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Variational and Geometric Methods for Nonlinear Differential Equations

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To my parents

Acknowledgements

"It seems that if one is working from the point of view of getting beauty in one's equation, and if one has really a sound insight, one is on a sure line of progress"
Paul Dirac, 1963

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Introduction

This thesis, which collects the results obtained during my Ph. D. [36, 37, 78, 79, 80, 81], is devoted to the study of different problems arising in nonlinear analysis. In this summary, we give only a brief account of the main issues considered in this work; having treated topics which are sometimes very different each one by the other, we prefer to endow each chapter with an own introduction, containing a detailed description of our research, of the methods we used and of the pertinent bibliography.

The work is divided in two parts. The first part regards the existence of entire solutions for different types of nonlinear differential equations:

- in Chapter 1, we study the scalar equation

$$\ddot{u} + g(u) = p(t),$$

  where the reaction term g is bounded and sufficiently regular. Under some additional assumptions, we provide a necessary and sufficient condition on the forcing term $p \in \mathcal{C}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ for the existence of infinitely many bounded solutions. This can be seen as a generalized Landesman-Lazer result, in a non-periodic framework. The proof is based entirely on variational methods.

- Chapter 2 regards the planar N-centre problem of celestial mechanics, that is, the study of the motion of a moving test particle under the gravitational force fields of N fixed heavy bodies. If $x = x(t) \in \mathbb{R}^2$ denotes the position of the particle at time $t \in \mathbb{R}$, and c_j ($j = 1, \dots, N$) denotes the position of the j-th centre, the motion equation is

$$\ddot{x} = -\sum_{j=1}^{N} \frac{m_j}{|x - c_j|^3} (x - c_j) . \qquad (0.1)$$

  We prove the existence of infinitely many collision-free periodic solutions with negative and small (in absolute value) energies. As a corollary, we characterize the associated dynamical system with a symbolic dynamics. In our proofs, we make use of perturbative, variational and geometric techniques; in particular, we exploit the fact that periodic solutions of equation (0.1) with a fixed energy are closed geodesics in a suitable Riemannian manifold. Moreover, to obtain collision-free solutions, we employ a local Levi-Civita regularization in a variational framework.

- In Chapter 3, by means of a perturbative approach, we extend the results of Chapter 2 in a more complicated situation, obtained by introducing a uniform circular motion for the centres of the problem. In a rotating frame of reference, this leads to the study of equation

$$\ddot{z} + 2\nu i \dot{z} = \nu^2 z - \sum_{j=1}^{N} \frac{m_j}{|z - c_j|^3} (z - c_j)\,.$$

  We show that, provided $|\nu|$ is sufficiently small, a *large quantity* of the collision-free solutions found in the previous chapter still exist. The main difficulty consists in ruling out the possibility that a solution has some collision, because in this setting the Levi-Civita regularization does not provide optimal results. To overcome this problem, we show that, at least locally (in a suitable sense), the variational structure of the problem "converges" to that of the N-centre problem as $\nu \to 0$, so that the fact that many solutions are collision-free is a consequence of the results of the previous chapter.

- In Chapter 4, we investigate the existence of solutions with super-algebraic growth to the nonlinear elliptic system

$$\begin{cases} \Delta u = u v^2 \\ \Delta v = u^2 v \quad \text{in } \mathbb{R}^N \\ u, v > 0, \end{cases} \tag{0.2}$$

  which appears in the analysis of phase-separation phenomena for Bose-Einstein condensates with multiple states. Our research is motivated by the fact that the known results concerning problem (0.2) involve only solutions with algebraic growth. For $N = 2$ (and hence also for every $N > 2$), we show that there exist also solutions with exponential growth, and we give a complete description of their geometry. In our proofs, the imposition of particular symmetries and the use of some Almgren-type monotonicity formulae play a central role.

In the aforementioned results, we obtain solutions exhibiting an "oscillatory behaviour", where the precise meaning of "oscillatory" depends on the peculiar problem we deal with. Concerning the first three cases, we always use the same abstract idea, which, as far as we know, can be ascribed to Seifert (who introduced, in a more geometric framework, the broken geodesics method, see [76]) and Nehari (who introduced the Nehari method in [66]); we roughly describe the idea in a general setting: let us consider a differential equation

$$\ddot{x} = F(t,x), \quad x : I \subset \mathbb{R} \to \mathbb{R}^N. \tag{0.3}$$

We divide the configuration space $\mathbb{R}^N$ in two disjoint subsets A and B. Let us assume that there exist two sets $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^2 \times \mathbb{R}^{2N}$ such that

- for any $(t_1, t_2, x_1, x_2) \in \mathcal{A}$ there exists a solution $x_{\mathcal{A}}(\cdot\,; t_1, t_2, x_1, x_2) : (t_1, t_2) \to \mathbb{R}^N$ of (0.3), such that

  $$x_{\mathcal{A}}(t_1; t_1, t_2, x_1, x_2) = x_1 \quad x_{\mathcal{A}}(t_2; t_1, t_2, x_1, x_2) = x_2,$$

  and $x_{\mathcal{A}}(t; t_1, t_2, x_1, x_2) \in A$ for every $t \in (t_1, t_2)$;

- for any $(t_1, t_2, x_1, x_2) \in \mathcal{B}$ there exists a solution $x_{\mathcal{B}}(\cdot\,; t_1, t_2, x_1, x_2) : (t_1, t_2) \to \mathbb{R}^N$ of (0.3), such that

  $$x_{\mathcal{B}}(t_1; t_1, t_2, x_1, x_2) = x_1 \quad x_{\mathcal{B}}(t_2; t_1, t_2, x_1, x_2) = x_2,$$

  and $x_{\mathcal{B}}(t; t_1, t_2, x_1, x_2) \in B$ for every $t \in (t_1, t_2)$.

The reader may think at the situations

- $N = 1$, $A = \{x > 0\}$ and $B = \{x < 0\}$ (cf. Chapter 1);
- $N = 2$, $A = \{|x| > R\}$ and $B = \{|x| < R\}$ for some $R > 0$ (cf. Chapters 2 and 3).

Now, let us assume that it is possible to juxtapose solutions corresponding to points of $\mathcal{A}$ and to points of $\mathcal{B}$, in the following sense: for any $(t_1, t_2, x_1, x_2) \in \mathcal{A}$, there exists $(t_2, t_3, x_2, x_3) \in \mathcal{B}$, and for any $(t_2, t_3, x_2, x_3) \in \mathcal{B}$, there exists $(t_3, t_4, x_3, x_4) \in \mathcal{A}$. Thus, given a finite sequence a points

$$(t_1, t_2, x_1, x_2) \in \mathcal{A},\ (t_2, t_3, x_2, x_3) \in \mathcal{B}, \\ \cdots,\ (t_{n-2}, t_{n-1}, x_{n-2}, x_{n-1}) \in \mathcal{A},\ (t_{n-1}, t_n, x_{n-1}, x_n) \in \mathcal{B}, \tag{0.4}$$

there is a well defined function $x : (t_1, t_n) \to \mathbb{R}^N$, as

$$x(t) = \begin{cases} x_{\mathcal{A}}(t; t_i, t_{i+1}, x_i, x_{i+1}) & \text{if } t \in [t_i, t_{i+1}) \text{ and } i \text{ is odd} \\ x_{\mathcal{B}}(t; t_i, t_{i+1}, x_i, x_{i+1}) & \text{if } t \in [t_i, t_{i+1}) \text{ and } i \text{ is even.} \end{cases}$$

Any x of this type is characterized by an "oscillatory" behaviour, in the sense that it passes alternatively from the set A to the set B, and vice versa; clearly, in general it is not a global solution of equation (0.3), because in the junction times t_i it is not necessarily $\mathcal{C}^1$. So, to find a solution defined in the whole $\mathbb{R}$, we have to complete the following program:

(*i*) we have to find at least one juxtaposed function which is regular, so that, by construction, it solves equation (0.3) in the time interval $[t_1, t_n]$. As we shall see, this can be done by means of a suitable choice of the sequence in (0.2): we show that, if such sequence is a solution of an appropriate variational problem, then the corresponding juxtaposed function is sufficiently regular. A remarkable fact is that, dealing with a finite sequence of points of $\mathbb{R}^2 \times \mathbb{R}^{2N}$, we passed to a problem in finite dimension;

(*ii*) we have to prove that in the previous line of reasoning it is possible to choose sequences (t_1^m) and $(t_{n_m}^m)$ such that $t_1^m \to -\infty$ and $t_{n_m}^m \to +\infty$ as $m \to \infty$, and the corresponding sequence of solutions defined in $[t_1^m, t_{n_m}^m]$ converges, in $\mathcal{C}^2_{\mathrm{loc}}(\mathbb{R})$, to a solution of (0.3).

Regarding the existence results of Chapter 4, we characterize the oscillatory behaviour of the solutions in a different way. We wish to prescribe an appropriate partition of $\mathbb{R}^N$ in two subsets A and B, to obtain a solution such that $u > v$ in A and $v > u$ in B. In our case, we choose

$$A = \bigcup_{k \in \mathbb{Z}} \mathbb{R} \times [2k\pi, (2k+1)\pi] \qquad B = \bigcup_{k \in \mathbb{Z}} \mathbb{R} \times [(2k+1)\pi, (2k+2)\pi].$$

Therefore, the oscillatory behaviour of (u, v) is given by the fact that the function $u - v$ changes sign when x passes alternatively from a strip $\mathbb{R} \times [k\pi, (k+1)\pi]$ (with $k \in \mathbb{Z}$) to another one. To succeed in finding solutions having the required properties, we consider system (0.2) in bounded cylinders $[-R, R] \times \mathbb{R}/(2\pi\mathbb{Z})$ instead of in $\mathbb{R}^2$, imposing suitable boundary conditions, which

- take into account the desired sign condition on $u - v$;
- are 2π-periodic in y.

By means of some Almgren-type monotonicity formulae, it is possible to show that, passing to the limit as $R \to +\infty$, we obtain a solution defined in the whole cylinder $\mathbb{R} \times \mathbb{R}/(2\pi\mathbb{Z})$, which can be extended by periodicity in the y variable in the whole $\mathbb{R}^2$. Moreover, thanks to an appropriate choice of the boundary conditions, $u - v > 0$ in A and $u - v < 0$ in B.

The second part of this thesis concerns the study of qualitative properties for solutions to some elliptic problems in unbounded domains. Chapter 5 is devoted to the classification of nonnegative solutions for

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) = u - g(x) & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases}$$

when $N = 2, 3$. We assume that the matrix A is of type

$$A(x') = \begin{pmatrix} \widehat{A}(x') & 0 \\ 0 & 1 \end{pmatrix},$$

and is such that the operator div $(A(x)\nabla(\cdot))$ is elliptic (not necessarily uniformly elliptic). By means of a combination of Liouville-type theorems and basic Fourier analysis, we provide a full classification of the admissible solutions for many types of inhomogeneous terms g. For instance, we show that:

- if $g(x) = 1$, then $u(x) = 1 - \cos x_N$;
- if $g(x) = \tilde{g}(x')$ for some $\tilde{g} \in \mathcal{C}(\mathbb{R}^{N-1})$, then a nonnegative solution does not exist;
- if $g(x) = \tilde{g}(x_N)$ for some $\tilde{g} \in \mathcal{C}(\mathbb{R})$ satisfying a suitable additional assumption, then u depends only on x_N, and is uniquely determined as the solution of

$$\begin{cases} -u''(x_N) = u(x_N) - g(x_N) & x_N > 0 \\ u(2k\pi) = u'(2k\pi) = 0 & \forall k \in \mathbb{Z}. \end{cases}$$

In Chapter 6, we consider again entire solutions of the nonlinear elliptic system (0.2):

$$\begin{cases} \Delta u = uv^2 \\ \Delta v = u^2 v & \text{in } \mathbb{R}^N \\ u, v > 0, \end{cases}$$

where $N \geq 2$. This time, we are interested in the study of monotonicity and 1-dimensional symmetry of algebraically growing solutions. The existence of a solution to (0.2) depending only on 1-variable has been proved in [12]. Therein, the authors formulated the following Gibbons-type conjecture:

Conjecture (Section 7 of [12])**.** Let $N \geq 2$, let (u, v) be a solution of (0.2) satisfying

$$\lim_{x_N \to -\infty} u(x', x_N) = 0 \quad \text{and} \quad \lim_{x_N \to +\infty} u(x', x_N) = +\infty$$
$$\lim_{x_N \to -\infty} v(x', x_N) = +\infty \quad \text{and} \quad \lim_{x_N \to +\infty} v(x', x_N) = 0,$$

the limits being uniform in $x' \in \mathbb{R}^{N-1}$. Then (u, v) is 1-dimensional.

The main result of Chapter 6 is the proof of the validity of this conjecture for algebraically growing solution. For such a proof, we make use of the moving planes method, which has to be suitably adapted in order to deal with a system of equation and with unbounded solutions. These facts introduce a lot of complications with respect to classical applications of the moving planes techniques to scalar equations. We can overcome these difficulties by means of some a priori estimates for solutions of (0.2) having algebraic growth, which follow from particular monotonicity formulae of Almgren-type and Alt-Caffarelli-Friedman-type.

Synthèse

Ce mémoire, qui réunit les résultats obtenus pendant ma thèse de doctorat [32, 33, 63, 64, 65, 66], est consacré à l'étude de divers problèmes d'analyse non-linéaire. Dans cette synthèse, nous donnons seulement un bref compte rendu des principales questions etudiées dans ce travail ainsi que des résultats obtenus. Ayant traité différents sujets, nous préférons doter chaque chapitre d'une propre introduction, contenant une description détaillée de notre recherche, des méthodes que nous avons utilisées ainsi que d'une bibliographie pertinente et exhaustive.

L'ouvrage est divisé en deux parties. La première partie de la thèse concerne l'existence de solutions entières pour différents type d'équations aux dérivées partielles non-linéaires.

- Dans le Chapitre 1, nous étudions l'équation

$$\ddot{u} + g(u) = p(t)$$

où le terme de réaction g est une fonction régulière et bornée. Sous des hypothèses adéquates, nous donnons une condition nécessaire et suffisante, sur le second membre $p \in C^0 \cap L^\infty$, pour l'existence d'une infinité de solutions bornées. Il s'agit d'un résultat à la Landesman-Lazer, mais dans un contexte non periodique. La preuve de ce résultat repose entièrement sur des méthodes variationnelles.

- Le Chapitre 2 concerne le problème des N-centres issue de la mécanique céleste. Si $x = x(t) \in \mathbb{R}^2$ désigne la position de la particule au temps $t \in \mathbb{R}$, et si c_j ($j = 1, ..., N$) désigne la position du $j-$ième centre, l'équation du mouvement est

$$\ddot{x} = -\sum_{j=1}^{N} \frac{m_j}{|x - c_j|^3}(x - c_j). \tag{0.1}$$

Nous démontrons l'existence d'une infinité de solutions périodiques (collision-free) d'énergie négative et petite. Comme corollaire, nous caractérisons le système dynamique associé avec une dynamique symbolique. Pour ces résultats, nous utilisons des techniques perturbatives, variationnelles et géométriques. En particulier, nous exploitons le fait que les solutions de (0.1) d'énergie fixée, sont des géodésiques fermées pour une certaine structure Riemannienne. De plus, pour obtenir les solutions, nous utilisons une régularisation de Levi-Civita locale dans un contexte variationnel.

- Dans le Chapitre 3, à l'aide d'une approche perturbative, nous étendons les résultats du Chapitre 2 à une situation beaucoup plus compliquée, obtenue en introduisant un mouvement circulaire uniforme pour les centres du problème. Ceci conduit à l'étude de l'équation

$$\ddot{z} + 2\nu i\dot{z} = \nu^2 z - \sum_{j=1}^{N} \frac{m_j}{|z - c_j|^3}(z - c_j).$$

  Nous prouvons que, si $|\nu|$ est suffisamment petit, une grande quantité de solutions (collision-free) trouvées au Chapitre 2 continuent à exister. La difficulté principale est celle d'éviter les collisions, car dans ce contexte la régularisation de Levi-Civita ne donne pas des résultats optimaux. Pour surmonter cette difficulté, nous montrons que, au moins localement, la structure variationnelle du problème "converge" vers celle du problème des N-centres. Le résultat précité est ainsi une conséquence des résultats obtenus dans le chapitre 2.

- Dans le Chapitre 4, nous étudions l'existence de solutions, avec croissance plus que polynômiale, du système elliptique non-linéaire

$$\begin{cases} \Delta u = uv^2 & \text{in } \mathbb{R}^N \\ \Delta v = vu^2 & \text{in } \mathbb{R}^N \\ u, v > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{0.2}$$

  qui apparaît dans l'analyse des phénomènes de séparation de phase pour les condensats de Bose-Einstein. Notre recherche est motivée par le fait que tous les résultats connus au sujet du problème (0.2) concernent seulement des solutions à croissance au plus algébrique. Pour $N = 2$ (et donc aussi pour $N > 2$), nous démontrons l'existence de solutions à croissance exponentielle et nous donnons aussi une déscription de leur géométrie. L'utilisation de formules de monotonie à la Almgren, ainsi que de symétries particulières, joue un rôle central dans la preuve de ce résultat.

Dans les résultats mentionnés ci-dessus, on obtient des solutions présentant un "comportement oscillatoire", où la signification précise de "oscillatoire" dépend du problème particulier que nous traitons. En ce qui concerne les trois premiers cas, nous utilisons la même idée abstraite, qui, à notre connaissance peut être attribuée à Seifert (voir [76]) et à Nehari (voir [66]). L'idée, dans un cadre général, est la suivante: considérons une équation différentielle

$$\ddot{x} = F(t,x), \quad x : I \subset \mathbb{R} \to \mathbb{R}^N. \tag{0.3}$$

On partitionne l'espace des configurations $\mathbb{R}^N$ en deux parties disjointes A et B. Supposons qu'il existe deux parties $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^2 \times \mathbb{R}^{2N}$ telles que:

- pour tout $(t_1, t_2, x_1, x_2) \in \mathcal{A}$ il existe une solution $x_{\mathcal{A}}(\cdot\,; t_1, t_2, x_1, x_2) : (t_1, t_2) \to \mathbb{R}^N$ de (0.3), telle que
$$x_{\mathcal{A}}(t_1; t_1, t_2, x_1, x_2) = x_1 \quad x_{\mathcal{A}}(t_2; t_1, t_2, x_1, x_2) = x_2,$$
et $x_{\mathcal{A}}(t; t_1, t_2, x_1, x_2) \in A$ pour tout $t \in (t_1, t_2)$;
- pour tout $(t_1, t_2, x_1, x_2) \in \mathcal{B}$ il existe une solution $x_{\mathcal{B}}(\cdot\,; t_1, t_2, x_1, x_2) : (t_1, t_2) \to \mathbb{R}^N$ de (0.3), telle que
$$x_{\mathcal{B}}(t_1; t_1, t_2, x_1, x_2) = x_1 \quad x_{\mathcal{B}}(t_2; t_1, t_2, x_1, x_2) = x_2,$$
et $x_{\mathcal{B}}(t; t_1, t_2, x_1, x_2) \in B$ pour tout $t \in (t_1, t_2)$.

Le lecteur peut penser aux situations suivantes:

- $N = 1$, $A = \{x > 0\}$ et $B = \{x < 0\}$ (cf. chapitre 1);
- $N = 2$, $A = \{|x| > R\}$ et $B = \{|x| < R\}$, avec $R > 0$ (cf. chapitres 2 et 3).

Maintenant, supposons qu'il est possible de juxtaposer des solutions correspondants aux points de $\mathcal{A}$ et aux points de $\mathcal{B}$, dans le sens suivant: pour tout $(t_1, t_2, x_1, x_2) \in \mathcal{A}$ il existe $(t_2, t_3, x_2, x_3) \in \mathcal{B}$ et pour tout $(t_2, t_3, x_2, x_3) \in \mathcal{B}$ il existe $(t_3, t_4, x_3, x_4) \in \mathcal{A}$. Ainsi, étant donné un nombre fini de points:

$$(t_1, t_2, x_1, x_2) \in \mathcal{A},\ (t_2, t_3, x_2, x_3) \in \mathcal{B}, \\ \cdots,\ (t_{n-2}, t_{n-1}, x_{n-2}, x_{n-1}) \in \mathcal{A},\ (t_{n-1}, t_n, x_{n-1}, x_n) \in \mathcal{B}, \tag{0.4}$$

la fonction $x : (t_1, t_n) \to \mathbb{R}^N$,

$$x(t) = \begin{cases} x_{\mathcal{A}}(t; t_i, t_{i+1}, x_i, x_{i+1}) & \text{si } t \in [t_i, t_{i+1}) \text{ et } i \text{ est impari} \\ x_{\mathcal{B}}(t; t_i, t_{i+1}, x_i, x_{i+1}) & \text{si } t \in [t_i, t_{i+1}) \text{ et } i \text{ est pari.} \end{cases}$$

est bien définie.

Toute fonction x de ce type est caractérisée par un comportement "oscillatoire", en ce sens, qu'elle passe alternativement de l'ensemble A à l'ensemble B, et viceversa; clairement, en général, elle n'est pas une solution globale de l'quation (0.3), parce qu'au temps t_i elle n'est pas nécessairement de classe $mathcalC^1$. Ainsi, pour trouver une solution définie sur $\mathbb{R}$, nous devons remplir les deux points du programme suivant:

(i) nous devons trouver au moins une fonction juxtaposée lisse, de sorte que, par construction, elle résout l'équation (0.3) dans l'intervalle de temps $[t_1, t_n]$. Comme nous allons le voir, cela peut se faire au moyen d'une caractérisation variationnelle appropriée de la suite en (0.4). Nous sommes ainsi passés à un problème en dimension finie.

(ii) nous devons démontrer que dans le raisonnement précédent il est possible de choisir les suites (t_1^m) et $(t_{n_m}^m)$ t.q. $t_1^m \to -\infty$ et $t_{n_m}^m \to +\infty$, lorsque $m \to +\infty$ et que, la suite de fonctions correspondante (définie sur $[t_1^m, t_{n_m}^m]$) converge vers une solution de (0.3) dans $\mathcal{C}^2_{loc}(\mathbb{R})$.

Au sujet du résultat d'existance du Chapitre 4, nous caractérisons le comportement oscillatoire des solutions d'une manière différente. Nous souhaitons prescrire une partition appropriée de $\mathbb{R}^N$ en deux sous-ensembles A et B, pour obtenir une solution telle que $u > v$ dans A et $v > u$ dans B. Dans notre cas, nous choisissons

$$A = \bigcup_{k \in \mathbf{Z}} \mathbb{R} \times [2k\pi, (2k+1)\pi] \qquad B = \bigcup_{k \in \mathbf{Z}} \mathbb{R} \times [(2k+1)\pi, (2k+2)\pi].$$

Par conséquent, le comportement oscillatoire de (u, v) est donnée par le fait que la fonction $u-v$ change de signe lorsque x passe alternativement d'une bande $\mathbb{R} \times [k\pi, (k+1)\pi]$ (avec $k \in \mathbb{Z}$) à une autre. Pour réussir à trouver des solutions ayant les propriétés requises, nous considérons le système (0.2) dans des cylindres bornés de la forme $[-R, R] \times \mathbb{R}/(2\pi\mathbb{Z})$, en imposant des conditions aux limites appropriées, qui

- tiennent en compte de la condition de signe désirée sur $u - v$;
- sont 2π-périodiques en y.

Par le biais de certaines formules de monotonie à la Almgren, nous passons à la limite pour $R \to +\infty$, en obtenant ainsi des solutions sur le cylindre $\mathbb{R} \times \mathbb{R}/(2\pi\mathbb{Z})$ qui peuvent être prolongées par périodicité (en la variable y) à $\mathbb{R}$ tout entier. En outre, grâce à un choix approprié des conditions aux limites, nous avons $u - v > 0$ dans A et $u - v < 0$ dans B.

La seconde partie de la thèse concerne l'étude des propriétés qualitatives des solutions de problèmes elliptiques non-linéaires posés dans des domaines non-bornés. Le Chapitre 5 est consacré à la classification des solutions positives d'équations elliptiques du type

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) = u - g(x) & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases}$$

où $N = 2, 3$. Nous supposons la matrice A de la forme suivante:

$$A(x') = \begin{pmatrix} \widehat{A}(x') & 0 \\ 0 & 1 \end{pmatrix},$$

et telle que l'opérateur $div(A(x)\nabla(\cdot))$ est elliptique, mais non nécessairement uniformement elliptique. En combinant des théorèmes de type Liouville et l'analyse de Fourier, nous donnons la classification complète des solutions admissibles pour une grande variété de termes non homogènes g. En particulier, nous démontrons que :

- si $g(x) = 1$, alors $u(x) = 1 - \cos(x_N)$;
- si $g(x) = \tilde{g}(x')$, avec $\tilde{g} \in C(\mathbb{R}^{N-1})$, alors le problème considéré ne possède aucune solution;
- si $g(x) = \tilde{g}(x_N)$, avec $\tilde{g} \in C(\mathbb{R})$ approprié, alors la solution u ne dépend que de la variable x_N et elle est donnée par l'unique solution de

$$\begin{cases} -u''(x_N) = u(x_N) - g(x_N) & x_N > 0 \\ u(2k\pi) = u'(2k\pi) = 0 & \forall k \in \mathbb{Z}. \end{cases}$$

Dans le Chapitre 6, nous considérons encore les solutions entières du système elliptique non-linéaire (0.2):

$$\begin{cases} \Delta u = uv^2 \\ \Delta v = u^2 v & \text{in } \mathbb{R}^N \\ u, v > 0, \end{cases}$$

où $N \geq 2$. Ici nous nous intéressons à l'étude de la monotonie et de la symétrie unidimensionnelle pour les solutions à croissance au plus algébrique. L'existence de solutions de (0.2) en dimension $N = 1$ a été prouvée in [12]. Dans ce travail, les auteurs formulent aussi la conjecture suivante (à la Gibbons) :

Conjecture (Section 7 of [12])**.** Let $N \geq 2$, let (u, v) be a solution of (0.2) satisfying

$$\lim_{x_N \to -\infty} u(x', x_N) = 0 \quad \text{and} \quad \lim_{x_N \to +\infty} u(x', x_N) = +\infty$$
$$\lim_{x_N \to -\infty} v(x', x_N) = +\infty \quad \text{and} \quad \lim_{x_N \to +\infty} v(x', x_N) = 0,$$

the limits being uniform in $x' \in \mathbb{R}^{N-1}$. Then (u, v) is 1-dimensional.

Le résultat principal du Chapitre 6 est la démonstration de cette conjecture pour les solutions à croissance au plus algébrique. Pour cette preuve, nous utilisons la méthode des hyperplans mobiles (moving planes method), qui doit être attentivement modifiée pour gérer le fait que nous travaillons ici avec un système d'équations et avec des solutions non-bornées. Ces faits introduisent de nombreuses nouvelles difficultés. Nous avons surmonté ces difficultés à l'aide d'estimations *a priori* pour les solutions de (0.2), à crossance au plus algébrique, qui sont obtenues par des formules de monotonie à la Almgren et à la Alt-Caffarelli-Friedmann.

Introduzione

Questa tesi, che contiene i risultati ottenuti durante i mie studi di dottorato [36, 37, 78, 79, 80, 81], è dedicata a diversi problemi emergenti nel contesto dell'analisi nonlineare. In questa introduzione diamo soltanto un breve resoconto dei principali argomenti considerati in questo lavoro; essendoci occupati di problemi anche molto diversi l'uno dall'altro, abbiamo preferito dotare ciascun capitolo di una propria introduzione, contenente una descrizione dettagliata della nostra ricerca, delle tecniche utilizzate e della bibliografia pertinente.

Il lavoro è diviso in due parti. La prima parte riguarda l'esistenza di soluzioni intere per diversi tipi di equazioni differenziali nonlineari:

- nel Capitolo 1, consideriamo l'equazione scalare

  $$\ddot{u} + g(u) = p(t),$$

  dove il termine di reazione g è limitato e sufficientemente regolare. Sotto alcune ipotesi supplementari, determiniamo una condizione necessaria e sufficiente sul termine forzante $p \in \mathcal{C}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ per l'esistenza di infinite soluzioni limitate. Questo può essere considerato un risultato alla Landesman-Lazer in un contesto non periodico. La dimostrazione si basa interamente su metodi variazionali.

- Il Capitolo 2 riguarda il problema planare degli N-centri, ossia lo studio del moto di una particella mobile sottoposta all'attrazione gravitazionale di N corpi pesanti, la cui posizione è fissata. Se $x = x(t) \in \mathbb{R}^2$ denota la posizione della particella all'istante $t \in \mathbb{R}$, e c_j $(j = 1, \dots, N)$ denota la posizione del j-esimo centro, l'equazione del moto è

  $$\ddot{x} = -\sum_{j=1}^{N} \frac{m_j}{|x - c_j|^3}\,(x - c_j)\,. \tag{0.1}$$

  Dimostriamo l'esistenza di infinite soluzioni periodiche prive di collisioni con energia negativa e piccola (in valore assoluto). Come corollario, caratterizziamo il sistema dinamico associato con una dinamica simbolica. Per le dimostrazioni dei risultati principali, facciamo uso di tecniche perturbative, variazionali e geometriche; in particolare, sfruttiamo il fatto che soluzioni periodiche dell'equazione (0.1) con un'energia fissata sono geodetiche chiuse di un'opportuna varietà riemanniana. Inoltre, per ottenere soluzioni prive di collisioni, utilizziamo una regolarizzazione locale alla Levi-Civita in un contesto variazionale.

- Nel Capitolo 3, attraverso un approccio perturbativo, estendiamo i risultati del Capitolo 2 ad una situazione più complicata, che si ottiene ponendo in rotazione, con moto circolare uniforme, i centri del problema. In un sistema di riferimento rotante, ciò porta allo studio dell'equazione

$$\ddot{z} + 2\nu i \dot{z} = \nu^2 z - \sum_{j=1}^{N} \frac{m_j}{|z - c_j|^3} (z - c_j) .$$

  Mostriamo che, a patto di segliere $|\nu|$ abbastanza piccolo, una *gran parte* delle soluzioni prive di collisioni trovate nel capitolo precedente continua ad esistere. La principale difficoltà consiste nell'escludere la possibilità che una soluzione abbia qualche collisione, perchè in questo caso la regolarizzazione di Levi-Civita non fornisce risultati ottimali. Per aggirare questo ostacolo, mostriamo che, almeno localmente (in un senso opportuno), la struttura variazionale del problema "converge" a quella del problema degli N-centri per $\nu \to 0$, cosicché il fatto che molte soluzioni siano prive di collisioni segue dai risultati del capitolo precedente.

- Nel Capitolo 4, studiamo l'esistenza di soluzioni con crescita super-polinomiale per il sistema ellittico nonlineare

$$\begin{cases} \Delta u = uv^2 \\ \Delta v = u^2 v \quad \text{in } \mathbb{R}^N \\ u, v > 0, \end{cases} \tag{0.2}$$

  che compare nell'analisi di fenomeni di separazione di fase per condensati di Bose-Einstein a più stati. Il nostro interesse deriva dal fatto che i risultati noti riguardanti il problema (0.2) riguardano esclusivamente soluzioni con crescita al più polinomiale. Per $N = 2$ (e quindi anche per $N > 2$), mostriamo che esistono anche soluzioni con crescita esponenziale, e diamo una descrizione completa delle loro proprietà geometriche. Nelle nostre dimostrazioni l'imposizione di particolari simmetrie e l'uso di formule di monotonia di tipo Almgren giocano un ruolo fondamentale.

Nei risultati menzionati, otteniamo soluzioni che esibiscono un "comportamento oscillatorio", dove il particolare significato di "oscillatorio" dipende dal particolare problema in esame. Per quanto riguarda i primi tre casi, usiamo sempre la stessa idea astratta, che, per quanto ci è noto, può essere attribuita a Seifert (il quale ha introdotto in un contesto geometrico il *metodo delle broken geodesics*, si veda [76]) e Nehari (il quale ha introdotto il *metodo di Nehari* in [66]); di seguito descriviamo sinteticamente l'idea in un contesto del tutto generale: consideriamo un'equazione differenziale

$$\ddot{x} = F(t,x), \quad x : I \subset \mathbb{R} \to \mathbb{R}^N. \tag{0.3}$$

Dividiamo lo spazio delle configurazioni $\mathbb{R}^N$ in due insiemi disgiunti A e B. Supponiamo che esistano $\mathcal{A}, \mathcal{B} \subset \mathbb{R}^2 \times \mathbb{R}^{2N}$ tali che

- per ogni $(t_1, t_2, x_1, x_2) \in \mathcal{A}$ esiste una soluzione $x_{\mathcal{A}}(\cdot\,; t_1, t_2, x_1, x_2) : (t_1, t_2) \to \mathbb{R}^N$ di (0.3), tale che
$$x_{\mathcal{A}}(t_1; t_1, t_2, x_1, x_2) = x_1 \quad x_{\mathcal{A}}(t_2; t_1, t_2, x_1, x_2) = x_2,$$
e $x_{\mathcal{A}}(t; t_1, t_2, x_1, x_2) \in A$ per ogni $t \in (t_1, t_2)$;
- per ogni $(t_1, t_2, x_1, x_2) \in \mathcal{B}$ esiste una soluzione $x_{\mathcal{B}}(\cdot\,; t_1, t_2, x_1, x_2) : (t_1, t_2) \to \mathbb{R}^N$ di (0.3), tale che
$$x_{\mathcal{B}}(t_1; t_1, t_2, x_1, x_2) = x_1 \quad x_{\mathcal{B}}(t_2; t_1, t_2, x_1, x_2) = x_2,$$
and $x_{\mathcal{B}}(t; t_1, t_2, x_1, x_2) \in B$ per ogni $t \in (t_1, t_2)$.

Il lettore può pensare alle situazioni

- $N = 1$, $A = \{x > 0\}$ e $B = \{x < 0\}$ (si veda il Capitolo 1);
- $N = 2$, $A = \{|x| > R\}$ e $B = \{|x| < R\}$ per qualche $R > 0$ (si vedano i Capitoli 2 e 3).

Ora, supponiamo che sia possibile giustapporre soluzioni corrispondenti a punti di $\mathcal{A}$ ed a punti di $\mathcal{B}$, nel modo seguente: per ogni $(t_1, t_2, x_1, x_2) \in \mathcal{A}$, esiste $(t_2, t_3, x_2, x_3) \in \mathcal{B}$, e per ogni $(t_2, t_3, x_2, x_3) \in \mathcal{B}$, esiste $(t_3, t_4, x_3, x_4) \in \mathcal{A}$. Pertanto, data una successione

$$\begin{aligned}(t_1, t_2, x_1, x_2) \in \mathcal{A},\ (t_2, t_3, x_2, x_3) \in \mathcal{B}, \\ \cdots,\ (t_{n-2}, t_{n-1}, x_{n-2}, x_{n-1}) \in \mathcal{A},\ (t_{n-1}, t_n, x_{n-1}, x_n) \in \mathcal{B},\end{aligned} \tag{0.4}$$

è ben definita la funzione $x : (t_1, t_n) \to \mathbb{R}^N$ definita da

$$x(t) = \begin{cases} x_{\mathcal{A}}(t; t_i, t_{i+1}, x_i, x_{i+1}) & \text{se } t \in [t_i, t_{i+1}) \text{ e } i \text{ è dispari} \\ x_{\mathcal{B}}(t; t_i, t_{i+1}, x_i, x_{i+1}) & \text{se } t \in [t_i, t_{i+1}) \text{ e } i \text{ è pari.} \end{cases}$$

Qualsiasi x di questo tipo è caratterizzata da un comportamento "oscillatorio", nel senso che passa alternativamente dall'insieme A all'insieme B, e viceversa; chiaramente, in genere non è una soluzione dell'equazione (0.3), perchè negli istanti di connessione t_i non è necessariamente $\mathcal{C}^1$. Quindi, per trovare una soluzione definita globalmente, occorre completare il seguente programma:

(i) bisogna trovare almeno una funzione giustapposta che sia regolare, cosicché, per costruzione, risolve l'equazione (0.3) nell'intervallo $[t_1, t_n]$. Come vedremo, questo può essere fatto attraverso un'opportuna scelta della successione in (0.2): mostriamo che se tale successione è soluzione di un problema variazionale convenientemente introdotto, allora la funzione giustapposta considerata è sufficientemente regolare. È significativo il fatto che, trattando con successioni finite di punti di $\mathbb{R}^2 \times \mathbb{R}^{2N}$, ci si riconduca ad un problema in dimensione finita;

(ii) bisogna provare che nel precedente ragionamento è possibile scegliere successioni (t_1^m) e $(t_{n_m}^m)$ tali che $t_1^m \to -\infty$ e $t_{n_m}^m \to +\infty$ per $m \to \infty$, e la corrispondente successione di soluzioni in $[t_1^m, t_{n_m}^m]$ converge, in $\mathcal{C}^2_{\text{loc}}(\mathbb{R})$, ad una soluzione di (0.3).

Per quanto riguarda i risultati di esistenza del Capitolo 4, caratterizziamo il comportamento oscillatorio della soluzione in un modo diverso. Il nostro obiettivo è prescrivere un'appropriata partizione di $\mathbb{R}^N$ in due sottoinsiemi A e B, per ottenere una soluzione tale che $u > v$ in A e $v > u$ in B. Nel caso che prenderemo in esame,

$$A = \bigcup_{k\in\mathbb{Z}} \mathbb{R} \times [2k\pi, (2k+1)\pi] \qquad B = \bigcup_{k\in\mathbb{Z}} \mathbb{R} \times [(2k+1)\pi, (2k+2)\pi].$$

Di conseguenza, il carattere oscillatorio di (u, v) è dato dal fatto che la funzione $u - v$ cambia segno quando x passa alternativamente da una striscia $\mathbb{R} \times [k\pi, (k+1)\pi]$ (con $k \in \mathbb{Z}$) ad un'altra. Per trovare soluzioni soddisfacenti una tale condizione, consideriamo il sistema (0.2) in cilindri limitati $[-R, R] \times \mathbb{R}/(2\pi\mathbb{Z})$ piuttosto che in $\mathbb{R}^2$, imponendo opportune condizioni al contorno che

- tengano in considerazione le proprietà di segno desiderate sulla funzione $u - v$;
- siano 2π-periodiche in y.

Per mezzo di alcune formule di monotonia di tipo Almgren, è possibile mostrare che, passando al limite per $R \to +\infty$, si ottiene una soluzione definita in tutto il cilindro $\mathbb{R} \times \mathbb{R}/(2\pi\mathbb{Z})$, la quale può essere estesa per periodicità nella variabile y in tutto $\mathbb{R}^2$. Inoltre, grazie ad un'opportuna scelta delle condizioni al contorno, $u - v > 0$ in A e $u - v < 0$ in B, come desiderato.

La seconda parte di questa tesi riguarda lo studio di proprietà qualitative per soluzioni di alcuni problemi ellittici posti in domini illimitati. Il Capitolo 5 è dedicato alla classificazione di soluzioni nonnegative per

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) = u - g(x) & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{su } \partial\mathbb{R}^N_+, \end{cases}$$

dove $N = 2, 3$. Supponiamo che la matrice A sia del tipo

$$A(x') = \begin{pmatrix} \widehat{A}(x') & 0 \\ 0 & 1 \end{pmatrix},$$

e sia tale che l'operatore div $(A(x)\nabla(\cdot))$ sia ellittico (non necessariamente uniformemente ellittico). Per mezzo di una combinazione di teoremi alla Liouville e analisi di Fourier di base, forniamo una classificazione completa delle soluzioni ammissibili per diversi tipi di termini non omogenei g. Per esempio, mostriamo che:

- se $g(x) = 1$, allora $u(x) = 1 - \cos x_N$;
- se $g(x) = \tilde{g}(x')$ con $\tilde{g} \in \mathcal{C}(\mathbb{R}^{N-1})$, allora non esiste alcuna soluzione nonnegativa;
- se $g(x) = \tilde{g}(x_N)$, con $\tilde{g} \in \mathcal{C}(\mathbb{R})$ soddisfacente un'opportuna ipotesi supplementare, allora u dipende solo da x_N, ed è univocamente determinata come soluzione di

$$\begin{cases} -u''(x_N) = u(x_N) - g(x_N) & x_N > 0 \\ u(2k\pi) = u'(2k\pi) = 0 & \forall k \in \mathbb{Z}. \end{cases}$$

Nel Capitolo 6, consideriamo ancora soluzioni intere del sistema ellittico nonlineare (0.2):

$$\begin{cases} \Delta u = uv^2 \\ \Delta v = u^2 v & \text{in } \mathbb{R}^N \\ u, v > 0, \end{cases}$$

dove $N \geq 2$. Ci occupiamo ora della monotonia e della simmetria 1-dimensionale delle soluzioni con crescita al più polinomiale. L'esistenza si una soluzione 1-dimensionale del problema (0.2) è stata dimostrata in [12]. Nel lavoro citato, gli autori hanno formulato la seguente congettura alla Gibbons:

Conjecture (Section 7 of [12])**.** Let $N \geq 2$, let (u, v) be a solution of (0.2) satisfying

$$\lim_{x_N \to -\infty} u(x', x_N) = 0 \quad \text{and} \quad \lim_{x_N \to +\infty} u(x', x_N) = +\infty$$
$$\lim_{x_N \to -\infty} v(x', x_N) = +\infty \quad \text{and} \quad \lim_{x_N \to +\infty} v(x', x_N) = 0,$$

the limits being uniform in $x' \in \mathbb{R}^{N-1}$. Then (u, v) is 1-dimensional.

Il principale risultato del Capitolo 6 è la dimostrazione della validità di questa congettura per soluzioni aventi crescita al più polinomiale. Nella dimostrazione di tale risultato, usiamo il metodo dei piani mobili, che deve essere adattato opportunamente per poter essere applicato ad un sistema di equazioni ed a soluzioni illimitate. Queste due caratteristiche introducono notevoli complicazioni rispetto alle classiche applicazioni del metodo dei piani mobili allo studio di soluzioni limitate di equazioni scalari. Possiamo aggirare queste difficoltà attraverso opportune stime a priori per soluzioni di (0.2) aventi crescita al più polinomiale, che seguono da formule di monotonia di tipo Almgren e di tipo Alt-Caffarelli-Friedman.

Contents

Part I

Existence of oscillating solutions for some nonlinear problems

Chapter 1

Bounded solutions for a forced bounded oscillator without friction

1.1 Introduction and main results

This chapter concerns the existence of solutions, bounded on the real line together with their first derivatives, for the differential equation

$$\ddot{u} + g(u) = p(t), \tag{1.1}$$

where $g \in \mathcal{C}^2(\mathbb{R})$ is bounded, increasing, and has exactly one inflection point, and $p \in \mathcal{C}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ admits asymptotic average $A(p) \in \mathbb{R}$, that is

$$\lim_{T \to +\infty} \frac{1}{T} \int_t^{t+T} p(s)\, ds = A(p),$$

uniformly in $t \in \mathbb{R}$. Such an equation describes the forced motions of an oscillator exhibiting saturation effects. As a model problem, the reader may think to the equation

$$\ddot{u} + \arctan u = p(t),$$

even though we do not require any symmetry assumption on the reaction term g. Under the above assumptions, the main result we prove is the following theorem.

Theorem 1.1.1. *Equation* (1.1) *admits a bounded solution if and only if*

$$g(-\infty) < A(p) < g(+\infty). \tag{1.2}$$

In such a case, equation (1.1) *admits a countable set of bounded solutions, having arbitrarily large L^∞ norm.*

The motivation for our investigation relies on the papers [1, 68], which in turn have been inspired by some classical results of Landesman-Lazer type holding in the periodic framework. Such studies concern the equation

$$\ddot{u} + c\dot{u} + g(u) = p(t), \tag{1.3}$$

where $c \in \mathbb{R}$ and the continuous function g, not necessarily monotone, admits limits at $\pm\infty$, with the property that

$$g(-\infty) < g(s) < g(+\infty)$$

for every s. Also the cases $g(\pm\infty) = \pm\infty$ can be considered, requiring g to be sublinear at infinity if $c = 0$. When p is T-periodic, it is nowadays well known that equation (1.1) admits a periodic solution if and only if the Landesman-Lazer condition

$$g(-\infty) < \frac{1}{T}\int_0^T p(s)\, ds < g(+\infty)$$

is satisfied, regardless of the constant c; this result was first proved by Lazer, using the Schauder fixed point theorem, see [55]. When p is merely bounded, one would like to find analogous conditions for the research of bounded solutions. This problem was first studied by Ahmad [1], under the assumption that p has asymptotic average, in the sense explained above; by means of techniques of the qualitative theory of dissipative equations, the existence of a bounded solution is characterized, whenever $c \neq 0$, by (1.2). The case in which p is an arbitrary continuous function was solved by Ortega [68], who assumes $c \neq 0$ and provides a sharp necessary and sufficient condition: (1.3) has a bounded solution if and only if p can be written as $p^* + p^{**}$, where p^* has bounded primitive and p^{**} assumes values strictly contained between $g(-\infty)$ and $g(+\infty)$. This result relies on the Krasnoselskii method of guiding functions, and was generalized by Ortega and Tineo [69] to equations of higher order, using the notions of lower and upper averages of p; again, the condition $c \neq 0$ sticks as a crucial assumption. Later, by means of the method of lower and upper solutions, Mawhin and Ward [61] achieved some results in the case $c = 0$, in the situation in which $g(-\infty) \geq g(+\infty)$; we remark that this case is complementary with respect to that we are considering. Up to our knowledge, this last is the unique extension of the Landesman-Lazer theory to second order equations without friction, and the question in the case $g(-\infty) < g(+\infty)$ is still open. Under this perspective, in this paper we go back to the setting originally considered by Ahmad, and we prove that its aforementioned result holds also in the case $c = 0$, at least for the particular class of g that we consider.

The proof of our result is variational: we use a dual Nehari method which was first introduced in [70] to obtain bounded solutions in the case of a sublinear reaction (i.e.

$g(s) = s^{1/3}$). The name "dual Nehari method" is motivated by the fact that it leads to the study of a max min problem, while in the "classical" Nehari method (see e.g. [51, 66]) one considers a min max problem.

Firstly, we search for a solution of

$$\begin{cases} \ddot{u} + g(u) = p(t) & t \in (a, b), \\ u(a) = 0 = u(b), \\ u(t) > 0 & t \in (a, b), \end{cases} \tag{1.4}$$

as a minimizer of the action functional

$$J_{(a,b)}(u) := \int_a^b \left[\frac{1}{2}\dot{u}^2(t) - G(u(t)) + p(t)u(t) \right] dt$$

in the weakly closed set $\{u \in H_0^1(a, b) : \; u \geq 0\}$. In Section 1.3 we obtain some general properties of the nonnegative minimizers of $J_{(a,b)}$ in any interval (a, b); in Section 1.4 we prove that, when $b-a$ is sufficiently large, the minimizer $u_+(\cdot\,; a, b)$ is unique and solves problem (1.4). The proof of these results is substantially different from the corresponding one in the sublinear case [70]: indeed in the present situation the nonlinearity g and the forcing term p have the same order of growth (they are both bounded), while, as far as $b - a$ is sufficiently large, the forced sub-linear problem can be considered as a small perturbation of the unforced one. This fact introduces a lot of complications, which we can overcome thanks to a careful analysis of the balance between g and p, via measure theory tools, and of the asymptotic properties of the functional $J_{(a,b)}$ as $b-a \to +\infty$. Of course, analogous results can be obtained for negative minimizers $u_-(\cdot\,; a, b)$. To proceed, it is necessary to prove that $u_\pm(\cdot\,; a, b)$ is non-degenerate and that $J_{(a,b)}(u_\pm(\cdot\,; a, b))$ is differentiable as a function of (a, b). This is the object of Sections 1.5, 1.6, and it is the only part which requires $g \in \mathcal{C}^2$. We believe that this assumption can be weakened by a suitable approximating procedure, but we prefer to avoid further technicalities at this point.

Once the existence of one-signed solutions is established, in Section 1.7 we juxtapose positive and negative minimizers with alternate signs to obtain oscillating solutions, in the following way: let us fix $k \geq 1$, a bounded interval $[A, B]$ sufficiently large, and let us consider the class of partitions

$$\mathcal{B}_k := \left\{ (t_1, \ldots, t_k) \in \mathbb{R}^k \;\middle|\; \begin{array}{l} A =: t_0 \leq t_1 \leq \cdots \leq t_k \leq t_{k+1} := B, \\ t_{i+1} - t_i \text{ is sufficiently large for any } i \end{array} \right\}.$$

For each partition $P = (t_1, \ldots, t_k)$ of $\mathcal{B}_k$ there is a function u_P obtained by juxtaposing $u_\pm(\cdot\,; t_i, t_{i+1})$ with alternate signs $+$ and $-$. In general, this function is not a solution of equation (1.1), because the derivatives $\dot{u}_P(t_i^{\pm})$ may not coincide. We prove that these corner points disappear for the partition maximizing the quantity

$$\psi(P) = \sum_{i=0}^{k} J_{(t_i,t_{i+1})}(u_{\pm}(\cdot\,; t_i, t_{i+1})).$$

This argument provides a solution of (1.1) having k zeros in $[A, B]$, together with some estimates on the $W^{1,\infty}$ norm of the solution which depend only on the ratio $(B-A)/k$. Therefore, taking $A \to -\infty$, $B \to +\infty$ and $k \to +\infty$ in an appropriate way, one can pass to the limit and obtain the desired bounded solution. In doing this, one has again to modify the corresponding arguments in the sub-linear case, indeed they do not allow to treat the non-symmetric case $g(+\infty) - A(p) \neq A(p) - g(-\infty)$.

Incidentally, assuming p to be T-periodic, a simple variation of the argument above allows to obtain the existence of infinitely many subharmonic solutions, i.e. solutions which have minimal period nT, $n \in \mathbb{N}$ (see Theorem 1.7.10 at the end of the chapter).

To conclude, we remark that also the case of infinite limits $g(\pm\infty)$ can be treated by variational methods. On one hand, as already mentioned, infinitely many bounded solutions for equation (1.1) were obtained in [70] when $g(s) = |s|^{q-1}s$, $0 < q < 1$, and $p \in L^\infty(\mathbb{R})$. On the other hand, the original Nehari method, together with a limiting procedure, allows to obtain an analogous result also when g is superlinear at infinity, as done in [84, 87].

1.2 Preliminaries

It is not difficult to check that if equation (1.1) admits a bounded solution with bounded derivative, then necessarily condition (1.2) is satisfied. Indeed, by integrating equation (1.1) in $(t, t+T)$, we obtain

$$\frac{\dot{u}(t+T) - \dot{u}(t)}{T} = \frac{1}{T}\int_t^{t+T} (p(s) - g(u(s)))\, ds.$$

Since $\dot{u}$ is bounded, passing to the limit as $T \to +\infty$ we deduce that the left hand side tends to 0, so that

$$\begin{aligned} 0 &= \lim_{T\to+\infty} \frac{1}{T}\int_t^{t+T} (p(s) - g(u(s)))\, ds \\ &= A(p) - \lim_{T\to+\infty} \frac{1}{T}\int_t^{t+T} g(u(s))\, ds. \end{aligned} \tag{1.5}$$

Now, the boundedness of u and the monotonicity of g implies also that for every $s \in \mathbb{R}$

$$g(-\infty) < g\left(-\|u\|_\infty\right) \leq g(u(s)) \leq g\left(\|u\|_\infty\right) < g(+\infty), \tag{1.6}$$

and a comparison between (1.5) and (1.6) gives the desired result (in fact, from this point of view, it is sufficient that $g(-\infty) < g(s) < g(+\infty)$ for every s).

We observe that, by means of suitable translations, it is not restrictive to assume that

$$\begin{aligned}&g(0) = 0,\ g \in \mathcal{C}^2(\mathbb{R}) \text{ is bounded, strictly increasing in } \mathbb{R},\\ &\text{strictly concave in } (0, +\infty) \text{ and strictly convex in } (-\infty, 0).\end{aligned} \tag{h1}$$

We denote as G the primitive of g vanishing in 0, and

$$\lim_{s\to\pm\infty} g(s) = g_\pm,$$

so that

$$\lim_{s\to\pm\infty} \frac{G(s)}{s} = g_\pm \quad \text{and} \quad g_- < \frac{G(s)}{s} < g_+ \quad \forall s \in \mathbb{R}.$$

As far as the function p is concerned, as we already mentioned, we assume that $p \in \mathcal{C}(\mathbb{R}) \cap L^\infty(\mathbb{R})$ is such that for every $\varepsilon > 0$ there exists $\bar{T} > 0$ such that if $T > \bar{T}$ then

$$\sup_{t\in\mathbb{R}} \left| \frac{1}{T} \int_t^{t+T} p(s)\, ds - A(p) \right| < \varepsilon,$$

in such a way that

$$\begin{aligned}&p \text{ is bounded and continuous in } \mathbb{R},\\ &\text{and has asymptotic average } g_- < A(p) < g_+.\end{aligned} \tag{h2}$$

Note that we do not make any assumption on the L^∞ norm of p.

In view of the previous considerations and notations, we can rephrase Theorem 1.1.1 as follows.

Theorem 1.2.1. *Under assumptions* (h1)-(h2)*, there exists a sequence* (u_m) *of solutions of* (1.1) *defined in* $\mathbb{R}$*, with* $u_m, \dot{u}_m \in L^\infty(\mathbb{R})$ *and* $\|u_m\|_\infty \to \infty$ *as* $m \to \infty$*. Moreover, each* u_m *has infinitely many zeros in* $\mathbb{R}$*.*

1.3 Existence and basic properties of nonnegative minimizers

In this section we deal with the boundary value problem (1.4):

$$\begin{cases} \ddot{u}(t) + g(u(t)) = p(t) & t \in (a, b),\\ u(a) = 0 = u(b), & \\ u(t) > 0 & t \in (a, b). \end{cases}$$

We seek solutions as minimizers of the related action functional

$$J_{(a,b)}(u) := \int_a^b \left[\frac{1}{2}\dot{u}^2(t) - G(u(t)) + p(t)u(t)\right] dt$$

in the H^1-weakly closed set

$$H_0^1(a,b)^+ := \{u \in H_0^1(a,b) : \; u \geq 0\}.$$

We introduce the value

$$\varphi^+(a,b) := \inf_{u \in H_0^1(a,b)^+} J_{(a,b)}(u).$$

Remark 1.3.1. Of course, even though in the following we focus on positive solutions, negative ones can be treated similarly as well, seeking solutions to the boundary value problem

$$\begin{cases} \ddot{u}(t) + g(u(t)) = p(t) & t \in (a,b) \\ u(a) = 0 = u(b) & \\ u(t) < 0 & t \in (a,b) \end{cases}$$

associated to the candidate critical value

$$\varphi^-(a,b) := \inf_{u \in H_0^1(a,b)^-} J_{(a,b)}(u),$$

where $H_0^1(a,b)^- := \{u \in H_0^1(a,b) : u \leq 0\}$. Indeed, the two problems are related by the change of variable $v = -u$, $\bar{g}(s) = -g(-s)$ and $\bar{p} = -p$, and $\bar{g}$, $\bar{p}$ satisfy (h1)-(h2) if and only if g, p do. In particular, when dealing with negative solutions, in all the explicit constants we will find the quantity $g_\pm$ should be replaced by $-g_\mp$, and $A(p)$ by $-A(p)$.

Lemma 1.3.2. *The value $\varphi^+(a,b)$ is a real number and it is achieved by $u_{(a,b)} \in H_0^1(a,b)^+$.*

Proof. It is not difficult to check that $J_{(a,b)}$ is weakly lower semi-continuous and coercive, so that the direct method of the calculus of variations applies. □

In what follows we are going to show that, if (a,b) is sufficiently large, a minimizer $u_{(a,b)}$ is an actual solution of (1.4); this is not obvious, because in principle $u_{(a,b)}$ could vanish somewhere. Having in mind to let (a,b) vary and wishing to catch the behaviour of the minimizers $u_{(a,b)}$ under variations of the domain, it is convenient to introduce suitable scaling to work on a common time-interval. To be precise, for every $u \in H_0^1(a,b)^+$ we can define

$$\widehat{u}(t) := \frac{1}{(b-a)^2} u(a + t(b-a)) \quad \Longleftrightarrow \quad u(t) = (b-a)^2 \widehat{u}\left(\frac{t-a}{b-a}\right), \tag{1.7}$$

and $\widehat{p}_{(a,b)}(t) := p(a + t(b - a))$. Of course, $\widehat{u} \in H_0^1(0,1)^+$ and

$$J_{(a,b)}(u) = (b-a)^3 \int_0^1 \left[\frac{1}{2}\dot{\widehat{u}}^{\,2}(t) - \frac{1}{(b-a)^2} G((b-a)^2 \widehat{u}(t)) + \widehat{p}_{(a,b)}(t)\widehat{u}(t) \right] dt$$
$$=: (b-a)^3 \widehat{J}_{(a,b)}(\widehat{u}). \quad (1.8)$$

This reveals that the minimizations of $J_{(a,b)}$ in $H_0^1(a,b)^+$ and of $\widehat{J}_{(a,b)}$ in $H_0^1(0,1)^+$ are equivalent; in particular, the function $\widehat{u}_{(a,b)}$ defined by (1.7) with $u = u_{(a,b)}$ is a minimizer of $\widehat{J}_{(a,b)}$ in $H_0^1(0,1)^+$.
The Euler-Lagrange equation associated to the functional $\widehat{J}_{(a,b)}$ leads to the research of solutions to

$$\begin{cases} \ddot{w}(t) + g((b-a)^2 w(t)) = \widehat{p}_{(a,b)}(t) & \text{in } (0,1) \\ w(0) = 0 = w(1) & \\ w(t) > 0 & \text{in } (0,1). \end{cases} \quad (1.9)$$

Our aim is to show that if $b - a$ is sufficiently large, then a minimizer $\widehat{u}_{(a,b)}$ is an actual solution of (1.9). We start showing that where it is positive it solves equation (1.1), and it is of class $\mathcal{C}^1$ in the whole $(0,1)$.

Lemma 1.3.3. *Let* $(c,d) \subset (0,1)$ *be such that*

$$\widehat{u}_{(a,b)} > 0 \qquad \text{in } (c,d).$$

Then $\widehat{u}_{(a,b)}$ *is a classical solution of the first equation in* (1.9) *in* (c,d). *Moreover, if* $c > 0$ *then* $\dot{\widehat{u}}_{(a,b)}(c^+) = 0$, *and if* $d < 1$ *then* $\dot{\widehat{u}}_{(a,b)}(d^-) = 0$.

