbf{G}) \\ (\alpha, g) & \longrightarrow & (\theta, p_g, f_g, 1) \end{array}$$

respectively, where $p_g(\sigma) = g\,{}^{\sigma}g^{-1}$, $f_g(\beta,\sigma) = {}^{g}h(\beta, {}^{\sigma}g^{-1})$ and $\theta(\sigma) = \alpha\,{}^{g\,{}^{\sigma}g^{-1}}({}^{\sigma}\alpha^{-1})$.
There are natural isomorphisms $\overline{\nu}$ and $\overline{\chi}$ such that $(\mathbf{G}, \overline{\mathbf{T}}, \overline{\nu}, \overline{\chi})$ is a categorical $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$-crossed module (see [14]).
In this case, we observe that the isomorphism $\overline{\chi}$ is given by the composition of the three morphisms:

$$ {}^{\overline{T}_0(g_1)}g_2\, g_1 = g_1\, g_1^{-1}\, g_2\, g_1 \xrightarrow[= ({}^{g_1}\{g_2, g_1^{-1}\},\, g_1\, g_1^{-1}\, g_2\, g_1)]{(1, g_1)\otimes(\{g_2, g_1^{-1}\},\, g_1^{-1}\, g_2)\otimes(1, g_1) =} g_1\, g_2\, g_1^{-1}\, g_1 = g_1\, g_2 $$

therefore $\overline{\chi}_{g_1,g_2} = ({}^{g_1}\{g_2, g_1^{-1}\}, g_1\, g_1^{-1}\, g_2\, g_1)$.
Thanks to this observation $\mathcal{H}^0(\mathbf{\Gamma}, \mathbf{G})$ can be equipped with a braiding (see Proposition 2.7 in [14]) given by

$$ g_2\, g_1 = g_2\, g_1 = {}^{\overline{T}_0(g_1)}g_2\, g_1 \xrightarrow{\overline{\chi}_{g_1,g_2}} g_1\, g_2 $$

Then $\overline{\overline{\partial}} : G_1 \to \mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}})$ is also a braided crossed module with braiding defined by

$$ \{(g_1, \theta_1), (g_2, \theta_2)\} = {}^{g_1}\{g_2, g_1^{-1}\} = \{g_2, g_1\}^{-1}. \tag{5.17} $$

Moreover, we have that this braiding is equivariant respect an action of $\partial' : \Gamma_1 \to \Gamma_0$ (see the following Proposition).

Proposition 5.3.3. *The crossed module $\overline{\overline{\partial}} : G_1 \to Ob(\mathbf{G}^{\mathbf{\Gamma}})$ is a braided crossed module equivariant respect the action of $\partial' : \Gamma_1 \to \Gamma_0$ defined as follows:*

- *the action of Γ_0 on G_1 is induced by the action of $\partial' : \Gamma_1 \to \Gamma_0$ on $\partial : G_1 \to G_0$;*
- *the action of Γ_0 on $Ob(\mathbf{G}^{\mathbf{\Gamma}})$ is defined by ${}^{\sigma}(g, \theta) = ({}^{\sigma}g, \overline{\theta})$, where $\overline{\theta}(\tau) = {}^{\sigma}\theta(\sigma^{-1}\,\tau\,\sigma)$;*
- *the map $\overline{h} : \Gamma_1 \times Ob(\mathbf{G}^{\mathbf{\Gamma}}) \to G_1$ is defined by $\overline{h}(\beta, (g, \theta)) = h(\beta, g)$ where the function h is given by the action of $\partial' : \Gamma_1 \to \Gamma_0$ on $\partial : G_1 \to G_0$.*

Proof.
First of all, we are going to show that the action of Γ_0 on $\mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}})$ is well defined.

- $({}^{\sigma}g, \overline{\theta}) \in \mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}})$, in fact, we have:

$$\begin{aligned}
\overline{\theta}(\tau\,\overline{\tau}) &= {}^{\sigma}\theta(\sigma^{-1}\,\tau\,\overline{\tau}\,\sigma) = {}^{\sigma}\theta(\sigma^{-1}\,\tau\,\sigma\,\sigma^{-1}\,\overline{\tau}\,\sigma) = \\
&= {}^{\sigma}\theta(\sigma^{-1}\,\tau\,\sigma)\,{}^{\tau\,\sigma}\theta(\sigma^{-1}\,\overline{\tau}\,\sigma) = \overline{\theta}(\tau)\,{}^{\tau}\overline{\theta}(\overline{\tau}); \\
\partial\overline{\theta}(\tau)^{-1}\,{}^{\sigma}g &= \partial(\,{}^{\sigma}\theta(\sigma^{-1}\,\tau\,\sigma))^{-1}\,{}^{\sigma}g = {}^{\sigma}\partial\theta(\sigma^{-1}\,\tau\,\sigma)^{-1}\,{}^{\sigma}g = \\
&= {}^{\sigma}(\,{}^{\sigma^{-1}\,\tau\,\sigma}g\,g^{-1})\,{}^{\sigma}g = {}^{\tau\,\sigma}g = {}^{\tau}(\,{}^{\sigma}g); \\
\overline{\theta}(\tau) &= {}^{\sigma}\theta(\sigma^{-1}\tau\,\sigma) = {}^{\sigma}\theta(\partial'(\,{}^{\sigma^{-1}}\beta)\,\sigma^{-1}\,\tau\,\sigma)\,{}^{\sigma}h(\,{}^{\sigma^{-1}}\beta,\,{}^{\sigma^{-1}\tau\,\sigma}g) = \\
&= {}^{\sigma}\theta(\sigma^{-1}\,\partial'(\beta)\,\tau\,\sigma)\,h(\beta,\,{}^{\tau\,\sigma}g) = \overline{\theta}(\partial'(\beta)\,\tau)\,h(\beta,\,{}^{\tau}(\,{}^{\sigma}g)).
\end{aligned}$$

- Γ_0 acts on $\mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}})$, in fact, we have:

$$\begin{aligned}
{}^{\sigma_1}(\,{}^{\sigma_2}(g,\theta)) &= {}^{\sigma_1}(\,{}^{\sigma_2}g,\overline{\theta}) = (\,{}^{\sigma_1}(\,{}^{\sigma_2}g),\overline{\overline{\theta}}) = (\,{}^{\sigma_1\,\sigma_2}g,\overline{\overline{\theta}}); \\
\overline{\theta}(\tau) &= {}^{\sigma_2}\theta(\sigma_2^{-1}\,\tau\,\sigma_2); \\
\overline{\overline{\theta}}(\tau) &= {}^{\sigma_1}\overline{\theta}(\sigma_1^{-1}\,\tau\,\sigma_1) = {}^{\sigma_1}(\,{}^{\sigma_2}\theta(\sigma_2^{-1}\,\sigma_1^{-1}\,\tau\,\sigma_1\sigma_2)) = \\
&= {}^{\sigma_1\,\sigma_2}\theta(\sigma_2^{-1}\,\sigma_1^{-1}\,\tau\,\sigma_1\sigma_2); \\
{}^{\sigma_1\,\sigma_2}(g,\theta) &= (\,{}^{\sigma_1\,\sigma_2}g,\widetilde{\theta}); \\
\widetilde{\theta}(\tau) &= {}^{\sigma_1\,\sigma_2}\theta(\,\sigma_2^{-1}\,\sigma_1^{-1}\,\tau\,\sigma_1\,\sigma_2) = \overline{\overline{\theta}}(\tau); \\
{}^{\sigma}(g_1,\theta_1)\,{}^{\sigma}(g_2,\theta_2) &= (\,{}^{\sigma}g_1,\overline{\theta}_1)\,(\,{}^{\sigma}g_2,\overline{\theta}_2) = (\,{}^{\sigma}g_1\,{}^{\sigma}g_2,\,{}^{\sigma g_1}\overline{\theta}_2\,\overline{\theta}_1); \\
{}^{\sigma g_1}\overline{\theta}_2\,\overline{\theta}_1(\tau) &= {}^{\sigma g_1}\overline{\theta}_2(\tau)\,\overline{\theta}_1(\tau) = {}^{\sigma g_1}(\,{}^{\sigma}\theta_2(\sigma^{-1}\,\tau\,\sigma))\,{}^{\sigma}\theta_1(\sigma^{-1}\,\tau\,\sigma); \\
{}^{\sigma}[(g_1,\theta_1)(g_2,\theta_2)] &= {}^{\sigma}(g_1\,g_2,\,{}^{g_1}\theta_2\,\theta_1) = (\,{}^{\sigma}(g_1\,g_2),\widehat{\theta}); \\
\widehat{\theta}(\tau) &= {}^{\sigma}({}^{g_1}\theta_2\,\theta_1)(\sigma^{-1}\,\tau\,\sigma) = {}^{\sigma}({}^{g_1}\theta_2(\sigma^{-1}\,\tau\,\sigma)\,\theta_1(\sigma^{-1}\,\tau\,\sigma)) = \\
&= {}^{\sigma g_1}(\,{}^{\sigma}\theta_2(\sigma^{-1}\,\tau\,\sigma))\,{}^{\sigma}\theta_1(\sigma^{-1}\,\tau\,\sigma) = ({}^{\sigma g_1}\overline{\theta}_2\,\overline{\theta}_1)(\tau).
\end{aligned}$$

Secondly, we are going to check the five conditions making the action of $\partial':\Gamma_1\to\Gamma_0$ on $\overline{\overline{\partial}}:G_1\to\mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}})$ a good action (see section 1.3):

(i) $\overline{h}(\beta_1\,\beta_2,(g,\theta)) = h(\beta_1\,\beta_2,g) = {}^{\beta_1}h(\beta_2,g)\,h(\beta_1,g) =$
$= {}^{\beta_1}\overline{h}(\beta_2,(g,\theta))\,\overline{h}(\beta_1,(g,\theta))$.

(ii) $\overline{h}(\beta,(g_1,\theta_1)\,(g_2,\theta_2)) = \overline{h}(\beta,(g_1\,g_2,\,{}^{g_1}\theta_2\,\theta_1) = h(\beta,g_1\,g_2) =$
$= h(\beta,g_1)\,{}^{g_1}h(\beta,g_2) = \overline{h}(\beta,(g_1,\theta_1))\,{}^{(g_1,\theta_1)}\overline{h}(\beta,(g_2,\theta_2))$.

(iv) $\overline{\overline{\partial}}(\,{}^{\sigma}\alpha) = (\partial(\,{}^{\sigma}\alpha),\theta_{\,{}^{\sigma}\alpha})$;
${}^{\sigma}\overline{\overline{\partial}}(\alpha) = {}^{\sigma}(\partial(\alpha),\theta_\alpha) = (\,{}^{\sigma}\partial(\alpha),\overline{\theta}_\alpha) = (\partial(\,{}^{\sigma}\alpha),\overline{\theta}_\alpha)$;
${}^{\sigma}({}^{(g,\theta)}\alpha) = {}^{\sigma}({}^{g}\alpha) = {}^{\sigma g}({}^{\sigma}\alpha) = {}^{({}^{\sigma}g,\overline{\theta})}({}^{\sigma}\alpha) = {}^{\sigma(g,\theta)}({}^{\sigma}\alpha)$;

where $\overline{\theta}_\alpha(\tau) = {}^{\sigma}\theta_\alpha(\sigma^{-1}\,\tau\,\sigma) = {}^{\sigma}(\alpha^{\,\sigma^{-1}\,\tau\,\sigma}\alpha^{-1}) = {}^{\sigma}\alpha^{\,\tau\,\sigma}\alpha^{-1} = \theta_{\,{}^{\sigma}\alpha}(\tau)$.

(v) $\overline{h}(\beta,\overline{\overline{\partial}}(\alpha)) = \overline{h}(\beta,(\partial(\alpha),\theta_\alpha)) = h(\beta,\partial(\alpha)) = {}^{\beta}\alpha\,\alpha^{-1}$;
$\overline{\overline{\partial}}\,\overline{h}(\beta,(g,\theta)) = \overline{\overline{\partial}}\,h(\beta,g) = (\partial\,h(\beta,g),\theta_{h(\beta,g)}) = (\,{}^{\partial'(\beta)}g\,g^{-1},\theta_{h(\beta,g)})$;
${}^{\partial'(\beta)}(g,\theta)\,(g,\theta)^{-1} = (\,{}^{\partial'(\beta)}g,\overline{\theta})\,(g^{-1},\theta^*) = (\,{}^{\partial'(\beta)}g\,g^{-1},\,{}^{\partial'(\beta)g}\theta^*\,\overline{\theta})$;

where

$$
\begin{aligned}
({}^{\partial'(\beta)g}\theta^{*}\,\overline{\theta})(\sigma) &= {}^{\partial'(\beta)g}\theta^{*}(\sigma)\,\overline{\theta}(\sigma) = \\
&= {}^{\partial'(\beta)g\,g^{-1}}\theta(\sigma)^{-1}\,{}^{\partial'(\beta)}\theta(\partial'(\beta)^{-1}\,\sigma\,\partial'(\beta)) = \\
&= {}^{\partial\theta(\partial'(\beta))^{-1}}\theta(\sigma)^{-1}\,{}^{\partial'(\beta)}\theta(\partial'(\beta)^{-1}\,\sigma\,\partial'(\beta)) = \\
&= \theta(\partial'(\beta))^{-1}\,\theta(\sigma)^{-1}\,\theta(\partial'(\beta))\,{}^{\partial'(\beta)}\theta(\partial'(\beta)^{-1})\,\theta(\sigma\,\partial'(\beta)) = \\
&= \theta(\partial'(\beta))^{-1}\,\theta(\sigma)^{-1}\,\theta(\partial'(\beta))\,\theta(\partial'(\beta))^{-1}\,\theta(\sigma\,\partial'(\beta)) = \\
&= \theta(\partial'(\beta))^{-1}\,\theta(\sigma)^{-1}\,\theta(\sigma\,\partial'(\beta)) = \theta(\partial'(\beta))^{-1}\,{}^{\sigma}\theta(\,\partial'(\beta)) = \\
&= h(\beta,g)\,{}^{\sigma}h(\beta,g)^{-1} = \theta_{h(\beta,g)}(\sigma).
\end{aligned}
$$

(vi) $\overline{h}({}^{\sigma}\beta,\,{}^{\sigma}(g,\theta)) = \overline{h}({}^{\sigma}\beta,({}^{\sigma}g,\overline{\theta})) = h({}^{\sigma}\beta,\,{}^{\sigma}g) = {}^{\sigma}h(\beta,g) =$
$= {}^{\sigma}\overline{h}(\beta,(g,\theta))$.

Finally, we are going to prove that the braiding defined in (5.17) is equivariant respect the action of $\partial' : \Gamma_1 \to \Gamma_0$ on $\overline{\overline{\partial}} : G_1 \to \mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}})$:

$$
\begin{aligned}
{}^{\sigma}\{(g_1,\theta_1),(g_2,\theta_2)\} &= {}^{\sigma}\{g_2,g_1\}^{-1} = \{{}^{\sigma}g_2,\,{}^{\sigma}g_1\}^{-1} = \\
&= \{({}^{\sigma}g_1,\overline{\theta}_1),({}^{\sigma}g_2,\overline{\theta}_2)\} = \{{}^{\sigma}(g_1,\theta_1),\,{}^{\sigma}(g_2,\theta_2)\}.
\end{aligned}
$$

□

In the symmetric case we have $\{(g_1,\theta_1),(g_2,\theta_2)\} = \{g_2,g_1\}^{-1} = \{g_1,g_2\}$ and $\overline{\overline{\partial}} : G_1 \to \mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}})$ becomes a symmetric crossed module.

Remark 5.3.2. *Here we want to show (as discussed in section 4.1) that $Ob(\mathbf{G}^{\mathbf{\Gamma}})$ coincides with the pullback of the pair of maps:*

$$
\begin{array}{ccc}
 & & App(\Gamma_0,G_1) \\
 & & \downarrow \overline{\partial} \\
G_0 & \xrightarrow{\overline{T}_0} & Ob(Der(\mathbf{\Gamma},\mathbf{G}))
\end{array}
$$

indeed, we have:

$$
\begin{aligned}
G_0\times_{Ob(Der(\mathbf{\Gamma},\mathbf{G}))}App(\Gamma_0,G_1) &= \{(g,\theta)\in G_0\times App(\Gamma_0,G_1)\ :\ \overline{T}_0(g)=\overline{\partial}(\theta)\} \\
&= \{(g,\theta)\in G_0\times App(\Gamma_0,G_1)\ : \\
&\qquad g\,{}^{\sigma}g^{-1} = \partial\,\theta(\sigma) \\
&\qquad {}^{g}h(\beta,\,{}^{\sigma}g^{-1}) = \theta(\partial'(\beta)\,\sigma)\,\theta(\sigma)^{-1} \\
&\qquad 1 = \theta(\sigma)\,{}^{\sigma}\theta(\tau)\,\theta(\sigma\,\tau)^{-1}\} = \\
&= \{(g,\theta)\in G_0\times App(\Gamma_0,G_1)\ : \\
&\qquad \partial\,\theta(\sigma) = g\,{}^{\sigma}g^{-1} \\
&\qquad \theta(\sigma\,\tau) = \theta(\sigma)\,{}^{\sigma}\theta(\tau) \\
&\qquad \theta(\partial'(\beta)\,\sigma) = {}^{g}h(\beta,\,{}^{\sigma}g^{-1})\,\theta(\sigma)\}.
\end{aligned}
$$

The third condition becomes:

$$\theta(\partial'(\beta)\,\sigma) = {}^{\partial\,\theta(\sigma)\,{}^{\sigma}g}h(\beta,\,{}^{\sigma}g^{-1})\,\theta(\sigma) = \theta(\sigma)\,h(\beta,\,{}^{\sigma}g)^{-1}.$$

Then we have $\theta(\sigma) = \theta(\partial'(\beta)\,\sigma)\,h(\beta,\,{}^{\sigma}g)$.

Now we can consider $\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G})$ as defined in [14]. $\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G})$ is a quotient categorical group defined in the following way:

$$\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G}) = \frac{\mathrm{Der}(\mathbf{\Gamma},\mathbf{G})}{<\mathbf{G},\overline{\overline{\mathbf{T}}}>}.$$

We have $\mathrm{Ob}(\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G})) = \mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma},\mathbf{G}))$ and the tensor product on objects in $\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G})$ is the same defined in $\mathrm{Der}(\mathbf{\Gamma},\mathbf{G})$. Then $\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G})$ is a strict categorical group because $\mathrm{Der}(\mathbf{\Gamma},\mathbf{G})$ is a strict categorical group.

We are going to describe the morphisms in $\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G})$.

Proposition 5.3.4. *A premorphism in* $\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G})$ *is uniquely specified by* $(g,\theta,p_2,f_2,\varepsilon_2)$ *with* $(p_2,f_2,\varepsilon_2) \in Ob(Der(\mathbf{\Gamma},\mathbf{G}))$, $g \in G_0$ *and a function* $\theta : \Gamma_0 \to G_1$. *The target of* $(g,\theta,p_2,f_2,\varepsilon_2)$ *is* (p_2,f_2,ε_2) *and the source of* $(g,\theta,p_2,f_2,\varepsilon_2)$ *is given by* (p_1,f_1,ε_1) *where*

$$p_1(\sigma) = \partial\,\theta(\sigma)^{-1}\,g\,{}^{\sigma}g^{-1}\,p_2(\sigma); \tag{5.18}$$

$$f_1(\beta,\sigma) = \theta(\partial'(\beta)\,\sigma)^{-1}\,{}^{g}h(\beta,\,{}^{\sigma}g^{-1})\,{}^{g\,{}^{\sigma}g^{-1}}f_2(\beta,\sigma)\,\theta(\sigma); \tag{5.19}$$

$$\begin{aligned}\varepsilon_1(\sigma,\tau) = {} & \theta(\sigma)^{-1}\,{}^{g\,{}^{\sigma}g^{-1}\,p_2(\sigma)}({}^{\sigma}\theta(\tau)^{-1})\,{}^{g\,{}^{\sigma}g^{-1}}\{p_2(\sigma),\,{}^{\sigma}g\,{}^{\sigma\tau}g^{-1}\}\\ & {}^{g\,{}^{\sigma\tau}g^{-1}}\varepsilon_2(\sigma,\tau)\,\theta(\sigma\,\tau).\end{aligned} \tag{5.20}$$

Proof.
A premorphism in $\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G})$ from (p_1,f_1,ε_1) to (p_2,f_2,ε_2) is a pair of an object $g \in G_0$ and an arrow in $\mathrm{Der}(\mathbf{\Gamma},\mathbf{G})$ from (p_1,f_1,ε_1) to $(p_g,f_g,1)(p_2,f_2,\varepsilon_2)$, that is (see Proposition 5.3.1) an arbitrary function $\theta : \Gamma_0 \to G_1$ such that:

$$\begin{aligned}& g\,{}^{\sigma}g^{-1}\,p_2(\sigma) = \partial\,\theta(\sigma)\,p_1(\sigma);\\ & {}^{g}h(\beta,\,{}^{\sigma}g^{-1})\,{}^{g\,{}^{\sigma}g^{-1}}f_2(\beta,\sigma) = \theta(\partial'(\beta)\,\sigma)\,f_1(\beta,\sigma)\,\theta(\sigma)^{-1};\\ & {}^{g\,{}^{\sigma}g^{-1}}\{p_2(\sigma),\,{}^{\sigma}g\,{}^{\sigma\tau}g^{-1}\}\,{}^{g\,{}^{\sigma\tau}g^{-1}}\varepsilon_2(\sigma,\tau) =\\ & = \theta(\sigma)\,{}^{p_1(\sigma)}({}^{\sigma}\theta(\tau))\,\varepsilon_1(\sigma,\tau)\,\theta(\sigma\,\tau)^{-1}.\end{aligned}$$

From this we obtain the three relations introduced in the Proposition with same easy computations.

$\square$

Definition 5.3.1. *A morphism in* $\mathcal{H}^1(\mathbf{\Gamma}, \mathbf{G})$ *from* $(p_1, f_1, \varepsilon_1)$ *to* $(p_2, f_2, \varepsilon_2)$ *is a class of premorphisms* $[g, \theta, p_2, f_2, \varepsilon_2]$ *where* $(g, \theta, p_2, f_2, \varepsilon_2)$ *and* $(g', \theta', p_2, f_2, \varepsilon_2)$ *are equivalent if there is an arrow in* $\mathbf{G}$ *from* g *to* g'*, that is an* $\alpha \in G_1$ *such that* $g' = \partial(\alpha)\, g$ *and the diagram*

$$\begin{array}{ccc} p_1(\sigma) & \xrightarrow{(\theta(\sigma), p_1(\sigma))} & g\,{}^{\sigma}g^{-1}\, p_2(\sigma) \\ & {\scriptstyle (\theta'(\sigma), p_1(\sigma))} \searrow \quad \swarrow {\scriptstyle (\alpha^{\,g\,{}^{\sigma}g^{-1}}({}^{\sigma}\alpha^{-1}),\, g\,{}^{\sigma}g^{-1}\, p_2(\sigma))} & \\ & g'\,{}^{\sigma}g'^{-1}\, p_2(\sigma) & \end{array}$$

commutes in $\mathbf{G}$*. Therefore we have:*

$$\theta'(\sigma) = \alpha^{\,g\,{}^{\sigma}g^{-1}}({}^{\sigma}\alpha^{-1})\,\theta(\sigma).$$

Given two morphisms

$$(p_1, f_1, \varepsilon_1) \xrightarrow{[g,\theta,p_2,f_2,\varepsilon_2]} (p_2, f_2, \varepsilon_2) \xrightarrow{[g',\theta',p_3,f_3,\varepsilon_3]} (p_3, f_3, \varepsilon_3)$$

we define their composition by $(p_1, f_1, \varepsilon_1) \xrightarrow{[g\,g',\overline{\theta},p_3,f_3,\varepsilon_3]} (p_3, f_3, \varepsilon_3)$ where $\overline{\theta}$ is given by:

$$p_1(\sigma) \xrightarrow{(\theta(\sigma),\, p_1(\sigma))} g\,{}^{\sigma}g^{-1}\, p_2(\sigma) \xrightarrow[= ({}^{g\,{}^{\sigma}g^{-1}}\theta'(\sigma),\, g\,{}^{\sigma}g^{-1}\, p_2(\sigma))]{(1,\, g\,{}^{\sigma}g^{-1}) \otimes (\theta'(\sigma),\, p_2(\sigma)) =} g\,{}^{\sigma}g^{-1}\, g'\,{}^{\sigma}g'^{-1}\, p_3(\sigma)$$

$$\xrightarrow[= ({}^{g}\{g'\,{}^{\sigma}g'^{-1},\, {}^{\sigma}g^{-1}\},\, g\,{}^{\sigma}g^{-1}\, g'\,{}^{\sigma}g'^{-1}\, p_3(\sigma))]{(1, g) \otimes (\{g'\,{}^{\sigma}g'^{-1},\, {}^{\sigma}g^{-1}\},\, {}^{\sigma}g^{-1}\, g'\,{}^{\sigma}g'^{-1}) \otimes (1, p_3(\sigma)) =} g\, g'\,{}^{\sigma}g'^{-1}\,{}^{\sigma}g^{-1}\, p_3(\sigma).$$

Therefore we obtain:

$$\overline{\theta}(\sigma) = {}^{g}\{g'\,{}^{\sigma}g'^{-1},\, {}^{\sigma}g^{-1}\}^{\,g\,{}^{\sigma}g^{-1}}\theta'(\sigma)\,\theta(\sigma).$$