Proof. The fact that $\widehat{u}_{(a,b)}$ is a (classical) solution in (c,d) follows from the extremality of $\widehat{u}_{(a,b)}$ with respect to variations with compact support in (c,d).
Now we assume that $c > 0$ and prove that $\dot{\widehat{u}}_{(a,b)}(c^+) = 0$. By contradiction, let $\dot{\widehat{u}}_{(a,b)}(c^+) = \xi > 0$. Given $\varepsilon > 0$ small enough such that $[c-\varepsilon, c+\varepsilon] \subset (0,d)$, we consider the set

$$\Lambda_\varepsilon := \left\{ v \in H^1(c-\varepsilon, c+\varepsilon) : v(c \pm \varepsilon) = \widehat{u}_{(a,b)}(c \pm \varepsilon) \right\}.$$

As

$$\|v\|_\infty \le \sqrt{2\varepsilon}\|\dot{v}\|_2 + \min\{\widehat{u}_{(a,b)}(c-\varepsilon), \widehat{u}_{(a,b)}(c+\varepsilon)\} \qquad \forall v \in \Lambda_\varepsilon,$$

the functional $\widehat{J}_{(a,b)}$ (considered on the interval $(c-\varepsilon, c+\varepsilon)$) is bounded below and coercive in the weakly closed set Λ_ε, so that there exists a minimizer v_ε. Clearly, $v_\varepsilon \in \mathcal{C}^2(c-\varepsilon, c+\varepsilon)$ and is a solution of

$$\ddot{v}_\varepsilon(t) + g((b-a)^2 v_\varepsilon(t)) = \widehat{p}_{(a,b)}(t). \quad (1.10)$$

Since the restriction $\widehat{u}_{(a,b)}$ is not differentiable in c, we deduce

$$\widehat{J}_{(a,b)}(v_\varepsilon) < \widehat{J}_{(a,b)}(\widehat{u}_{(a,b)}|_{(c-\varepsilon,c+\varepsilon)}).$$

We claim that $v_\varepsilon \geq 0$ in $(c-\varepsilon, c+\varepsilon)$. If v_ε is monotone, this follows from its boundary conditions. If it is not monotone, there exists $\tau \in (c-\varepsilon, c+\varepsilon)$ such that $\dot{v}_\varepsilon(\tau) = 0$. As a consequence, from equation (1.10) it follows that

$$\|\dot{v}\|_\infty \leq (\|g\|_\infty + \|p\|_\infty)\, 2\varepsilon,$$

and hence, for every $t \in (c-\varepsilon, c+\varepsilon)$, we have

$$v_\varepsilon(t) \geq v_\varepsilon(c+\varepsilon) - |v_\varepsilon(c+\varepsilon) - v_\varepsilon(t)| \geq \widehat{u}_{(a,b)}(c+\varepsilon) - (\|g\|_\infty + \|p\|_\infty)\, 4\varepsilon^2.$$

Now, $\widehat{u}_{(a,b)}(c+\varepsilon) = \xi\varepsilon + O(\varepsilon^2)$, so that at least for ε small enough we have $v_\varepsilon(t) \geq 0$ in $(c-\varepsilon, c+\varepsilon)$, as announced. This implies that the function

$$\widetilde{u}(t) := \begin{cases} \widehat{u}_{(a,b)}(t) & t \in [0, c-\varepsilon) \cup (c+\varepsilon, 1], \\ v_\varepsilon(t) & t \in (c-\varepsilon, c+\varepsilon) \end{cases}$$

stays in $H_0^1(0,1)^+$ and, clearly, $\widehat{J}_{(a,b)}(\widetilde{u}) < \widehat{J}_{(a,b)}(\widehat{u}_{(a,b)})$, in contradiction with the minimality of $\widehat{u}_{(a,b)}$. □

In the following lemma we prove that the family of the minimizers $\{\widehat{u}_{(a,b)}\}$ is uniformly bounded and equi-Lipschitz-continuous.

Lemma 1.3.4. *For every $(a,b) \subset \mathbb{R}$ and any $\widehat{u}_{(a,b)}$, it holds*

$$|\widehat{u}_{(a,b)}(t)| \leq (\|g\|_\infty + \|p\|_\infty) \qquad \forall t \in (0,1)$$
$$|\dot{\widehat{u}}_{(a,b)}(t)| \leq (\|g\|_\infty + \|p\|_\infty) \qquad \forall t \in (0,1).$$

Proof. Let $(c,d) \subset [0,1]$ be such that $\widehat{u}_{(a,b)} > 0$ in (c,d), vanishing at c and d. From Lemma 1.3.3 it follows that

$$|\ddot{\widehat{u}}_{(a,b)}(t)| \leq \left|g((b-a)^2\widehat{u}_{(a,b)}(t))\right| + |p(t)| \leq \|g\|_\infty + \|p\|_\infty \qquad \forall t \in (c,d).$$

Since $\widehat{u}_{(a,b)}(c) = 0 = \widehat{u}_{(a,b)}(d)$ and $\widehat{u}_{(a,b)} \in \mathcal{C}^1(0,1)$, there exists $\tau \in (c,d)$ such that $\dot{\widehat{u}}_{(a,b)}(\tau) = 0$. Hence

$$|\dot{\widehat{u}}_{(a,b)}(t)| \leq |\dot{\widehat{u}}_{(a,b)}(\tau)| + \|g\|_\infty + \|p\|_\infty = \|g\|_\infty + \|p\|_\infty \qquad \forall t \in (c,d).$$

Since this relation holds in each interval (c,d) as before, one can easily conclude by recalling that, being $u \in H^1$, it holds

$$\int_{\{u(t)=0\}} |\dot{u}(t)|\, dt = 0.$$

□

Let

$$s(t)=\sum_{k=0}^{n-1} y_k \chi_{[t_k,t_{k+1})}(t)$$

denote a simple function. We define the quantity

$$\delta(s) := \inf\{t_{k+1}-t_k:\ k=0,\dots,n-1\}. \tag{1.11}$$

Given any measurable and bounded function $u \in \mathcal{M}(0,1)$, it is well known that for every $\varepsilon>0$ there is a simple function s_u such that $\|u-s_u\|_\infty<\varepsilon$. In general the quantity $\delta(s_u)$ depends on u and ε. The following lemma says that if we consider the family of the minimizers $\{\widehat{u}_{(a,b)}\}$, given $\varepsilon>0$ it is possible to find a family of approximating simple functions $\{s_{(a,b)}\}$ such that $\delta(s_{(a,b)})$ is bounded below uniformly with respect to (a,b).

Lemma 1.3.5. *For every $\varepsilon>0$, let $m\in\mathbb{N}$ be such that $m>(\|g\|_\infty+\|p\|_\infty)/\varepsilon$. Then for every $(a,b)\subset\mathbb{R}$*

$$s_{(a,b)}(t) := \sum_{k=0}^{m-1} \widehat{u}_{(a,b)}\left(\frac{k}{m}\right)\chi_{[\frac{k}{m},\frac{k+1}{m})}(t)$$

is such that

$$\|\widehat{u}_{(a,b)}-s_{(a,b)}\|_\infty<\varepsilon \quad \text{and} \quad \delta(s_{(a,b)})=\bar{\delta}:=\frac{1}{m}.$$

In particular, m can be chosen only depending on ε and $\|p\|_\infty$, and not on p.

Proof. For every $t\in(0,1)$ there exists $k\in\{0,\dots,m-1\}$ such that $t\in[k/m,(k+1)/m)$, so that by Lemma 1.3.4

$$|\widehat{u}_{(a,b)}(t)-s_{(a,b)}(t)| = \left|\int_{\frac{k}{m}}^{t} \dot{\widehat{u}}_{(a,b)}(\tau)\,d\tau\right| \le \frac{1}{m}\left(\|g\|_\infty+\|p\|_\infty\right) \qquad \forall t\in(0,1). \qquad \square$$

1.4 The boundary value problem for large intervals

Here and in the next section we consider the minimizer $u_{(a,b)}$ as function of a,b and p. For this reason, we write

- $u(\cdot\,;a,b;p)$ and $\widehat{u}(\cdot\,;a,b;p)$ instead of $u_{(a,b)}$ and $\widehat{u}_{(a,b)}$ respectively,
- $J_{(a,b),p}$ and $\widehat{J}_{(a,b),p}$ instead of $J_{(a,b)}$ and $\widehat{J}_{(a,b)}$ respectively,
- $\varphi^+(a,b;p)$ instead of $\varphi^+(a,b)$,

to emphasize the dependence we are considering. As we have already mentioned, we can introduce an auxiliary problem which carries the asymptotic behaviour of (1.9) for $b-a$ greater than a sufficiently large threshold (which depends on p). Let us consider

$$\begin{cases} \ddot{w}(t) = -\left(g_+ - A(p)\right) =: -k \quad \text{in } (0,1) \\ w(0) = 0 = w(1), \end{cases} \tag{1.12}$$

with $k > 0$ thanks to (h2). Of course, this problem has the unique solution

$$w_k(t) = \frac{k}{2} t(1-t). \tag{1.13}$$

The related action functional is

$$J_k^\infty(w) := \int_0^1 \left[\frac{1}{2}\dot{w}^2(t) - kw(t)\right] dt, \tag{1.14}$$

which has the unique minimizer w_k in $H_0^1(0,1)^+$ (the uniqueness follows from the strict convexity of J_k^∞). A direct computation gives

$$J_k^\infty(w_k) = -\frac{k^2}{24}.$$

Having in mind to compare minimizers related to different forcing terms, for any p satisfying (h2) it is convenient to introduce a subset $\mathcal{P}$ of $L^\infty(\mathbb{R})$ such that the mentioned threshold can be chosen independently of $q \in \mathcal{P}$. To this aim, first of all we recall the following result.

Lemma 1.4.1 ([68, Lemma 2.2])**.** *Let p satisfy* (h2)*. For every $\varepsilon > 0$ there exists a decomposition $p = p_{1,\varepsilon} + \dot{p}_{2,\varepsilon}$, where $\|p_{1,\varepsilon} - A(p)\|_\infty < \varepsilon/2$ and $p_{2,\varepsilon} \in L^\infty(\mathbb{R})$.*

This means that if p has asymptotic average it can be written as a sum between a term $p_{1,\varepsilon}$ which is arbitrarily close to the average $A(p)$, plus a term $\dot{p}_{2,\varepsilon}$ which has bounded primitive.

Given $p \in L^\infty(\mathbb{R})$, we compute $\|p\|_\infty$ and $A(p)$, and for any $0 < \varepsilon < 1$ we consider a decomposition as in Lemma 1.4.1; we introduce

$$M_1 := \|p\|_\infty + 1 \quad \text{and} \quad M_\varepsilon := \|p_{2,\varepsilon}\|_\infty + 1.$$

We define

$$\mathcal{P} := \left\{ q \in L^\infty(\mathbb{R}) \;\middle|\; \begin{array}{l} \|q\|_\infty < M_1,\ q \text{ has asymptotic average,} \\ A(q) = A(p), \text{ and for any } \varepsilon \in (0,1) \\ \text{there exists a decomposition } q = q_{1,\varepsilon} + \dot{q}_{2,\varepsilon} \\ \text{as in Lemma 1.4.1, with } \|q_{2,\varepsilon}\|_\infty < M_\varepsilon \end{array} \right\}. \tag{1.15}$$

Remark 1.4.2. Note that given any p satisfying assumption (h2) we can define the set $\mathcal{P}$, whose definition depends on p. Clearly, $p \in \mathcal{P}$ and the constant function $A(p)$ belongs to $\mathcal{P}$. Moreover, if q is of type

$$q(t) = A(p) + \dot{q}_2(t) \quad \text{or} \quad q(t) = p(t) + \dot{q}_2(t),$$

with $\|q_2\|_\infty, \|\dot{q}_2\|_\infty < 1$, then $q \in \mathcal{P}$.

We are ready to show that problem (1.12) is the limit problem of (1.4) as $b-a \to +\infty$, in the following sense.

Proposition 1.4.3. *Let p satisfy assumption* (h2)*, and let $\mathcal{P}$ be defined by* (1.15)*. For every $0 < \varepsilon < (g_+ - A(p))^2/24$ there exists $L_1 > 0$ depending only on ε such that if $b - a \geq L_1$, then*

$$-\underline{\alpha} \leq \widehat{J}_{(a,b),q}(\widehat{u}(\cdot\,; a, b; q)) \leq -\overline{\alpha} \qquad \forall q \in \mathcal{P},$$

where

$$\underline{\alpha} := \frac{(g_+ - A(p))^2}{24} + \varepsilon \quad \textit{and} \quad \overline{\alpha} := \frac{(g_+ - A(p))^2}{24} - \varepsilon. \tag{1.16}$$

Remark 1.4.4. The upper bound on ε implies that $\widehat{u}(\cdot\,; a, b; q)$ cannot vanish identically whenever $b - a > L_1$.

To prove Proposition 1.4.3 we need some intermediate results.

Lemma 1.4.5. *Let $\mathfrak{F} \subset H_0^1(0,1)^+$ and $M > 0$ be such that*

$$\|u\|_{L^1(0,1)} \leq M \qquad \forall u \in \mathfrak{F}.$$

For every $\varepsilon > 0$ there exists $L_2 = L_2(\varepsilon) > 0$ such that, if $b - a > L_2$, then

$$\left| \int_0^1 \left[\frac{1}{(b-a)^2} G((b-a)^2 u) - g_+ u \right] \right| < \varepsilon$$

$$\left| \int_0^1 \left[g((b-a)^2 u) u - g_+ u \right] \right| < \varepsilon$$

$$\left| \int_0^1 \left[g((b-a)^2 u) u - \frac{1}{(b-a)^2} G((b-a)^2 u) \right] \right| < \varepsilon,$$

for every $u \in \mathfrak{F}$.

Proof. Let $K_1 := 2(1 + M g_+)$ and $\varepsilon > 0$ be fixed. By assumption (h1) we infer the existence of $\bar{s} > 0$ such that

$$s > \bar{s} \quad \Longrightarrow \quad \left(1 - \frac{\varepsilon}{K_1}\right) g_+ \leq \frac{G(s)}{s} \leq g_+.$$

For every (a,b) and for every $u \in \mathfrak{F}$ we can write

$$\int_0^1 \frac{G((b-a)^2 u)}{(b-a)^2} = \int_{\{(b-a)^2 u \le \bar{s}\}} \frac{G((b-a)^2 u)}{(b-a)^2 u} u + \int_{\{(b-a)^2 u > \bar{s}\}} \frac{G((b-a)^2 u)}{(b-a)^2 u} u. \quad (1.17)$$

As far as the first integral on the right hand side is concerned, since $s > 0$ implies $0 \le G(s)/s \le g_+$, it results

$$0 \le \int_{\{(b-a)^2 u \le \bar{s}\}} \frac{G((b-a)^2 u)}{(b-a)^2 u} u \le \int_{\{(b-a)^2 u \le \bar{s}\}} g_+ u \le \frac{g_+ \bar{s}}{(b-a)^2} < \frac{\varepsilon}{K_1}, \quad (1.18)$$

whenever $b - a > L_2$ sufficiently large, for every $u \in \mathfrak{F}$. Note also that the same choice of L_2 gives

$$b - a > L_2 \quad \Longrightarrow \quad 0 \le g_+ \left(\int_0^1 u - \int_{\{(b-a)^2 u > \bar{s}\}} u \right) < \frac{\varepsilon}{K_1} \qquad \forall u \in \mathfrak{F}.$$

Let us consider the second integral on the right hand side of (1.17). Our choice of $\bar{s}$ and the previous relation imply that, if $b - a > L_2$, then

$$\begin{aligned} -\left(1 - \frac{\varepsilon}{K_1}\right) \frac{\varepsilon}{K_1} + g_+ \left(1 - \frac{\varepsilon}{K_1}\right) \int_0^1 u \le g_+ \left(1 - \frac{\varepsilon}{K_1}\right) \int_{\{(b-a)^2 u > \bar{s}\}} u \\ \le \int_{\{(b-a)^2 u > \bar{s}\}} \frac{G((b-a)^2 u)}{(b-a)^2 u} u \le g_+ \int_{\{(b-a)^2 u > \bar{s}\}} u \le g_+ \int_0^1 u, \end{aligned}$$

for every $u \in \mathfrak{F}$. Due to the boundedness of the family $\mathfrak{F}$ in $L^1(0,1)$, it results

$$0 \le g_+ \int_0^1 u - \int_{\{(b-a)^2 u > \bar{s}\}} \frac{G((b-a)^2 u)}{(b-a)^2 u} u \le (1 + M g_+) \frac{\varepsilon}{K_1} = \frac{\varepsilon}{2}, \quad (1.19)$$

for every $u \in \mathfrak{F}$. Collecting together (1.17), (1.18) and (1.19), we obtain the first estimate of the thesis. To prove the second one, we can adapt the same argument because of assumption (h1). The third estimate follows easily. □

Lemma 1.4.6. *Let $\mathfrak{F} \subset H_0^1(0,1)^+$ be such that*

$$\|u\|_{L^1(0,1)} \le M \qquad \forall u \in \mathfrak{F}.$$

For $\varepsilon > 0$, $\delta_1 > 0$ and for every $u \in \mathfrak{F}$, let us assume the existence of a simple function s_u such that

$$\|u - s_u\|_\infty < \varepsilon_1 \quad \text{and} \quad \delta(s_u) \ge \delta_1,$$

where $\delta(\cdot)$ is defined as in (1.11) *and* $\varepsilon_1 := \varepsilon/(M_1 + M + \|g\|_\infty + 1)$. *Then there exists $L_3 > 0$, depending on ε, δ_1 but independent of $q \in \mathcal{P}$, such that, if $b - a > L_3$, then*

$$\left| \int_0^1 \left(\widehat{q}_{(a,b)} - A(p) \right) u \right| < \varepsilon,$$

for every $u \in \mathfrak{F}$ and $q \in \mathcal{P}$.

Proof. Let $K_2 := (M_1 + M + \|g\|_\infty + 1)$, and let us assume that $(a, b) = (0, L)$ to ease the notation. It is straightforward to apply the following argument for a general $(a, b) \subset \mathbb{R}$. Let us consider, for $(c, d) \subset [0, 1]$,

$$\int_c^d \widehat{q}_L(t)\, dt = \frac{1}{L} \int_{cL}^{dL} q(t)\, dt = \frac{d - c}{L(d - c)} \int_{cL}^{dL} q(t)\, dt.$$

For any $\varepsilon > 0$ sufficiently small, we consider the decomposition $q = q_{1,\varepsilon} + \dot{q}_{2,\varepsilon}$ given by Lemma 1.4.1. By definition of $\mathcal{P}$, we know that

$$\begin{aligned} \sup_{t \in \mathbb{R}} \left| \frac{1}{T} \int_t^{t+T} q(\sigma)\, d\sigma - A(p) \right| & \\ \leq \sup_{t \in \mathbb{R}} \left(\frac{1}{T} \int_t^{t+T} |q_{1,\varepsilon}(\sigma) - A(p)|\, d\sigma + \left| \frac{1}{T} \int_t^{t+T} \dot{q}_{2,\varepsilon}(\sigma)\, d\sigma \right| \right) & \\ < \frac{\varepsilon}{2} + \frac{2}{T} \|q_{2,\varepsilon}\|_\infty < \frac{\varepsilon}{2} + \frac{2}{T} M_\varepsilon < \varepsilon & \end{aligned}$$

whenever $T > \bar{T}(\varepsilon) := 4M_\varepsilon/\varepsilon$, independently of $q \in \mathcal{P}$. Therefore, if $(d - c)L > \bar{T}\,(\varepsilon/K_2)$, then

$$\left| \frac{1}{L(d - c)} \int_{Lc}^{Ld} q(t)\, dt - A(p) \right| < \frac{\varepsilon}{K_2} \qquad \forall q \in \mathcal{P}.$$

Let us consider the family of simple functions $\{s_u : u \in \mathfrak{F}\}$. Let us set $L_3 := (1/\delta_1)\bar{T}\,(\varepsilon/K_2)$; for $s_u = \sum_{k=0}^{n-1} y_k \chi_{[t_k, t_{k+1})}$, we note that if $L > L_3$, then

$$(t_{k+1} - t_k)L \geq \delta_1 L_3 = \bar{T}\left(\frac{\varepsilon}{K_2} \right),$$

so that

$$\begin{aligned} \left| \int_0^1 (\widehat{q}_L - A(p))\, s_u \right| &\leq \sum_{k=0}^{n-1} |y_k|(t_{k+1} - t_k) \left| \frac{1}{L(t_{k+1} - t_k)} \int_{Lt_k}^{Lt_{k+1}} q(\sigma)\, d\sigma - A(p) \right| \\ &< \frac{\varepsilon}{K_2} \int_0^1 |s_u| < \frac{\varepsilon}{K_2}(M + 1), \end{aligned}$$

independently of $u \in \mathfrak{F}$ and on $q \in \mathcal{P}$, where for the last inequality we use the boundedness of $\mathfrak{F}$ in $L^1(0,1)$. Therefore, if $L \geq L_3$, then

$$\begin{aligned}\left|\int_0^1 (\widehat{q}_L - A(p))\, u\right| &\leq \int_0^1 |\widehat{q}_L + A(p)|\, |u - s_u| + \left|\int_0^1 (\widehat{q}_L - A(p))\, s_u\right| \\ &< (\|q\|_\infty + \|g\|_\infty)\, \|u - s_u\|_\infty + \frac{\varepsilon}{K_2}(M+1) \\ &< \varepsilon,\end{aligned}$$

for every $u \in \mathfrak{F}$ and for every $q \in \mathcal{P}$ (for the reader's convenience, we recall that by definition $M_1 > \|q\|_\infty$ for every $q \in \mathcal{P}$). □

We are in position to prove Proposition 1.4.3.

Proof of Proposition 1.4.3. Let us consider the family

$$\mathfrak{F} := \{\widehat{u}(\cdot\,; a, b; q) : (a,b) \subset \mathbb{R}, q \in \mathcal{P}\} \cup \{w_{(g_+ - A(p))}\},$$

where we recall that $\widehat{u}(\cdot; a, b; q)$ is the minimizer of $\widehat{J}_{(a,b),q}$ (defined by (1.8)), and $w_{(g_+ - A(p))}$ has been defined by (1.13). In light of Lemmas 1.3.4 and 1.3.5, the family satisfies the assumptions of Lemmas 1.4.5 and 1.4.6.

Let $L_1 := \max\{L_2(\varepsilon/2), L_3(\varepsilon/2)\}$, where L_2 and L_3 have been defined in the quoted statements, and we recall that L_3 is independent of $q \in \mathcal{P}$. By definition, if $b - a > L_1$, then

$$\begin{aligned}\widehat{J}_{(a,b),q}(\widehat{u}(\cdot\,; a, b; q)) &> \int_0^1 \left[\frac{1}{2}\dot{\widehat{u}}^{\,2}(t; a, b; q) - (g_+ - A(p))\, \widehat{u}(t; a, b; q)\right] dt - \varepsilon \\ &\geq \inf_{H_0^1(0,1)^+} J^\infty_{(g_+ - A(p))} - \varepsilon = -\frac{(g_+ - A(p))^2}{24} - \varepsilon,\end{aligned}$$

for every $q \in \mathcal{P}$, where we recall that J_k^∞ has been defined in (1.14) for any $k \in \mathbb{R}$. Moreover, by minimality,

$$\begin{aligned}\widehat{J}_{(a,b),q}(\widehat{u}(\cdot\,; a, b; q)) &\leq \widehat{J}_{(a,b),q}(w_{(g_+ - A(p))}) \\ &< \int_0^1 \left[\frac{1}{2}\dot{w}^2_{(g_+ - A(p))}(t) - (g_+ - A(p))\, w_{(g_+ - A(p))}(t)\right] dt + \varepsilon \\ &= \inf_{H_0^1(0,1)^+} J^\infty_{(g_+ - A(p))} + \varepsilon = -\frac{(g_+ - A(p))^2}{24} + \varepsilon,\end{aligned}$$

whenever $b - a > L_1$. □

Now we can come back on the time interval $[a,b]$: due to the explicit relations (1.7) and (1.8), we can summarize the previous results in the following statement.

Corollary 1.4.7. *For $0<\varepsilon<(1-A(p))^2/24$, let $L_1(\varepsilon)$ be defined as in Proposition 1.4.3. If $b-a>L_1(\varepsilon)$ then*

$$-\underline{\alpha}(b-a)^3 \le \varphi^+(a,b;q) \le -\overline{\alpha}(b-a)^3,$$

for every $q\in\mathcal{P}$, where $\underline{\alpha}$, $\overline{\alpha}$ are defined as in equation (1.16).

Remark 1.4.8. By definition, $L_1\ge L_2, L_3$. Therefore, if $b-a>L_1$, Lemmas 1.4.5 and 1.4.6 hold true; in particular, we deduce that for every $0<\varepsilon<(1-A(p))^2/24$, if $b-a>L_1(\varepsilon)$, then

$$\left|\int_a^b \left[g\left(u(t;a,b;q)\right)u(t;a,b;q)-G\left(u(t;a,b;q)\right)\right]dt\right|<\varepsilon(b-a)^3$$

for every $q\in\mathcal{P}$.

In the next statement and in the rest the symbol $\|\cdot\|$ denotes the Dirichlet H_0^1 norm on the considered interval, that is,

$$\|u\|=\left(\int_a^b \dot u^2(t)\,dt\right)^{1/2} \qquad \forall u\in H_0^1(a,b).$$

Corollary 1.4.9. *There exists $L_4>0$ and a positive constant $C_1>0$ such that, if $b-a\ge L_4$, then $\|u(\cdot\,;a,b;q)\|\ge C_1(b-a)^{3/2}$ and $\|u(\cdot\,;a,b;q)\|_\infty\ge C_1(b-a)^2$ for every $q\in\mathcal{P}$.*

Proof. Since the function $\lambda\mapsto J_{(a,b),q}(\lambda u(\cdot\,;a,b;q))$ reaches its minimum at $\lambda=1$, it results

$$\int_a^b \left[\dot u^2(t;a,b;q)-g\left(u(t;a,b;q)\right)u(t;a,b;q)+q(t)u(t;a,b;q)\right]dt=0.$$

We can solve this identity for the last term, and substitute into the expression of $J_{(a,b),q}(u(\cdot;a,b;q))$:

$$J_{(a,b),q}(u(\cdot\,;a,b;q))=-\int_a^b \frac{1}{2}\dot u^2(t;a,b;q)\,dt \\ +\int_a^b \left[g\left(u(t;a,b;q)\right)u(t;a,b;q)-G\left(u(t;a,b;q)\right)\right]dt.$$

Given $\varepsilon > 0$ sufficiently small, if $b - a > L_1(\varepsilon)$ defined in Proposition 1.4.3, we have (we refer also to Corollary 1.4.7 and to Remark 1.4.8)

$$J_{(a,b),q}(u(\cdot\,; a, b; q)) > -\frac{1}{2}\|\dot{u}(\cdot\,; a, b; q)\|^2 - \varepsilon(b-a)^3 \quad \text{and}$$

$$J_{(a,b),q}(u(\cdot\,; a, b; q)) \leq \left(-\frac{(g_+ - A(p))^2}{24} + \varepsilon\right)(b-a)^3,$$

for every $q \in \mathcal{P}$, from which we deduce

$$\|\dot{u}(\cdot\,; a, b; q)\|^2 > \left(\frac{(g_+ - A(p))^2}{12} - 4\varepsilon\right)(b-a)^3 \qquad \forall q \in \mathcal{P}.$$

We choose $\bar{\varepsilon} = (g_+ - A(p))^2/96$ and set $L_4 = L_1(\bar{\varepsilon})$. Hence

$$\|u(\cdot\,; a, b; q)\| \geq \frac{(g_+ - A(p))}{\sqrt{24}}(b-a)^{\frac{3}{2}} \qquad \forall q \in \mathcal{P},$$

and

$$\begin{aligned}\frac{(g_+ - A(p))^2}{24}(b-a)^3 &\leq \int_a^b \dot{u}^2(t; a, b; q)\, dt \\ &= \int_a^b \left[g\left(u(t; a, b; q)\right) u(t; a, b; q) - q(t)u(t; a, b; q)\right] dt \\ &\leq \left(\|g\|_\infty + M_1\right) \|u(\cdot; a, b; q)\|_\infty (b-a),\end{aligned}$$

which gives the desired result for

$$C_1 := \frac{(g_+ - A(p))^2}{24(\|g\|_\infty + M_1)}. \qquad \square$$

Finally, we can prove that if $b-a$ is sufficiently large, then any minimizer $u(\cdot\,; a, b; q)$ with $q \in \mathcal{P}$ is an actual solution of the boundary problem (1.4).

Proposition 1.4.10 (Existence)**.** *Let p satisfy assumption* (h2)*, and let $\mathcal{P}$ be defined by* (1.15)*. There exists $\tilde{L} \geq L_4$ such that, if $b - a \geq \tilde{L}$, then $u(t; a, b; q) > 0$ for every $t \in (a, b)$, $q \in \mathcal{P}$. Hence, $u(\cdot\,; a, b; q)$ is a solution of* (1.4).

Proof. For $q \in \mathcal{P}$, let

$$\{t \in (a, b) : u(t; a, b; q) > 0\} = \bigcup_{i \in I} (a_i, b_i),$$

where I is a family of indexes and $u(t; a, b; q) > 0$ for $t \in (a_i, b_i)$ (thus the (a_i, b_i) are disjoint intervals). By continuity, there exists $j \in I$ such that in (a_j, b_j) there exists a point τ of global maximum for $u(\cdot; a, b; q)$. By Corollary 1.4.9, we know that $u(\tau; a, b; q) \geq C_1(b-a)^2$ whenever $b - a \geq L_4$, for every $q \in \mathcal{P}$. Assume by contradiction that $(a_j, b_j) \neq (a, b)$; say, for instance, $a_j > a$. In order to obtain a contradiction, we consider separately the cases $A(p) > 0$ or $A(p) \leq 0$.

The case $A(p) > 0$**.** We choose $0 < \varepsilon < \min\{C_1, 2A(p)/3\}$, where we recall that C_1 has been defined in Corollary 1.4.9, and we consider the decomposition of Lemma 1.4.1 for the forcing term q. By the monotonicity of g, assumption (h1), there exists $s_\varepsilon := g^{-1}(A(p) - 3\varepsilon/2)$. Assuming $b - a$ sufficiently large in such a way that $u(\tau) > s_\varepsilon$ we can introduce

$$T := \inf\{\bar{t} > a_j : u(t; a, b; q) > s_\varepsilon \text{ for every } t \in (\bar{t}, \tau)\},$$
$$a' := \inf\{\bar{t} \leq T : \dot{u}(t; a, b; q) \geq 0 \text{ for every } t \in [\bar{t}, T]\}$$

(in particular, if $\dot{u}(T; a, b; q) = 0$, then $a' := T$). Note that, by definition,

$$\begin{cases} 0 \leq u(t; a, b; q) \leq s_\varepsilon & \text{if } t \in [a', T] \\ u(t; a, b; q) \geq s_\varepsilon & \text{if } t \in [T, \tau]. \end{cases} \tag{1.20}$$

As $u(\cdot\,; a, b; q) \in \mathcal{C}^1(a, b)$, $a' \geq a_j > a$ necessarily implies $\dot{u}(a'; a, b; q) = 0$. As a consequence, if we reach a contradiction, we deduce that both $a' = a_j = a$ and $\dot{u}(a; a, b; q) > 0$.

Step 1) *there exists* $C_2 > 0$ *independent of* $q \in \mathcal{P}$ *such that* $T - a' \leq C_2$.
By the monotonicity of g and (1.20), we deduce that, for every $t \in (a', T)$,

$$\begin{aligned} \ddot{u}(t; a, b; q) &= -g(u(t; a, b; q)) + q_{1,\varepsilon}(t) + \dot{q}_{2,\varepsilon}(t) \\ &\geq -g(s_\varepsilon) + A(p) - \frac{\varepsilon}{2} + \dot{q}_{2,\varepsilon}(t) = \varepsilon + \dot{q}_{2,\varepsilon}(t). \end{aligned}$$

By integrating twice in (a', t), and using the fact that $\dot{u}(a'; a, b; q) = 0$, we obtain

$$s_\varepsilon \geq u(T; a, b; q) - u(a'; a, b; q) \geq \frac{\varepsilon}{2}(T - a')^2 - 2M_\varepsilon(T - a'),$$

which provides the desired estimate.

Step 2) *There exists* $C_3 > 0$ *independent of* $q \in \mathcal{P}$ *such that* $\dot{u}(T; a, b; q) \leq C_3$.
As $g(s) \geq 0$ for $s \geq 0$, we see that, for every $t \in (a', T)$,

$$\begin{aligned} \ddot{u}(t; a, b; q) &= -g(u(t; a, b; q)) + q_{1,\varepsilon}(t) + \dot{q}_{2,\varepsilon}(t) \\ &\leq A(p) + \frac{\varepsilon}{2} + \dot{q}_{2,\varepsilon}(t). \end{aligned}$$

By integrating in (a', T), we deduce that

$$\dot{u}(T; a, b; q) \leq \left(A(p) + \frac{\varepsilon}{2}\right)(T - a') + 2M_\varepsilon \leq C_3,$$

where we use the first step and the fact that $\dot{u}(a'; a, b; q) = 0$.

Step 3) *Conclusion of the proof in case $A(p) > 0$.*
By the monotonicity of g (assumption (h1)) and (1.20), we deduce that, for every $t \in (T, \tau)$,

$$\begin{aligned}\ddot{u}(t;a,b;q) &= -g(u(t;a,b;q)) + q_{1,\varepsilon}(t) + \dot{q}_{2,\varepsilon}(t)\\ &\leq -g(s_\varepsilon) + A(p) + \frac{\varepsilon}{2} + \dot{q}_{2,\varepsilon}(t) = 2\varepsilon + \dot{q}_{2,\varepsilon}(t).\end{aligned}$$

By integrating twice in (T, t) and evaluating in τ, we deduce

$$\begin{aligned}u(\tau;a,b;q) &\leq \varepsilon(b-a)^2 + (\dot{u}(T;a,b;q) + 2M_\varepsilon)\,(b-a) + u(T;a,b;q)\\ &\leq \varepsilon(b-a)^2 + (C_3 + 2M_\varepsilon)\,(b-a) + s_\varepsilon,\end{aligned}$$

where we used the result of the previous step and the definition of T. The choice $\varepsilon < C_1$ gives a contradiction with Corollary 1.4.9 for $b-a$ sufficiently large (greater than a constant $\tilde{L}$ depending only on $\mathcal{P}$ and not on the particular choice of q).

The case $A(p) \leq 0$. We choose $0 < \varepsilon < C_1$, where we recall that C_1 has been defined in Corollary 1.4.9, and consider the decomposition of Lemma 1.4.1 for the forcing term q. For every $t \in (a_j, b_j)$ we have

$$\ddot{u}(t;a,b;q) = -g\left(u(t;a,b;q)\right) + q_{1,\varepsilon}(t) + \dot{q}_{2,\varepsilon}(t) \leq \frac{\varepsilon}{2} + \dot{q}_{2,\varepsilon}(t),$$

where we used the fact that $g(s) \geq 0$ for $s \geq 0$. By integrating twice in (a_j, t) with $t \in (a_j, b_j)$, and evaluating in τ, we obtain

$$u(\tau;a,b;q) \leq \varepsilon(b-a)^2 + 2M_\varepsilon(b-a).$$

Having chosen $\varepsilon < C_1$, this immediately contradicts Corollary 1.4.9 for $b-a$ sufficiently large. □

For the results of the next sections it is important to prove the uniqueness of the minimizer of the functional $J_{(a,b),q}$ with $q \in \mathcal{P}$. In light of the previous and the next statements, this uniqueness is guaranteed provided $b - a > \tilde{L}$. In the following proposition the forcing term p is fixed; therefore, we will use the simplified notation of the previous section.

Proposition 1.4.11 (Uniqueness)**.** *Let u and v be functions in $\mathcal{C}^2(a,b) \cap H^1_0(a,b)$ such that $u > 0$ and $v > 0$ in (a,b). Assume that*

$$J_{(a,b)}(u) = J_{(a,b)}(v) = \varphi^+(a,b).$$

Then $u \equiv v$ in $[a,b]$.

Proof. Let us consider the function

$$\Phi(\lambda) := J_{(a,b)}((1-\lambda)u + \lambda v).$$

We note that $\Phi \in \mathcal{C}^1(\mathbb{R})$ and

$$\Phi'(\lambda) = dJ_{(a,b)}((1-\lambda)u + \lambda v)[v-u].$$

As $\Phi(0) = \Phi(1)$, there exists $\bar\lambda \in (0,1)$ such that $\Phi'(\bar\lambda) = 0$, that is,

$$\int_a^b \left[\left(1-\bar\lambda\right)\dot u + \bar\lambda \dot v\right](\dot v - \dot u) - g\left(\left(1-\bar\lambda\right)u + \bar\lambda v\right)(v-u) + p(v-u) = 0. \tag{1.21}$$

Also, by minimality we know that $\Phi'(0) = \Phi'(1) = 0$, that is

$$\int_a^b \dot u(\dot v - \dot u) - g(u)(v-u) + p(v-u) = 0 \tag{1.22}$$

$$\int_a^b \dot v(\dot v - \dot u) - g(v)(v-u) + p(v-u) = 0. \tag{1.23}$$

If we consider (1.21) and subtract $(1-\bar\lambda)$ times (1.22) and $\bar\lambda$ times (1.23), we obtain

$$\int_a^b \left[\left(1-\bar\lambda\right)g(u) + \bar\lambda g(v) - g\left(\left(1-\bar\lambda\right)u + \lambda v\right)\right](v-u) = 0. \tag{1.24}$$

We claim that

$$\text{either } u \equiv v \text{ or the function } v-u \text{ changes sign in } (a,b). \tag{1.25}$$

Indeed, assume $u \not\equiv v$ and, w.l.o.g., $v \geq u$ in (a,b). The set $A := \{t \in (a,b) : v(t) > u(t)\}$ is not empty and has positive measure. Hence, by (1.24) and the strict concavity of g in $(0,+\infty)$, assumption (h1), we deduce that

$$\begin{aligned} 0 &= \int_a^b \left[\left(1-\bar\lambda\right)g(u) + \bar\lambda g(v) - g\left(\left(1-\bar\lambda\right)u + \lambda v\right)\right](v-u) \\ &= \int_A \left[\left(1-\bar\lambda\right)g(u) + \bar\lambda g(v) - g\left(\left(1-\bar\lambda\right)u + \lambda v\right)\right](v-u) < 0, \end{aligned}$$

a contradiction. This proves the claim (1.25), so that it remains to show that $v-u$ cannot change sign in (a,b). By contradiction again, assume that $v-u$ changes sign in (a,b), so that in particular there exists τ in (a,b) such that $u(\tau) = v(\tau)$. Say, for instance, that

$$\int_a^\tau \left(\frac{1}{2}\dot u^2 - G(u) + pu\right) \leq \int_a^\tau \left(\frac{1}{2}\dot v^2 - G(v) + pv\right);$$

necessarily it results

$$\int_\tau^b \left(\frac{1}{2}\dot{u}^2 - G(u) + pu\right) \geq \int_\tau^b \left(\frac{1}{2}\dot{v}^2 - G(v) + pv\right).$$

Let

$$\tilde{u}(t) := \begin{cases} u(t) & \text{if } t \in (a,\tau) \\ v(t) & \text{if } t \in [\tau, b). \end{cases}$$

By definition $\tilde{u} \in H_0^1(a,b)^+$, $\tilde{u} > 0$ in (a,b) and $J_{(a,b)}(\tilde{u}) \leq J_{(a,b)}(u) = \varphi^+(a,b)$, that is, $\tilde{u}$ is a minimizer of $J_{(a,b)}$ in $H_0^1(a,b)^+$ which is strictly positive in (a,b); hence, it solves the boundary value problem (1.4) and has to be of class $\mathcal{C}^2(a,b)$. This implies that $\dot{u}(\tau) = \dot{v}(\tau)$, and recalling that $u(\tau) = v(\tau)$, we can apply the uniqueness theorem for the initial value problems, proving that $u \equiv v$ in (a,b). □

Let $p \in \mathcal{P}$, and let $\mathcal{P}$ be defined by (1.15). Collecting together the results of Propositions 1.4.10 and 1.4.11, we can conclude that there exists $\tilde{L} > 0$ such that for every $(a,b) \subset \mathbb{R}$ with $b-a \geq \tilde{L}$ and for every $q \in \mathcal{P}$ there exists a unique minimizer $u(\cdot\,; a, b; q)$ of the functional $J_{(a,b),q}$ in $H_0^1(a,b)^+$, which is strictly positive in (a,b) and hence solves problem (1.4) with forcing term q. It is then possible to define a map which associates to each triple (a,b,q), with $b-a \geq \tilde{L}$ and $q \in \mathcal{P}$, the unique minimizer $u(\cdot; a, b; q)$. We conclude this section proving that this map is continuous.

Lemma 1.4.12. *Let p satisfy* (h2), *and let $\mathcal{P}$ be defined by* (1.15). *Let A and B be fixed and let*

$$\mathcal{I} := \left\{(t,a,b) \in \mathbb{R}^3 : b - a > \tilde{L},\ A < a \leq t \leq b < B\right\},$$

where $\tilde{L}$ has been defined in Proposition 1.4.10. Let us consider the metric space $\mathcal{P}$ endowed with the distance $d(q_1,q_2) = \|q_1 - q_2\|_{L^2(A,B)}$. The map

$$(t,a,b,q) \in \overline{\mathcal{I}} \times \mathcal{P} \mapsto (u(t;a;b;q), \dot{u}(t;a;b;q)) \in \mathbb{R}^2$$

is continuous.

Proof. Let $(a_n, b_n, p_n) \to (a^*, b^*, p^*)$ in $\overline{\mathcal{I}} \times \mathcal{P}$. Thanks to the explicit relations (1.7), we can consider the scaled functions $\widehat{u}_n := \widehat{u}(\cdot\,; a_n, b_n; \widehat{p}_n)$ and $\widehat{u}^* := \widehat{u}(\cdot\,; a^*, b^*; \widehat{p}^*)$. Having chosen $b - a > \tilde{L}$ and $(p_n) \subset \mathcal{P}$, from the previous results we deduce that each $\widehat{u}_n$ solves problem (1.9) with a_n, b_n, $\widehat{p}_n$ instead of a, b, $\widehat{p}$. By Lemma 1.3.4, we know that the sequence $(\widehat{u}_n)$ is bounded in $H_0^1(0,1)$, so that, up to a subsequence, it is weakly convergent in $H_0^1(0,1)$ to some $\widetilde{u} \in H_0^1(0,1)^+$. This, together with the fact that, up to a subsequence, $\widehat{p}_n \to \widehat{p}^*$ almost everywhere in $[0,1]$ (this follows from the convergence of $\widehat{p}_n$ to $\widehat{p}^*$ in $L^2(0,1)$), implies that $\widehat{u}_n \to \widetilde{u}$ in $H^2(0,1) \cap H_0^1(0,1)$, and

$$\begin{cases} \ddot{\widetilde{u}}(t) + g\left((b^* - a^*)^2 \widetilde{u}(t)\right) = \widehat{p}^*(t) & t \in (0,1) \\ \widetilde{u}(0) = 0 = \widetilde{u}(1) & \\ \widetilde{u}(t) \geq 0 & t \in (0,1). \end{cases}$$

We aim at proving that $\widetilde{u} \equiv \widehat{u}^*$; if this is not true, then the variational characterization of $\widehat{u}^*$ and Proposition 1.4.11 imply that

$$\widehat{J}_{(a^*,b^*),p^*}(\widehat{u}^*) < \widehat{J}_{(a^*,b^*),p^*}(\widetilde{u}). \tag{1.26}$$

By the continuity of $\widehat{J}$ with respect to u, p, a and b, we have also

$$\widehat{J}_{(a_n,b_n),p_n}(\widehat{u}_n) \to \widehat{J}_{(a^*,b^*),p^*}(\widetilde{u}) \quad \text{and} \quad \widehat{J}_{(a_n,b_n),p_n}(\widehat{u}^*) \to \widehat{J}_{(a^*,b^*),p^*}(\widehat{u}^*). \tag{1.27}$$

A comparison between (1.26) and (1.27) for n sufficiently large gives a contradiction with the fact $\widehat{u}_n$ reaches the minimum of $\widehat{J}_{(a_n,b_n),p_n}$ in $H_0^1(0,1)^+$, so that necessarily $\widetilde{u} \equiv \widehat{u}^*$. Since this argument holds for any subsequence, we deduce the convergence of the whole sequence, and to obtain the desired result it is sufficient to observe that, since $\widehat{u}_n \to \widehat{u}^*$ in $H^2(0,1) \cap H_0^1(0,1)$, then $\widehat{u}_n \to \widehat{u}^*$ in $\mathcal{C}^1([0,1])$. □

1.5 Non-degeneracy of positive minimizers

Assume that u solves (1.4) in (a,b); we can consider the variational equation

$$\begin{cases} \ddot{\psi}(t) + g'(u(t))\psi(t) = 0 & t \in (a,b) \\ \psi(a) = 0 = \psi(b). \end{cases} \tag{1.28}$$

Definition 1.5.1. We say that u is *non-degenerate* as solution of (1.4) if problem (1.28) has only the trivial solution $\psi \equiv 0$ in $H^2(a,b) \cap H_0^1(a,b)$.

The main result of this section is the following.

Proposition 1.5.2. *Let p satisfy* (h2), *$\mathcal{P}$ be defined by* (1.15), *and $\tilde{L}$ be defined as in Proposition 1.4.10, and let us assume that $b - a \geq \tilde{L}$. The function $u(\cdot; a, b; p)$ is non-degenerate as solution of the boundary value problem* (1.4).

For the proof, we will use some known results in singularity theory, which we recall here and for which we refer to Section 3.2 of the book by Ambrosetti and Prodi [4].

Definition 1.5.3. Let $\Phi : \Omega \subset E \to F$ be of class $\mathcal{C}^2(\Omega)$, where Ω is open, E and F are Banach spaces and $u_0 \in \Omega$. We say that u_0 is *singular* if $d\Phi(u_0)$ is not invertible. It is *ordinary singular* if it is singular and

(i) Ker $(d\Phi(u_0))$ is one-dimensional:

$$\mathrm{Ker}\,(d\Phi(u_0))) = \mathbb{R}\psi_0 \quad \text{for some } \psi_0 \in E \setminus \{0\};$$

Range $(d\Phi(u_0))$ is closed and has codimension 1:

$$\mathrm{Range}\,(d\Phi(u_0)) = \{q \in F : \langle \gamma_0, q \rangle = 0\} \quad \text{with } \gamma_0 \in F^* \setminus \{0\}.$$

(ii) $\langle \gamma_0, d^2\Phi(u_0)[\psi_0, \psi_0] \rangle \neq 0.$

Theorem 1.5.4 (Ambrosetti-Prodi)**.** *Let u_0 be an ordinary singular point for Φ, and, say,*

$$\langle \gamma_0, d^2\Phi(u_0)[\psi_0, \psi_0] \rangle > 0;$$

let $q_0 = \Phi(u_0)$, and let $q \in F$ be such that $\langle \gamma_0, q \rangle > 0$. Then there exists a neighbourhood U of u_0 in E and a positive number ε^* such that the equation

$$\Phi(u) = q_0 + \varepsilon q, \qquad u \in U$$

has exactly two solutions for $0 < \varepsilon < \varepsilon^*$ and no solution for $-\varepsilon^* < \varepsilon < 0$.

We are ready to show that $u(\cdot; a, b; p)$ is non-degenerate.

Proof of Proposition 1.5.2. Let

$$X := H^2(a, b) \cap H_0^1(a, b), \quad \|u\|_X := \|\ddot{u}\|_2, \quad Y := L^2(a, b).$$

We introduce the map $\mathcal{F} : X \to Y$ defined by

$$\mathcal{F}(u) = -\ddot{u} - g(u).$$

Under assumption (h1), it is immediate to see that $\mathcal{F} \in \mathcal{C}^2(X, Y)$ and

$$d\mathcal{F}(u)\psi = -\ddot{\psi} - g'(u)\psi, \qquad d^2\mathcal{F}(u)[\psi_1, \psi_2] = -g''(u)\psi_1\psi_2.$$

By the Fredholm alternative, $u(\cdot; a, b; p)$ is degenerate as solution of (1.4) if and only if it is singular for $\mathcal{F}$. So, let us assume by contradiction that $u(\cdot; a, b; p)$ is degenerate as solution of (1.4).

Step 1) *$u(\cdot;a,b;p)$ is ordinary singular for $\mathcal{F}$.*
We have to show that $u(\cdot;a,b;p)$ satisfies points (i) and (ii) of Definition 1.5.3. By degeneracy, problem

$$\begin{cases} \ddot\psi(t) + g'(u(t;a,b;p))\psi(t) = 0 & t \in (a,b) \\ \psi(a) = 0 = \psi(b) \end{cases}$$

has a nontrivial solution ψ_0, that is, 0 is an eigenvalue for the operator $d\mathcal{F}(u(t;a,b;p))$; this is a Sturm-Liouville operator with Dirichlet boundary conditions, hence all its eigenvalues are simple, and in particular $\mathrm{Ker}\,(d\mathcal{F}(u(t;a,b;p))) = \mathbb{R}\psi_0$. Moreover, in light of the Fredholm alternative, $d\mathcal{F}(u(t;a,b;p))$ is a Fredholm operator with index 0, so that property (i) in Definition 1.5.3 follows.

As far as point (ii) is concerned, first of all we claim that 0 is the first eigenvalue of $d\mathcal{F}(u(t;a,b;p))$; if not, there exists $\lambda_1 < 0$ and $\psi_1 \in X \setminus \{0\}$ such that

$$\begin{cases} \ddot\psi_1(t) + g'(u(t;a,b;p))\psi_1(t) = -\lambda_1\psi_1(t) & t \in (a,b) \\ \psi_1(a) = 0 = \psi_1(b). \end{cases}$$

On the other hand, since $u(t;a,b;p)$ is a local minimizer for $J_{(a,b),p}$, we know that $d^2J_{(a,b),p}(u(t;a,b;p))$ is a positive semi-definite quadratic form; this implies that

$$0 \le d^2J_{(a,b),p}(u(t;a,b;p))[\psi_1,\psi_1] = -\int_a^b \left(\ddot\psi_1 + g'(u(t;a,b;p))\psi_1\right)\psi_1$$
$$= \lambda_1 \int_a^b \psi_1^2 < 0,$$

a contradiction. Having proved that 0 is the first eigenvalue of $d\mathcal{F}(u(t;a,b;p))$, we can assume that $\psi_0 > 0$ in (a,b). By the Fredholm alternative, $\mathrm{Range}\,(d\mathcal{F}(u(t;a,b;p))) = \{q \in Y : \langle \gamma_0, q\rangle = 0\}$, where $\langle \gamma_0, q\rangle = \int_a^b \psi_0 q$. Hence

$$\langle \gamma_0, d^2\mathcal{F}(u(t;a,b;p))[\psi_0,\psi_0]\rangle = -\int_a^b g''(u(t;a,b;p))\psi_0^3 \neq 0$$

being $g'' < 0$ in $(0,+\infty)$ and $\psi_0 > 0$ in (a,b).

Step 2) *Conclusion of the proof.*
By definition, $\mathcal{F}(u(t;a,b;p)) = p$. We can choose $q \in Y$ such that

- $\int_a^b q\psi_0 > 0$;
- $p + \varepsilon q \in \mathcal{P}$ for every $|\varepsilon|$ sufficiently small.

Indeed, let $\phi \in \mathcal{C}_c^\infty(a,b) \setminus \{0\}$ be negative. Taking $q = \ddot{\phi}$, we obtain

$$\int_a^b \ddot{\phi}\psi_0 = \int_a^b \phi\ddot{\psi}_0 = -\int_a^b g'\left(u(t;a,b;q)\right)\phi\psi_0 > 0,$$

because $-g' < 0$ in $\mathbb{R}$ and $\psi_0 > 0$ in (a,b). Also, it is easy to check that the function $p + \varepsilon q \in \mathcal{P}$ whenever $|\varepsilon|$ is sufficiently small (see Remark 1.4.2). So, by definition, $\mathcal{F}\left(u(\cdot;a,b;p+\varepsilon q)\right) = p + \varepsilon q$ (to ensure that $u(\cdot;a,b;p+\varepsilon q)$ solves (1.4) with forcing term $p + \varepsilon q$, it is essential to know that $p + \varepsilon q \in \mathcal{P}$), and by Lemma 1.4.12 it results $u(\cdot;a,b;p+\varepsilon q) \to u(\cdot;a,b;p)$ in X as $\varepsilon \to 0^-$. On the other hand, by Theorem 1.5.4 there exists a neighbourhood U of $u(\cdot;a,b;p)$ in X such that the equation $\mathcal{F}(u) = p + \varepsilon q$ has no solution in U for $\varepsilon < 0$ sufficiently small, a contradiction. □

As an easy consequence of the Fredholm alternative, we obtain also the following corollary.

Corollary 1.5.5. *Let p satisfy* (h2)*, let $\mathcal{P}$ be defined by* (1.15)*, let $\tilde{L}$ be defined in Proposition 1.4.10, and assume that $b - a \geq \tilde{L}$. The boundary value problem*

$$\begin{cases} \ddot{\psi}(t) + g'(u(t;a,b;q))\psi(t) = 0 & t \in (a,b) \\ \psi(a) = \psi_a, \quad \psi(b) = \psi_b \end{cases}$$

has a unique solution for every $q \in \mathcal{P}$.

1.6 Differentiability of $\varphi^+(a,b)$

In this section we will show that $\varphi^+(a,b) = J_{(a,b),p}(u(\cdot\,;a,b;p))$ is differentiable as function of a and b.

Lemma 1.6.1. *Let p satisfy* (h2)*, and let $\mathcal{P}$ be defined by* (1.15)*. Let A and B be fixed and let*

$$\mathcal{I} := \left\{(t,a,b) \in \mathbb{R}^3 : b - a > \tilde{L},\ A < a \leq t \leq b < B\right\},$$

where $\tilde{L}$ has been defined in Proposition 1.4.10. If $q \in \mathcal{P}$ is of class $\mathcal{C}^1$, then the map

$$(t,a,b) \in \mathcal{I} \mapsto (u(t;a,b;q), \dot{u}(t;a,b;q)) \in \mathbb{R}^2$$

is of class $\mathcal{C}^1$, too. More precisely,

$$\frac{\partial u}{\partial a}(t;a,b;q) = \xi_1(t) \qquad \frac{\partial \dot{u}}{\partial a}(t;a,b;q) = \dot{\xi}_1(t)$$
$$\frac{\partial u}{\partial b}(t;a,b;q) = \xi_2(t) \qquad \frac{\partial \dot{u}}{\partial b}(t;a,b;q) = \dot{\xi}_2(t),$$

where ξ_1 and ξ_2 are the solutions (unique by Corollary 1.5.5) of

$$\ddot{\xi}(t) + g'(u(t;a,b;q))\xi(t) = 0$$

with the boundary conditions

$$\begin{cases} \xi_1(a) = -\dot{u}(a^+;a,b;q) \\ \xi_1(b) = 0 \end{cases} \qquad or \qquad \begin{cases} \xi_2(a) = 0 \\ \xi_2(b) = -\dot{u}(b^-;a,b;q), \end{cases}$$

respectively.

Proof. In light of the results of the previous sections, it is not difficult to adapt the proof of Lemma 5.1 in [70]. We report the sketch of the proof for the sake of completeness. Thanks to the explicit relations (1.7)-(1.9), the first part of the thesis follows if we prove the differentiability of $\widehat{u}(\cdot\,;a,b;q)$ with respect to (t,a,b). Let $\Delta := \{(a,b) \in \mathbb{R}^2 : b-a > \tilde{L}, A < a < b < B\}$, $X = H_0^1(0,1) \cap H^2(0,1)$, and consider the map $\Phi : \Delta \times X \to L^2(0,1)$ defined by

$$\Phi(w,a,b) = -\ddot{w} - g\left((b-a)^2 w\right) + q(a + t(b-a)).$$

By definition, $\Phi\left(\widehat{u}(\cdot\,;a,b;q);a,b\right) = 0$; we wish to show that the implicit function theorem applies to Φ in a neighbourhood of $\widehat{u}(\cdot\,;a,b;q)$. Having chosen $q \in \mathcal{C}^1(\mathbb{R})$, it is not difficult to check that $\Phi \in \mathcal{C}^1(\Delta \times X, Y)$, and that in particular

$$\partial_w \Phi\left(\widehat{u}(\cdot\,;a,b;p);a,b\right)[\psi] = -\ddot{\psi} - (b-a)^2 g'\left((b-a)^2 \widehat{u}(\cdot\,;a,b;p)\right)\psi,$$

which is invertible thanks to Proposition 1.5.2. Therefore, the implicit function theorem applies and the map $(a,b) \mapsto \widehat{u}(\cdot\,;a,b;q)$ is of class $\mathcal{C}^1(\Delta, X)$. By looking at the topology of X, this means that the map

$$(t,a,b) \in \mathcal{I} \mapsto (u(t;a,b;q), \dot{u}(t;a,b;q)) \in \mathbb{R}^2$$

has partial derivatives with respect to a and b, and that they are continuous in the three variables. The differential equation for $\widehat{u}(\cdot\,;a,b;q)$ reveals that also the partial derivative with respect to t exists and is continuous, that is, the map is $\mathcal{C}^1$, which completes the first part of the proof.

Now we have to compute the partial derivatives; to do this, we firstly use the boundary conditions on $u(\cdot\,;a,b;q)$:

$$\begin{aligned} u(a;a,b;q) = 0 \quad &\Longrightarrow \quad \xi_1(a) = -\dot{u}(a^+;a,b;q), \\ u(b;a,b;q) = 0 \quad &\Longrightarrow \quad \xi_1(b) = 0 \end{aligned}$$

Moreover, we note that for any $\phi \in \mathcal{C}_c^\infty((a,b))$ it results

$$\int_a^b \left[u(t;a,b;q)\ddot{\phi}(t) + \left(g(u(t;a,b;q)) - q(t)\right)\phi(t)\right] dt = 0.$$

Differentiating with respect to a we deduce

$$\int_a^b \left[\xi_1(t)\ddot{\phi}(t) + g'(u(t;a,b;q))\xi_1(t)\phi(t)\right] dt = 0 \qquad \forall \phi \in \mathcal{C}_c^\infty(a,b),$$

which completes the proof for ξ_1. □

Proposition 1.6.2. *For every p satisfying* (h2)*, the function* $\varphi^+(a,b) = \varphi^+(a,b;p)$ *is of class $\mathcal{C}^1$ with respect to a and b in $\{b-a > \tilde{L}\}$, with derivatives*

$$\frac{\partial \varphi^+}{\partial a}(a,b) = \frac{1}{2}\dot{u}^2(a^+;a,b;p) \quad \textit{and} \quad \frac{\partial \varphi^+}{\partial b}(a,b) = -\frac{1}{2}\dot{u}^2(b^-;a,b;p).$$

Proof. If $p \in \mathcal{C}^1(\mathbb{R})$, then by Lemma 1.6.1 $\varphi^+(a,b) = J_{(a,b),p}(u(\cdot\,;a,b;p))$ is differentiable. In such case, the expressions of its derivatives follow by direct computation. In the general case, we claim that

$$\text{there exists } (q_n) \subset \mathcal{P} \cap \mathcal{C}^1(\mathbb{R}) \text{ such that } q_n \to p \text{ in } L^2(A,B). \tag{1.29}$$

This is not straightforward, since $\mathcal{P}$ is defined as in (1.15). Let $\varepsilon_n \to 0$ as $n \to \infty$, an let us consider the decomposition

$$p = p_{1,\varepsilon_n} + \dot{p}_{2,\varepsilon_n}$$

given by Lemma 1.4.1. For any fixed n, we consider

$$q_{n,m} = A(p) + \frac{d}{dt}\left(\rho_m * p_{2,\varepsilon_n}\right) = A(p) + \rho_m * \dot{p}_{2,\varepsilon_n},$$

where (ρ_m) is a family of mollifiers, $*$ denotes the usual product of convolution, and the last identity follows from the fact that $p_{2,\varepsilon} \in \mathcal{C}^1(\mathbb{R})$. It is not difficult to check that $q_{n,m} \in \mathcal{P} \cap \mathcal{C}^1(\mathbb{R})$ for any m, n, and that for any n there exists m_n sufficiently large such that

$$\|q_{n,m_n} - p\|_{L^2(A,B)} < \varepsilon_n.$$

Hence, the sequence (q_{n,m_n}) has the desired properties, and claim (1.29) follows.
We introduce $\varphi_n(a,b) := \varphi^+(a,b;q_n)$ and $\varphi(a,b) := \varphi^+(a,b;p)$, and observe that, thanks to the previous step, each φ_n is of class $\mathcal{C}^1(\mathbb{R})$. Let $\Delta := \{(a,b) : b-a > \tilde{L}, A < a < b < B\}$. We claim that

$$\varphi_n \to \varphi \text{ uniformly for } (a,b) \in \overline{\Delta}. \tag{1.30}$$

If not,

$$\sup_{(a,b)\in\overline{\Delta}} |\varphi_n(a,b) - \varphi(a,b)| = \sup_{(a,b)\in\overline{\Delta}} \left|\varphi^+(a,b;q_n) - \varphi^+(a,b;p)\right| = c_n \geq \bar{c} > 0.$$

By Lemma 1.4.12 and the continuity of $J_{(a,b),p}(u)$ as function of (u, a, b, p), the function φ^+ is continuous in the three variables, so that by compactness for every n the supremum is achieved by $(a_n, b_n) \in \overline{\Delta}$. Therefore, if (1.30) does not hold, then

$$\left|\varphi^+(a_n, b_n; q_n) - \varphi^+(a_n, b_n; p)\right| \geq \bar{c}$$

for any n. Since, up to a subsequence, both a_n and b_n converge, this contradicts the continuity of φ^+.

With a similar argument we see also that $\dot{u}(\tau; a, b; q_n) \to \dot{u}(\tau; a, b; p)$ for $\tau = a, b$, uniformly in $\overline{\Delta}$, so that

$$\frac{\partial \varphi_n}{\partial a}(a, b) \to \frac{1}{2}\dot{u}^2(a^+; a, b; p) \quad \text{and} \quad \frac{\partial \varphi_n}{\partial b}(a, b) \to -\frac{1}{2}\dot{u}^2(b^-; a, b; p),$$

uniformly in $\overline{\Delta}$. The convergence of (φ_n) and of the sequences of the derivatives reveals that φ is of class $\mathcal{C}^1$ in Δ, and the thesis follows. □

1.7 Sign-changing solutions

In this section we complete the proof of Theorem 1.2.1. Firstly, we prove the existence of sign-changing solutions of (1.1) in bounded (sufficiently large) intervals; then, by an exhaustion procedure, we pass to the whole real line. To do this, we juxtapose positive and negative solutions on adjacent intervals, the latter existing and satisfying analogous properties of the former ones, as enlightened in Remark 1.3.1. To distinguish between positive and negative solutions, and since the forcing term p is now fixed, we change our notations accordingly, denoting such solutions as $u_\pm(\cdot\,; a, b)$. To sum up, we have the following result.

Proposition 1.7.1. *For every $\varepsilon > 0$ there exists $L > 0$ such that, if $b - a \geq L$, then the value $\varphi^\pm(a, b)$ is achieved by a unique $u_\pm(\cdot\,; a, b) \in H_0^1(a, b)$, which is strictly positive/negative and solves equation* (1.1) *in (a, b). Moreover,*

$$\begin{gathered}\|u_\pm(\cdot\,; a, b)\| \leq (\|g\|_\infty + \|p\|_\infty)(b-a)^{\frac{3}{2}} \\ -\underline{\alpha}(b-a)^3 \leq \varphi^+(a, b) \leq -\overline{\alpha}(b-a)^3 \\ -\underline{\beta}(b-a)^3 \leq \varphi^-(a, b) \leq -\overline{\beta}(b-a)^3,\end{gathered}$$

where $\underline{\alpha}, \overline{\alpha}$ have been defined in (1.16) *and*

$$\underline{\beta} := \frac{(-g_- + A(p))^2}{24} + \varepsilon \quad \text{and} \quad \overline{\beta} := \frac{(-g_- + A(p))^2}{24} - \varepsilon.$$

Proof. The proposition directly follows from Proposition 1.4.10, Lemma 1.3.4, Corollary 1.4.7 and Remark 1.3.1. □

By assumption (h2), there are two possibilities:

$$\text{either} \quad g_+ - A(p) = -g_- + A(p) \quad \text{or} \quad g_+ - A(p) \neq -g_- + A(p).$$

In the former case, we observe that for a given ε it results $\underline{\alpha} = \underline{\beta}$ and $\overline{\alpha} = \overline{\beta}$. Otherwise, it is possible to choose ε sufficiently small in such a way that

$$\text{either} \quad \underline{\alpha} < \overline{\beta} \quad \text{or} \quad \underline{\beta} < \overline{\alpha}.$$

To fix the ideas, in the following we consider the case

$$\overline{\beta} < \underline{\beta} < \overline{\alpha} < \underline{\alpha}. \tag{1.31}$$

The reader can easily adapt the arguments below in order to cover also the other situations (actually, if $g_+ - A(p) = -g_- + A(p)$, the problem is considerably simplified).

Firstly, we start by choosing $\varepsilon > 0$ sufficiently small in Proposition 1.7.1 in such a way that

$$\frac{\underline{\beta}}{\left(1 + \sqrt{\underline{\beta}/\underline{\alpha}}\right)^2} < \overline{\beta}; \tag{1.32}$$

by definition, one can easily check that this choice is possible.

Remark 1.7.2. Let $\nu := \underline{\beta}/\underline{\alpha}$. It is useful to observe that equation (1.32) implies that

$$\underline{\alpha}\left(\frac{\sqrt{\nu}}{1+\sqrt{\nu}}\right)^3 + \underline{\beta}\left(\frac{1}{1+\sqrt{\nu}}\right)^3 - \overline{\beta} < 0$$
$$\underline{\alpha}\left(\frac{\sqrt{\nu}}{1+\sqrt{\nu}}\right)^3 + \underline{\beta}\left(\frac{1}{1+\sqrt{\nu}}\right)^3 - \overline{\alpha} < 0.$$

First of all, by (1.31) we immediately see that the second of these relations is automatically satisfied provided the first one holds. And for the first one it is sufficient to note that

$$\underline{\alpha}\left(\frac{\sqrt{\nu}}{1+\sqrt{\nu}}\right)^3 + \underline{\beta}\left(\frac{1}{1+\sqrt{\nu}}\right)^3 = \underline{\alpha}\left(\frac{1}{1+\sqrt{\nu}}\right)^3 \left[\left(\sqrt{\nu}\right)^3 + \nu\right]$$
$$= \frac{\underline{\alpha}\nu}{(1+\sqrt{\nu})^2} = \frac{\underline{\beta}}{\left(1 + \sqrt{\underline{\beta}/\underline{\alpha}}\right)^2}.$$

Let $(A, B) \subset \mathbb{R}$ and $k \in \mathbb{N}$ be such that $(k+1)L \leq B - A$; hence, it is possible to divide the interval (A, B) in $k+1$ sub-intervals, in such a way that each of them is larger than L. We define the set of admissible partitions of (A, B) in $(k+1)$ sub-intervals as

$$\mathcal{B}_k := \left\{(t_1, \dots, t_k) \in \mathbb{R}^k : A =: t_0 \leq t_1 \leq \cdots \leq t_k \leq t_{k+1} := B, t_{i+1} - t_i \geq L\right\};$$

also, we introduce the function $\psi : \mathcal{B}_k \to \mathbb{R}$ defined by

$$\psi(t_1, \dots, t_k) := \sum_{i=0}^{k} \varphi^{\sigma(i)}(t_i, t_{i+1}), \quad \text{where} \quad \sigma(i) = \begin{cases} + & \text{if } i \text{ is even} \\ - & \text{if } i \text{ is odd.} \end{cases} \tag{1.33}$$

We consider the maximization problem

$$c_k(A, B) := \sup \{\psi(t_1, \dots, t_k) : (t_1, \dots, t_k) \in \mathcal{B}_k\}. \tag{1.34}$$

Remark 1.7.3. It is possible to consider also the maximization problem for the function having opposite $\sigma(i)$. The situation is essentially the same.

Lemma 1.7.4. *The value $c_k(A, B)$ is achieved by a partition $(\bar{t}_1, \dots, \bar{t}_k) \in \mathcal{B}_k$.*

Proof. This follows from the continuity of $\varphi^{\sigma(i)}$ (in fact $\varphi^{\sigma(i)}$ is differentiable, Proposition 1.6.2), and from the compactness of $\mathcal{B}_k$. □

To each interval $(\bar{t}_i, \bar{t}_{i+1})$ we associate

$$u_i := u_{\sigma(i)}(\cdot\,; \bar{t}_i, \bar{t}_{i+1}).$$

In this way, it is defined on the whole $[A, B]$ a function

$$u_{(A,B),k}(t) := u_i(t) \quad \text{if } t \in [\bar{t}_i, \bar{t}_{i+1}], \tag{1.35}$$

which is a solution of (1.1) in $(A, B) \setminus \{\bar{t}_1, \dots, \bar{t}_k\}$, and has exactly k zeros in (A, B). If we show that it is differentiable in each $\bar{t}_i$, then $u_{(A,B),k}$ will be a solution in the whole (A, B). To prove the smoothness of $u_{(A,B),k}$, we wish to exploit the knowledge of the explicit expression of the derivatives of $\varphi^{\sigma(i)}$, given in Proposition 1.6.2. Having this in mind, we observe that, if $(\bar{t}_1, \dots, \bar{t}_k)$ is an inner point of $\mathcal{B}_k$, then by maximality it results $\nabla\psi(\bar{t}_1, \dots, \bar{t}_k) = 0$, where the partial derivatives of ψ can be expressed in terms of the partial derivatives of $\varphi^{\sigma(i)}$. Therefore, the next step consists in the proof of the following lemma.

Lemma 1.7.5. *There exists H, depending only on L and on p, such that for any $(A, B) \subset \mathbb{R}$, $k \in \mathbb{N}$ with*

$$B - A \geq H(k+1),$$

the corresponding maximizing partition $(\bar{t}_1, \dots, \bar{t}_k) \in \mathcal{B}_k$ is an inner point of $\mathcal{B}_k$, that is, $\bar{t}_{i+1} - \bar{t}_i > L$ for every i.

We need two intermediate results. The first one says that the ratio between two adjacent sub-intervals of a maximizing partition can be controlled by means of a positive constant depending only on L and on p.

Lemma 1.7.6. *Let* $(\bar t_1, \dots, \bar t_k) \in \mathcal{B}_k$ *be a maximizing partition for* (1.34)*. There exists* $\bar h \geq 1$*, depending only on* L *and on* p*, such that*

$$\frac{1}{\bar h}(\bar t_i - \bar t_{i-1}) \leq \bar t_{i+1} - \bar t_i \leq \bar h(\bar t_i - \bar t_{i+1})$$

for every $i = 1, \dots, k$.

Proof. For an arbitrary i, let $\lambda = \bar t_i - \bar t_{i-1}$ and $h\lambda = \bar t_{i+1} - \bar t_i$. We wish to show that h is bounded from below and from above by two positive constants depending only on L and on p. Let $\nu := \underline{\beta}/\underline{\alpha}$, which belongs to $(0,1)$ by (1.31). If both λ and $h\lambda$ are smaller than or equal to $L/\sqrt{\nu}$, then $\sqrt{\nu} \leq h \leq 1/\sqrt{\nu}$. Otherwise, at least one between λ and $h\lambda$ is greater than $L/\sqrt{\nu}$, so that

$$(1+h)\lambda > \left(1 + \frac{1}{\sqrt{\nu}}\right) L. \tag{1.36}$$

Firstly, let us consider the case $\sigma(i-1) = +$, that is, $i-1$ is even. Let

$$s := \bar t_{i-1} + \frac{\sqrt{\nu}}{1+\sqrt{\nu}}(\bar t_{i+1} - \bar t_{i-1}) \in (\bar t_{i-1}, \bar t_{i+1}).$$

We consider the variation of $(\bar t_1, \dots, \bar t_k)$ obtained replacing $\bar t_i$ with s. This is an admissible partition in $\mathcal{B}_k$, as by (1.36) we have

$$\begin{aligned} s - \bar t_{i-1} &= \frac{\sqrt{\nu}}{1+\sqrt{\nu}}(1+h)\lambda > \frac{\sqrt{\nu}}{1+\sqrt{\nu}}\left(1 + \frac{1}{\sqrt{\nu}}\right) L = L \\ \bar t_{i+1} - s &= \frac{1}{1+\sqrt{\nu}}(1+h)\lambda > \frac{1}{1+\sqrt{\nu}}\left(1 + \frac{1}{\sqrt{\nu}}\right) L > L. \end{aligned}$$

The variational characterization of $(\bar t_1, \dots, \bar t_k)$ implies that

$$\psi(\bar t_1, \dots, \bar t_{i-1}, s, \bar t_{i+1}, \dots, \bar t_k) \leq \psi(\bar t_1, \dots, \bar t_{i-1}, \bar t_i, \bar t_{i+1}, \dots, \bar t_k);$$

by definition, this means

$$\varphi^{\sigma(i-1)}(\bar t_{i-1}, s) + \varphi^{\sigma(i)}(s, \bar t_{i+1}) \leq \varphi^{\sigma(i-1)}(\bar t_{i-1}, \bar t_i) + \varphi^{\sigma(i)}(\bar t_i, \bar t_{i+1}).$$

Therefore, recalling that we are considering the case $\sigma(i-1) = +$, by Proposition 1.7.1 we deduce

$$-\underline{\alpha}\left(\frac{\sqrt{\nu}}{1+\sqrt{\nu}}\right)^3 (1+h)^3\lambda^3 - \underline{\beta}\left(\frac{1}{1+\sqrt{\nu}}\right)^3 (1+h)^3\lambda^3 \leq -\overline{\alpha}\lambda^3 - \overline{\beta}h^3\lambda^3,$$

that is,

$$\left[\underline{\alpha}\left(\frac{\sqrt{\nu}}{1+\sqrt{\nu}}\right)^3+\underline{\beta}\left(\frac{1}{1+\sqrt{\nu}}\right)^3-\overline{\beta}\right]h^3$$
$$+3\left[\underline{\alpha}\left(\frac{\sqrt{\nu}}{1+\sqrt{\nu}}\right)^3+\underline{\beta}\left(\frac{1}{1+\sqrt{\nu}}\right)^3\right](h^2+h)$$
$$\left[\underline{\alpha}\left(\frac{\sqrt{\nu}}{1+\sqrt{\nu}}\right)^3+\underline{\beta}\left(\frac{1}{1+\sqrt{\nu}}\right)^3-\overline{\alpha}\right]\geq 0.$$

As observed in Remark 1.7.2, thanks to the choice (1.32), the coefficient of h^3 and the last term are negative, so that this relation cannot be satisfied if h is too small or too large: this implies that necessarily $1/\bar{h}_1 \leq h \leq \bar{h}_1$ for a positive constant $\bar{h}_1 > 1$, which depends only on L and on p.

In case $\sigma(i-1) = -$, one can follow the same line of reasoning, replacing the previous definition of s with

$$s := \bar{t}_{i-1} + \frac{1}{1+\sqrt{\nu}}(\bar{t}_{i+1} - \bar{t}_{i-1}) \in (\bar{t}_{i-1}, \bar{t}_{i+1}).$$

Again, the relation

$$\psi(\bar{t}_1, \ldots, \bar{t}_{i-1}, s, \bar{t}_{i+1}, \ldots, \bar{t}_k) \leq \psi(\bar{t}_1, \ldots, \bar{t}_{i-1}, \bar{t}_i, \bar{t}_{i+1}, \ldots, \bar{t}_k)$$

implies that for the quantity $\bar{h}_1 > 1$ previously introduced it results $1/\bar{h}_1 \leq h \leq \bar{h}_1$, and the desired result follows choosing $\bar{h} := \max\{1/\sqrt{\nu}, \bar{h}_1\}$. □

Now we can show that, in a maximizing partition, the ratio between the larger subinterval and the smaller one is bounded by a constant depending only on L and on p.

Lemma 1.7.7. *Let*

$$\underline{\lambda} := \min_i (\bar{t}_{i+1} - \bar{t}_i) \quad \textit{and} \quad \overline{\lambda} := \max_i (\bar{t}_{i+1} - \bar{t}_i).$$

Then there exists $h^* \geq 1$, depending only on L and on p, such that

$$\overline{\lambda} \leq h^* \underline{\lambda}.$$

Proof. Let us denote with $i \neq j$, $0 \leq i, j \leq k$, two indexes such that

$$\overline{\lambda} = \bar{t}_{i+1} - \bar{t}_i \quad \text{and} \quad \underline{\lambda} = \bar{t}_{j+1} - \bar{t}_j.$$

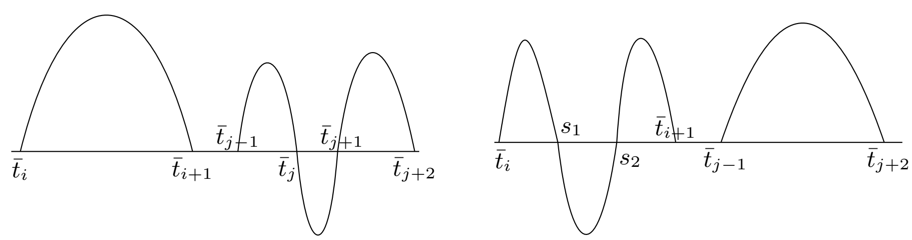
To fix the ideas we consider the case $i < j$. As the previous lemma asserts that the length of any interval is comparable with the one of its neighbours, we can assume without loss of generality i and j to be even, $k \geq 5$ and $j - i \geq 4$, i.e. $i + 2 \leq j - 2$. Let us set again $\nu := \underline{\beta}/\underline{\alpha}$, and let

$$\bar{\sigma} := \frac{1}{2}\left(\frac{1}{2} + \frac{1}{\sqrt[3]{2}}\right)\frac{\sqrt{\nu}}{1+\sqrt{\nu}}.$$

If $\overline{\lambda} \leq \max\{L/\bar{\sigma}, L/(1-2\bar{\sigma})\}$, we can choose $h^* = \max\{1/\bar{\sigma}, 1/(1-2\bar{\sigma})\}$. Otherwise, we consider a variation of $(\bar{t}_1, \ldots, \bar{t}_k)$ introducing two points

$$s_1 := \bar{t}_i + \bar{\sigma}(\bar{t}_{i+1} - \bar{t}_i) \quad \text{and} \quad s_2 := \bar{t}_i + (1-\bar{\sigma})(\bar{t}_{i+1} - \bar{t}_i).$$

between $\bar{t}_i$ and $\bar{t}_{i+1}$, and eliminating $\bar{t}_j$ and $\bar{t}_{j+1}$ if $j < k$; if $j = k$, we eliminate $\bar{t}_{k-1}$ and $\bar{t}_k$. For the reader's convenience, we explicitly observe that, since $\nu \in (0,1)$, it results $\bar{t}_i < s_1 < s_2 < \bar{t}_{i+1}$.



In what follows, the notation corresponds to the case $j < k$.
As $\overline{\lambda} > \max\{L/\bar{\sigma}, L/(1-2\bar{\sigma})\}$, the new partition is in $\mathcal{B}_k$: indeed

$$\begin{aligned} s_1 - \bar{t}_i &= \bar{\sigma}(\bar{t}_{i+1} - \bar{t}_i) = \bar{\sigma}\overline{\lambda} > L \\ s_2 - s_1 &= (1-2\bar{\sigma})(\bar{t}_{i+1} - \bar{t}_i) = (1-2\bar{\sigma})\overline{\lambda} > L \\ \bar{t}_{i+1} - s_2 &= \bar{\sigma}(\bar{t}_{i+1} - \bar{t}_i) = \bar{\sigma}\overline{\lambda} > L. \end{aligned} \tag{1.37}$$

As a consequence, by maximality,

$$\psi(\bar{t}_1, \ldots, \bar{t}_i, s_1, s_2, \bar{t}_{i+1}, \ldots, \bar{t}_{j-1}, \bar{t}_{j+2}, \ldots, \bar{t}_k) \leq \psi(\bar{t}_1, \ldots, \bar{t}_k),$$

that is,

$$\begin{aligned} &\varphi^+(\bar{t}_i, s_1) + \varphi^-(s_1, s_2) + \varphi^+(s_2, \bar{t}_{i+1}) + \varphi^+(\bar{t}_{j-1}, \bar{t}_{j+2}) \\ &\qquad \leq \varphi^+(\bar{t}_i, \bar{t}_{i+1}) + \varphi^+(\bar{t}_{j-1}, \bar{t}_j) + \varphi^-(\bar{t}_j, \bar{t}_{j+1}) + \varphi^+(\bar{t}_{j+1}, \bar{t}_{j+2}). \end{aligned}$$

We know that $\varphi^+(\bar{t}_i, \bar{t}_{i+1}) \leq -\overline{\alpha}\overline{\lambda}^3$, and the other terms on the right hand side are negative; on the other hand, for the left hand side we can use the expressions (1.37) and the fact that, by Lemma 1.7.6, $\bar{t}_{j+2} - \bar{t}_{j-1} \leq (2\bar{h}+1)\underline{\lambda}$. Therefore

$$-2\underline{\alpha}\bar{\sigma}^3\overline{\lambda}^3 - \underline{\beta}(1-2\bar{\sigma})^3\overline{\lambda}^3 - \underline{\alpha}(2\bar{h}+1)^3\underline{\lambda}^3 \leq -\overline{\alpha}\overline{\lambda}^3,$$

which gives

$$\left[\overline{\alpha} - 2\underline{\alpha}\bar{\sigma}^3 - \underline{\beta}(1-2\bar{\sigma})^3\right]\left(\frac{\overline{\lambda}}{\underline{\lambda}}\right)^3 \leq \underline{\alpha}(2\bar{h}+1)^3.$$

We claim that

$$\overline{\alpha} - 2\underline{\alpha}\bar{\sigma}^3 - \underline{\beta}(1-2\bar{\sigma})^3 > 0;$$

as a consequence, the thesis will follows. To prove the claim, we note that, by definition of $\bar{\sigma}$, it results

$$2\bar{\sigma}^3 < \left(\frac{\sqrt{\nu}}{1+\sqrt{\nu}}\right)^3 \quad \text{and} \quad (1-2\bar{\sigma})^3 < \left(\frac{1}{1+\sqrt{\nu}}\right)^3;$$

Thanks to the choice (1.32), recalling also Remark 1.7.2, we easily deduce

$$\overline{\alpha} - 2\underline{\alpha}\bar{\sigma}^3 - \underline{\beta}(1-2\bar{\sigma})^3 > \overline{\alpha} - \underline{\alpha}\left(\frac{\sqrt{\nu}}{1+\sqrt{\nu}}\right)^3 - \underline{\beta}\left(\frac{1}{1+\sqrt{\nu}}\right)^3 > 0,$$

which completes the proof. □

End of the proof of Lemma 1.7.5. Let $H = h^*(L+1)$, with h^* introduced in Lemma 1.7.7. Then any partition of an interval of length $B - A \geq H(k+1)$ in $k+1$ sub-intervals has a sub-interval larger than $h^*(L+1)$, and in particular $\overline{\lambda} \geq h^*(L+1)$. Applying Lemma 1.7.7, we immediately deduce $\underline{\lambda} \geq L+1$. □

We are ready to prove the existence of sign-changing solutions of (1.1) in large intervals.

Proposition 1.7.8. *There exists H, depending only on L and on p, such that if $B-A \geq H(k+1)$ and $(\bar{t}_1, \ldots, \bar{t}_k)$ is a maximizing partition for* (1.34)*, then the function $u_{(A,B),k}$ defined by* (1.35) *is a solution of* (1.1)*.*

Proof. By construction, $u_{(A,B),k}$ solves (1.1) in $(A,B)\setminus\{\bar{t}_1, \ldots, \bar{t}_k\}$. Moreover, by Lemma 1.7.5, $(\bar{t}_1, \ldots, \bar{t}_k)$ is a free critical point of the function ψ, so that $\nabla\psi(\bar{t}_1, \ldots, \bar{t}_k) = 0$. In view of Proposition 1.6.2, this writes

$$-\frac{1}{2}\dot{u}_{i-1}^2(\bar{t}_i^-) + \frac{1}{2}\dot{u}_i^2(\bar{t}_i^+) = 0 \qquad i = 1, \ldots, k.$$

But then $u_{(A,B),k}$ is $\mathcal{C}^1$ across each $\bar{t}_i$, and the proposition follows. □

Remark 1.7.9. Directly from the construction of $u_{(A,B),k}$, it is possible to obtain some estimates which will be useful in the next proof; we keep here the notation previously introduced. First of all, we note that for every $t \in (A, B)$ there exists i such that $t \in [\bar{t}_i, \bar{t}_{i+1})$. Thanks to Lemma 1.3.4, we deduce that

$$|u_{(A,B),k}(t)| = |u_i(t)| \le C(\bar{t}_{i+1} - \bar{t}_i)^2 \le C\overline{\lambda}^2$$
$$|\dot{u}_{(A,B),k}(t)| = |\dot{u}_i(t)| \le C(\bar{t}_{i+1} - \bar{t}_i) \le C\overline{\lambda},$$

where C is a positive constant depending only on g and p. As a consequence

$$\|u_{(A,B),k}\|_{L^\infty(A,B)} \le C\overline{\lambda}^2 \quad \text{and} \quad \|\dot{u}_{(A,B),k}\|_{L^\infty(A,B)} \le C\overline{\lambda}.$$

On the other hand, let τ be a point of maximum of $|u_{(A,B),k}|$. There exists $j \in \{0, \ldots, k\}$ such that $\tau \in (\bar{t}_j, \bar{t}_{j+1})$, so that by Corollary 1.4.9 it results

$$\|u_{(A,B),k}\|_{L^\infty(A,B)} = |u_j(\tau)| \ge C_1(\bar{t}_{j+1} - \bar{t}_j) \ge C_1\underline{\lambda},$$

where C_1 is a positive constant depending only on g and p.

It is now possible to complete the proof of the main result.

Proof of Theorem 1.2.1. For a fixed $L > \bar{L}$, let $\bar{h}$, h^* and H be as in Lemmas 1.7.6, 1.7.7 and Proposition 1.7.8 respectively. Let $\mu \ge H$ be fixed (we explicitly remark that h^* is independent of μ). For every $n \in \mathbb{N}$ we have $2n\mu \ge 2nH$, so that by Proposition 1.7.8 there exists $u_{\mu,n} := u_{(-\mu n, \mu n), 2n-1}$ which is a solution of (1.1) in $(-\mu n, \mu n)$ with $2n - 1$ zeros, and its zeros correspond to a partition

$$-\mu n =: \bar{t}_0 < \bar{t}_1 < \ldots, \bar{t}_{2n-1} < \bar{t}_{2n} := \mu n,$$

maximizing for $c_{2n-1}(-\mu n, \mu n)$, defined by (1.34). At least one of the sub-intervals of the partition has to be smaller than or equal to μ; recalling that $\underline{\lambda} := \min_i(\bar{t}_{i+1} - \bar{t}_i)$ and $\overline{\lambda} = \max_i(\bar{t}_{i+1} - \bar{t}_i)$, it results $\underline{\lambda} \le \mu$; this implies, by means of Lemma 1.7.7, that $\overline{\lambda} \le h^*\mu$, where h^* does not depend on n or on μ. Analogously, from the fact that at least one of the sub-intervals of the partition has to be larger than or equal to μ, it is possible to deduce that $\underline{\lambda} \ge \mu/h^*$.

By using the estimates of Remark 1.7.9, it is immediate to obtain

$$C_1 \left(\frac{\mu}{h^*}\right)^2 \le \|u_{\mu,n}\|_{L^\infty(-\mu n, \mu n)} \le C(h^*\mu)^2 \quad \text{and} \quad \|\dot{u}_{\mu,n}\|_{L^\infty(-\mu n, \mu n)} \le C(h^*\mu).$$

Furthermore, being $u_{\mu,n}$ a solution of (1.1), it results

$$\|\ddot{u}_{\mu,n}\|_{L^\infty(-\mu n, \mu n)} \le \|g\|_\infty + \|p\|_\infty.$$

The previous estimates reveals that the sequence $(u_{\mu,n})_{n\in\mathbb{N}}$ is uniformly bounded in $W^{2,\infty}_{\text{loc}}(\mathbb{R})$, so that by the Ascoli-Arzelà theorem it converges in $\mathcal{C}^1_{\text{loc}}(\mathbb{R})$, up to a subsequence, to a function u_μ which is a solution of (1.1) in the whole $\mathbb{R}$, and satisfies

$$C_1\left(\frac{\mu}{h^*}\right)^2 \le \|u_\mu\|_{L^\infty(\mathbb{R})} \le C(h^*\mu)^2 \quad \text{and} \quad \|\dot{u}_\mu\|_{L^\infty(\mathbb{R})} \le C(h^*\mu) \tag{1.38}$$

By construction, u_μ has infinitely many zeros tending to infinity in both the directions; indeed, if this were not true, then $|u_\mu(t)| \ge C > 0$ on an interval of length greater than $h^*\mu$, and by the $\mathcal{C}^1_{\text{loc}}$ convergence the same should hold also for $u_{\mu,n}$ when n is sufficiently large, which is not possible.

We have constructed a solution of (1.1) defined in $\mathbb{R}$, which is bounded together with its first derivative. Now, we can obtain the sequence of bounded solutions $u_m = u_{\mu_m}$ simply repeating the same procedure for a sequence of parameters μ_m such that $\mu_m \to +\infty$ and

$$\mu_m > \sqrt{\frac{C}{C_1}}(h^*)^2\mu_{m-1}$$

for every m. Indeed, thanks to equation (1.38), we deduce

$$\|u_{m-1}\|_{L^\infty(\mathbb{R})} \le C(h^*\mu_{m-1})^2 < C_1\left(\frac{\mu_m}{h^*}\right)^2 \le \|u_m\|_{L^\infty(\mathbb{R})},$$

so that $u_{m-1} \not\equiv u_m$ and $\|u_m\|_\infty \to +\infty$ as $m \to \infty$. □

To conclude, as we mentioned in the introduction, we turn to the periodic framework. We keep the previous notations, in particular H is defined as in Lemma 1.7.5. We have the following.

Theorem 1.7.10. *Let g satisfy* (h1)*, and let p be a continuous T-periodic function such that*

$$g_- < A(p) = \frac{1}{T}\int_0^T p(t)\,dt < g_+.$$

Then, for any $(k,n) \in \mathbb{N}^2$ with k odd and $nT \ge H(k+1)$, there exist a nT-periodic solution of (1.1)*, having exactly $k+1$ zeros in $[0,nT)$.*

Remark 1.7.11. The nodal characterization of the solutions ensures that, whenever T is the minimal period of p, and n and $(k+1)/2$ are coprime integers, then nT is the minimal period of the corresponding solution. This ensures the existence of an infinite sequence of subharmonic solutions, with diverging minimal period.

Proof. Let

$$\mathcal{A}_k := \left\{ (t_0, t_1, \dots, t_k) \in \mathbb{R}^k \;\middle|\; \begin{array}{l} t_0 \le t_1 \le \cdots \le t_k \le t_{k+1} := t_0 + nT, \\ t_{i+1} - t_i \ge L,\ t_0 \in [-T, 2T] \end{array} \right\},$$

and let $\psi : \mathcal{A}_k \to \mathbb{R}$ defined as in (1.33) (we point out that now t_0 is not fixed). There exists a maximizer $(\bar{t}_0, \bar{t}_1 \dots, \bar{t}_k)$ for ψ. Since p is T-periodic, we can assume $\bar{t}_0 \in [0, T)$. As a consequence, it results $\nabla\psi(\bar{t}_0, \bar{t}_1, \dots, \bar{t}_k) = 0$. The expression of the partial derivatives of ψ with respect to t_i, $i = 1, \dots, k$, says that the function $u_{(\bar{t}_0, \bar{t}_0+nT),k}$ (defined as in (1.35)) is a solution of (1.1) in $(\bar{t}_0, \bar{t}_0 + nT)$; also, the fact that $\partial_{t_0}\psi(\bar{t}_0, \bar{t}_1, \dots, \bar{t}_k) = 0$ implies that

$$-\frac{1}{2}\dot{u}^2_{(\bar{t}_0,\bar{t}_0+nT),k}(\bar{t}_0^+) + \frac{1}{2}\dot{u}^2_{(\bar{t}_0,\bar{t}_0+nT),k}((\bar{t}_0 + nT)^-) = 0,$$

that is, $u_{(\bar{t}_0,\bar{t}_0+nT),k}$ can be extended by nT-periodicity as a (smooth) solution of (1.1) in the whole $\mathbb{R}$. □

Chapter 2

Symbolic dynamics for the N-centre problem at negative energies

2.1 Introduction and main results

In the N-body problem, one investigates the motion of an arbitrary (finite) number of heavy bodies which move in the space $\mathbb{R}^3$ under their mutual gravitational attraction. Describing the position of each body as a vector valued function of the time,

$$x_k : I \subset \mathbb{R} \to \mathbb{R}^3 \qquad k = 1, \dots, N,$$

according to the second Kepler law the motion of the bodies is described by the system of second order ordinary differential equations

$$\ddot{x}_k(t) = -\sum_{j \neq k} \frac{m_j}{|x_k(t) - x_j(t)|^3}(x_k(t) - x_j(t)),$$

where m_j is the mass of the j-th particle.

Whereas the 1-body problem correspond to the free motion and the 2-body problem was solved by Newton, the N-body problem for $N \geq 3$ is analytically non-integrable, and, in its complete generality, is still far away from being well understood. For this reason, a wide part of the research has been devoted to some relevant simplified problems. The N-centre problem fits exactly in this contest.

Let us consider a $(N+1)$-body problem. If one body is much faster then the others, then we may approximate its motion by assuming that the N other bodies are fixed: this leads to the N-centre problem, which consists in the study of the motion of a test particle of null mass under the gravitational force fields of N *fixed* heavy bodies, the *centres* of the problem. The motion equation is

$$\ddot{x}(t) = -\sum_{j=1}^{N} \frac{m_j}{|x(t) - c_j|^3} \left(x(t) - c_j \right), \tag{2.1}$$

where $x = x(t) \in \mathbb{R}^3$ denotes the position of the particle at time $t \in \mathbb{R}$, and c_j $(j = 1, \ldots, N)$ is the position of the j-th centre. Introduced the potential

$$V(x) = \sum_{j=1}^{N} \frac{m_j}{|x - c_j|}, \qquad x \in \mathbb{R}^3 \setminus \{c_1, \ldots, c_N\},$$

we can rewrite equation (2.1) as $\ddot{x} = \nabla V(x)$, which has Hamiltonian structure, of Hamiltonian

$$\frac{1}{2}|v|^2 - V(x) = h(v, x).$$

The 1-centre problem is nothing but the Kepler problem, probably the most famous integrable problem in celestial mechanics. The 2-centre problem was investigated and solved by Euler. Moreover, Jacobi provided explicit solutions in his celebrated work "Vorlesungen über dynamik". These solutions are particularly relevant for calculating the motion of artificial satellites, since the gravitational force field of the Earth can be approximated by one of the two centres in a 2-centre problem. This has been used by Vinti in [88], see also [46] for an application. Moreover, we point out that the 2-centre problem plays a role as a physic model of a diatomic molecule in which the centres are atomic nuclei, and the test particle is an electron; this approach was introduced by Pauli, and we refer to [89] for more details.

Contrarily to the cases $N = 1$ and $N = 2$, if $N \geq 3$ the dynamical system of the N-centre problem exhibits a chaotic and non-integrable structure. The first contribution in this direction is due to Bolotin in [14]: he proved the analytic non-integrability of the *planar* N-centre problem for $N \geq 3$ on any energy shell

$$\mathcal{U}_h := \left\{ (x, v) \in \mathbb{R}^2 \setminus \{c_1, \ldots, c_N\} \times \mathbb{R}^2 : \frac{1}{2}|v|^2 - V(x) = h \right\}$$

such that $h > 0$. The question of analytic non-integrability has been faced also for the spatial problem by the same author and Negrini [15], and by Knauf and Taimanov in [54]. In [15], it has been proved that if $N \geq 3$, then the restriction of the dynamical system on the energy shell $\mathcal{U}_h$ has positive topological entropy for every $h \geq 0$; this fact suggests the analytic non-integrability of the dynamical system, which has been rigorously proved by Knauf and Taimanov under the assumption that the energy is larger than a positive threshold h_{th}.

Concerning the qualitative description of the dynamical system, a crucial reference for the planar problem and for the positive energy case is the contribution by Klein and Knauf [52]: therein the authors provided an accurate description of the planar scattering phenomena for a wide class of problems of N-centre type; in particular, they showed that the planar N-centre problem on positive energy levels has a symbolic dynamics. Regarding the spatial problem, we refer to Knauf [53], in which the author extended some results of [52] assuming that the energy is positive and sufficiently large.

As far as the negative energy case is concerned, the literature shows very few works, and the most remarkable results are obtained under strongly restrictive assumptions: in [16] Bolotin and Negrini proved the occurrence of chaotic dynamics for the 3-centre problem on the energy level $\mathcal{U}_h$, assuming that the third centre is far away from the others two and that the absolute value of h is sufficiently small; in [31], Dimare obtained a similar result for $h < 0$, $|h|$ small enough, when one centre has small mass with respect to the others. In both papers the problem is approached with a perturbation argument. In a general setting, almost nothing is known.

To complete this bibliographic introduction, we mention also the contribution of Castelli [19], who deals with the planar problem and proved the existence of infinitely many periodic solutions having a common fixed period for a suitable displacement of the centres in the plane. In this work there is no information on the energy of the solutions.

In this chapter, which is based on [79], we study the planar N-centre problem, considering the more general situational of gravitational potentials of degree $-\alpha$:

$$V(x) = \sum_{j=1}^{N} \frac{m_j}{\alpha\,|x - c_j|^{\alpha}} \qquad x \in \mathbb{R}^2 \setminus \{c_1, \dots, c_N\},$$

with $\alpha \in [1, 2)$; the equation for the motion of the test particle modifies as

$$\ddot{x}(t) = \nabla V(x(t)) = -\sum_{j=1}^{N} \frac{m_j}{|x(t) - c_j|^{\alpha+2}}\,(x(t) - c_j)\,, \tag{2.2}$$

where we recall that $c_j \in \mathbb{R}^2$ denotes the position of the j-th centre at time t. We explicitly remark that, when $\alpha = 1$, equation (2.2) coincides with equation (2.1).

To describe our main results, we need some notation. We say that $x : I \subset \mathbb{R} \to \mathbb{R}^2$ is a *collision-free solution of* (2.2) if $x \in \mathcal{C}^2(I)$ solves equation (2.2) in a classical sense: in particular, $x(t) \neq c_j$ for every $t \in I$, for every $j = 1, \dots, N$. On the other hand, we say that $x : I \subset \mathbb{R} \to \mathbb{R}^2$ is a *collision solution of* (2.2) if $x \in H^1(I)$, and there exists a *collision set* $T_c(x) \subset I$ such that

- for every $t \in T_c(x)$, it results $x(t) = c_j$ for some $j = 1, \dots, N$;

- $T_c(x)$ has null measure;
- if $(a,b) \subset I \setminus T_c(x)$, then x is a collision-free solution of (2.2) in (a,b);
- there exists $h \in \mathbb{R}$ such that the energy function is constant in I:

$$\frac{1}{2}|\dot{x}(t)|^2 - V(x(t)) = h \qquad \forall t \in I \setminus T_c(x).$$

Let us consider the possible partitions of the set of the centres $\{c_1, \ldots, c_N\}$ in two disjoint (non ordered) non-empty sets. Here non ordered means that for us

$$\{\{c_1\}, \{c_2, \ldots, c_N\}\} = \{\{c_2, \ldots, c_N\}, \{c_1\}\}.$$

There are exactly

$$\frac{1}{2}\left(\binom{N}{1} + \ldots + \binom{N}{N-1}\right) = \frac{1}{2}\left(\sum_{k=0}^{N}\binom{N}{k} - 2\right) = 2^{N-1} - 1$$

such partitions, which we label within the set

$$\mathcal{P} := \{P_j : j = 1, \ldots, 2^{N-1} - 1\}.$$

It is convenient to distinguish those partitions which separate a single c_j from the others: we call

$$P_j := \{\{c_j\}, \{c_1, \ldots, c_N\} \setminus \{c_j\}\} \qquad j = 1, \ldots, N.$$

This special kind of partitions defines a subset of labels

$$\mathcal{P}_1 := \{P_j \in \mathcal{P} : j = 1, \ldots, N\} \subset \mathcal{P}.$$

Remark 2.1.1. The notation can appear not so clear. Indeed, with P_j we denote both symbols in $\mathcal{P}$ and in the subset $\mathcal{P}_1$. For this reason, in what follows we always emphasize if we consider a partition in the entire set of symbols $\mathcal{P}$, or only in the subclass $\mathcal{P}_1$.

We define the *right shift* $T_r : \mathcal{P}^n \to \mathcal{P}^n$ as

$$T_r(P_{j_1}, P_{j_2}, \ldots, P_{j_n}) = (P_{j_n}, P_{j_1}, \ldots, P_{j_{n-1}}),$$

and we say that $(P_{j_1}, \ldots, P_{j_n}) \in \mathcal{P}^n$ *is equivalent to* $(P'_{j_1}, \ldots, P'_{j_n}) \in \mathcal{P}^n$ if there exists $m \in \mathbb{N}$ such that

$$(P'_{j_1}, \ldots, P'_{j_n}) = T_r^m\left((P_{j_1}, \ldots, P_{j_n})\right).$$

Periodic solutions. Our first goal is the existence of infinitely many periodic solutions with negative energy:

Theorem 2.1.2. *Let $\alpha \in [1, 2)$, $N \geq 3$, $c_1, \ldots, c_N \in \mathbb{R}^2$, $m_1, \ldots, m_N \in \mathbb{R}^+$. There exists $\bar{h} < 0$ such that, for every $h \in (\bar{h}, 0)$, $n \in \mathbb{N}$ and $(P_{j_1}, \ldots, P_{j_n}) \in \mathcal{P}^n$, there exists a periodic (possibly collision) solution $x_{((P_{j_1},\ldots,P_{j_n}),h)}$ of the N-centre problem (2.2) with energy h, which depends on $(P_{j_1}, \ldots, P_{j_n})$ in the following way: there exist $\bar{R}$ sufficiently large (in particular, we require that $c_j \in B_{\bar{R}}(0)$ for every j), and $\bar{\delta} > 0$ sufficiently small (both $\bar{R}$ and $\bar{\delta}$ depending only on h), such that $x_{((P_{j_1},\ldots,P_{j_n}),h)}$ crosses $2n$ times within one period the circle $\partial B_{\bar{R}}(0)$, at times $(t_k)_{k=0,\ldots,2n-1}$, in such a way that:*

- *in (t_{2k}, t_{2k+1}) the solution stays outside $B_{\bar{R}}(0)$, and*

$$|x_{((P_{j_1},\ldots,P_{j_n}),h)}(t_{2k}) - x_{((P_{j_1},\ldots,P_{j_n}),h)}(t_{2k+1})| < \bar{\delta};$$

- *in (t_{2k+1}, t_{2k+2}) the solution lies inside $B_{\bar{R}}(0)$, and, if it does not collide against any centre, then it separates them according to the partition P_{j_k}.*

Concerning the possibility to have a collision, it results:

(i) if $\alpha \in (1, 2)$ then $x_{((P_{j_1},\ldots,P_{j_n}),h)}$ is collision-free;

(ii) if $\alpha = 1$ there are three possibilities:

(a) either $x_{((P_{j_1},\ldots,P_{j_n}),h)}$ is collision-free;

(b) or $x_{((P_{j_1},\ldots,P_{j_n}),h)}$ collides first with one centre c_j, covers a certain trajectory, then falls back on a second centre c_k (it may happen that $c_j = c_k$) bouncing and coming back along the same trajectory. This is possible only when n is even and $(P_{j_1}, \ldots, P_{j_n})$ is equivalent to $(P'_{j_1}, \ldots, P'_{j_n})$ such that

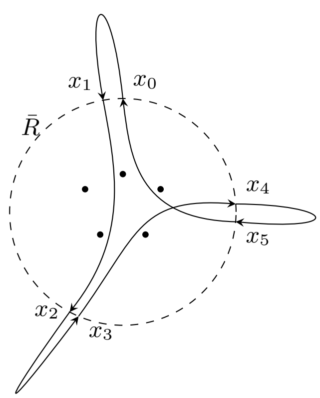
$$P'_{j_1} \in \mathcal{P}_1, \quad P'_{j_{n/2+1}} \in \mathcal{P}_1 \quad \text{and} \quad (\text{if } n > 2)$$
$$P'_{j_n} = P'_{j_2},\ P'_{j_{n-1}} = P'_{j_3},\ \ldots,\ P'_{j_{n/2+2}} = P'_{j_{n/2}};$$

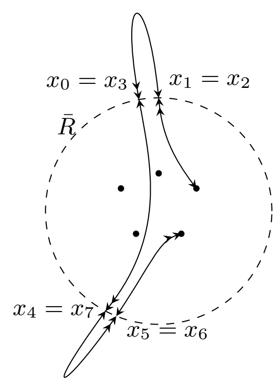
(c) or else $x_{((P_{j_1},\ldots,P_{j_n}),h)}$ has a collision against one centre c_j, covers a certain trajectory, "bounces" against the curve $\{x \in \mathbb{R}^2 : V(x) = -h\}$ with null velocity and comes back along the same trajectory. This is possible only if n is odd and $(P_{j_1}, \ldots, P_{j_n})$ is equivalent to $(P'_{j_1}, \ldots, P'_{j_n})$ such that

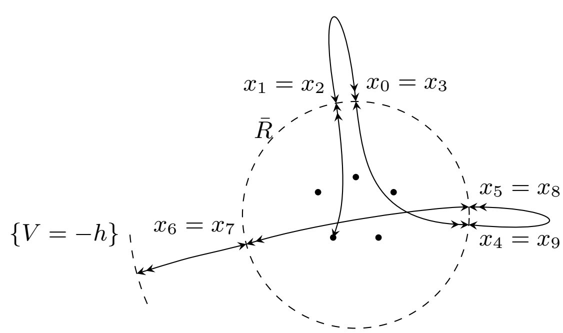
$$P'_{j_1} \in \mathcal{P}_1 \quad \text{and} \quad (\text{if } n > 1)$$
$$P'_{j_n} = P'_{j_2},\ P'_{j_{n-1}} = P'_{j_3},\ \ldots,\ P'_{j_{(n+1)/2+1}} = P'_{j_{(n+1)/2}}.$$

Of course, by varying both the number $n \in \mathbb{N}$ and the choice of the partitions $(P_{j_1}, \ldots, P_{j_n}) \in \mathcal{P}^n$, we find infinitely many periodic solutions for every fixed value $h \in (\bar{h}, 0)$. Note that we cannot completely rule out the occurrence of collision solutions in the case of the classical Newtonian interaction $\alpha = 1$; however, we observe that the symbol sequences of the collision solutions exhibits a peculiar symmetry; by choosing non symmetric sequences, it is then possible to find infinitely many collision-free periodic solutions even for $\alpha = 1$.

The following pictures represent the case (i) or (ii)-(a), (ii)-(b), (ii)-(c) respectively.







Remark 2.1.3. 1) The assumption "$|h|$ is sufficiently small" is substantial. Indeed, every solution of the N-centre problem with energy h is confined in the *Hill region* $\{x \in \mathbb{R}^2 : V(x) > -h\}$; when h becomes very large, due to the singular nature of the potential V this set can be even disconnected, so that it is impossible to find solutions having the behaviour described in the statement. However, we stress that in general the threshold $|\bar{h}|$ is strictly smaller than the infimum $\tilde{h} > 0$ such that $c_j \in \{V(x) > \tilde{h}\}$ for every j.

On the other hand, we remark that we do not make any assumption on the position of the centres or on the values of their masses.
2) We point out that $\bar{R}$ will be explicitly chosen in a convenient way in the next section, see equation (2.12) and Proposition 2.2.1.
3) The solutions of type (ii)-(c) determine a particular type of *brake solutions.* A brake solution x is characterized by the existence of an instant $\bar{t}$ such that $\dot{x}(\bar{t}) = 0$; in such a situation, by the energy integral necessarily $x(\bar{t}) \in \{V(x) = -h\}$, that is, the solution is allowed to have zero velocity only on the boundary of the Hill region. Moreover, due to the reversibility of the motion equation with respect to the time involution $t \mapsto -t$, and the uniqueness theorem for the solutions of regular initial values problems, it results $x(\bar{t}+t) = x(\bar{t}-t)$, and for this reason we can say that any brake solution "bounces" against the boundary of the Hill region.
4) In some sense, the solutions of Theorem 2.1.2 can be related to the syzygies in the 3-body problem, see [64]. A solution of the N-body or of the N-centre problem suffers a *syzygy* when three bodies become collinear. In [64], the author proved that any bounded solution (that is, any solution such that the distance between bodies remains bounded by a positive constant for all time) of the 3-body problem with zero angular momentum and no triple collision suffers infinitely many syzygies. Analogously, any periodic solution given by Theorem 2.1.2 suffers infinitely many syzygies; even more, the prescription of a particular sequence of symbols $(P_{j_1}, \dots, P_{j_n})$ implies that the moving particle describes a corresponding sequence of syzygies, which can be distinguish by the ordered couple of centres which becomes collinear with the particle itself. A challenging issue at this point becomes the possibility of translating Theorem 2.1.2, and the forthcoming Corollary 2.1.7, in terms of the N-body problem with zero angular momentum.

Fixed ends problems. To prove Theorem 2.1.2, we use of a broken geodesics argument, finally leading to a finite dimensional reduction. The fact that the juxtaposition of geodesics defined in small interval in a convenient way can lead to a construction of a global geodesics has been introduced, in a completely different setting, by Seifert [76].

A key step consists in solving fixed ends problems having the desired topological characterization with respect to the centres. This leads to a constrained minimization for the Maupertuis functional (which we define in the sequel of the section), the main difficulty being the possible collisions with the centres.

At this point some remarks are in order. Collisions are the main obstruction in order to apply minimization arguments in singular problems of celestial mechanics, and in the last decades several arguments to insure that a minimizing path is collision-free have been developed. One of the most powerful tool can be ascribed to Marchal, see [23, 59], and it can be summarized as follows: let Q_1, Q_2 two different configurations in the N-body problem, and let $H(Q_1, Q_2, T)$ be the space of H^1-functions connecting Q_1 and

Q_2 in time T. The so called Marchal's lemma establishes that any action minimizer in $H(Q_1, Q_2, T)$ is collision-free (a version of such a result for the Maupertuis functional has been recently proved in [63], exploiting the relationship between the action functional and the Maupertuis one; we refer to Subsection 2.4.1 for more details); we briefly recall the sketch of the proof. Let $\bar{q}$ be a collision trajectory; let us construct a family of rigid variations moving one of the colliding masses away from the collision, parametrized over the sphere. Let us evaluate the action as the parameter varies on the sphere. Afterwards, we estimate the average of the perturbed action over all the perturbations, and show that this average is less than the action of $\bar{q}$, concluding that $\bar{q}$ cannot be a minimizer. This idea has been widely generalized in [40, 41] under some symmetry assumptions, where it has been employed to proof the existence of a plethora of periodic solutions.

On the other hand, in several practical situations one seeks minimizers of the action in a set of functions sharing a prescribed topological behaviour, see e.g. [9, 19, 21, 22, 42, 83]. In such a situations, Marchal's lemma cannot be employed because the average argument does not take into account any topological constraint, and this, usually, makes impossible to deduce any conclusive information. As a typical example, one can think at the hip-hop trajectories constructed in [83], where the authors introduced a method, adapted also in a different situation in [19], to prove that minimizers of the action functional are, under suitable topological constraints, collision-free. Here we give a further generalization, proving the following intermediate result of independent interest.

Theorem 2.1.4. *Let $\alpha \in [1, 2)$, $N \geq 3$, $c_1, \dots, c_N \in \mathbb{R}^2$, $m_1, \dots, m_N \in \mathbb{R}^+$. There exist $\bar{h} < 0$ and $\bar{R} > 0$ such that, for every $h \in (\bar{h}, 0)$, every pair of points $p_1, p_2 \in \partial B_R(0)$, and every partition $P_j \in \mathcal{P}$, there are two solutions $x_{P_j}(\cdot\,; p_1, p_2; h)$ and $x_2^{P_j}(\cdot\,; p_1, p_2; h)$ of* (2.2) *with energy h, both defined in a finite time-interval, starting from p_1 and arriving at p_2, and depending on P_j in the following way. One of this function, say x_{P_j}, minimizes globally the length associated to the Jacobi metric in the closure, with respect to the weak topology of H^1, of the set of paths separating the centres according to the partition P_j, while $x_2^{P_j}$ is a local minimizer in the same class. Moreover:*

- *(i) if $\alpha \in (1, 2)$, then x_{P_j} and $x_2^{P_j}$ are collision-free and self-intersection-free.*
- *(ii) if $\alpha = 1$, we have to distinguish among three cases:*
  - *a) assume $p_1 \neq p_2$; then x_{P_j} and $x_2^{P_j}$ are collision-free and self-intersection-free.*
  - *b) assume $p_1 = p_2$ and $P_j \in \mathcal{P} \setminus \mathcal{P}_1$; then x_{P_j} and $x_2^{P_j}$ are collision-free and self-intersection-free.*
  - *c) assume $p_1 = p_2$ and $P_j \in \mathcal{P}_1$; then x_{P_j} can be a collision-free and self-intersection-free solution, or can be an ejection-collision solution, with a unique collision against c_j. The same holds true for $x_2^{P_j}$.*

Whenever it is collision-free, x_{P_j} separates the centres according to the partition P_j. The same holds for $x_2^{P_j}$.
Moreover, if at least one between x_{P_j} and $x_2^{P_j}$ is collision-free, then they are different solutions. Otherwise they can coincide.

We remind the reader to Section 2.4 for the definitions of "Jacobi metric" and of "ejection-collision solution".

Symbolic dynamics. Let us consider a discrete and finite set S, with at least two elements. It is a metric space, when endowed with the trivial distance: $d_1(s_j, s_k) := \delta_{jk}$ $\forall s_j, s_k \in S$, where δ_{jk} is the Kronecker delta; consider the bi-infinite sequences of elements of S:

$$S^{\mathbb{Z}} := \{(s_m)_{m\in\mathbb{Z}} : s_m \in S \ \forall m\}.$$

It is a metric space, too, with respect to the distance

$$d((s_m), (t_m)) := \sum_{m\in\mathbb{Z}} \frac{1}{2^{|m|}} d_1(s_m, t_m), \qquad \forall (s_m), (t_m) \in S^{\mathbb{Z}}.$$

Of course, we can introduce a right shift letting

$$T_r((s_m)) := (s_{m+1}) \qquad \forall (s_m) \in S^{\mathbb{Z}}.$$

Definition 2.1.5. Let Σ be a metric space, $\sigma : \Sigma \to \Sigma$ a continuous map, S a finite set. We say that the dynamical system (Σ, σ) has a *symbolic dynamics with set of symbols* S if there exist a σ-invariant subset Π of Σ, and a continuous and surjective map $\pi : \Pi \to S^{\mathbb{Z}}$, such that the diagram

$$\begin{array}{ccc} \Pi & \xrightarrow{\sigma} & \Pi \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ S^{\mathbb{Z}} & \xrightarrow{T_r} & S^{\mathbb{Z}} \end{array}$$

commutes; that is, the restriction $\sigma|_{\Pi}$ is topologically semi-conjugate to the right shift in the metric space $(S^{\mathbb{Z}}, d)$.

Remark 2.1.6. The topological semi-conjugacy relates the topological properties of the first system with those of the second system; the definition of symbolic dynamics is particularly relevant since a lot of features of the discrete dynamical system $T_r : S^{\mathbb{Z}} \to S^{\mathbb{Z}}$ are known, e.g.

- it has a countable dense set of periodic points;

- there is high sensitivity with respect to initial conditions;
- there is positive topological entropy.

Therefore, the fact that a system has a symbolic dynamics reflects an extremely complicated behaviour of its trajectories. This is true even though, in general, the properties of the dynamical system T_r are not necessarily preserved by semi-conjugacy (because π is not necessarily invertible, and the inverse, if exists, could not be continuous). Usually, one proves the occurrence of symbolic dynamics and afterwards, exploiting this fact, shows that some of the other properties are fulfilled.