Given two morphisms

$$(p_1, f_1, \varepsilon_1) \xrightarrow{[g,\theta,p_1',f_1',\varepsilon_1']} (p_1', f_1', \varepsilon_1') \quad (p_2, f_2, \varepsilon_2) \xrightarrow{[g',\theta',p_2',f_2',\varepsilon_2']} (p_2', f_2', \varepsilon_2')$$

their tensor product is given by:

$$[g^{\,(p_1', f_1', \varepsilon_1')} g', \overline{\theta}, p, f, \varepsilon] = [g\, p_1'(1)\, g', \overline{\theta}, p, f, \varepsilon]$$

where $(p, f, \varepsilon) = (p_1', f_1', \varepsilon_1')(p_2', f_2', \varepsilon_2')$ (see (5.11), (5.12), (5.13)). The function $\overline{\theta}$ is given by the composition of the following three morphisms:

$$
\begin{array}{c}
p_1(\sigma)p_2(\sigma) \\
\Big\downarrow \; {\scriptstyle (\theta(\sigma),\, p_1(\sigma)) \otimes (\theta'(\sigma),\, p_2(\sigma)) = } \\
{\scriptstyle = (\theta(\sigma)\, {}^{p_1(\sigma)}\theta'(\sigma),\, p_1(\sigma)\, p_2(\sigma))} \\
g\, {}^{\sigma}g^{-1}\, p_1'(\sigma)\, g'\, {}^{\sigma}g'^{-1}\, p_2'(\sigma) \\
\Big\downarrow \; {\scriptstyle (1,\, g\, {}^{\sigma}g^{-1}) \otimes \overline{\nu}^{-1}_{((p_1',f_1',\varepsilon_1'),g')\sigma} \otimes (1,\, p_2'(\sigma))} \\
g\, {}^{\sigma}g^{-1}\, p_1'(1)\, g'\, {}^{\sigma}g'^{-1}\, {}^{\sigma}p_1'(1)^{-1}\, p_1'(\sigma)\, p_2'(\sigma) \\
\Big\downarrow \; {\scriptstyle (1,\, g) \otimes \chi_{p_1'(1)\, g'\, {}^{\sigma}g'^{-1}\, {}^{\sigma}p_1'(1)^{-1},\, {}^{\sigma}g^{-1}} \otimes (1,\, p_1'(\sigma)\, p_2'(\sigma)) =} \\
{\scriptstyle = (1,\, g) \otimes (\{p_1'(1)\, g'\, {}^{\sigma}g'^{-1}\, {}^{\sigma}p_1'(1)^{-1},\, {}^{\sigma}g^{-1}\},\, {}^{\sigma}g^{-1}\, p_1'(1)\, g'\, {}^{\sigma}g'^{-1}\, {}^{\sigma}p_1'(1)^{-1}) \otimes (1,\, p_1'(\sigma)\, p_2'(\sigma)) =} \\
{\scriptstyle = ({}^{g}\{p_1'(1)\, g'\, {}^{\sigma}g'^{-1}\, {}^{\sigma}p_1'(1)^{-1},\, {}^{\sigma}g^{-1}\},\, g\, {}^{\sigma}g^{-1}\, p_1'(1)\, g'\, {}^{\sigma}g'^{-1}\, {}^{\sigma}p_1'(1)^{-1}\, p_1'(\sigma)\, p_2'(\sigma))} \\
g\, p_1'(1)\, g'\, {}^{\sigma}g'^{-1}\, {}^{\sigma}p_1'(1)^{-1}\, {}^{\sigma}g^{-1}\, p_1'(\sigma)\, p_2'(\sigma)
\end{array}
$$

where $\overline{\nu}_{((p_1',f_1',\varepsilon_1'),g')\sigma}$ is one of the natural isomorphisms given in the structure of $\mathbf{G}$ as a categorical $\mathrm{Der}(\mathbf{\Gamma},\mathbf{G})$-crossed module (see [14]) and it is given by:

$$
\begin{array}{c}
p_1'(1)\, g'\, {}^{\sigma}g'^{-1}\, {}^{\sigma}p_1'(1)^{-1}\, p_1'(\sigma) \\
\Big\downarrow \; {\scriptstyle (1,\, p_1'(1)\, g') \otimes (\{p_1'(\sigma),\, {}^{\sigma}g'^{-1}\, {}^{\sigma}p_1'(1)^{-1}\},\, {}^{\sigma}g'^{-1}\, p_1'(1)^{-1}\, p_1'(\sigma))=} \\
{\scriptstyle = ({}^{p_1'(1)\, g'}\{p_1'(\sigma),\, {}^{\sigma}g'^{-1}\, {}^{\sigma}p_1'(1)^{-1}\},\, p_1'(1)\, g'\, {}^{\sigma}g'^{-1}\, p_1'(1)^{-1}\, p_1'(\sigma))} \\
p_1'(1)\, g'\, p_1'(\sigma)\, {}^{\sigma}g'^{-1}\, {}^{\sigma}p_1'(1)^{-1} \\
\Big\downarrow \; {\scriptstyle (1,\, p_1'(1)) \otimes (\{g',\, p_1'(\sigma)\}^{-1},\, g'\, p_1'(\sigma)) \otimes (1,\, {}^{\sigma}g'^{-1}\, {}^{\sigma}p_1'(1)^{-1})=} \\
{\scriptstyle = ({}^{p_1'(1)}\{g',\, p_1'(\sigma)\}^{-1},\, p_1'(1)\, g'\, p_1'(\sigma)\, {}^{\sigma}g'^{-1}\, {}^{\sigma}p_1'(1)^{-1})} \\
p_1'(1)\, p_1'(\sigma)\, g'\, {}^{\sigma}g'^{-1}\, {}^{\sigma}p_1'(1)^{-1} \\
\Big\downarrow \; {\scriptstyle (1,\, p_1'(1)\, p_1'(\sigma)) \otimes (\{g'\, {}^{\sigma}g'^{-1},\, {}^{\sigma}p_1'(1)^{-1}\}^{-1},\, g'\, {}^{\sigma}g'^{-1}\, {}^{\sigma}p_1'(1)^{-1})=} \\
{\scriptstyle = ({}^{p_1'(1)\, p_1'(\sigma)}\{g'\, {}^{\sigma}g'^{-1},\, {}^{\sigma}p_1'(1)^{-1}\}^{-1},\, p_1'(1)\, p_1'(\sigma)\, g'\, {}^{\sigma}g'^{-1}\, {}^{\sigma}p_1'(1)^{-1})} \\
p_1'(1)\, p_1'(\sigma)\, {}^{\sigma}p_1'(1)^{-1}\, g'\, {}^{\sigma}g'^{-1} \\
\Big\downarrow \; {\scriptstyle (1,\, p_1'(1)\, p_1'(\sigma)) \otimes {}^{\sigma}(\varepsilon_1'(1,1)\, \varepsilon_1'(1,1),\, p_1'(1)^{-1}) \otimes (1,\, g'\, {}^{\sigma}g'^{-1})=} \\
{\scriptstyle = ({}^{p_1'(1)\, p_1'(\sigma)}[{}^{\sigma}(\varepsilon_1'(1,1)\, \varepsilon_1'(1,1))],\, p_1'(1)\, p_1'(\sigma)\, {}^{\sigma}p_1'(1)^{-1}\, g'\, {}^{\sigma}g'^{-1})} \\
p_1'(1)\, p_1'(\sigma)\, {}^{\sigma}p_1'(1)\, g'\, {}^{\sigma}g'^{-1}
\end{array}
$$

$$\begin{array}{c} \Big| \\ (1, {p_1}'(1)) \otimes ({\varepsilon_1}'(\sigma,1)^{-1}, {p_1}'(\sigma)\, {}^{\sigma}{p_1}'(1)) \otimes (1, g'\, {}^{\sigma}g'^{-1}) = \\ = ({}^{{p_1}'(1)}{\varepsilon_1}'(\sigma,1)^{-1}, {p_1}'(1)\, {p_1}'(\sigma)\, {}^{\sigma}{p_1}'(1)\, g'\, {}^{\sigma}g'^{-1}) \\ \Big\downarrow \\ {p_1}'(1)\, {p_1}'(\sigma)\, g'\, {}^{\sigma}g'^{-1} \\ \Big| \\ ({\varepsilon_1}'(1,\sigma)^{-1}, {p_1}'(1)\, {p_1}'(\sigma)) \otimes (1, g'\, {}^{\sigma}g'^{-1}) = \\ = ({\varepsilon_1}'(1,\sigma)^{-1}, {p_1}'(1)\, {p_1}'(\sigma)\, g'\, {}^{\sigma}g'^{-1}) \\ \Big\downarrow \\ {p_1}'(\sigma)\, g'\, {}^{\sigma}g'^{-1}. \end{array}$$

Finally, we have:

$$\begin{aligned} \overline{\theta}(\sigma) \;=\; & {}^{g}\{{p_1}'(1)\, g'\, {}^{\sigma}g'^{-1}\, {}^{\sigma}{p_1}'(1)^{-1},\, {}^{\sigma}g^{-1}\} \\ & {}^{g\, {}^{\sigma}g^{-1}}\big[\, {}^{{p_1}'(1)\, g'}\{{p_1}'(\sigma),\, {}^{\sigma}g'^{-1}\, {}^{\sigma}{p_1}'(1)^{-1}\}^{-1} \\ & {}^{{p_1}'(1)}\{g', {p_1}'(\sigma)\}\, {}^{{p_1}'(1)\, {p_1}'(\sigma)}\{g'\, {}^{\sigma}g'^{-1},\, {}^{\sigma}{p_1}'(1)^{-1}\} \\ & {}^{{p_1}'(1)\, {p_1}'(\sigma)}[\, {}^{\sigma}({\varepsilon_1}'(1,1)\, {\varepsilon_1}'(1,1))]^{-1}\, {}^{{p_1}'(1)}{\varepsilon_1}'(\sigma,1)\, {\varepsilon_1}'(1,\sigma)]\, \theta(\sigma) \\ & {}^{p_1(\sigma)}\theta'(\sigma). \end{aligned}$$

Because $\mathcal{H}^1(\mathbf{\Gamma}, \mathbf{G})$ is a strict categorical group it corresponds to the crossed module constructed as follows:

$$d : \mathrm{Ker}t \;\to\; \mathrm{Ob}(\mathcal{H}^1(\mathbf{\Gamma}, \mathbf{G})) = \mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G}))$$

with $d = s_{|\mathrm{Ker}t}$, where s and t are the source and target maps, respectively, of the underlying groupoid $\mathcal{H}^1(\mathbf{\Gamma}, \mathbf{G})$. We denote with $\mathcal{H}^1_1(\mathbf{\Gamma}, \mathbf{G})$ the set of arrows in $\mathcal{H}^1(\mathbf{\Gamma}, \mathbf{G})$ and we consider the target map:

$$\begin{aligned} t : \mathcal{H}^1_1(\mathbf{\Gamma}, \mathbf{G}) &\longrightarrow \mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})) \\ (g, \theta, p_2, f_2, \varepsilon_2) &\longrightarrow (p_2, f_2, \varepsilon_2) \end{aligned}$$

while the source map:

$$\begin{aligned} s : \mathcal{H}^1_1(\mathbf{\Gamma}, \mathbf{G}) &\longrightarrow \mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})) \\ (g, \theta, p_2, f_2, \varepsilon_2) &\longrightarrow (p_1, f_1, \varepsilon_1) \end{aligned}$$

where $(p_1, f_1, \varepsilon_1)$ as in Proposition 5.3.4.
Thus we have:

$$\begin{aligned} d : \mathrm{Ker}t &\to \mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})) \\ (g, \theta, 1, 1, 1) &\to (p, f, \varepsilon) \end{aligned}$$

where

- $p(\sigma) = \partial\,\theta(\sigma)^{-1}\, g\, {}^{\sigma}g^{-1}$;
- $f(\beta, \sigma) = \theta(\partial'(\beta)\,\sigma)^{-1}\, {}^{g}h(\beta, {}^{\sigma}g^{-1})\,\theta(\sigma)$;
- $\varepsilon(\sigma, \tau) = \theta(\sigma)^{-1}\, {}^{g\,{}^{\sigma}g^{-1}}({}^{\sigma}\theta(\tau)^{-1})\,\theta(\sigma\,\tau)$.

The product of two arrows $[g, \theta, 1, 1, 1]$ and $[g', \theta', 1, 1, 1]$ in $\mathrm{Ker}t$ is $[g\,g', \tilde{\theta}, 1, 1, 1]$, where $\tilde{\theta}$ is given by:

$$\begin{aligned} \tilde{\theta}(\sigma) &= {}^{g}\{g'\,{}^{\sigma}g'^{-1\,\sigma}, {}^{\sigma}g^{-1}\}\,\theta(\sigma)^{\partial\,\theta(\sigma)^{-1}\,g\,{}^{\sigma}g^{-1}}\theta'(\sigma) = \\ &= {}^{g}\{g'\,{}^{\sigma}g'^{-1\,\sigma}, {}^{\sigma}g^{-1}\}\,{}^{g\,{}^{\sigma}g^{-1}}\theta'(\sigma)\;\theta(\sigma). \end{aligned} \tag{5.21}$$

The action of the group $\mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G}))$ on $\mathrm{Ker}t$ is given by:

$${}^{(p,f,\varepsilon)}[g, \theta, 1, 1, 1] = i(p, f, \varepsilon)\,[g, \theta, 1, 1, 1]\, i(p, f, \varepsilon)^{-1}.$$

We recall that the map i for the groupoid $\mathcal{H}^1(\mathbf{\Gamma}, \mathbf{G})$ is defined by:

$$\begin{aligned} i : \mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})) &\longrightarrow \mathcal{H}^1_1(\mathbf{\Gamma}, \mathbf{G}) \\ (p, f, \varepsilon) &\longrightarrow (1, 1, p, f, \varepsilon) \end{aligned}$$

Therefore, using the multiplication defined above on $\mathcal{H}^1_1(\mathbf{\Gamma}, \mathbf{G})$ and the inverse in $\mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G}))$, we have:

$$\begin{aligned} {}^{(p,f,\varepsilon)}[g, \theta, 1, 1, 1] &= [1, 1, p, f, \varepsilon]\,[g, \theta, 1, 1, 1]\,[1, 1, p, f, \varepsilon]^{-1} = \\ &= [1, 1, p, f, \varepsilon]\,[g, \theta, 1, 1, 1]\,[1, 1, p^*, f^*, \varepsilon^*] = \\ &= [p(1)\,g, \hat{\theta}, p, f, \varepsilon]\,[1, 1, p^*, f^*, \varepsilon^*] \end{aligned}$$

where

$$\begin{aligned} \hat{\theta}(\sigma) = \; & {}^{p(1)\,g}\{p(\sigma), {}^{\sigma}g^{-1\,\sigma}p(1)^{-1}\}^{-1}\, {}^{p(1)}\{g, p(\sigma)\}\, {}^{p(1)p(\sigma)}\{g\,{}^{\sigma}g^{-1}, {}^{\sigma}p(1)^{-1}\} \\ & {}^{p(1)p(\sigma)}[{}^{\sigma}\varepsilon(1,1)\,{}^{\sigma}\varepsilon(1,1)]^{-1}\, {}^{p(1)}\varepsilon(\sigma, 1)\,\varepsilon(1, \sigma)\, {}^{p(\sigma)}\theta(\sigma). \end{aligned}$$

$(p, f, \varepsilon) \in \mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G}))$ then $p(1) = \partial\varepsilon(1, 1)$ and we are going to prove that

$$[p(1)\,g, \hat{\theta}, p, f, \varepsilon] \sim [g, \widetilde{\theta}, p, f, \varepsilon]$$

where $\widetilde{\theta}(\sigma) = {}^{g}\{p(\sigma), {}^{\sigma}g^{-1}\}^{-1}\,\{g, p(\sigma)\}\, {}^{p(\sigma)}\theta(\sigma)$.

$$\begin{aligned} \hat{\theta}(\sigma) &= \varepsilon(1,1)\, {}^{g}\{p(\sigma), {}^{\sigma}g^{-1\,\sigma}\partial\varepsilon(1,1)^{-1}\}^{-1}\{g, p(\sigma)\} \\ & \quad {}^{p(\sigma)}\{g\,{}^{\sigma}g^{-1}, {}^{\sigma}\partial\varepsilon(1,1)^{-1}\}\, {}^{p(\sigma)}[{}^{\sigma}\varepsilon(1,1)\,{}^{\sigma}\varepsilon(1,1)]^{-1}\,\varepsilon(\sigma, 1)\,\varepsilon(1,1)^{-1} \\ & \quad \varepsilon(1, \sigma)\, {}^{p(\sigma)}\theta(\sigma) = \\ &= \varepsilon(1,1)\, {}^{g\,{}^{\sigma}g^{-1}}\{p(\sigma), {}^{\sigma}\partial\varepsilon(1,1)^{-1}\}^{-1}\, {}^{g}\{p(\sigma), {}^{\sigma}g^{-1}\}^{-1}\{g, p(\sigma)\} \\ & \quad {}^{p(\sigma)\,g\,{}^{\sigma}g^{-1}}({}^{\sigma}\varepsilon(1,1)^{-1})\, {}^{p(\sigma)}({}^{\sigma}\varepsilon(1,1))^{p(\sigma)}({}^{\sigma}\varepsilon(1,1)^{-1})\, {}^{p(\sigma)}({}^{\sigma}\varepsilon(1,1)^{-1}) \\ & \quad \varepsilon(\sigma, 1)\, {}^{p(\sigma)}\theta(\sigma) = \end{aligned}$$

$$
\begin{aligned}
&= \varepsilon(1,1)\, {}^{g\,{}^\sigma g^{-1}}({}^\sigma\varepsilon(1,1)^{-1})\, {}^{g\,{}^\sigma g^{-1} p(\sigma)}({}^\sigma\varepsilon(1,1))\ {}^g\{p(\sigma), {}^\sigma g^{-1}\}^{-1} \\
&\quad \{g, p(\sigma)\}\, {}^{p(\sigma)\,g\,{}^\sigma g^{-1}}({}^\sigma\varepsilon(1,1)^{-1})\, {}^{p(\sigma)}\theta(\sigma) = \\
&= \varepsilon(1,1)\, {}^{g\,{}^\sigma g^{-1}}({}^\sigma\varepsilon(1,1)^{-1})\, {}^g\{p(\sigma), {}^\sigma g^{-1}\}^{-1}\{g, p(\sigma)\} \\
&\quad {}^{p(\sigma)\,g\,{}^\sigma g^{-1}}({}^\sigma\varepsilon(1,1))\, {}^{p(\sigma)\,g\,{}^\sigma g^{-1}}({}^\sigma\varepsilon(1,1)^{-1})\, {}^{p(\sigma)}\theta(\sigma) = \\
&= \varepsilon(1,1)\, {}^{g\,{}^\sigma g^{-1}}({}^\sigma\varepsilon(1,1)^{-1})\, {}^g\{p(\sigma), {}^\sigma g^{-1}\}^{-1}\{g, p(\sigma)\}\, {}^{p(\sigma)}\theta(\sigma).
\end{aligned}
$$

In the last equalities, we have used the following relations:

1. since $(p, f, \varepsilon) \in \mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G}))$ (in particular $(p, \varepsilon) \in Z^2_B(\Gamma_0, \partial : G_1 \to G_0)$) then:

$$
\begin{aligned}
&{}^{p(\sigma)}({}^\sigma\varepsilon(1,1)) = \varepsilon(\sigma, 1); \\
&\varepsilon(1,\sigma) = \varepsilon(1,1);
\end{aligned}
$$

2. since $\partial : G_1 \to G_0$ is a braided crossed module we have:

$$
\begin{aligned}
g\,{}^\sigma g^{-1}\, p(\sigma) &= \partial[\,{}^g\{p(\sigma),\ {}^\sigma g^{-1}\}^{-1}]\, g\, p(\sigma)\, {}^\sigma g^{-1} \\
&= \partial[\,{}^g\{p(\sigma),\ {}^\sigma g^{-1}\}^{-1}\, \{g, p(\sigma)\}]\, p(\sigma)\, g\, {}^\sigma g^{-1}.
\end{aligned}
$$

Thus we obtain:

$$
\begin{aligned}
{}^{(p,f,\varepsilon)}[g, \theta, 1, 1, 1] &= [g, \widetilde{\theta}, p, f, \varepsilon]\, [1, 1, p^*, f^*, \varepsilon^*] = \\
&= [g\, p(1), \overline{\theta}, 1, 1, 1]
\end{aligned}
$$

where $g\, p(1) = g\, \partial\varepsilon(1,1) = \partial(\,{}^g\varepsilon(1,1))\, g$ and

$$
\begin{aligned}
\overline{\theta}(\sigma) &= {}^g\{p(1)\, {}^\sigma p(1)^{-1},\ {}^\sigma g^{-1}\}\, {}^{g\,{}^\sigma g^{-1}}[\,{}^{p(1)}\{p(\sigma),\ {}^\sigma p(1)^{-1}\}^{-1}\, {}^{p(1)\,p(\sigma)}[{}^\sigma\varepsilon(1,1) \\
&\quad {}^\sigma\varepsilon(1,1)]^{-1}\, {}^{p(1)}\varepsilon(\sigma,1)\, \varepsilon(1,\sigma)]\, \widetilde{\theta}(\sigma) = \\
&= {}^g\{\partial[\varepsilon(1,1)\, {}^\sigma\varepsilon(1,1)^{-1}],\ {}^\sigma g^{-1}\}\, {}^{g\,{}^\sigma g^{-1}}[\,\varepsilon(1,1)\, \{p(\sigma),\ {}^\sigma\partial\varepsilon(1,1)^{-1}\}^{-1} \\
&\quad {}^{p(\sigma)}({}^\sigma\varepsilon(1,1)^{-1})\, {}^{p(\sigma)}({}^\sigma\varepsilon(1,1)^{-1})\, \varepsilon(\sigma,1)\, \varepsilon(1,1)^{-1}\, \varepsilon(1,\sigma)] \\
&\quad {}^g\{p(\sigma),\ {}^\sigma g^{-1}\}^{-1}\, \{g, p(\sigma)\}\, {}^{p(\sigma)}\theta(\sigma) = \\
&= {}^g(\varepsilon(1,1)\, {}^\sigma\varepsilon(1,1)^{-1})\, {}^{g\,{}^\sigma g^{-1}}({}^\sigma\varepsilon(1,1)\, \varepsilon(1,1)^{-1})\, {}^{g\,{}^\sigma g^{-1}}(\varepsilon(1,1) \\
&\quad {}^\sigma\varepsilon(1,1)^{-1})\, {}^{g\,{}^\sigma g^{-1}}[\,{}^{p(\sigma)}({}^\sigma\varepsilon(1,1))\, {}^{p(\sigma)}({}^\sigma\varepsilon(1,1)^{-1})]\, {}^g\{p(\sigma),\ {}^\sigma g^{-1}\}^{-1} \\
&\quad \{g, p(\sigma)\}\, {}^{p(\sigma)}\theta(\sigma) = \\
&= {}^g(\varepsilon(1,1)\, {}^\sigma\varepsilon(1,1)^{-1})\, {}^g\{p(\sigma),\ {}^\sigma g^{-1}\}\, \{g, p(\sigma)\}\, {}^{p(\sigma)}\theta(\sigma) = \\
&= {}^g\varepsilon(1,1)\, {}^{g\,{}^\sigma g^{-1}}(\,{}^\sigma(\,{}^g\varepsilon(1,1)^{-1})\, {}^g\{p(\sigma),\ {}^\sigma g^{-1}\}^{-1}\, \{g, p(\sigma)\}\, {}^{p(\sigma)}\theta(\sigma).
\end{aligned}
$$

So we have:

$$
{}^{(p,f,\varepsilon)}[g, \theta, 1, 1, 1] = [g, \widetilde{\theta}, 1, 1, 1].
$$

We can prove that:

$$
\mathrm{Ker}t \quad \text{is isomorphic to} \quad \frac{C^1_B(\Gamma_0, \partial : G_1 \to G_0)}{B^1_B(\Gamma_0, \partial : G_1 \to G_0)}.
$$

Thanks to the definition of the product in $\operatorname{Ker} t$ (see (5.21)), we have:

$$[\partial(\alpha), \theta_\alpha, 1, 1, 1]\,[g, \theta, 1, 1, 1] = [\partial(\alpha)\, g, \hat{\theta}, 1, 1, 1]$$

where:

$$\begin{aligned} \hat{\theta}(\sigma) &= {}^{\partial(\alpha)}\{g\,{}^\sigma g^{-1},\, {}^\sigma\partial(\alpha)^{-1}\}\, \theta_\alpha(\sigma)\,{}^{\partial\,\theta_\alpha(\sigma)^{-1}\,\partial(\alpha)\,{}^\sigma\partial(\alpha)^{-1}}\theta(\sigma) = \\ &= \alpha\,{}^{g\,{}^\sigma g^{-1}}({}^\sigma\alpha^{-1})\,{}^\sigma\alpha\,\alpha^{-1}\,\alpha\,{}^\sigma\alpha^{-1}\,\theta(\sigma) = \\ &= \alpha\,{}^{g\,{}^\sigma g^{-1}}({}^\sigma\alpha^{-1})\,\theta(\sigma). \end{aligned}$$

When restricted to $Z^1_B(\Gamma_0, \partial : G_1 \to G_0)$ the product defined in (5.21) coincides with the Borovoi product.
It is clear the isomorphism between d and a homomorphism

$$d : \frac{C^1_B(\Gamma_0, \partial : G_1 \to G_0)}{B^1_B(\Gamma_0, \partial : G_1 \to G_0)} \;\to\; \mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G}))$$

which, by abuse of notation, we have denoted again by d.

Remark 5.3.3. *[26] If $\partial : G_1 \to G_0$ is a symmetric crossed module, then $d : \frac{C^1_B(\Gamma_0,\partial:G_1\to G_0)}{B^1_B(\Gamma_0,\partial:G_1\to G_0)} \to Ob(Der(\mathbf{\Gamma}, \mathbf{G}))$ is a symmetric crossed module where the braiding*

$$\{-,-\} : Ob(Der(\mathbf{\Gamma}, \mathbf{G})) \times Ob(Der(\mathbf{\Gamma}, \mathbf{G})) \to \frac{C^1_B(\Gamma_0, \partial : G_1 \to G_0)}{B^1_B(\Gamma_0, \partial : G_1 \to G_0)}$$

is determined by $\{(p_1, f_1, \varepsilon_1), (p_2, f_2, \varepsilon_2)\}(\sigma) = [1, \{p_1, p_2\}]$ where $\{p_1, p_2\}(\sigma) = \{p_1(\sigma), p_2(\sigma)\}$. However, if $\{-,-\}$ is just a braiding in $\partial : G_1 \to G_0$ (but not a symmetry), then $d : \frac{C^1_B(\Gamma_0,\partial:G_1\to G_0)}{B^1_B(\Gamma_0,\partial:G_1\to G_0)} \to Ob(Der(\mathbf{\Gamma}, \mathbf{G}))$ is not a braided crossed module.

Remark 5.3.4. *If $\Gamma_1 = 1$ then $\partial : G_1 \to G_0$ is a Γ_0-equivariant braided crossed module (case considered by Noohi). In this case, $Der(\Gamma_0[0], \mathbf{G})$ is associated with crossed module:*

$$\begin{aligned} \overline{\partial} : App(\Gamma_0, G_1) &\to Z^2_B(\Gamma_0, \partial : G_1 \to G_0) \\ \theta &\to (p, \varepsilon) \end{aligned}$$

- $p(\sigma) = \partial\,\theta(\sigma)$;
- $\varepsilon(\sigma, \tau) = \theta(\sigma)\,{}^\sigma\theta(\tau)\,\theta(\sigma\,\tau)^{-1}$.