Coming back to the N-centre problem, let us rewrite equation (2.2) as a first order autonomous Hamiltonian system:

$$\begin{cases} \dot{x}(t) = v(t) \\ \dot{v}(t) = \nabla V(x(t)) \end{cases} \qquad \Longleftrightarrow \qquad \dot{z} = J \nabla_z h(z) \ , \tag{2.3}$$

where h is the Hamiltonian function, and

$$z = \begin{pmatrix} x \\ v \end{pmatrix}, \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}.$$

For the reader's convenience, we recall that $\mathcal{U}_h$ denotes the energy shell

$$\left\{ (x, v) \in \mathbb{R}^2 \setminus \{c_1, \dots, c_N\} : \frac{1}{2}|v|^2 - V(x) = h \right\}.$$

To each finite sequence of partitions in a set $\mathcal{P}^n$, we can associate a bi-infinite periodic sequence in $\mathcal{P}^{\mathbb{Z}}$. Starting from Theorem 2.1.2, it is possible to prove the following result.

Corollary 2.1.7. *Let $\bar{h}$ be introduced in Theorem 2.1.2, let $h \in (\bar{h}, 0)$. Then there exist a subset Π_h of the energy shell $\mathcal{U}_h$, a first return map $\mathfrak{R} : \Pi_h \to \Pi_h$, and a continuous and surjective map $\pi : \Pi_h \to \mathcal{P}^{\mathbb{Z}}$, such that the diagram*

$$\begin{array}{ccc} \Pi_h & \xrightarrow{\mathfrak{R}} & \Pi_h \\ \downarrow{\scriptstyle \pi} & & \downarrow{\scriptstyle \pi} \\ \mathcal{P}^{\mathbb{Z}} & \xrightarrow{T_r} & \mathcal{P}^{\mathbb{Z}}, \end{array}$$

commutes; namely, for every $h \in (\bar{h}, 0)$, the N-centre problem on the energy shell $\mathcal{U}^h$ has a symbolic dynamics, with set of symbols $\mathcal{P}$.

Some readers might rightly object to the admissibility of collisions in our symbolic dynamics. This objection concerns the Newtonian potential $\alpha = 1$, the most relevant for physical applications. However, with some small adjustment, the same construction leads to a collision-free symbolic dynamics. To be convinced, we focus our attention on those sequences that are not reflectionally symmetric with respect to one partition of the class $\mathcal{P}_1$ (in the sense specified by cases (ii)-(b) and (ii)-(c) of Theorem 2.1.2). With these sequences, semi-conjugacy with the right shift still holds. Now, to be rigorous, we have to distinguish between the cases $N = 3$ and $N \geq 4$. In the latter one the occurrence of symbolic dynamics is a simple consequence of the previous corollary, once we ruled out the partitions of $\mathcal{P}_1$ (the only ones allowing (possibly) collision solutions). Note that, for every $N \geq 4$, $\mathcal{P} \setminus \mathcal{P}_1$ has at least two elements. The case $N = 3$ is more delicate, since $\mathcal{P} = \mathcal{P}_1$. Hence we have to use a little trick: we take as symbols two blocks of 4 partitions (e.g. $P_1P_1P_2P_3$ and $P_2P_2P_3P_1$) in such a way that no composed sequence has the reflection symmetry which characterizes the collision trajectories, see Remark 2.6.4 (recall that P_j denotes the partition which isolates the centre c_j).

Corollary 2.1.8. *Let $\alpha = 1$, $N \geq 3$, $c_1, \ldots, c_N \in \mathbb{R}^2$ and $m_1, \ldots, m_N > 0$; let $\bar{h}$ be introduced in Theorem 2.1.2. For every $h \in (\bar{h}, 0)$, the N-centre problem on the energy shell $\mathcal{U}^h$ has a collision-free symbolic dynamics. To be precise:*

(i) if $N \geq 4$ the set of symbols is $\mathcal{P} \setminus \mathcal{P}_1$.

(ii) if $N = 3$ the set of symbols is $\{P_1P_1P_2P_3, P_2P_2P_3P_1\}$, and the semi-conjugacy with the right shift holds for the fourth return map $\mathfrak{R}^4$.

Strategy of the proofs. In Section 2.2, we show that it is equivalent to search for solutions of (2.2) having a prescribed energy $h < 0$, and to search for solutions of

$$\begin{cases} \ddot{y}(t) = -\displaystyle\sum_{j=1}^{N} \frac{m_j}{|y(t) - c'_j|^{\alpha+2}} \left(y(t) - c'_j\right) = \nabla V_\varepsilon(y(t)) \\ \\ \dfrac{1}{2}|\dot{y}(t)|^2 - V_\varepsilon(y(t)) = -1, \end{cases} \tag{2.4}$$

where

$$V_\varepsilon(y) = \sum_{k=1}^{N} \frac{m_k}{\alpha|y - c'_k|^\alpha} \quad \text{and} \quad c'_j = \varepsilon(h)c_j = (-h)^{\frac{1}{\alpha}}c_j.$$

This is a different N-centre problem, where the energy is normalized to -1 and $c_j \neq c'_j$. In this change of perspective, we see that the new centres stay inside a ball of radius $\varepsilon = \varepsilon(h)$, with $\varepsilon \to 0^+$ as $h \to 0^-$. This means that, if we choose $|h|$ sufficiently small, we can assume that the centres are arbitrarily close to the origin. In this way, given $\bar{\varepsilon} > 0$ and for any $\varepsilon \in (0, \bar{\varepsilon})$, outside a ball of radius $R > \bar{\varepsilon}$ problem (2.4) is a small perturbation of the Kepler problem with homogeneity degree $-\alpha < 0$ (we will call it "α-Kepler problem"). This consideration leads to the research of periodic solutions to (2.4) separating the investigation inside/outside a ball of radius $R > \bar{\varepsilon} > 0$.

In Section 2.3, we find arcs of solutions to (2.4) with image outside $B_R(0)$, connecting any pair of points $(p_0, p_1) \in (\partial B_R(0))^2$ such that the distance $|p_0 - p_1|$ is sufficiently small; this is done by means of perturbative techniques.

In Section 2.4 we study the dynamics inside the ball $B_R(0)$, finding solutions of (2.4) which connect any $p_1, p_2 \in \partial B_R(0)$; in this step, we use a variational approach based on the Maupertuis principle.

In Section 2.5, we collect the previous results, obtaining periodic solutions of (2.4) which pass alternatively outside and inside $B_R(0)$; to juxtapose the arcs of solutions found in the previous sections, we perform a convenient finite dimensional reduction.

Finally, using the results of Section 2.2.1, we obtain a periodic solution of the original problem with energy h.

Once we proved Theorem 2.1.2, we focus on the symbolic dynamics, proving Corollaries 2.1.7 and 2.1.8 in Section 2.6.

Further notation. We often identify a function u with its image $u([a,b]) \subset \mathbb{R}^2$, with some abuse of notation.

It is convenient to introduce the polar coordinates for a point $x \in \mathbb{R}^2$:

$$x = re^{i\theta}, \qquad r > 0 \text{ and } \theta \in \mathbb{R}.$$

The angle θ is counted in counterclockwise sense, and $\theta = 0$ if $x = (1,0)$. For every continuous function $x : I \subset \mathbb{R} \to \mathbb{R}^2 \setminus \{0\}$, there exist continuous functions $r : I \to \mathbb{R}^+$ and $\theta : I \to \mathbb{R}$ such that

$$x(t) = r(t)e^{i\theta(t)}.$$

Dealing with the angular momentum of a $\mathcal{C}^1$ function x, we write

$$\mathfrak{C}_x(t) := |x(t) \wedge \dot{x}(t)| = |r^2(t)\dot{\theta}(t)|$$

We use the notations $\|\cdot\|_{L^p([a,b])}$ for the $L^p\left([a,b], \mathbb{R}^2\right)$-norm and $\|\cdot\|_{H^1([a,b])}$ for the $H^1\left([a,b], \mathbb{R}^2\right)$-norm; when there will not be possibility of misunderstanding, we briefly write $\|\cdot\|_p$ or $\|\cdot\|$, respectively. The symbol $\rightharpoonup$ denotes the weak convergence in H^1.

2.2 Preliminaries

Let us fix $\alpha \in [1, 2)$, $N \geq 3$, $c_1, \ldots, c_N \in \mathbb{R}^2$, $m_1, \ldots, m_N > 0$, and fix the origin in the center of mass. Here and in what follows $M := \sum_{j=1}^N m_j$. In this section we prove that solving (2.2) with energy $h<0$ is equivalent to solving a rescaled N-centre problem on the energy level -1. In this perspective the quadratic mean of the centres will replace the energy as a parameter. To be precise, we state the following elementary result.

Proposition 2.2.1. *Let $x \in \mathcal{C}^2((a,b))$ be a classical solution of (2.2) with energy $h<0$. Then the function*

$$y(t) = (-h)^{\frac{1}{\alpha}}\, x\left((-h)^{-\frac{\alpha+2}{2\alpha}}\, t\right), \qquad t \in \left((-h)^{\frac{\alpha+2}{2\alpha}}\, a, (-h)^{\frac{\alpha+2}{2\alpha}}\, b\right) \tag{2.5}$$

is a solution of energy -1 of a N-centre problem with centres

$$c_j' = (-h)^{\frac{1}{\alpha}}\, c_j, \qquad j=1,\dots,N.$$

The converse holds true: let $y \in \mathcal{C}^2((a',b'))$ be a classical solution of energy -1 of a N-centres problem, with centres c_j'. Let us set

$$c_j = (-h)^{-\frac{1}{\alpha}}\, c_j', \qquad j=1,\dots,N.$$

Then

$$x(t) = (-h)^{-\frac{1}{\alpha}}\, y\left((-h)^{\frac{\alpha+2}{2\alpha}}\, t\right), \qquad t \in \left((-h)^{-\frac{\alpha+2}{2\alpha}}\, a', (-h)^{-\frac{\alpha+2}{2\alpha}}\, b'\right)$$

is a classical solution of (2.2) with energy $h<0$.

Proof. We show how it is possible to infer the form of y arguing by homogeneity. Let $x=x(t)$ be a solution of (2.2) on $(a,b) \subset \mathbb{R}$ with energy h. We define

$$x_{\lambda,\beta}(t) := \lambda^{\beta} x(\lambda t), \qquad \text{where } \lambda \text{ and } \beta \text{ will be determined in what follows.}$$

The function $x_{\lambda,\beta}(\cdot)$, defined on $(a/\lambda, b/\lambda)$, satisfies

$$\begin{aligned}\ddot{x}_{\lambda,\beta}(t) &= -\lambda^{\beta+2} \sum_{j=1}^{N} \frac{m_j}{|x(\lambda t)-c_j|^{\alpha+2}}\,(x(\lambda t)-c_j) \\ &= -\lambda^{(\alpha+2)\beta+2} \sum_{j=1}^{N} \frac{m_j}{|x_{\lambda,\beta}(t)-\lambda^{\beta}c_j|^{\alpha+2}}\left(x_{\lambda,\beta}(t)-\lambda^{\beta}c_j\right).\end{aligned}$$

We fix

$$\beta(\alpha) := -\frac{2}{\alpha+2} < 0,$$

so that

$$\ddot{x}_{\lambda,\beta(\alpha)}(t) = -\sum_{j=1}^{N} \frac{m_j}{\left|x_{\lambda,\beta(\alpha)}(t)-\lambda^{-\frac{2}{\alpha+2}}c_j\right|^{\alpha+2}}\left(x_{\lambda,\beta(\alpha)}(t)-\lambda^{-\frac{2}{\alpha+2}}c_j\right);$$

namely, with this choice of β we get a new N-centres problem with centres $c'_j := \lambda^{-\frac{2}{\alpha+2}} c_j$. This implies the conservation of the energy

$$h' = \frac{1}{2}|\dot{x}_{\lambda,\beta(\alpha)}(t)|^2 - \sum_{j=1}^{N} \frac{m_j}{\alpha|x_{\lambda,\beta(\alpha)}(t) - \lambda^{-\frac{2}{\alpha+2}} c_j|^\alpha} = \lambda^{\frac{2\alpha}{\alpha+2}} h.$$

With the choice

$$\lambda(h) := \left(-\frac{1}{h}\right)^{\frac{\alpha+2}{2\alpha}} \quad \Longrightarrow \quad h' = -1$$

the function $y(t) := x_{\lambda(h),\beta(\alpha)}$ is the solution of the transformed problem with energy -1.

The same arguments apply for the converse. □

From now on we will refer to the problem "to find a periodic solutions of equation (2.2) with energy h" as to *the original problem*. The spatial variable for the original problem will be denoted by x. On the other hand, we will refer to the problem "to find periodic solutions of system (2.4)" as to *the transformed* (or *normalized*) *problem*; the spatial variable in this case will be denoted by y.

Corollary 2.2.2. *For every $\varepsilon > 0$ there exists $\zeta(\varepsilon) > 0$ such that if $h = -\zeta(\varepsilon)$, then*

$$\max_{1 \le j \le N} |c'_j| = \varepsilon.$$

The function $\varepsilon \mapsto -\zeta(\varepsilon)$ is strictly decreasing in ε.

Proof. Given $\varepsilon > 0$ we find

$$\zeta(\varepsilon) = \left(\frac{\varepsilon}{\max_{1 \le j \le N} |c_j|}\right)^\alpha. \qquad \square$$

Remark 2.2.3. Of course, periodic solutions of problem (2.4) for every $\varepsilon \in (0, \bar{\varepsilon})$ corresponds, via Proposition 2.2.1 and Corollary 2.2.2, to periodic solutions of (2.2) of energy $h = -\zeta(\varepsilon)$ for every $h \in (-\zeta(\bar{\varepsilon}), 0)$. Two corresponding solutions exhibit the same topological behaviour, as showed by equation (2.5).

As already observed, if ε is chosen sufficiently small, outside a ball of radius $R > \varepsilon > 0$ we can consider the new problem as a small perturbation of the α-Kepler problem, whose potential is

$$V_0(y) := \frac{M}{\alpha|y|^\alpha} \qquad y \in \mathbb{R}^2 \setminus \{0\}.$$

We aim at proving that

$$\|V_\varepsilon - V_0\|_{\mathcal{C}^1(B_R(0)^c)} = o(\varepsilon) \qquad \text{for } \varepsilon \to 0^+. \tag{2.6}$$

Note that for $y \in \mathbb{R}^2 \setminus B_R(0)$ and $\varepsilon < R$ the function V_ε is smooth in y, and is smooth also as function of c'_k. If $\varepsilon \to 0^+$, then the centres c'_k collapse in the origin, therefore we perform a Taylor expansion in a neighbourhood of the point $(c'_1, \dots, c'_k) = (0, \dots, 0)$. It results

$$V_\varepsilon(y) = \frac{M}{\alpha |y|^\alpha} + \sum_{k=1}^{N} \frac{m_k}{|y|^{\alpha+2}} \langle y, c'_k \rangle + o(|c'_k|) = \frac{M}{\alpha |y|^\alpha} + o(\varepsilon)$$

uniformly in $y \in \mathbb{R}^2 \setminus B_R(0)$, where the first order term vanishes because we fixed the centre of mass in the origin: $\sum_{k=1}^N m_k c'_k = 0$. The previous equation implies

$$\|V_\varepsilon - V_0\|_{\mathcal{C}^0(B_R(0)^c)} = o(\varepsilon) \qquad \text{for } \varepsilon \to 0^+. \tag{2.7}$$

Using again the position of the centre of mass we have also

$$\begin{aligned} \nabla V_\varepsilon(y) &= -\sum_{k=1}^{N} \frac{m_k}{|y - c_k|^{\alpha+2}} (y - c_k) \\ &= -\frac{M}{|y|^{\alpha+2}} y - (\alpha+2) \sum_{k=1}^{N} \frac{m_k}{|y|^{\alpha+4}} \langle y, c'_k \rangle y + \sum_{k=1}^{N} \frac{m_k c'_k}{|y|^{\alpha+2}} + o(|c'_k|) \\ &= -\frac{M}{|y|^{\alpha+2}} y + o(\varepsilon) \end{aligned}$$

uniformly in $y \in \mathbb{R}^2 \setminus B_R(0)$, so that

$$\|\nabla V_\varepsilon - \nabla V_0\|_{\mathcal{C}^0(B_R(0)^c)} = o(\varepsilon) \qquad \text{for } \varepsilon \to 0^+. \tag{2.8}$$

Collecting together (2.7) and (2.8), we obtain the (2.6). We will denote by W_ε the perturbation term:

$$W_\varepsilon(y) := V_\varepsilon(y) - V_0(y), \qquad y \in \mathbb{R}^2.$$

Remark 2.2.4. If y is a solution of $\ddot{y} = \nabla V_\varepsilon(y)$ with energy -1 over an interval $I \subset \mathbb{R}$, it holds

$$V_\varepsilon(y(t)) \geq 1 \qquad \forall t \in I,$$

so that to exploit the previous argument we have to check that, for every $\varepsilon > 0$ sufficiently small, there exists $R > 0$ such that

$$B_\varepsilon(0) \subset B_R(0) \subset \left\{ y \in \mathbb{R}^2 : V_\varepsilon(y) \geq 1 \right\}. \tag{2.9}$$

Proposition 2.2.5. *Let $\varepsilon > 0$. Let $R > 0$ such that*

$$\varepsilon < R < \left(\frac{M}{\alpha} \right)^{\frac{1}{\alpha}} - \varepsilon.$$

Then (2.9) holds true. There exists $\varepsilon_1 > 0$ such that, for every $0 < \varepsilon < \varepsilon_1$, this choice is possible.

Proof. Under our assumptions on R for every $y \in B_R(0)$ and for every j

$$|y - c_j'| \leq R + \varepsilon < \left(\frac{M}{\alpha}\right)^{\frac{1}{\alpha}} - \varepsilon \quad \Longrightarrow \quad V_\varepsilon(y) > \frac{M}{\alpha\left(\left(\frac{M}{\alpha}\right)^{\frac{1}{\alpha}}\right)^{\alpha}} = 1.$$

There exists $\varepsilon_1 > 0$ such that

$$0 < \varepsilon < \varepsilon_1 \quad \Longrightarrow \quad \left(\frac{M}{\alpha}\right)^{\frac{1}{\alpha}} - \varepsilon > \varepsilon. \qquad \square$$

Remark 2.2.6. Actually, we make the further request

$$\varepsilon < \frac{R}{2} < R < \left(\frac{M}{\alpha}\right)^{\frac{1}{\alpha}} - \varepsilon,$$

which is satisfied for every $\varepsilon \in (0, \varepsilon_1/2)$.

For reasons which appear clear in Section 2.4, *it is convenient to choose R such that $\partial B_R(0)$ is the image of the circular solution of the α-Kepler problem with energy* -1: hence, R has to be chosen so that $y(t) = R\exp\{i\omega t\}$ is a solution to

$$\ddot{y}(t) = -M\frac{y(t)}{|y(t)|^{\alpha+2}} \quad \Longleftrightarrow \quad R\omega^2 = \frac{M}{R^{\alpha+1}}. \tag{2.10}$$

Moreover, the conservation of the angular momentum $\mathfrak{C}_y(t) = |y(t) \wedge \dot{y}(t)|$ gives

$$R^2\omega = R\sqrt{2\left(-1 + \frac{M}{\alpha R^\alpha}\right)}. \tag{2.11}$$

Collecting (2.10) and (2.11), we obtain

$$R := \left(\frac{(2-\alpha)M}{2\alpha}\right)^{\frac{1}{\alpha}}. \tag{2.12}$$

This is consistent with the previous restriction on R, if ε_1 is sufficiently small (if this was not true, it is sufficient to replace ε_1 with a smaller quantity).
We end this remark pointing out that there exists $C > 0$ such that

$$V_\varepsilon(y) - 1 \geq C \qquad \forall y \in \overline{B_R(0)}, \tag{2.13}$$

see the proof of Proposition 2.2.5.

2.3 Outer dynamics

We are going to use a perturbation argument in order to find particular solutions of problem (2.4) lying in $\mathbb{R}^2 \setminus B_R(0)$, connecting pairs of neighbouring points of $\partial B_R(0)$ with a *close to brake* arc. Recall that a brake solution y is characterized by the existence of an instant $\bar{t}$ in its time interval of definition such that $\dot{y}(\bar{t}) = 0$; due to the reversibility of the motion equation with respect to the time involution $t \mapsto -t$, and the uniqueness theorem for the solutions of regular initial values problems, we remark that for a brake solution

$$y(\bar{t} + t) = y(\bar{t} - t).$$

Proposition 2.3.1. *There exist $\delta > 0$ and $\varepsilon_2 > 0$ such that for every $\varepsilon \in (0, \varepsilon_2)$, for every $p_0, p_1 \in \partial B_R(0) : |p_1 - p_0| < 2\delta$, there exist $T > 0$ and a unique solution $y_{ext}(\cdot\,; p_0, p_1; \varepsilon)$ of (2.4) such that $|y(t)| > R$ for $t \in (0, T)$, and $y(0) = p_0$, $y(T) = p_1$. Moreover, y depends in a $\mathcal{C}^1$ way on the endpoints p_0 and p_1.*

The proof requires some preliminary results. We start from the analysis of the unperturbed problem

$$\begin{cases} \ddot{y}(t) = -M \frac{y(t)}{|y(t)|^{\alpha+2}} & t \in [0, T], \\ \frac{1}{2}|\dot{y}(t)|^2 - \frac{M}{\alpha |y(t)|^{\alpha}} = -1 & t \in [0, T], \\ |y(t)| > R & t \in (0, T). \end{cases} \tag{2.14}$$

Let us solve the Cauchy problem

$$\begin{cases} \ddot{y}(t) = -M \frac{y(t)}{|y(t)|^{\alpha+2}} \\ y(0) = p_0 = R \exp\{i\theta_0\}, \quad \dot{y}(0) = \sqrt{2\left(-1 + \frac{M}{\alpha R^{\alpha}}\right)} \left(\frac{p_0}{R}\right). \end{cases}$$

The solution returns to the point p_0 after a certain time $\bar{T} > 0$, having swept the portion of the rectilinear brake orbit starting from p_0 and lying in $\mathbb{R}^2 \setminus B_R(0)$. Our aim is to catch the behaviour of the solutions under small variations of the boundary conditions. Hence, we consider

$$\begin{cases} \ddot{y}(t) = -M \frac{y(t)}{|y(t)|^{\alpha+2}} \\ y(0) = p_0, \qquad \dot{y}(0) = \dot{r}_0 e^{i\theta_0} + R\dot{\theta}_0 i e^{i\theta_0}, \end{cases} \tag{2.15}$$

where $\dot{r}_0$ is assigned as function of $\dot{\theta}_0$ by means of the energy integral:

$$\dot{r}_0 = \dot{r}_0(\dot{\theta}_0) = \sqrt{2\left(\frac{M}{\alpha R^{\alpha}} - 1\right) - R^2 \dot{\theta}_0^2}.$$

We denote as $y(\cdot\,;\theta_0,\dot\theta_0)$ the solution of (2.15). For the brake orbit $y\,(\cdot\,;\theta_0,0)$, it results

$$\theta(t;\theta_0,0)\equiv\theta_0 \qquad \forall t\in[0,\bar T].$$

Let us fix $p_0\in\partial B_R(0)$. We define

$$\begin{aligned}\psi :&\Theta\times I\to\mathbb{R}^2\\ &(\dot\theta_0,T)\mapsto y(T;\theta_0,\dot\theta_0),\end{aligned}$$

where $\Theta\times I\subset S^1\times\mathbb{R}$ is a neighbourhood of $(0,\bar T)$ on which ψ is well defined (such a neighbourhood exists). We can assume

$$\max\left\{\sup_{(\dot\theta_0,T)\in\Theta\times I}4|T\dot\theta_0|,\ \sup_{(\dot\theta_0,T)\in\Theta\times I}\left|\left(\frac{\alpha}{M}\right)^{\frac{2}{\alpha}}R^2T\dot\theta_0\right|\right\}<\frac{\pi}{2}, \tag{2.16}$$

otherwise it is sufficient to replace $\Theta\times I$ with a smaller neighbourhood.

Lemma 2.3.2. *The Jacobian of ψ in $(0,\bar T)$ is invertible.*

Proof. Since the α-Kepler problem is invariant under rotations, it isn't restrictive to suppose $\theta_0=\pi/2$, so that $\exp\{i\theta_0\}=(0,1)=:e_2$. The function $\psi\in\mathcal{C}^1(\Theta\times I)$ satisfies

$$\frac{\partial\psi}{\partial T}\left(0,\bar T\right)=\dot y(\bar T;\theta_0,0)=-\sqrt{2\left(\frac{M}{\alpha R^\alpha}-1\right)}e_2.$$

Hence the Jacobian matrix of ψ is invertible in $(0,\bar T)$ if

$$\left\langle\frac{\partial\psi}{\partial\dot\theta_0}(0,\bar T),e_1\right\rangle\neq 0,$$

where $e_1:=(1,0)$. By the continuous dependence with respect to initial data we have, for every $(\dot\theta_0,T)\in\Theta\times I$, and for every $t\in[0,T]$,

$$r(t;\theta_0,\dot\theta_0)\geq\frac{R}{2}. \tag{2.17}$$

We use the conservation of the angular momentum: for every $t\in[0,T]$ there holds

$$\mathfrak{C}_y:=\mathfrak{C}_y(t)=\mathfrak{C}_y(0)\quad\Longleftrightarrow\quad\mathfrak{C}_y=r^2(t)|\dot\theta(t)|=R^2|\dot\theta_0|.$$

Assume $\dot\theta_0>0$; one has

$$\theta(t;\theta_0,\dot\theta_0)=\frac{\pi}{2}+\int_0^t\frac{d\theta}{ds}(s)\,ds=\frac{\pi}{2}+\int_0^t\frac{R^2\dot\theta_0}{r^2(s)}\,ds.$$

If $(\dot\theta_0, T) \in \Theta \times I$, from (2.16), (2.17), and the fact that $r(s) \le (M/\alpha)^{1/\alpha}$, it follows

$$\frac{\pi}{2} < \left(\frac{\alpha}{M}\right)^{\frac{2}{\alpha}} R^2 T\dot\theta_0 + \frac{\pi}{2} \le \theta(T;\theta_0,\dot\theta_0) \le 4T\dot\theta_0 + \frac{\pi}{2} < \pi.$$

The function $\xi \mapsto \cos\xi$ being decreasing over $(\pi/2, \pi)$, we obtain

$$\left\langle \frac{\psi(\dot\theta_0,\bar T) - \psi(0,\bar T)}{\dot\theta_0}, e_1 \right\rangle = \frac{r(\bar T;\theta_0,\dot\theta_0)\cos\Big(\theta(\bar T;\theta_0,\dot\theta_0)\Big)}{\dot\theta_0}$$
$$\le \frac{r(\bar T;\theta_0,\dot\theta_0)\cos\Big(\left(\frac{\alpha}{M}\right)^{\frac{2}{\alpha}} R^2 T\dot\theta_0 + \frac{\pi}{2}\Big)}{\dot\theta_0} = -r(\bar T;\theta_0,\dot\theta_0)\left(\frac{\alpha}{M}\right)^{\frac{2}{\alpha}} R^2 T + o(\dot\theta_0) < 0,$$

for $\dot\theta_0 \to 0$. Passing to the limit for $\dot\theta_0 \to 0$ the strict inequality is preserved. Since the same argument works for $\dot\theta_0 < 0$, the thesis follows. □

The previous discussion has to be refined in order to include the variations of the potential due to the presence of the centres, which are now included in the ε-disk. Recall that we fixed $p_0 \in \partial B_R(0)$. We know that

$$\lim_{\varepsilon\to 0^+} V_\varepsilon(y) = \frac{M}{\alpha|y|^\alpha} \qquad \text{uniformly in } y \in \mathbb{R}^2 \setminus B_R(0).$$

So we define

$$\begin{aligned}\Psi : &\Theta \times I \times \left[0, \frac{\varepsilon_1}{2}\right) \times \partial B_R(0) \to \mathbb{R}^2\\ &(\dot\theta_0, T, \varepsilon, p_1) \mapsto y(T;\theta_0,\dot\theta_0;\varepsilon) - p_1,\end{aligned}$$

where $y(\cdot\,;\theta_0,\dot\theta_0;\varepsilon)$ is the solution of

$$\begin{cases} \ddot y(t) = \nabla V_\varepsilon(y(t)) \\ y(0) = p_0, \qquad \dot y(0) = \dot r_\varepsilon e^{i\theta_0} + R\dot\theta_0 i e^{i\theta_0}, \end{cases} \tag{2.18}$$

and

$$\dot r_\varepsilon = \dot r_\varepsilon(\dot\theta_0;\varepsilon) = \sqrt{2\left(V_\varepsilon(p_0) - 1\right) - R^2\dot\theta_0^2}.$$

Lemma 2.3.3. *There exist $\delta > 0$ and $0 < \varepsilon_2 < \varepsilon_1/2$ such that for every $\varepsilon \in (0, \varepsilon_2)$, for every $p_1 \in \partial B_R(0) : |p_1 - p_0| < 2\delta$, there exists a unique solution $y(\cdot;\theta_0,\dot\theta_0;\varepsilon)$ of (2.18) defined in $[0,T]$ for a certain T, and satisfying*

$$\frac{1}{2}|\dot y(t;\theta_0,\dot\theta_0;\varepsilon)|^2 - V_\varepsilon(y(t;\theta_0,\dot\theta_0;\varepsilon)) = -1 \qquad t \in [0,T],$$
$$|y(t;\theta_0,\dot\theta_0;\varepsilon)| > R \qquad t \in (0,T), \qquad y(T;\theta_0,\dot\theta_0;\varepsilon) = p_1.$$

Moreover, it is possible to choose δ and ε_2 independent of $p_0 \in \partial B_R(0)$.

Proof. We apply the implicit function theorem to the function Ψ, which is $\mathcal{C}^1$ in the variables $\dot\theta_0$ and T for the differentiable dependence of the solutions by time and initial data. Since $y(\cdot\,;\theta_0,0;0)$ is the solution of (2.14), it holds

$$\Psi(0,\bar T,0,p_0)=0,\quad \frac{\partial\Psi}{\partial\dot\theta_0}\left(0,\bar T,0,p_0\right)=\frac{\partial\psi}{\partial\dot\theta_0}\left(0,\bar T\right),\quad \frac{\partial\Psi}{\partial T}\left(0,\bar T,0,p_0\right)=\frac{\partial\psi}{\partial T}\left(0,\bar T\right),$$

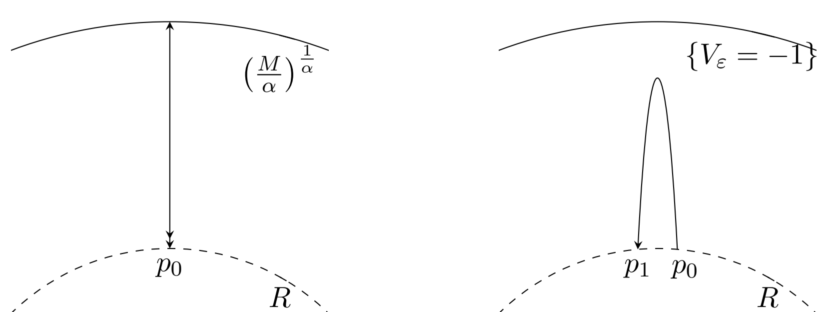
so that from Lemma 2.3.2 we deduce that the Jacobian matrix of Ψ with respect to $(\dot\theta_0,T)$ is invertible; hence the assumptions of the implicit function theorem are satisfied, and we can find a neighbourhood $\Theta'\times J\subset\Theta\times I$ of $(0,\bar T)$, a neighbourhood $[0,\varepsilon_2)\times B_{2\delta}(p_0)\subset[0,\varepsilon_1/2)\times\mathbb{R}^2$ of $(0,p_0)$, and a unique function $\eta:[0,\varepsilon_2)\times B_{2\delta}(p_0)\to\Theta'\times J$ such that

$$\begin{aligned}
&1)\eta\left(0,p_0\right)=(0,\bar T),\\
&2)\Psi\left(\eta_1(\varepsilon,p_1),\eta_2(\varepsilon,p_1),\varepsilon,p_1\right)=0\quad\text{for every }(\varepsilon,p_1)\in[0,\varepsilon_2)\times B_{2\delta}(p_0),\\
&3)\Psi(\dot\theta_0,T,\varepsilon,p_1)=0\quad\text{with }(\dot\theta_0,T,\varepsilon,p_1)\in\Theta'\times J\times[0,\varepsilon_2)\times B_{2\delta}(p_0)\\
&\quad\Longrightarrow\quad(\dot\theta_0,T)=\eta(\varepsilon,p_1).
\end{aligned}$$

This means that, if we fix $\varepsilon\in(0,\varepsilon_2)$, for every $p_1\in\partial B_R(0)\cap B_{2\delta}(p_0)$ we can find a solution $y(\cdot\,;\theta_0,\dot\theta_0;\varepsilon)$ of (2.18). This solution has constant energy -1 because of the definition of $\dot r_\varepsilon$; moreover, $y(T;\theta_0,\dot\theta_0;\varepsilon)=p_1$. We remark that outside $B_R(0)$ the potential V_ε is a small perturbation of the α-Kepler one, so that $|y(t;\theta_0,\dot\theta_0;\varepsilon)|>R$ for every $t\in(0,T)$. It remains to prove that one can choose δ and ε_2 independent on p_0. This is a consequence of the proof of the implicit function theorem: the wideness of the neighbourhood of $(0,p_0)$ in $[0,\varepsilon_1/2)\times\mathbb{R}^2$ in which we can guarantee the definition of the implicit function depends on the norm of

$$\left(J_{(\dot\theta_0,T)}\Psi(0,\bar T,0,p_0)\right)^{-1},$$

and for every $p_0\in\partial B_R(0)$ this matrix is the same, up to rotations. □



The picture represents the portion of the rectilinear brake solution for the α-Kepler problem in comparison with a "perturbed" solution obtained for the potential V_ε via the implicit function theorem.

Proposition 2.3.1 is a straightforward consequence of this lemma. The solutions obtained are uniquely determined and depend in a smooth way on the ends p_0 and p_1.

In the next sections, we adopt the following notation: for any $\varepsilon \in (0, \varepsilon_2)$ and $p_0, p_1 \in \partial B_R(0)$ such that $|p_1 - p_0| < \delta$, the outer solution of (2.4), found in Proposition 2.3.1, will be denoted by $y_{\text{ext}}(\cdot\,; p_0, p_1; \varepsilon)$; it is uniquely determined $T_{\text{ext}}(p_0, p_1; \varepsilon) > 0$ such that

$$y_{\text{ext}}\left(T_{\text{ext}}(p_0, p_1; \varepsilon); p_0, p_1; \varepsilon\right) = p_1.$$

Lemma 2.3.4. *For any $\varepsilon \in (0, \varepsilon_2)$, there exist $C_1, C_2 > 0$ such that*

$$C_1 \leq T_{ext}(p_0, p_1; \varepsilon) \leq C_2$$

for every $(p_0, p_1) \in (\partial B_R(0))^2$ such that $|p_0 - p_1| < \delta$.

Proof. It is a straightforward consequence of the continuous dependence of the solutions on initial data and of the construction of $y_{\text{ext}}(\cdot\,; p_0, p_1; \varepsilon)$ as a perturbed solution. □

2.4 Inner dynamics

In this section we are going to seek arcs of solutions of (2.4) connecting two points $p_1, p_2 \in \partial B_R(0)$ and lying inside the disk $B_R(0)$. We admit the case $p_1 = p_2$. Close to the center of the ball, the potential V_ε cannot be seen as a small perturbation of the α-Kepler one, so that we are lead to use variational methods rather than perturbative techniques. The first step is to introduce a suitable functional whose critical points are weak solutions of (2.4); this is the object of Subsection 2.4.1. Our trajectories will be local minimizers of the Maupertuis functional or, equivalently, of the Jacobi length. In Subsection 2.4.2 we rigorously define the functional setting, determining weakly closed sets in which we search for minimizers of the Maupertuis functional, and we state the main theorem of the section. The proof of such a theorem is given in Subsections 2.4.3 and 2.4.4; in the first one we show that the direct method of the calculus of variations applies to provide weak solutions of (2.4), while in the latter one we discuss the possibility that a weak solution has some collisions.

In what follows we will consider $\varepsilon \in (0, \varepsilon_1/2)$ fixed, and we will write c_j instead of

c_j' to ease the notation. We seek solutions of

$$\begin{cases} \ddot{y}(t) = \nabla V_\varepsilon(y(t)) & t \in [0,T], \\ \frac{1}{2}|\dot{y}(t)|^2 - V_\varepsilon(y(t)) = -1 & t \in [0,T], \\ |y(t)| < R & t \in (0,T), \\ y(0) = p_1, \quad y(T) = p_2, \end{cases} \tag{2.19}$$

with $p_1, p_2 \in \partial B_R(0)$, and $T > 0$ to be determined.

2.4.1 The Maupertuis principle

Dealing with a singular potential, we introduce the spaces of non-collision H^1 paths

$$\widehat{H}_{p_1 p_2}([a,b]) := \left\{ u \in H^1\left([a,b],\mathbb{R}^2\right) \;\middle|\; \begin{array}{l} u(a) = p_1,\ u(b) = p_2, \\ u(t) \neq c_j\ \forall t \in [a,b],\ \forall j \end{array} \right\},$$

the set of collision H^1 paths

$$\mathfrak{Coll}_{p_1 p_2}([a,b]) := \left\{ u \in H^1\left([a,b],\mathbb{R}^2\right) \;\middle|\; \begin{array}{l} u(a) = p_1,\ u(b) = p_2,\ u(t) = c_j \\ \text{for some } t \in [a,b] \text{ and } j \in \{1,\dots,N\} \end{array} \right\},$$

and their union

$$\begin{aligned} H_{p_1 p_2}([a,b]) :&= \widehat{H}_{p_1 p_2}([a,b]) \cup \mathfrak{Coll}_{p_1 p_2}([a,b]) \\ &= \left\{ u \in H^1\left([a,b],\mathbb{R}^2\right) : u(a) = p_1,\ u(b) = p_2 \right\}. \end{aligned}$$

We write $\widehat{H}$, $\mathfrak{Coll}$ and H instead of $\widehat{H}_{p_1 p_2}([a,b])$, $\mathfrak{Coll}_{p_1 p_2}([a,b])$ and $H_{p_1 p_2}([a,b])$ when there is not possibility of misunderstanding, to simplify the notation. Since the weak H^1 convergence implies the uniform one, it is immediate to check that $H_{p_1 p_2}([a,b])$ is the closure of $\widehat{H}_{p_1 p_2}([a,b])$ in the weak topology of H^1.
Let us define the *Maupertuis functional* $M_h([a,b];\cdot) : H_{p_1 p_2}([a,b]) \to \mathbb{R} \cup \{+\infty\}$ as

$$M_h([a,b];u) := \frac{1}{2}\int_a^b |\dot{u}(t)|^2\, dt \int_a^b \left(V(u(t)) + h\right)\, dt.$$

We often write M_h instead of $M_h([a,b];\cdot)$. If $M_h([a,b];u) > 0$, both its factors are strictly positive and it makes sense to set

$$\omega^2 := \frac{\int_a^b (V(u) + h)}{\frac{1}{2}\int_a^b |\dot{u}|^2}. \tag{2.20}$$

The Maupertuis functional is differentiable over $\widehat{H}$ (seen as an affine space on $H_0^1(a,b)$), and its critical points, suitably re-parametrized, are solutions to our fixed energy problem (we report the proof of the following known fact for the sake of completeness, see also Theorem 4.1 of [3]).

Theorem 2.4.1. *Let* $u \in \widehat{H}_{p_1 p_2}([a,b])$ *be a critical point of* M_h *at a positive level, i.e.*

$$dM_h([a,b];u)[v] = 0 \quad \forall v \in H_0^1\left([a,b],\mathbb{R}^2\right), \qquad \text{and} \qquad M_h([a,b];u) > 0,$$

and let ω *be given by* (2.20)*. Then* $x(t) := u(\omega t)$ *is a classical solution of*

$$\begin{cases} \ddot{x}(t) = \nabla V(x(t)) & t \in \left[\frac{a}{\omega}, \frac{b}{\omega}\right], \\ \frac{1}{2}|\dot{x}(t)|^2 - V(x(t)) = h & t \in \left[\frac{a}{\omega}, \frac{b}{\omega}\right], \\ x\left(\frac{a}{\omega}\right) = p_1, \qquad x\left(\frac{b}{\omega}\right) = p_2, \end{cases} \tag{2.21}$$

while u *itself is a classical solution of*

$$\begin{cases} \omega^2 \ddot{u}(t) = \nabla V(u(t)) & t \in [a,b], \\ \frac{1}{2}|\dot{u}(t)|^2 - \frac{V(u(t))}{\omega^2} = \frac{h}{\omega^2} & t \in [a,b], \\ u(a) = p_1, \qquad u(b) = p_2. \end{cases} \tag{2.22}$$

Proof. Since $dM_h(u) = 0$ we have

$$\int_a^b \langle \dot{u}, \dot{v} \rangle \int_a^b (V(u) + h) + \frac{1}{2}\int_a^b |\dot{u}|^2 \int_a^b \langle \nabla V(u), v \rangle = 0 \qquad \forall v \in H_0^1([a,b]).$$

Since $M_h(u) > 0$ this is equivalent to

$$\omega^2 \int_a^b \langle \dot{u}, \dot{v} \rangle + \int_a^b \langle \nabla V(u), v \rangle = 0 \qquad \forall v \in H_0^1([a,b]),$$

namely u is a (weak, and by regularity strong) solution of

$$\omega^2 \ddot{u} = \nabla V(u). \tag{2.23}$$

As a consequence, $x(t) = u(\omega t)$ solves $\ddot{x} = \nabla V(x)$ in $[a/\omega, b/\omega]$; furthermore, from equation (2.23) we deduce the existence of $k \in \mathbb{R}$ such that

$$\frac{\omega^2}{2}|\dot{u}(t)|^2 = V(u(t)) + k \quad \Longleftrightarrow \quad \frac{1}{2}|\dot{x}(s)|^2 = V(x(s)) + k,$$

respectively for every $t \in [a,b]$ and for every $s \in [a/\omega, b/\omega]$. Integrating the first equation in $[a,b]$ and comparing with (2.20), we obtain $k = h$. □

Remark 2.4.2. The converse of Theorem 2.4.1 is also true: if $x \in \mathcal{C}^2((a',b'))$ is a collisions-free solution of (2.21), setting $\omega = 1/(b'-a')$ and $u(t) := x(t/\omega)$, u is a classical solution of (2.22) defined in $[a'/(b'-a'), b'/(b'-a')] =: [a,b]$ and hence a critical point of $M_h([a,b];\cdot)$ at a strictly positive level. Also, the identity

$$\omega^2 = \frac{\int_a^b (V(u) + h)}{\int_a^b |\dot{u}|^2}$$

is fulfilled.

In order to use variational methods it is worth working in H rather than in $\widehat{H}$, because $\widehat{H}$ is not weakly closed. The disadvantage is that we will need some *ad hoc* argument to rule out the occurrence of collisions in order to apply Theorem 2.4.1 and to obtain a classical solution of the motion equation. Nevertheless, although collision minimizers are not true critical points of the Maupertuis functional in H, the following result allows to recover the conservation of the energy.

Lemma 2.4.3. *If $u \in H$ is a local minimizer of M_h at a strictly positive level, then*

$$\frac{1}{2}|\dot{u}(t)|^2 - \frac{V(u(t))}{\omega^2} = \frac{h}{\omega^2} \qquad a.e.\ t \in [a,b].$$

Remark 2.4.4. The lemma says that the energy is constant almost everywhere even if u has collisions. Of course, in this case u could be not of class $\mathcal{C}^1$.

Proof. It is a classical result and it is a consequence of the extremality of u with respect to time re-parametrization keeping the ends fixed: if $\varphi \in \mathcal{C}_c^\infty((a,b),\mathbb{R})$, setting $u_\lambda(t) := u(t+\lambda\varphi(t))$, it holds

$$\frac{d}{d\lambda}M_h(u_\lambda)\Big|_{\lambda=0} = 0.$$

Note that for λ sufficiently small the function $t \mapsto t + \lambda\varphi(t)$ is increasing in $[a,b]$, so that in particular it is invertible. Now,

$$\begin{aligned} M_h(u_\lambda) &= \frac{1}{2}\int_a^b |\dot{u}\left(t+\lambda\varphi(t)\right)|^2 \left(1+\lambda\dot{\varphi}(t)\right)^2\, dt \int_a^b \left[V\left(u\left(t+\lambda\varphi(t)\right)\right)+h\right]\, dt \\ &= \frac{1}{2}\int_a^b |\dot{u}(s)|^2 \left(1+\lambda\dot{\varphi}(t(s))\right)\, ds \int_a^b \frac{V\left(u\left(s\right)\right)+h}{1+\lambda\dot{\varphi}(t(s))}\, ds. \end{aligned}$$

Observe that, letting $\lambda \to 0$, the family of functions $t(s) = t_\lambda(s)$ uniformly converges to s in $[a,b]$:

$$|t_\lambda(s) - s| = |s - \lambda\varphi(t(s)) - s| \le |\lambda|\|\varphi\|_\infty \qquad \forall s \in [a,b].$$

Hence

$$\begin{aligned} \frac{d}{d\lambda}M_h(u_\lambda)\Big|_{\lambda=0} &= \frac{1}{2}\int_a^b |\dot{u}(s)|^2\, \dot{\varphi}(s)\, ds \int_a^b \left[V\left(u\left(s\right)\right)+h\right]\, ds + \\ &\quad - \frac{1}{2}\int_a^b |\dot{u}(s)|^2\, ds \int_a^b \left[V\left(u\left(s\right)\right)+h\right]\dot{\varphi}(s)\, ds \\ &= \int_a^b \left[\frac{1}{2}\left(\int_a^b V(u)+h\right)|\dot{u}(s)|^2 - \frac{1}{2}\|\dot{u}\|_2^2\left(V(u(s))+h\right)\right]\dot{\varphi}(s)\, ds. \end{aligned}$$

Since this vanishes for every $\varphi \in \mathcal{C}_c^\infty$ it results (see for instance [17])

$$\frac{1}{2}\left(\int_a^b V(u)+h\right)|\dot{u}(s)|^2-\frac{1}{2}\|\dot{u}\|_2^2\left(V(u(s))+h\right)=k \qquad \text{a.e. } s\in[a,b],$$

for some $k \in \mathbb{R}$. This gives (here we use the fact that the minimum is attained at a positive level)

$$\frac{\omega^2}{2}|\dot{u}(s)|^2 = V(u(s))+h+k \qquad \text{a.e. } s\in[a,b].$$

Integrating over $[a,b]$ we obtain

$$\omega^2=\frac{\int_a^b\left(V(u)+h+k\right)}{\frac{1}{2}\int_a^b|\dot{u}|^2}.$$

A comparison with definition (2.20) gives $k=0$. □

The Jacobi metric. Another version of the Maupertuis principle, well described for instance in [63], states that solutions of (2.21) are obtained, after a suitable re-parametrization, as non-constant critical points of the functional

$$L_h(u)=L_h\left([a,b];u\right):=\int_a^b\sqrt{|\dot{u}(t)|^2\left(V(u(t))+h\right)}\,dt,$$

which is defined on the set of the functions $u \in H_{p_1p_2}([a,b])$ such that $V(u(t)) \geq -h$ for every $t \in [a,b]$. We define

$$H_h=H_h^{p_1p_2}\left([a,b]\right):=\left\{u\in H: V(u(t))>-h, |\dot{u}(t)|>0 \text{ for every } t\in[a,b]\right\};$$

the domain of L_h is the closure of $H_h^{p_1p_2}([a,b])$ in the weak topology of H^1.
The functional L_h has an important geometric meaning: the value $L_h(\gamma)$ is the length of the curve parametrized by $\gamma \in H_h$ with respect to the *Jacobi metric*:

$$g_{ij}(x):=\left(V(x)+h\right)\delta_{ij}, \quad \text{where } \delta_{ij} \text{ is the Kronecker delta.}$$

This metric makes the Hill region $\{V(x)+h>0\}$ a Riemannian manifold.

The explicit expression of the re-parametrization needed to pass from critical points of L_h to solution of (2.21) is given in the following theorem. For $u \in H^1\left([a,b];\mathbb{R}^2\right)$, let us set

$$\Gamma_u:=\left\{\left([a',b'],f\right)\;\middle|\; \begin{array}{l} f:[a',b']\to[a,b], f\in\mathcal{C}^1\left([a',b']\right) \text{ and increasing,}\\ \text{such that } u\circ f\in H^1\left(a',b'\right)\end{array}\right\}.$$

Since L_h is a length, it is invariant under re-parametrization: for every $u \in H_h^{p_1p_2}([a,b])$ and for every $([a',b'],f)\in\Gamma_u$ it results

$$L_h\left([a,b];u\right)=L_h\left([a',b'];u\circ f\right).$$

Theorem 2.4.5. *Let $u \in H_h^{p_1 p_2}([a,b]) \cap \widehat{H}_{p_1 p_2}([a,b])$ be a non-constant critical point of $L_h([a,b];\cdot)$. Then there exist a re-parametrization x of u which is a classical solution of (2.21) in a certain interval $[0, T/\sqrt{2}]$.*

Proof. It is well known that $u \in H_h$ is a critical point of L_h with respect to variations with compact support if and only if u solves the Euler-Lagrange equation

$$\frac{d}{dt}\left(\dot{u}(t)\sqrt{\frac{V(u(t))+h}{|\dot{u}(t)|^2}}\right) - \frac{1}{2}\sqrt{\frac{|\dot{u}(t)|^2}{V(u(t))+h}}\nabla V(u(t)) = 0 \tag{2.24}$$

for almost every $t \in [a,b]$. The function u is a collisions-free weak solution of (2.24), hence it is a strong solution. Define, for $t \in [a,b]$,

$$\theta(t) := \int_a^t \sqrt{\frac{|\dot{u}(z)|^2}{V(u(z))+h}}\, dz,$$

and set $T = \theta(b)$. It results $([0,T],\theta) \in \Gamma_u$ and for every $s \in [0,T]$ (we denote with " $'$ " the differentiation with respect to the new independent variable s)

$$\frac{dt}{ds}(s) = \left(\frac{d\theta}{dt}(t)\bigg|_{t=t(s)}\right)^{-1} = \sqrt{\frac{V(u(t(s)))+h}{|\dot{u}(t(s))|^2}}.$$

With this change of variable, setting $y(s) = u(t(s))$, the (2.24) becomes

$$\frac{1}{t'(s)}\frac{d}{ds}\left(\frac{y'(s)}{t'(s)}t'(s)\right) - \frac{1}{2t'(s)}\nabla V(y(s)) = 0,$$

i.e.

$$y''(s) = \frac{1}{2}\nabla V(y(s)).$$

Setting $x(s) := y(\sqrt{2}s)$, x is a solution of the first equation in (2.21) in $[0, T/\sqrt{2}]$. As far as the second equation is concerned, it results

$$|y'(s)|^2 = |\dot{u}(t(s))t'(s)|^2 = V(u(t(s))) + h = V(y(s)) + h,$$

so that for every $s \in [0, T/\sqrt{2}]$

$$\frac{1}{2}|x'(s)|^2 = V(x(s)) + h$$

which completes the proof. □

Remark 2.4.6. If $h \geq 0$, since $V > 0$ in $\mathbb{R}^2$, the Hill region is the punctured plane $\mathbb{R}^2 \setminus \{c_1, \dots, c_N\}$. Therefore, in order to get closed geodesics with respect to the Jacobi metric (i.e. periodic solutions of the N-centre problem with energy h), it is possible to apply global arguments based on the fundamental group of the punctured plane itself; we refer to [52] for more details. If $h < 0$ the problem is considerably more complicated, because the Hill region is a proper subset of the punctured plane; this introduce a degeneration, because an arc of the curve $\{V(x) + h = 0\}$ has null length in the Jacobi metric, so that it is a minimizer of the Jacobi metric for the related fixed ends problem, but of course is not a solution of the N-centre problem; this degeneration makes impossible to use global argument based on the fundamental group. To avoid this complication, we separated the study of the dynamics outside/inside $B_R(0)$: while inside the ball we will show that the research of geodesics can be carried on, outside the ball (when we could have problems coming from the degeneracy of the metric) we used a completely analytic argument which has nothing to do with the geometric interpretation of the problem, see Section 2.3.

Relationship between L_h and M_h. It is convenient to establish a correspondence between minimizers of M_h at positive level and minimizers of L_h. This can be done through the Cauchy-Schwarz inequality: for every $u \in H_h$

$$L_h^2(u) = \left(\int_a^b \sqrt{|\dot{u}|^2 \left(V(u) + h\right)} \right)^2 \leq \int_a^b |\dot{u}|^2 \int_a^b \left(V(u) + h\right) = 2M_h(u), \qquad (2.25)$$

with equality if and only if there exists $\lambda \in \mathbb{R}$ such that for almost every $t \in [a, b]$

$$|\dot{u}(t)|^2 = \lambda \left(V(u(t)) + h\right).$$

Proposition 2.4.7. *Let $u \in H_h \cap H$ be a non-constant minimizer of M_h. Then u is a minimizer of L_h in $H_h \cap H$.*

Proof. Even if u is not a critical point of M_h, from Lemma 2.4.3 we know that

$$|\dot{u}(t)|^2 = \frac{2}{\omega^2} \left(V(u(t)) + h\right) \qquad \text{a.e. } t \in [a, b]$$

(note that, since $u \in H_h$, we have $M_h(u) > 0$). Hence there is equality in (2.25). If there existed $v \in H_h \cap H$ such that $L_h(v) < L_h(u)$, then we could re-parametrize v to obtain a function (still denoted by v) satisfying

$$|\dot{v}(t)|^2 = V(v(t)) + h;$$

indeed since $v \in H_h$ we can perform the same re-parametrization introduced in Theorem 2.4.5. So,

$$0 < 2M_h(v) = L_h^2(v) < L_h^2(u) = 2M_h(u),$$

a contradiction. □

Proposition 2.4.8. *If $u \in H_h \cap H$ is a non-constant minimizer of L_h then, up to a re-parametrization, u is a minimizer of M_h on $H_h \cap H$.*

Proof. We can assume from the beginning that there exists $\lambda \in \mathbb{R}$ such that for every $t \in [0,1]$

$$|\dot{u}(t)|^2 = \lambda \left(V(u(t)) + h\right).$$

Otherwise it is sufficient to perform the re-parametrization introduce in Theorem 2.4.5. Then there is equality in (2.25). Assume by contradiction that there existed $v \in H_h \cap H$ such that $M_h(v) < M_h(u)$. We can re-parametrize v so that there is equality in (2.25). Therefore, we deduce

$$L_h^2(v) = 2M_h(v) < 2M_h(u) = L_h^2(u),$$

a contradiction. □

Final comments. We will use both M_h and L_h. It is clear that the Maupertuis functional M_h is easier to treat, so that it is convenient to use it whenever possible. On the other hand the geometric meaning of the functional L_h will be extremely useful. Indeed, as already mentioned, the couple set-metric given by

$$N = \left\{x \in \mathbb{R}^2 : V(x) > -h\right\}, \quad g_{ij}(x) = \left(V(x) + h\right)\delta_{ij}$$

defines a Riemaniann manifold and we will take advantage of this structure, in spite to the degeneration of the metric on the boundary of the Hill region. More precisely, we will often make use of the following known facts (see [32]):

1) If $\gamma : [a,b] \to N$ is a piecewise differentiable curve, it is always possible to re-parametrize it so that the length of the tangent vector

$$\sqrt{|\dot{\gamma}(t)|^2 \left(V(\gamma(t)) + h\right)}$$

is a constant $C \in \mathbb{R}^+ \cup \{0\}$.

2) If a piecewise differentiable curve $\gamma : [a,b] \to N$, with parameter proportional to arc length, has length less or equal to the length of any other piecewise differentiable curve joining $\gamma(a)$ and $\gamma(b)$, then γ is a geodesic. In particular, γ is regular (recall that a geodesic is a curve satisfying the geodesics equation).

3) Let $p \in N$. We say that a subset $A \subset N$ is a *totally normal neighbourhood* of p if for every $p_1, p_2 \in \bar{A}$ there exists a unique minimizing geodesic γ joining p_1 and p_2. If this geodesic is contained in A, we say that A is a *strongly convex* neighbourhood.

For any $p \in N$ there exists a totally normal neighbourhood U of p. It is possible to choose U in such a way that U is strongly convex. If γ is the minimizing geodesic connecting p_1 and p_2 in U, γ depends smoothly on p_1 and p_2.

Furthermore we will strongly use the fact that, in contrast with M_h, *the functional* L_h *is addictive*. This is essential for the (easy) proof of the following Proposition.

Proposition 2.4.9. *Let* $u \in H_{p_1p_2}([a,b])$ *be a minimizer of* $L_h([a,b];\cdot)$*, let* $[c,d] \subset [a,b]$*. Then* $u|_{[c,d]}$ *is a minimizer of* $L_h([c,d];\cdot)$ *in* $H_{u(c)u(d)}([c,d])$*. Moreover, if* u *is a minimizer of* $M_h([a,b];\cdot)$ *in* $H_{p_1p_2}([a,b])$*, then, for any subinterval* $[c,d] \subset [a,b]$*, the restriction* $u|_{[c,d]}$ *is a minimizer of* $M_h([c,d];\cdot)$ *in* $H_{u(c)u(d)}([c,d])$*.*

2.4.2 The existence theorem

As announced, in order to find weak solutions of (2.19), we are going to minimize the Maupertuis functional with some topological constraints. To this aim, the first step is to introduce suitable (weakly closed) sets of functions which take into account the desired topological features. Let us fix $[a,b] \subset \mathbb{R}$ and $p_1, p_2 \in \partial B_R(0)$, $p_1 = R\exp\{i\theta_1\}$, $p_2 = R\exp\{i\theta_2\}$ for $\theta_1, \theta_2 \in [0, 2\pi)$. The paths in $\widehat{H}$ can be classified according to their winding numbers with respect to each centre. This can be done by artificially closing them, in the following way: for every $u \in \widehat{H}$ we define

$$\Gamma_u(t) := \begin{cases} \begin{cases} u(t) & t \in [a,b] \\ Re^{i(t-b+\theta_2)} & t \in (b, b+\theta_1+2\pi-\theta_2) \end{cases} & \text{if } \theta_1 < \theta_2 \\ u(t) \qquad t \in [a,b] & \text{if } \theta_1 = \theta_2 \\ \begin{cases} u(t) & t \in [a,b] \\ Re^{i(t-b+\theta_2)} & t \in (b, b+\theta_1-\theta_2) \end{cases} & \text{if } \theta_1 > \theta_2, \end{cases}$$

i.e. if $p_1 \neq p_2$ we close the path u with the arc of $\partial B_R(0)$ connecting p_2 and p_1 in counterclockwise sense. Then it is well defined the usual winding number

$$\operatorname{Ind}(u([a,b]), c_j) = \frac{1}{2\pi i}\int_{\Gamma_u} \frac{dz}{z - c_j}.$$

Given $l = (l_1, \ldots, l_N) \in \mathbb{Z}^N$, a connected component of $\widehat{H}$ is

$$\widehat{\mathfrak{H}}_l := \left\{ u \in \widehat{H} : \operatorname{Ind}(u([a,b]), c_j) = l_j \quad \forall j = 1, \ldots, N \right\}.$$

Remark 2.4.10. 1) In general $\widehat{\mathfrak{H}}_l$ may contain paths with self-intersections. Actually, $\widehat{\mathfrak{H}}_l$ contains self-intersections-free paths lying completely in $B_R(0)$ if and only if $l_j \in \{0, \pm 1\}$ for every j.
2) For every $l \in \mathbb{Z}^N$ the set $\widehat{\mathfrak{H}}_l$ is not weakly closed in H^1.

In the next subsection it will be useful to work on sets containing self-intersections-free paths. For this reason we consider $l \in \mathbb{Z}_2^N$ instead of $l \in \mathbb{Z}^N$ and we set

$$\widehat{H}_l := \left\{ u \in \widehat{H} : \operatorname{Ind}(u([a,b]), c_j) \equiv l_j \mod 2 \quad \forall j = 1, \dots, N \right\},$$

namely we collect together the components having winding numbers having the same parity with respect to each centre. We also assume that

$$\exists j, k \in \{1, \dots, N\}, \; j \neq k, \text{ such that } l_j \neq l_k \mod 2. \tag{2.26}$$

With this choice of l, if $u \in \widehat{H}_l$, then u has to pass through the ball $B_\varepsilon(0)$ which contains the centres. In particular $u \in \widehat{H}_l$ cannot be constant even if $p_1 = p_2$, so that all the results stated in Subsection 2.4.1 hold true even in this case (there are no constant functions in $\widehat{H}_l$). From now on, we say that $l \in \mathbb{Z}_2^N$ is a *winding vector*, and we term $\mathfrak{I}^N := \{l \in \mathbb{Z}_2^N : l \text{ satisfies } (2.26)\}$.

In order to succeed in minimizing, we need to close $\widehat{H}_l$ with respect to the weak H^1 topology. To this aim, we need to allow collisions with the centres. For $j \in \{1, \dots, N\}$, let us set

$$\begin{aligned} \mathfrak{Coll}_l^j := \{ u \in H : \operatorname{Ind}(u([a,b]), c_k) \equiv l_k \mod 2 \\ \forall k \in \{1, \dots, j-1, j+1, \dots, N\}, \text{ and there exists } t \in [a,b] : u(t) = c_j \}. \end{aligned}$$

A path $u \in \mathfrak{Coll}_l^j$ behaves as a path of $\widehat{H}_l$ with respect to c_k for any $k \neq j$, and collides in c_j at a certain instant. Analogously, for $j_1, j_2 \in \{1, \dots, N\}$ we define

$$\begin{aligned} \mathfrak{Coll}_l^{j_1,j_2} = \{ u \in H : \operatorname{Ind}(u([a,b]), c_k) \equiv l_k \mod 2, \forall k \in \{1, \dots, N\} \setminus \{j_1, j_2\}, \\ \text{and there are } t_1, t_2 \in [a,b] : u(t_1) = c_{j_1}, u(t_2) = c_{j_2} \}, \end{aligned}$$

the set of the paths behaving as paths of $\widehat{H}_l$ with respect to c_k for $k \in \{1, \dots N\} \setminus \{j_1, j_2\}$ and colliding in c_{j_1} and c_{j_2}; in the same way

$$\begin{aligned} &\mathfrak{Coll}_l^{j_1,j_2,j_3} := \dots, \\ &\vdots \\ &\mathfrak{Coll}_l^{1,\dots,N} = \mathfrak{Coll}^{1,\dots,N} := \{ u \in H : u \text{ collides in each centre} \}. \end{aligned}$$

Finally, we name

$$\mathfrak{Coll}_l := \bigcup_{j=1}^N \mathfrak{Coll}_l^j \cup \bigcup_{1 \le j_1 < j_2 \le N} \mathfrak{Coll}_l^{j_1,j_2} \cup \dots \cup \mathfrak{Coll}_l^{1,\dots,N}.$$

Proposition 2.4.11. *The set*

$$H_l := \widehat{H}_l \cup \mathfrak{Coll}_l$$

is weakly closed in $H^1\left([a,b],\mathbb{R}^2\right)$.

Proof. Let $(u_n) \subset H_l$, $u_n \rightharpoonup u$ in H^1. Since the weak convergence in H^1 implies the uniform one, if u has a collision

$$(u_n) \subset H_l \quad \Longrightarrow \quad u \in \mathfrak{Coll}_l.$$

If u is collisions-free, the uniform convergence implies the existence of $n_0 \in \mathbb{N}$ such that

$$u_n \in \widehat{H}_l \quad \forall n \geq n_0 \quad \Longrightarrow \quad u \in \widehat{H}_l. \qquad \square$$

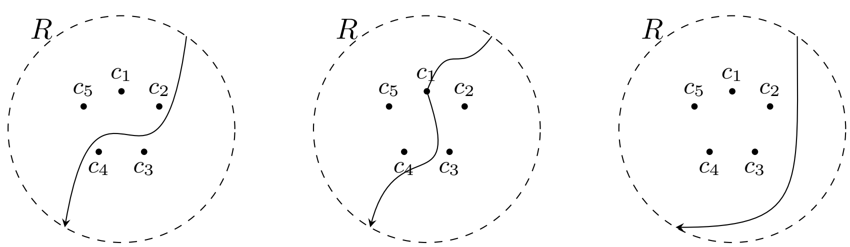
To complete the choice of suitable sets, it is convenient to add a further requirement: since we search functions lying in $B_R(0)$, let us set

$$\widehat{K}_l = \widehat{K}_l^{p_1 p_2}([a,b]) := \left\{u \in \widehat{H}_l : |u(t)| \leq R \ \forall t \in [a,b]\right\}$$
$$K_l = K_l^{p_1 p_2}([a,b]) := \left\{u \in H_l : |u(t)| \leq R \ \forall t \in [a,b]\right\}.$$

Proposition 2.4.12. *The set* K_l *is weakly closed in* $H^1\left([a,b],\mathbb{R}^2\right)$.

Proof. K_l is a subset of the weakly closed set H_l, and it is stable under uniform convergence. $\square$

Some examples of paths of K_l: the first path is a collisions-free path with winding vector $(0,0,1,1,0)$; the second one is a collision path of K_l with $l = (0,1,1,0,0)$ or $l = (1,1,1,0,0)$; the third one is a path of K_l with $l = (0,0,0,0,0)$, which does not satisfy (2.26).



We recall the following definition.

Definition 2.4.13. An *ejection-collision solution* of

$$\ddot{y}(t) = \nabla V_\varepsilon(y(t))$$

is an H^1 function $y : I \subset \mathbb{R} \to \mathbb{R}^2$ such that:

- there exists a collision set of null measure $T_c(y) \subset I$ such that for every $t^* \in T_c(y)$ there holds $y(t^*) = c_k$ for some $k = 1, \dots, N$;
- the restriction $y|_{I \setminus T_c(x)}$ is a classical solution of

$$\ddot{y}(t) = \nabla V_\varepsilon(y(t));$$

- the energy function

$$t \mapsto \frac{1}{2}|\dot{y}(t)|^2 - V_\varepsilon(y(t)),$$

  which is defined almost everywhere in I, is constant;
- at a collision instant, the trajectory is reflected:

$$y(t + t^*) = y(t^* - t) \qquad \forall t^* \in T_c(y), \forall t \in I \setminus T_c(y).$$

We are in position to state the main result of this section.

Theorem 2.4.14. *There exists $\varepsilon_3 > 0$ such that for every $\varepsilon \in (0, \varepsilon_3)$, $p_1, p_2 \in \partial B_R(0)$ and $l \in \mathfrak{I}^N$, there exist $T > 0$ and a solution $y \in K_l^{p_1 p_2}([0, T])$ of problem (2.19), which is a re-parametrization of a local minimizer of the Maupertuis functional in $K_l^{p_1 p_2}([0, 1])$. Moreover:*

(i) if $\alpha \in (1, 2)$ then y is collision-free and self-intersection-free;

(ii) if $\alpha = 1$ we have to distinguish among:

(a) $p_1 \neq p_2$; then y is collision-free and self-intersection-free;

(b) $p_1 = p_2$ and l is such that there exist $j_1, j_2, k_1, k_2 \in \{1, \dots, N\}$:

$$l_{j_1} = l_{j_2} \equiv 0 \mod 2 \quad l_{k_1} = l_{k_2} \equiv 1 \mod 2;$$

then y is collision-free and self-intersection-free;

(c) $p_1 = p_2$ and l is such that there exists $j \in \{1, \dots, N\}$:

$$l_1 = \cdots = l_{j-1} = l_{j+1} = \cdots = l_N \neq l_j \mod 2; \tag{2.27}$$

then either y is collision-free and self-intersection-free, it is an ejection-collision solution, with a unique collision against the centre c_j.

Remark 2.4.15. The statement motivates us to say that an element $l \in \mathfrak{I}^N$ is a *collision winding vector* if it satisfies the (2.27). Let us also observe that the condition of case (ii)-(b) can be fulfilled only for $N \geq 4$.

Before proceeding into the proof, we translate Theorem 2.4.14 in the language of partitions. To do this, we note that if $u \in \widehat{K}_l$ is self-intersection-free, then it separates the centres in two different groups, which are determined by the particular choice of $l \in \mathfrak{I}^N$; namely, a self-intersection-free path in a class $\widehat{K}_l$ induces a partition of the centres in two sets. Since we imposed (2.26), these sets are both non-empty. Hence it is well-defined an application $\mathcal{A} : \mathfrak{I}^N \to \mathcal{P}$ which associates to a winding vector

$$l = (l_1, \dots, l_N) \text{ with } \begin{cases} l_k \equiv 0 \mod 2 & k \in A_0 \subset \{1, \dots, N\} \\ l_k \equiv 1 \mod 2 & k \in A_1 \subset \{1, \dots, N\} \end{cases}$$

the partition

$$\mathcal{A}(l) := \{\{c_k : l_k \in A_0\}, \{c_k : l_k \in A_1\}\}.$$

This map is surjective but non injective, since for each couple $l, \widetilde{l} \in \mathfrak{I}^N$ such that

$$l_k \neq \widetilde{l}_k \mod 2 \qquad \forall k = 1, \dots, N,$$

it results $\mathcal{A}(l) = \mathcal{A}(\widetilde{l})$. In particular, for each $P_j \in \mathcal{P}$, there are two $l \in \mathfrak{I}^N$ such that $\mathcal{A}(l) = P_j$.
Now it is natural to define

$$\widehat{K}_{P_j} = \widehat{K}_{P_j}^{p_1 p_2}([a,b]) := \left\{ u \in \widehat{K}_l : l \in \mathcal{A}^{-1}(P_j) \right\},$$
$$K_{P_j} = K_{P_j}^{p_1 p_2}([a,b]) := \left\{ u \in K_l : l \in \mathcal{A}^{-1}(P_j) \right\}.$$

They are respectively the set of the paths which connect p_1 and p_2 dividing the centres according to the partitions P_j, and its closure in the weak topology of H^1.
From Theorem 2.4.14, we obtain

Corollary 2.4.16. *Let ε_3 be introduced in Theorem 2.4.14. For every $\varepsilon \in (0, \varepsilon_3)$, $p_1, p_2 \in \partial B_R(0)$ and $P_j \in \mathcal{P}$, there exist two solutions $y_1 \in K_{P_j}^{p_1 p_2}([0, T_1])$ and $y_2 \in K_{P_j}^{p_1 p_2}([0, T_2])$ of problem (2.19), for some $T_1, T_2 > 0$. One of them, say y_1, is a re-parametrization of a global minimizer of the Maupertuis functional M_{-1} in $K_{P_j}^{p_1 p_2}([0, 1])$, while the other is a re-parametrization of a local minimizer in the same class. Moreover:*

(i) if $\alpha \in (1, 2)$ then they are collision-free and self-intersection-free;

(ii) if $\alpha = 1$ we have to distinguish among:

(a) $p_1 \neq p_2$; then they are collision-free and self-intersection-free;

(b) $p_1 = p_2$ and $P_j \in \mathcal{P} \setminus \mathcal{P}_1$; then they are collision-free and self-intersection-free;

(c) $p_1 = p_2$ and $P_j \in \mathcal{P}_1$; then either y_1 is collision-free and self-intersection-free, or it is an ejection-collision solution, with a unique collision against c_j. The same holds true for y_2.

If one between y_1 and y_2 is collision-free, then they are two different solutions.

This last observation is a straightforward consequence of the fact that $\widehat{H}_{l_1} \cap \widehat{H}_{l_2} = \emptyset$ if $l_1 \neq l_2$. From this result, recalling Proposition 2.2.1 and Remark 2.2.2, we obtain Theorem 2.1.4.

2.4.3 Minimization inside $B_R(0)$

Let us fix $p_1, p_2 \in \partial B_R(0)$ and $l \in \mathfrak{I}^N$ satisfying (2.26), and consider the restriction of the Maupertuis functional M_{-1} to the set $K_l^{p_1 p_2}([0,1])$. We are going to provide weak solutions of (2.19) applying the direct method of the calculus of variations to M_{-1}. We write M and L instead of M_{-1} and L_{-1}, respectively.

Remark 2.4.17. In the statement of Theorem 2.4.14 the value ε_3 depends neither on $p_1, p_2 \in \partial B_R(0)$, nor on $l \in \mathfrak{I}^N$, while here we fixed p_0, p_1 and l before finding ε_3. Actually, when we will find ε_3, we will see that it is independent on these quantities.

Lemma 2.4.18. *There exists a constant $C > 0$ such that*

$$M(u) \geq C > 0 \qquad \forall u \in K_l.$$

Proof. If $u \in K_l$, for every $j = 1, \dots, N$ and every $t \in [0,1]$, we have

$$V_\varepsilon(u(t)) - 1 \geq C > 0,$$

see (2.13). Therefore the proof will be complete when we show the existence of $C > 0$ such that, for every $u \in K_l$, there holds

$$\|\dot{u}\|_2 \geq C. \tag{2.28}$$

If not, there exists $(u_n) \subset K_l$ such that $\|\dot{u}_n\|_2 \to 0$. In particular $(\|\dot{u}_n\|_2) \subset \mathbb{R}$ is bounded. The sequence $(\|u_n\|_2)$ is bounded, too:

$$\int_0^1 |u_n(t)|^2 \, dt \leq R^2.$$

Then the sequence (u_n) is bounded in H^1, and this implies that up to subsequence $u_n \rightharpoonup v \in K_l$. Let's write

$$u_n(t) = \xi_n + w_n(t), \quad \text{where} \quad \xi_n := \int_0^1 u_n(t) \, dt \in \mathbb{R}^2.$$

It results

$$\dot{u}_n(t) = \dot{w}_n(t) \quad \text{a.e. } t \in [0,1] \quad \Longrightarrow \quad \|\dot{w}_n\|_2 \to 0.$$

Also, from the Poincaré-Wirtinger inequality

$$\|w_n\|_2 \leq \|w_n\|_\infty \leq \|\dot{w}_n\|_1 \leq \|\dot{w}_n\|_2,$$

so that $\|w_n\| \to 0$: (w_n) strongly converges to 0 in H^1. As a consequence (w_n) uniformly converges to 0, and therefore $v(t) \equiv \xi$ with $\xi \in \mathbb{R}^2$; but we have already observed that, because of (2.26), $K_l \subset H_l$ does not contain constant functions; a contradiction. □

The following statements are by now standard results and can be proved by routine applications of Poincaré inequality, Fatou lemma and weak compactness arguments (see, for instance, [47, 83, 8]). We report them for the sake of completeness.

Lemma 2.4.19. *The functional M is weakly lower semi-continuous (w.l.s.c.) on K_l.*

Proof. It is well known (see for instance [17]) that M is w.l.s.c. on K_l if and only if for every $C \in \mathbb{R}$

$$M^C := \{u \in K_l : M(u) \leq C\} \text{ is weakly closed in } H^1.$$

Let $(u_n) \subset M^C$, $u_n \rightharpoonup u \in K_l$. The H^1 norm is w.l.s.c., so that

$$\|u\|_2^2 + \|\dot{u}\|_2^2 \leq \liminf_{n\to\infty} \|u_n\|_2^2 + \|\dot{u}_n\|_2^2.$$

Hence, the uniform convergence of u_n to u implies

$$\|\dot{u}\|_2^2 \leq \liminf_{n\to\infty} \|\dot{u}_n\|_2^2. \tag{2.29}$$

Now, since $(u_n) \subset M^C$, $V_\varepsilon(u_n) \in L^1(0,1)$ for every n. This implies that the set of all t such that $u(t) = c_j$ for some j has null measure in $[0,1]$. Therefore

$$u_n \rightharpoonup u \quad \Longrightarrow \quad \|u_n - u\|_\infty \to 0 \quad \Longrightarrow \quad V_\varepsilon(u_n(t)) \to V_\varepsilon(u(t)) \quad \text{a.e. } t \in [0,1].$$

Also, $V_\varepsilon(u_n(t)) - 1 \geq 0$ for every $t \in [0,1]$, for every n. Then, from the Fatou lemma it follows that $V_\varepsilon(u) \in L^1(0,1)$ and

$$\int_0^1 (V_\varepsilon(u) - 1) \leq \liminf_{n\to\infty} \int_0^1 (V_\varepsilon(u_n) - 1)\,. \tag{2.30}$$

Collecting (2.29) and (2.30) we obtain

$$M(u) \leq \left(\liminf_{n\to\infty} \|\dot{u}_n\|_2^2\right) \left(\liminf_{n\to\infty} \int_0^1 (V_\varepsilon(u_n) - 1)\right) \leq \limsup_{n\to\infty} M(u_n) \leq C,$$

i.e. $u \in M^C$. □

Lemma 2.4.20. *The functional M is coercive on K_l.*

Proof. Let $(u_n) \subset K_l$, $\|u_n\| \to +\infty$ for $n \to \infty$, namely

$$\lim_{n\to\infty} \|u_n\|_2^2 + \|\dot{u}_n\|_2^2 = +\infty.$$

We know that $u_n \in K_l$ implies $\int_0^1 (V_\varepsilon(u_n) - 1) \geq C > 0$, so that it is sufficient to prove $\|\dot{u}_n\|_2^2 \to +\infty$. Assume by contradiction that $\limsup_n \|\dot{u}_n\|_2^2 < +\infty$; then $\lim_n \|u_n\|_2^2 = +\infty$. But since $|u_n(t)| \leq R$ for every t, it results

$$\int_0^1 |u_n(t)|^2 \, dt \leq R^2 \quad \forall n,$$

a contradiction. □

Proposition 2.4.21. *Let $p_1, p_2 \in \partial B_R(0)$ (it is admissible $p_1 = p_2$), let $l \in \mathfrak{I}^N$. Then there exists a minimum of M on K_l at a positive level.*

Proof. Apply the direct method of the calculus of variations to the functional M defined on K_l: use Proposition 2.4.12 and Lemmas 2.4.18, 2.4.19, 2.4.20. □

Let $l \in \mathfrak{I}^N$ be fixed. If we show that the minimizer $u \in K_l$ is collision-free and $|u(t)| < R$ for every $t \in (0,1)$, we can say that for every $\varphi \in \mathcal{C}_c^\infty((0,1), \mathbb{R}^2)$ there holds

$$\frac{d}{d\lambda} M(u + \lambda\varphi)\bigg|_{\lambda=0} = 0,$$

so that u is critical with respect to variations with compact support in $(0,1)$ and Theorem 2.4.1 applies. In order to prove that $|u(t)| < R$, we apply a result in [60] which concerns the regularity of solutions to some "obstacle problems".
Let us introduce the set of the collision times of u:

$$T_c(u) := \{t \in [0,1] : u(t) = c_j \text{ for some } j \in \{1, \ldots, N\}\},$$

and let us also term

$$T_R(u) := \{t \in [0,1] : |u(t)| = R\}, \quad T_{R/2}^+(u) := \left\{t \in [0,1] : |u(t)| > \frac{R}{2}\right\}.$$

A connected component of $T_{R/2}^+(u)$ is an open interval; a component of $T_c(u)$ or of $T_R(u)$ is a closed interval (possibly a single point). The complement of $[0,1] \setminus T_c(u)$ or $T_{R/2}^+(u) \setminus T_R(u)$ is the union of a finite or countable number of open intervals.

Proposition 2.4.22. *Let $u \in K_l$ be a minimizer of M. If (a,b) is a connected component of $T^+_{R/2}(u) \setminus T_R(u)$, then u is of class $\mathcal{C}^1$ in (a,b).*

Proof. Let $\bar{p}_1 := u(a)$, $\bar{p}_2 := u(b)$. Let us term

$$K = \left\{ v \in H^1([a,b],\mathbb{R}^2) \,\middle|\, \begin{array}{l} v(a) = \bar{p}_1,\ v(b) = \bar{p}_2,\ R/2 \le |v(t)| \le R \ \forall t \in (a,b), \\ v = w|_{[a,b]} \text{ for some } w \in K_l^{p_1 p_2}([0,1]) \end{array} \right\}.$$

It is a closed set in the weak H^1 topology. From Proposition 2.4.9 it follows that $u|_{[a,b]}$ is a minimizer of $M_{-1}([a,b];\cdot)$ in K. Let ω be defined by (2.20):

$$\omega^2 = \frac{\int_0^1 V_\varepsilon(u) - 1}{\frac{1}{2}\int_0^1 |\dot{u}|^2},$$

and let

$$K' = \left\{ x \in H^1\left(\left[\frac{a}{\omega}, \frac{b}{\omega}\right], \mathbb{R}^2 \right) : x(s) = v(\omega s) \text{ for some } v \in K \right\}.$$

There is a bijective correspondence between K and K' given by

$$v(t) \in K \longleftrightarrow x(s) = v(\omega s) \in K'.$$

We claim that since u is a minimizer of M in K, then $\bar{x}(t) = u(\omega t)$ minimizes the action

$$A_{[a/\omega, b/\omega]}(x) := \int_{a/\omega}^{b/\omega} \left(\frac{1}{2}|\dot{x}(s)|^2 + V_\varepsilon(x(s)) - 1 \right) ds$$

in the set K'. We point out that here ω is fixed and is determined by u.

For every $x \in K_l^{p_1 p_2}([0,1/\omega])$, $x \leftrightarrow v$, we have

$$\begin{aligned} A_{[a/\omega,b/\omega]}(x) &\ge 2 \left(\int_{a/\omega}^{b/\omega} \frac{1}{2}|\dot{x}(s)|^2 \, ds \right)^{\frac{1}{2}} \left(\int_{a/\omega}^{b/\omega} (V_\varepsilon(x(s)) - 1) \, ds \right)^{\frac{1}{2}} \\ &= 2 \left(\int_0^1 \frac{\omega^2}{2}|\dot{v}(t)|^2 \frac{dt}{\omega} \int_0^1 (V_\varepsilon(v(t)) - 1) \, \frac{dt}{\omega} \right)^{\frac{1}{2}} = 2\sqrt{M(v)} \ge 2\sqrt{M(u)}. \end{aligned}$$

The first inequality is nothing but $a^2 + b^2 \ge 2ab$ for every $a, b \in \mathbb{R}^+$. The second inequality follows from the minimality of u. As a consequence of the conservation of the energy for u, we have

$$A_{[a/\omega,b/\omega]}(x) \ge 2\sqrt{M(u)} = A_{[a/\omega,b/\omega]}(\bar{x}) \qquad \forall x \in K',$$

which proves the claim. Note that $\bar{x} \in \mathcal{C}^1((a/\omega, b/\omega))$ if and only if $u \in \mathcal{C}^1((a,b))$. According to Theorem 1.6 of [60], if we prove that

$$\max\left\{0, \limsup_{\substack{\|\bar{x}-x\|_{L^2}\to 0\\ x\in K'}} \frac{\frac{1}{2}\int_{a/\omega}^{b/\omega}|\dot{\bar{x}}|^2 - \frac{1}{2}\int_{a/\omega}^{b/\omega}|\dot{x}|^2}{\|\bar{x}-x\|_{L^2}}\right\} < +\infty,$$

then $\bar{x} \in H^2(a/\omega, b/\omega)$ and the proof is complete. We point out that in [60], one of the initial assumption is that the fixed ends (which in our case are $u(a)$ and $u(b)$) are in the unbounded connected component of $\mathbb{R}^n$ without the obstacle; but here we are in the bounded component $B_R(0)$. This is not a problem since the proof of Theorem 1.6 does not use the quoted assumption.

We consider variations of compact support of the form $x = \bar{x} + \varphi$, with $\varphi \in H_0^1(a,b)$ such that $\bar{x} + \varphi \in K'$. For these φ we have

$$\frac{\frac{1}{2}\int_{a/\omega}^{b/\omega}|\dot{\bar{x}}|^2 - \frac{1}{2}\int_{a/\omega}^{b/\omega}|\dot{x}|^2}{\|\bar{x}-x\|_{L^2}} = \frac{A_{[a/\omega,b/\omega]}(\bar{x}) - A_{[a/\omega,b/\omega]}(x)}{\|\bar{x}-x\|_{L^2}} + \frac{\int_{a/\omega}^{b/\omega} V_\varepsilon(x) - V_\varepsilon(\bar{x})}{\|\bar{x}-x\|_{L^2}}.$$

The first term on the right hand side is less then 0 because of the variational characterization of $\bar{x}$; as far as the second term is concerned, we use the fact that $|\bar{x}(t)+\varphi(t)| \geq R/2$ for every $t \in (a,b)$; in $B_R(0)\backslash B_{R/2}(0)$ the potential is regular and bounded with bounded gradient, so that for every φ

$$\int_{a/\omega}^{b/\omega} V_\varepsilon(x) - V_\varepsilon(\bar{x}) \leq C\int_{a/\omega}^{b/\omega}|\bar{x}-x| \leq C\|\bar{x}-x\|_{L^2},$$

and the thesis follows. □

Before proceeding, it is convenient to recall a well known property of the solutions of the α-Kepler problem.

Proposition 2.4.23. *Let $\alpha \in [1,2)$ and let $x : (a,b) \subset \mathbb{R} \to \mathbb{R}^2$, $x = r\exp\{i\theta\}$, be a collision solution of the α-Kepler problem with energy $h < 0$; in particular, we assume that*

$$\lim_{t\to b^-} x(t) = 0.$$

Then the angular momentum $\mathfrak{C}_x = r^2(t)\dot{\theta}(t)$ of x is 0.

Proof. In polar coordinates the energy is

$$\frac{1}{2}\dot{r}^2(t) + \frac{\mathfrak{C}_x^2}{2r^2(t)} - \frac{M}{\alpha r^\alpha(t)} = h \qquad \forall t \in (a,b).$$

In particular

$$h - \frac{\mathfrak{C}_x^2}{2r^2(t)} + \frac{M}{\alpha r^\alpha(t)} \geq 0 \qquad \forall t \in (a,b),$$

but if $\mathfrak{C}_x \neq 0$ then

$$\lim_{t\to b^-} \left(h - \frac{\mathfrak{C}_x^2}{2r^2(t)} + \frac{M}{\alpha r^\alpha(t)} \right) = -\infty,$$

a contradiction. Necessarily $\mathfrak{C}_x = 0$. □

We are ready to collect some properties of the minimizers of M in K_l. Recall that in polar coordinates $u(t) = r(t)\exp\{i\theta(t)\}$.

Lemma 2.4.24. *A minimizer $u \in K_l$ of M has the following properties:*

(i) If (a,b) is a connected component of $[0,1] \setminus (T_c(u) \cup T_R(u))$, then $u|_{(a,b)}$ is of class $\mathcal{C}^2$ and is a solution of

$$\omega^2 \ddot{u}(t) = \nabla V_\varepsilon(u(t)), \quad \textit{where} \quad \omega^2 := \frac{\int_0^1 (V_\varepsilon(u) - 1)}{\frac{1}{2}\int_0^1 |\dot{u}|^2};$$

(ii) the energy function

$$t \mapsto \frac{1}{2}|\dot{u}(t)|^2 - \frac{V_\varepsilon(u(t))}{\omega^2}$$

is constant in $[0,1]$, and it is equal to $-1/\omega^2$;

(iii) If $[t_1,t_2]$ is a connected component of $T_R(u)$, then $\theta|_{(t_1,t_2)}$ is $\mathcal{C}^2$, strictly monotone, and solves

$$\ddot{\theta}(t) = \frac{1}{R\omega^2}\left\langle \nabla V_\varepsilon(Re^{i\theta(t)}), ie^{i\theta(t)} \right\rangle; \tag{2.31}$$

(iv) There exist $\varepsilon_3 > 0$ and $\tau > 0$ such that, if $\varepsilon \in (0,\varepsilon_3)$, for t_3 and t_4 satisfying

$$|u(t_3)| = R, \quad |u(t_4)| = \frac{R}{2}, \quad \frac{R}{2} < |u(t)| < R \quad \forall t \in \begin{cases} (t_3,t_4) & \textit{if } t_3 < t_4 \\ (t_4,t_3) & \textit{if } t_3 > t_4 \end{cases},$$

there holds $|t_4 - t_3| \leq \tau$.

Proof. (i) It is a consequence of the minimality of u with respect to variations $u_\lambda = u + \lambda\varphi$, with $|\lambda|$ sufficiently small and $\varphi \in \mathcal{C}_c^\infty(a,b)$. These variations are compatible with the constraints: $u_\lambda \in K_l$, so that

$$\frac{d}{d\lambda} M(u_\lambda)\bigg|_{\lambda=0} = 0.$$

A direct computation gives the desired result, as in the proof of Theorem 2.4.1.
(ii) It is a consequence of the minimality of u with respect to time re-parametrizations keeping the ends fixed, see the proof of Lemma 2.4.3.
(iii) For $t \in (t_1, t_2)$, the energy integral reads

$$R^2\dot{\theta}^2(t) = -\frac{2}{\omega^2} + \frac{2}{\omega^2} V_\varepsilon\left(Re^{i\theta(t)}\right) \qquad \forall t \in [t_1, t_2]; \tag{2.32}$$

as a consequence $\theta \in \mathcal{C}^2((t_1, t_2))$. Since $V_\varepsilon\left(R\exp\{i\theta\}\right) > 1$ for every $\theta \in [0, 2\pi]$, equation (2.32) implies that $\dot{\theta}(t) \neq 0$ for every $t \in (t_1, t_2)$. To get (2.31) it is sufficient to differentiate (2.32) with respect to t.
(iv) In polar coordinates the energy integral reads

$$\frac{1}{2}\dot{r}^2(t) + \frac{\mathfrak{C}_u^2(t)}{2r^2(t)} - \frac{V_\varepsilon\left(r(t)e^{i\theta(t)}\right)}{\omega^2} = -\frac{1}{\omega^2} \qquad \forall t \in [0, 1]. \tag{2.33}$$

It results

$$\frac{2}{\omega^2}\left(-1 + V_\varepsilon\left(r(t)e^{i\theta(t)}\right)\right) - \frac{\mathfrak{C}_u^2(t)}{r^2(t)} \geq \frac{2}{\omega^2}\left(-1 + \frac{M}{\alpha(R+\varepsilon)^\alpha}\right) + o(1) \quad \text{for } \varepsilon \to 0^+;$$

The last equality is due to the fact that if we pass to the limit as $\varepsilon \to 0^+$, then V_ε uniformly converges in the circular crown $R/2 \leq |x| \leq R$ to the potential of the Kepler problem with homogeneity degree $-\alpha$. In particular, since u has to pass through the ball $B_\varepsilon(0)$, which collapses in the origin, the angular momentum of u uniformly converges to 0 over the interval $[t_3, t_4]$ (or $[t_4, t_3]$, see Proposition 2.4.23). From (2.33) we infer

$$|t_4 - t_3| \leq \int_{R/2}^{R} \frac{dr}{\sqrt{\frac{2}{\omega^2}\left(-1 + \frac{M}{\alpha(R+\varepsilon)^\alpha}\right) + o(1)}} \qquad \text{for } \varepsilon \to 0^+.$$

Since $-1 + \frac{M}{\alpha(R+\varepsilon)^\alpha} > 0$ for every $\varepsilon \in (0, \varepsilon_1/2)$, there exists $0 < \varepsilon_3 \leq \varepsilon_1/2$ such that

$$\frac{2}{\omega^2}\left(-1 + \frac{M}{\alpha(R+\varepsilon)^\alpha}\right) + o(1) \geq C > 0 \qquad \forall \varepsilon \in (0, \varepsilon_3),$$

so that for such values of ε

$$|t_4 - t_3| \leq \frac{R}{2C} =: \tau. \qquad \square$$

Remark 2.4.25. From the proof of point *(iv)* it follows that ε_3 does not depend on $p_1, p_2 \in \partial B_R(0)$ or on $l \in \mathfrak{I}^N$, cf. Remark 2.4.17.

Proposition 2.4.26. *If $u \in K_l$ is the minimizer found in Proposition 2.4.21, then*

$$|u(t)| < R \quad \forall t \in (0,1).$$

Proof. Let $[t_1, t_2]$ be a connected component of $T_R(u)$, let (a,b) be the connected component of $T^+_{R/2}$ such that $[t_1,t_2] \subset (a,b)$. Let us consider $y(t) := u(\omega t)$. Since $y \in \mathcal{C}^1\left((a/\omega, b/\omega)\right)$, it can lean against the circle $\left\{y \in \mathbb{R}^2 : |y| = R\right\}$ with tangential velocity, and for every $\nu > 0$ there exists $t_5 > t_2$ (or $t_5 < t_1$, and in this case the following inequality has to be changed in obvious way) such that

$$\left|y\left(\frac{t_5}{\omega}\right) - Re^{i\theta(t_2/\omega)}\right| < \nu \quad \text{and} \quad \left|\dot{y}\left(\frac{t_5}{\omega}\right) - R\dot{\theta}\left(\frac{t_2}{\omega}\right) ie^{i\theta(t_2/\omega)}\right| < \nu.$$

Summing up, we have

- R is the radius of the circular solution of energy -1 for the Kepler problem with homogeneity degree $-\alpha$:
- outside $B_{R/2}(0)$, the N-centres problem can be seen as a small perturbation of the α-Kepler one: $V_\varepsilon(y) = \frac{M}{\alpha|y|^\alpha} + W_\varepsilon(y)$;
- y is a solution of

$$\begin{cases} \ddot{y}(t) = \nabla V(y(t)) \\ y\left(\frac{t_5}{\omega}\right) \simeq Re^{i\theta(t_2/\omega)}, \quad \dot{y}\left(\frac{t_5}{\omega}\right) \simeq R\dot{\theta}\left(\frac{t_2}{\omega}\right) ie^{i\theta(t_2/\omega)} \end{cases}$$

  in an open neighbourhood of t_5/ω; these initial data are "more or less" the initial data of a circular solution;
- the theorem of continuous dependence of the solutions with respect to the vector field and the initial data holds true for our problem outside $B_{R/2}(0)$.

Therefore, provided ε_3 has been chosen sufficiently small (otherwise we can replace it with a smaller quantity, independent on p_0, p_1 or l), for every $\varepsilon \in (0, \varepsilon_3)$ the function y cannot enter (or exit from) the ball $B_{R/2}(0)$ in a finite time, in contradiction with the choice of l and point (iv) of Lemma 2.4.24. □

Note that, as $T_R(u) = \emptyset$, point (i) of Lemma 2.4.24 says that if the interval (a,b) is a connected component of $[0,1] \setminus T_c(u)$, then $u|_{(a,b)} \in \mathcal{C}^2((a,b))$ and

$$\omega^2 \ddot{u}(t) = \nabla V_\varepsilon(u(t)) \qquad \forall t \in (a,b). \tag{2.34}$$

2.4.4 Classification of the minimizers

So far, we obtained a set of extremals of the Maupertuis functional M at positive levels. In what follows, we try to understand if it is possible that these minimizers are collision-free or not.

Let us fix $l \in \mathfrak{I}^N$. We assume that a minimizer $u \in K_l$ has at least one collision; developing a blow-up analysis, we will reach a contradiction in case $\alpha \in (1, 2)$; in case $\alpha = 1$, we will have to distinguish many possibilities according to the fact that $p_0 = p_1$ or not, and to the choice of l.

Step 1) We prove that *the set $T_c(u)$ of the collision times of u is discrete and finite*; moreover, *either u has no self-intersections at points different from the centres, or it has at least one collision and at one of them there is a reflection.*

Since $M(u) < +\infty$, it follows immediately that $T_c(u)$ is a closed set of null measure. Hence $[0, 1] \setminus T_c(u)$ is the union of a finite or countable number of open intervals. We recall that the energy of u is constant and equal to $-1/\omega^2$, see point (ii) of Lemma 2.4.24.

The following result is a generalization in our particular setting of a known fact (see e.g. [8]).

Lemma 2.4.27. *The collision set $T_c(u)$ is discrete and has a finite number of elements.*

Proof. Assume by contradiction that t_0 is an accumulation point in the set $T_c(u)$, with $u(t_0) = c_j$. By continuity, only collisions in c_j can accumulate in t_0. In this case there exists a sequence of intervals $((a_n, b_n))$ with $(a_n, b_n) \subset [0, 1]$, $a_n \to t_0$ and $b_n \to t_0$ as $n \to \infty$, $u(a_n) = c_j = u(b_n)$ for every n, and

$$|u(t) - c_j| > 0 \quad \forall t \in (a_n, b_n).$$

On each of these intervals, since u is close to c_j (at least for n sufficiently large),

$$|u(t) - c_k| \geq C > 0 \qquad \text{for every } k \in \{1, \dots, N\},\ k \neq j.$$

Let us set $I(t) := |u(t) - c_j|^2$. Since $t \mapsto u(t)$ is a classical solution of (2.34) for $t \in (a_n, b_n)$, by differentiating twice $I(t)$ we obtain a modified Lagrange-Jacobi identity:

$$\ddot{I}(t) = -\frac{4}{\omega^2} + \frac{2}{\omega^2}(2 - \alpha)\frac{m_j}{\alpha|u(t) - c_j|^\alpha} + \frac{2}{\omega^2}\sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{m_k}{|u(t) - c_k|^\alpha}\left(\frac{2}{\alpha} - \frac{\langle u(t) - c_k, u(t) - c_j\rangle}{|u(t) - c_k|^2}\right).$$

Let $\xi_n \in (a_n, b_n)$ the maximizer of I in (a_n, b_n). It results $\ddot{I}(\xi_n) \leq 0$ for every n. Since in a neighbourhood of t_0 the second term in the expression of $\ddot{I}$ becomes arbitrarily large, while the other terms are bounded, we also get

$$\lim_{n\to\infty} \ddot{I}(\xi_n) = +\infty,$$

a contradiction. The collisions are isolated and, by compactness, the interval $[0, 1]$ contains only a finite number of them. □

Remark 2.4.28. The previous proof shows that, *if u collides in c_j, in a sufficiently small neighbourhood of c_j the function $I(t) = |u(t) - c_j|^2$ is strictly convex.*

Proposition 2.4.29. *If u is a minimizer of M in K_l, then one of the following situation occurs:*

(i) u parametrizes a path without self-intersections in points different from the centres c_j $(j = 1, \ldots, N)$,

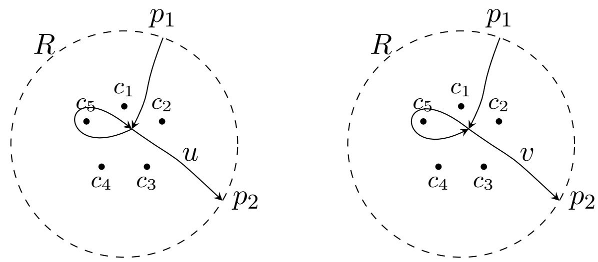
(ii) u parametrizes a path with at least one self-intersection in a point different from the centres; in such a situation, u has at least one collision, and at one collision-time $\bar{t}$ there is a reflection:

$$u(\bar{t} + t) = u(\bar{t} - t)$$

for t in a neighbourhood of $\bar{t}$.

Proof. Assume that we are not in case (ii). Then either we are in case (i), or u has a self-intersection at a point $p \neq c_j$ for every j: there exist $0 < t_* < t_{**} < 1$ such that $p = u(t_*) = u(t_{**})$; in this case, if u has a collision, then there is not any reflection with respect to a collision-time. Assume by contradiction that we are in this latter situation. Let (a, b) the connected component of $[0, 1] \setminus T_c(u)$ containing t_*. We know that $u|_{(a,b)}$ is a classical solution of (2.34), in particular it is of class $\mathcal{C}^2$. First we notice that, by the energy integral, $|\dot{u}(t)| > 0$ for every t such that $u(t) \in B_R(0)$, hence both $\dot{u}(t_*)$ and $\dot{u}(t_{**})$ are different from 0. One of the following alternatives has to occur: $\dot{u}(t_*)$ is transversal to $\dot{u}(t_{**})$, or $\dot{u}(t_*)$ is tangential to $\dot{u}(t_{**})$ with same or opposite direction. In the first two cases, let us define $v : [0, 1] \to \mathbb{R}^2$ as follows:

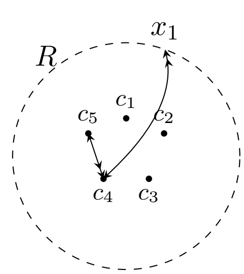
$$v(t) = \begin{cases} u(t) & t \in [0, t_*] \cup (t_{**}, 1], \\ u\left(\frac{t-t_*}{t_{**}-t_*} t_* + \left(1 - \frac{t-t_*}{t_{**}-t_*}\right) t_{**}\right) & t \in (t_*, t_{**}]. \end{cases}$$



The function v parametrizes a path with $u([0,1]) = v([0,1])$, but it goes along the loop connecting $u(t_*)$ and $u(t_{**})$ with the reversed orientation, see the above figure. The key observation is that this operation does not change the parity of the winding numbers with respect to the centres. Hence $v \in K_l$. Note that v is also an extremal for M, since $M(u) = M(v)$. On the other hand, it is trivially checked that, unless $\dot{u}(t_*) = \dot{u}(t_{**}) = 0$, v isn't $\mathcal{C}^1$ at those instants. So we have a new minimizer of M on K_l, which is collision-free in an interval $(a,d) \ni t_*$, and hence here should be a classical solution of (2.34); but this isn't possible since $v|_{(a,d)} \notin \mathcal{C}^1((a,d))$.

It could still be possible that the minimizer u has a tangential self-intersection with $\dot{u}(t_*)$ opposite to $\dot{u}(t_{**})$; this situation can be easily ruled out by the uniqueness theorem for initial value problem, taking into account the reversibility of the equation for u with respect to the involution $t \mapsto -t$: indeed, it turns out that $u(t_* + t) = u(t_{**} - t)$, but, since $B_R(0) \subset\subset \{V(u) > -h\}$, so that $\dot{u}(t) \neq 0$ for almost every $t \in (0,1)$, this is possible only if we are in case (ii) of the statement. □

The following picture represents the second alternative. Note that, due to the reversibility of the differential equation $\ddot{y} = \nabla V_\varepsilon(y)$ with respect to the involution $t \mapsto -t$, we don't reach a contradiction if we try to apply the proof above to the situation described in case (ii). In particular, if $p_1 = p_2$, it is possible that u is an ejection-collision minimizer.



Step 2) We would pass from a global analysis of the minimizer u to a local study in a neighbourhood of a collision. This is possible thanks to step 1: *u has an isolated*

collision at t_0 in a centre c_j, $j \in \{1, \dots, N\}$. In particular there exist $c, d \in [0,1]$ such that

- $c < t_0 < d$ and t_0 is the unique collision time in $[c,d]$,
- the function $I(t) = |u(t) - c_j|^2$ is strictly convex in $[c,d]$.

Let us set $\bar{p}_1 := u(c)$, $\bar{p}_2 = u(d)$. Since u is continuous, there exists $\mu > 0$ such that

$$|u(t) - c_k| \geq 2\mu > 0 \quad \text{for every } t \in [c,d] \text{ and for every } k \in \{1, \dots, N\} \setminus \{j\}.$$

This motivates us to write

$$V_\varepsilon(y) = \frac{m_j}{\alpha |y - c_j|^\alpha} + V_\varepsilon^j(y), \quad \text{where} \quad V_\varepsilon^j(y) := \sum_{\substack{k=1 \\ k \neq j}}^{N} \frac{m_k}{\alpha |y - c_k|^\alpha}.$$

Indeed, in a neighbourhood U_j of c_j such that $\mathrm{dist}(U_j, c_k) \geq \mu$ for every k, the potential V_ε splits in a principal component due to the attraction of c_j, and a perturbation term V_ε^j due to the attraction of the other centres. Of course, V_ε^j is smooth and bounded in U_j.

We define

$$\widehat{\mathcal{K}}_l^{\bar{p}_1\bar{p}_2} := \left\{ v \in H^1\left([c,d], \mathbb{R}^2\right) \;\middle|\; \begin{array}{l} v(c) = \bar{p}_1, v(d) = \bar{p}_2, \\ v(t) \neq c_j \ \forall t \in [c,d], \ \forall j, \\ \text{the function } \Gamma_v(t) := \begin{cases} u(t) & t \in [0,c) \cup (d,1] \\ v(t) & t \in [c,d] \end{cases} \\ \text{belongs to } K_l \end{array} \right\},$$

and

$$\mathcal{K}_l^{\bar{p}_1\bar{p}_2} := \widehat{\mathcal{K}}_l^{\bar{p}_1\bar{p}_2} \cup \left\{ v \in H^1([c,d], \mathbb{R}^2) : v(c) = \bar{p}_1, v(d) = \bar{p}_2, \Gamma_v \in \mathfrak{Coll}_l \right\}.$$

The set $\mathcal{K}_l^{\bar{p}_1\bar{p}_2}$ is weakly closed. We define the restriction of the Maupertuis functional to $\mathcal{K}_l^{\bar{p}_1\bar{p}_2}$ as

$$M_l^{\bar{p}_1\bar{p}_2} : \mathcal{K}_l^{\bar{p}_1\bar{p}_2} \to \mathbb{R} \cup \{+\infty\} \qquad M_l^{\bar{p}_1\bar{p}_2}(v) = \frac{1}{2} \int_c^d |\dot{v}(t)|^2 \, dt \int_c^d \left(V_\varepsilon(v(t)) - 1\right) \, dt.$$

It inherits the properties of weak lower semi-continuity and coercivity from M, then it has a minimum on $\mathcal{K}_l^{\bar{p}_1\bar{p}_2}$ at a positive level. Since u is a minimizer of M on K_l, from Proposition 2.4.9 it follows that $u|_{[c,d]}$ is a minimizer of $M_l^{\bar{p}_1\bar{p}_2}$ on $\mathcal{K}_l^{\bar{p}_1\bar{p}_2}$.

Step 3) We introduce further notations. For $\rho \geq 0$, we define

$$d(\rho) := \min \left\{ M_l^{\bar{p}_1\bar{p}_2}(v) : v \in \mathcal{K}_l^{\bar{p}_1\bar{p}_2}, \min_{t\in[c,d]} |v(t) - c_j| = \rho \right\}.$$

The value $d(0)$ is the minimum of $M_l^{\bar{p}_1\bar{p}_2}$ on the elements of $\mathcal{K}_l^{\bar{p}_1\bar{p}_2}$ which collide in c_j; as a consequence of our absurd assumption, $d(0)$ is achieved by $u|_{[c,d]}$.

Lemma 2.4.30. *The function $\rho \mapsto d(\rho)$ is continuous in $\rho = 0$.*

Proof. The proof is a slightly modification of that of Lemma 17 in [83]. We have to take into account that in our case collisions occur in c_j and not in 0, and that we are dealing with the Maupertuis functional and not with the action functional; nevertheless the same argument works, because we can rely on the same asymptotic estimates.
We want to prove that $\lim_{\rho\to 0^+} d(\rho) = d(0)$. Since the weak H^1 convergence implies the uniform one, the set

$$\left\{ v \in \mathcal{K}_l^{\bar{p}_1\bar{p}_2}, \min_{t\in[c,d]} |v(t) - c_j| = \rho \right\}$$

is weakly closed for every $\rho \geq 0$, and therefore the value $d(\rho)$ is achieved by an element of $\mathcal{K}_l^{\bar{p}_1\bar{p}_2}$; note in particular that $d(0)$ is achieved by $u|_{[c,d]}$. We know that $u(t_0) = c_j$ and t_0 is the unique collision time of u in $[c,d]$; by classical asymptotic estimates (see e.g. [8] or [19]) we deduce

$$\begin{cases} |u(t) - c_j| \simeq C|t - t_0|^{\frac{2}{\alpha+2}} \\ |\dot{u}(t)| \simeq C|t - t_0|^{-\frac{\alpha}{\alpha+2}}. \end{cases} \tag{2.35}$$

For $\rho > 0$ sufficiently small, let $\zeta_\pm(\rho)$ be positive solutions of

$$|u(t_0 + \zeta_+(\rho)) - c_j| = \rho, \qquad |u(t_0 - \zeta_-(\rho)) - c_j| = \rho.$$

From the asymptotic estimates (2.35) we infer

$$\zeta_\pm(\rho) \simeq C\rho^{\frac{\alpha+2}{2}}. \tag{2.36}$$

Let also $\theta_\pm$ be such that $u(t_+ \pm \zeta_\pm(\rho)) = c_j + \rho \exp\{i\theta_\pm\}$. We can define

$$u_\rho(t) := \begin{cases} u(t) & t \in [c,d] \setminus J(\rho) \\ c_j + \rho \exp\{i\theta(t)\} & t \in J(\rho), \end{cases}$$

where $J(\rho) := [t_0 - \zeta_-(\rho), t_0 + \zeta_+(\rho)]$ and $\theta(t)$ parametrizes an arc of the circle $\partial B_\rho(0)$, chosen in such a way that $u_\rho \in \mathcal{K}_l^{\bar{p}_1\bar{p}_2}$. To fix our minds, we suppose

$$\theta(t) = \frac{\theta_+(t - t_0 + \zeta_-(\rho)) - \theta_-(t - t_0 - \zeta_+(\rho))}{\zeta_+(\rho) + \zeta_-(\rho)} \quad \forall t \in J(\rho).$$

Note that $\min_t |u_\rho(t) - c_j| = \rho$.
We want to estimate the difference $M_l^{\bar{p}_1\bar{p}_2}(u) - M_l^{\bar{p}_1\bar{p}_2}(u_\rho)$; this can be done through a direct computation:

$$\begin{aligned} M_l^{\bar{p}_1\bar{p}_2}&(u) - M_l^{\bar{p}_1\bar{p}_2}(u_\rho) \\ &= \frac{1}{2}\left(\int_{[c,d]\setminus J(\rho)} |\dot{u}|^2 + \int_{J(\rho)} |\dot{u}|^2\right)\left(\int_{[c,d]-J(\rho)} (V_\varepsilon(u)-1) + \int_{J(\rho)} (V_\varepsilon(u)-1)\right) - \\ &- \frac{1}{2}\left(\int_{[c,d]\setminus J(\rho)} |\dot{u}|^2 + \int_{J(\rho)} |\dot{u}_\rho|^2\right)\left(\int_{[c,d]-J(\rho)} (V_\varepsilon(u)-1) + \int_{J(\rho)} (V_\varepsilon(u_\rho)-1)\right) \\ &\qquad\qquad = \frac{1}{2}\int_{[c,d]\setminus J(\rho)} |\dot{u}|^2 \left(\int_{J(\rho)} V_\varepsilon(u) - V_\varepsilon(u_\rho)\right) + \\ &\qquad\qquad + \frac{1}{2}\int_{[c,d]-J(\rho)} (V_\varepsilon(u)-1)\left(\int_{J(\rho)} |\dot{u}|^2 - |\dot{u}_\rho|^2\right) + \\ &\qquad + \frac{1}{2}\int_{J(\rho)} |\dot{u}|^2 \int_{J(\rho)} (V_\varepsilon(u)-1) - \frac{1}{2}\int_{J(\rho)} |\dot{u}_\rho|^2 \int_{J(\rho)} (V_\varepsilon(u_\rho)-1)\,. \end{aligned} \tag{2.37}$$

For every $t \in J(\rho)$ we have, using (2.35),

$$\begin{aligned} |V_\varepsilon(u(t)) - V_\varepsilon(u_\rho(t))| &\le \left|\frac{m_j}{\alpha}\left(\frac{1}{|u(t)-c_j|^\alpha} - \frac{1}{\rho^\alpha}\right)\right| + |V_\varepsilon^j(u(t)) - V_\varepsilon^j(u_\rho(t))| \\ &\le C(|t-t_0|^{-\frac{2\alpha}{\alpha+2}} + \rho^{-\alpha}) + C|u(t)-u_\rho(t)| \le C(|t-t_0|^{-\frac{2\alpha}{\alpha+2}} + \rho^{-\alpha}) + C\rho, \end{aligned}$$

so that taking into account the estimate (2.36), for every $\rho \ge 0$ small enough we have

$$\begin{aligned} \int_{J(\rho)} |V_\varepsilon(u(t)) - V_\varepsilon(u_\rho(t))|\,dt &\le \int_{J(\rho)} (C(|t-t_0|^{-\frac{2\alpha}{\alpha+2}} + \rho^{-\alpha}) + C\rho)\,dt \\ = C\left(\zeta_+(\rho)^{\frac{2-\alpha}{\alpha+2}} + \zeta_-(\rho)^{\frac{2-\alpha}{\alpha+2}}\right) &+ \left(\rho^{-\alpha} + c\rho\right)(\zeta_+(\rho) + \zeta_-(\rho)) \le C\rho^{\frac{2-\alpha}{\alpha}}. \end{aligned} \tag{2.38}$$

Also, for $\rho \ge 0$ sufficiently small

$$\begin{aligned} \int_{J(\rho)} |\dot{u}(t)|^2\,dt &\le C\int_{J(\rho)} |t-t_0|^{-\frac{2\alpha}{\alpha+2}}\,dt = C\rho^{\frac{2-\alpha}{\alpha}}, \\ \int_{J(\rho)} |\dot{u}_\rho(t)|^2\,dt &= C\int_{J(\rho)} \left(\frac{\rho}{\zeta_+(\rho)+\zeta_-(\rho)}\right)^2 dt \le C\rho^{\frac{2-\alpha}{2}}. \end{aligned} \tag{2.39}$$

We can come back to equation (2.37): collecting (2.38) and (2.39), for every $\rho \ge 0$ sufficiently small we obtain

$$|M_l^{\bar{p}_1\bar{p}_2}(u) - M_l^{\bar{p}_1\bar{p}_2}(u_\rho)| \le C\rho^{\frac{2-\alpha}{2}}.$$

In particular

$$d(\rho) \leq M_l^{\bar{p}_1\bar{p}_2}(u_\rho) \leq M_l^{\bar{p}_1\bar{p}_2}(u) + C\rho^{\frac{2-\alpha}{2}} = d(0) + C\rho^{\frac{2-\alpha}{2}} \qquad \forall \rho \ll 1,$$

which implies

$$\limsup_{\rho\to 0^+} d(\rho) \leq d(0). \tag{2.40}$$

It is not difficult to conclude the proof: let (ρ_n) be a sequence of positive real numbers such that $\rho_n \to 0$ and $d(\rho_n) \to \liminf_{\rho\to 0^+} d(\rho)$ as $n \to \infty$; we can find $(u_n) \subset \mathcal{K}_l^{\bar{p}_1\bar{p}_2}$ such that

$$\min_{t\in[c,d]} |u_n(t) - c_j| = \rho_n \quad \text{and} \quad M_l^{\bar{p}_1\bar{p}_2}(u_n) = d(\rho_n).$$

Since $(M_l^{\bar{p}_1\bar{p}_2}(u_n))$ is bounded and $M_l^{\bar{p}_1\bar{p}_2}$ is coercive, the sequence (u_n) is bounded in H^1 and therefore, up to a subsequence, it is weakly convergent in H^1 (and hence uniformly, too) to a $\bar{u} \in M_l^{\bar{p}_1\bar{p}_2}$; note that $\bar{u}$ has a collision. Using the weak lower semi-continuity of $M_l^{\bar{p}_1\bar{p}_2}$, we obtain

$$d(0) \leq M_l^{\bar{p}_1\bar{p}_2}(\bar{u}) \leq \liminf_{n\to\infty} M_l^{\bar{p}_1\bar{p}_2}(u_n) = \liminf_{\rho\to 0^+} d(\rho),$$

which together with (2.40) gives the thesis. □

Now, given $0 < \rho_1 < \rho_2$, we set

$$\mathcal{K}_l^{\bar{p}_1\bar{p}_2}(\rho_1,\rho_2) := \left\{ v \in \mathcal{K}_l^{\bar{p}_1\bar{p}_2} : \min_{t\in[c,d]} |v(t) - c_j| \in [\rho_1, \rho_2] \right\}.$$

It is a weakly closed subset of $\mathcal{K}_l^{\bar{p}_1\bar{p}_2}$, so it is well defined

$$m(\rho_1,\rho_2) := \min_{v\in\mathcal{K}_l^{\bar{p}_1\bar{p}_2}(\rho_1,\rho_2)} M_l^{\bar{p}_1\bar{p}_2}(v).$$

We also set

$$\mathcal{M}_{\rho_1\rho_2} := \left\{ v \in \mathcal{K}_l^{\bar{p}_1\bar{p}_2}(\rho_1,\rho_2) : M_l^{\bar{p}_1\bar{p}_2}(v) = m(\rho_1,\rho_2) \text{ and } \min_{t\in[c,d]} |v(t) - c_j| < \rho_2 \right\}.$$

In this step we aim at proving the following result. In light of the notation introduced in this section, Theorem 2.4.14 follows.