The product in $Z^2_B(\Gamma_0, \partial : G_1 \to G_0)$ is given by:

$$\begin{aligned} &(p_1, \varepsilon_1)(p_2, \varepsilon_2) = (p, \varepsilon); \\ &p(\sigma) = p_1(\sigma)\, p_2(\sigma); \\ &\varepsilon(\sigma, \tau) = {}^{p_1(\sigma)}\{p_2(\sigma),\, {}^\sigma p_1(\tau)\}\, \varepsilon_1(\sigma, \tau)\,{}^{p_1(\sigma\,\tau)}\varepsilon_2(\sigma, \tau); \end{aligned}$$

and the product in $App(\Gamma_0, G_1)$ *is given by:*

$$(\theta_1 \cdot \theta_2)(\sigma) = \theta_1(\sigma)\, \theta_2(\sigma).$$

The action of the group $Z^2_B(\Gamma_0, \partial : G_1 \to G_0)$ *on the group* $App(\Gamma_0, G_1)$ *is defined by:*

$$^{(p,\varepsilon)}\theta = \hat{\theta} \quad \text{where} \quad \hat{\theta}(\sigma) = {}^{p(\sigma)}\theta(\sigma).$$

As we have already introduced at the beginning of this section, we can easily observe that $\mathcal{H}^0(\Gamma_0[0], \mathbf{G})$, $\mathcal{H}^1(\Gamma_0[0], \mathbf{G})$ *are associated respectively with crossed modules* $\overline{\overline{\partial}} : G_1 \to Z^1_B$, $d : C^1_B/B^1_B \to Z^2_B$, *presented in section 5.1.3.*

Noohi proves in the Lemma 5.1.1 that H^1_B *is abelian. From the categorical point of view,* $\mathcal{H}^0(\Gamma_0[0], \mathbf{G})$ *can be equipped with a braiding and consequently* $\overline{\overline{\partial}} : G_1 \to Z^1_B$ *can be seen as a braided crossed module. Then* $G_1 \xrightarrow{\overline{\overline{\partial}}} Z^1_B \xrightarrow{1} 1$ *is a 2 crossed module (example (b) in section 2.3) and* $1 : H^1_B \to 1$ *is a crossed module (consequence of Proposition 2.3.1) then* H^1_B *is abelian.*

5.4 Cohomology in 2-crossed modules

In this section, we want to revisit the cohomology with coefficients in categorical crossed modules for another particular case. We have already seen that if

$$G_1 \xrightarrow{\partial} G_0 \xrightarrow{p_0} \Gamma_0 \tag{5.22}$$

is a 2-crossed module, then this is an example of special semistrict $\Gamma_0[0]$-categorical crossed modules (see example (b) of the section 3.4). We use $\mathbf{\Gamma}$ to denote $\Gamma_0[0]$ and $\mathbf{G}$ for $\mathbf{G}(\partial)$.

In this case, we consider the monoidal category $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$. The associativity a, left unit l and right unit r of the monoidal structure of $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$ are defined by using the canonical isomorphisms of $\mathbf{\Gamma}$, $\mathbf{G}$ and the strict functor $\mathbf{T}$ (induced by the morphism p_0) from $\mathbf{\Gamma}$ to $\mathbf{G}$ (see [14]) and they are all identity maps. Thus $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$ is a strict monoidal category.

We are going to describe the objects and arrows of $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$.

Lemma 5.4.1. *A derivation from* $\mathbf{\Gamma}$ *into* $\mathbf{G}$ *is uniquely specified by a pair of functions* $p : \Gamma_0 \to G_0$ *and* $\varepsilon : \Gamma_0 \times \Gamma_0 \to G_1$ *satisfying*

$$p(\sigma)\, {}^{\sigma}p(\tau) = \partial(\varepsilon(\sigma, \tau))\, p(\sigma\, \tau) \tag{5.23}$$

$$^{p(\sigma)}({}^{\sigma}\varepsilon(\tau, \upsilon))\, \varepsilon(\sigma, \tau\, \upsilon) = \varepsilon(\sigma, \tau)\, \varepsilon(\sigma\, \tau, \upsilon) \tag{5.24}$$

Thus $Ob(Der(\mathbf{\Gamma}, \mathbf{G})) = Z^2_B(\Gamma_0, \partial : G_1 \to G_0)$.

Proof. As in the braided case.

□

Proposition 5.4.1. *An arrow in the categorical group* $Der(\mathbf{\Gamma}, \mathbf{G})$ *is uniquely specified by a triple* $(\theta, p_1, \varepsilon_1)$ *with* (p_1, ε_1) *as in Lemma 5.4.1 and an arbitrary function* $\theta : \Gamma_0 \to G_1$. *The source of* $(\theta, p_1, \varepsilon_1)$ *is the derivation from* $\mathbf{\Gamma}$ *into* $\mathbf{G}$ *given by* (p_1, ε_1); *the target of* $(\theta, p_1, \varepsilon_1)$ *is the derivation from* $\mathbf{\Gamma}$ *into* $\mathbf{G}$ *given by the pair of functions* $p_2(\sigma) = \partial\theta(\sigma)\, p_1(\sigma)$ *and* $\varepsilon_2(\sigma, \tau) = \theta(\sigma)\, {}^{p_1(\sigma)}({}^{\sigma}\theta(\tau))\, \varepsilon_1(\sigma, \tau)\, \theta(\sigma\,\tau)^{-1}$.

Proof. As in the braided case.

□

$\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$ is a strict monoidal category and the tensor product on objects (see Theorem 5.2 in [14]) is given by:

$$\begin{aligned} &(p_1, \varepsilon_1)(p_2, \varepsilon_2) = (p, \varepsilon); \\ &p(\sigma) = p_1(p_0(p_2(\sigma))\sigma)\, p_2(\sigma); \end{aligned} \tag{5.25}$$

where ε is defined by the composition of the following sequence of arrows in $\mathbf{G}$:

$$p_1(p_0(p_2(\sigma\,\tau))\sigma\,\tau)\, p_2(\sigma\,\tau)$$

$$\Vert$$

$$p_1(p_0(p_2(\sigma))\,\sigma\, p_0(p_2(\tau))\,\tau)\, p_2(\sigma\,\tau)$$

$$\Big\downarrow \begin{array}{c} (\varepsilon_1(p_0(p_2(\sigma))\,\sigma, p_0(p_2(\tau))\,\tau), p_1(p_0(p_2(\sigma))\,\sigma\, p_0(p_2(\tau))\,\tau)) \otimes (\varepsilon_2(\sigma,\tau), p_2(\sigma\,\tau)) = \\ (\varepsilon_1(p_0(p_2(\sigma))\,\sigma, p_0(p_2(\tau))\,\tau)\, {}^{p_1(p_0(p_2(\sigma))\,\sigma\, p_0(p_2(\tau))\,\tau)}\varepsilon_2(\sigma,\tau), p_1(p_0(p_2(\sigma))\,\sigma\, p_0(p_2(\tau))\,\tau)\, p_2(\sigma\,\tau)) \end{array}$$

$$p_1(p_0(p_2(\sigma))\,\sigma)\, {}^{p_0(p_2(\sigma))\,\sigma}p_1(p_0(p_2(\tau))\,\tau)\, p_2(\sigma)\, {}^{\sigma}p_2(\tau)$$

$$\Vert$$

$$p_1(p_0(p_2(\sigma))\,\sigma)\, {}^{p_0(p_2(\sigma))}[\, {}^{\sigma}p_1(p_0(p_2(\tau))\,\tau)]\, p_2(\sigma)\, {}^{\sigma}p_2(\tau)$$

$$\Big\downarrow \begin{array}{c} (1, p_1(p_0(p_2(\sigma))\,\sigma)) \otimes (\{p_2(\sigma), {}^{\sigma}p_1(p_0(p_2(\tau))\,\tau)\}, {}^{p_0(p_2(\sigma))}[\, {}^{\sigma}p_1(p_0(p_2(\tau))\,\tau)]\, p_2(\sigma)) \otimes (1, {}^{\sigma}p_2(\tau)) = \\ = ({}^{p_1(p_0(p_2(\sigma))\,\sigma)}\{p_2(\sigma), {}^{\sigma}p_1(p_0(p_2(\tau))\,\tau)\}, p_1(p_0(p_2(\sigma))\,\sigma)\, {}^{p_0(p_2(\sigma))}[\, {}^{\sigma}p_1(p_0(p_2(\tau))\,\tau)]\, p_2(\sigma)\, {}^{\sigma}p_2(\tau)) \end{array}$$

$$p_1(p_0(p_2(\sigma))\,\sigma)\, p_2(\sigma)\, {}^{\sigma}p_1(p_0(p_2(\tau))\,\tau)\, {}^{\sigma}p_2(\tau)$$

$$\Vert$$

$$p_1(p_0(p_2(\sigma))\,\sigma)\, p_2(\sigma)\, {}^{\sigma}(p_1(p_0(p_2(\tau))\,\tau)\, p_2(\tau)).$$

Therefore, we have:

$$\begin{aligned} \varepsilon(\sigma,\tau) &= {}^{p_1(p_0(p_2(\sigma))\,\sigma)}\{p_2(\sigma),\, {}^{\sigma}p_1(p_0(p_2(\tau))\,\tau)\} \\ &\quad \varepsilon_1(p_0(p_2(\sigma))\,\sigma, p_0(p_2(\tau))\,\tau)\, {}^{p_1(p_0(p_2(\sigma))\,\sigma\, p_0(p_2(\tau))\,\tau)}\varepsilon_2(\sigma,\tau). \end{aligned} \tag{5.26}$$

Let

$$(p_1,\varepsilon_1) \xrightarrow{(\theta_1,p_1,\varepsilon_1)} (\overline{p}_1,\overline{\varepsilon}_1) \quad \text{and} \quad (p_2,\varepsilon_2) \xrightarrow{(\theta_2,p_2,\varepsilon_2)} (\overline{p}_2,\overline{\varepsilon}_2)$$

be two arrows in $\mathrm{Der}(\mathbf{\Gamma},\mathbf{G})$, where $(\overline{p}_i,\overline{\varepsilon}_i)$ are determined by $(\theta_i,p_i,\varepsilon_i)$ under the Proposition 5.4.1 for $i=1,2$, the tensor product of these two arrows (defined in the Theorem 5.2 in [14]) is given by:

$$\begin{array}{c} p_1(p_0(p_2(\sigma))\,\sigma)\,p_2(\sigma) \\ \Big\downarrow {\scriptstyle (\theta_1(p_0(p_2(\sigma))\,\sigma), p_1(p_0(p_2(\sigma))\,\sigma))\otimes(1,p_2(\sigma)) = (\theta_1(p_0(p_2(\sigma))\,\sigma), p_1(p_0(p_2(\sigma))\,\sigma)\,p_2(\sigma))} \\ \overline{p}_1(p_0(p_2(\sigma))\,\sigma)\,p_2(\sigma) \\ \Big\downarrow {\scriptstyle (1,\overline{p}_1(p_0(p_2(\sigma))\,\sigma))\otimes(\theta_2(\sigma),p_2(\sigma)) = ({}^{\overline{p}_1(p_0(p_2(\sigma))\,\sigma)}\theta_2(\sigma), \overline{p}_1(p_0(p_2(\sigma))\,\sigma)\,p_2(\sigma))} \\ \overline{p}_1(p_0(p_2(\sigma))\,\sigma)\,\overline{p}_2(\sigma). \end{array}$$

Therefore, we obtain:

$$(\theta_1,p_1,\varepsilon_1)(\theta_2,p_2,\varepsilon_2) = (\theta,p,\varepsilon);$$

where

$$\begin{aligned} \theta(\sigma) &= {}^{\overline{p}_1(p_0(p_2(\sigma))\,\sigma)}\theta_2(\sigma)\,\theta_1(p_0(p_2(\sigma))\,\sigma) = \\ &= {}^{\partial\theta_1(p_0(p_2(\sigma))\,\sigma)\,p_1(p_0(p_2(\sigma))\,\sigma)}\theta_2(\sigma)\,\theta_1(p_0(p_2(\sigma))\,\sigma) = \\ &= \theta_1(p_0(p_2(\sigma))\,\sigma)\,{}^{p_1(p_0(p_2(\sigma))\,\sigma)}\theta_2(\sigma) \end{aligned}$$

and $(p,\varepsilon) = (p_1,\varepsilon_1)(p_2,\varepsilon_2)$ as in (5.25) and (5.26).

Now we consider the categorical group $\mathrm{Der}^*(\mathbf{\Gamma},\mathbf{G})$ and we observe (thanks to the Theorem 4.7 in [25]) that a strict inverse exists for any object in $\mathrm{Der}^*(\mathbf{\Gamma},\mathbf{G})$. In the Theorem 4.7, we have the following equivalent statements:

a) $(p,\varepsilon) \in \mathrm{Der}^*(\mathbf{\Gamma},\mathbf{G})$;

b) $\sigma_p : \Gamma_0 \to \Gamma_0 \in \mathrm{Aut}(\Gamma_0)$.

In the proof b) $\Rightarrow$ a), we observe that the inverse of (p,ε) is strict. In fact: let $(p,\varepsilon) \in \mathrm{Der}(\mathbf{\Gamma},\mathbf{G})$ such that σ_p is an automorphism (in this case σ_p is given by $\sigma_p(\sigma) = p_0(p(\sigma))\,\sigma)$, an inverse $(p,\varepsilon)^* = (p^*,\varepsilon^*)$ for (p,ε) is obtained as follows. $p^*(\sigma) = (p(\sigma_p^{-1}(\sigma)))^{-1}$ and ε^* is determined by the composition of the following morphisms:

$$\begin{array}{c}
p^*(\sigma\,\tau) = (p(\sigma_p^{-1}(\sigma\,\tau)))^{-1} = (p(\sigma_p^{-1}(\sigma)\,\sigma_p^{-1}(\tau)))^{-1} \\
\Big\downarrow {\scriptstyle ({}^{(p(\sigma_p^{-1}(\sigma)\,\sigma_p^{-1}(\tau)))^{-1}}\varepsilon(\sigma_p^{-1}(\sigma),\sigma_p^{-1}(\tau))^{-1},(p(\sigma_p^{-1}(\sigma)\,\sigma_p^{-1}(\tau)))^{-1})} \\
[p(\sigma_p^{-1}(\sigma))\,{}^{\sigma_p^{-1}(\sigma)}p(\sigma_p^{-1}(\tau))]^{-1} \\
\Big\downarrow {\scriptstyle ({}^{p(\sigma_p^{-1}(\sigma))^{-1}\,\bar{p}_0(p(\sigma_p^{-1}(\sigma)))\,({}^{\sigma_p^{-1}(\sigma)}p(\sigma_p^{-1}(\tau)))^{-1}}\{p(\sigma_p^{-1}(\sigma)),\,{}^{\sigma_p^{-1}(\sigma)}p(\sigma_p^{-1}(\tau))\},\ [p(\sigma_p^{-1}(\sigma))\,{}^{\sigma_p^{-1}(\sigma)}p(\sigma_p^{-1}(\tau))]^{-1})} \\
[{}^{p_0(p(\sigma_p^{-1}(\sigma)))}({}^{\sigma_p^{-1}(\sigma)}p(\sigma_p^{-1}(\tau)))\,p(\sigma_p^{-1}(\sigma))]^{-1} \\
\Big\| \\
[{}^{\sigma_p(\sigma_p^{-1}(\sigma))}p(\sigma_p^{-1}(\tau))\,p(\sigma_p^{-1}(\sigma))]^{-1} \\
\Big\| \\
[{}^{\sigma}p(\sigma_p^{-1}(\tau))\,p(\sigma_p^{-1}(\sigma))]^{-1} \\
\Big\| \\
p(\sigma_p^{-1}(\sigma))^{-1}\,{}^{\sigma}p(\sigma_p^{-1}(\tau))^{-1}
\end{array}$$

therefore

$$\begin{aligned}
\varepsilon^*(\sigma,\tau) \quad = \quad & {}^{p(\sigma_p^{-1}(\sigma))^{-1}\,\bar{p}_0(p(\sigma_p^{-1}(\sigma)))\,({}^{\sigma_p^{-1}(\sigma)}p(\sigma_p^{-1}(\tau)))^{-1}}\{p(\sigma_p^{-1}(\sigma)),\,{}^{\sigma_p^{-1}(\sigma)}p(\sigma_p^{-1}(\tau))\} \\
& {}^{(p(\sigma_p^{-1}(\sigma)\,\sigma_p^{-1}(\tau)))^{-1}}\varepsilon(\sigma_p^{-1}(\sigma),\sigma_p^{-1}(\tau))^{-1}.
\end{aligned}$$

And we have:

$$\begin{aligned}
& (p,\varepsilon)^*\,(p,\varepsilon) = (p^*,\varepsilon^*)\,(p,\varepsilon) = (\widehat{p},\widehat{\varepsilon}); \\
& \widehat{p}(\sigma) = p^*(\sigma_p(\sigma))\,p(\sigma) = (p(\sigma_p^{-1}\circ\sigma_p(\sigma)))^{-1}\;p(\sigma) = (p(\sigma))^{-1}\;p(\sigma) = 1; \\
& \widehat{\varepsilon}(\sigma,\tau) = {}^{p^*(\sigma_p(\sigma))}\{p(\sigma),\,{}^{\sigma}p^*(\sigma_p(\tau))\}\,\varepsilon^*(\sigma_p(\sigma),\sigma_p(\tau)) \\
& {}^{p^*(\sigma_p(\sigma)\,\sigma_p(\tau))}\varepsilon(\sigma,\tau) = \\
& = {}^{p(\sigma)^{-1}}\{p(\sigma),\,{}^{\sigma}p(\tau)^{-1}\}\,{}^{p(\sigma)^{-1}\,\bar{p}_0(p(\sigma))\,{}^{\sigma}p(\tau)^{-1}}\{p(\sigma),\,{}^{\sigma}p(\tau)\} \\
& {}^{p(\sigma\,\tau)^{-1}}\varepsilon(\sigma,\tau)^{-1}\,{}^{p(\sigma\,\tau)^{-1}}\varepsilon(\sigma,\tau) = \\
& = {}^{p(\sigma)^{-1}}[\{p(\sigma),\,{}^{\sigma}p(\tau)^{-1}\}\,{}^{\bar{p}_0(p(\sigma))\,{}^{\sigma}p(\tau)^{-1}}\{p(\sigma),\,{}^{\sigma}p(\tau)\}] = \\
& = {}^{p(\sigma)^{-1}}[\{p(\sigma),\,{}^{\sigma}p(\tau)^{-1}\,{}^{\sigma}p(\tau)\}] = {}^{p(\sigma)^{-1}}\{p(\sigma),1\} = 1;
\end{aligned}$$

Thus $\mathrm{Der}^*(\mathbf{\Gamma},\mathbf{G})$ is a strict categorical group and it corresponds to the crossed module constructed as follows:

$$\overline{\partial} : \mathrm{Ker}s \quad \to \quad \mathrm{Ob}(\mathrm{Der}^*(\mathbf{\Gamma},\mathbf{G}))$$

with $\overline{\partial} = t_{|\mathrm{Ker}s}$, where s and t are the source and target maps, respectively, of the underlying groupoid $\mathrm{Der}^*(\mathbf{\Gamma}, \mathbf{G})$. We denote with $\mathrm{Der}^*_1(\mathbf{\Gamma}, \mathbf{G})$ the set of arrows in $\mathrm{Der}^*(\mathbf{\Gamma}, \mathbf{G})$ and we recall the source map:

$$\begin{array}{rcl} s : \mathrm{Der}^*_1(\mathbf{\Gamma}, \mathbf{G}) & \longrightarrow & {Z^2_B}^*(\Gamma_0, \partial : G_1 \to G_0) \\ (\theta_1, p_1, \varepsilon_1) & \longrightarrow & (p_1, \varepsilon_1) \end{array}$$

where ${Z^2_B}^*(\Gamma_0, \partial : G_1 \to G_0)$ is the group of invertible elements of the monoid $Z^2_B(\Gamma_0, \partial : G_1 \to G_0)$ under the product defined above. The target map is given by:

$$\begin{array}{rcl} t : \mathrm{Der}^*_1(\mathbf{\Gamma}, \mathbf{G}) & \longrightarrow & {Z^2_B}^*(\Gamma_0, \partial : G_1 \to G_0) \\ (\theta_1, p_1, \varepsilon_1) & \longrightarrow & (p_2, \varepsilon_2) \end{array}$$

where (p_2, ε_2) as in Proposition 5.4.1.
Thus we have

$$\begin{array}{rcl} \overline{\partial} : \mathrm{Ker}s & \to & {Z^2_B}^*(\Gamma_0, \partial : G_1 \to G_0) \\ (\theta, 1, 1) & \to & (p, \varepsilon) \end{array}$$

where

- $p(\sigma) = \partial\, \theta(\sigma)$;
- $\varepsilon(\sigma, \tau) = \theta(\sigma)\, {}^{\sigma}\theta(\tau)\, \theta(\sigma\, \tau)^{-1}$.

The product of two arrows $(\theta_1, 1, 1)$ and $(\theta_2, 1, 1)$ in $\mathrm{Ker}s$ is $(\theta, 1, 1)$ where $\theta(\sigma) = \theta_1(\sigma)\, \theta_2(\sigma)$. The action of the group ${Z^2_B}^*(\Gamma_0, \partial : G_1 \to G_0)$ on $\mathrm{Ker}s$ is given by:

$$^{(p,\varepsilon)}(\theta, 1, 1) = i(p, \varepsilon)(\theta, 1, 1)(i(p, \varepsilon))^{-1}.$$

We recall that the map i for the groupoid $\mathrm{Der}^*(\mathbf{\Gamma}, \mathbf{G})$ is given by:

$$\begin{array}{rcl} i : {Z^2_B}^*(\Gamma_0, \partial : G_1 \to G_0) & \longrightarrow & \mathrm{Der}^*_1(\mathbf{\Gamma}, \mathbf{G}) \\ (p, \varepsilon) & \longrightarrow & (1, p, \varepsilon). \end{array}$$

Therefore, using the multiplication defined above on $\mathrm{Der}^*_1(\mathbf{\Gamma}, \mathbf{G})$, we have:

$$\begin{array}{rcl} ^{(p,\varepsilon)}(\theta, 1, 1) & = & (1, p, \varepsilon)\, (\theta, 1, 1)\, (1, p, \varepsilon)^{-1} \\ & = & (1, p, \varepsilon)\, (\theta, 1, 1)\, (1, p^*, \varepsilon^*) \\ & = & (\widehat{\theta}, p, \varepsilon)\, (1, p^*, \varepsilon^*) \\ & = & (\widehat{\widehat{\theta}}, 1, 1) \end{array}$$

$$\text{where} \quad \begin{array}{rcl} \widehat{\theta}(\sigma) & = & {}^{p(\sigma)}\theta(\sigma) \\ \widehat{\widehat{\theta}}(\sigma) & = & \widehat{\theta}(p_0(p^*(\sigma))\, \sigma) = {}^{p(p_0(p^*(\sigma))\, \sigma)}\theta(p_0(p^*(\sigma))\, \sigma) = \\ & = & {}^{p^*(\sigma)^{-1}}\theta(p_0(p^*(\sigma))\, \sigma) \end{array}$$

and the pair (p^*, ε^*) is the strict inverse of (p, ε).
Because $\mathrm{Ker} s$ is isomorphic to $\mathrm{App}(\Gamma_0, G_1)$, it is clear the isomorphism between $\overline{\partial}$ and a homomorphism

$$\overline{\partial} : \mathrm{App}(\Gamma_0, G_1) \quad \to \quad Z_B^{2\,*}(\Gamma_0, \partial : G_1 \to G_0)$$

which, by abuse of notation, we have denoted again by $\overline{\partial}$.

Now we are going to describe the structure of $\mathcal{H}^0(\mathbf{\Gamma}, \mathbf{G})$. $\mathcal{H}^0(\mathbf{\Gamma}, \mathbf{G})$ corresponds to the categorical group of $\mathbf{\Gamma}$-invariant objects $\mathbf{G}^{\mathbf{\Gamma}}$ (see [26]). The associativity a, left unit l and right unit r of the monoidal structure of $\mathbf{G}^{\mathbf{\Gamma}}$ are given by the respective constraints a, l and r of $\mathbf{G}$ and they are all identity maps. Furthermore for any object in $\mathbf{G}^{\mathbf{\Gamma}}$ there exists a strict inverse. Thus $\mathbf{G}^{\mathbf{\Gamma}}$ is a strict categorical group.

Lemma 5.4.2. *A $\mathbf{\Gamma}$-invariant object of $\mathbf{G}$ is uniquely specified by a pair (g, θ), with $g \in G_0$ and a function $\theta : \Gamma_0 \to G_1$ satisfying*

$$\partial\, \theta(\sigma) = g\, {}^{\sigma}g^{-1} \tag{5.27}$$

$$\theta(\sigma\, \tau) = \theta(\sigma)\, {}^{\sigma}\theta(\tau) \tag{5.28}$$

Thus $Ob(\mathbf{G}^{\mathbf{\Gamma}}) = Z_B^1(\Gamma_0, \partial : G_1 \to G_0)$.

Proof. As in the braided case.

□

Proposition 5.4.2. *An arrow in the categorical group $\mathbf{G}^{\mathbf{\Gamma}}$ is uniquely specified by a triple (α, g_1, θ_1) with (g_1, θ_1) as in Lemma 5.4.2 and an element $\alpha \in G_1$. The source of (α, g_1, θ_1) is the $\mathbf{\Gamma}$-invariant object of $\mathbf{G}$ given by (g_1, θ_1); the target of (α, g_1, θ_1) is the $\mathbf{\Gamma}$-invariant object of $\mathbf{G}$ given by (g_2, θ_2) where $g_2 = \partial(\alpha)\, g_1$ and $\theta_2(\sigma) = \alpha\, \theta_1(\sigma)\, {}^{\sigma}\alpha^{-1}$.*

Proof. As in the braided case.