Proposition 2.4.31. *(i) If $\alpha \in (1,2)$, there exists $\bar{\rho} > 0$ such that for ρ_1, $\rho_2 : 0 < \rho_1 < \rho_2 \leq \bar{\rho}$ implies $\mathcal{M}_{\rho_1\rho_2} = \emptyset$; in this case u is collision-free;*
(ii) if $\alpha = 1$, then one of the following alternatives occurs:

(a) there exists $\bar{\rho} > 0$ *such that for* $\rho_1,\ \rho_2 : 0 < \rho_1 < \rho_2 \leq \bar{\rho}$ *implies* $\mathcal{M}_{\rho_1\rho_2} = \emptyset$*; in this case* u *is collision-free;*

(b) u *is an ejection-collision minimizer, with a unique collision. This is possible only if* $p_1 = p_2$ *and* l *satisfies condition* (2.27).

Remark 2.4.32. The proposition states that, if $\alpha \in (1,2)$ or $\alpha = 1$ and $p_1 \neq p_2$, if we force the functions to go very close to c_j, i.e.

$$\min_{t\in[c,d]} |v(t) - c_j| < \bar{\rho},$$

then the minima $m(\rho_1, \rho_2)$ are achieved by elements of $\mathcal{K}_l^{\bar{p}_1\bar{p}_2}(\rho_1, \rho_2)$ which stay as far as possible from c_j. If $\alpha = 1$, $p_1 = p_2$, and l satisfies condition (2.27), a minimizer can have a collision, but in such a situation it is an ejection-collision minimizer.

The proof of this proposition occupy the rest of the section. We will use a lot of intermediate results which hold true both for $\alpha \in (1,2)$ and $\alpha = 1$; so, unless otherwise specified, we will consider $\alpha \in [1,2)$. We will explicitly point out the moment in which we will assume $\alpha \in (1,2)$ or $\alpha = 1$.

Assume that

- either $\alpha \in (1,2)$ and our statement is false;
- or $\alpha = 1$ and we are not in case (ii)-(a).

Then there are two sequences (ρ_n), $(\bar{\rho}_n)$ such that

$$\begin{gathered} 0 < \rho_n < \bar{\rho}_n \quad \forall n, \qquad \rho_n \to 0, \bar{\rho}_n \to 0, \quad \text{for } n \to \infty, \\ \forall n\ \exists u_n \in \mathcal{K}_l^{\bar{p}_1\bar{p}_2} : \min_{t\in[c,d]} |u_n(t) - c_j| = \rho_n, \\ M_l^{\bar{p}_1\bar{p}_2}(u_n) = m(\rho_n, \bar{\rho}_n) = d(\rho_n). \end{gathered} \tag{2.41}$$

We can assume also that for every $n \in \mathbb{N}$

$$\max\left\{ \inf_{y\in\partial B_{\rho_n}(c_j)} |\bar{p}_1 - y|, \inf_{y\in\partial B_{\rho_n}(c_j)} |\bar{p}_2 - y| \right\} > 0.$$

Thanks to Lemma 2.4.30, $M_l^{\bar{p}_1\bar{p}_2}(u_n) \to d(0)$ for $n \to \infty$, namely (u_n) is a minimizing sequence in $\mathcal{K}_l^{\bar{p}_1\bar{p}_2}$ (we are assuming that the minimum of $M_l^{\bar{p}_1\bar{p}_2}$ in $\mathcal{K}_l^{\bar{p}_1\bar{p}_2}$ is achieved over collisions). Since $M_l^{\bar{p}_1\bar{p}_2}$ is coercive, (u_n) is bounded and up to a subsequence is weakly convergent to a function $\widetilde{u} \in \mathcal{K}_l^{\bar{p}_1\bar{p}_2}$, which is a minimizer of $M_l^{\bar{p}_1\bar{p}_2}$ (possibly different from $u|_{[c,d]}$) due to the weak lower semi-continuity of $M_l^{\bar{p}_1\bar{p}_2}$. We point out that $\widetilde{u}$ has

to collide in c_j and could collide in centres different from c_j as well.
By Lemma 2.4.3, the energy of $\widetilde{u}$ is constant and equal to $-1/\widetilde{\omega}^2$, where

$$\widetilde{\omega}^2 := \frac{\int_c^d V_\varepsilon(\widetilde{u}) - 1}{\frac{1}{2}\int_c^d |\dot{\widetilde{u}}|^2}.$$

Now, the same discussion of step 1 shows that the set $T_c(\widetilde{u})$ of collision times of $\widetilde{u}$ contains a finite number of elements, and we can assume that

- there exists a unique collision time t_0 in $[c,d]$ such that $\widetilde{u}(t_0) = c_j$;
- there exists $\mu > 0$ such that $|\widetilde{u}(t) - c_k| \geq 2\mu > 0$ for every $t \in [c,d]$, for every $k \neq j$;
- the function $|\widetilde{u}(t) - c_j|^2$ is strictly convex in $[c,d]$.

Otherwise we can replace $[c,d]$ with a smaller interval.

The paths u_n enjoy some common properties. Firstly, since the weak convergence in H^1 implies the uniform one, there exists $n_0 \in \mathbb{N}$ such that

$$n \geq n_0 \quad \Longrightarrow \quad |u_n(t) - c_k| \geq \mu \quad \forall t \in [c,d],\ \forall k \neq j. \tag{2.42}$$

We rename as (u_n) the sequence obtained by dropping the first $(n_0 - 1)$-terms. Let us set

$$T_{\rho_n}(u_n) = \{t \in [c,d] : |u_n(t) - c_j| = \rho_n\}.$$

We also introduce the polar coordinates and the (absolute value of the) angular momentum of u_n with respect to the centre c_j:

$$u_n(t) = c_j + w_n(t)e^{i\phi_n(t)},$$
$$\mathfrak{C}_n^j(t) := |(u_n(t) - c_j) \wedge \dot{u}_n(t)|.$$

Here $w_n : [c,d] \to \mathbb{R}^+$ and $\phi_n : [c,d] \to \mathbb{R}$.

Lemma 2.4.33. *For every $n \in \mathbb{N}$, the function u_n has the following properties:*

(i) every u_n is of class $\mathcal{C}^1((c,d))$, and if (c',d') is a connected component of $[c,d] \setminus T_{\rho_n}(u_n)$, then $u_n|_{(c',d')}$ is $\mathcal{C}^2$ and solves

$$\omega_n^2 \ddot{u}_n(t) = \nabla V_\varepsilon(u_n(t)) \quad \textit{where} \quad \omega_n^2 := \frac{\int_c^d V_\varepsilon(u_n) - 1}{\frac{1}{2}\int_c^d |\dot{u}_n|^2}; \tag{2.43}$$

(ii) for every $n \in \mathbb{N}$*, there exist* $t_n^- \leq t_n^+$ *such that:*

$$\begin{aligned} |u_n(t) - c_j| &> \rho_n \qquad t \in [c, t_n^-) \cup (t_n^+, d] \\ |u_n(t) - c_j| &= \rho_n \qquad t \in [t_n^-, t_n^+], \end{aligned}$$

that is, $T_{\rho_n}(u_n) = [t_n^-, t_n^+]$*;*

(iii) the sequence (ω_n^2) *is bounded above and below by strictly positive constants. Hence there exist a subsequence of* (u_n) *(still denoted* (u_n)*) and* $\Omega > 0$ *such that*

$$\lim_{n\to\infty} \omega_n = \Omega;$$

(iv) the energy of the function u_n *is constant in* $[c, d]$*:*

$$\frac{1}{2}|\dot{u}_n(t)|^2 - \frac{V_\varepsilon(u_n(t))}{\omega_n^2} = -\frac{1}{\omega_n^2} \qquad \forall t \in [c, d].$$

Moreover, the sequence $(-1/\omega_n^2)$ *is bounded in* $\mathbb{R}$*;*

(v) the function $\phi_n|_{(t_n^-, t_n^+)}$ *is* $\mathcal{C}^2$*, strictly monotone and is a solution of*

$$\ddot{\phi}_n(t) = \frac{1}{\rho_n \omega_n^2} \left\langle \nabla V_\varepsilon \left(c_j + \rho_n e^{i\phi_n(t)} \right), ie^{i\phi_n(t)} \right\rangle.$$

Proof. To show that each u_n is $\mathcal{C}^1$, we can slightly modify the proof of Proposition 2.4.22; to complete the point (i), and to prove point (v), it is sufficient to adapt the proof of (i) and (ii) of Lemma 2.4.24.
(ii) On every interval $(c', d') \subset (c, d) \setminus T_{\rho_n}(u_n)$, the function u_n solves equation (2.43); hence the uniform convergence of (u_n) to $\widetilde{u}$ and the computation of the derivative

$$\frac{d^2}{dt^2}|u_n(t) - c_j|^2$$

(see the proof of Lemma 2.4.27) imply that the function $|u_n(t) - c_j|^2$ is strictly convex over such an interval. Therefore, if there exist $t_1 < t_2$ such that $|u_n(t_1) - c_j| = |u_n(t_2) - c_j| = \rho_n$ then $|u_n(t) - c_j| = \rho_n$ for every $t \in (t_1, t_2)$.
(iii) We have

$$\omega_n^2 = \frac{M_l^{\bar{p}_1 \bar{p}_2}(u_n)}{\frac{1}{4}\left(\int_c^d |\dot{u}_n|^2\right)^2} = \frac{d(\rho_n)}{\frac{1}{4}\|\dot{u}_n\|_2^4}. \tag{2.44}$$

We know that $0 < d(0) < d(\rho_n)$ and $d(\rho_n) \to d(0)$, so that

$$\exists C_1, C_2 > 0 : \; C_1 \leq d(\rho_n) \leq C_2 \; \forall n. \tag{2.45}$$

As far as the denominator of (2.44) is concerned, we observe that, for every n, the path u_n covers at least a fixed distance; therefore, as showed in the proof of Lemma 2.4.18, there exists $C_3 > 0$ such that

$$\|\dot{u}_n\|_2 \geq C_3 \qquad \forall n. \tag{2.46}$$

Moreover, being (u_n) a minimizing sequence of a coercive functional, (u_n) is bounded in the H^1 norm and a fortiori there exists $C_4 > 0$ such that

$$\|\dot{u}_n\|_2 \leq C_4 \qquad \forall n. \tag{2.47}$$

Altogether, (2.44), (2.45), (2.46) and (2.47) imply the assertion.
(iv) It is a slightly modification of the proof of point *(ii)* of Lemma 2.4.24, and of point *(iii)*. □

We are now in a position to prove the following result.

Lemma 2.4.34. *The minimizer u_n is free of self-intersections in $[c, d]$. In particular, the total variation of the angle ϕ_n is smaller then 2π.*

Proof. The function u_n has no self-intersections for $t \in [c, t_n^-) \cup (t_n^+, d]$. The prove is the same of that of Proposition 2.4.29. If u_n has a self-intersection on the obstacle $\{|y - c_j| = \rho_n\}$, the monotonicity of ϕ_n implies that u_n makes a complete wind around it. But then we can consider the function v which parametrizes the same path of u_n, but reverses the orientation on the obstacle. One has $M_l^{\bar{p}_1\bar{p}_2}(u_n) = M_l^{\bar{p}_1\bar{p}_2}(v)$, so that v is a local minimizer of $M_l^{\bar{p}_1\bar{p}_2}$ with $\min_{t\in[c,d]} |v(t) - c_j| = \rho_n$. For the minimality, v satisfies the energy integral and cannot approach the obstacle with velocity 0. Therefore it should be a minimizer which is not $\mathcal{C}^1$, a contradiction. □

Lemma 2.4.35. *The estimates*

$$\mathfrak{C}_n^j(t) = \rho_n^{\frac{2-\alpha}{2}} \sqrt{\frac{2m_j}{\omega_n^2 \alpha}} \left(1 + O(\rho_n^\alpha)\right) \quad \forall t \in [t_n^-, t_n^+], \qquad t_n^+ - t_n^- = O(\rho_n^{\frac{\alpha+2}{2}})$$

hold for $n \to \infty$.

Proof. Since $u_n \in \mathcal{C}^1((c, d))$, it can lean against the obstacle $\{|y - c_j| = \rho_n\}$ with velocity $\dot{u}_n(t)$ orthogonal to the radial segment joining c_j and $u_n(t)$. Therefore for every $t \in [t_n^-, t_n^+]$ there holds

$$\rho_n |\dot{u}_n(t)| = \mathfrak{C}_n^j(t) = \rho_n^2 \dot{\phi}_n(t). \tag{2.48}$$

From the expression of the energy and the uniform boundedness of $(V_\varepsilon^j(u_n))$ (see (2.42)), we deduce that

$$\rho_n|\dot{u}_n(t)| = \rho_n\sqrt{\frac{2}{\omega_n^2}\left(\frac{m_j}{\alpha\rho_n^\alpha} + V_\varepsilon^j\,(u_n(t)) - 1\right)}$$
$$= \rho_n^{\frac{2-\alpha}{2}}\sqrt{\frac{2m_j}{\omega_n^2\alpha} + \frac{2\rho_n^\alpha}{\omega_n^2}\left(V_\varepsilon^j(u_n(t)) - 1\right)} = \rho_n^{\frac{2-\alpha}{2}}\sqrt{\frac{2m_j}{\omega_n^2\alpha}}\,(1 + O(\rho_n^\alpha))\,.$$

Plugging in (2.48) we have

$$\dot{\phi}_n(t) = \rho_n^{-\frac{2+\alpha}{2}}\sqrt{\frac{2m_j}{\omega_n^2\alpha}}\,(1 + O(\rho_n^\alpha))\,,$$

and the total variation of ϕ_n on the obstacle is

$$\phi_n(t_n^+) - \phi_n(t_n^-) = \rho_n^{-\frac{2+\alpha}{2}}\sqrt{\frac{2m_j}{\omega_n^2\alpha}}\,(1 + O(\rho_n^\alpha))\,(t_n^+ - t_n^-).$$

This variation is bounded by 2π, so that $t_n^+ - t_n^- = O(\rho_n^{\frac{\alpha+2}{2}})$. □

In order to exploit a careful analysis of the behaviour of $\widetilde{u}$ in a neighbourhood of the collision time t_0, we consider a blow-up of our sequence.
For every $n \in \mathbb{N}$, let us fix $t_n \in [t_n^-, t_n^+]$. By the previous lemma, the sequence (t_n) tends to a limit, which by continuity is the unique collision time t_0 of $\widetilde{u}$ in (c, d). Let us set

$$c_n := \rho_n^{-\frac{\alpha+2}{2}}(c - t_n), \qquad d_n := \rho_n^{-\frac{\alpha+2}{2}}(d - t_n).$$

We also define

$$s_n^- := \rho_n^{-\frac{\alpha+2}{2}}(t_n^- - t_n), \qquad s_n^+ := \rho_n^{-\frac{\alpha+2}{2}}(t_n^+ - t_n)$$

We note that $c_n \to -\infty$, $d_n \to +\infty$ as $n \to \infty$. As far as (s_n^-) and (s_n^+) are concerned, they are two bounded sequences thanks to Lemma 2.4.35, so that there exists a subsequence of (ρ_n) (which we still denote (ρ_n)) such that they converge to limits s^- and s^+ respectively.

Remark 2.4.36. Consider the change of variable

$$s^n(t) = \rho_n^{-\frac{\alpha+2}{2}}(t - t_n) \quad\Longleftrightarrow\quad t^n(s) = t_n + \rho_n^{\frac{\alpha+2}{2}}s.$$

One has

$$s^n(t) \in [c_n, d_n] \quad\Longleftrightarrow\quad t^n(s) \in [c, d], \qquad s^n(t) \in [s_n^-, s_n^+] \quad\Longleftrightarrow\quad t^n(s) \in [t_n^-, t_n^+].$$

We introduce the blow-up sequence of paths $v_n : [c_n, d_n] \to \mathbb{R}^2$ defined by

$$v_n(s) := c_j + \frac{1}{\rho_n}\left(u_n\left(t_n + \rho_n^{\frac{\alpha+2}{2}} s\right) - c_j\right).$$

In polar coordinates with respect to the centre c_j we write

$$v_n(s) = c_j + \bar{w}_n(s) e^{i\bar{\phi}_n(s)},$$

where

$$\bar{w}_n(s) = \frac{1}{\rho_n} w_n\left(t_n + \rho_n^{\frac{\alpha+2}{2}} s\right), \qquad \bar{\phi}_n(s) = \phi_n\left(t_n + \rho_n^{\frac{\alpha+2}{2}} s\right).$$

Each v_n is of class $\mathcal{C}^1$ and

$$\begin{aligned} |v_n(s) - c_j| &= 1 \qquad \text{for } s \in [s_n^-, s_n^+], \\ |v_n(s) - c_j| &> 1 \qquad \text{for } s \in [c_n, s_n^-) \cup (s_n^+, d_n]. \end{aligned}$$

The restriction $v_n|_{[c_n, s_n^-) \cup (s_n^+, d_n]}$ is of class $\mathcal{C}^2$ and satisfies the equation

$$\begin{aligned} \ddot{v}_n(s) &= -\frac{\rho_n^{2+\alpha}}{\omega_n^2 \rho_n} \sum_{k=1}^{N} \frac{m_k}{|u_n(t^n(s)) - c_k|^{\alpha+2}} \left(u_n(t^n(s)) - c_k\right) \\ &= -\frac{m_j\left[\frac{1}{\rho_n}\left(u_n(t^n(s)) - c_j\right) \pm c_j\right]}{\omega_n^2\left|\frac{1}{\rho_n}\left(u_n(t^n(s)) - c_j\right) \pm c_j\right|^{\alpha+2}} + \frac{\rho_n^{\alpha+1}}{\omega_n^2} \nabla V_\varepsilon^j\left(u_n(t^n(s))\right) \\ &= -\frac{m_j}{\omega_n^2 |v_n(s) - c_j|^{\alpha+2}} \left(v_n(s) - c_j\right) + O(\rho_n^{\alpha+1}). \end{aligned}$$

This suggests to consider the quantity

$$\bar{h}_n(s) := \frac{1}{2}|\dot{v}_n(s)|^2 - \frac{m_j}{\omega_n^2 \alpha |v_n(s) - c_j|^{\alpha}},$$

the energy of the function v_n for the potential of the α-Kepler problem with centre in c_j. This is not a constant function in $[c_n, d_n]$, however it can be easily controlled:

$$\begin{aligned} \bar{h}_n(s) &= \rho_n^\alpha\left[\frac{1}{2}\left|\dot{u}_n(t^n(s))\right|^2 - \frac{m_j}{\omega_n^2 \alpha \left|u_n(t^n(s)) - c_j\right|^\alpha}\right] \\ &= \rho_n^\alpha\left[-\frac{1}{\omega_n^2} + \frac{1}{\omega_n^2} V_\varepsilon^j\left(u_n(t^n(s))\right)\right]. \end{aligned}$$

Therefore, from the point (iv) of Lemma 2.4.33 we deduce

$$\lim_{n\to\infty} \bar{h}_n(s) = 0 \quad \text{for every } s \in \mathbb{R}.$$

The uniform boundedness of $(V_\varepsilon^j(u_n))$ makes the convergence uniform on every closed interval $[a,b] \subset \mathbb{R}$.

Let us also define the (absolute value of the) angular momentum of v_n with respect to the centre c_j:

$$\bar{\mathfrak{C}}_n^j(s) := |(v_n(s) - c_j) \wedge \dot{v}_n(s)| \,.$$

If $s \in [s_n^-, s_n^+]$, using Lemma 2.4.35 we obtain

$$\bar{\mathfrak{C}}_n^j(s) = \rho_n^{\frac{\alpha+2}{2}} \dot{\phi}_n\left(t^n(s)\right) = \rho_n^{\frac{\alpha-2}{2}} \mathfrak{C}_n^j\left(t^n(s)\right) = \sqrt{\frac{2m_j}{\omega_n^2\alpha}}\left(1 + O(\rho_n^\alpha)\right).$$

Hence

$$\lim_{n\to\infty} \bar{\mathfrak{C}}_n^j(s) = \sqrt{\frac{2m_j}{\Omega^2\alpha}}, \quad \text{for every } s \in [s^-, s^+], \tag{2.49}$$

with uniform convergence in $[s^-, s^+]$. For the reader's convenience, we recall that $\Omega = \lim_n \omega_n$. The previous computation implies that the sequence $(\bar{\mathfrak{C}}_n^j|_{[s^-,s^+]})$ is uniformly bounded.

Recalling the point (v) of Lemma 2.4.33, we obtain an equation for $\bar{\phi}_n$ when $s \in (s_n^-, s_n^+)$:

$$\begin{aligned}
\ddot{\bar{\phi}}_n(s) &= \frac{\rho_n^{\alpha+1}}{\omega_n^2}\left\langle \nabla V_\varepsilon\left(c_j + \rho_n e^{i\bar{\phi}_n(s)}\right), ie^{i\bar{\phi}_n(s)}\right\rangle \\
&= -\frac{1}{\omega_n^2}\left\langle m_j e^{i\bar{\phi}_n(s)}, ie^{i\bar{\phi}_n(s)}\right\rangle + \frac{\rho_n^{\alpha+1}}{\omega_n^2}\left\langle \nabla V_\varepsilon^j\left(c_j + \rho_n e^{i\bar{\phi}_n(s)}\right), ie^{i\bar{\phi}_n(s)}\right\rangle \\
&= 0 + O(\rho_n^{\alpha+1}).
\end{aligned}$$

As a consequence, the restriction $v_n|_{(s_n^-,s_n^+)}$ is of class $\mathcal{C}^2$ and satisfies

$$\begin{aligned}
\ddot{v}_n(s) &= \ddot{\bar{\phi}}_n(s) ie^{i\bar{\phi}_n(s)} - \left(\dot{\bar{\phi}}_n(s)\right)^2 e^{i\bar{\phi}_n(s)} \\
&= \ddot{\bar{\phi}}_n(s) i\left(v_n(s) - c_i\right) - \left(\bar{\mathfrak{C}}_n^j(s)\right)^2 \left(v_n(s) - c_i\right) \\
&= -\left(\bar{\mathfrak{C}}_n^j(s)\right)^2 \left(v_n(s) - c_i\right) + i\left(v_n(s) - c_i\right) O(\rho_n^{\alpha+1}).
\end{aligned}$$

Summing up

$$\ddot{v}_n = \begin{cases} -\frac{m_j(v_n - c_i)}{\omega_n^2|v_n - c_i|^{\alpha+2}} + O(\rho_n^{\alpha+1}) & \text{in } [c_n, s_n^-) \cup (s_n^+, d_n] \\ \\ -\left(\bar{\mathfrak{C}}_n^j\right)^2 (v_n - c_i) + i\left(v_n - c_i\right) O(\rho_n^{\alpha+1}) & \text{in } (s_n^-, s_n^+). \end{cases} \tag{2.50}$$

This shows that v_n is not necessarily of class $\mathcal{C}^2$ in s_n^- and s_n^+; anyway there exist the right and left limits of the second derivative at these points.

Lemma 2.4.37. *Let $[a,b] \subset \mathbb{R}$, $a \leq 0 \leq b$. There exists a subsequence of (v_n) which converges in the $\mathcal{C}^1$ topology on $[a,b]$.*

Proof. There is uniform convergence to 0 of the energies $\bar{h}_n$ over $[a,b]$; thus the restrictions $(\bar{h}_n|_{[a,b]})$ define a bounded sequence in the uniform topology. Since for every n

$$\inf_{s\in[a,b]} |v_n(s) - c_j| = |v_n(0) - c_j| = 1,$$

for every $s \in [a,b]$

$$|\dot{v}_n(s)|^2 = 2\bar{h}_n(s) + \frac{2m_j}{\omega_n^2\alpha\,|v_n(s) - c_j|^\alpha} \leq 2\|\bar{h}_n|_{[a,b]}\|_\infty + 2\frac{m_j}{\omega_n^2\alpha}.$$

Therefore

$$\|\dot{v}_n|_{[a,b]}\|_\infty \leq \sqrt{2}\sup_n \left(\|\bar{h}_n|_{[a,b]}\|_\infty + \frac{m_j}{\omega_n^2\alpha}\right)^{\frac{1}{2}} < +\infty,$$

i.e. $\left(\dot{v}_n|_{[a,b]}\right)$ is uniformly bounded. Now,

1. $(v_n|_{[a,b]})$ is equi-continuous: for every $s_1, s_2 \in [a,b]$, for every $n \in \mathbb{N}$
$$|v_n(s_1) - v_n(s_2)| \leq \|\dot{v}_n|_{[a,b]}\|_\infty |s_1 - s_2| \leq C|s_1 - s_2|.$$
2. $\left(v_n|_{[a,b]}\right)$ is uniformly bounded: for every $s \in [a,b]$, for every $n \in \mathbb{N}$:
$$|v_n(s)| \leq |v_n(0)| + C|s| \leq \varepsilon + 1 + C\max\{|a|,|b|\}.$$

Hence we can apply the Ascoli-Arzelà theorem, to obtain a uniformly converging subsequence (still denoted by (v_n)). From equation (2.50) we see also that $(\ddot{v}_n|_{[a,b]})$ is uniformly bounded. Indeed

$$\begin{aligned}
&|\ddot{v}_n(s)| \leq \frac{m_j}{\omega_n^2} + O(\rho_n^{\alpha+1}) \leq C < +\infty \qquad \text{for every } s \in [c_n, s_n^-) \cup (s_n^+, d_n],\\
&|\ddot{v}_n(s)| \leq \left(\bar{\mathfrak{C}}_n^j(s)\right)^2 + O(\rho_n^{\alpha+1}) \leq C < +\infty \qquad \text{for every } s \in (s_n^-, s_n^+)\\
&\max\left\{\lim_{s\to\left(s_n^\pm\right)^\pm} |\ddot{v}(s)|\right\} = C < +\infty,
\end{aligned}$$

(recall (2.49) for the second bound) and immediately $\sup_n \|\ddot{v}_n|_{[a,b]}\|_\infty < +\infty$. Moreover

$$\lim_{n\to\infty} \frac{1}{2}|\dot{v}_n(0)|^2 = \lim_{n\to\infty} \bar{h}_n(0) + \frac{m_j}{\omega_n^2\alpha} = \frac{m_j}{\Omega^2\alpha}.$$

In particular, $(\dot{v}_n(0))$ is bounded, too. Now it is sufficient to repeat the previous argument and use the Ascoli-Arzelà theorem for $(\dot{v}_n)$. □

Applying the Lemma on each interval $[-k, k]$ we obtain a subsequence of (v_n) (still denoted by (v_n)) which converges in the $\mathcal{C}^1$ topology on every closed interval of $\mathbb{R}$ (this is a standard diagonal selection). We call $v : \mathbb{R} \to \mathbb{R}^2$ its limit. We write

$$v(s) = c_j + w(s) \exp\{i\phi(s)\}.$$

By equation (2.50), the sequence $(\ddot{v}_n)$ uniformly converges on every compact subset of $\mathbb{R} \setminus \{s^-, s^+\}$, so $v \in \mathcal{C}^2 (\mathbb{R} \setminus \{s^-, s^+\})$ and

- v is a classical solution of the α-Kepler problem
$$\ddot{v}(s) = -\frac{m_j}{\Omega^2 |v(s) - c_j|^{\alpha+2}} (v(s) - c_j) \quad \text{for } s \in (-\infty, s^-) \cup (s^+, +\infty);$$
- v has constant energy equal to 0 (even in $[s^-, s^+]$);
- v has constant angular momentum with respect to c_j, whose modulus is $\bar{\mathfrak{C}}^j = \sqrt{\frac{2m_j}{\Omega^2 \alpha}}$ (even in $[s^-, s^+]$); indeed from the equation for v and the (2.49) it follows that $\bar{\mathfrak{C}}^j$ is constant in the three intervals $(-\infty, s^-)$, $[s^-, s^+]$ and $(s^+, +\infty)$, and $\bar{\mathfrak{C}}^j = \sqrt{\frac{2m_j}{\Omega^2 \alpha}}$ in $[s^-, s^+]$. Let us consider $s \in (-\infty, s^-)$; using the conservation of the energy and the fact that $v \in \mathcal{C}^1(\mathbb{R})$ we have
$$\begin{aligned}\left(\bar{\mathfrak{C}}^j(s)\right)^2 = \frac{2m_j}{\alpha\Omega^2} w(s)^{2-\alpha} - w(s)^2 \dot{w}^2(s) &= \lim_{s \to (s^-)^-} \frac{2m_j}{\alpha\Omega^2} w(s)^{2-\alpha} - w(s)^2 \dot{w}^2(s) \\ &= \lim_{s \to (s^-)^+} \frac{2m_j}{\alpha\Omega^2} w(s)^{2-\alpha} - w(s)^2 \dot{w}^2(s) = \frac{2m_j}{\alpha\Omega^2},\end{aligned}$$
for every $s \in (-\infty, s^-)$. The same argument works for $s \in (s^+, +\infty)$. Hence $\bar{\mathfrak{C}}^j$ is constant in $\mathbb{R}$, and the conservation of the angular momentum follows;
- $|v(s) - c_j| = 1$ for $s \in [s^-, s^+]$;
- $|v(s) - c_j| > 1$ for $s \in (-\infty, s^-) \cup (s^+, +\infty)$.

Let $\phi^- := \phi(s^-)$, $\phi^+ := \phi(s^+)$. Thanks to the conservation of the angular momentum, the function $s \mapsto \phi(s)$ is strictly monotone; it is not restrictive to assume that it is increasing, and it makes sense to write

$$\phi(+\infty) = \lim_{s \to +\infty} \phi(s), \qquad \phi(-\infty) = \lim_{s \to -\infty} \phi(s).$$

Writing the energy in polar coordinates we get

$$ds = \frac{dw}{\sqrt{2\left(\frac{m_j}{\alpha\Omega^2 w^\alpha} - \frac{\left(\bar{\mathfrak{C}}^j\right)^2}{w^2}\right)}}.$$

Hence

$$\phi(+\infty)-\phi^+ = \int_{s^+}^{+\infty} \frac{d\phi}{ds}\,ds = \int_1^{+\infty} \frac{\bar{\mathfrak{C}}^j\,dw}{w^2\sqrt{\frac{2m_j}{\alpha\Omega^2 w^\alpha}-\frac{\left(\bar{\mathfrak{C}}^j\right)^2}{w^2}}}$$
$$= \int_1^{+\infty} \frac{dw}{w^2\sqrt{\frac{1}{w^\alpha}-\frac{1}{w^2}}} = \int_0^1 \frac{d\xi}{\sqrt{\xi^\alpha-\xi^2}}.$$

The same computation holds true for $\phi^- - \phi(-\infty)$. With the change of variable $\xi = \eta^{\frac{2}{2-\alpha}}$ we obtain

$$\phi(+\infty)-\phi^+ = \phi^- - \phi(-\infty) = \frac{2}{2-\alpha}\int_0^1 \frac{\eta^{\frac{\alpha}{2-\alpha}}}{\sqrt{\eta^{\frac{2\alpha}{2-\alpha}}-\eta^{\frac{4}{2-\alpha}}}}\,d\eta$$
$$= \frac{2}{2-\alpha}\int_0^1 \frac{d\eta}{\sqrt{1-\eta^2}} = \frac{\pi}{2-\alpha}.$$

We deduce the following estimate for the total variation of the angle ϕ:

$$\phi(+\infty)-\phi(-\infty) = \frac{2\pi}{2-\alpha} + \phi^+ - \phi^- \geq \frac{2\pi}{2-\alpha}. \tag{2.51}$$

On the other hand, we know that $\bar{\phi}_n$ uniformly converges to ϕ on every closed interval $[a,b]$ of $\mathbb{R}$. For n sufficiently large

$$\bar{\phi}_n(b) - \bar{\phi}_n(a) \leq \bar{\phi}_n(d_n) - \bar{\phi}_n(c_n) \leq 2\pi$$

by Lemma 2.4.34. Passing to the limit as $n \to \infty$, we deduce that

$$\phi(b) - \phi(a) \leq 2\pi.$$

Since a and b are arbitrarily chosen, we can take $a \to -\infty$, $b \to +\infty$ to obtain

$$\phi(+\infty) - \phi(-\infty) \leq 2\pi. \tag{2.52}$$

If $\alpha \in (1,2)$ (this is the first time we need the assumption $\alpha \in (1,2)$), (2.51) and (2.52) give a contradiction, and the proof of Proposition 2.4.31 is complete. Otherwise, we deduce the following.

Lemma 2.4.38. *Let $\alpha = 1$. If we are not in case (ii)-(a) of Proposition 2.4.31, then necessarily*

$$|\bar{\phi}^+ - \bar{\phi}^-| = 0.$$

We remark that in the proof of (2.51) and (2.52) we supposed (it is not restrictive) the angle ϕ increasing. This is why we omitted the absolute value.

Step 4) Conclusion of the proof of points (i) and (ii)-(a) of Proposition 2.4.31. Since there exists $\bar{\rho} > 0$ such that, if $0 < \rho_1 < \rho_2 < \rho^* \leq \bar{\rho}$, we have

$$u \text{ is a minimizer of } M_l^{\bar{p}_1\bar{p}_2}|_{\mathcal{K}_l(\rho_2,\rho^*)} \quad \Longrightarrow \quad \min_{t\in[c,d]} |u(t) - c_i| = \rho^*,$$

and also

$$u \text{ is a minimizer of } M_l^{\bar{p}_1\bar{p}_2}|_{\mathcal{K}_l(\rho_1,\rho_2)} \quad \Longrightarrow \quad \min_{t\in[c,d]} |u(t) - c_i| = \rho_2.$$

Hence $d(\rho^*) < d(\rho_2) < d(\rho_1)$. We recall that the function $d(\cdot)$ is continuous in 0, so that taking $\rho_1 \to 0^+$ we obtain $d(\rho^*) < d(0)$: this is a contradiction, since we are assuming that the minimum of $M_l^{\bar{p}_1\bar{p}_2}$ on $\mathcal{K}_l^{p_1p_2}$ is achieved over collision paths.

Step 5) Proof of point (ii)-(b) of Proposition 2.4.31. We have to prove the following.

Proposition 2.4.39. *If a minimizer $u \in K_l$ of M has a collision, then the collision set $T_c(u)$ consists of a unique instant, and $y(t) := u(\omega t)$ is an ejection-collision solution of* (2.19)*. In particular, this implies that necessarily $p_1 = p_2$ and l satisfies* (2.27)*.*

We show that if u is a collision minimizer in $K_l^{p_1p_2}(0,1)$, then there is a possibly different ejection-collision minimizer in the same class of type

$$\widehat{u}(t) = \begin{cases} u(t) & t \in [0,1] \setminus [c,d] \\ \widetilde{u}(t) & t \in [c,d], \end{cases}, \tag{2.53}$$

where $\widetilde{u}$, which has been introduced in step 3, is an ejection-collision function with a unique collision in c_j. This implies the thesis, because

$$\text{if } \widehat{u} \text{ is a minimizer, then } u = \widehat{u}, \tag{2.54}$$

so that u itself is an ejection-collision minimizer. To prove this claim, let t_0 be the unique collision time of $\widehat{u}$, t_1 be the first collision time of u, and t_2 be the last collision time of u. Recall that for each local minimizer of M (even if it has some collisions) the conservation of the energy holds true (see Lemma 2.4.3):

$$\frac{1}{2}|\dot{u}(t)|^2 - \frac{V_\varepsilon(u(t))}{\omega^2} = -\frac{1}{\omega^2} \quad \text{and} \quad \frac{1}{2}|\dot{\widehat{u}}(t)|^2 - \frac{V_\varepsilon(\widehat{u}(t))}{\widehat{\omega}^2} = -\frac{1}{\widehat{\omega}^2}$$

a.e. in $[0,1]$, where

$$\omega^2 = \frac{\int_0^1 V_\varepsilon(u) - 1}{\frac{1}{2}\int_0^1 |\dot{u}|^2} \quad \text{and} \quad \widehat{\omega}^2 = \frac{\int_0^1 V_\varepsilon(\widehat{u}) - 1}{\frac{1}{2}\int_0^1 |\dot{\widehat{u}}|^2}.$$

Since $u = \widehat{u}$ in $[0, c)$, $\omega^2 = \widehat{\omega}^2$. Let $t_* \in (0, c)$; both u and $\widehat{u}$ are $\mathcal{C}^2$ solutions of the regular Cauchy problem

$$\begin{cases} \omega^2 \ddot{v}(t) = \nabla V_\varepsilon(v(t)) \\ v(t_*) = u(t_*) \quad \dot{v}(t_*) = \dot{u}(t_*); \end{cases}$$

They can be extended in a unique manner to a solution in $(0, \min\{t_0, t_1\})$; by continuity $t_0 = t_1$, so that $u|_{(0,t_0)} = \widehat{u}|_{(0,t_0)}$. Analogously, it is possible to check that $t_0 = t_2$ and $u|_{(t_0,1)} = \widehat{u}|_{(t_0,1)}$. This proves (2.54).

Now, the goal is to show that we the function $\widetilde{u} \in H^1(c, d)$, introduced in step 3, is such that $\widehat{u}$ defined by (2.53) is an ejection-collision minimizer of M in $K_l^{p_1 p_2}([0, 1])$; we know that t_0 is the first collision time of $\widetilde{u}$, and that $\widetilde{u}(t_0) = c_j$.

We introduce a transformation of time and of the space in order to regularize the flow in a neighbourhood of the singularity c_j. An argument of this type has been firstly introduced in 1920 by Levi-Civita in [56]. Actually, to take advantage of the careful blow-up analysis developed in the previous steps, we consider a sequence of local Levi-Civita regularization.

Definition 2.4.40. (Local Levi-Civita transform). For every complex-valued continuous function u we define the set $\Lambda(u)$ of the continuous function q such that

$$u(t) = q^2(\tau(t)) + c_j,$$

where we re-parametrize the time as

$$dt = |q(\tau)|^2 \, d\tau.$$

The symbols " $'$ " and " ∇_q " denote the differentiation with respect to τ and the gradient in the Levi-Civita space, respectively. We remark that, if a path u does not collide in c_j, then $\Lambda(u)$ consists in two elements $\pm\sqrt{u(t(\tau)) - c_j}$.

We perform the Levi-Civita-type transform along the sequence (u_n) defined in (2.41). So, it is convenient to define

$$S_n := \int_c^d \frac{dt}{|u_n(t) - c_j|}.$$

Lemma 2.4.41. *The sequence (S_n) is bounded above and bounded below by a strictly positive constant. Hence there exist a subsequence (still denoted (S_n)) and $\widetilde{S} > 0$ such that*

$$\lim_{n \to \infty} S_n = \widetilde{S}.$$

Proof. Assume by contradiction that (S_n) is not bounded above:

$$\limsup_{n\to\infty} \int_c^d \frac{dt}{|u_n(t) - c_j|} = +\infty.$$

In the proof of point (iii) of Lemma 2.4.33 we showed that

$$\liminf_{n\to\infty} \int_c^d |\dot{u}_n(t)|^2 \, dt > 0,$$

and hence $(M_l^{\bar{p}_1\bar{p}_2}(u_n))$ is unbounded, in contradiction with the fact that (u_n) is a minimizing sequence of a coercive functional. Furthermore, since

$$\int_c^d \frac{dt}{|u_n(t) - c_j|} \geq \frac{d-c}{R+\varepsilon} > 0,$$

(S_n) is also bounded below by a positive constant. $\square$

For every n, we define the set $\Lambda(u_n)$ of the continuous function q_n such that

$$\begin{aligned} u_n(t) &= q_n^2(\tau(t)) + c_j \\ dt &= S_n |q_n(\tau)|^2 \, d\tau. \end{aligned}$$

We also set

$$\begin{aligned} \widetilde{u}(t) &= \widetilde{q}^{\,2}(\tau(t)) + c_j \\ dt &= \widetilde{S} |\widetilde{q}(\tau)|^2 \, d\tau. \end{aligned}$$

We point out that the new time τ depends on n (we keep in mind this dependence, but we don't write it down to ease the notation). Note that the time parameters are suitably normalized to work in a common time interval: setting $\tau(c) = 0$ for every n, the right end of the interval of definition of each function q_n is

$$\int_0^{\tau(d)} d\tau = \frac{1}{S_n} \int_c^d \frac{dt}{|u_n(t) - c_j|} = 1,$$

so that q_n is defined over $[0, 1]$.
For $q_n \in \Lambda(u_n)$, we set $\tau_n^- := \tau(t_n^-)$ and $\tau_n^+ := \tau(t_n^+)$ (recall that $t_n^- = \inf\{t \in [c,d] : |u_n(t) - c_j| = \rho_n\}$, $t_n^+ = \sup\{t \in [c,d] : |u_n(t) - c_j| = \rho_n\}$).
The constraint $B_{\rho_n}(c_j)$ corresponds through the transformation to the ball $B_{\sqrt{\rho_n}}(0)$, so that q_n satisfies

$$\begin{aligned} |q_n(\tau)| &> \sqrt{\rho_n} \qquad \tau \in [0, \tau_n^-) \cup (\tau_n^+, 1] \\ |q_n(\tau)| &= \sqrt{\rho_n} \qquad \tau \in [\tau_n^-, \tau_n^+]. \end{aligned}$$

In polar coordinates we write

$$q_n(\tau) = \kappa_n(\tau) e^{i\sigma_n(\tau)},$$

where $\kappa_n : [0,1] \to \mathbb{R}^+$, $\sigma_n : [0,1] \to \mathbb{R}$.

For every $\rho_1, \rho_2 > 0$, each $u \in \mathcal{K}_l^{\bar{p}_1\bar{p}_2}(\rho_1,\rho_2)$ does not collide in c_j, so that $\Lambda(u) = \{\pm\sqrt{u - c_j}\}$; setting $\Lambda_+(u) = \{+\sqrt{u-c_j}\}$, the Levi-Civita transform is a bijective correspondance between the spaces $(\mathcal{K}_l^{\bar{p}_1\bar{p}_2}(\rho_1,\rho_2), dt)$ and $(\Lambda_+(\mathcal{K}_l^{\bar{p}_1\bar{p}_2}(\rho_1,\rho_2)), d\tau)$. In particular, for every n, writing q_n we denote the function $+\sqrt{u_n - c_j}$. In this way, it is possible to choose $\widetilde{q} \in \Lambda(\widetilde{u})$ such that $q_n \to \widetilde{q}$ uniformly in $[0,1]$.

The next lemma establishes the relationship between the variational properties of a function and its Levi-Civita transform.

Lemma 2.4.42. *Every $q_n \in \Lambda_+(u_n)$ is a minimizer of*

$$\widetilde{M}(q) := 4\int_0^1 |q'|^2 \int_0^1 \left[m_j + \left(V_\varepsilon^j(q^2 + c_j) - 1\right)|q|^2\right]$$

in the set $\Lambda_+(\mathcal{K}_l^{\bar{p}_1\bar{p}_2}(\rho_1,\rho_2))$ at a strictly positive level.

Proof. Since $(\mathcal{K}_l^{\bar{p}_1\bar{p}_2}(\rho_1,\rho_2), dt)$ and $(\Lambda_+(\mathcal{K}_l^{\bar{p}_1\bar{p}_2}(\rho_1,\rho_2)), d\tau)$ are in bijective correspondence, it is sufficient to write the factors of M in terms of τ and q_n:

$$|\dot{u}_n(t)|^2\, dt = \left|2q_n(\tau(t))q_n'(\tau(t))\frac{d\tau}{dt}(t)\right|^2 dt = \frac{4}{S_n}|q_n'(\tau)|^2\, d\tau,$$

and

$$\begin{aligned}(V_\varepsilon(u_n(t)) - 1)\; dt &= \left(\frac{m_j}{|q_n(\tau(t))|^2} + V_\varepsilon^j(q_n^2(\tau(t)) + c_j) - 1\right) dt \\ &= S_n\left[m_j + \left(V_\varepsilon^j(q_n^2(\tau) + c_j) - 1\right)|q_n(\tau)|^2\right] d\tau.\end{aligned}$$

□

Remark 2.4.43. We get a functional of Maupertuis-type. In this case the potential is no more singular in c_j, and the mass m_j plays the role of the energy.

Now a technical result:

Lemma 2.4.44. *For every n, let*

$$\widetilde{\omega}_n^2 := \frac{\int_0^1 \left[m_j + \left(V_\varepsilon^j(q_n^2 + c_j) - 1\right)|q_n|^2\right]}{\frac{1}{2}\int_0^1 |q_n'|^2}.$$

The sequence $(\widetilde{\omega}_n^2)$ is bounded above and bounded below by a strictly positive constant. Hence there exist a subsequence (still denoted $(\widetilde{\omega}_n)$) and $\widetilde{\Omega} > 0$ such that

$$\lim_{n\to\infty} \widetilde{\omega}_n = \widetilde{\Omega}.$$

Proof. There holds

$$\widetilde{\omega}_n^2 = \frac{\frac{1}{S_n}\int_c^d V_\varepsilon(u_n) - 1}{\frac{S_n}{8}\int_c^d |\dot{u}_n|^2} = \frac{4}{S_n^2}\omega_n^2.$$

Now it is sufficient to recall Lemma 2.4.41 and the fact that $\omega_n^2 \to \Omega^2 > 0$. □

From now on, we always consider the subsequence introduced in this statement. We are ready to prove the main features of the functions q_n.

Lemma 2.4.45. *For every n:*

(i) the function q_n is of class $\mathcal{C}^1((0,1))$;

(ii) the restrictions $q_n|_{[0,\tau_n^-)}$ and $q_n|_{(\tau_n^+,1]}$ are $\mathcal{C}^2$ solutions of

$$\widetilde{\omega}_n^2 q_n''(\tau) = \nabla_{q_n}\left(V_\varepsilon^j(q_n^2(\tau)+c_j)|q_n(\tau)|^2\right) - 2q_n(\tau);$$

(iii) the energy of q_n is constant in $[0,1]$:

$$\frac{1}{2}|q_n'(\tau)|^2 - \frac{1}{\widetilde{\omega}_n^2}\left(V_\varepsilon^j(q_n^2(\tau)+c_j)-1\right)|q_n(\tau)|^2 = \frac{m_j}{\widetilde{\omega}_n^2} \qquad \forall \tau \in [0,1];$$

(iv) the variation of the angle on the constraint tends to 0 as $n \to \infty$:

$$\lim_{n\to\infty} |\sigma_n(\tau_n^+) - \sigma_n(\tau_n^-)| = 0;$$

(v) the time interval on the constraint tends to 0 for $n \to \infty$:

$$\lim_{n\to\infty} (\tau_n^+ - \tau_n^-) = 0.$$

Proof. The point (i) is obvious, the points (ii) and (iii) are consequence of the variational property of q_n, Lemma 2.4.42.
(iv) We can use the results already obtained in step 3 (recall in particular the expression of u_n in polar coordinates, the definition of the sequence (v_n), the expression of v_n in polar coordinates, and Lemma 2.4.38). The angle of the function q_n with respect to the origin is exactly half of the angle of u_n with respect to c_j:

$$q_n^2 = w_n e^{i\phi_n} \quad \Longrightarrow \quad q_n = \sqrt{w_n}e^{i\frac{\phi_n}{2}}.$$

Hence we can we prove that

$$\lim_{n\to\infty} |\phi_n(t_n^+) - \phi_n(t_n^-)| = 0.$$

or equivalently

$$\lim_{n\to\infty} |\bar{\phi}_n(s_n^+) - \bar{\phi}_n(s_n^-)| = |\phi^+ - \phi^-| = 0,$$

which is given by Lemma 2.4.38.
(v) It is a consequence of the same property for u_n, Lemma 2.4.35:

$$\begin{aligned}\tau_n^+ - \tau_n^- &= \int_{\tau_n^-}^{\tau_n^+} d\,\tau = \int_{t_n^-}^{t_n^+} \frac{dt}{S_n|q_n(\tau(t))|^2} = \frac{t_n^+ - t_n^-}{S_n\rho_n} \\ &= \frac{O(\rho_n^{\frac{3}{2}})}{S_n\rho_n} \simeq \frac{\rho_n^{\frac{1}{2}}}{S_n} \to 0\end{aligned}$$

for $n\to\infty$, where we used the boundedness of the sequence (S_N), Lemma 2.4.41. □

Lemma 2.4.46. *The path $\widetilde{q}$ is a classical solution of*

$$\widetilde{\Omega}^2\widetilde{q}\,''(\tau) = \nabla_{\widetilde{q}}\left(V_\varepsilon^j(\widetilde{q}^{\,2}(\tau)+c_j)|\widetilde{q}(\tau)|^2\right) - 2\widetilde{q}(\tau) \qquad \textit{for } \tau\in(0,1). \tag{2.55}$$

Proof. Point (v) of the previous lemma implies that the sequences (τ_n^-) and (τ_n^+) converge to some $\tau_0\in(0,1)$, such that $\widetilde{q}(\tau_0)=0$. This instant τ_0 corresponds to the unique collision time $t_0\in(c,d)$ of the function $\widetilde{u}$. We know that q_n uniformly converges to $\widetilde{q}$ in $[0,1]$, and it is not difficult to see that $q_n\to\widetilde{q}$ in the $\mathcal{C}^1$-topology in any compact subset of $[0,\tau_1)\cup(\tau_1,1]$ (one can easily modify the proof of Proposition 2.4.37). Since every q_n is $\mathcal{C}^1$, the vectors $q_n(\tau)$ is tangent to the circle $\{w\in\mathbb{C}:|w|=\sqrt{\rho}_n\}$ in the time interval $[\tau_n^-,\tau_n^+]$. So, using the fact that the variation of the angle σ_n on the constraint tends to 0 (we refer to point (iv) of Lemma 2.4.45), we deduce that

$$\lim_{\tau\to\tau_0^-}\widetilde{q}\,'(\tau) = \lim_{\tau\to\tau_0^+}\widetilde{q}\,'(\tau),$$

that is, $\widetilde{q}$ passes trough the origin without any change of direction. As a consequence $\widetilde{q}\in\mathcal{C}^1((0,1))$, and it turns out to be a (weak, and by regularity strong) solution of (2.55). □

Conclusion of the proof of Proposition 2.4.39. We wish to show that $\widetilde{u}(t_0+t)=\widetilde{u}(t_0-t)$. Let us consider the functions

$$\widetilde{q}_1(\tau)=\widetilde{q}(\tau_0+\tau), \qquad \widetilde{q}_2(\tau)=-\widetilde{q}(\tau_0-\tau).$$

They are both solutions of (2.55) (for $\widetilde{q}_1$ this is immediate, for $\widetilde{q}_2$ it is not difficult to check, observing that since the function $q\mapsto(V_\varepsilon^j(q^2+c_j)-1)|q|^2$ is even, then $q\mapsto\nabla_q(V_\varepsilon^j(q^2+c_j)-1)|q|^2)$ is odd. Moreover, $\widetilde{q}_1$ and $\widetilde{q}_2$ have the same initial values.

Thanks to the regularity of (2.55), the uniqueness theorem for initial values problem and the definition of the Levi-Civita transform give

$$\widetilde{q}(\tau_0 + \tau) = -\widetilde{q}(\tau_0 - \tau) \quad \Longrightarrow \quad \widetilde{u}(t_0 + t) = \widetilde{u}(t_0 - t) :$$

if the function $\widetilde{u}$ has a collision, then necessarily bounce against one centre and comes back along the same trajectory. Now, we observed that in this case also u in an ejection-collision minimizer, with a unique collision in c_j. This implies that $p_1 = p_2$. Note also that a function of this type belongs to K_l only if l satisfies condition (2.27). □

We end this section with some remarks about our peculiar use of the Levi-Civita regularization.

Remark 2.4.47. We proved that, if the minimum of the restriction of $\mathcal{M}$ over K_l is achieved by a collision function $\widetilde{u}$, then $\widetilde{u}$ is an ejection-collision minimizer. To do this, we considered the minimizing sequence (u_n), defined by means of the introduction of the obstacle problems, and then we passed to the limit in the Levi-Civita space. Thanks to the regularity of the transformed problem, we obtained an equation satisfied by the limit, and this implied the ejection-collision condition for the function $\widetilde{u}$. A natural question is the following: why did we pass to $q_n \in \Lambda(u_n)$ instead of considering directly a function in $\Lambda(\widetilde{u})$? The answer is that, since $|\widetilde{u}(t_1)| = c_j$, the set $\Lambda(\widetilde{u})$ has not two connected components, so that it is not so clear to give a variational characterization of an arbitrary function in $\Lambda(\widetilde{u})$ (and hence to deduce an equation for an element of this set). On the other hand, the fact that we fixed the choice $q_n = \sqrt{u_n - c_j}$ and the uniform convergence of u_n to $\widetilde{u}$ allows to show that the sequence (q_n) converges to a uniquely determined $\widetilde{q} \in \Lambda(\widetilde{u})$.

On the Levi-Civita transform 2.4.48. As clearly explained in [52], the N-centre problem admits a global Levi-Civita regularization. It consists in extending the pullback of the Jacobi metric on the Riemann surface

$$\mathcal{R} = \left\{ (u, Q) \ : \ Q^2 = \prod_{j=1}^{N} (u - c_j) \right\}$$

to a smooth metric. The projection from $\mathcal{R} \to \mathbb{C}$ on the first factor is a branched covering of $\mathbb{C}$ whose ramification points $C_j = (c_j, 0)$ are of order one and project on the centres $\{c_j\}$. The Riemannian surface $\tilde{\mathcal{R}} = \mathcal{R} \setminus \{C_j\}$ doubly covers the configuration space $\mathbb{C} \setminus \{c_1, \ldots, c_N\}$; moreover, there is a unique way of lifting the Jacobi metric to $\tilde{\mathcal{R}}$ and this extend in an unique way to a smooth metric on $\mathcal{R}$. Geodesics on $\mathcal{R}$ can be classified according with the fundamental group $\pi_1(\mathcal{R})$, which is known to be isomorphic to the free group on $N-1$ generators. The main reason why we choose to use the local

L-C transform is that we want to keep track of the topology of the true configuration space, and specially, of the winding number of the functions u_n with respect to the centres. This explains why we are led to swinging back and forth from the configuration space to the Riemannian surface.

Of course, also the local Levi-Civita transform induces a regularization of the flow associated with the first order system (2.3). Indeed, let us consider an ejection-collision solution $\widehat{y}$ of (2.4) starting from $p_0 \in \partial B_R(0)$, coming from an ejection-collision minimizer $\widehat{u} \in K_l^{p_0 p_0}([0,1])$. The Levi-Civita transform $\widehat{q}$ of $\widehat{u}$ is a regular solution of (2.55). Let us define the re-parametrization $\widehat{\mathfrak{q}}(\tau) := \widehat{q}(\widetilde{\Omega}\tau)$; it is a regular solution of

$$\mathfrak{q}''(\tau) = \nabla_{\mathfrak{q}} \left(V_\varepsilon^j(\mathfrak{q}(\tau)^2 + c_j)|\mathfrak{q}(\tau)|^2 \right) - 2\mathfrak{q}(\tau) \tag{2.56}$$

with energy m_j, starting from $\widehat{\mathfrak{x}}_0 \in \Lambda(x_0)$ and arriving to $\widetilde{\mathfrak{x}}_0 \in \Lambda(x_0)$, with $\widehat{\mathfrak{x}}_0 \neq \widehat{\mathfrak{x}}_0$. Now let us consider a collisions-free solution y_l of problem (2.4), with initial data $(p_1, \dot{x}_l(0))$ close to the initial data of $\widehat{y}$. This solution comes from a collisions-free minimizer $u_l \in K_l^{p_1 p_2}([0,1])$ of M, for some $p_2 \in \partial B_R(0)$ (see Remark 2.4.2). Even in this case we can consider the Levi-Civita transform $\Lambda(u_l)$ (centred in c_j), given by

$$\begin{aligned} u_l(t) &= q_l^2(\tau(t)) + c_j \\ dt &= S|q_l(\tau)|^2\, d\tau \\ S &= \int_0^1 \frac{dt}{|u_l(t) - c_j|}. \end{aligned}$$

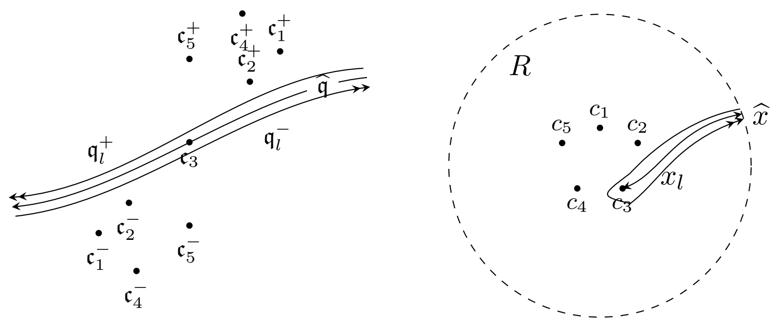
Each component $q_l \in \Lambda(u_l)$ is a local minimizer of $\widetilde{M}$ at a positive level. Setting,

$$\omega_l^2 := \frac{\int_0^1 \left[m_j + \left(V_\varepsilon^j(q_l^2 + c_j) - 1 \right) |q_l|^2 \right]}{\frac{1}{2}\int_0^1 |q_l|^2},$$

we infer

$$\begin{aligned} \omega_q^2 q_l''(\tau) &= \nabla_q \left(V_\varepsilon^j(q_l(\tau)^2 + c_j)|q_l(\tau)|^2 \right) - 2q_l(\tau) \qquad \forall \tau \in [0,1] \\ \frac{1}{2}|q_l'(\tau)|^2 - \frac{1}{\widetilde{\omega}_n^2} &\left(V_\varepsilon^j(q_l^2(\tau) + c_j) - 1 \right) |q_l(\tau)|^2 = \frac{m_j}{\widetilde{\omega}_n^2} \qquad \forall \tau \in [0,1]. \end{aligned}$$

The re-parametrization $\mathfrak{q}_l(\tau) = q_l(\omega_l \tau)$ is a solution of equation (2.56) with energy m_j and initial data close to those of $\widehat{\mathfrak{q}}$. This is a smooth equation, hence the continuous dependence of the solutions holds true: since the initial values of $\mathfrak{q}_l$ and of $\widehat{\mathfrak{q}}$ are close together, these solutions stays close together in a right neighbourhood of 0. In particular it is not difficult to see that this continuous dependence holds true if the solutions stay in the set which corresponds to $B_R(0)$ through the Levi-Civita transform.



The picture represents a comparison between the Levi-Civita space (on the left), centred in $\mathfrak{c}_3 = \Lambda(c_3)$, and the configuration space (on the right) of the true N-centre problem. We have the ejection-collision solution $\widehat{x}$, with its collision against c_3. In the Levi-Civita space, the corresponding path $\widehat{\mathfrak{q}}$ solves the regular differential equation (2.56) (we fix an orientation of this solution given by the arrow). If we take one solution $\mathfrak{q}_l^+$ with similar initial data, we can apply the continuous dependence theorem: hence $\widehat{\mathfrak{q}}$ is close (in the uniform topology) to $\mathfrak{q}_l$. If we had chosen the inverse orientation for $\widehat{\mathfrak{q}}$, we would got the $\mathfrak{q}_l^-$. Coming back to the physical space, this means that if we take a solution with initial data close to those of a collision-ejection one, a there is continuous dependence despite the lack of regularity of the potential!.
Let us note that, with the exception of c_3, each point of $\mathbb{R}^2$ corresponds to two points of the Levi-Civita space. For instance $\mathfrak{c}_i^\pm \in \Lambda(c_i)$ for $i = 1, 2, 4, 5$. This is due to the fact that in the Levi-Civita space two points which are poles apart are identified when we come back to the physical space. Therefore, our ejection-collision solution correspond to a path crossing the origin and showing a central symmetry, connecting two points which are identified in $p_0 \in \mathbb{R}^2$. We could choose both the orientations for $\widehat{\mathfrak{q}}$, and the identification would give the same path in the physical space.
Let us also note that the angles with respect to the point c_3 in the physical space are cut by half in the Levi Civita one.

To conclude this section, we introduce a different notation for the solutions found in Corollary 2.4.16. Given any $\varepsilon \in (0, \varepsilon_3)$, $P_j \in \mathcal{P}$ and $p_1, p_2 \in \partial B_R(0)$, let $y_{P_j}(\cdot\,; p_1, p_2; \varepsilon)$ be a solution of (2.19) coming from a global minimizer $u_{P_j}(\cdot\,; p_1, p_2; \varepsilon) \in K_{P_j}^{p_1 p_2}([0,1])$ of the Maupertuis functional M_{-1}. Let $T_{P_j}(p_1, p_2; \varepsilon) > 0$ be such that

$$y_{P_j}\left(T_{P_j}(p_1, p_2; \varepsilon); p_1, p_2; \varepsilon\right) = p_2.$$

Lemma 2.4.49. *For any $\varepsilon \in (0, \varepsilon_3)$, there exist $C_3, C_4 > 0$ such that*

$$C_3 \leq T_{P_j}(p_1, p_2; \varepsilon) \leq C_4$$

for every $p_0, p_1 \in \partial B_R(0)$, for every $P_j \in \mathcal{P}$.

Proof. We recall that $T_{P_j}(p_1,p_2;\varepsilon) = 1/\omega_{P_j}(p_1,p_2;\varepsilon)$, where

$$\omega_{P_j}(p_1,p_2;\varepsilon) = \frac{\int_0^1 \left(V_\varepsilon(u_{P_j}(t;p_1,p_2;\varepsilon)) - 1\right)\, dt}{\frac{1}{2}\int_0^1 |\dot u_{P_j}(t;p_1,p_2;\varepsilon)|^2\, dt}.$$

Therefore we can prove that there exist $C_3, C_4 > 0$ such that

$$\frac{1}{C_4} \le \omega_{P_j}(p_1,p_2;\varepsilon) \le \frac{1}{C_3} \qquad \forall (p_1,p_2) \in (\partial B_R(0))^2\,,\ \forall P_j \in \mathcal{P}.$$

Since $\mathcal{P}$ is a discrete and finite set, we can fix $P_j \in \mathcal{P}$ and apply the same reasoning for every j. Let us fix $\widetilde p_1, \widetilde p_2 \in \partial B_R(0)$. There exist $\widetilde u_* \in \widehat K_{P_j}^{\widetilde p_1 \widetilde p_2}([0,1])$ and $C, \mu > 0$ such that

- $|\dot{\widetilde u}_*(t)| = C$ for every $t \in [0,1]$;
- $|\widetilde u_*(t) - c_k| \ge \mu$ for every $t \in [0,1]$, for every $k = 1,\dots,N$.

It results

$$\begin{aligned} M(\widetilde u_*) &= \frac{1}{2}\int_0^1 |\dot{\widetilde u}_*|^2 \int_0^1 (V_\varepsilon(\widetilde u_*) - 1) = \frac{C^2}{2}\int_0^1 \left(\sum_{k=1}^N \frac{m_k}{\alpha|\widetilde u_* - c_k|^\alpha} - 1\right) \\ &\le \frac{C^2}{2}\left(\frac{M}{\alpha\mu^\alpha} - 1\right) =: C_5 > 0. \end{aligned}$$

Also, for every $u \in \bigcup_{p_1,p_2 \in \partial B_R(0)} K_{P_j}^{p_1p_2}([0,1])$,

$$\int_0^1 (V_\varepsilon(u) - 1) \ge \frac{M}{\alpha\,(R+\varepsilon)} - 1 =: C_6 \tag{2.57}$$

with $C_6 > 0$ for our choice of R. For a minimizer $\widetilde u = \widetilde u_{P_j}(\,\cdot\,;\widetilde p_1,\widetilde p_2;\varepsilon) \in K_{P_j}^{\widetilde p_1 \widetilde p_2}([0,1])$, one has

$$M(\widetilde u) = \frac{1}{2}\int_0^1 |\dot{\widetilde u}|^2 \int_0^1 (V_\varepsilon(\widetilde u) - 1) \le M(\widetilde u_*),$$

which together with (2.57) gives

$$\int_0^1 |\dot{\widetilde u}|^2 \le \frac{2C_5}{C_6}.$$

Starting from this bound for one single minimizer, it is not difficult to obtain a uniform bound (with respect to the ends) for every minimizers. Indeed if $(p_1,p_2) \ne (\widetilde p_1, \widetilde p_2)$, we consider

$$\widehat u_*(t) := \begin{cases} \zeta_R(3t; p_1, \widetilde p_1) & t \in [0,1/3] \\ \widetilde u_*(3t-1) & t \in (1/3,2/3] \\ \zeta_R(3t-2; \widetilde p_2, p_2) & t \in (2/3,1], \end{cases}$$

where, for $p_*, p_{**} \in \partial B_R(0)$, $\zeta_R(\cdot\,; p_*, p_{**}) : [0,1] \to \mathbb{R}^2$ parametrizes the shorter (in the Euclidean metric) arc of $\partial B_R(0)$ connecting p_* and p_{**} with constant angular velocity. As far as the angular velocity is concerned, it is easy to see that it is uniformly bounded with respect to p_*, p_{**}. This, together with the assumptions on $\widetilde{u}_*$, implies that also the velocity of $\widehat{u}_*$ is bounded in $[0,1]$, and

$$M(\widehat{u}_*) \leq \frac{C^2}{2} \int_0^1 (V_\varepsilon(\widehat{u}_*) - 1) = C + 3C \int_0^1 (V_\varepsilon(\widetilde{u}_*) - 1) =: C_7.$$

This (positive) constant does not depend on the ends p_1 and p_2, so that for the family of the minimizers there holds

$$M(u_{P_j}(\cdot\,; p_1, p_2; \varepsilon)) \leq C_7 \qquad \forall p_1, p_2 \in \partial B_R(0). \tag{2.58}$$

Collecting (2.57) and (2.58) we obtain

$$\int_0^1 |\dot{u}_{P_j}(\cdot\,; p_1, p_2; \varepsilon)|^2 \leq \frac{2C_7}{C_6} =: C_8 \qquad \forall p_1, p_2 \in \partial B_R(0). \tag{2.59}$$

A few more observations: as we have already repeated many times, the functions in the set $\bigcup_{p_1,p_2 \in \partial B_R(0)} K_{P_j}^{p_1 p_2}([0,1])$ are uniformly non-constant, since they have to cover at least a distance $R - \varepsilon > 0$. Thus, there exists $C_9 > 0$ such that

$$\|\dot{u}\|_2^2 \geq C_9 \qquad \forall u \in \bigcup_{p_1,p_2 \in \partial B_R(0)} K_{P_j}^{p_1 p_2}([0,1]). \tag{2.60}$$

From (2.58) and (2.60) it follows

$$\int_0^1 \big(V_\varepsilon(u_{P_j}(\cdot\,; p_1, p_2; \varepsilon)) - 1\big) \leq \frac{C_4}{C_6} =: C_{10} \qquad \forall p_1, p_2 \in \partial B_R(0). \tag{2.61}$$

Collecting (2.57), (2.59), (2.60) and (2.61), we obtain

$$C_9 \leq \inf_{p_1,p_2 \in \partial B_R(0)} \|\dot{u}_{P_j}(\cdot\,; p_1, p_2; \varepsilon)\|_2^2 \leq \sup_{p_1,p_2 \in \partial B_R(0)} \|\dot{u}_{P_j}(\cdot\,; p_1, p_2; \varepsilon)\|_2^2 \leq C_8$$

and

$$C_6 \leq \inf_{p_1,p_2 \in \partial B_R(0)} \int_0^1 \big(V_\varepsilon(u_{P_j}(\cdot\,; p_1, p_2; \varepsilon)) - 1\big)$$
$$\leq \sup_{p_1,p_2 \in \partial B_R(0)} \int_0^1 \big(V_\varepsilon(u_{P_j}(\cdot\,; p_1, p_2; \varepsilon)) - 1\big) \leq C_{10}.$$

The thesis is now an immediate consequence of the definition of $\omega_{P_j}(p_1, p_2; \varepsilon)$. □

2.5 A finite dimensional reduction

In this section we glue outer and inner solutions in order to construct periodic orbits of the N-centre problem in the whole plane. Our building blocks are the fixed ends trajectories found in Proposition 2.3.1 and Corollary 2.4.16, which we have to juxtapose in a convenient way. In order to obtain smooth junctions, we are going to use a variational argument.

For $0 < \varepsilon < \min\{\varepsilon_2, \varepsilon_3\}$, $n \in \mathbb{N}$, let us choose a finite sequence of partitions $(P_{k_1}, \dots, P_{k_n}) \in \mathcal{P}^n$. For the reader's convenience, we recall that ε_2 and ε_3 have been introduced in Proposition 2.3.1 and Theorem 2.4.14, respectively. We define

$$D = \left\{ (p_0, \dots, p_{2n}) \in (\partial B_R(0))^{2n+1} \;\middle|\; \begin{array}{l} |p_{2j+1} - p_{2j}| \leq \delta \\ \text{for } j = 0, \dots, n-1, \\ p_{2n} = p_0 \end{array} \right\},$$

where δ has been introduced in Proposition 2.3.1. Let $(p_0, \dots, p_{2n}) \in D$. For every $j \in \{0, \dots, n-1\}$, we can apply Proposition 2.3.1 to obtain the uniquely determined outer solution

$$y_{2j}(t) := y_{\text{ext}}(t; p_{2j}, p_{2j+1}; \varepsilon) \qquad t \in [0, T_{2j}],$$

where $T_{2j} := T_{\text{ext}}(p_{2j}, p_{2j+1}; \varepsilon)$. Namely

$$\begin{cases} \ddot{y}_{2j}(t) = \nabla V_\varepsilon(y_{2j}(t)) & t \in [0, T_{2j}], \\ \frac{1}{2}|\dot{y}_{2j}(t)|^2 - V_\varepsilon(y_{2j}(t)) = -1 & t \in [0, T_{2j}], \\ |y_{2j}(t)| > R & t \in (0, T_{2j}), \\ y_{2j}(0) = p_{2j}, \quad y_{2j}(T_{2j}) = p_{2j+1}. \end{cases}$$

We recall that y_{2j} depends on p_{2j} and p_{2j+1} in a $\mathcal{C}^1$ manner.

On the other hand, for every $j = 0, \dots, n-1$, we can find through Corollary 2.4.16 an inner solution

$$y_{2j+1}(t) := y_{P_{k_{j+1}}}(t; p_{2j+1}, p_{2j+2}; \varepsilon) \qquad t \in [0, T_{2j+1}],$$

where $T_{2j+1} := T_{P_{k_{j+1}}}(p_{2j+1}, p_{2j+2}; \varepsilon)$, which is a re-parametrization of a global minimizer of M in $K_{P_j}^{p_{2j+1}p_{2j+2}}([0,1])$. Namely, $y_{2j+1} \in K_{P_{k_{j+1}}}^{p_{2j+1}p_{2j+2}}([0, T_{2j+1}])$ is such that

$$\begin{cases} \ddot{y}_{2j+1}(t) = \nabla V_\varepsilon(y_{2j+1}(t)) & t \in [0, T_{2j+1}], \\ \frac{1}{2}|\dot{y}_{2j+1}(t)|^2 - V_\varepsilon(y_{2j+1}(t)) = -1 & t \in [0, T_{2j+1}], \\ |y_{2j+1}(t)| < R & t \in (0, T_{2j+1}), \\ y_{2j+1}(0) = p_{2j+1}, \quad y_{2j+1}(T_{2j+1}) = p_{2j+2}, \end{cases}$$

where, in general, the first equation has to be understood in a weak sense. We know that if $\alpha \neq 1$ or $\alpha = 1$ and $p_{2j+1} \neq p_{2j+2}$ then y_{2j+1} is collisions-free, while if $\alpha = 1$, $p_{2j+1} = p_{2j+2}$ and $P_j \in \mathcal{P}_1$, then y_{2j+1} can be an ejection-collision solution. Due to the invariance under re-parametrizations of any length, y_{2j+1} is a minimizer of the functional $L\left([0, T_{2j+1}]\,; \cdot\right)$ in $K_{P_j}^{p_{2j+1}p_{2j+2}}([0, T_{2j+1}])$.

Let us set $\mathfrak{T}_k := \sum_{j=0}^{k} T_j$, for $k = 0, \dots, 2n-1$. We define

$$\gamma_{(p_0,\dots,p_{2n})}(s) := \begin{cases} y_0(s) & s \in [0, \mathfrak{T}_0] \\ y_1(s - \mathfrak{T}_0) & s \in [\mathfrak{T}_0, \mathfrak{T}_1] \\ \quad\vdots & \\ y_{2n-2}\left(s - \mathfrak{T}_{2n-3}\right) & s \in [\mathfrak{T}_{2n-3}, \mathfrak{T}_{2n-2}] \\ y_{2n-1}\left(s - \mathfrak{T}_{2n-2}\right) & s \in [\mathfrak{T}_{2n-2}, \mathfrak{T}_{2n-1}]. \end{cases} \tag{2.62}$$

The function $\gamma_{(p_0,\dots,p_{2n})}$ is a piecewise differentiable $\mathfrak{T}_{2n-1}$-periodic function; to be precise, if $\alpha \in (1,2)$ it is a classical solution of the N-centre problem (2.4) with energy -1 in $[0, \mathfrak{T}_{2n-1}] \setminus \{0, \mathfrak{T}_0, \dots, \mathfrak{T}_{2n-1}\}$; in general, it is not $\mathcal{C}^1$ in the junction instants $\{0, \mathfrak{T}_0, \dots, \mathfrak{T}_{2n-1}\}$, but the right and left limits of the derivative in these times are finite, so that it is a function of $H^1([0, \mathfrak{T}_{2n-1}])$. If $\alpha = 1$, it is possible that $\gamma_{(p_0,\dots,p_{2n})}$ has a finite number of collisions. Let us observe that, thanks to Lemmas 2.3.4 and 2.4.49, we are sure that the time interval of $\gamma_{(p_0,\dots,p_{2n})}$ is bounded above and bounded below by a positive constant for every $(p_0, \dots, p_{2n})$, so that the period is neither trivial, nor infinite.

We consider the function $F_{((P_{k_1},\dots,P_{k_n});\varepsilon)} : D \to \mathbb{R}$ defined by

$$\begin{aligned} F_{((P_{k_1},\dots,P_{k_n});\varepsilon)}(p_0, \dots, p_{2n}) :&= L\left([0, \mathfrak{T}_{2n-1}]; \gamma_{(p_0,\dots,p_{2n})}\right) \\ &= \sum_{j=0}^{2n-1} \int_0^{T_j} \sqrt{\left(V_\varepsilon(y_j) - 1\right)|\dot{y}_j|^2} = \sum_{j=0}^{2n-1} L\left([0, T_j]; y_j\right), \end{aligned}$$

which we simply denote as F when there is not possibility of misunderstanding, to simplify the notation. It associates to each point of D the length, in the Jacobi metric, of the curve parametrized by $\gamma_{(p_0,\dots,p_{2n})}$. We point out that F depends on $(P_{k_1}, \dots, P_{k_n})$ and ε through the dependence on these quantities of $\{y_j\}$ and V_ε. Also, we explicitly remark that F is a function defined in a finite dimensional domain.

The main goal of this section is to prove the following theorem.

Theorem 2.5.1. *There exists $(\bar{p}_0, \dots, \bar{p}_{2n}) \in D$ which minimizes F. There exists $\bar{\varepsilon} > 0$ such that, if $\varepsilon \in (0, \bar{\varepsilon})$, then the associated function $\gamma_{(\bar{p}_0,\dots,\bar{p}_{2n})}$ is a periodic solution of the N-centre problem* (2.4) *with energy -1. The value $\bar{\varepsilon}$ depends neither on n, nor on $(P_{k_1}, \dots, P_{k_n}) \in \mathcal{P}^n$. Moreover:*

(i) if $\alpha \in (1,2)$ *then* $\gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n})}$ *is collision-free;*

(ii) if $\alpha = 1$ *there are three possibilities:*

a) $\gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n})}$ *is collision-free;*

b) $\gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n})}$ *has a collision against one centre* c_j*, covers a certain trajectory, falls again on one centre* c_k *(it can occur* $c_j = c_k$*) and bounces, coming back along the same trajectory. This is possible only when* n *is even and* $(P_{k_1},\ldots,P_{k_n})$ *is equivalent to* $(P'_{k_1},\ldots,P'_{k_n})$ *such that*

$$P'_{k_1} \in \mathcal{P}_1, \quad P'_{j_{n/2+1}} \in \mathcal{P}_1 \quad \textit{and (if } n>2\textit{)}$$
$$P'_{k_n} = P'_{k_2},\ P'_{k_{n-1}} = P'_{k_3},\ \ldots,\ P'_{k_{n/2+2}} = P'_{k_{n/2}};$$

c) $\gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n})}$ *has a collision against one centre* c_j*, covers a certain path, bounces against the surface* $\left\{x \in \mathbb{R}^2 : V_\varepsilon(x) = 1\right\}$ *with null velocity and comes back along the same trajectory. This is possible only if* n *is odd and* $(P_{k_1},\ldots,P_{k_n})$ *is equivalent to* $(P'_{k_1},\ldots,P'_{k_n})$ *such that*

$$P'_{k_1} \in \mathcal{P}_1 \quad \textit{and (if } n>1\textit{)}$$
$$P'_{k_n} = P'_{k_2},\ P'_{k_{n-1}} = P'_{k_3},\ \ldots,\ P'_{k_{(n+1)/2+1}} = P'_{k_{(n+1)/2}}.$$

Remark 2.5.2. Theorem 2.1.2 follows directly from this result, see also Remark 2.2.3: given $0<\varepsilon<\bar{\varepsilon}$, for every $n \in \mathbb{N}$ and for every $(P_{k_1},\ldots,P_{k_n}) \in \mathcal{P}^n$ there exists a periodic solution $\gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n})}$ of (2.4), whose behaviour is determined by $(P_{k_1},\ldots,P_{k_n})$. Let us set $\bar{h} = -\zeta(\bar{\varepsilon})$. Now, given $\bar{h} < h < 0$, for every $n \in \mathbb{N}$ and $(P_{k_1},\ldots,P_{k_n}) \in \mathcal{P}^n$ we obtain a periodic solution $x_{((P_{k_1},\ldots,P_{k_n}),h)}$ of the problem (2.2) with energy h, via Proposition 2.2.1. As we pointed out at the end of Section 2.2, the shape of the orbits parametrized by $x_{((P_{k_1},\ldots,P_{k_n}),h)}$ and by $\gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n})}$ is the same.

We reach the result through a series of lemmas. Let us fix $\varepsilon \in (0,\bar{\varepsilon})$, $n \in \mathbb{N}$, $(P_{k_1},\ldots,P_{k_n}) \in \mathcal{P}^n$.

Lemma 2.5.3. *There exists* $(\bar{p}_0,\ldots,\bar{p}_{2n})$ *which minimizes* F*.*

Proof. The set D is compact since it is a closed subset of the compact set $(\partial B_R(0))^{2n+1}$. It remains to show that F is continuous. Let $((p_0^m,\ldots,p_{2n}^m))$ a convergent sequence in D: $(p_0^m,\ldots,p_{2n}^m) \to (p_0,\ldots,p_{2n}) \in D$ as $m \to +\infty$. Let us consider

$$F(p_0^m,\ldots,p_{2n}^m) = \sum_{j=0}^{n-1} L\left([0,T_{2j}^m]; y_{2j}^m\right) + \sum_{j=0}^{n-1} L\left([0,T_{2j+1}^m]; y_{2j+1}^m\right).$$

Here y_{2j}^m (resp. y_{2j+1}^m) is defined as y_{2j} (resp. y_{2j+1}); it has boundary values p_{2j}^m, p_{2j+1}^m (resp. p_{2j+1}^m, p_{2j+2}^m), and domain $[0, T_{2j}^m]$ (resp. $[0, T_{2j+1}^m]$).
The first sum is continuous in D, since the function y_{2j}^m depends in a differentiable way on its ends. As far as the second sum is concerned, we can treat the first addendum and repeat the reasoning for the others. For $p_*, p_{**} \in \partial B_R(0)$, we consider again $\zeta_R(\cdot\,; p_*, p_{**})$ (the shorter (in the Euclidean metric) arc of $\partial B_R(0)$ connecting p_* and p_{**} with constant angular velocity, parametrized in $[0,1]$). Obviously,

$$\forall\lambda > 0\ \exists\varrho > 0 :\ |p_* - p_{**}| < \varrho \quad \Longrightarrow \quad L\left([0,1]; \zeta_R\left(\cdot\,; p_*, p_{**}\right)\right) < \lambda.$$

Since y_1 minimizes L among the paths connecting p_1 and p_2 which separate the centres according to P_{k_1}, we have

$$L\left([0,T_1]; y_1\right) \leq L\left([0,T_1^m]; y_1^m\right) + L\left([0,1]; \zeta_R\left(\cdot\,; p_1^m, p_1\right)\right) + L\left([0,1]; \zeta_R\left(\cdot\,; p_2^m, p_2\right)\right). \quad (2.63)$$

Here we use the invariance of L under re-parametrizations, which permits to compare the values of L for functions defined over different time-intervals.
Analogously, the minimal property of y_1^m implies

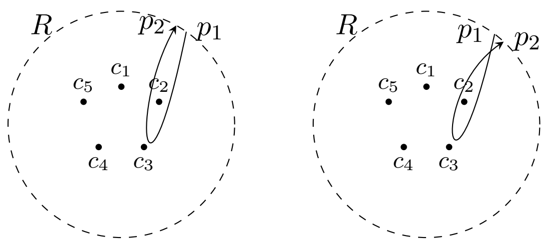
$$L\left([0,T_1^m]; y_1^m\right) \leq L\left([0,T_1]; y_1\right) + L\left([0,1]; \zeta_R\left(\cdot\,; p_1^m, p_1\right)\right) + L\left([0,1]; \zeta_R\left(\cdot\,; p_2^m, p_2\right)\right). \quad (2.64)$$

Passing to the lim inf as $m \to +\infty$ in (2.63), and to the lim sup as $m \to +\infty$ in the (2.64), we finally deduce

$$\lim_{m\to\infty} L\left([0,T_1^m]; y_1^m\right) = L\left([0,T_1]; y_1\right).$$

Therefore F is continuous on D, and has a minimum. □

Remark 2.5.4. The main existence result of inner solutions, Proposition 2.4.14, is stated in terms of winding vectors rather than in terms of partitions. Thus, it could seem reasonable to prescribe a finite sequence of winding vectors $(l_1, \ldots, l_n) \in \mathbb{Z}_2^N$ and try to prove the existence of a periodic solution associated to this sequence in the same way as $\gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n})}$ is associated to $(P_{k_1}, \ldots, P_{k_n})$. This, clearly, would lead to a larger class of periodic solutions. But such a generalization does not seem possible, for the following reason. For the proof of Theorem 2.5.1 we consider variations of an inner minimizer with respect to its endpoints p_1, p_2; the function $\mathrm{Ind}(u([a,b]), c_j)$ is not continuous in u with respect to the uniform convergence topology if we let p_1 and p_2 vary on $\partial B_R(0)$, and this makes impossible to prove the continuity of a function like F. Note that the discontinuity occurs when $p_1 = p_2$:



When p_2 moves continuously on $\partial B_R(0)$ and crosses p_1, although the two represented arcs remains "close" in the uniform topology, the winding vector drastically changes, passing from $(1,0,1,1,1)$ to $(0,1,0,0,0)$ (recall that to compute the winding vector we close the arc with the portion of $\partial B_R(0)$ connecting p_2 with p_1 in counterclockwise sense). On the contrary, the partition which is determined by the inner arc does not change when p_2 crosses p_1. This makes possible to prove Lemma 2.5.3 only when working with prescribed sequences of partitions, and not of winding vectors: indeed, if we had fixed a sequence of winding vectors, the choice of the *shorter* arc of $\partial B_R(0)$ connecting p_i^m and p_i in the proof of Lemma 2.5.3 could have implied a change in the assigned winding vector, so that the comparisons (2.63) and (2.64) would not have been justified.

We wish to show that the Euler equation $\nabla F(\bar{p}_0, \ldots, \bar{p}_{2n}) = 0$ gives a smoothness condition for the function $\gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n})}$. Unfortunately, due to the lack of uniqueness of the inner minimizers of the Maupertuis functional, we cannot say at this moment that F has partial derivatives. However, we can overcome the problem with the introduction of a family of functions which are strictly related to F.

Let $k \in \{0, \ldots, 2n-1\}$. To fix our minds, let $k = 2j+1$ for some $j \in \{0, \ldots, n-1\}$. We introduce a strongly convex neighbourhood U_{2j+1} of the point $\bar{p}_{2j+1}$ with respect to the Jacobi metric. Let us choose $t_* \in (0, T_{2j+1})$ such that

$$\widetilde{p}_{2j+1} := y_{2j+1}(t_*) \in U_{2j+1}, \quad |\widetilde{p}_{2j+1}| < R, \quad y\left([0,t_*]\right) \subset \left(B_R(0) \setminus B_{R/2}(0)\right);$$

in this way, in $[0, t_*]$ the function y_{2j+1} does not interact with the singularities of the potential. There exists a unique minimal geodesic $\widehat{y}(\cdot; \bar{p}_{2j+1}, \widetilde{p}_{2j+1}; \varepsilon)$ for the Jacobi metric, parametrized with respect to the arc length, connecting p_{2j+1} and $\widetilde{p}_{2j+1}$ and lying in U_{2j+1}, which depends smoothly on its ends. We know that y_{2j+1} is a minimizer of the length L connecting p_{2j+1} and p_{2j+2}, therefore (Proposition 2.4.9) this geodesic has to be a re-parametrization of y_{2j+1}. Note that if $p_{2j+1} \in U_{2j+1}$, then there exists a unique minimal geodesics $\widehat{y}(\cdot; p_{2j+1}, \widetilde{p}_{2j+1}; \varepsilon)$ for the Jacobi metric, parametrized with respect to the arc length, which connects p_{2j+1} and $\widetilde{p}_{2j+1}$. We will consider the re-parametrization $\widetilde{y}(\cdot\,; p_{2j+1}, \widetilde{p}_{2j+1}; \varepsilon)$ of $\widehat{y}(\cdot\,; p_{2j+1}, \widetilde{p}_{2j+1}; \varepsilon)$ such that

$$\begin{cases} \ddot{\widetilde{y}}(t) = \nabla V_\varepsilon(\widetilde{y}(t)) \\ \frac{1}{2}|\dot{\widetilde{y}}(t)|^2 - V_\varepsilon(\widetilde{y}(t)) = -1, \end{cases}$$

denoting by $[0, T(p_{2j+1}, \widetilde{p}_{2j+1})]$ its domain. Due to the minimality of $\widehat{y}(\cdot\,; p_{2j+1}, \widetilde{p}_{2j+1}; \varepsilon)$ for L, such a re-parametrization exists, see Theorem 2.4.5. In this way

$$\widetilde{y}(\cdot\,; \bar{p}_{2j+1}, \widetilde{p}_{2j+1}; \varepsilon) \equiv y_{P_{k_{j+1}}}(\cdot\,; \bar{p}_{2j+1}, \widetilde{p}_{2j+1}; \varepsilon)|_{[0, T(\bar{p}_{2j+1}, \widetilde{p}_{2j+1})]}. \tag{2.65}$$

Let

$$D_{2j+1} := \{p_{2j+1} \in (\partial B_R(0) \cap U) : |\bar{p}_{2j} - p_{2j+1}| \leq \delta\}.$$

We define $G_{2j+1} : D_{2j+1} \to \mathbb{R}$ as

$$G_{2j+1}(p_{2j+1}) := L\left([0, T(p_{2j+1})]; y_{\text{ext}}(\cdot\,; \bar{p}_{2j}, p_{2j+1}; \varepsilon)\right) + L\left([0, T(p_{2j+1}, \widetilde{p}_{2j+1})]; \widetilde{y}(\cdot\,; p_{2j+1}, \widetilde{p}_{2j+1}; \varepsilon)\right),$$

where we write (and we will adopt this notation from now on) $T(p_{2j+1})$ for $T_{\text{ext}}(\bar{p}_{2j}, p_{2j+1}; \varepsilon)$. Of course, with minor changes we can also define a function G_{2j}, for every $j \in \{0, \ldots, 2n\}$.
Note that G_k is continuous (for every k), since it is a sum of terms which are both continuous with respect to p_k. As a consequence, G_k has a minimum.

Lemma 2.5.5. *If $(\bar{p}_0, \ldots, \bar{p}_{2n})$ is a minimizer for F, then $\bar{p}_k$ is a minimizer for G_k.*

Proof. We consider the case $k = 1$. Assume by contradiction that there exists $p_1 \neq \bar{p}_1$, $p_1 \in D_1$, such that $G_1(p_1) < G_1(\bar{p}_1)$. We show that we can produce a variation of $(\bar{p}_0, \ldots, \bar{p}_{2n})$ such that F decreases along this variation, which gives an absurd. To be precise, let us consider the function

$$\widehat{y}(t) := \begin{cases} \widetilde{y}(t; p_1, \widetilde{p}_1; \varepsilon) & \text{if } t \in [0, T(p_1, \widetilde{p}_1)] \\ y_{P_{k_1}}(t - T(p_1, \widetilde{p}_1); \bar{p}_1, \bar{p}_2; \varepsilon) & \\ \qquad \text{if } t \in [T(p_1, \widetilde{p}_1) + T(\bar{p}_1, \widetilde{p}_1), T(p_1, \widetilde{p}_1) + T_{P_{k_1}}(\bar{p}_1, \bar{p}_2; \varepsilon)] \end{cases}$$

i.e. $\widehat{y}$ is obtained as the junction between the minimal geodesic connecting p_1 and $\widetilde{p}_1$, and the arc of $y_{P_1}(\cdot\,; \bar{p}_1, \bar{p}_2; \varepsilon)$ starting from $\widetilde{p}_1$ and arriving at $\bar{p}_2$. By construction it follows that

$$\widehat{y} \in K_{P_{k_1}}^{p_1 \bar{p}_2}([0, T(p_1, \widetilde{p}_1) + T_{P_{k_1}}(\bar{p}_1, \bar{p}_2; \varepsilon)]).$$

The assumption $G_1(p_1) < G_1(\bar{p}_1)$ implies that

$$L(y_{\text{ext}}(\cdot\,; \bar{p}_0, p_1; \varepsilon)) + L(\widehat{y}) < L(y_{\text{ext}}(\cdot\,; \bar{p}_0, \bar{p}_1; \varepsilon)) + L(y_{P_1}(\cdot\,; \bar{p}_1, \bar{p}_2; \varepsilon)),$$

where we omitted the dependence of the functional L by the time interval to ease the notation. Since $L(\widehat{y})$ is greater than or equal to $L(y_{P_{k_1}}(\cdot\,; p_1, \bar{p}_2; \varepsilon))$, we have

$$F(\bar{p}_0, p_1, \ldots, \bar{p}_{2n}) < F(\bar{p}_0, \bar{p}_1, \ldots, \bar{p}_{2n}),$$

in contradiction with the minimality of $(\bar{p}_0, \ldots, \bar{p}_{2n})$. □

The main reason to pass from the study of F to the study of the functions G_k is that G_k is clearly differentiable for every k: let us think at $k = 2j+1$; the value $L\left([0, T(p_{2j+1})]; y_{\text{ext}}(\cdot\,; \bar{p}_{2j}, p_{2j+1}; \varepsilon)\right)$ depends smoothly on p_{2j+1} for the differentiable dependence of outer solutions with respect to the ends, and
$L\left([0, T(p_{2j+1}, \widetilde{p}_{2j})]; \widetilde{y}(\cdot\,; p_{2j+1}, \widetilde{p}; \varepsilon)\right)$ depends smoothly on p_{2j+1} for the differentiable dependence of minimal geodesics in a strongly convex neighbourhood with respect to the ends. Therefore the minimality of $\bar{p}_{2j+1}$ implies that if $\bar{p}_{2j+1} \in D^{\circ}_{2j+1}$ (the inner of D_{2j+1}), then

$$\frac{\partial G_{2j+1}}{\partial p_{2j+1}}(\bar{p}_{2j+1}) = 0.$$

We point out that this partial derivative is a linear operator from the tangent space $T_{\bar{p}_{2j+1}}(\partial B_R(0))$ into $\mathbb{R}$.

In what follows we show that, provided ε is small enough, $\bar{p}_k \in D^{\circ}_k$ for every k, and that this Euler equation is nothing but a regularity condition for the functions

$$\zeta_{2j}(t) := \begin{cases} y_{P_{k_{j-1}}}(t; \bar{p}_{2j-1}, \bar{p}_{2j}; \varepsilon) & \text{if } t \in [0, T(\widetilde{p}_{2j}, \bar{p}_{2j})] \\ y_{\text{ext}}(t - T(\widetilde{p}_{2j}, \bar{p}_{2j}); \bar{p}_{2j}, \bar{p}_{2j+1}; \varepsilon) & \\ \qquad \text{if } t \in [T(\widetilde{p}_{2j}, \bar{p}_{2j}), T(\widetilde{p}_{2j}, \bar{p}_{2j}) + T(\bar{p}_{2j+1})] \end{cases}$$

and

$$\zeta_{2j+1}(t) := \begin{cases} y_{\text{ext}}(t; \bar{p}_{2j}, \bar{p}_{2j+1}; \varepsilon) & \text{if } t \in [0, T(\bar{p}_{2j+1})] \\ y_{P_{k_{j+1}}}(t - T(\bar{p}_{2j+1}); \bar{p}_{2j}, \bar{p}_{2j+1}; \varepsilon) & \\ \qquad \text{if } t \in [T(\bar{p}_{2j+1}), T(\bar{p}_{2j+1}) + T(\bar{p}_{2j+1}, \widetilde{p}_{2j+1})]. \end{cases}$$

At that point the proof of Theorem 2.5.1 will be almost complete: by taking into account that ζ_k is (up to a time translation) the restriction of $\gamma_{(\bar{p}_0, \ldots, \bar{p}_{2n})}$ on a neighbourhood of the junction time $\mathfrak{T}_{k-1}$, we obtain $\mathcal{C}^1$ regularity for $\gamma_{(\bar{p}_0, \ldots, \bar{p}_{2n})}$. Then it will not be difficult to conclude the proof of 2.5.1.

Lemma 2.5.6. *For every $p_{2j} \in D_{2j}$ and for every $\varphi \in T_{p_{2j}}(\partial B_R(0))$ we have*

$$\frac{\partial G_{2j}}{\partial p_{2j}}(p_{2j})[\varphi] = \frac{1}{\sqrt{2}} \langle \dot{\widetilde{y}}(T(\widetilde{p}_{2j}, p_{2j}); \widetilde{p}_{2j}, p_{2j}; \varepsilon) - \dot{y}_{ext}(0; p_{2j}, \bar{p}_{2j+1}; \varepsilon), \varphi \rangle.$$

For every $p_{2j+1} \in D_{2j+1}$ and for every $\varphi \in T_{p_{2j+1}}(\partial B_R(0))$ we have

$$\frac{\partial G_{2j+1}}{\partial p_{2j+1}}(p_{2j+1})[\varphi] = \frac{1}{\sqrt{2}} \langle \dot{y}_{ext}(T(p_{2j+1}); \bar{p}_{2j}, p_{2j+1}; \varepsilon) - \dot{\widetilde{y}}(0; p_{2j+1}, \widetilde{p}_{2j+1}; \varepsilon), \varphi \rangle.$$

Proof. It is not restrictive to consider the derivative of G_1 to ease the notation. The same calculations work for the other cases. It holds

$$\frac{\partial G_1}{\partial p_1}(p_1) = \frac{\partial}{\partial p_1} L\left([0, T(p_1)]; y_{\text{ext}}(\cdot\,; \bar{p}_0, p_1; \varepsilon)\right) + \frac{\partial}{\partial p_1} L\left([0, T(p_1, \widetilde{p}_1)]; \widetilde{y}(\cdot\,; p_1, \widetilde{p}_1; \varepsilon)\right). \tag{2.66}$$

Let us consider the first term on the right hand side, writing simply y_0 instead of $y_{\text{ext}}(\cdot\,; \bar{p}_0, p_1; \varepsilon)$; we consider $u_0(t) = y_0(T_0 t)$, defined in $[0, 1]$. It results

$$\begin{aligned}
\frac{\partial}{\partial p_1} L\left([0, T(p_1)]; y_0\right) &= \frac{\partial}{\partial p_1} L\left([0, 1]; u_0\right) = dL\left([0, 1]; u_0\right)\left[\frac{\partial u_0}{\partial p_1}\right] \\
&= \frac{1}{\sqrt{2}} \int_0^1 \left[\frac{1}{T_0}\left\langle \dot{u}_0, \frac{d}{dt}\frac{\partial u_0}{\partial p_1}\right\rangle + T_0 \left\langle \nabla V_\varepsilon(u_0), \frac{\partial u_0}{\partial p_1}\right\rangle\right] \\
&= \frac{1}{\sqrt{2}} \int_0^1 \left(\left\langle -\frac{1}{T_0}\ddot{u}_0 + T_0 \nabla V_\varepsilon(u_0), \frac{\partial u_0}{\partial p_1}\right\rangle\right) + \frac{1}{\sqrt{2}T_0}\left[\left\langle \dot{u}_0(t), \frac{\partial u_0}{\partial p_1}(t)\right\rangle\right]_0^1 \\
&= \frac{1}{\sqrt{2}}\left[\left\langle \dot{y}_0(t), \frac{\partial y_0}{\partial p_1}(t)\right\rangle\right]_0^{T(p_1)}
\end{aligned}$$

In the second equality we use the conservation of the energy for y_0, in the last one we use the fact that, by definition, u_0 is a solution of

$$\frac{1}{T_0^2}\ddot{u}_0(t) = \nabla V_\varepsilon(u_0(t)).$$

Every $\varphi \in T_{p_1}(\partial B_R(0))$ is of the form

$$\varphi = \beta'(0) \quad \text{for some } \beta : I \to \partial B_R(0) \text{ of class } \mathcal{C}^1,\ \beta(0) = p_1.$$

For $\varphi = \beta'(0) \in T_{p_1}(\partial B_R(0))$ it results

$$\frac{\partial}{\partial p_1} y_0(0)[\beta'(0)] = \lim_{\lambda \to 0} \frac{y_{\text{ext}}(0; \bar{p}_0, \beta(\lambda); \varepsilon) - y_{\text{ext}}(0; \bar{p}_0, p_1; \varepsilon)}{\lambda} = 0,$$

and

$$\begin{aligned}
\frac{\partial}{\partial p_1} y_0(T(p_1))[\beta'(0)] &= \lim_{\lambda \to 0} \frac{y_{\text{ext}}(T(\beta(\lambda)); \bar{p}_0, \beta(\lambda); \varepsilon) - y_{\text{ext}}(T(p_1); \bar{p}_0, p_1; \varepsilon)}{\lambda} \\
&= \beta'(0),
\end{aligned}$$

where $y_{\text{ext}}(\cdot\,; p_0, \beta(\lambda); \varepsilon)$ is the exterior solution of (2.4) connecting p_0 and $\beta(\lambda)$ in time $T(\beta(\lambda))$. Therefore, for every $\varphi \in T_{p_1}(\partial B_R(0))$

$$\frac{\partial}{\partial p_1} L\left([0, T(p_1)]; y_{\text{ext}}(\cdot\,; \bar{p}_0, p_1; \varepsilon)\right)[\varphi] = \frac{1}{\sqrt{2}}\left\langle \dot{y}_{\text{ext}}(T(p_1); \bar{p}_0, p_1; \varepsilon), \varphi\right\rangle.$$

As far as the second term in the right side of the (2.66) is concerned, we can repeat the same computations with minor changes; note that in principle this wouldn't have been possible if we had considered the whole $y_{P_{k_1}}(\cdot\,;p_1,p_2;\varepsilon)$ instead of $\widetilde{y}$, since it is not evident that $y_{P_{k_1}}$ depends smoothly on p_1. □

Lemma 2.5.7. *There exists $\bar{\varepsilon} > 0$ such that for every $\varepsilon \in (0, \bar{\varepsilon})$*

$$\bar{p}_k \text{ minimizes } G_k \quad \Longrightarrow \quad \bar{p}_k \in D_k^\circ \qquad \forall k.$$

The value $\bar{\varepsilon}$ is independent on the sequence of partitions $(P_{k_1}, \ldots, P_{k_n}) \in \mathcal{P}^n$.

Proof. Assume that there exists $k \in \{0, \ldots, 2n\}$ such that

$$\begin{cases} |\bar{p}_k - \bar{p}_{k+1}| = \delta & \text{if } k \text{ is even} \\ |\bar{p}_k - \bar{p}_{k-1}| = \delta & \text{if } k \text{ is odd.} \end{cases}$$

To fix our minds, let $k = 1$. We can produce an explicit variation of $\bar{p}_1$ such that G_1 decreases along this variation, in contradiction with the minimality of $\bar{p}_1$. We write

$$\begin{aligned} &y_{\text{ext}}(t;p_0,p_1;\varepsilon) = r_{\text{ext}}(t;p_0,p_1;\varepsilon)\exp\{i\theta_{\text{ext}}(t;p_0,p_1;\varepsilon)\}, \\ &y_{P_{k_1}}(t;p_1,p_2;\varepsilon) = r_{P_{k_1}}(t;p_1,p_2;\varepsilon)\exp\{i\theta_{P_{k_1}}(t;p_1,p_2;\varepsilon)\}, \\ &\widetilde{y}(t;p_1,\widetilde{p}_1;\varepsilon) = \widetilde{r}(t;p_1,\widetilde{p}_1;\varepsilon)\exp\{i\widetilde{\theta}(t;p_1,\widetilde{p}_1;\varepsilon)\}, \end{aligned}$$

where we recall that $\widetilde{y}(\cdot\,;p_1,p_2;\varepsilon)$ is characterized by (2.65). The first step consists in proving that there exist $C_1 > 0$ and $\varepsilon_4 > 0$ such that, if $0 < \varepsilon < \varepsilon_4$, then

$$|\dot{\theta}_{\text{ext}}(T_{\text{ext}}(p_*,p_{**};\varepsilon);p_*,p_{**};\varepsilon)| \geq C_1 \qquad \text{for every} \\ (p_*,p_{**}) \in \{(p_*,p_{**}) \in (\partial B_R(0))^2 : |p_* - p_{**}| = \delta\}. \tag{2.67}$$

This means that, if the distance between (p_*, p_{**}) is δ, for ε small enough the outer solution connecting these two points arrive in p_{**} with an angular momentum which cannot be too small. To show it, we observe that, since the unperturbed problem ($\varepsilon = 0$) is invariant under rotations, there is $C_2 > 0$ such that

$$|\dot{\theta}_{\text{ext}}(T_{\text{ext}}(p_*,p_{**};0);p_*,p_{**};0)| = C_2 \qquad \text{for every} \\ (p_*,p_{**}) \in \{(p_*,p_{**}) \in (\partial B_R(0))^2 : |p_* - p_{**}| = \delta\}.$$

Now, assume by contradiction that (2.67) does not hold. Then there exist two sequences (λ_n) and (ε_n) of positive numbers and a sequence of points $(p_*^n, p_{**}^n) \in (\partial B_R(0))^2$, with $|p_*^n - p_{**}^n| = \delta$ for every n, such that

$$\lambda_n \to 0 \quad \varepsilon_n \to 0 \quad |\dot{\theta}_{\text{ext}}(T_{\text{ext}}(p_*^n,p_{**}^n;\varepsilon_n);p_*^n,p_{**}^n;\varepsilon_n)| < \lambda_n.$$

Since the set $\{(p_*, p_{**}) \in (\partial B_R(0))^2 : |p_* - p_{**}| = \delta\}$ is compact, up to a subsequence (p_*^n, p_{**}^n) converges to a point $(\bar{p}_*, \bar{p}_{**})$, and thanks to the continuous dependence of any outer solutions with respect to variations of the vector field and initial data, we obtain

$$|\dot{\theta}_{\text{ext}}(T_{\text{ext}}(\bar{p}_*, \bar{p}_{**}; 0); \bar{p}_*, \bar{p}_{**}; 0)| = 0,$$

a contradiction. This proves (2.67).

On the other hand, we can prove that any inner trajectory (for every p_1 and p_2 on $\partial B_R(0)$, for every $P_j \in \mathcal{P}$) starts with a small angular momentum, if ε is sufficiently small; to be precise

$$\forall \lambda > 0 \ \exists \varepsilon_5 > 0 : \ 0 < \varepsilon < \varepsilon_5 \quad \Longrightarrow \quad |\dot{\theta}_{P_j}(0; p_1, p_2; \varepsilon)| < \lambda, \tag{2.68}$$

for every $p_1, p_2 \in \partial B_R(0)$, for every $P_j \in \mathcal{P}$. To show it, we define $S = S(p_1, p_2; \varepsilon) > 0$ by

$$t \in (0, S) \quad \Longrightarrow \quad \frac{R}{2} < |y_{P_j}(t; p_1, p_2; \varepsilon)| < R \quad \text{and} \quad |y_{P_j}(S; p_1, p_2; \varepsilon)| = \frac{R}{2}.$$

The energy integral makes this quantity uniformly bounded from below by a positive constant C, as function of ε. Letting $\varepsilon \to 0^+$ the centres collapse in the origin, so that for the angular momentum of $y_{P_{k_1}}(\cdot\,; p_1, p_2; \varepsilon)$ it results

$$\mathfrak{C}_{y_{P_{k_1}}(\cdot\,; p_1, p_2; \varepsilon)}(t) = o(1) \quad \text{for} \quad \varepsilon \to 0^+,$$

uniformly in $[0, C]$ (recall Proposition 2.4.23). This limit is uniform in p_1, p_2 and P_{k_1}: indeed, since the curve parametrized by $y_{P_{k_1}}(\cdot\,; p_1, p_2; \varepsilon)$ has to pass inside the ball or radius ε, the function $y_{P_{k_1}}(\cdot\,; p_1, p_2; \varepsilon)$ uniformly converges in $[0, C]$, for $\varepsilon \to 0$, to the same (up to a rotation) piece of collision solution of the Kepler problem. This proves the estimate(2.68). The choice $\lambda = C_1/2$ in (2.68) gives

$$|\dot{\theta}_{P_j}(0; p_1, p_2; \varepsilon)| < \frac{C_1}{2} \qquad \text{if } 0 < \varepsilon < \varepsilon_5,$$

for every $p_1, p_2 \in \partial B_R(0)$, for every $P_j \in \mathcal{P}$. Recalling the relation (2.65), we deduce that

$$|\dot{\widetilde{\theta}}(0; \bar{p}_1, \widetilde{p}_1; \varepsilon)| < \frac{C_1}{2} \qquad \text{if } 0 < \varepsilon < \varepsilon_5. \tag{2.69}$$

Assume now that $\bar{p}_0 = R\exp\{i\bar{\theta}_0\}$, $\bar{p}_1 = R\exp\{i\bar{\theta}_1\}$, with $\bar{\theta}_0, \bar{\theta}_1 \in [0, 2\pi)$ and $\bar{\theta}_0 < \bar{\theta}_1$ (if $\bar{\theta}_0 < \bar{\theta}_1$ a very similar argument works). We consider a variation $\varphi \in T_{\bar{p}_1}(\partial B_R(0))$ of $\bar{p}_1$ directed towards $\bar{p}_0$ on $\partial B_R(0)$. Since $\bar{\theta}_0 < \bar{\theta}_1$, this variation is a positive multiple of $-i\exp\{i\bar\theta_1\}$. Collecting (2.67), (2.69) and using Lemma 2.5.6, for any $0<\varepsilon<\min\{\varepsilon_2,\varepsilon_3,\varepsilon_4,\varepsilon_5\}=:\bar\varepsilon$ we have that if $|\bar p_0-\bar p_1|=\delta$ then

$$\begin{aligned}\frac{\partial G_1}{\partial p_1}(\bar p_1)[\varphi] &= \frac{CR}{\sqrt{2}}\left\langle\left(\dot\theta_{\mathrm{ext}}\left(T_{\mathrm{ext}}(\bar p_0,\bar p_1;\varepsilon);\bar p_0,\bar p_1;\varepsilon\right)-\dot{\tilde\theta}\left(0;\bar p_1,\widetilde p_1;\varepsilon\right)\right)ie^{i\theta_1},-ie^{i\theta_1}\right\rangle\\ &< \frac{CR}{\sqrt{2}}\left(\frac{C_1}{2}-C_1\right)<0,\end{aligned}$$

against the minimality of $(\bar p_0,\dots,\bar p_{2n})$. We point out that $\bar\varepsilon$ does not depend neither on $n\in\mathbb{N}$ nor on $(P_{k_1},\dots,P_{k_n})\in\mathcal{P}^n$. □

Lemma 2.5.8. *Each function ζ_k is $\mathcal{C}^1$.*

Proof. In light of the previous lemma, we know that

$$\frac{\partial G_k}{\partial p_k}(\bar p_k)=0\qquad\forall k.$$

Without loss of generality, we consider the case $k=1$. For every $\varphi\in T_{p_1}(\partial B_R(0))$

$$\frac{1}{\sqrt{2}}\langle\dot y_{\mathrm{ext}}(T(p_1);\bar p_0,\bar p_1;\varepsilon)-\dot{\tilde y}(0;\bar p_1,\widetilde p_1;\varepsilon),\varphi\rangle=0.$$

If $\bar p_1=Re^{i\bar\theta_1}$, the tangent space $T_{p_1}(\partial B_R(0))$ is spanned by $ie^{i\bar\theta_1}$. We deduce that

$$\begin{aligned}&|\dot y_{\mathrm{ext}}(T(\bar p_1);\bar p_0,\bar p_1;\varepsilon)|\cos\left(\widehat{\dot y_{\mathrm{ext}}(T(\bar p_1);\bar p_0,\bar p_1;\varepsilon),ie^{i\bar\theta_1}}\right)\\ &\qquad\qquad=|\dot{\tilde y}(0;\bar p_1,\widetilde p_1;\varepsilon)|\cos\left(\widehat{\dot{\tilde y}(0;\bar p_1,\widetilde p_1;\varepsilon),ie^{i\bar\theta_1}}\right).\end{aligned}$$

Here $(\widehat{v_1,v_2})$ denotes the angle between the vectors v_1 and v_2. As a consequence of the conservation of the energy

$$|\dot y_{\mathrm{ext}}(T(\bar p_1);\bar p_0,\bar p_1;\varepsilon)|=|\dot{\tilde y}(0;\bar p_1,\widetilde p_1;\varepsilon)|,\tag{2.70}$$

so that

$$\cos\left(\widehat{\dot y_{\mathrm{ext}}(T(\bar p_1);\bar p_0,\bar p_1;\varepsilon),ie^{i\bar\theta_1}}\right)=\cos\left(\widehat{\dot{\tilde y}(0;\bar p_1,\widetilde p_1;\varepsilon),ie^{i\bar\theta_1}}\right).$$

Both $\dot y_{\mathrm{ext}}(T(\bar p_1);\bar p_0,\bar p_1;\varepsilon)$ and $\dot{\tilde y}(0;\bar p_1,\widetilde p_1;\varepsilon)$ point towards the interior of $B_{R/2}(0)$, so that

$$\left(\widehat{\dot y_{\mathrm{ext}}(T(\bar p_1);\bar p_0,\bar p_1;\varepsilon),ie^{i\bar\theta_1}}\right)=\left(\widehat{\dot{\tilde y}(0;\bar p_1,\widetilde p_1;\varepsilon),ie^{i\bar\theta_1}}\right).\tag{2.71}$$

From (2.70) and (2.71), we easily deduce

$$\dot y_{\mathrm{ext}}(T(\bar p_1);\bar p_0,\bar p_1;\varepsilon)=\dot{\tilde y}(0;\bar p_1,\widetilde p_1;\varepsilon).$$

□

Conclusion of the proof of Theorem 2.5.1. Let $(\bar{p}_0, \ldots, \bar{p}_{2n})$ be a minimizer of F in D°, and let $\gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n})}$ the associated periodic function defined by (2.62). Due to Lemma 2.5.8 we can say that $\gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n})}$ has $\mathcal{C}^1$ junctions in each time $\mathfrak{T}_k$.

If $\alpha \in (1,2)$, it is also a classical solution of the N-centres problem with energy -1 in $[0, \mathfrak{T}_{2n-1}] \setminus \{0, \mathfrak{T}_0, \ldots, \mathfrak{T}_{2n-1}\}$. Since

$$\gamma_{(p_0,\ldots,p_{2n})}(0) = \gamma_{(p_0,\ldots,p_{2n})}(\mathfrak{T}_{2n-1}), \qquad \dot{\gamma}_{(p_0,\ldots,p_{2n})}(0) = \dot{\gamma}_{(p_0,\ldots,p_{2n})}(\mathfrak{T}_{2n-1}),$$

it can be defined in the whole $\mathbb{R}$ by periodicity. If we prove that it is of class $\mathcal{C}^2$, we can say that $\gamma_{(p_0,\ldots,p_{2n})}$ is a classical periodic solution. Let us fix $k = 2j+1$, $j \in \{0, \ldots, n-1\}$ (for k even the same reasoning applies). It results

$$\begin{aligned}\lim_{t \to \mathfrak{T}_{2j+1}^-} \ddot{\gamma}_{(\bar{p}_0,\ldots,\bar{p}_{2n})}(t) &= \lim_{t \to T_{2j+1}^-} \ddot{y}_{2j+1}(t) = \lim_{t \to T_{2j+1}^-} \nabla V(y_{2j+1}(t)) = \\ &= \lim_{t \to 0^+} \nabla V(y_{2j+2}(t)) = \lim_{t \to 0^+} \ddot{y}_{2j+2}(t) = \lim_{t \to \mathfrak{T}_{2j+1}^+} \ddot{\gamma}_{(\bar{p}_0,\ldots,\bar{p}_{2n})}(t);\end{aligned}$$

this completes the proof for $\alpha \in (1,2)$.

If $\alpha = 1$, it is possible that $\gamma_{(p_0,\ldots,p_{2n})}$ is collision-free; in such a case the same line of reasoning leads to alternative (ii)-(a) in Theorem 2.5.1. If a collision occurs, we aim at showing that necessarily we are in cases (ii)-(b) or (ii)-(c). From Corollary 2.4.16, a necessary condition for the presence of collisions is the existence of $P_{k_j} \in \mathcal{P}_1$ for some $j \in \{1, \ldots, n\}$; by possibly applying the right shift a number of times, it is not restrictive to assume that $j = 1$. First of all we prove that $\gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n})}$ has to bounce again against a centre or against the curve $\{y \in \mathbb{R}^2 : V_\varepsilon(y) = 1\}$. Let t^* its first collision time. Since y_1 is an ejection-collision trajectory, $\gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n})}$ has the same property:

$$\gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n})}(t^* + t) = \gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n})}(t^* - t) \qquad \forall t \in \mathbb{R};$$

this is a consequence of the uniqueness theorem for regular initial values problems. On the other hand, since $\gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n-1})}$ has period $\mathfrak{T}_{2n-1}$, it has a reflectional symmetry also with respect to $t^* + \mathfrak{T}_{2n-1}/2$: indeed

$$\gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n-1})}(t^* + t) = \gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n-1})}(t^* - t) = \gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n-1})}(t^* - t + \mathfrak{T}_{2n-1}),$$

so that

$$\gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n-1})}\left(t^* + \frac{\mathfrak{T}_{2n-1}}{2} + t\right) = \gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n-1})}\left(t^* + \frac{\mathfrak{T}_{2n-1}}{2} - t\right).$$

The function can be smooth at this second reflection time only if

$$\dot{\gamma}_{(\bar{p}_0,\ldots,\bar{p}_{2n})}(t^* + \mathfrak{T}_{2n-1}/2) = 0 \quad \Longleftrightarrow \quad V_\varepsilon\left(\gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n})}\left(t^* + \frac{\mathfrak{T}_{2n-1}}{2}\right)\right) = 1;$$

otherwise $t^* + \mathfrak{T}_{2n-1}/2$ has to be another collision instant.

To conclude, we note that the reflectional symmetry of the solution impose some restrictions on the sequence $(P_{k_1}, \ldots, P_{k_n})$, as specified in Theorem 2.5.1. □

2.6 Symbolic dynamics

In this section we fix $\alpha \in [1, 2)$ and $h \in (\bar{h}, 0)$. Let us rewrite some partial results obtained for the normalized problem (energy -1 with parameter $\varepsilon \in (0, \bar{\varepsilon})$) in term of the "original" N-centre problem (to find solution of (2.2) with energy h).