□

$\mathbf{G}^{\mathbf{\Gamma}}$ is a strict categorical group and the tensor product on objects (see [26]) is given by:

$$(g_1, \theta_1)\,(g_2, \theta_2) = (g_1\, g_2, \theta)$$

$$\begin{array}{ccc} {}^{\sigma}g_1 & {}^{\sigma}g_2 & {}^{\sigma}g_1\, {}^{\sigma}g_2 \\ \Big\downarrow {\scriptstyle (\theta_1(\sigma),\, {}^{\sigma}g_1)} & \Big\downarrow {\scriptstyle (\theta_2(\sigma),\, {}^{\sigma}g_2)} & \Big\downarrow {\scriptstyle (\theta_1(\sigma)\, {}^{\sigma g_1}\theta_2(\sigma),\, {}^{\sigma}g_1\, {}^{\sigma}g_2)} \\ g_1 & g_2 & g_1\, g_2 \end{array}$$

where

$$\begin{aligned} \theta(\sigma) &= \theta_1(\sigma)\, {}^{\sigma g_1}\theta_2(\sigma) = {}^{\partial\theta_1(\sigma)\, \sigma g_1}\theta_2(\sigma)\, \theta_1(\sigma) = {}^{g_1\, \sigma g_1^{-1}\, \sigma g_1}\theta_2(\sigma)\, \theta_1(\sigma) = \\ &= {}^{g_1}\theta_2(\sigma)\, \theta_1(\sigma). \end{aligned}$$

Thus this product in $Z^1_B(\Gamma_0, \partial : G_1 \to G_0)$ is the Borovoi product defined on 1-cochains.

Let

$$(g_1,\theta_1) \xrightarrow{(\alpha_1,g_1,\theta_1)} (\overline{g}_1,\overline{\theta}_1) \quad \text{and} \quad (g_2,\theta_2) \xrightarrow{(\alpha_2,g_2,\theta_2)} (\overline{g}_2,\overline{\theta}_2)$$

be two arrows in $\mathbf{G}^{\mathbf{\Gamma}}$, where $(\overline{g}_i,\overline{\theta}_i)$ are determined by $(\alpha_i, g_i, \theta_i)$ under the Proposition 5.4.2 for $i = 1, 2$, the tensor product of these two arrows is given by:

$$(\alpha_1, g_1, \theta_1)\,(\alpha_2, g_2, \theta_2) = (\alpha_1\, {}^{g_1}\alpha_2, g_1\, g_2, \theta)$$

where θ is defined as above.

Because $\mathbf{G}^{\mathbf{\Gamma}}$ is a strict categorical group it corresponds to the following crossed module:

$$\begin{array}{rcl} \overline{\overline{\partial}} : G_1 & \to & Z^1_B(\Gamma_0, \partial : G_1 \to G_0) \\ \alpha & \to & (\partial(\alpha), \theta_\alpha) \end{array}$$

where $\theta_\alpha(\sigma) = \alpha\, {}^{\sigma}\alpha^{-1}$ and the action of $Z^1_B(\Gamma_0, \partial : G_1 \to G_0)$ on G_1 is given by ${}^{(g,\theta)}\alpha = {}^{g}\alpha$ (the calculations are similar to the braided case).

$\mathcal{H}^0(\mathbf{\Gamma},\mathbf{G})$ is defined by the kernel of $\overline{\mathbf{T}} : \mathbf{G} \to \mathrm{Der}^*(\mathbf{\Gamma},\mathbf{G})$. The last one is defined on objects and on arrows

$$\begin{array}{rcl} \overline{T}_0 : G_0 & \longrightarrow & Z^{2\,*}_B(\Gamma_0, \partial : G_1 \to G_0) \\ g & \longrightarrow & (p_g, 1) \end{array} \quad \begin{array}{rcl} \overline{T}_1 : G_1 \rtimes G_0 & \longrightarrow & \mathrm{Der}^*_1(\mathbf{\Gamma},\mathbf{G}) \\ (\alpha, g) & \longrightarrow & (\theta, p_g, 1) \end{array}$$

respectively, where $p_g(\sigma) = g\, {}^{\sigma}g^{-1}$ and $\theta(\sigma) = \alpha\, {}^{g\, {}^{\sigma}g^{-1}}({}^{\sigma}\alpha^{-1})$.
There are natural isomorphisms $\overline{\nu}$ and $\overline{\chi}$ such that $(\mathbf{G}, \overline{\mathbf{T}}, \overline{\nu}, \overline{\chi})$ is a categorical $\mathrm{Der}^*(\mathbf{\Gamma},\mathbf{G})$-crossed module (see [14]).
In this case, we observe that the isomorphism $\overline{\chi}$ is given by the composition of the three morphisms:

$${}^{\overline{T}_0(g_1)}g_2\, g_1 = g_1\, {}^{p_0(g_2)}g_1^{-1}\, g_2\, g_1 \xrightarrow[= ({}^{g_1}\{g_2,g_1^{-1}\},\, g_1\, {}^{p_0(g_2)}g_1^{-1}\, g_2\, g_1)]{(1,g_1)\otimes(\{g_2,g_1^{-1}\},\, {}^{p_0(g_2)}g_1^{-1}\, g_2)\otimes(1,g_1) =} g_1\, g_2\, g_1^{-1}\, g_1 = g_1\, g_2$$

therefore $\overline{\chi}_{g_1,g_2} = ({}^{g_1}\{g_2, g_1^{-1}\}, g_1\, {}^{p_0(g_2)}g_1^{-1}\, g_2\, g_1)$. Thanks to this observation $\mathcal{H}^0(\mathbf{\Gamma},\mathbf{G})$ can be equipped with a braiding (Proposition 2.7 in [14]) given by:

$$g_2\, g_1 = g_2\, g_1 \xrightarrow{(\theta_1(p_0(g_2)),\, g_2\, g_1)} {}^{\overline{T}_0(g_1)}g_2\, g_1 \xrightarrow{\overline{\chi}_{g_1,g_2}} g_1\, g_2.$$

Then $\overline{\overline{\partial}} : G_1 \to Z^1_B(\Gamma_0, \partial : G_1 \to G_0)$ is also a braided crossed module with a braiding defined by:

$$\begin{array}{rcl} \{(g_1,\theta_1),(g_2,\theta_2)\} & = & {}^{g_1}\{g_2, g_1^{-1}\}\,\theta_1(p_0(g_2)) = \\ & = & {}^{\partial\theta_1(p_0(g_2))\, {}^{p_0(g_2)}g_1}\{g_2, g_1^{-1}\}\,\theta_1(p_0(g_2)) = \\ & = & \theta_1(p_0(g_2))\,\{g_2, g_1\}^{-1}. \end{array}$$

Moreover we can define a structure of 2-crossed module on $\overline{\overline{\partial}} : G_1 \to Z^1_B(\Gamma_0, \partial : G_1 \to G_0)$.

Proposition 5.4.3. *The complex of groups*

$$G_1 \xrightarrow{\overline{\overline{\partial}}} Z^1_B(\Gamma_0, \partial : G_1 \to G_0) \xrightarrow{p_0} \Gamma_0 \qquad (5.29)$$

together with:

- *the action of* Γ_0 *on* G_1 *determined by the 2-crossed module structure of (5.22);*
- *the action of* Γ_0 *on* $Z^1_B(\Gamma_0, \partial : G_1 \to G_0)$ *defined by* ${}^{\sigma}(g, \theta) = ({}^{\sigma}g, \overline{\theta})$*; where* $\overline{\theta}(\tau) = {}^{\sigma}\theta(\sigma^{-1}\,\tau\,\sigma)$*;*
- *the Peiffer lifting* $\{-,-\} : Z^1_B(\Gamma_0, \partial : G_1 \to G_0) \times Z^1_B(\Gamma_0, \partial : G_1 \to G_0) \to G_1$ *given by* $\{(g_1, \theta_1), (g_2, \theta_2)\} = \{g_1, g_2\}$*;*
- *the map* $p_0 : Z^1_B(\Gamma_0, \partial : G_1 \to G_0) \to \Gamma_0$*, by abuse of notation, given by* $p_0(g, \theta) = p_0(g)$*;*

is a 2-crossed module.

Proof. The calculations to show that the action of Γ_0 on $Z^1_B(\Gamma_0, \partial : G_1 \to G_0)$ is well defined, are similar to those used to prove that Γ_0 acts on $\mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}})$ in the braided case (see Proposition 5.3.3).

$G_1 \xrightarrow{\overline{\overline{\partial}}} Z^1_B(\Gamma_0, \partial : G_1 \to G_0) \xrightarrow{p_0} \Gamma_0$ is a normal complex of groups; in fact, we have:

- $p_0(\overline{\overline{\partial}}(\alpha)) = p_0(\partial(\alpha), \theta_\alpha) = p_0(\partial(\alpha)) = 1$;
- $\overline{\overline{\partial}}G_1 = B^1_B(\Gamma_0, \partial : G_1 \to G_0)$ is a normal subgroup of $Z^1_B(\Gamma_0, \partial : G_1 \to G_0)$;
- $p_0(Z^1_B(\Gamma_0, \partial : G_1 \to G_0)) \le p_0(G_0)$ and the last one is a normal subgroup of Γ_0.

We have already seen that $\overline{\overline{\partial}} : G_1 \to Z^1_B(\Gamma_0, \partial : G_1 \to G_0)$ is a group homomorphism and a crossed module. $p_0 : Z^1_B(\Gamma_0, \partial : G_1 \to G_0) \to \Gamma_0$ is a morphism of groups because $p_0 : G_0 \to \Gamma_0$ is.

Now we want to check the seven properties making (5.29) a 2-crossed module.

(i) the maps $\overline{\overline{\partial}}$ and p_0 are Γ_0-equivariant:

$$\begin{aligned} \overline{\overline{\partial}}({}^{\sigma}\alpha) &= (\partial({}^{\sigma}\alpha), \theta_{{}^{\sigma}\alpha}) = ({}^{\sigma}\partial(\alpha), \theta_{{}^{\sigma}\alpha}); \\ \theta_{{}^{\sigma}\alpha}(\tau) &= {}^{\sigma}\alpha\, {}^{\tau\sigma}\alpha^{-1}; \end{aligned}$$

$$\begin{aligned}
{}^{\sigma}\overline{\overline{\partial}}(\alpha) &= {}^{\sigma}(\partial(\alpha),\theta_\alpha) = ({}^{\sigma}\partial(\alpha),\overline{\theta}_\alpha);\\
\overline{\theta}_\alpha(\tau) &= {}^{\sigma}\theta_\alpha(\sigma^{-1}\,\tau\,\sigma) = {}^{\sigma}(\alpha\,{}^{\sigma^{-1}\tau\,\sigma}\alpha^{-1}) = {}^{\sigma}\alpha\,{}^{\tau\,\sigma}\alpha^{-1} =\\
&= \theta_{\,{}^{\sigma}\alpha}(\tau);\\
p_0({}^{\sigma}(g,\theta)) &= p_0({}^{\sigma}g) = {}^{\sigma}p_0(g) = {}^{\sigma}p_0(g,\theta).
\end{aligned}$$

(ii)

$$\begin{aligned}
&\overline{\overline{\partial}}\{(g_1,\theta_1),(g_2,\theta_2)\} = \overline{\overline{\partial}}\{g_1,g_2\} = (\partial\{g_1,g_2\},\theta_{\{g_1,g_2\}}) =\\
&= (g_1\,g_2\,g_1^{-1}\,{}^{p_0(g_1)}g_2^{-1},\theta_{\{g_1,g_2\}});\\
&(g_1,\theta_1)\,(g_2,\theta_2)\,(g_1,\theta_1)^{-1}\,{}^{p_0(g_1,\theta_1)}(g_2,\theta_2)^{-1} = (g_1,\theta_1)\,(g_2,\theta_2)\\
&(g_1^{-1},\theta_1^*)\,({}^{p_0(g_1)}g_2^{-1},\overline{\theta}_2^*) = (g_1\,g_2\,g_1^{-1}\,{}^{p_0(g_1)}g_2^{-1},\widehat{\theta}) \quad \text{where}\\
&\widehat{\theta}(\sigma) = {}^{g_1\,g_2\,g_1^{-1}}[\,{}^{p_0(g_1)}(\,{}^{g_2^{-1}}\theta_2(p_0(g_1)^{-1}\,\sigma\,p_0(g_1))^{-1})]\,{}^{g_1\,g_2\,g_1^{-1}}\theta_1(\sigma)^{-1}\\
&\qquad {}^{g_1}\theta_2(\sigma)\,\theta_1(\sigma).
\end{aligned}$$

Because (5.22) is a 2-crossed module, we can observe that

$$\begin{aligned}
p_0(g_1)^{-1}\,\sigma\,p_0(g_1) &= p_0(g_1^{-1})\,p_0({}^{\sigma}g_1)\,\sigma = p_0(g_1^{-1}\,{}^{\sigma}g_1)\,\sigma =\\
&= p_0(\partial(\theta_1^*(\sigma)))\,\sigma = \sigma
\end{aligned}$$

thus $\widehat{\theta}(\sigma)$ becomes:

$$\begin{aligned}
\widehat{\theta}(\sigma) &= {}^{g_1\,g_2\,g_1^{-1}}[\,{}^{p_0(g_1)}(\,{}^{g_2^{-1}}\theta_2(\sigma)^{-1})]\,{}^{g_1\,g_2\,g_1^{-1}}\theta_1(\sigma)^{-1}\,{}^{g_1}\theta_2(\sigma)\,\theta_1(\sigma) =\\
&= {}^{g_1\,g_2\,g_1^{-1}\,{}^{p_0(g_1)}g_2^{-1}}(\,{}^{p_0(g_1)}\theta_2(\sigma)^{-1})\,{}^{g_1\,g_2\,g_1^{-1}}\theta_1(\sigma)^{-1}\,{}^{g_1}\theta_2(\sigma)\\
&\quad\ \theta_1(\sigma) =\\
&= {}^{\partial\{g_1,g_2\}}(\,{}^{p_0(g_1)}\theta_2(\sigma)^{-1}\,{}^{p_0(g_1)g_2}\theta_1(\sigma)^{-1})\,{}^{g_1}\theta_2(\sigma)\,\theta_1(\sigma) =\\
&= \{g_1,g_2\}\,{}^{p_0(g_1)}\theta_2(\sigma)^{-1}\,{}^{p_0(g_1)g_2}\theta_1(\sigma)^{-1}\{g_1,g_2\}^{-1}\,{}^{g_1}\theta_2(\sigma)\\
&\quad\ \theta_1(\sigma) =\\
&= \{g_1,g_2\}\,\{\partial\theta_1(\sigma)^{-1}\,g_1,\partial\theta_2(\sigma)^{-1}\,g_2\}^{-1}\,\theta_1(\sigma)^{-1}\,{}^{g_1}\theta_2(\sigma)^{-1}\\
&\quad\ {}^{g_1}\theta_2(\sigma)\,\theta_1(\sigma) =\\
&= \{g_1,g_2\}\{\,{}^{\sigma}g_1,{}^{\sigma}g_2\}^{-1} = \{g_1,g_2\}\,{}^{\sigma}\{g_1,g_2\}^{-1} = \theta_{\{g_1,g_2\}}(\sigma)
\end{aligned}$$

(iii)

$$\begin{aligned}
&\{\overline{\overline{\partial}}(\alpha_1),\overline{\overline{\partial}}(\alpha_2)\} = \{(\partial(\alpha_1),\theta_{\alpha_1}),(\partial(\alpha_2),\theta_{\alpha_2})\} = \{\partial(\alpha_1),\partial(\alpha_2)\} =\\
&= \alpha_1\,\alpha_2\,\alpha_1^{-1}\,\alpha_2^{-1}.
\end{aligned}$$

(iv)

$$\{\overline{\overline{\partial}}(\alpha),(g,\theta)\}\,\{(g,\theta),\overline{\overline{\partial}}(\alpha)\} = \{(\partial(\alpha),\theta_\alpha),(g,\theta)\}\,\{(g,\theta),(\partial(\alpha),\theta_\alpha)\} =$$

$$= \{\partial(\alpha),g\}\,\{g,\partial(\alpha)\} = \alpha\,{}^{p_0(g)}\alpha^{-1} = \alpha\,{}^{p_0(g,\theta)}\alpha^{-1}.$$

(v)

$$\begin{aligned}
&\{(g_1,\theta_1),(g_2,\theta_2)\,(g_3,\theta_3)\} = \{(g_1,\theta_1),(g_2\,g_3,\,{}^{g_2}\theta_3\,\theta_2)\} = \\
&= \{g_1, g_2\,g_3\} = \{g_1,g_2\}^{\,p_0(g_1)g_2}\,\{g_1,g_3\} = \\
&= \{(g_1,\theta_1),(g_2,\theta_2)\}^{\,p_0(g_1,\theta_1)(g_2,\theta_2)}\,\{(g_1,\theta_1),(g_3,\theta_3)\}.
\end{aligned}$$

(vi)

$$\begin{aligned}
&\{(g_1,\theta_1)\,(g_2,\theta_2),(g_3,\theta_3)\} = \{(g_1\,g_2,\,{}^{g_1}\theta_2\,\theta_1),(g_3,\theta_3)\} = \\
&= \{g_1\,g_2, g_3\} = {}^{g_1}\{g_2,g_3\}\,\{g_1,\,{}^{p_0(g_2)}g_3\} = \\
&= {}^{(g_1,\theta_1)}\{(g_2,\theta_2),(g_3,\theta_3)\}\,\{(g_1,\theta_1),\,{}^{p_0(g_2,\theta_2)}(g_3,\theta_3)\}.
\end{aligned}$$

The previous relation holds because:

$$\begin{aligned}
{}^{(g,\theta)}\alpha &:= \alpha\,\{\overline{\overline{\partial}}\alpha^{-1},(g,\theta)\} = \alpha\,\{(\partial\alpha^{-1},\theta_{\alpha^{-1}}),(g,\theta)\} = \\
&= \alpha\,\{\partial\alpha^{-1},g\} =: {}^{g}\alpha
\end{aligned}$$

(vii) ${}^{\sigma}\{(g_1,\theta_1),(g_2,\theta_2)\} = {}^{\sigma}\{g_1,g_2\} = \{{}^{\sigma}g_1,{}^{\sigma}g_2\} = \{{}^{\sigma}(g_1,\theta_1),{}^{\sigma}(g_2,\theta_2)\}.$

□

Remark 5.4.1. *It is easy to observe (as discussed in section 4.1) that $Ob(\mathbf{G}^{\mathbf{\Gamma}}) = Z^1_B(\Gamma_0, \partial : G_1 \to G_0)$ coincides with the pullback of the pair of maps:*

$$\begin{array}{ccc}
 & & App(\Gamma_0, G_1) \\
 & & \downarrow \overline{\partial} \\
G_0 & \xrightarrow{\overline{T}_0} & Z^{2\,*}_B(\Gamma_0, \partial : G_1 \to G_0).
\end{array}$$

Now we are going to analyze $\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G})$, introduced in the generale case in [14]. $\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G})$ is a quotient categorical group defined in the following way:

$$\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G}) = \frac{\mathrm{Der}^*(\mathbf{\Gamma},\mathbf{G})}{<\mathbf{G},\overline{\mathbf{T}}>}.$$

We have $\mathrm{Ob}(\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G})) = \mathrm{Ob}(\mathrm{Der}^*(\mathbf{\Gamma},\mathbf{G})) = Z^{2\,*}_B(\Gamma_0, \partial : G_1 \to G_0)$ and the tensor product on objects in $\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G})$ is the same defined in $\mathrm{Der}^*(\mathbf{\Gamma},\mathbf{G})$. Then $\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G})$ is a strict categorical group because $\mathrm{Der}^*(\mathbf{\Gamma},\mathbf{G})$ is.

We are going to describe the morphisms in $\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G})$.

Proposition 5.4.4. *A premorphism in $\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G})$ is uniquely specified by $(g,\theta,p_2,\varepsilon_2)$ with $(p_2,\varepsilon_2) \in Ob(Der^*(\mathbf{\Gamma},\mathbf{G}))$, $g \in G_0$ and a function θ :*

$\Gamma_0 \to G_1$. The target of $(g, \theta, p_2, \varepsilon_2)$ is (p_2, ε_2) and the source of $(g, \theta, p_2, \varepsilon_2)$ is given by (p_1, ε_1) where

$$p_1(\sigma) = \partial\,\theta(\sigma)^{-1}\, g^{\,p_0(p_2(\sigma))\,\sigma} g^{-1}\, p_2(\sigma) \tag{5.30}$$

$$\begin{aligned}\varepsilon_1(\sigma,\tau) = \theta(\sigma)^{-1}\, {}^{g^{\,p_0(p_2(\sigma))\,\sigma} g^{-1}\, p_2(\sigma)}({}^{\sigma}\theta(\tau)^{-1})\\ {}^{g^{\,p_0(p_2(\sigma))\,\sigma} g^{-1}}\{p_2(\sigma),\, {}^{\sigma} g^{\,\sigma\, p_0(p_2(\tau))\,\tau} g^{-1}\}\\ {}^{g^{\,p_0(p_2(\sigma))\,\sigma\, p_0(p_2(\tau))\,\tau} g^{-1}}\varepsilon_2(\sigma,\tau)\,\theta(\sigma\,\tau)\end{aligned} \tag{5.31}$$

Proof. As in the braided case.

□

Definition 5.4.1. *A morphism in $\mathcal{H}^1(\mathbf{\Gamma}, \mathbf{G})$ from (p_1, ε_1) to (p_2, ε_2) is a class of premorphisms $[g, \theta, p_2, \varepsilon_2]$ where $(g, \theta, p_2, \varepsilon_2)$ and $(g', \theta', p_2, \varepsilon_2)$ are equivalent if there is an arrow in $\mathbf{G}$ from g to g', that is an $\alpha \in G_1$ such that $g' = \partial(\alpha)\, g$ and the diagram*

$$\begin{array}{ccc} p_1(\sigma) & \xrightarrow{(\theta(\sigma),p_1(\sigma))} & g^{\,p_0(p_2(\sigma))\,\sigma} g^{-1}\, p_2(\sigma) \\ & \searrow^{(\theta'(\sigma),p_1(\sigma))} \quad \swarrow_{(\alpha\, {}^{g^{\,p_0(p_2(\sigma))\,\sigma} g^{-1}}({}^{p_0(p_2(\sigma))\,\sigma}\alpha^{-1}),\, g^{\,p_0(p_2(\sigma))\,\sigma} g^{-1}\, p_2(\sigma))} & \\ & g'^{\,p_0(p_2(\sigma))\,\sigma} g'^{-1}\, p_2(\sigma) & \end{array}$$

commutes in $\mathbf{G}$. Therefore, we have:

$$\theta'(\sigma) = \alpha\, {}^{g^{\,p_0(p_2(\sigma))\,\sigma} g^{-1}}({}^{p_0(p_2(\sigma))\,\sigma}\alpha^{-1})\,\theta(\sigma).$$

Given two morphisms $(p_1, \varepsilon_1) \xrightarrow{[g,\theta,p_2,\varepsilon_2]} (p_2, \varepsilon_2) \xrightarrow{[g',\theta',p_3,\varepsilon_3]} (p_3, \varepsilon_3)$, we define their composition by:

$$(p_1, \varepsilon_1) \xrightarrow{[g\,g',\overline{\theta},p_3,\varepsilon_3]} (p_3, \varepsilon_3)$$

where $\overline{\theta}$ is given by:

$$\begin{array}{c} p_1(\sigma) \\ \big\downarrow {\scriptstyle (\theta(\sigma),\, p_1(\sigma))} \\ g^{\,p_0(p_2(\sigma))\,\sigma} g^{-1}\, p_2(\sigma) \\ \big\downarrow {\scriptstyle (1,\, g^{\,p_0(p_2(\sigma))\,\sigma} g^{-1}) \otimes (\theta'(\sigma),\, p_2(\sigma)) = = ({}^{g^{\,p_0(p_2(\sigma))\,\sigma} g^{-1}}\theta'(\sigma),\, g^{\,p_0(p_2(\sigma))\,\sigma} g^{-1}\, p_2(\sigma))} \\ \downarrow \end{array}$$

$$
\begin{array}{c}
g^{\,p_0(p_2(\sigma))\,\sigma}g^{-1}\, g'^{\,p_0(p_3(\sigma))\,\sigma}g'^{-1}\, p_3(\sigma) = \\
= g^{\,p_0(g'^{\,p_0(p_3(\sigma))\,\sigma}g'^{-1}\, p_3(\sigma))\,\sigma}g^{-1}\, g'^{\,p_0(p_3(\sigma))\,\sigma}g'^{-1}\, p_3(\sigma) \\
\Big| \\
{\scriptstyle (1,g)\otimes(\{g'^{\,p_0(p_3(\sigma))\,\sigma}g'^{-1},\ {}^{p_0(p_3(\sigma))\,\sigma}g^{-1}\},\ {}^{p_0(g'^{\,p_0(p_3(\sigma))\,\sigma}g'^{-1}\, p_3(\sigma))\,\sigma}g^{-1}\, g'^{\,p_0(p_3(\sigma))\,\sigma}g'^{-1})} \\
{\scriptstyle \otimes(1,p_3(\sigma)) = ({}^{g}\{g'^{\,p_0(p_3(\sigma))\,\sigma}g'^{-1},\ {}^{p_0(p_3(\sigma))\,\sigma}g^{-1}\},\, g^{\,p_0(p_2(\sigma))\,\sigma}g^{-1}\, g'^{\,p_0(p_3(\sigma))\,\sigma}g'^{-1}\, p_3(\sigma))} \\
\downarrow \\
g\, g'^{\,p_0(p_3(\sigma))\,\sigma}g'^{-1}\, {}^{p_0(p_3(\sigma))\,\sigma}g^{-1}\, p_3(\sigma).
\end{array}
$$

Therefore, we obtain:

$$\overline{\theta}(\sigma) = {}^{g}\{g'^{\,p_0(p_3(\sigma))\,\sigma}g'^{-1},\ {}^{p_0(p_3(\sigma))\,\sigma}g^{-1}\}\ {}^{g\,p_0(p_2(\sigma))\,\sigma}g^{-1}\theta'(\sigma)\,\theta(\sigma).$$

Given two morphisms

$$(p_1,\varepsilon_1) \xrightarrow{[g,\theta,p_1{}',\varepsilon_1{}']} (p_1{}',\varepsilon_1{}') \quad \text{and} \quad (p_2,\varepsilon_2) \xrightarrow{[g',\theta',p_2{}',\varepsilon_2{}']} (p_2{}',\varepsilon_2{}')$$

their tensor product is given by:

$$[g^{\,(p_1{}',\varepsilon_1{}')}g',\overline{\theta},p,\varepsilon] = [g\,p_1{}'(p_0(g'))\,g',\overline{\theta},p,\varepsilon]$$

where $(p,\varepsilon) = (p_1{}',\varepsilon_1{}')(p_2{}',\varepsilon_2{}')$ as in (5.25), (5.26). The function $\overline{\theta}$ is given by the composition of the three complicated morphisms. When we will describe the crossed module associated with the strict categorical group $\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G})$, we will calculate this product for particular morphisms.

$\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G})$ is a strict categorical group and it corresponds to the crossed module constructed as follows:

$$d : \mathrm{Ker}t \ \rightarrow \ \mathrm{Ob}(\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G})) = \mathrm{Ob}(\mathrm{Der}^*(\mathbf{\Gamma},\mathbf{G}))$$

with $d = s_{|\mathrm{Ker}t}$, where s and t are the source and target maps, respectively, of the underlying groupoid $\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G})$. We denote with $\mathcal{H}^1_1(\mathbf{\Gamma},\mathbf{G})$ the set of arrows in $\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G})$ and we consider the target map:

$$\begin{array}{rcl} t : \mathcal{H}^1_1(\mathbf{\Gamma},\mathbf{G}) & \longrightarrow & Z^{2\ *}_B(\Gamma_0,\partial : G_1 \to G_0) \\ (g,\theta,p_2,\varepsilon_2) & \longrightarrow & (p_2,\varepsilon_2) \end{array}$$

while the source map:

$$\begin{array}{rcl} s : \mathcal{H}^1_1(\mathbf{\Gamma},\mathbf{G}) & \longrightarrow & Z^{2\ *}_B(\Gamma_0,\partial : G_1 \to G_0) \\ (g,\theta,p_2,\varepsilon_2) & \longrightarrow & (p_1,\varepsilon_1) \end{array}$$

where (p_1,ε_1) as in Proposition 5.4.4.
Thus we have:

$$\begin{array}{rcl} d : \mathrm{Ker}t & \rightarrow & Z^{2\ *}_B(\Gamma_0,\partial : G_1 \to G_0) \\ (g,\theta,1,1) & \rightarrow & (p,\varepsilon) \end{array}$$

where:

- $p(\sigma) = \partial\,\theta(\sigma)^{-1}\, g\, {}^{\sigma}g^{-1}$,
- $\varepsilon(\sigma,\tau) = \theta(\sigma)^{-1\;g\,{}^{\sigma}g^{-1}}({}^{\sigma}\theta(\tau)^{-1})\,\theta(\sigma\,\tau)$.

Given two arrows in $\mathrm{Ker}t$

$$(p_1,\varepsilon_1) \xrightarrow{[g,\theta,1,1]} (1,1) \qquad\qquad (p_2,\varepsilon_2) \xrightarrow{[g',\theta',1,1]} (1,1)$$

where:

- $p_1(\sigma) = \partial\,\theta(\sigma)^{-1}\, g\, {}^{\sigma}g^{-1}$,
- $p_2(\sigma) = \partial\,\theta'(\sigma)^{-1}\, g'\, {}^{\sigma}g'^{-1}$,
- $\varepsilon_1(\sigma,\tau) = \theta(\sigma)^{-1\;g\,{}^{\sigma}g^{-1}}({}^{\sigma}\theta(\tau)^{-1})\,\theta(\sigma\,\tau)$,
- $\varepsilon_2(\sigma,\tau) = \theta'(\sigma)^{-1\;g'\,{}^{\sigma}g'^{-1}}({}^{\sigma}\theta'(\tau)^{-1})\,\theta'(\sigma\,\tau)$,

their product is given by:

$$(p_1,\varepsilon_1)\,(p_2,\varepsilon_2) \xrightarrow{[g\,^{(1,1)}g',\tilde{\theta},1,1]=[g\,g',\tilde{\theta},1,1]} (1,1)\ .$$

The map $\tilde{\theta}$ is defined by the composition of the following two morphisms:

$$\begin{array}{c}
p_1(p_0(p_2(\sigma))\,\sigma)\,p_2(\sigma)\\
\Big\downarrow\ {\scriptstyle (\theta(p_0(p_2(\sigma))\,\sigma),\,p_1(p_0(p_2(\sigma))\,\sigma)\otimes(\theta'(\sigma),\,p_2(\sigma))) = \;=(\theta(p_0(p_2(\sigma))\,\sigma)\,{}^{p_1(p_0(p_2(\sigma))\,\sigma)}\theta'(\sigma),\,p_1(p_0(p_2(\sigma))\,\sigma)\,p_2(\sigma))}\\
g^{\,p_0(p_2(\sigma))\,\sigma}g^{-1}\,g'\,{}^{\sigma}g'^{-1} = g^{\,p_0(g'\,{}^{\sigma}g'^{-1})\,\sigma}g^{-1}\,g'\,{}^{\sigma}g'^{-1}\\
\Big\downarrow\ {\scriptstyle (1,g)\otimes(\{g'\,{}^{\sigma}g'^{-1},\ {}^{\sigma}g^{-1}\},\ {}^{p_0(g'\,{}^{\sigma}g'^{-1})\,\sigma}g^{-1}\,g'\,{}^{\sigma}g'^{-1}) = \;=({}^{g}\{g'\,{}^{\sigma}g'^{-1},\ {}^{\sigma}g^{-1}\},\,g\,{}^{p_0(g'\,{}^{\sigma}g'^{-1})\,\sigma}g^{-1}\,g'\,{}^{\sigma}g'^{-1})}\\
g\,g'\,{}^{\sigma}g'^{-1}\,{}^{\sigma}g^{-1}
\end{array}$$

therefore we have:

$$\begin{aligned}
\tilde{\theta}(\sigma) &= {}^{g}\{g'\,{}^{\sigma}g'^{-1},\ {}^{\sigma}g^{-1}\}\,\theta(p_0(p_2(\sigma))\,\sigma)\,{}^{p_1(p_0(p_2(\sigma))\,\sigma)}\theta'(\sigma) =\\
&= {}^{g}\{g'\,{}^{\sigma}g'^{-1},\ {}^{\sigma}g^{-1}\}\,\theta(p_0(g'\,{}^{\sigma}g'^{-1})\,\sigma)\,{}^{p_1(p_0(g'\,{}^{\sigma}g'^{-1})\,\sigma)}\theta'(\sigma) =\\
&= {}^{g}\{g'\,{}^{\sigma}g'^{-1},\ {}^{\sigma}g^{-1}\}\,{}^{g\,{}^{p_0(g'\,{}^{\sigma}g'^{-1})\,\sigma}g^{-1}}\theta'(\sigma)\,\theta(p_0(g'\,{}^{\sigma}g'^{-1})\,\sigma). \qquad (5.32)
\end{aligned}$$

The action of the element $(p,\varepsilon) \in Z_B^{2\,*}(\Gamma_0, \partial : G_1 \to G_0)$ on $\mathrm{Ker}t$ is given by:

$$^{(p,\varepsilon)}[g,\theta,1,1] = i(p,\varepsilon)\,[g,\theta,1,1]\,i(p,\varepsilon)^{-1}.$$

We recall that the map i for the groupoid $\mathcal{H}^1(\mathbf{\Gamma}, \mathbf{G})$ is defined by:

$$\begin{array}{rcl} i : {Z_B^2}^*(\Gamma_0, \partial : G_1 \to G_0) & \longrightarrow & \mathcal{H}^1_1(\mathbf{\Gamma}, \mathbf{G}) \\ (p, \varepsilon) & \longrightarrow & (1, 1, p, \varepsilon). \end{array}$$

Therefore, using the tensor product on the arrows of $\mathcal{H}^1_1(\mathbf{\Gamma}, \mathbf{G})$ and the inverse in ${Z_B^2}^*(\Gamma_0, \partial : G_1 \to G_0)$, we have:

$$\begin{array}{rcl} {}^{(p,\varepsilon)}[g, \theta, 1, 1] & = & [1, 1, p, \varepsilon]\,[g, \theta, 1, 1]\,[1, 1, p, \varepsilon]^{-1} = \\ & = & [1, 1, p, \varepsilon]\,[g, \theta, 1, 1]\,[1, 1, p^*, \varepsilon^*] = \\ & = & [p(p_0(g))\,g, \hat{\theta}, p, \varepsilon]\,[1, 1, p^*, \varepsilon^*] = \\ & = & [p(p_0(g))\,g\,p(1), \overline{\theta}, 1, 1]. \end{array}$$

The map $\hat{\theta}$ is obtained by the composition of the following morphisms:

$$\begin{array}{c} p(p_0(\partial\,\theta(\sigma)^{-1}\,g\,{}^{\sigma}g^{-1})\,\sigma)\,\partial\,\theta(\sigma)^{-1}\,g\,{}^{\sigma}g^{-1} = \\ = p(p_0(g\,{}^{\sigma}g^{-1})\,\sigma)\,\partial\,\theta(\sigma)^{-1}\,g\,{}^{\sigma}g^{-1} \\ \Big\downarrow\ {\scriptstyle (1,\, p(p_0(g\,{}^{\sigma}g^{-1})\,\sigma)) \otimes (\theta(\sigma),\, \partial\,\theta(\sigma)^{-1}\,g\,{}^{\sigma}g^{-1}) = \; = ({}^{p(p_0(g\,{}^{\sigma}g^{-1})\,\sigma)}\theta(\sigma),\, p(p_0(g\,{}^{\sigma}g^{-1})\,\sigma)\,\partial\,\theta(\sigma)^{-1}\,g\,{}^{\sigma}g^{-1})} \\ p(p_0(g\,{}^{\sigma}g^{-1})\,\sigma)\,g\,{}^{\sigma}g^{-1} \\ \Big\downarrow\ {\scriptstyle ((\overline{\nu}_{(p,\varepsilon),g})_{\sigma})^{-1}} \\ p(p_0(g))\,g\,{}^{p_0(p(\sigma))\,\sigma}(p(p_0(g))\,g)^{-1}\,p(\sigma) \end{array}$$

where $(\overline{\nu}_{(p,\varepsilon),g})_{\sigma}$ for any $\sigma \in \Gamma_0$ is given by the following composition (see Proposition 5.6. in [14]):

$$\begin{array}{c} p(p_0(g))\,g\,{}^{p_0(p(\sigma))\,\sigma}(p(p_0(g))\,g)^{-1}\,p(\sigma) \\ \Big\downarrow\ {\scriptstyle (1,\, p(p_0(g))\,g) \otimes (\{p(\sigma),\, {}^{\sigma}g^{-1}\,{}^{\sigma}p(p_0(g))^{-1}\},\, {}^{p_0(p(\sigma))\,\sigma}(p(p_0(g))\,g)^{-1}\,p(\sigma)) = \; = ({}^{p(p_0(g))\,g}\{p(\sigma),\, {}^{\sigma}g^{-1}\,{}^{\sigma}p(p_0(g))^{-1}\},\, p(p_0(g))\,g\,{}^{p_0(p(\sigma))\,\sigma}(p(p_0(g))\,g)^{-1}\,p(\sigma))} \\ p(p_0(g))\,g\,p(\sigma)\,{}^{\sigma}g^{-1}\,{}^{\sigma}p(p_0(g))^{-1} \\ \Big\downarrow\ {\scriptstyle (1,\, p(p_0(g))) \otimes (\{g, p(\sigma)\},\, {}^{p_0(g)}p(\sigma)\,g)^{-1} \otimes (1,\, {}^{\sigma}g^{-1}\,{}^{\sigma}p(p_0(g))^{-1}) = \; = ({}^{p(p_0(g))}\{g, p(\sigma)\}^{-1},\, p(p_0(g))\,g\,p(\sigma)\,{}^{\sigma}g^{-1}\,{}^{\sigma}p(p_0(g))^{-1})} \\ p(p_0(g))\,{}^{p_0(g)}p(\sigma)\,g\,{}^{\sigma}g^{-1}\,{}^{\sigma}p(p_0(g))^{-1} \\ \Big\downarrow\ {\scriptstyle (1,\, p(p_0(g))\,{}^{p_0(g)}p(\sigma)) \otimes (\{g\,{}^{\sigma}g^{-1},\, {}^{\sigma}p(p_0(g))^{-1}\},\, {}^{p_0(g\,{}^{\sigma}g^{-1})\sigma}p(p_0(g))^{-1}\,g\,{}^{\sigma}g^{-1})^{-1} = \; = ({}^{p(p_0(g))\,{}^{p_0(g)}p(\sigma)}\{g\,{}^{\sigma}g^{-1},\, {}^{\sigma}p(p_0(g))^{-1}\}^{-1},\, p(p_0(g))\,{}^{p_0(g)}p(\sigma)\,g\,{}^{\sigma}g^{-1}\,{}^{\sigma}p(p_0(g))^{-1})} \end{array}$$

$$p(p_0(g))\,{}^{p_0(g)}p(\sigma)\,{}^{p_0(g\,{}^{\sigma}g^{-1})\sigma}p(p_0(g))^{-1}\,g\,{}^{\sigma}g^{-1}$$

$$\Vert$$

$$p(p_0(g))\,{}^{p_0(g)}[p(\sigma)\,{}^{p_0({}^{\sigma}g^{-1})\sigma}p(p_0(g))^{-1}]\,g\,{}^{\sigma}g^{-1}$$

$$\Vert$$

$$p(p_0(g))\,{}^{p_0(g)}[p(\sigma)\,{}^{\sigma\,p_0(g^{-1})}p(p_0(g))^{-1}]\,g\,{}^{\sigma}g^{-1}$$

$$\Big\downarrow \; (1,p(p_0(g)))\otimes{}^{p_0(g)}[(1,p(\sigma))\otimes{}^{\sigma}(\varepsilon(p_0(g)^{-1},p_0(g))\,\varepsilon(1,1),\,{}^{p_0(g)^{-1}}p(p_0(g))^{-1})]\otimes(1,g\,{}^{\sigma}g^{-1}) =$$
$$= ({}^{p(p_0(g))\,p_0(g)}p(\sigma)[\,{}^{p_0(g)\,\sigma}\varepsilon(p_0(g)^{-1},p_0(g))\,{}^{p_0(g)\,\sigma}\varepsilon(1,1)],\,p(p_0(g))\,{}^{p_0(g)}[p(\sigma)\,{}^{\sigma\,p_0(g^{-1})}p(p_0(g))^{-1}]\,g\,{}^{\sigma}g^{-1})$$

$$p(p_0(g))\,{}^{p_0(g)}[p(\sigma)\,{}^{\sigma}p(p_0(g^{-1}))]\,g\,{}^{\sigma}g^{-1}$$

$$\Big\downarrow \; (1,p(p_0(g)))\otimes{}^{p_0(g)}(\varepsilon(\sigma,p_0(g^{-1})),p(\sigma\,p_0(g^{-1})))^{-1}\otimes(1,g\,{}^{\sigma}g^{-1}) =$$
$$= ({}^{p(p_0(g))}[\,{}^{p_0(g)}\varepsilon(\sigma,p_0(g^{-1}))^{-1}],\,p(p_0(g))\,{}^{p_0(g)}[p(\sigma)\,{}^{\sigma}p(p_0(g^{-1}))]\,g\,{}^{\sigma}g^{-1})$$

$$p(p_0(g))\,{}^{p_0(g)}p(\sigma\,p_0(g^{-1}))\,g\,{}^{\sigma}g^{-1}$$

$$\Vert$$

$$p(p_0(g))\,{}^{p_0(g)}p(p_0({}^{\sigma}g^{-1})\,\sigma)\,g\,{}^{\sigma}g^{-1}$$

$$\Big\downarrow \; (\varepsilon(p_0(g),p_0({}^{\sigma}g^{-1})\,\sigma),p(p_0(g)\,p_0({}^{\sigma}g^{-1})\,\sigma))^{-1}\otimes(1,g\,{}^{\sigma}g^{-1}) =$$
$$= (\varepsilon(p_0(g),p_0({}^{\sigma}g^{-1})\,\sigma)^{-1},\,p(p_0(g))\,{}^{p_0(g)}p(p_0({}^{\sigma}g^{-1})\,\sigma)\,g\,{}^{\sigma}g^{-1})$$

$$p(p_0(g)\,p_0({}^{\sigma}g^{-1})\,\sigma)\,g\,{}^{\sigma}g^{-1}$$

$$\Vert$$

$$p(p_0(g\,{}^{\sigma}g^{-1})\,\sigma)\,g\,{}^{\sigma}g^{-1}.$$

We obtain:

$$\begin{aligned}
(\overline{\nu}_{(p,\varepsilon),g})_{\sigma} &= (\varepsilon(p_0(g),p_0({}^{\sigma}g^{-1})\,\sigma)^{-1}\,{}^{p(p_0(g))}[\,{}^{p_0(g)}\varepsilon(\sigma,p_0(g^{-1}))^{-1}] \\
&\quad {}^{p(p_0(g))\,p_0(g)}p(\sigma)[\,{}^{p_0(g)\,\sigma}\varepsilon(p_0(g)^{-1},p_0(g))\,{}^{p_0(g)\,\sigma}\varepsilon(1,1)] \\
&\quad {}^{p(p_0(g))\,p_0(g)}p(\sigma)\{g\,{}^{\sigma}g^{-1},\,{}^{\sigma}p(p_0(g))^{-1}\}^{-1}\,{}^{p(p_0(g))}\{g,p(\sigma)\}^{-1} \\
&\quad {}^{p(p_0(g))\,g}\{p(\sigma),\,{}^{\sigma}g^{-1}\,{}^{\sigma}p(p_0(g))^{-1}\},p(p_0(g))\,g \\
&\quad {}^{p_0(p(\sigma))\,\sigma}(p(p_0(g))\,g)^{-1}\,p(\sigma)); \qquad (5.33) \\
((\overline{\nu}_{(p,\varepsilon),g})_{\sigma})^{-1} &= ({}^{p(p_0(g))\,g}\{p(\sigma),\,{}^{\sigma}g^{-1}\,{}^{\sigma}p(p_0(g))^{-1}\}^{-1}\,{}^{p(p_0(g))}\{g,p(\sigma)\} \\
&\quad {}^{p(p_0(g))\,p_0(g)}p(\sigma)\{g\,{}^{\sigma}g^{-1},\,{}^{\sigma}p(p_0(g))^{-1}\} \\
&\quad {}^{p(p_0(g))\,p_0(g)}p(\sigma)[\,{}^{p_0(g)\,\sigma}\varepsilon(p_0(g)^{-1},p_0(g))\,{}^{p_0(g)\,\sigma}\varepsilon(1,1)]^{-1} \\
&\quad {}^{p(p_0(g))}[\,{}^{p_0(g)}\varepsilon(\sigma,p_0(g^{-1}))]\,\varepsilon(p_0(g),p_0({}^{\sigma}g^{-1})\,\sigma),
\end{aligned}$$

$$p(p_0(g\,{}^{\sigma}g^{-1})\,\sigma)\,g\,{}^{\sigma}g^{-1}); \tag{5.34}$$

and accordingly we have

$$\begin{aligned}\hat{\theta}(\sigma) \quad = \quad & {}^{p(p_0(g))\,g}\{p(\sigma),{}^{\sigma}\,g^{-1}\,{}^{\sigma}p(p_0(g))^{-1}\}^{-1}\,{}^{p(p_0(g))}\{g, p(\sigma)\} \\ & {}^{p(p_0(g))\,{}^{p_0(g)}p(\sigma)}\{g\,{}^{\sigma}g^{-1},\ {}^{\sigma}p(p_0(g))^{-1}\} \\ & {}^{p(p_0(g))\,{}^{p_0(g)}p(\sigma)}[{}^{p_0(g)\,\sigma}\varepsilon(p_0(g)^{-1}, p_0(g))\,{}^{p_0(g)\,\sigma}\varepsilon(1,1)]^{-1} \\ & {}^{p(p_0(g))}[{}^{p_0(g)}\varepsilon(\sigma, p_0(g)^{-1})]\,\varepsilon(p_0(g), p_0({}^{\sigma}g^{-1})\sigma)\,{}^{p(p_0(g\,{}^{\sigma}g^{-1})\sigma)}\theta(\sigma).\end{aligned}$$

$\overline{\theta}$ is obtained by the composition of following morphisms:

$$p(p_0(g^{\,p_0(p^*(\sigma))\,\sigma}g^{-1})\,p_0(p^*(\sigma))\,\sigma)\,\partial\,\theta(p_0(p^*(\sigma))\,\sigma)^{-1}\,g^{\,p_0(p^*(\sigma))\,\sigma}g^{-1}\,p^*(\sigma)$$

$$\Big\downarrow\ {\scriptstyle (\hat{\theta}(p_0(p^*(\sigma))\,\sigma),\, p(p_0(g^{\,p_0(p^*(\sigma))\,\sigma}g^{-1})\,p_0(p^*(\sigma))\,\sigma)\,\partial\,\theta(p_0(p^*(\sigma))\,\sigma)^{-1}\,g^{\,p_0(p^*(\sigma))\,\sigma}g^{-1})\otimes(1, p^*(\sigma)) =}$$
$${\scriptstyle = (\hat{\theta}(p_0(p^*(\sigma))\,\sigma),\, p(p_0(g^{\,p_0(p^*(\sigma))\,\sigma}g^{-1})\,p_0(p^*(\sigma))\,\sigma)\,\partial\,\theta(p_0(p^*(\sigma))\,\sigma)^{-1}\,g^{\,p_0(p^*(\sigma))\,\sigma}g^{-1}\,p^*(\sigma))}$$

$$\begin{aligned}& p(p_0(g))\,g^{\,p_0(p(p_0(p^*(\sigma))\,\sigma))\,p_0(p^*(\sigma))\,\sigma}(p(p_0(g))\,g)^{-1}\,p(p_0(p^*(\sigma))\,\sigma)\,p^*(\sigma) = \\ &= p(p_0(g))\,g^{\,p_0(p(p_0(p^*(\sigma))\,\sigma)\,p^*(\sigma))\,\sigma}(p(p_0(g))\,g)^{-1} \\ &= p(p_0(g))\,g\,{}^{\sigma}(p(p_0(g))\,g)^{-1}\end{aligned}$$

$$\Big\downarrow\ {\scriptstyle (1, p(p_0(g))\,g\,{}^{\sigma}(p(p_0(g))\,g)^{-1})\otimes((\overline{\nu}_{(p,\varepsilon),1})_{\sigma})^{-1}\otimes(1, p^*(\sigma))}$$

$$\begin{aligned}& p(p_0(g))\,g^{\,p_0(p(1)\,{}^{\sigma}p(1)^{-1})\sigma}(p(p_0(g))\,g)^{-1}\,p(1)\,{}^{\sigma}p(1)^{-1} = \\ &= p(p_0(g))\,g\,{}^{\sigma}(p(p_0(g))\,g)^{-1}\,p(1)\,{}^{\sigma}p(1)^{-1}\end{aligned}$$

$$\Big\downarrow\ {\scriptstyle (1, p(p_0(g))\,g)\otimes(\{p(1)\,{}^{\sigma}p(1)^{-1},\ {}^{\sigma}(p(p_0(g))\,g)^{-1}\},\ {}^{\sigma}(p(p_0(g))\,g)^{-1}\,p(1)\,{}^{\sigma}p(1)^{-1}) =}$$
$${\scriptstyle = ({}^{p(p_0(g))\,g}\{p(1)\,{}^{\sigma}p(1)^{-1},\ {}^{\sigma}(p(p_0(g))\,g)^{-1}\},\, p(p_0(g))\,g\,{}^{\sigma}(p(p_0(g))\,g)^{-1}\,p(1)\,{}^{\sigma}p(1)^{-1})}$$

$$p(p_0(g))\,g\ p(1)^{\sigma}(p(p_0(g))\,g\,p(1))^{-1}.$$

The domain of the second arrow is simplified because we know that

$$p(p_0(p^*(\sigma))\,\sigma)\,p^*(\sigma) = 1$$

where p^* is the inverse of p. The codomain is simplified because we can easily observe that $p(1) = \partial\,\varepsilon(1,1)$ (with $(p,\varepsilon) \in {Z_B^2}^*(\Gamma_0, \partial : G_1 \to G_0)$) and in a 2-crossed module it holds $p_0 \circ \partial = 1$.

In the second morphism, the first component of the arrow $((\overline{\nu}_{(p,\varepsilon),1})_{\sigma})^{-1}$, thanks to (5.34), is given by:

$$\begin{aligned}\pi_{G_1}[((\overline{\nu}_{(p,\varepsilon),1})_{\sigma})^{-1}] \quad &= \quad {}^{p(1)}\{p(\sigma),\ {}^{\sigma}p(1)^{-1}\}^{-1}\,{}^{p(1)\,p(\sigma)}[{}^{\sigma}\varepsilon(1,1)\,{}^{\sigma}\varepsilon(1,1)]^{-1} \\ & \quad\ {}^{p(1)}\varepsilon(\sigma,1)\,\varepsilon(1,\sigma) = \\ &= \quad {}^{\partial\,\varepsilon(1,1)}\{p(\sigma),\ {}^{\sigma}\partial\,\varepsilon(1,1)^{-1}\}^{-1}\,{}^{\partial\,\varepsilon(1,1)\,p(\sigma)}[{}^{\sigma}\varepsilon(1,1) \\ & \quad\ {}^{\sigma}\varepsilon(1,1)]^{-1}\,{}^{\partial\,\varepsilon(1,1)}\varepsilon(\sigma,1)\,\varepsilon(1,\sigma) =\end{aligned}$$

$$\begin{aligned}
&= \varepsilon(1,1)\,\{p(\sigma), \partial({}^{\sigma}\varepsilon(1,1)^{-1})\}^{-1}\,{}^{p(\sigma)}({}^{\sigma}\varepsilon(1,1))^{-1} \\
&\quad {}^{p(\sigma)}({}^{\sigma}\varepsilon(1,1)^{-1})\,\varepsilon(\sigma,1)\,\varepsilon(1,1)^{-1}\,\varepsilon(1,\sigma) = \\
&= \varepsilon(1,1)\,{}^{p_0(p(\sigma))}({}^{\sigma}\varepsilon(1,1)^{-1})\,{}^{p(\sigma)}({}^{\sigma}\varepsilon(1,1)) \\
&\quad {}^{p(\sigma)}({}^{\sigma}\varepsilon(1,1))^{-1} = \\
&= \varepsilon(1,1)\,{}^{p_0(p(\sigma))\,\sigma}\varepsilon(1,1)^{-1}.
\end{aligned}$$

In the last equalities, we have used the following relations:

1. since $(p,\varepsilon) \in Z_B^{2\,*}(\Gamma_0, \partial : G_1 \to G_0)$ then:

$$\begin{aligned}
&p(1) = \partial\,\varepsilon(1,1); \\
&{}^{p(\sigma)}({}^{\sigma}\varepsilon(1,1)) = \varepsilon(\sigma,1); \\
&\varepsilon(1,\sigma) = \varepsilon(1,1);
\end{aligned}$$

2. since $G_1 \xrightarrow{\partial} G_0 \xrightarrow{p_0} \Gamma_0$ is a 2-crossed module, we have

$$\{g, \partial(\alpha)\} = {}^{g}\alpha\,{}^{p_0(g)}\alpha^{-1}.$$

Therefore we have:

$$\begin{aligned}
\overline{\theta}(\sigma) &= {}^{p(p_0(g))\,g}\{p(1)\,{}^{\sigma}p(1)^{-1},\ {}^{\sigma}(p(p_0(g))\,g)^{-1}\} \\
&\quad {}^{p(p_0(g))\,g\,{}^{\sigma}(p(p_0(g))\,g)^{-1}}(\varepsilon(1,1)\,{}^{p_0(p(\sigma))\,\sigma}\varepsilon(1,1)^{-1})\,\hat{\theta}(p_0(p^*(\sigma))\,\sigma) = \\
&= {}^{p(p_0(g))\,g}\{\partial(\varepsilon(1,1)\,{}^{\sigma}\varepsilon(1,1)^{-1}),\ {}^{\sigma}(p(p_0(g))\,g)^{-1}\} \\
&\quad {}^{p(p_0(g))\,g\,{}^{\sigma}(p(p_0(g))\,g)^{-1}}(\varepsilon(1,1)\,{}^{p_0(p(\sigma))\,\sigma}\varepsilon(1,1)^{-1})\,\hat{\theta}(p_0(p^*(\sigma))\,\sigma) = \\
&= {}^{p(p_0(g))\,g}(\varepsilon(1,1)\,{}^{\sigma}\varepsilon(1,1)^{-1})\,{}^{p(p_0(g))\,g\,{}^{\sigma}(p(p_0(g))\,g)^{-1}}(\varepsilon(1,1) \\
&\quad {}^{p_0(p(\sigma))\,\sigma}\varepsilon(1,1)^{-1})^{-1}\,{}^{p(p_0(g))\,g\,{}^{\sigma}(p(p_0(g))\,g)^{-1}}(\varepsilon(1,1)\,{}^{p_0(p(\sigma))\,\sigma}\varepsilon(1,1)^{-1}) \\
&\quad \hat{\theta}(p_0(p^*(\sigma))\,\sigma) = \\
&= {}^{p(p_0(g))\,g}(\varepsilon(1,1)\,{}^{\sigma}\varepsilon(1,1)^{-1})\,\hat{\theta}(p_0(p^*(\sigma))\,\sigma) = \\
&= {}^{p(p_0(g))\,g}\varepsilon(1,1)\,{}^{p(p_0(g))\,g\,{}^{\sigma}(p(p_0(g))\,g)^{-1}}[{}^{\sigma}(\,{}^{p(p_0(g))\,g}\varepsilon(1,1)^{-1})] \\
&\quad \hat{\theta}(p_0(p^*(\sigma))\,\sigma).
\end{aligned}$$

In the last equalities, we have used together with the relations above the following:

- since $G_1 \xrightarrow{\partial} G_0 \xrightarrow{p_0} \Gamma_0$ is a 2 crossed module we have

$$\{\partial(\alpha), g\} = \alpha\,{}^{g}\alpha^{-1}.$$

Because $p(p_0(g))\,g\,p(1) = p(p_0(g))\,g\,\partial\varepsilon(1,1) = \partial(\,{}^{p(p_0(g))\,g}\varepsilon(1,1))p(p_0(g))\,g$ we can observe that:

$${}^{(p,\varepsilon)}[g,\theta,1,1] = [p(p_0(g))\,g\,p(1), \overline{\theta}, 1, 1] = [p(p_0(g))\,g, \widetilde{\theta}, 1, 1]$$

where $\widetilde{\theta}(\sigma) = \hat{\theta}(p_0(p^*(\sigma))\sigma)$.

We can prove that:

$$\mathrm{Ker}t \quad \text{is isomorphic to} \quad \frac{C^1_B(\Gamma_0, \partial : G_1 \to G_0)}{B^1_B(\Gamma_0, \partial : G_1 \to G_0)}.$$

Thanks to the definition of the product in $\mathrm{Ker}t$, we have:

$$[\partial(\alpha), \theta_\alpha, 1, 1][g_1, \theta_1, 1, 1] = [\partial(\alpha)\, g_1, \hat{\theta}, 1, 1]$$

where

$$\begin{aligned} \hat{\theta}(\sigma) &= {}^{\partial(\alpha)}\{g_1\, {}^\sigma g_1^{-1}, {}^\sigma\partial(\alpha)^{-1}\}\, {}^{\partial(\alpha)\, {}^{p_0(g_1\, {}^\sigma g_1^{-1})\sigma}\partial(\alpha)^{-1}}\theta_1(\sigma) \\ &\quad \theta_\alpha(p_0(g_1\, {}^\sigma g_1^{-1})\sigma) = \\ &= \alpha\, \{g_1\, {}^\sigma g_1^{-1}, \partial({}^\sigma\alpha^{-1})\}\, {}^{p_0(g_1\, {}^\sigma g_1^{-1})\sigma}\alpha^{-1}\theta_1(\sigma) = \\ &= \alpha\, {}^{g_1\, {}^\sigma g_1^{-1}}({}^\sigma\alpha^{-1})\theta_1(\sigma). \end{aligned}$$

We want to emphasize that in the third passage we have used the property (c) (see section 2.3) of the 2-crossed module (5.22).
When restricted to $Z^1_B(\Gamma_0, \partial : G_1 \to G_0)$, the product defined in (5.32) coincides with the Borovoi product.
It is clear the isomorphism between d and a homomorphism

$$d : \frac{C^1_B(\Gamma_0, \partial : G_1 \to G_0)}{B^1_B(\Gamma_0, \partial : G_1 \to G_0)} \quad \to \quad Z^{2\,*}_B(\Gamma_0, \partial : G_1 \to G_0)$$

which, by abuse of notation, we have denoted again by d.

Remark 5.4.2. *The Noohi cohomology is a particular case of the cohomology in 2-crossed modules because every Γ_0-equivariant braided crossed module can be seen as a 2-crossed module as in (5.22) with $p_0 = 1$.*

5.5 Cohomology in crossed squares

Let

$$\begin{array}{ccc} G_1 & \xrightarrow{\bar{p}_1} & \Gamma_1 \\ {\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial'} \\ G_0 & \xrightarrow{\bar{p}_0} & \Gamma_0 \end{array} \tag{5.35}$$

be a crossed square. If we call $\mathbf{G}$ the strict categorical group associated with $\partial : G_1 \to G_0$ and $\mathbf{\Gamma}$ the strict categorical group associated with $\partial' : \Gamma_1 \to \Gamma_0$, then $\mathbf{G}$ is a strict categorical $\mathbf{\Gamma}$-crossed module (see the example (d) in 3.4).

In this case (see [14]), $\mathrm{Der}(\mathbf{\Gamma},\mathbf{G})$ is just a strict monoidal category. The associativity a, left unit l and right unit r of $\mathrm{Der}\,(\mathbf{\Gamma},\mathbf{G})$ are defined by using the canonical isomorphisms of $\mathbf{\Gamma}$, $\mathbf{G}$ and the strict functor $\mathbf{T}:\mathbf{\Gamma}\to\mathbf{G}$, so that they are all identity maps. Then we consider the categorical group $\mathrm{Der}^*(\mathbf{\Gamma},\mathbf{G})$. The last one is not a strict categorical group because every object in $\mathrm{Der}^*(\mathbf{\Gamma},\mathbf{G})$ has an inverse but this inverse is not necessarily strict (see Proposition 5.5 in [14]).

Lemma 5.5.1. *A derivation from $\mathbf{\Gamma}$ into $\mathbf{G}$ is uniquely specified by a triple of functions $p:\Gamma_0\to G_0$, $f:\Gamma_1\rtimes\Gamma_0\to G_1$ and $\varepsilon:\Gamma_0\times\Gamma_0\to G_1$ satisfying*

$$p(\partial'(\beta)\,\sigma)=\partial f(\beta,\sigma)\,p(\sigma); \tag{5.36}$$

$$f(\beta_1\,\beta_2,\sigma)=f(\beta_1,\partial'(\beta_2)\,\sigma)\,f(\beta_2,\sigma); \tag{5.37}$$

$$\begin{aligned}&f(\beta,\sigma)^{\,p(\sigma)}h(\beta,{}^{\sigma}p(\partial'(\beta')\,\sigma'))^{\,p(\sigma)}({}^{\sigma}f(\beta',\sigma'))\,\varepsilon(\sigma,\sigma')=\\&=\varepsilon(\partial'(\beta)\,\sigma,\partial'(\beta')\,\sigma')\,f(\beta\,{}^{\sigma}\beta',\sigma\,\sigma');\end{aligned} \tag{5.38}$$

$$p(\sigma)\,{}^{\sigma}p(\tau)=\partial(\varepsilon(\sigma,\tau))\,p(\sigma\,\tau); \tag{5.39}$$

$${}^{p(\sigma)}({}^{\sigma}\varepsilon(\tau,\upsilon))\,\varepsilon(\sigma,\tau\,\upsilon)=\varepsilon(\sigma,\tau)\,\varepsilon(\sigma\,\tau,\upsilon). \tag{5.40}$$

Proof.
As in the braided case.

□

Proposition 5.5.1. *An arrow in the categorical group Der$(\mathbf{\Gamma},\mathbf{G})$ is uniquely specified by a quadruple $(\theta,p_1,f_1,\varepsilon_1)$ with (p_1,f_1,ε_1) as in Lemma 5.5.1 and an arbitrary function $\theta:\Gamma_0\to G_1$. The source of $(\theta,p_1,f_1,\varepsilon_1)$ is the derivation from $\mathbf{\Gamma}$ into $\mathbf{G}$ given by (p_1,f_1,ε_1); the target of $(\theta,p_1,f_1,\varepsilon_1)$ is the derivation from $\mathbf{\Gamma}$ into $\mathbf{G}$ given by the triple of functions $p_2(\sigma)=\partial\theta(\sigma)\,p_1(\sigma)$, $f_2(\beta,\sigma)=\theta(\partial'(\beta)\,\sigma)\,f_1(\beta,\sigma)\,\theta(\sigma)^{-1}$ and $\varepsilon_2(\sigma,\tau)=\theta(\sigma)$ ${}^{p_1(\sigma)}({}^{\sigma}\theta(\tau))\,\varepsilon_1(\sigma,\tau)\theta(\sigma\,\tau)^{-1}$.*

Proof.
As in the braided case.

□

$\mathrm{Der}(\mathbf{\Gamma},\mathbf{G})$ is a strict monoidal category and the tensor product on objects (see Theorem 5.2 in [14]) is given by:

$$(p_1,f_1,\varepsilon_1)(p_2,f_2,\varepsilon_2)=(p,f,\varepsilon)\quad\text{where}$$

$$p(\sigma)=p_1(\bar p_0(p_2(\sigma))\,\sigma)\,p_2(\sigma), \tag{5.41}$$

$$f(\beta,\sigma)=f_1(\bar p_1(f_2(\beta,\sigma))^{\,\bar p_0(p_2(\sigma))}\beta,\bar p_0(p_2(\sigma))\,\sigma)^{\,p_1(\bar p_0(p_2(\sigma))\,\sigma)}f_2(\beta,\sigma). \tag{5.42}$$

ε is defined by the composition of the following sequence of arrows in $\mathbf{G}$:

$$p_1(\bar{p}_0(p_2(\sigma\,\tau))\,\sigma\,\tau)\,p_2(\sigma\,\tau)$$

$$\Big\downarrow {\scriptstyle (f_1(\bar{p}_1(\varepsilon_2(\sigma,\tau)),\bar{p}_0(p_2(\sigma\,\tau))\,\sigma\,\tau),\,p_1(\bar{p}_0(p_2(\sigma\,\tau))\,\sigma\,\tau))\otimes(1,\,p_2(\sigma\,\tau)) = (f_1(\bar{p}_1(\varepsilon_2(\sigma,\tau)),\bar{p}_0(p_2(\sigma\,\tau))\,\sigma\,\tau),\,p_1(\bar{p}_0(p_2(\sigma\,\tau))\,\sigma\,\tau)\,p_2(\sigma\,\tau))}$$

$$p_1(\bar{p}_0(p_2(\sigma))\,\sigma\,\bar{p}_0(p_2(\tau))\,\tau)\,p_2(\sigma\,\tau)$$

$$\Big\downarrow {\scriptstyle (\varepsilon_1(\bar{p}_0(p_2(\sigma))\,\sigma,\bar{p}_0(p_2(\tau))\,\tau),\,p_1(\bar{p}_0(p_2(\sigma))\,\sigma\,\bar{p}_0(p_2(\tau))\,\tau))\otimes(\varepsilon_2(\sigma,\tau),\,p_2(\sigma\,\tau)) = (\varepsilon_1(\bar{p}_0(p_2(\sigma))\,\sigma,\bar{p}_0(p_2(\tau))\,\tau)\,{}^{p_1(\bar{p}_0(p_2(\sigma))\,\sigma\,\bar{p}_0(p_2(\tau))\,\tau)}\varepsilon_2(\sigma,\tau),\,p_1(\bar{p}_0(p_2(\sigma))\,\sigma\,\bar{p}_0(p_2(\tau))\,\tau)\,p_2(\sigma\,\tau))}$$

$$p_1(\bar{p}_0(p_2(\sigma))\,\sigma)\,{}^{\bar{p}_0(p_2(\sigma))\,\sigma}p_1(\bar{p}_0(p_2(\tau))\,\tau)\,p_2(\sigma)\,{}^{\sigma}p_2(\tau)$$

$$\|$$

$$p_1(\bar{p}_0(p_2(\sigma))\,\sigma)\,{}^{\bar{p}_0(p_2(\sigma))}(\,{}^{\sigma}p_1(\bar{p}_0(p_2(\tau))\,\tau))\,p_2(\sigma)\,{}^{\sigma}p_2(\tau)$$

$$\|$$

$$p_1(\bar{p}_0(p_2(\sigma))\,\sigma)\,p_2(\sigma)\,{}^{\sigma}p_1(\bar{p}_0(p_2(\tau))\,\tau))\,{}^{\sigma}p_2(\tau)$$

$$\|$$

$$p_1(\bar{p}_0(p_2(\sigma))\,\sigma)\,p_2(\sigma)\,{}^{\sigma}[p_1(\bar{p}_0(p_2(\tau))\,\tau))\,p_2(\tau)]$$

therefore we have:

$$\begin{aligned}\varepsilon(\sigma,\tau) \;&=\; \varepsilon_1(\bar{p}_0(p_2(\sigma))\,\sigma,\bar{p}_0(p_2(\tau))\,\tau)\,{}^{p_1(\bar{p}_0(p_2(\sigma))\,\sigma\,\bar{p}_0(p_2(\tau))\,\tau)}\varepsilon_2(\sigma,\tau)\\ &\quad\; f_1(\bar{p}_1(\varepsilon_2(\sigma,\tau)),\bar{p}_0(p_2(\sigma\,\tau))\,\sigma\,\tau). \qquad (5.43)\end{aligned}$$

Since $\mathrm{Der}(\mathbf{\Gamma},\mathbf{G})$ is a strict monoidal category the set of objects of $\mathrm{Der}(\mathbf{\Gamma},\mathbf{G})$ is a monoid.

Let

$$(p_1,f_1,\varepsilon_1)\xrightarrow{(\theta_1,p_1,f_1,\varepsilon_1)}(\widetilde{p}_1,\widetilde{f}_1,\widetilde{\varepsilon}_1) \quad\text{and}\quad (p_2,f_2,\varepsilon_2)\xrightarrow{(\theta_2,p_2,f_2,\varepsilon_2)}(\widetilde{p}_2,\widetilde{f}_2,\widetilde{\varepsilon}_2)$$

be two arrows in $\mathrm{Der}(\mathbf{\Gamma},\mathbf{G})$, where $(\widetilde{p}_i,\widetilde{f}_i,\widetilde{\varepsilon}_i)$ are determined by $(\theta_i,p_i,f_i,\varepsilon_i)$ under the Proposition 5.5.1 for $i=1,2$; the tensor product of these two arrows, is defined in the Theorem 5.2 in [14] by:

$$p_1(\bar{p}_0(p_2(\sigma))\,\sigma)\,p_2(\sigma)$$

$$\Big\downarrow {\scriptstyle (\theta_1(\bar{p}_0(p_2(\sigma))\,\sigma),\,p_1(\bar{p}_0(p_2(\sigma))\,\sigma))\otimes(1,\,p_2(\sigma)) = (\theta_1(\bar{p}_0(p_2(\sigma))\,\sigma),\,p_1(\bar{p}_0(p_2(\sigma))\,\sigma)\,p_2(\sigma))}$$

$$\widetilde{p}_1(\bar{p}_0(p_2(\sigma))\,\sigma)\,p_2(\sigma)$$

$$\Big\downarrow$$

$$(\widetilde{f}_1(\bar{p}_1(\theta_2(\sigma)), \bar{p}_0(p_2(\sigma))\,\sigma), \widetilde{p}_1(\bar{p}_0(p_2(\sigma))\,\sigma)) \otimes (\theta_2(\sigma), p_2(\sigma)) =$$
$$= (\widetilde{f}_1(\bar{p}_1(\theta_2(\sigma)), \bar{p}_0(p_2(\sigma))\,\sigma)\,{}^{\widetilde{p}_1(\bar{p}_0(p_2(\sigma))\,\sigma)}\theta_2(\sigma), \widetilde{p}_1(\bar{p}_0(p_2(\sigma))\,\sigma)\,p_2(\sigma))$$

$$\Big\downarrow$$

$$\widetilde{p}_1(\bar{p}_0(\widetilde{p}_2(\sigma))\,\sigma)\,\widetilde{p}_2(\sigma)$$

therefore we obtain:

$(\theta_1, p_1, f_1, \varepsilon_1)(\theta_2, p_2, f_2, \varepsilon_2) = (\theta, p, f, \varepsilon)$ with

$$\theta(\sigma) = \widetilde{f}_1(\bar{p}_1(\theta_2(\sigma)), \bar{p}_0(p_2(\sigma))\,\sigma)\,{}^{\widetilde{p}_1(\bar{p}_0(p_2(\sigma))\,\sigma)}\theta_2(\sigma)\,\theta_1(\bar{p}_0(p_2(\sigma))\,\sigma) =$$
$$= \widetilde{f}_1(\bar{p}_1(\theta_2(\sigma)), \bar{p}_0(p_2(\sigma))\,\sigma)\,{}^{\partial\theta_1(\bar{p}_0(p_2(\sigma))\,\sigma)\,p_1(\bar{p}_0(p_2(\sigma))\,\sigma)}\theta_2(\sigma)$$
$$\theta_1(\bar{p}_0(p_2(\sigma))\,\sigma) =$$
$$= \widetilde{f}_1(\bar{p}_1(\theta_2(\sigma)), \bar{p}_0(p_2(\sigma))\,\sigma)\,\theta_1(\bar{p}_0(p_2(\sigma))\,\sigma)\,{}^{p_1(\bar{p}_0(p_2(\sigma))\,\sigma)}\theta_2(\sigma),$$

and $(p, f, \varepsilon) = (p_1, f_1, \varepsilon_1)(p_2, f_2, \varepsilon_2)$ as in (5.41), (5.42), (5.43).

Now we are going to describe the structure of $\mathcal{H}^0(\mathbf{\Gamma}, \mathbf{G})$. $\mathcal{H}^0(\mathbf{\Gamma}, \mathbf{G})$ corresponds to the categorical group of $\mathbf{\Gamma}$-invariant objects $\mathbf{G}^{\mathbf{\Gamma}}$ (see [14]). The associativity a, left unit l and right unit r of the monoidal structure of $\mathbf{G}^{\mathbf{\Gamma}}$ are given by the respective constraints a, l and r of $\mathbf{G}$, so that they are all identity maps. Furthermore, for any object in $\mathbf{G}^{\mathbf{\Gamma}}$ an strict inverse exists. Thus $\mathbf{G}^{\mathbf{\Gamma}}$ is a strict categorical group associated with a crossed module (analogous to what has already done in the braided case):

$$\begin{array}{rcl} \overline{\overline{\partial}} : G_1 & \to & \mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}}) \\ \alpha & \to & (\partial(\alpha), \theta_\alpha) \end{array}$$

where $\theta_\alpha(\sigma) = \alpha\,{}^\sigma\alpha^{-1}$ and

$$\mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}}) = \{(g, \theta) \in Z^1_B(\Gamma_0, \partial : G_1 \to G_0) \;/\; \theta(\partial'(\beta)\sigma)\,h(\beta, {}^\sigma g) = \theta(\sigma)\}.$$

The product in G_1 is the usual product and the product in $\mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}})$ is the Borovoi product. The action of $(g, \theta) \in \mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}})$ on $\alpha \in G_1$ is given by ${}^{(g,\theta)}\alpha = {}^g\alpha$.

$\mathcal{H}^0(\mathbf{\Gamma}, \mathbf{G})$ is defined by the kernel of $\overline{\mathbf{T}} : \mathbf{G} \to \mathrm{Der}^*(\mathbf{\Gamma}, \mathbf{G})$, the last one is defined on objects and on arrows

$$\begin{array}{rclrcl} \overline{T}_0 : G_0 & \longrightarrow & \mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})) & \overline{T}_1 : G_1 \rtimes G_0 & \longrightarrow & \mathrm{Der}_1(\mathbf{\Gamma}, \mathbf{G}) \\ g & \longrightarrow & (p_g, f_g, 1) & (\alpha, g) & \longrightarrow & (\theta, p_g, f_g, 1) \end{array}$$

respectively, where $p_g(\sigma) = g\,{}^\sigma g^{-1}$, $f_g(\beta, \sigma) = {}^g h(\beta, {}^\sigma g^{-1})$ and $\theta(\sigma) = \alpha\,{}^{g\,{}^\sigma g^{-1}}({}^\sigma\alpha^{-1})$.
There are natural isomorphisms $\overline{\nu}$ and $\overline{\chi}$ such that $(\mathbf{G}, \overline{\mathbf{T}}, \overline{\nu}, \overline{\chi})$ is a categorical $\mathrm{Der}^*(\mathbf{\Gamma}, \mathbf{G})$-crossed module (see [14]).

In this case, we observe that the isomorphism $\overline{\chi}$ is given by the composition of the three morphisms:

$$^{\overline{T}_0(g_1)}g_2\, g_1 = = = g_1\, ^{\bar{p}_0(g_2)}g_1^{-1}\, g_2\, g_1 = = = g_1\, g_2\, g_1^{-1}\, g_1 = = = g_1\, g_2$$

therefore $\overline{\chi}$ is the identity map. Thanks to this observation $\mathcal{H}^0(\mathbf{\Gamma}, \mathbf{G})$ can be equipped with a braiding (Proposition 2.7 in [14]) given by

$$g_2\, g_1 = = = g_2\, g_1 \xrightarrow{(\theta_1(\bar{p}_0(g_2)), g_2\, g_1)} {}^{\overline{T}_0(g_1)}g_2\, g_1 \overset{\overline{\chi}_{g_1,g_2}}{= = =} g_1\, g_2.$$

Then $\overline{\overline{\partial}} : G_1 \to \mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}})$ is also a braided crossed module with a braiding defined by

$$\{(g_1, \theta_1), (g_2, \theta_2)\} = \theta_1(p_0(g_2)).$$

In this case, we can do even better.

Proposition 5.5.2. *The following diagram*

$$\begin{array}{ccc} G_1 & \xrightarrow{\bar{p}_1} & \Gamma_1 \\ {\scriptstyle \overline{\overline{\partial}}}\downarrow & & \downarrow{\scriptstyle \partial'} \\ Ob(\mathbf{G}^{\mathbf{\Gamma}}) & \xrightarrow{p_0} & \Gamma_0 \end{array} \tag{5.44}$$

is a crossed square with actions, group homomorphism p_0 and function $\overline{h}$: $\Gamma_1 \times Ob(\mathbf{G}^{\mathbf{\Gamma}}) \to G_1$ defined as following:

- *the action of Γ_0 on G_1 is induced by the action of $\partial' : \Gamma_1 \to \Gamma_0$ on $\partial : G_1 \to G_0$;*
- *the action of Γ_0 on Γ_1 is the action of the crossed module $\partial' : \Gamma_1 \to \Gamma_0$;*
- *the action of Γ_0 on $Ob(\mathbf{G}^{\mathbf{\Gamma}})$ is defined by $^{\sigma}(g, \theta) = (^{\sigma}g, \overline{\theta})$ where $\overline{\theta}(\tau) = {}^{\sigma}\theta(\sigma^{-1}\, \tau\, \sigma)$;*
- *$p_0 : Ob(\mathbf{G}^{\mathbf{\Gamma}}) \to \Gamma_0$ is determined by $p_0(g, \theta) = \bar{p}_0(g)$;*
- *$\overline{h}(\beta, (g, \theta)) = h(\beta, g)$ where the function h is given by the crossed square structure of (5.35).*

Proof. The action of Γ_0 on $\mathrm{Ob}(\mathbf{G}^{\mathbf{\Gamma}})$ is well defined and the proof is the same as in the Proposition 5.3.3. p_0 is a group homomorphism because $\bar{p}_0$ is and the diagram (5.44) commutes:

$$p_0(\overline{\overline{\partial}}(\alpha)) = p_0(\partial(\alpha), \theta_\alpha) = \bar{p}_0(\partial(\alpha)) = \partial'(\bar{p}_1(\alpha)).$$

Now we want to check the five properties making the diagram (5.44) a crossed square (see Definition 2.2.1 in the section 2.2).

(i) The map $\overline{\overline{\partial}}$ preserves the actions of Γ_0 and the proof is the same as in the Proposition 5.4.3. The map $\bar{p}_1$ preserves the actions of Γ_0 because (5.35) is a crossed square. ∂' is a crossed module because (5.35) is a crossed square and we want to prove that p_0 is a crossed module. The pre-crossed module property is shown the same way as in the Proposition 5.4.3. Now, also the Peiffer condition holds:

$$\begin{aligned}
&{}^{p_0(g_1,\theta_1)}(g_2,\theta_2) = {}^{\bar{p}_0(g_1)}(g_2,\theta_2) = ({}^{\bar{p}_0(g_1)}g_2, \hat{\theta}) = (g_1\, g_2\, g_1^{-1}, \hat{\theta}),\\
&(g_1,\theta_1)\,(g_2,\theta_2)\,(g_1,\theta_1)^{-1} = (g_1,\theta_1)\,(g_2,\theta_2)\,(g_1^{-1},\theta_1^*) =\\
&= (g_1\, g_2\, g_1^{-1}, \theta),
\end{aligned}$$

where

$$\begin{aligned}
\theta(\sigma) &= {}^{g_1\, g_2}\theta_1^*(\sigma)\, {}^{g_1}\theta_2(\sigma)\,\theta_1(\sigma) = {}^{g_1\, g_2\, g_1^{-1}}\theta_1(\sigma)^{-1}\, {}^{g_1}\theta_2(\sigma)\,\theta_1(\sigma),\\
\hat{\theta}(\sigma) &= {}^{\bar{p}_0(g_1)}\theta_2(\bar{p}_0(g_1)^{-1}\,\sigma\,\bar{p}_0(g_1)) = {}^{\bar{p}_0(g_1)}\theta_2(\bar{p}_0(g_1^{-1}\,{}^{\sigma}g_1)\,\sigma) =\\
&= {}^{\bar{p}_0(g_1)}\theta_2(\bar{p}_0\,\partial\theta_1^*(\sigma)\,\sigma) = {}^{\bar{p}_0(g_1)}\theta_2(\partial'\,\bar{p}_1\,\theta_1^*(\sigma)\,\sigma) =\\
&= {}^{\bar{p}_0(g_1)}[\theta_2(\sigma)\,h(\bar{p}_1\,\theta_1^*(\sigma),\, {}^{\sigma}g_2)^{-1}] =\\
&= {}^{g_1}\theta_2(\sigma)\, {}^{g_1\,{}^{\sigma}g_2}\theta_1^*(\sigma)\, {}^{g_1}\theta_1^*(\sigma)^{-1} =\\
&= {}^{g_1}\theta_2(\sigma)\, {}^{g_1\,{}^{\sigma}g_2\, g_1^{-1}}\theta_1(\sigma)^{-1}\,\theta_1(\sigma) =\\
&= {}^{\partial(\,{}^{g_1}\theta_2(\sigma))\, g_1\,{}^{\sigma}g_2\, g_1^{-1}}\theta_1(\sigma)^{-1}\, {}^{g_1}\theta_2(\sigma)\,\theta_1(\sigma) =\\
&= {}^{g_1\, g_2\,{}^{\sigma}g_2^{-1}\, g_1^{-1}\, g_1\,{}^{\sigma}g_2\, g_1^{-1}}\theta_1(\sigma)^{-1}\, {}^{g_1}\theta_2(\sigma)\,\theta_1(\sigma) =\\
&= {}^{g_1\, g_2\, g_1^{-1}}\theta_1(\sigma)^{-1}\, {}^{g_1}\theta_2(\sigma)\,\theta_1(\sigma) = \theta(\sigma).
\end{aligned}$$

$p_0\,\overline{\overline{\partial}} = \partial'\,\bar{p}_1$ is a crossed module because (5.35) is a crossed square.

(ii) $\bar{p}_1(\overline{h}(\beta,(g,\theta))) = \bar{p}_1(h(\beta,g)) = \beta\,{}^{g}\beta^{-1} = \beta\,{}^{\bar{p}_0(g)}\beta^{-1} = \beta\,{}^{p_0(g,\theta)}\beta^{-1} =$
$= \beta\,{}^{(g,\theta)}\beta^{-1}$.
$\overline{\overline{\partial}}\,\overline{h}(\beta,(g,\theta)) = {}^{\beta}(g,\theta)\,(g,\theta)^{-1}$ (as in the Proposition 5.3.3).

(iii) $\overline{h}(\bar{p}_1(\alpha),(g,\theta)) = h(\bar{p}_1(\alpha),g) = \alpha\,{}^{g}\alpha^{-1} = \alpha\,{}^{(g,\theta)}\alpha^{-1}$.
$\overline{h}(\beta,\overline{\overline{\partial}}(\alpha)) = {}^{\beta}\alpha\,\alpha^{-1}$ (as in the Proposition 5.3.3).

(iv) $\overline{h}(\beta_1\,\beta_2,(g,\theta)) = {}^{\beta_1}\overline{h}(\beta_2,(g,\theta))\,\overline{h}(\beta_1,(g,\theta))$ (as in the Proposition 5.3.3).
$\overline{h}(\beta,(g_1,\theta_1)\,(g_2,\theta_2)) = \overline{h}(\beta,(g_1,\theta_1))\,{}^{(g_1,\theta_1)}\overline{h}(\beta,(g_2,\theta_2))$ (as in the Proposition 5.3.3).

(v) $\overline{h}({}^{\sigma}\beta,{}^{\sigma}(g,\theta)) = \overline{h}({}^{\sigma}\beta,({}^{\sigma}g,\overline{\theta})) = h({}^{\sigma}\beta,{}^{\sigma}g) = {}^{\sigma}h(\beta,g) = {}^{\sigma}\overline{h}(\beta,(g,\theta))$.

□

Now we want to generalize what happens in the context of crossed modules with objects groups.

It is known that given a crossed module of groups $\partial : G_1 \to G_0$, the homomorphism $\gamma : G_1 \to \mathrm{Der}^*(G_0, G_1)$ becomes a crossed module of groups (see section 5.1).

At this point, we want to interpret what happens in the new context of crossed modules with objects crossed modules.

Given a crossed square (5.35), that is a crossed module of crossed modules, we can consider the morphism of categorical groups $\overline{\mathbf{T}} : \mathbf{G} \to \mathrm{Der}^*(\mathbf{\Gamma}, \mathbf{G})$, previously defined. The last one is a categorical $\mathrm{Der}^*(\mathbf{\Gamma}, \mathbf{G})$-crossed module (see [14]) but it is certainly not strict because $\mathrm{Der}^*(\mathbf{\Gamma}, \mathbf{G})$ is not a strict categorical group.

In this section, we want to define a category $\mathbf{D}$ included in $\mathrm{Der}^*(\mathbf{\Gamma}, \mathbf{G})$ such that we can consider a restriction of the homomorphism $\overline{\mathbf{T}}$:

$$\overline{\mathbf{T}} : \mathbf{G} \to \mathbf{D}$$

and this is a strict categorical $\mathbf{D}$-crossed module (that is equivalent to a crossed square, that is a crossed module of crossed modules).

One condition to have a strict categorical crossed module $(\mathbf{G}, \overline{\mathbf{T}}, \nu, \chi)$ is that the maps ν and χ are identity maps. Then, we analyze these maps for the categorical crossed module $\overline{\mathbf{T}} : \mathbf{G} \to \mathrm{Der}^*(\mathbf{\Gamma}, \mathbf{G})$. Thanks to [14] we observe that the map $\overline{\nu}_{(p,f,\varepsilon),g}$ is given by:

$$\begin{array}{c}
p(\bar{p}_0(g))\, g^{\,\bar{p}_0(p(\sigma))\,\sigma}(p(\bar{p}_0(g))\, g)^{-1}\, p(\sigma) \\
\Vert \\
p(\bar{p}_0(g))\, g\, p(\sigma)\, {}^{\sigma}g^{-1}\, {}^{\sigma}p(\bar{p}_0(g))^{-1} \\
\Vert \\
p(\bar{p}_0(g))\, {}^{\bar{p}_0(g)}p(\sigma)\, g\, {}^{\sigma}g^{-1}\, {}^{\sigma}p(\bar{p}_0(g))^{-1} \\
\Vert \\
p(\bar{p}_0(g))\, {}^{\bar{p}_0(g)}p(\sigma)\, {}^{\bar{p}_0(g\, {}^{\sigma}g^{-1})\sigma}p(\bar{p}_0(g))^{-1}\, g\, {}^{\sigma}g^{-1} \\
\Vert \\
p(\bar{p}_0(g))\, {}^{\bar{p}_0(g)}[p(\sigma)\, {}^{\bar{p}_0({}^{\sigma}g^{-1})\sigma}p(\bar{p}_0(g))^{-1}]\, g\, {}^{\sigma}g^{-1} \\
\Vert \\
p(\bar{p}_0(g))\, {}^{\bar{p}_0(g)}[p(\sigma)\, {}^{\sigma\,\bar{p}_0(g^{-1})}p(\bar{p}_0(g))^{-1}]\, g\, {}^{\sigma}g^{-1} \\
\Big\downarrow {\scriptstyle ({}^{p(\bar{p}_0(g))\,\bar{p}_0(g)}p(\sigma)[{}^{\bar{p}_0(g)\,\sigma}\varepsilon(\bar{p}_0(g)^{-1}, \bar{p}_0(g))\; {}^{\bar{p}_0(g)\,\sigma}\varepsilon(1,1)],\, p(\bar{p}_0(g))\, {}^{\bar{p}_0(g)}[p(\sigma)\, {}^{\sigma\,\bar{p}_0(g^{-1})}p(\bar{p}_0(g))^{-1}]\, g\, {}^{\sigma}g^{-1})} \\
p(\bar{p}_0(g))\, {}^{\bar{p}_0(g)}[p(\sigma)\, {}^{\sigma}p(\bar{p}_0(g^{-1}))]\, g\, {}^{\sigma}g^{-1} \\
\Big\downarrow {\scriptstyle ({}^{p(\bar{p}_0(g))}({}^{\bar{p}_0(g)}\varepsilon(\sigma, \bar{p}_0(g^{-1}))^{-1}),\, p(\bar{p}_0(g))\, {}^{\bar{p}_0(g)}[p(\sigma)\, {}^{\sigma}p(\bar{p}_0(g^{-1}))]\, g\, {}^{\sigma}g^{-1})}
\end{array}$$

$$\begin{array}{c}
p(\bar{p}_0(g))\, {}^{\bar{p}_0(g)}p(\sigma\, \bar{p}_0(g^{-1}))\, g\, {}^{\sigma}g^{-1} \\
\Vert \\
p(\bar{p}_0(g))\, {}^{\bar{p}_0(g)}p(\bar{p}_0({}^{\sigma}g^{-1})\, \sigma)\, g\, {}^{\sigma}g^{-1} \\
\Big\downarrow {\scriptstyle (\varepsilon(\bar{p}_0(g),\, \bar{p}_0({}^{\sigma}g^{-1})\, \sigma)^{-1},\, p(\bar{p}_0(g))\, {}^{\bar{p}_0(g)}p(\bar{p}_0({}^{\sigma}g^{-1})\, \sigma)\, g\, {}^{\sigma}g^{-1})} \\
p(\bar{p}_0(g)\bar{p}_0({}^{\sigma}g^{-1})\, \sigma)\, g\, {}^{\sigma}g^{-1} \\
\Vert \\
p(\bar{p}_0(g\, {}^{\sigma}g^{-1})\, \sigma)\, g\, {}^{\sigma}g^{-1}.
\end{array}$$

Therefore we have:

$$\begin{aligned}
(\overline{\nu}_{(p,f,\varepsilon),g})(\sigma) \;=\;& (\varepsilon(\bar{p}_0(g), \bar{p}_0({}^{\sigma}g^{-1})\, \sigma)^{-1}\, {}^{p(\bar{p}_0(g))}({}^{\bar{p}_0(g)}\varepsilon(\sigma, \bar{p}_0(g^{-1}))^{-1}) \\
& {}^{p(\bar{p}_0(g))\, {}^{\bar{p}_0(g)}p(\sigma)}[{}^{\bar{p}_0(g)\, \sigma}\varepsilon(\bar{p}_0(g)^{-1}, \bar{p}_0(g))\; {}^{\bar{p}_0(g)\, \sigma}\varepsilon(1,1)], \\
& p(\bar{p}_0(g))\, g\, {}^{\bar{p}_0(p(\sigma))\, \sigma}(p(\bar{p}_0(g))\, g)^{-1}\, p(\sigma))
\end{aligned}$$

while $\overline{\chi}$ is the identity (as previously seen).

Then we considers $\mathrm{Der}_N(\mathbf{\Gamma}, \mathbf{G})$ as the full subcategory of $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$ with objects satisfying the conditions:

$$\varepsilon(\sigma, \bar{p}_0(g_1)) = \varepsilon(\bar{p}_0(g_1), \sigma) = 1 \quad \forall g_1 \in G_0, \forall \sigma \in \Gamma \tag{5.45}$$

$\mathrm{Ob}(\mathrm{Der}_N(\mathbf{\Gamma}, \mathbf{G}))$ is a submonoid of $\mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G}))$; in fact, given the product in $\mathrm{Ob}(\mathrm{Der}_N(\mathbf{\Gamma}, \mathbf{G}))$ of two objects $(p_1, f_1, \varepsilon_1)(p_2, f_2, \varepsilon_2) = (p, f, \varepsilon)$, we have:

$$\begin{aligned}
\varepsilon(\sigma, \bar{p}_0(g_1)) \;=\;& \varepsilon_1(\bar{p}_0(p_2(\sigma))\, \sigma, \bar{p}_0(p_2(\bar{p}_0(g_1)))\, \bar{p}_0(g_1)) \\
& {}^{p_1(\bar{p}_0(p_2(\sigma))\, \sigma\, \bar{p}_0(p_2(\bar{p}_0(g_1)))\, \bar{p}_0(g_1))}\varepsilon_2(\sigma, \bar{p}_0(g_1)) \\
& f_1(\bar{p}_1(\varepsilon_2(\sigma, \bar{p}_0(g_1))), \bar{p}_0(p_2(\sigma\, \bar{p}_0(g_1)))\, \sigma\, \bar{p}_0(g_1)) = 1; \\
\varepsilon(\bar{p}_0(g_1), \sigma) \;=\;& \varepsilon_1(\bar{p}_0(p_2(\bar{p}_0(g_1)))\, \bar{p}_0(g_1), \bar{p}_0(p_2(\sigma))\, \sigma) \\
& {}^{p_1(\bar{p}_0(p_2(\bar{p}_0(g_1)))\, \bar{p}_0(g_1)\, \bar{p}_0(p_2(\sigma))\, \sigma)}\varepsilon_2(\bar{p}_0(g_1), \sigma) \\
& f_1(\bar{p}_1(\varepsilon_2(\bar{p}_0(g_1), \sigma)), \bar{p}_0(p_2(\bar{p}_0(g_1)\, \sigma))\, \bar{p}_0(g_1)\, \sigma) = 1.
\end{aligned}$$

Then $\mathrm{Der}_N(\mathbf{\Gamma}, \mathbf{G})$ is a monoidal subcategory of $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$, because $\mathrm{Der}_N(\mathbf{\Gamma}, \mathbf{G})$ is a subcategory of $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$, closed under the tensor product of objects and morphisms (because $\mathrm{Der}_N(\mathbf{\Gamma}, \mathbf{G})$ is a full subcategory of $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$). $\mathrm{Der}_N(\mathbf{\Gamma}, \mathbf{G})$ is a strict monoidal subcategory of $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})$.
Now we consider $\mathrm{Der}^{**}_N(\mathbf{\Gamma}, \mathbf{G})$ the subcategory of $\mathrm{Der}_N(\mathbf{\Gamma}, \mathbf{G})$ given by strict invertible objects and isomorphisms between them. Then $\mathrm{Der}^{**}_N(\mathbf{\Gamma}, \mathbf{G})$ is a strict categorical group and it corresponds to the crossed module constructed as follows:

$$\overline{\partial} : \mathrm{Ker} s \;\to\; \mathrm{Ob}(\mathrm{Der}^{**}_N(\mathbf{\Gamma}, \mathbf{G}))$$

with $\overline{\partial} = t_{|\mathrm{Ker}s}$, where s and t are the source and target maps, respectively, of the underlying groupoid $\mathrm{Der}^{**}_N(\mathbf{\Gamma}, \mathbf{G})$.
We can observe that $\mathrm{Ob}(\mathrm{Der}^{**}_N(\mathbf{\Gamma}, \mathbf{G}))$ is the group of invertible elements of the monoid $\mathrm{Ob}(\mathrm{Der}_N(\mathbf{\Gamma}, \mathbf{G}))$, and usually it is denoted by $\mathrm{Ob}(\mathrm{Der}_N(\mathbf{\Gamma}, \mathbf{G}))^*$.
We denote with $\mathrm{Der}^{**}_N(\mathbf{\Gamma}, \mathbf{G})_1$ the set of arrows in $\mathrm{Der}^{**}_N(\mathbf{\Gamma}, \mathbf{G})$ and we recall the source map:

$$\begin{array}{rcl} s : \mathrm{Der}^{**}_N(\mathbf{\Gamma}, \mathbf{G})_1 & \longrightarrow & \mathrm{Ob}(\mathrm{Der}_N(\mathbf{\Gamma}, \mathbf{G}))^* \\ (\theta_1, p_1, f_1, \varepsilon_1) & \longrightarrow & (p_1, f_1, \varepsilon_1) \end{array}$$

while the target map:

$$\begin{array}{rcl} t : \mathrm{Der}^{**}_N(\mathbf{\Gamma}, \mathbf{G})_1 & \longrightarrow & \mathrm{Ob}(\mathrm{Der}_N(\mathbf{\Gamma}, \mathbf{G}))^* \\ (\theta_1, p_1, f_1, \varepsilon_1) & \longrightarrow & (p_2, f_2, \varepsilon_2) \end{array}$$

where $(p_2, f_2, \varepsilon_2)$ as in Proposition 5.5.1. Because $(p_2, f_2, \varepsilon_2)$ has to belong to $\mathrm{Ob}(\mathrm{Der}_N(\mathbf{\Gamma}, \mathbf{G}))^*$, θ_1 has to satisfy the following conditions:

$$\theta_1(\bar{p}_0(g_1)\,\sigma) = \theta_1(\bar{p}_0(g_1))^{\,p_1(\bar{p}_0(g_1))}(\,^{\bar{p}_0(g_1)}\theta_1(\sigma)); \qquad (5.46)$$
$$\theta_1(\sigma\,\bar{p}_0(g_1)) = \theta_1(\sigma)^{\,p_1(\sigma)}(\,^{\sigma}\theta_1(\bar{p}_0(g_1))). \qquad (5.47)$$

Thus we have

$$\begin{array}{rcl} \overline{\partial} : \mathrm{Ker}s & \to & \mathrm{Ob}(\mathrm{Der}_N(\mathbf{\Gamma}, \mathbf{G}))^* \\ (\theta, 1, 1, 1) & \to & (p, f, \varepsilon) \end{array}$$

where

- $p(\sigma) = \partial\,\theta(\sigma)$,
- $f(\beta, \sigma) = \theta(\partial'(\beta)\,\sigma)\,\theta(\sigma)^{-1}$,
- $\varepsilon(\sigma, \tau) = \theta(\sigma)\,^{\sigma}\theta(\tau)\,\theta(\sigma\,\tau)^{-1}$.

The product of two arrows $(\theta_1, 1, 1, 1)$ and $(\theta_2, 1, 1, 1)$ in $\mathrm{Der}^{**}_N(\mathbf{\Gamma}, \mathbf{G})$ is $(\theta, 1, 1, 1)$ where $\theta(\sigma) = \theta_1(\bar{p}_0(\partial(\theta_2(\sigma)))\,\sigma)\,\theta_2(\sigma)$ and the product on objects of $\mathrm{Der}^{**}_N(\mathbf{\Gamma}, \mathbf{G})$ is the same as defined in $\mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G}))$. The action of the group $\mathrm{Ob}(\mathrm{Der}_N(\mathbf{\Gamma}, \mathbf{G}))^*$ on $\mathrm{Ker}s$ is given by:

$$^{(p,f,\varepsilon)}(\theta, 1, 1, 1) = i(p, f, \varepsilon)(\theta, 1, 1, 1)(i(p, f, \varepsilon))^{-1}.$$

We recall that the map i for the groupoid $\mathrm{Der}^{**}_N(\mathbf{\Gamma}, \mathbf{G})$ is given by:

$$\begin{array}{rcl} i : \mathrm{Ob}(\mathrm{Der}_N(\mathbf{\Gamma}, \mathbf{G}))^* & \longrightarrow & \mathrm{Der}^{**}_N(\mathbf{\Gamma}, \mathbf{G})_1 \\ (p, f, \varepsilon) & \longrightarrow & (1, p, f, \varepsilon). \end{array}$$

Therefore, using the multiplication defined above on arrows in $\mathrm{Der}(\mathbf{\Gamma},\mathbf{G})$, we have:

$$\begin{aligned} {}^{(p,f,\varepsilon)}(\theta,1,1,1) &= (1,p,f,\varepsilon)\,(\theta,1,1,1)\,(1,p,f,\varepsilon)^{-1} \\ &= (1,p,f,\varepsilon)\,(\theta,1,1,1)\,(1,p^*,f^*,\varepsilon^*) \\ &= (\widehat{\theta},p,f,\varepsilon)\,(1,p^*,f^*,\varepsilon^*) \\ &= (\widehat{\widehat{\theta}},1,1,1) \end{aligned}$$

where
$$\begin{aligned} \widehat{\theta}(\sigma) &= f(\bar p_1(\theta(\sigma)),\sigma)\,{}^{p(\sigma)}\theta(\sigma), \\ \widehat{\widehat{\theta}}(\sigma) &= \widehat{\theta}(\bar p_0(p^*(\sigma))\,\sigma) = \\ &= f(\bar p_1(\theta(\bar p_0(p^*(\sigma))\,\sigma)),\bar p_0(p^*(\sigma))\,\sigma) \\ &\quad {}^{p(\bar p_0(p^*(\sigma))\,\sigma)}\theta(\bar p_0(p^*(\sigma))\,\sigma) = \\ &= f(\bar p_1(\theta(\bar p_0(p^*(\sigma))\,\sigma)),\bar p_0(p^*(\sigma))\,\sigma)\,{}^{p^*(\sigma)^{-1}}\theta(\bar p_0(p^*(\sigma))\,\sigma). \end{aligned}$$

$\mathrm{Ker}s$ is isomorphic to D^*, that is the group of the invertible elements of the following monoid:

$$\begin{aligned} D = \{\theta\in \mathrm{App}(\Gamma_0,G_1)\ /\ \ & \theta(\bar p_0(g_1)\,\sigma) = \theta(\bar p_0(g_1))\,{}^{\bar p_0(g_1)}\theta(\sigma) \\ & \theta(\sigma\,\bar p_0(g_1)) = \theta(\sigma)\,{}^{\sigma}\theta(\bar p_0(g_1)) \quad \} \end{aligned}$$

under the product $(\theta_1\cdot\theta_2)(\sigma) = \theta_1(\bar p_0(\partial(\theta_2(\sigma)))\,\sigma)\,\theta_2(\sigma)$. It is clear the isomorphism between $\overline{\partial}$ and a homomorphism

$$\overline{\partial} : D^* \ \to\ \mathrm{Ob}(\mathrm{Der}_N(\mathbf{\Gamma},\mathbf{G}))^*$$

which, by abuse of notation, we have denoted again by $\overline{\partial}$.

Now we want to show that:

$$\overline{\mathbf{T}} : \mathbf{G} \to \mathrm{Der}^{**}_N(\mathbf{\Gamma},\mathbf{G})$$

is a strict categorical $\mathrm{Der}^{**}_N(\mathbf{\Gamma},\mathbf{G})$-crossed module.
Firstly $\overline{\mathbf{T}} : \mathbf{G}\to \mathrm{Der}^*(\mathbf{\Gamma},\mathbf{G})$ is strict monoidal functor:

$$\begin{aligned} &\overline{T}(1) = (1,1,1); \\ &\overline{T}(g_1\,g_2) = (p_{g_1\,g_2}, f_{g_1\,g_2},1); \\ &\overline{T}(g_1)\,\overline{T}(g_2) = (p_{g_1},f_{g_1},1)\,(p_{g_2},f_{g_2},1) = (p,f,1); \end{aligned}$$

where

$$\begin{aligned} p_{g_1\,g_2}(\sigma) &= g_1\,g_2\,{}^{\sigma}(g_1\,g_2)^{-1} = g_1\,g_2\,{}^{\sigma}g_2^{-1}\,{}^{\sigma}g_1^{-1}, \\ f_{g_1\,g_2}(\beta,\sigma) &= {}^{g_1\,g_2}h(\beta,{}^{\sigma}(g_1\,g_2)^{-1}), \\ p(\sigma) &= p_{g_1}(\bar p_0(p_{g_2}(\sigma))\,\sigma)\,p_{g_2}(\sigma) = g_1\,{}^{\bar p_0(g_2\,{}^{\sigma}g_2^{-1})\,\sigma}g_1^{-1}\,g_2\,{}^{\sigma}g_2^{-1} = \end{aligned}$$

$$
\begin{aligned}
&= g_1\, g_2\, {}^{\sigma}g_2^{-1}\, {}^{\sigma}g_1^{-1} = p_{g_1\, g_2}(\sigma), \\
f(\beta,\sigma) &= f_{g_1}(\bar{p}_1(f_{g_2}(\beta,\sigma))\, {}^{\bar{p}_0(p_{g_2}(\sigma))}\beta, \bar{p}_0(p_{g_2}(\sigma))\,\sigma) \\
&\quad {}^{p_{g_1}(\bar{p}_0(p_{g_2}(\sigma))\,\sigma)}f_{g_2}(\beta,\sigma) = \\
&= f_{g_1}(\bar{p}_1({}^{\bar{p}_0(g_2)}h(\beta, {}^{\sigma}g_2^{-1}))\, {}^{\bar{p}_0(g_2\, {}^{\sigma}g_2^{-1})}\beta, \bar{p}_0(g_2\, {}^{\sigma}g_2^{-1})\,\sigma) \\
&\quad {}^{g_1\, {}^{\bar{p}_0(g_2\, {}^{\sigma}g_2^{-1})\,\sigma}\, g_1^{-1}}({}^{g_2}h(\beta, {}^{\sigma}g_2^{-1})) = \\
&= f_{g_1}({}^{\bar{p}_0(g_2)}(\beta\, {}^{\bar{p}_0({}^{\sigma}g_2^{-1})}\beta^{-1})\, {}^{\bar{p}_0(g_2\, {}^{\sigma}g_2^{-1})}\beta, \bar{p}_0(g_2\, {}^{\sigma}g_2^{-1})\,\sigma) \\
&\quad {}^{g_1\, g_2\, {}^{\sigma}g_2^{-1}\, {}^{\sigma}g_1^{-1}\, {}^{\sigma}g_2}h(\beta, {}^{\sigma}g_2^{-1}) = \\
&= f_{g_1}({}^{\bar{p}_0(g_2)}\beta, \bar{p}_0(g_2\, {}^{\sigma}g_2^{-1})\,\sigma)\, {}^{g_1\, g_2\, {}^{\sigma}g_2^{-1}\, {}^{\sigma}g_1^{-1}\, {}^{\sigma}g_2}h(\beta, {}^{\sigma}g_2^{-1}) = \\
&= {}^{g_1}h({}^{\bar{p}_0(g_2)}\beta, {}^{\bar{p}_0(g_2\, {}^{\sigma}g_2^{-1})\,\sigma}g_1^{-1})\, {}^{g_1\, g_2\, {}^{\sigma}g_2^{-1}\, {}^{\sigma}g_1^{-1}\, {}^{\sigma}g_2}h(\beta, {}^{\sigma}g_2^{-1}) = \\
&= {}^{g_1\, g_2}(h(\beta, {}^{\bar{p}_0({}^{\sigma}g_2^{-1})\,\sigma}g_1^{-1})\, {}^{{}^{\sigma}g_2^{-1}\, {}^{\sigma}g_1^{-1}\, {}^{\sigma}g_2}h(\beta, {}^{\sigma}g_2^{-1})) = \\
&= {}^{g_1\, g_2}(h(\beta, {}^{\sigma}g_2^{-1}\, {}^{\sigma}g_1^{-1}\, {}^{\sigma}g_2)\, {}^{{}^{\sigma}g_2^{-1}\, {}^{\sigma}g_1^{-1}\, {}^{\sigma}g_2}h(\beta, {}^{\sigma}g_2^{-1})) = \\
&= {}^{g_1\, g_2}h(\beta, {}^{\sigma}g_2^{-1}\, {}^{\sigma}g_1^{-1}) = {}^{g_1\, g_2}h(\beta, {}^{\sigma}(g_1\, g_2)^{-1}) = \\
&= f_{g_1\, g_2}(\beta,\sigma).
\end{aligned}
$$

It is easy to see that $\mathrm{Im}(\overline{T}) \subseteq \mathrm{Der}^{**}_N(\mathbf{\Gamma},\mathbf{G})$. Then we can consider a categorical $\mathrm{Der}^{**}_N(\mathbf{\Gamma},\mathbf{G})$-crossed module $(\mathbf{G},\overline{T},\overline{\nu},\overline{\chi})$. In this case, $\overline{\nu}$ and $\overline{\chi}$ are the identities. Furthermore, the action of $\mathrm{Der}^{**}_N(\mathbf{\Gamma},\mathbf{G})$ on $\mathbf{G}$ defined by ${}^{(p,f,\varepsilon)}g = p(\bar{p}_0(g))\, g$ is strict, in fact:

$$
\begin{aligned}
{}^{(p,f,\varepsilon)}(g_1\, g_2) &= p(\bar{p}_0(g_1\, g_2))\, g_1\, g_2; \\
{}^{(p,f,\varepsilon)}g_1\, {}^{(p,f,\varepsilon)}g_2 &= p(\bar{p}_0(g_1))\, g_1\, p(\bar{p}_0(g_2))\, g_2 = \\
&= p(\bar{p}_0(g_1))\, {}^{\bar{p}_0(g_1)}p(\bar{p}_0(g_2))\, g_1\, g_2 = \\
&= \partial\varepsilon(\bar{p}_0(g_1), \bar{p}_0(g_2))p(\bar{p}_0(g_1\, g_2))\, g_1\, g_2 = \\
&= p(\bar{p}_0(g_1\, g_2))\, g_1\, g_2 = {}^{(p,f,\varepsilon)}(g_1\, g_2); \\
{}^{((p_1,f_1,\varepsilon_1)\,(p_2,f_2,\varepsilon_2))}g &= {}^{(p,f,\varepsilon)}g = p(\bar{p}_0(g))\, g = \\
&= p_1(\bar{p}_0(p_2(\bar{p}_0(g)))\, \bar{p}_0(g))\, p_2(\bar{p}_0(g))\, g; \\
{}^{(p_1,f_1,\varepsilon_1)}[\, {}^{(p_2,f_2,\varepsilon_2)}g] &= {}^{(p_1,f_1,\varepsilon_1)}[p_2(\bar{p}_0(g))\, g] = \\
&= p_1(\bar{p}_0(p_2(\bar{p}_0(g))\, g))\, p_2(\bar{p}_0(g))\, g = \\
&= p_1(\bar{p}_0(p_2(\bar{p}_0(g)))\, \bar{p}_0(g))\, p_2(\bar{p}_0(g))\, g = \\
&= {}^{((p_1,f_1,\varepsilon_1)\,(p_2,f_2,\varepsilon_2))}g.
\end{aligned}
$$

Thus we have the following crossed square:

$$
\begin{array}{ccc}
G_1 & \xrightarrow{\overline{T}_1} & D^* \\
{\scriptstyle\partial}\downarrow & & \downarrow{\scriptstyle\partial} \\
G_0 & \xrightarrow{\overline{T}_0} & \mathrm{Ob}(\mathrm{Der}_N(\mathbf{\Gamma},\mathbf{G}))^*
\end{array}
$$

where $\overline{T}_1(\alpha) = \theta_\alpha$ (by abuse of notation, we have denoted again by $\overline{T}_1$), $\theta_\alpha(\sigma) = \alpha\,{}^\sigma\alpha^{-1}$, the action of the group $\mathrm{Ob}(\mathrm{Der}_N(\mathbf{\Gamma},\mathbf{G}))^*$ on D^* and on G_0 given above, the action of $\mathrm{Ob}(\mathrm{Der}_N(\mathbf{\Gamma},\mathbf{G}))^*$ on G_1 is defined by

$$^{(p,f,\varepsilon)}\alpha = f(\bar{p}_1(\alpha),1)\,^{p_1(1)}\alpha = f(\bar{p}_1(\alpha),1)\,\alpha$$

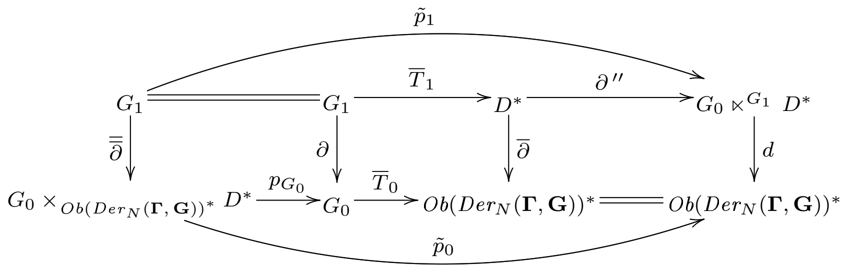
and the function $h : D^* \times G_0 \to G_1$ is given by $h(\theta, g) = \theta(\bar{p}_0(g))$.

Finally, we conclude, as we have seen in the Chapter 4, that $\mathcal{H}^1_N(\mathbf{\Gamma},\mathbf{G})$ constructed as follows:

$$\mathcal{H}^1_N(\mathbf{\Gamma},\mathbf{G}) = \frac{\mathrm{Der}^{**}_N(\mathbf{\Gamma},\mathbf{G})}{<\mathbf{G},\overline{\mathbf{T}}>}$$

corresponds to the crossed module $d : G_0 \ltimes^{G_1} D^* \to \mathrm{Ob}(\mathrm{Der}_N(\mathbf{\Gamma},\mathbf{G}))^*$ where $d(g,\theta) = \overline{\partial}(\theta)\cdot\overline{T}_0(g)$ and the action of $\mathrm{Ob}(\mathrm{Der}_N(\mathbf{\Gamma},\mathbf{G}))^*$ on $G_0 \ltimes^{G_1} D^*$ is given by $^{(p,f,\varepsilon)}(g,\theta) = (\,^{(p,f,\varepsilon)}g,\,^{(p,f,\varepsilon)}\theta)$.
Using Proposition 4.4.1 we easily obtain the following result.

Proposition 5.5.3. *The following outer diagram*



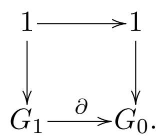
is a crossed square.

Examples. The following standard examples show when $\mathcal{H}^1_N(\mathbf{\Gamma},\mathbf{G})$ is isomorphic to $\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G})$ defined in [14] as follows:

$$\mathcal{H}^1(\mathbf{\Gamma},\mathbf{G}) = \frac{\mathrm{Der}^*(\mathbf{\Gamma},\mathbf{G})}{<\mathbf{G},\overline{\mathbf{T}}>}.$$

Notice that the last one is, in general, a categorical group not strict.
$\mathcal{H}^1_N(\mathbf{\Gamma},\mathbf{G})$ is valid as a generalization of the known cohomologies theory.

(a) If $\partial : G_1 \to G_0$ is a crossed module, then we can see it as the crossed square:



In this case, we denote by $\mathbf{\Gamma} = G_1[0]$, $\mathbf{G} = G_0[0]$ and $\mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G}))$ is the set $\mathrm{Der}(G_0, G_1)$ of all derivations from G_0 to G_1. The tensor product on derivations is the Whitehead product. $\mathrm{Ob}(\mathrm{Der}^*(\mathbf{\Gamma}, \mathbf{G}))$ becomes the Whitehead group $\mathrm{Der}^*(G_0, G_1)$, that is the group of units of $\mathrm{Der}(G_0, G_1)$. The set of arrows of $\mathcal{H}^1(\mathbf{\Gamma}, \mathbf{G})$ is isomorphic to G_1 and the tensor product is the usual product in G_1. Then we have:

$$\begin{aligned}
&\pi_1(\mathcal{H}^1(\mathbf{\Gamma}, \mathbf{G})) = {G_1}^{G_0} = H^0_L(G_0, G_1);\\
&\pi_0(\mathcal{H}^1(\mathbf{\Gamma}, \mathbf{G})) = \frac{\mathrm{Der}^*(G_0, G_1)}{\mathrm{Ider}(G_0, G_1)} = H^1_L(G_0, G_1);\\
&\pi_1(\mathcal{H}^0(\mathbf{\Gamma}, \mathbf{G})) = 1;\\
&\pi_0(\mathcal{H}^0(\mathbf{\Gamma}, \mathbf{G})) = {G_1}^{G_0} = H^0_L(G_0, G_1);
\end{aligned}$$

where L stands for Lue (see the recalls in the section 5.2 and the references [35], [29]). Moreover, we can easy observe that:

$$\mathcal{H}^1(\mathbf{\Gamma}, \mathbf{G}) = \mathcal{H}^1_N(\mathbf{\Gamma}, \mathbf{G}).$$

In the particular case where G_1 is a left G_0-module, we can see it as the trivial crossed module $1 : G_1 \to G_0$ and as the crossed square

$$\begin{array}{ccc}
1 & \longrightarrow & 1\\
\downarrow & & \downarrow\\
G_1 & \xrightarrow{1} & G_0.
\end{array}$$

Then we find the abelian cohomology (see A.1):

$$\begin{aligned}
&\pi_1(\mathcal{H}^1(\mathbf{\Gamma}, \mathbf{G})) = {G_1}^{G_0} = H^0(G_0, G_1);\\
&\pi_0(\mathcal{H}^1(\mathbf{\Gamma}, \mathbf{G})) = \frac{\mathrm{Der}(G_0, G_1)}{\mathrm{Ider}(G_0, G_1)} = H^1(G_0, G_1);\\
&\pi_1(\mathcal{H}^0(\mathbf{\Gamma}, \mathbf{G})) = 1;\\
&\pi_0(\mathcal{H}^0(\mathbf{\Gamma}, \mathbf{G})) = {G_1}^{G_0} = H^0(G_0, G_1).
\end{aligned}$$

(b) If $\partial : G_1 \to G_0$ is a Γ_0-equivariant braided crossed module, then we can see it as the crossed square:

$$\begin{array}{ccc}
G_1 & \longrightarrow & 1\\
{\scriptstyle\partial}\downarrow & & \downarrow\\
G_0 & \xrightarrow{1} & \Gamma_0.
\end{array} \tag{5.48}$$

where G_1 and G_0 are abelian groups and the action of G_0 on G_1 is trivial.

In this case, $\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G}) = \mathrm{Der}^*(\mathbf{\Gamma}, \mathbf{G})$ and we have $\mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G})) = Z^2_B(\Gamma_0, \partial : G_1 \to G_0)$ (the set of 2-cocycles defined by Borovoi). The tensor product on $\mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G}))$ is simplified to:

$$\begin{aligned}
&(p_1, \varepsilon_1)\,(p_2, \varepsilon_2) = (p, \varepsilon) \quad \text{where} \\
&p(\sigma) = p_1(\sigma)\, p_2(\sigma), \\
&\varepsilon(\sigma, \tau) = \varepsilon_1(\sigma, \tau)\, \varepsilon_2(\sigma, \tau).
\end{aligned}$$

Then we have:

$$\begin{aligned}
&\pi_1(\mathcal{H}^1(\mathbf{\Gamma}, \mathbf{G})) = H^1_B(\Gamma_0, \partial : G_1 \to G_0); \\
&\pi_0(\mathcal{H}^1(\mathbf{\Gamma}, \mathbf{G})) = H^2_B(\Gamma_0, \partial : G_1 \to G_0); \\
&\pi_1(\mathcal{H}^0(\mathbf{\Gamma}, \mathbf{G})) = \ker\partial^{\Gamma_0} = H^0_B(\Gamma_0, \partial : G_1 \to G_0); \\
&\pi_0(\mathcal{H}^0(\mathbf{\Gamma}, \mathbf{G})) = H^1_B(\Gamma_0, \partial : G_1 \to G_0).
\end{aligned}$$

Moreover, we can easy observe that:

$$\mathcal{H}^1(\mathbf{\Gamma}, \mathbf{G}) \cong \mathcal{H}^1_N(\mathbf{\Gamma}, \mathbf{G}).$$

In the particular case where $G_0 = 1$, the crossed square (5.48) becomes:

$$\begin{array}{ccc}
G_1 & \longrightarrow & 1 \\
\downarrow & & \downarrow \\
1 & \longrightarrow & \Gamma_0
\end{array}$$

and this implies that G_1 is a Γ_0-module.

In this case, $\mathrm{Ob}(\mathrm{Der}(\mathbf{\Gamma}, \mathbf{G}))$ is the set $Z^2(\Gamma_0, G_1)$ of 2-cocycles defined by Mac Lane (see A.1), with the obviously product. Then we find the cohomology in the abelian context:

$$\begin{aligned}
&\pi_1(\mathcal{H}^1(\mathbf{\Gamma}, \mathbf{G})) = H^1(\Gamma_0, G_1); \\
&\pi_0(\mathcal{H}^1(\mathbf{\Gamma}, \mathbf{G})) = H^2(\Gamma_0, G_1); \\
&\pi_1(\mathcal{H}^0(\mathbf{\Gamma}, \mathbf{G})) = {G_1}^{\Gamma_0} = H^0(\Gamma_0, G_1); \\
&\pi_0(\mathcal{H}^0(\mathbf{\Gamma}, \mathbf{G})) = H^1(\Gamma_0, G_1).
\end{aligned}$$

Appendix A

Cohomology of groups

A.1 Group cohomology with abelian coefficients via cochains

From K. S. Brown [3] "The cohomology theory of groups arose from both topological and algebraic sources. The starting point for topological aspect of the theory was the work of Hurewicz (1936) on aspherical spaces. [...] A few years later there was a rapid development of this subject by Eckmann, Eilenberg-Mac Lane, Freudenthal and Hopf. In particular, one had by the mid-1940's purely algebraic definition of group homology and cohomology, from which it became clear that the subject was of interest to algebraists as well as topologists. Indeed, the low dimensional cohomology groups were seen to coincide with groups which had been introduced much earlier in connection with various algebraic problems. H^1, for instance, consists of equivalence classes of derivations. And H^2 consists of equivalence classes of factor sets, the study of which goes back to Schur (1904), Schreier (1926), and Brauer (1926). Even H^3 had appeared in algebraic context (Teichmüller 1940)."

The cohomology of a group Γ with coefficients in a Γ-module G can be defined by using various constructions. One approach is to treat Γ-modules as modules over the group ring $\mathbb{Z}[G]$, which allows one to define group cohomology via Ext functors. This is a formal definition of group cohomology.

Another simpler way is to define $H^n(\Gamma, G)$ via cochains. These group cohomology is defined in terms of the standard "bar resolution".
Let Γ be a group. A Γ-module is an abelian group G together with an action of Γ on G. We shall denote this action by writing ${}^{\sigma}g$, where $\sigma \in \Gamma$ and $g \in G$.
The group of n-cochains of Γ with coefficients in G is the set of functions from Γ^n to G:

$$C^n(\Gamma, G) = \{f : \Gamma^n \to G\}.$$

$C^0(\Gamma, G)$ is taken simply to be G, as Γ^0 is a singleton set. The nth differential $\partial^n = \partial^n_G : C^n(\Gamma, G) \to C^{n+1}(\Gamma, G)$ is the map

$$\partial^n(f)(\sigma_1, \sigma_2, ..., \sigma_{n+1}) = {}^{\sigma_1} f(\sigma_2, ..., \sigma_{n+1}) \prod_{i=1}^{n} (f(\sigma_1, ..., \sigma_i\, \sigma_{i+1}, ..., \sigma_{n+1}))^{(-1)^i}$$
$$(f(\sigma_1, ..., \sigma_n))^{(-1)^{n+1}}.$$

Lemma A.1.1. *For any $n \geq 0$, one has $\partial^{n+1} \circ \partial^n = 1$.*

This Lemma shows that $C(\Gamma, G) = (C^n(\Gamma, G), \partial^n)$ is a cochain complex. Then we can consider the cohomology groups of $C(\Gamma, G)$.

For $n \geq 0$, we set $Z^n(\Gamma, G) = \ker(\partial^n)$, the group of n-cocycles of Γ with coefficients in G. We set $B^0(\Gamma, G) = 1$ and $B^n(\Gamma, G) = \mathrm{Im}(\partial^{n-1})$ for $n \geq 1$. We refer to $B^n(\Gamma, G)$ as the group of n-coboundaries of Γ with coefficients in G. Finally, because $C(\Gamma, G)$ is a cochain complex, we may make the following definition:

$$H^n(\Gamma, G) = \frac{Z^n(\Gamma, G)}{B^n(\Gamma, G)}.$$

The cohomology groups measure how far the cochain complex $C(\Gamma, G)$ is from being exact.

So the cohomology groups in low degree are:

- $Z^0(\Gamma, G) = \{\alpha \in G \;/\; {}^{\sigma}\alpha = \alpha \quad \forall \sigma \in \Gamma\} = G^{\Gamma}$
- $B^0(\Gamma, G) = 1$
- $H^0(\Gamma, G) = \dfrac{Z^0(\Gamma, G)}{B^0(\Gamma, G)} = G^{\Gamma}$
- $Z^1(\Gamma, G) = \{\theta : \Gamma \to G \;/\; \theta(\sigma\,\tau) = \theta(\sigma)\, {}^{\sigma}\theta(\tau)\} = \mathrm{Der}(\Gamma, G)$
- $B^1(\Gamma, G) = \{\theta : \Gamma \to G \;/\; \exists \mu \in G \;:\; \theta(\sigma) = {}^{\sigma}\mu\, \mu^{-1}\} = \mathrm{Ider}(\Gamma, G)$
- $H^1(\Gamma, G) = \dfrac{Z^1(\Gamma, G)}{B^1(\Gamma, G)} = \dfrac{\mathrm{Der}(\Gamma, G)}{\mathrm{Ider}(\Gamma, G)}$
- $Z^2(\Gamma, G) = \{\varepsilon : \Gamma \times \Gamma \to G \;/\; {}^{\sigma}\varepsilon(\tau, \upsilon)\, \varepsilon(\sigma, \tau\, \upsilon) = \varepsilon(\sigma, \tau)\, \varepsilon(\sigma\, \tau, \upsilon)\}$
- $B^2(\Gamma, G) = \{\varepsilon \in C^2 \;/\; \exists t : \Gamma \to G \;:\; \varepsilon(\sigma, \tau) = {}^{\sigma}t(\tau)\, t(\sigma\tau)^{-1}\, t(\sigma)\}$
- $H^2(\Gamma, G) = \dfrac{Z^2(\Gamma, G)}{B^2(\Gamma, G)}$

A.2 Serre cohomology

Serre (see [45]) was the first one to construct a low-dimensional cohomology theory for a group Γ with coefficients in a non-abelian group. Let G be a Γ-group, i.e. G is a group with an action of Γ on G, then he set:

- $H^0_S(\Gamma, G) = G^\Gamma$ the group of Γ-invariant elements;
- $Z^1_S(\Gamma, G) = \mathrm{Der}(\Gamma, G)$ the set of 1-cocycles of Γ with coefficients in G.

There exists an equivalence relation on $Z^1_S(\Gamma, G)$:

$$\theta_1 \sim \theta_2 \qquad \Leftrightarrow \qquad \theta_2(\sigma) = \mu^{-1}\, \theta_1(\sigma)\, {}^\sigma\mu$$

for every θ_1, $\theta_2 \in Z^1_S(\Gamma, G)$. The quotient set of this relation is $H^1_S(\Gamma, G)$.

The sets of cohomology $H^0_S(\Gamma, G)$ and $H^1_S(\Gamma, G)$ are functorial on G. In particular, if G is abelian then G is a Γ-module and we find the usual cohomology, defined in A.1.

Appendix B

B.1 Groupoids [7]

A groupoid $\mathcal{G}$ is a small category in which every morphism is an isomorphism. Thus $\mathcal{G}$ has a set of arrows, denoted by G, and a set G_0 of objects or vertices, together with functions $s, t : G \to G_0$, $i : G_0 \to G$ such that $s\,i = t\,i = id_{G_0}$.

$$G \underset{t}{\overset{s}{\rightrightarrows}} G_0, \qquad i : G_0 \to G$$

The functions s, t are sometimes called the source and target maps respectively. If $f, g \in G$ and $t(f) = s(g)$, $(g, f) \in G \times_\circ G$, where the latter set is defined by the following pullback diagram

$$\begin{array}{ccc} G \times_\circ G & \xrightarrow{\pi_2} & G \\ {\scriptstyle \pi_1}\downarrow & & \downarrow{\scriptstyle t} \\ G & \xrightarrow{s} & G_0. \end{array}$$

The composition of arrows: $\circ : G \times_\circ G \to G$ denoted by $g \circ f = g\,f$, is such that $s(g\,f) = s(f)$, $t(g\,f) = t(g)$. Furthermore, this composition is associative; the elements $i(x)$, $x \in G_0$, act as identities; and each arrow f has an inverse f^{-1} with

$$s(f^{-1}) = t(f), \qquad t(f^{-1}) = s(f), \qquad f\,f^{-1} = i(t(f)), \qquad f^{-1}\,f = i(s(f)).$$

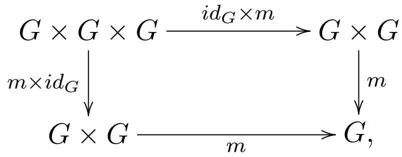
An element f is often written as an arrow $f : s(f) \to t(f)$.

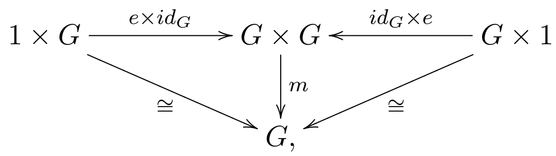
$$\underset{s(f)}{\bullet} \xrightarrow{\quad f \quad} \underset{t(f)=s(g)}{\bullet} \xrightarrow{\quad g \quad} \underset{t(g)}{\bullet} \qquad\qquad \underset{s(f)}{\bullet} \xrightarrow{\quad g\,f \quad} \underset{t(g)}{\bullet}$$

A morphism of groupoids is just a functor, and the category of groupoids will be denoted by $\mathcal{GPD}$.

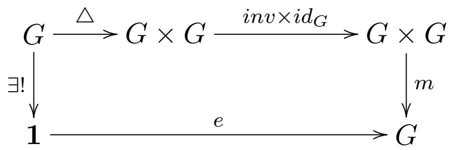
B.2 Groups in a category

Let $\mathcal{C}$ be a category with finite products and a terminal object 1. Let G be an object of $\mathcal{C}$. Then a monoid in $\mathcal{C}$ [37] is a triple $< G, m : G \times G \to G, e : 1 \to G >$ such that the following diagrams





commute. A group in a category $\mathcal{C}$ is a monoid $< G, m : G \times G \to G, e : 1 \to G >$ together with an arrow $inv : G \to G$ such that the diagram commutes



where $\triangle : G \to G \times G$ is the diagonal morphism (i.e. $p\triangle = q\triangle = id_G$, where p, q are the projections from the product to its components).

B.3 Simplicial groups

Definition B.3.1. *[38] A simplicial set $K_\bullet$ is a graded set indexed on the non-negative integers together with maps $\partial_i : K_q \to K_{q-1}$ and $s_i : K_q \to K_{q+1}, 0 \leq i \leq q$, which satisfy the following identities:*

(i) $\partial_i \partial_j = \partial_{j-1} \partial_i$ *if* $i < j$,

(ii) $s_i s_j = s_{j+1} s_i$ *if* $i \leq j$,

(iii) $\partial_i s_j = s_{j-1} \partial_i$ *if* $i < j$,
$\partial_j s_j = identity = \partial_{j+1} s_j$,
$\partial_i s_j = s_j \partial_{i-1}$ *if* $i > j + 1$.

The element of K_q are called q-simplices. The ∂_i and s_i are called face and degeneracy operators.

If $\mathcal{C}$ is any category, a simplicial object in $\mathcal{C}$ is given by a family of objects of $\mathcal{C}$, $\{K_n, n \geq 0\}$ and morphisms ∂_i and s_i as above. A simplicial group is a simplicial object in the category of groups.

Definition B.3.2. *Given a simplicial group $G_\bullet$, the Moore complex $((NG)_\bullet, d)$ is the normal chain complex defined by*

- $$(NG)_n = \cap_{i=1}^{n} ker(\partial_i^n)$$

  that is the joint kernel in degree n of all face maps except the 0-face,

- *and the differential maps given by the remaining 0-face*

$$d_n := \partial^n_{0_{|(NG)_n}} : (NG)_n \to (NG)_{n-1}.$$

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