From Corollary 2.2.2 we obtain a unique $\varepsilon \in (0, \bar{\varepsilon})$ such that $h = -\zeta(\varepsilon)$; let $\bar{R} = \bar{R}(h) = (-h)^{-1/\alpha} R$. In Section 2.3 we found a solution $y_{\text{ext}}(\cdot\,; p_0, p_1; \varepsilon)$ of (2.4) which stays outside $\partial B_R(0)$, and connects two points $p_0, p_1 \in \partial B_R(0)$ if their distance is smaller then δ. Via Proposition 2.2.1 we find a correspondent solution $x_{\text{ext}}(\cdot\,; x_0, x_1; h)$ for equation (2.2) with energy $h = -\zeta(\varepsilon)$, defined over an interval $[0, T_{\text{ext}}(x_0, x_1; h)]$. This solution connects $x_0, x_1 \in \partial B_{\bar{R}}$ close together (whose distance is smaller then $\bar{\delta}$), too, and stays outside $\partial B_{\bar{R}}(0)$.

In Section 2.4 we found a solution $y_{P_j}(\cdot\,; p_1, p_2; \varepsilon)$ of (2.4) connecting $p_1, p_2 \in \partial B_R(0)$, which comes from a minimizer u of the Maupertuis functional (with energy -1 and potential V_ε) in the class $K_{P_j}^{p_1 p_2}([0,1])$. Via Proposition 2.2.1 we find a correspondent solution $x_{P_j}(\cdot\,; x_1, x_2; h)$ for equation (2.2) with energy $h = -\zeta(\varepsilon)$, connecting $x_1, x_2 \in \partial B_{\bar{R}}(0)$, and defined over an interval $[0, T_{P_j}(x_1, x_2; h)]$. We set $T_{P_j}(x_1, x_2; h) = 1/\omega(x_1, x_2; h)$. As we mentioned in Remark 2.4.2, $x_{P_j}(\cdot\,; x_1, x_2; h)$ is a re-parametrization of a critical point $\mathfrak{u}_{P_j}(\cdot\,; x_1, x_2; h)$ of the Maupertuis functional (with energy h and with potential V) at a positive level. To be precise, for a fixed $h \in (\bar{h}, 0)$, let $x_1, x_2 \in \partial B_{\bar{R}}(0)$. We set

$$\widehat{\mathcal{H}}_{x_1 x_2}([a,b]) := \left\{ v \in H^1\left([a,b], \mathbb{R}^2\right) \;\middle|\; \begin{array}{l} v(a) = x_1,\ v(b) = x_2, \\ v(t) \neq c_j \ \forall t \in [a,b],\ \forall j \end{array} \right\}$$

and

$$\mathcal{H}_{x_1 x_2}([a,b]) := \left\{ v \in H^1([a,b], \mathbb{R}^2) : v(a) = x_1, v(b) = x_2, \right\}.$$

We point out that $\widehat{\mathcal{H}}_{x_1 x_2}([a,b]) \neq \widehat{H}_{x_1 x_2}([a,b])$, because the definition of the first set depends on the centres c_j of the original problem, while the definition of the second one is based on the the centres c_j' of the transformed problem (recall that, in Section 2.4, where we defined $\widehat{H}_{x_1 x_2}([a,b])$, we write c_j for the centres of the transformed problem; actually we should have written c_j', but we preferred to omit the " $'$ " to simplify the notation). Now, for $p_1, p_2 \in \partial B_R(0)$ we can set $x_1 = (-h)^{-1/\alpha} p_1 \in \partial B_{\bar{R}}(0)$, $x_2 = (-h)^{-1/\alpha} p_2 \in \partial B_{\bar{R}}(0)$; it is defined a bijective correspondence

$$J : u(t) \in H_{p_1 p_2}([0,1]) \mapsto (-h)^{-1/\alpha} u(t) \in \mathcal{H}_{x_1 x_2}([0,1]).$$

Note that the topological properties of a path in $\widehat{H}_{x_1 x_2}([0,1])$ with respect to a centre c'_j are the same of the associated path $\mathcal{H}_{x_1 x_2}([0,1])$ with respect to c'_j. In particular, for every $P_j \in \mathcal{P}$ we can set

$$\widehat{\mathfrak{K}}_{P_j}^{x_1 x_2}([0,1]) := J\left(\widehat{K}_{P_j}^{p_1 p_2}([0,1])\right) \quad \text{and} \quad \mathfrak{K}_{P_j}^{x_1 x_2}([0,1]) := J\left(K_{P_j}^{p_1 p_2}([0,1])\right).$$

Due to the characterization of $y_{P_j}(\cdot; p_1, p_2, \varepsilon)$ as re-parametrization of a minimizer of M_{-1} in $K_{P_j}^{p_1 p_2}([0,1])$, it is immediate to deduce that $\mathfrak{u}_{P_j}(\cdot\,; x_1, x_2; h)$ is a minimizer of M_h in $\mathfrak{K}_{P_j}^{x_1 x_2}([0,1])$.

In what follows *we consider* $h \in (\bar{h}, 0)$ *and fixed.* Hence we omit the dependence on h for the pieces of solutions of equation (2.2), to ease the notation. As we stated in Corollary 2.1.7, Theorem 2.1.2 enables us to characterized the dynamical system of the N-centre problem restricted on the energy shell

$$\mathcal{U}_h = \left\{(x,v) \in \mathbb{R}^2 \setminus \{c_1, \ldots, c_N\} \times \mathbb{R}^2 : \frac{1}{2}|v|^2 - V(x) = h\right\}$$

with a symbolic dynamics, where the symbols are the elements of $\mathcal{P}$. Let us rewrite the Hamilton's equations

$$\begin{cases} \dot{x}(t) = v(t) \\ \dot{v}(t) = \nabla V(x(t)). \end{cases} \tag{2.72}$$

Such a system defines the vector field

$$\begin{aligned} X :&\mathbb{R}^2 \setminus \{c_1, \ldots, c_N\} \times \mathbb{R}^2 \to \mathbb{R}^2 \times \mathbb{R}^2 \\ &(x,v) \mapsto (v, \nabla V(x)), \end{aligned}$$

which in turn generates the flow

$$\begin{aligned} \varphi^t :&\mathbb{R}^2 \setminus \{c_1, \ldots, c_N\} \times \mathbb{R}^2 \to \mathbb{R}^2 \setminus \{c_1, \ldots, c_N\} \times \mathbb{R}^2 \\ &(x_0, v_0) \mapsto (x(t; x_0, v_0), v(t; x_0, v_0)). \end{aligned}$$

It associates to (x_0, v_0) the solution of (2.72) having initial value $(x(0) = x_0, v(0) = v_0)$ evaluated at time t, and it is well defined for t in an open neighbourhood of 0. In general the flow is not complete (i.e. given (x_0, v_0) the solution $(x(t; x_0, v_0), v(t; x_0, v_0))$ is not defined for every $t \in \mathbb{R}$), due to the collisions; if $\alpha = 1$ we can complete it with the agreement that if there exists $t_* \in \mathbb{R}$ such that $x(\cdot\,; x_0, v_0)$ has a collision at t_*, then we extend the corresponding solution as an ejection-collision solution:

$$\varphi^{t_*+t}(x_0, v_0) := \varphi^{t_*-t}(x_0, v_0) \quad \forall t \in \mathbb{R}.$$

This implies in particular that at most two collisions occurs for every $(x,v) \in \mathbb{R}^2 \setminus \{c_1,\dots,c_N\} \times \mathbb{R}^2$, up to periodicity. Furthermore the resulting flow is the same given by the Levi-Civita regularization (see Remark 2.4.48); hence φ^t *is continuous for every* t.

If $\alpha \in (1,2)$ a similar completion is not necessary; it is sufficient to notice that φ^t is globally defined for every (x_0,v_0) such that the corresponding solution does not collide; in this case we have also continuity of the flow.

The energy shell $\mathcal{U}_h$ is a 3-dimensional submanifold of $\mathbb{R}^2 \setminus \{c_1,\dots,c_N\} \times \mathbb{R}^2$, which is invariant for X; hence it makes sense to consider the restriction $X_h := X|_{\mathcal{U}_h}$, for every $h \in (\bar{h},0)$. We consider

$$\mathcal{U}^{\pm}_{h,\bar{R}} := \left\{(x,v) \in \mathcal{U}^h : |x| = \bar{R} \text{ and } \langle v,x\rangle \gtrless 0\right\},$$

which are some sort of cylinders in $\mathbb{R}^4$; thinking at (x,v) as a pair position-velocity, $\mathcal{U}^+_{h,\bar{R}}$ (respectively $\mathcal{U}^-_{h,\bar{R}}$) is the set of pair with position $x \in \partial B_{\bar{R}}(0)$, and velocity which points towards the outer of (resp. towards the inner of) the ball $B_{\bar{R}}(0)$ and is not tangent to $\partial B_{\bar{R}}(0)$. For a point $(x,v) \in \mathcal{U}^+_{h,\bar{R}}$, the normal field to $\mathcal{U}^+_{h,\bar{R}}$ is

$$\mathcal{N}_{h,\bar{R}}(x,v) = \left(\frac{x}{\bar{R}},0\right).$$

The vector field X_h is transverse to $\mathcal{U}^+_{h,\bar{R}}$, in the sense that for every $(x,v) \in \mathcal{U}^+_{h,\bar{R}}$

$$\left\langle X_h(x,v), \mathcal{N}_{h,\bar{R}}(x,v)\right\rangle = \frac{\langle x,v\rangle}{\bar{R}} > 0.$$

For every $(x,v) \in \mathcal{U}^+_{h,\bar{R}}$ we can define

$$\mathfrak{T}^{\pm}(x,v) := \left\{t \in (0,+\infty) : \varphi^t(x,v) \in \mathcal{U}^{\pm}_{h,\bar{R}}\right\}$$

which in general can be empty. Let us term

$$\left(\mathcal{U}^+_{h,\bar{R}}\right)^{\pm} := \left\{(x,v) \in \mathcal{U}^+_{h,\bar{R}} : \mathfrak{T}^{\pm}(x,v) \neq \emptyset\right\}.$$

The set $\left(\mathcal{U}^+_{h,\bar{R}}\right)^{\pm}$ is not empty, since the periodic solutions we found in Theorem 2.1.2 cross the circle $\{|x| = \bar{R}\}$ with velocity $\dot{x}$ satisfying the transversality condition $\langle x,\dot{x}\rangle \gtrless 0$ an infinite number of times. The continuous dependence of the solution on initial data and the transversality of $\mathcal{U}^+_{h,\bar{R}}$ with respect to X_h implies that $\left(\mathcal{U}^+_{h,\bar{R}}\right)^+$ is open in $\mathcal{U}_{h,\bar{R}}$.

For $(x,v) \in \left(\mathcal{U}^+_{h,\bar{R}}\right)^{\pm}$, let

$$T^{\pm}_{\min} := \inf \mathfrak{T}^{\pm}(x,v).$$

For every $(x,v) \in \left(\mathcal{U}^+_{h,\bar{R}}\right)^{+} \cap \left(\mathcal{U}^+_{h,\bar{R}}\right)^{-}$ such that $T^-_{\min} < T^+_{\min}$, we consider the restriction of the trajectory starting from (x,v) to the first time interval needed to cross $B_{\bar{R}}(0)$, that is, $\{\varphi^t(x,v)\}_{t\in[T^-_{\min},T^+_{\min}]}$. We define

$$\mathcal{U}^{\mathcal{P}}_{h,\bar{R}} := \left\{ (x,v) \in \left(\mathcal{U}^+_{h,\bar{R}}\right)^+ \;\middle|\; \begin{array}{l} T^-_{\min} < T^+_{\min},\ \{\varphi^t(x,v)\}_{t\in[T^-_{\min},T^+_{\min}]} \text{ parametrizes} \\ \text{a minimizer of } L_h \text{ in } \mathfrak{K}^{x(T^-_{\min})x(T^+_{\min})}_{P_j}(T^-_{\min},T^+_{\min}), \\ \text{for some } P_j \in \mathcal{P} \end{array} \right\}.$$

It is non-empty, since the periodic solutions found in Theorem 2.1.2 provide an infinite number of points satisfying these conditions. It is possible to define a *first return map* on $\mathcal{U}^{\mathcal{P}}_{h,\bar{R}}$ as

$$\mathcal{R}(x,v) := \varphi^{T^+_{\min}}(x,v).$$

Note that $\mathcal{R}$ is continuous. We can also introduce an application $\chi : \mathcal{U}^{\mathcal{P}}_{h,\bar{R}} \to \mathcal{P}$ given by

$$\chi(x,v) := P_j \quad \text{if } \{\varphi^t(x,v)\}_{t\in[T^-_{\min},T^+_{\min}]} \in \mathfrak{K}^{x(T^-_{\min})x(T^+_{\min})}_{P_j}(T^-_{\min},T^+_{\min}).$$

Finally, let us term

$$\Pi_h := \bigcap_{j\in\mathbb{Z}} \mathcal{R}^j(\mathcal{U}^{\mathcal{P}}_{h,\bar{R}}),$$

the set of initial data such that the corresponding solutions cross the circle $\partial B_{\bar{R}}(0)$ with velocity directed towards the exterior of the ball $B_{\bar{R}}(0)$ an infinite number of time in the future and in the past, parametrizing a path of $\mathfrak{K}_{P_j}$ in each of its passages inside $B_R(0)$. In case $\alpha \in (1,2)$, we require also that the solution starting for a point $(x_0,v_0) \in \Pi_h$ is collision-free. In any case, the periodic solutions found in Theorem 2.1.2 provide an infinite number of points in Π_h. Now, for every $(x,v) \in \Pi_h$, we set $\pi : \Pi_h \to \mathcal{P}^{\mathbb{Z}}$ as

$$\pi(x,v) = (P_{j_k})_{k\in\mathbb{Z}} \quad \text{where} \quad P_{j_k} := \chi(\mathcal{R}^k(x,p)).$$

Introduced the restriction $\mathfrak{R} := \mathcal{R}|_{\Pi_h}$, we can re-formulate Corollary 2.1.7 as follows.

Proposition 2.6.1. *Under the assumption of Theorem 2.1.2, the map π is continuous and surjective, and the diagram*

$$\begin{array}{ccc} \Pi_h & \xrightarrow{\mathfrak{R}} & \Pi_h \\ \downarrow \pi & & \downarrow \pi \\ \mathcal{P}^{\mathbb{Z}} & \xrightarrow{T_r} & \mathcal{P}^{\mathbb{Z}}, \end{array}$$

commutes.

We need some preliminary results. The first step is to obtain uniform bounds, below and above, for the time interval of the pieces of outer and inner solutions.

Lemma 2.6.2. *There exist $C_1, C_2 > 0$ such that for every $(x_0, x_1) \in (\partial B_{\bar{R}}(0))^2$ such that $|x_1 - x_0| < \bar{\delta}$, and for every $(x_2, x_3) \in (\partial B_R(0))^2$, for every $P_j \in \mathcal{P}$, there holds*

$$C_1 \le T_{ext}(x_0, x_1) \le C_2$$
$$C_1 \le T_{P_j}(x_2, x_3) \le C_2.$$

Proof. It is a straightforward consequence of Lemmas 2.3.4 and 2.4.49, and of Proposition 2.2.1. □

It is useful to prove that, for a sequence of minimizers of M_h which separate the centres according to the same partition P_j, the convergence of the ends to $(\bar{x}_1, \bar{x}_2)$ is sufficient for the weak convergence in H^1 of the minimizers themselves; the limit path turns out to be minimal for M_h in $\mathfrak{K}_{P_j}^{\bar{x}_1 \bar{x}_2}([0,1])$.

Lemma 2.6.3. *Let $(x_1^n, x_2^n) \subset (\partial B_{\bar{R}}(0))^2$ such that $(x_1^n, x_2^n) \to (\bar{x}_1, \bar{x}_2)$, let $P_j \in \mathcal{P}$; let $\mathfrak{u}_n$ be a local minimizers of M_h in $\mathfrak{K}_{P_j}^{x_1^n x_2^n}([0,1])$. Then there exists a subsequence $(\mathfrak{u}_{n_k})$ of $(\mathfrak{u}_n)$ and a minimizer $\bar{\mathfrak{u}} \in \mathfrak{K}_{P_j}^{\bar{x}_1 \bar{x}_2}([0,1])$ of M_h such that $\mathfrak{u}_{n_k} \rightharpoonup \bar{\mathfrak{u}}$ in H^1.*

Proof. In order to prove that, up to subsequence, $(\mathfrak{u}_n)$ is weakly convergent, it is sufficient to show that $(\mathfrak{u}_n)$ is bounded in H^1. We know that

$$\|\mathfrak{u}_n\|_2^2 \le \bar{R}^2 \qquad \forall n,$$

hence it remains to check that there exists $C > 0$ such that

$$\|\dot{\mathfrak{u}}_n\|_2^2 \le C \qquad \forall n.$$

Repeating the same arguments explained in Lemma 2.4.49 to prove equation (2.59), we see that the minimality of $\mathfrak{u}_n$ implies this inequality.
Now let us prove that the limit $\mathfrak{u}$ is a minimizer of M_h. Arguing as in the proof of Lemma 2.5.3, it is not difficult to deduce that

$$(x_1^n, x_2^n) \to (\bar{x}_1, \bar{x}_2) \quad \Longrightarrow \quad L_h(\mathfrak{u}_{P_j}(\cdot; x_1^n, x_2^n; P_j)) \to L_h(\mathfrak{u}_{P_j}(\cdot; \bar{x}_1, \bar{x}_2; P_j)),$$

namely $L_h(\mathfrak{u}_n) \to L_h(\bar{\mathfrak{u}})$. Assume by contradiction that $\bar{\mathfrak{u}}$ is not a local minimizer of M_h; by Proposition 2.4.8 it follows that $\mathfrak{u}$ cannot be a minimizer also of L_h, then there exists a path $\mathfrak{v} \in L_{P_j}^{\bar{x}_1 \bar{x}_2}([0,1])$ such that $L_h(\mathfrak{v}) < L_h(\mathfrak{u})$. As usual, let $\zeta_{\bar{R}}(\cdot\,; x_*, x_{**})$ be the shorter (in the Euclidean metric) arc of $\partial B_{\bar{R}}(0)$ connecting x_* with x_{**}, parametrized with constant velocity in the time interval $[0,1]$. We have $L_h(\zeta_{\bar{R}}(\cdot;x_*,x_{**})) \to 0$ as $|x_* - x_{**}| \to 0$. Hence there exists $n_0 \in \mathbb{N}$ such that if $n > n_0$ then

$$\widehat{\mathfrak{v}}_n(t) := \begin{cases} \zeta_{\bar{R}}(3t; x_1^n, \bar{x}_1) & t \in [0,1/3] \\ \mathfrak{v}(3t-1) & t \in (1/3,2/3] \\ \zeta_{\bar{R}}(3t-2; \bar{x}_2, x_2^n) & t \in (2/3,1], \end{cases}$$

is a path of $\mathfrak{K}_{P_j}^{x_1^n x_2^n}([0,1])$ such that $L_h(\mathfrak{v}_n) < L_h(\mathfrak{u}_n)$, in contradiction with the minimality of $\mathfrak{u}_n$. □

Proof of Proposition 2.6.1. **Step 1)** We start with surjectivity. Let $(P_{j_n})_{n\in\mathbb{Z}} \subset \mathcal{P}^{\mathbb{Z}}$. We can consider the finite sequences

$$(P_{j_0}), \quad (P_{j_{-1}}, P_{j_0}, P_{j_1}), \quad \dots \quad (P_{j_{-n}}, \dots, P_{j_{-1}}, P_{j_0}, P_{j_1}, \dots, P_{j_n}), \quad \dots .$$

To each sequence we associate the corresponding periodic solution of equation (2.2) with energy h given by Theorem 2.1.2, according to the notation

$$(P_{j_{-n}}, \dots, P_{j_{-1}}, P_{j_0}, P_{j_1}, \dots, P_{j_n}) \leftrightsquigarrow x^n(\cdot).$$

Up to a time translation, we can take initial data $(x^n(0), \dot{x}^n(0)) \in \Pi_h$, in such a way that the first partition (or collision) determined by the solution $x^n(\cdot)$ is P_{j_0}, for every n. The path parametrized by $x^n(\cdot)$ detects a sequence of points $(x_k^n)_{k\in\mathbb{Z}}$ of $\partial B_{\bar{R}}(0)$ given by the intersections of the trajectories in $\mathbb{R}^2$ with the circle itself, taken in the temporal order (of course, since $x^n(\cdot)$ is periodic, the sequence will be periodic, too).
We consider the sequence of sequences:

$$(x_k^n)_{n\in\mathbb{N}} \subset \partial B_{\bar{R}}(0) \qquad \forall k \in \mathbb{Z}.$$

Now, since $\partial B_{\bar{R}}(0)$ is compact, we can extract a subsequence $(x_0^{n_0})_{n_0}$ which converges to $\bar{x}_0$. Analogously, as $(x_1^{n_0})_{n_0}$ stays in $\partial B_{\bar{R}}(0)$, therefore we can extract a subsequence $(x_1^{n_1})_{n_1}$ which converges to $\bar{x}_1$. Proceeding in this way, for every $k \in \mathbb{Z}$ we have a sequence $\left(x_k^{n_k}\right)_{n_k}$ which converges to $\bar{x}_k$. Then we relabel as $(x_k^n)_n$ the diagonal sequence, namely $(x_k^{n_n})_n$. It results

$$\lim_{n\to\infty} x_k^n = \bar{x}_k \qquad \forall k \in \mathbb{Z}. \tag{2.73}$$

For every $k \in \mathbb{Z}$, we connect the points $\bar{x}_{2k}$, $\bar{x}_{2k+1}$ with the unique outer solution of (2.2) given by Theorem 2.3.1. Analogously, we connect $\bar{x}_{2k+1}$ and $\bar{x}_{2k+2}$ with an inner solution given by Theorem 2.4.14. A collision can occur just if $\alpha = 1$ and $\bar{x}_{2k+1} = \bar{x}_{2k+2}$. We can juxtapose these paths in a continuous manner, following the same gluing procedure already carried on in Section 2.5 to define $\gamma_{(p_0,\dots,p_{2n})}$; in this way we obtain a continuous function $\bar{x}(\cdot) : \mathbb{R} \to \mathbb{R}^2$. It is important to note that, as we do not have uniqueness for the minimizers of the Maupertuis' functional in $\mathfrak{K}_{P_j}$, given the sequence of points $(\bar{x}_k)$, the function $\bar{x}$ is not uniquely determined. In what follows, we show that it is possible to build it in such a way that $\bar{x}$ is a solution of (2.2) (in case $\alpha = 1$, it can be an ejection-collision solution), with $(\bar{x}(0), \dot{\bar{x}}(0)) \in \Pi_h$ and $\pi((\bar{x}_0, \dot{\bar{x}}(0))) = (P_{j_k})_k$.

Let

$$T_c(\bar{x}) := \{t \in \mathbb{R} : \bar{x}(t) = c_j \text{ for some } j \in \{1, \dots, N\}\}.$$

The aim is to show that $x^n(\cdot) \to \bar{x}(\cdot)$ in $\mathcal{C}^2_{\text{loc}}(\mathbb{R} \setminus T_c(\bar{x}))$.
The first step consists in proving that, up to a subsequence, $(x^n(\cdot))$ converges to $\bar{x}(\cdot)$ uniformly on every compact set of $\mathbb{R}$. If $[a, b] \subset \mathbb{R}$ such that $\bar{x}(a) = \bar{x}_{2k}$ and $\bar{x}(b) = \bar{x}_{2k+1}$, with $k \in \mathbb{Z}$, then the uniform convergence in $[a, b]$ is a straightforward consequence of the continuous dependence of the external solutions by the end points (Theorem 2.3.1). On the other hand, if $[c, d] \subset \mathbb{R}$ with $\bar{x}(c) = \bar{x}_{2k+1}$ and $\bar{x}(d) = \bar{x}_{2k+2}$, then the uniform convergence has been proved in Lemma 2.6.3. We remark that this convergence uniquely determine the correct choice of the inner solution connecting $\bar{x}_{2k+1}$ and $\bar{x}_{2k+2}$, so that in turn $\bar{x}$ results uniquely determined.

From this, it is easy to obtain the uniform convergence for every compact subset of $\mathbb{R}$. Let us observe that since $\bar{x}|_{[c,d]}$ is a uniform limit of minimizers of L_h (and hence, up to re-parametrizations, also of M_h), if $\bar{x}|_{[c,d]}$ has a collision, necessarily $\bar{x}|_{[c,d]}$ parametrizes an ejection-collision path.

Now we show that the derivatives $\dot{x}^n(\cdot)$ are $\mathcal{C}^1$-convergent to the derivative of $\dot{\bar{x}}$. Assume first that $\bar{x}(\cdot)$ has no collisions in $\mathbb{R}$. Let $[a, b] \subset \mathbb{R}$ be compact. In this case there exists $\bar{n} \in \mathbb{N}$ such that $x^n(\cdot)$ is collision-free in $[a, b]$, as well. The function $V(\bar{x}(\cdot))$ is well defined in $\mathbb{R}$, and by regularity

$$\lim_{n\to\infty} \ddot{x}^n(t) = \lim_{n\to\infty} \nabla V(x^n(t)) = \nabla V(\bar{x}(t)), \tag{2.74}$$

with uniform convergence in $[a, b]$. Moreover, $\dot{x}^n(\cdot)$ is uniformly bounded in $[a, b]$ for the conservation of the energy:

$$|\dot{x}^n(t)| = \sqrt{2(V(x^n(t)) + h)} \leq \sqrt{2\,(C + h)} \qquad \forall t \in [a, b], \forall n \geq \bar{n}.$$

Hence, up to subsequence, for an arbitrary $\bar{t} \in (a, b)$ the sequence $(\dot{x}_n(t))$ is convergent in $\mathbb{R}^2$. This fact, together with (2.74), implies that $(\dot{x}^n(\cdot))$ converges in $\mathcal{C}^1([a, b])$, and hence $(x^n(\cdot))$ converges in $\mathcal{C}^2([a, b])$ to $\bar{x}(\cdot)$, for every compact subset $[a, b] \in \mathbb{R}$. This means that $\bar{x}$ is a $\mathcal{C}^2$ solution of (2.2) with energy h on $[a, b]$ and this argument works in every compact subset of $\mathbb{R}$. We point out that the uniform convergence is sufficient to say that, in its k-th passage inside $B_{\bar{R}}(0)$, $\bar{x}(\cdot)$ separates the centres according to P_{j_k}, namely $\pi(\bar{x}(0), \dot{\bar{x}}(0))) = (P_{j_k})_{k\in\mathbb{Z}}$.

We are left to examine what happens if a collision occurs. The $\mathcal{C}^2$-convergence of $(x^n(\cdot))$ to $\bar{x}(\cdot)$ is still true in every compact subset of $\mathbb{R} \setminus T_c(\bar{x})$, hence we obtain an

ejection-collision solution of (2.2) with energy h and $\pi(\bar{x}_0, \dot{\bar{x}}(0))) = (P_{j_k})_{k\in\mathbb{Z}}$.
We point out that this last case is possible just for $\alpha = 1$ and $(P_{j_k}) \in \mathcal{P}^{\mathbb{Z}}$ such that

- (P_{j_k}) is periodic and satisfies the conditions of points (*ii-b*) or (*ii-c*) of Theorem 2.1.2.
- up to a finite number of applications of the right shift, $P_{j_0} \in \mathcal{P}_1$ and the sequence is symmetric, i.e. $P_{j_{-n}} = P_{j_n}$ for every n.

Step 2) It remains to show that π is continuous. Let $(x_0, v_0) \in \Pi_h$. We would like to prove that given $\lambda > 0$ there exists $\varrho > 0$ such that for every $(x, v) \in \Pi_h$:

$$|(x,v) - (x_0, v_0)| < \varrho \quad \Longrightarrow \quad \sum_{m\in\mathbb{Z}} \frac{d_1(\pi_m(x,v), \pi_m(x_0,v_0))}{2^{|m|}} < \lambda,$$

where π_m is the projection $\pi_m : \Pi_h \to \mathcal{P}$ defined by

$$\pi_m(x,v) := \chi(\mathfrak{R}^m(x,v)),$$

i.e. π_m associate to (x, v) the partition that the corresponding solution induces in its m-th passage inside $B_{\bar{R}}(0)$. Let us observe that there exists $m_0 \in \mathbb{N}$ such that

$$\sum_{|m|>m_0} \frac{1}{2^{|m|}} < \lambda.$$

Hence it is sufficient to show that, if we take two initial data sufficiently close, then the corresponding solutions induce the same partitions P_{j_k} of the centres, for $k \in \{-m_0, \dots, m_0\}$. Thanks to lemma 2.6.2, we can fix a time interval $[-a, a]$ such that each solution with initial data in Π_h passes at least $2m_0 + 1$-times inside $B_{\bar{R}}(0)$ in $[-a, a]$. If the solution of (2.2) with starting point (x_0, v_0) is collision-free, then there exists $\mu > 0$ such that

$$|x(t; x_0, v_0) - c_j| \geq \mu \qquad \forall t \in [-a, a], \forall j.$$

If (x, v) is sufficiently close to (x_0, v_0), then the continuous dependence applies:

$$\exists \varrho > 0 : |(x,v) - (x_0, v_0)| < \varrho \quad \Longrightarrow \quad |x(t; x, v) - x(t; x_0, v_0)| < \frac{\mu}{2}.$$

This implies that $x(\cdot\,; x, v)$ is collision-free and detects the same partitions of $x(\cdot; x_0, v_0)$ in $[-a, a]$. In particular, $\pi_m(x, v) = \pi_m(x_0, v_0)$ for every $m \in \{-m_0, \dots, m_0\}$. This proves the continuity for non-collision initial data. But nothing change if we consider $(x_0, v_0) \in \Pi_h$ such that $x(\cdot; x_0, v_0)$ has a collision: this is possible only if $\alpha = 1$ (recall that for the case $\alpha \in (1, 2)$ we impose that the solution corresponding to $(x_0, v_0) \in \Pi_h$ is collision-free), and in this case we introduced a regularization trough the Levi-Civita transform (see Remark 2.4.48 on the Levi-Civita transform), which permits to apply the continuous dependence theorem also in such a situation. □

2.6.1 Proof of Corollary 2.1.8.

As we already mentioned, the case $N > 3$ is nothing but a consequence of the general case. If $N = 3$ it is sufficient to note that juxtaposing an arbitrary number of time the groups

$$G_1 := P_1P_1P_2P_3 \quad \text{and} \quad G_2 := P_2P_2P_3P_1,$$

we do not obtain a periodic sequence showing the symmetry of cases (ii)-(b) or (ii)-(c) of Theorem 2.1.2This is the object of the following remark.

Remark 2.6.4. A possible way to explicitly check that there aren't collisions for solutions to the 3-centre problem associated to sequences of partitions of $\mathcal{G}$ is the following. Let $x_{((P_{k_1},\dots,P_{k_{4n}}),h)}$ be the periodic solution of the N-centre problem found in Theorem 2.1.2. Writing $(P_{k_1},\dots,P_{k_{4n}}) \in \mathcal{G}^n$ as an infinite periodic sequence, a group of 5 consecutive partitions is one of the following:

$$\begin{aligned} &P_1P_1P_2P_3P_1 \quad P_1P_1P_2P_3P_2 \quad P_1P_2P_3P_1P_1 \quad P_1P_2P_3P_2P_2 \quad P_2P_3P_1P_1P_2 \quad P_2P_3P_2P_2P_3 \\ &P_3P_1P_1P_2P_3 \quad P_3P_2P_2P_3P_1 \quad P_2P_2P_3P_1P_1 \quad P_2P_2P_3P_1P_2 \quad P_2P_3P_1P_1P_1 \\ &P_2P_3P_1P_2P_2 \quad P_3P_1P_1P_1P_2 \quad P_3P_1P_2P_2P_3 \quad P_1P_1P_1P_2P_3 \quad P_1P_2P_2P_3P_1. \end{aligned} \tag{2.75}$$

Assume that the considered solution has a collision with the centre c_1. According to the periodicity of $x_{((P_{k_1},\dots,P_{k_{4n}}),h)}$, and recalling that any collision solution is a collision-ejection solution, this means that there exists a group of five consecutive partitions $(P_{k_1},\dots,P_{k_5})$ in (2.75) such that

- $P_{k_3} = P_1$;
- $P_{k_1} = P_{k_5}$ and $P_{k_2} = P_{k_4}$.

It is immediate to check that none of the groups in (2.75) satisfies both the requirements. Analogously, it is possible to check that $x_{((P_{k_1},\dots,P_{k_{4n}}),h)}$ does not collide against c_2 or c_3.

Chapter 3

Symbolic dynamics: from the N-centre to the $(N+1)$-body problem, a preliminary study

3.1 Introduction and main results

In this chapter we present the paper [78], which concerns the generalization of the results of the previous chapter for a perturbed N-centre problem. We assume that the centres are not fixed, but rotate according to the law $\xi_k(t) := \exp\{i\nu t\} c_k$; here $\nu \in \mathbb{R}$ is a parameter describing the angular velocity of the rotation, so that the equation for the motion of the test particle becomes

$$\ddot{x}(t) = -\sum_{k=1}^{N} \frac{m_k}{|x(t) - e^{i\nu t} c_k|^3} \left(x(t) - e^{i\nu t} c_k \right). \tag{3.1}$$

The motivation leading to this problem is the following: the N-centre problem can be considered as a simplified version of the $(N+1)$-body problem, when one of the bodies is much faster then the others. Therefore, in order to understand if the broken geodesics method introduced in Chapter 2 can be extended to find solutions of the $(N+1)$-body problem, it seems reasonable to start considering an "easy test motion" for the centres, such as the uniformly circular one. This is strictly related to the study of the circular restricted $(N+1)$-body problem, which we briefly recall; assigned N positive masses $m_1, \ldots, m_N$, let us consider any planar central configuration $(c_1, \ldots, c_N)$ of the N-body problem, that is, any critical point of the potential of the N-body problem

$$U(x_1, \ldots, x_N) = \sum_{1 \le j < k \le N} \frac{m_j m_k}{|x_j - x_k|}$$

constrained on the *inertial ellipsoid*, defined by

$$I(x_1,\dots,x_N)=1, \quad \text{where} \quad I(x_1,\dots,x_N)=\frac{1}{2}\sum_{k=1}^{N} m_k|x_k|^2.$$

A relative equilibrium of the N-body problem is a motion of type $\xi_k(t):=\exp\{i\nu t\}c_k$ $(k=1,\dots,N)$, with $\nu\in\mathbb{R}$, i.e. an equilibrium point in a rotating frame of reference with angular velocity ν. The restricted problem consists in studying the motion of a test particle of null mass under the gravitational force field of N bodies (the *primaries*) which move according to a motion of relative equilibrium. This leads to the research of solutions of equation (3.1), with the difference that the value of ν is uniquely determined by the prescribed central configuration $(c_1,\dots,c_N)$ through the relation

$$\nu^2=\frac{U(c_1,\dots,c_N)}{2I(c_1,\dots,c_N)},$$

see Meyer [62]. As a toy model towards the restricted $(N+1)$-body problem, we consider the perturbed problem previously defined; we point out that the motivation for its study is prevalently mathematical: our goal is to understand if the techniques introduced in Chapter 2 are sufficiently robust to survive when we perturb the N-centre problem by letting the centres move; the answer is yes, but the extension of the broken geodesics method is not trivial and requires new ideas, especially concerning the possibility of obtaining collision-free solutions. Therefore, the generalization to the real restricted problem seems possible, but extremely complicated.

As in the previous chapter, we study the case of α-gravitational potentials ($\alpha\in[1,2)$), so that the equation for the motion of the test particle becomes

$$\ddot{x}(t)=-\sum_{k=1}^{N}\frac{m_k}{|x(t)-e^{i\nu t}c_k|^{\alpha+2}}\left(x(t)-e^{i\nu t}c_k\right). \tag{3.2}$$

We refer to the research of solutions to this equation as to *the rotating N-centre problem* (briefly, the rotating problem). It is convenient to introduce a different frame of reference for x, taking into account the rotation of the centres: setting $x(t)=\exp\{i\nu t\}z(t)$, equation (3.2) becomes

$$\ddot{z}(t)+2\nu i\dot{z}(t)=\nu^2 z(t)-\sum_{k=1}^{N}\frac{m_k}{|z(t)-c_k|^{\alpha+2}}\left(z(t)-c_k\right). \tag{3.3}$$

We introduce $\Phi_\nu(z):=\nu^2|z|^2/2+V(z)$, so that (3.3) can be written as

$$\ddot{z}(t)+2\nu i\dot{z}(t)=\nabla\Phi_\nu(z(t)).$$

Since the terms in z and $\dot{z}$ are multiplied by powers of ν, the idea is that if $|\nu|$ is sufficiently small, then equation (3.3) can be regarded as a perturbation of the planar N-centre problem. We observe that the energy function of a solution of equation (3.3) is not constant; however, it is possible to find a first integral defining

$$J_\nu(z, \dot{z}) := \frac{1}{2}|\dot{z}|^2 - \Phi_\nu(z).$$

The value $h = J_\nu(z(t), \dot{z}(t))$, which is the same for every $t \in I$, is called the *Jacobi constant*, in analogy with the same first integral of the circular restricted $(N+1)$-body problem. Let us note the similarity between J_ν and the usual energy function $H(z, \dot{z}) = |\dot{z}|^2/2 - V(z)$: it results $H = J_0$.

We prove the existence of infinitely many collision-free periodic solutions of equation (3.3) with negative and small (in absolute value) Jacobi constant, provided the angular velocity $|\nu|$ is sufficiently small. As a consequence, for those values of h and ν we can characterize the dynamical system induced by (3.3) on the level sets

$$\mathcal{U}_{h,\nu} := \left\{(z, v) \in \mathbb{R}^4 : J_\nu(z, v) = h\right\}$$

with a symbolic dynamics, where the symbols are some selected partitions of the centres in two different non-empty sets. Coming back to equation (3.2), this means that, for $h < 0$ and $|h|, |\nu|$ sufficiently small, we have infinitely many collision-free *relative periodic solutions* (i.e. periodic solutions in the rotating frame of reference); this existence result allows to prove the occurrence of symbolic dynamics in a proper submanifold of the phase space (which correspond to $\mathcal{U}_{h,\nu}$ through the transformation $x \leftrightsquigarrow z$).

Periodic solutions. We keep the same notations of Chapter 2. To describe the first main result, we refer to Theorem 2.1.2; therein, we proved the existence of $\bar{h} < 0$ such that, for any $h \in (\bar{h}, 0)$ we can associate to any finite sequence of partition $(P_{j_1}, \dots, P_{j_n}) \in \mathcal{P}^n$ a periodic solution $x_{((P_{j_1},\dots,P_{j_n}),h)}$ of the N-centre problem (2.2) with energy h. Under particular assumptions on $(P_{j_1}, \dots, P_{j_n})$, assumptions which are specified in points (ii)-(b) or (ii)-(c) of the quoted statement, we have to allow collision solutions, but it is always possible (for every $N \geq 3$) to build infinitely many collision-free solutions. We would like to repeat this program associating to a finite sequence of partitions, for sufficiently small values of the absolute value of the Jacobi constant $|h|$ and of the angular velocity $|\nu|$, a periodic solution of equation (3.3). To accomplish such a result, we put some restrictions on the sequences of partitions which we want to consider; this is motivated by the fact that the rotation of the centres makes impossible the application of the blow-up technique introduce in Section 2.4. So, we start from the observation that, concerning the possibility to have a collision solution, there is a distinction among

$$1)\ \alpha = 1 \text{ and } N \geq 4, \qquad 2)\ \alpha = 1 \text{ and } N = 3, \qquad 3)\ \alpha \in (1, 2).$$

We start from the first case.

Theorem 3.1.1. *Let $\alpha = 1$, $N \geq 4$, $c_1, \ldots, c_N \in \mathbb{R}^2$, $m_1, \ldots, m_N \in \mathbb{R}^+$. There exists $\bar{h}_1$ such that, given $h \in (\bar{h}_1, 0)$, there is $\bar{\nu}_1 = \bar{\nu}_1(h) > 0$ such that, to each $\nu \in (-\bar{\nu}_1, \bar{\nu}_1)$, $n \in \mathbb{N}$ and $(P_{j_1}, \ldots, P_{j_n}) \in (\mathcal{P} \setminus \mathcal{P}_1)^n$, we can associate a collision-free periodic solution $z_{((P_{j_1},\ldots,P_{j_n}),h,\nu)}$ of*

$$\begin{cases} \ddot{z}(t) + 2\nu i \dot{z}(t) = \nabla \Phi_\nu(z(t)) \\ \frac{1}{2}|\dot{z}(t)|^2 - \Phi_\nu(z(t)) = h, \end{cases} \tag{3.4}$$

which depends on $(P_{j_1}, \ldots, P_{j_n})$ in the following way. There exist $\bar{R}, \bar{\delta} > 0$ (depending on h only) such that $z_{((P_{j_1},\ldots,P_{j_n}),h,\nu)}$ crosses $2n$ times within one period the circle $\partial B_{\bar{R}}(0)$, at times $(t_k)_{k=0,\ldots,2n-1}$, and

- *in (t_{2k}, t_{2k+1}) the solution stays outside $B_{\bar{R}}(0)$, and*

$$|z_{((P_{j_1},\ldots,P_{j_n}),h,\nu)}(t_{2k}) - z_{((P_{j_1},\ldots,P_{j_n}),h,\nu)}(t_{2k+1})| < \bar{\delta};$$

- *in (t_{2k+1}, t_{2k+2}) the solution lies inside $B_{\bar{R}}(0)$, and separates the centres according to the partition P_{j_k}.*

We remark the analogy with Theorem 2.1.2: if $\alpha = 1$ and $N \geq 4$, we can easily find a condition on $(P_{j_1}, \ldots, P_{j_n})$ in order to ensure that the periodic solution $z_{((P_{j_1},\ldots,P_{j_n}),h,0)}$ of the N-centre problem

$$\begin{cases} \ddot{z}(t) = \nabla V(z(t)) \\ \frac{1}{2}|\dot{z}(t)|^2 - V(z(t)) = h \end{cases}$$

is collision-free; it is sufficient to impose that $P_{j_k} \in (\mathcal{P} \setminus \mathcal{P}_1)$ for every k.

If $N = 3$ then $\mathcal{P} = \mathcal{P}_1$, so that if in addition $\alpha = 1$, we used a little trick to find collision-free solutions: let

$$(P_1, P_1, P_2, P_3) = G_1, \qquad (P_2, P_2, P_3, P_1) = G_2,$$

and let $\mathcal{G} := \{G_1, G_2\}$. We observed (Remark 2.6.4) that no composed sequence obtained by the juxtaposition of G_1 and G_2 satisfies the symmetry conditions of cases (ii)-(b) or (ii)-(c) of Theorem 2.1.2; this implies that a solution of the N-centre problem associated to $(P_{k_1}, \ldots, P_{k_{4n}}) \in \mathcal{G}^n \subset \mathcal{P}^{4n}$ is collision-free. Coming back to the rotating problem, this fact allows to prove the following statement.

Theorem 3.1.2. *Replacing the assumption $N \geq 4$ in Theorem 3.1.1 with $N = 3$, the same statement holds true replacing $(\mathcal{P} \setminus \mathcal{P}_1)^n$ with $\mathcal{G}^n$.*

If $\alpha \neq 1$ it is not necessary to put any restriction on the sequences of partitions which we want to consider, since in such a case $z_{((P_{j_1},\ldots,P_{j_n}),h,0)}$ was proved to be always collision-free.

Theorem 3.1.3. *Replacing the assumptions $\alpha = 1$ and $N \geq 4$ in Theorem 3.1.1 with $\alpha \in (1, 2)$ and $N \geq 3$, the previous statement holds true, replacing the set $\mathcal{P} \setminus \mathcal{P}_1$ with $\mathcal{P}$.*

Remark 3.1.4. The assumption "$|h|$ is sufficiently small" is substantial, as already observed in Remark 2.1.3.

Symbolic dynamics. Similarly to Corollary 2.1.7, as a consequence of Theorems 3.1.1, 3.1.2, 3.1.3, we obtain the following result.

Corollary 3.1.5. *Let $\alpha \in [1, 2)$, $N \geq 3$, $m_1, \dots, m_N \in \mathbb{R}^+$ and $c_1, \dots, c_N \in \mathbb{R}^2$. Let $h \in (\bar{h}_1, 0)$ and $\nu \in (-\bar{\nu}_1(h), \bar{\nu}_1(h))$, where $\bar{h}_1$ and $\bar{\nu}_1(h)$ have been introduced in Theorem 3.1.1, 3.1.2, 3.1.3. There exists a subset $\Pi_{h,\nu}$ of the level set $\mathcal{U}_{h,\nu}$, a return map $\mathfrak{R} : \Pi_{h,\nu} \to \Pi_{h,\nu}$ for the dynamical system associated to equation (3.3), a set of symbols $\widehat{\mathcal{P}}$ and a continuous and surjective map $\pi : \Pi_{h,\nu} \to \widehat{\mathcal{P}}^{\mathbb{Z}}$, such that the diagram*

$$
\begin{array}{ccc}
\Pi_{h,\nu} & \xrightarrow{\mathfrak{R}} & \Pi_{h,\nu} \\
\downarrow \pi & & \downarrow \pi \\
\widehat{\mathcal{P}}^{\mathbb{Z}} & \xrightarrow{T_r} & \widehat{\mathcal{P}}^{\mathbb{Z}},
\end{array}
$$

commutes (here T_r demotes the right shift in $\widehat{\mathcal{P}}^{\mathbb{Z}}$); namely for every $h \in (\bar{h}_1, 0)$ and $\nu \in (-\bar{\nu}_1(h), \bar{\nu}_1(h))$, the restriction of the dynamical system associated to the rotating problem on the level set $\mathcal{U}_{h,\nu}$ has a symbolic dynamics.

Strategy of the proofs. We follow the general strategy already developed for the proof of Theorem 2.1.2. In Section 3.2, we introduce a suitable rescaling in order to pass from problem (3.4) to an equivalent problem, where the parameter "Jacobi constant" is replaced by the parameter given by the maximal distance of the centres from the origin. This leads to the study of a rotating problem with a rescaled potential

$$V_\varepsilon(y) = \sum_{k=1}^{N} \frac{m_k}{|y - c'_k|^\alpha} \quad \text{where} \quad \max_{1 \leq k \leq N} |c'_k| = \varepsilon, \tag{3.5}$$

and a different angular velocity ν'; we are interested in solutions with Jacobi constant equal to -1. In this way, outside a ball or radius $R > \varepsilon > 0$, and for $|\nu'|$ sufficiently small, the equivalent problem

$$\begin{cases} \ddot{y}(t) + 2\nu' i \dot{y}(t) = \nabla \left(\frac{(\nu')^2}{2} |y|^2 + V_\varepsilon(y) \right) \\ \frac{1}{2} |\dot{y}(t)|^2 - \frac{(\nu')^2}{2} |y(t)|^2 - V_\varepsilon(y(t)) = -1 \end{cases} \tag{3.6}$$

is a small perturbation of the Kepler problem with homogeneity degree $-\alpha < 0$, $\alpha \in [1,2)$.

We face the research of periodic solutions of (3.6) splitting the study of the dynamics outside/inside a ball $B_R(0)$ (R will be conveniently chosen). As in Section 2.3, outside $B_R(0)$ we find arcs of solutions of (3.6) connecting two points $p_0, p_1 \in \partial B_R(0)$, provided their distance is sufficiently small, via perturbative techniques. Although in the present setting we have to take into account the new parameter ν', the argument is substantially the same.

In Section 3.4, we study the problem inside $B_R(0)$; we search minimizers of the Jacobi-type functional

$$L_{h,\nu} := \int_0^1 |\dot{u}|\sqrt{\Phi_\nu(u) - 1} + \frac{\nu}{\sqrt{2}}\int_0^1 \langle iu, \dot{u}\rangle,$$

under suitable constraints, in order to connect any pair $p_1, p_2 \in \partial B_R(0)$ with arcs of solution of (3.6) which separate the centres according to any prescribed partition in $\mathcal{P}$. The functional $L_{h,\nu}$, contrarily to the classical Jacobi length, does not come from a Riemaniann structure, but from a Finslerian one. A main consequence is the lack of reversibility of the problem, and this marks a significant difference in the argument we used to rule out the possibility of having collisions for its minimizers. The alternative "collision less" or "ejection-collision", valid for the N-centre problem, does not hold any more. Consequently, at this stage we can only find inner arcs of possibly colliding solutions.

The collection of the outer and inner dynamics is done in Section 3.5, using a variation of the method employed in Section 2.5. Clearly, in this case we obtain the existence of periodic collision solutions.

In Sections 3.6 and 3.7, we complete the proof of Theorems 3.1.1, 3.1.2 and 3.1.3, providing sufficient conditions on the sequences $(P_{j_1}, \ldots, P_{j_n})$ in order to have collision-free solutions; this is done through a kind of Gamma-convergence argument: we show that the minimizers of $L_{-1,\nu'}$ are weakly convergent in H^1, as $\nu' \to 0$, to the minimizers of $L_{-1,0}$, which is the classical Jacobi functional. Therefore we can exploit the description of the behaviour of such minimizers given in Theorem 2.1.2.

Remark 3.1.6. If $\alpha = 1$, the existence of periodic solutions to problem (3.4) can be obtained by means of a perturbation argument in the following way: the Poincaré map associated to the N-center problem ($N \geq 3$) admits a compact hyperbolic invariant set of periodic points on any energy level $J_{h,0}$ with $h \geq 0$ (see Klein and Knauf [52]); the corresponding closed trajectories are global minimizers of the Jacobi length in suitable sets of functions, and lies in a bounded region surrounding the centres. Due to the stability under perturbations of compact hyperbolic invariant sets, if $h < 0$ and $|h|$ and $|\nu|$ are small enough, periodic solutions of problem (3.4) still exist.

On the other hand, the results of Chapter 2 are not achieved through a perturbation argument from the case $h = 0$. Actually, the periodic solutions we found tend, as $h \nearrow 0$, to a "concatenation" of parabolic unbounded orbits. In particular, since they were build by the gluing of constrained minimizers (near the centres) and perturbed Keplerian ellipses interacting with the boundary of the Hill region (which, clearly, do not carry any hyperbolicity property), the previous discussion does not apply. This is why we have to adapt step by step the construction already carried on in the previous chapter in order to show that almost the entire set of periodic solutions we found in the previous chapter survives. Of course, compared with those obtained by perturbing the trajectories found by Klein and Knauf, we obtain different periodic solutions yielding a new symbolic dynamics.

3.2 Preliminaries

Let us fix $N \geq 3$, $\alpha \in [1,2)$, $c_1, \ldots, c_N \in \mathbb{R}^2$ and $m_1, \ldots, m_N > 0$, and let $M = \sum_{k=1}^N m_k$; we fix the origin in the centre of mass. In this section we prove that to find a periodic solution of the rotating problem (3.3) with Jacobi constant $h < 0$ is equivalent to find a periodic solution of a different rotating problem with Jacobi constant equal to -1. In this perspective the maximal distance of the centres from the origin replaces h as parameter, and the angular velocity changes as well. To be precise, with a slightly modification of the proof of Proposition 2.2.1 we obtain:

Proposition 3.2.1. *Let $z \in \mathcal{C}^2((a,b))$ be a classical solution of (3.3) with Jacobi constant $h < 0$. Then the function*

$$y(t) = (-h)^{\frac{1}{\alpha}} z\left((-h)^{-\frac{\alpha+2}{2\alpha}} t\right), \qquad t \in \left((-h)^{\frac{\alpha+2}{2\alpha}} a, (-h)^{\frac{\alpha+2}{2\alpha}} b\right) \tag{3.7}$$

is a solution of a rotating problem with

$$c_j' = (-h)^{\frac{1}{\alpha}} c_j, \quad j = 1, \ldots, N \quad \text{and} \quad \nu' = (-h)^{-\frac{\alpha+2}{2\alpha}} \nu; \tag{3.8}$$

the Jacobi constant of y as solution of the new problem is -1. Conversely: let $y \in \mathcal{C}^2((a',b'))$ be a classical solution with Jacobi constant -1 of a rotating problem with initial configuration of the centres $\{c_j'\}$ and angular velocity ν'. Let us set

$$c_j = (-h)^{-\frac{1}{\alpha}} c_j', \quad j = 1, \ldots, N \quad \text{and} \quad \nu = (-h)^{\frac{\alpha+2}{2\alpha}} \nu'.$$

Then

$$z(t) = (-h)^{-\frac{1}{\alpha}} y\left((-h)^{\frac{\alpha+2}{2\alpha}} t\right), \qquad t \in \left((-h)^{-\frac{\alpha+2}{2\alpha}} a', (-h)^{-\frac{\alpha+2}{2\alpha}} b'\right)$$

is a classical solution of (3.3) with Jacobi constant $h < 0$.

Corollary 3.2.2. *For every $\varepsilon > 0$ and for every $\widetilde{\nu} \in \mathbb{R}$ there exist $\zeta_1(\varepsilon)$ and $\zeta_2(\varepsilon, \widetilde{\nu}) \in \mathbb{R}$ such that if $h = -\zeta_1(\varepsilon)$ and $\nu = \zeta_2(\varepsilon, \widetilde{\nu})$ then*

$$\max_{1 \le k \le N} |c'_k| = \varepsilon, \qquad \nu' = \widetilde{\nu}.$$

The function $-\zeta_1$ is strictly decreasing in ε, the function ζ_2 is strictly increasing both in ε and $\widetilde{\nu}$.

Proof. Given $\varepsilon > 0$, from (3.8) we obtain

$$\zeta_1(\varepsilon) = \left(\frac{\varepsilon}{\max_{1 \le k \le N} |c_k|} \right)^{\alpha}.$$

Plugging this value of h in the expression of $\nu' = (-h)^{-\frac{\alpha+2}{2\alpha}} \nu$ we obtain

$$\nu' = \left(\frac{\max_{1 \le k \le N} |c_k|}{\varepsilon} \right)^{\frac{\alpha+2}{2}} \nu.$$

It is immediate to deduce

$$\zeta_2(\varepsilon, \widetilde{\nu}) = \left(\frac{\varepsilon}{\max_{1 \le k \le N} |c_k|} \right)^{\frac{\alpha+2}{2}} \widetilde{\nu}.$$

□

Remark 3.2.3. Problem (3.6) for $(\varepsilon, \nu') \in (0, \bar{\varepsilon}) \times (-\bar{\nu}', \bar{\nu}')$ is equivalent, through Proposition 3.2.1 and Corollary 3.2.2, to equation (3.3) associated with Jacobi constant $h < 0$ and angular velocity ν for $(h, \nu) \in (-\zeta_1(\bar{\varepsilon}), 0) \times (-\zeta_2(\bar{\varepsilon}, \bar{\nu}), \zeta_2(\bar{\varepsilon}, \bar{\nu}))$. Two corresponding solutions exhibit the same topological behaviour, as showed by equation (3.7). Note that the more the Jacobi constant is small, more the admissible angular velocities have to be small.

Let us fix $\varepsilon > 0$, $\nu' \in \mathbb{R}$, and $K := \overline{B_{R_2}(0)} \setminus B_{R_1}(0)$, with $R_2 > R_1 > \varepsilon$. In K we can consider the new problem as a small perturbation of the α-Kepler problem, whose potential is

$$V_0(y) := \frac{M}{\alpha |y|^\alpha} \qquad y \in \mathbb{R}^2 \setminus \{0\}.$$

Indeed, setting

$$\Phi_{\nu', \varepsilon}(y) := \frac{(\nu')^2}{2} |y|^2 + V_\varepsilon(y),$$

(V_ε has been already defined in (3.5)), it is not difficult to check that

$$\|\Phi_{\nu', \varepsilon} - V_0\|_{\mathcal{C}^1(K)} = o(\varepsilon) + o(\nu') \qquad \text{for } \varepsilon \to 0^+,\ \nu' \to 0. \tag{3.9}$$

Let us observe that if y is a solution of $\ddot{y} + 2\nu' i\dot{y} = \nabla\Phi_{\nu',\varepsilon}(y)$ with Jacobi constant -1 over an interval $I \subset \mathbb{R}$, then

$$\Phi_{\nu',\varepsilon}(y(t)) \geq 1 \qquad \forall t \in I.$$

To exploit the perturbative nature of the problem outside a ball $B_R(0)$, we have to check that, for $\varepsilon > 0$ sufficiently small and for ν' in a neighbourhood of 0, there exists $R > 0$ such that

$$B_\varepsilon(0) \subset B_R(0) \subset \left\{ y \in \mathbb{R}^2 : \Phi_{\nu',\varepsilon}(y) \geq 1 \right\}. \tag{3.10}$$

Then, considering any compact set $B_R(0) \subset A \subset \{\Phi_{\nu',\varepsilon}(y) \geq 1\}$, we will be able to use (3.9) in $A \setminus B_R(0)$.

Proposition 3.2.4. *Let $\varepsilon > 0$, $\nu' \in \mathbb{R}$. Let $R > 0$ such that $\varepsilon < R < (M/\alpha)^{1/\alpha} - \varepsilon$. Then* (3.10) *holds true. There exists $\varepsilon_1 > 0$ such that, for every $0 < \varepsilon < \varepsilon_1$, this choice is possible.*

Proof. See the proof of Proposition 2.2.5. □

Actually, we make the further request $\varepsilon < R/2 < R < (M/\alpha)^{1/\alpha} - \varepsilon$ which is satisfied for every $\varepsilon \in (0, \varepsilon_1/2)$.
Moreover, as in the first chapter, *we select R so that $\partial B_R(0)$ is the image of the circular solution of the α-Kepler problem with energy -1*:

$$R := \left(\frac{(2-\alpha)M}{2\alpha} \right)^{\frac{1}{\alpha}}.$$

This is consistent with the previous restriction on R, if ε_1 is sufficiently small (if this was not true, it is sufficient to replace ε_1 with a smaller quantity).

Remark 3.2.5. For future convenience, note that for every $y \in \overline{B_R(0)}$

$$V_\varepsilon(y) - 1 \geq \frac{M}{\alpha\left(\left(\frac{(2-\alpha)M}{2\alpha}\right)^{\frac{1}{\alpha}} + \varepsilon\right)^\alpha} - 1 \geq \frac{M}{\alpha\left(\left(\frac{(2-\alpha)M}{2\alpha}\right)^{\frac{1}{\alpha}} + \varepsilon_1\right)^\alpha} - 1 =: M_1 > 0, \tag{3.11}$$

and hence $\Phi_{\nu',\varepsilon}(y) - 1 \geq M_1$. This value is independent on $\varepsilon \in (0, \varepsilon_1/2)$. From now on we will use M_1 to denote this positive constant.

3.3 Outer dynamics

We are going to use a perturbative approach in order to find solutions of

$$\begin{cases} \ddot{y}(t) + 2\nu' i\dot{y}(t) = \nabla\Phi_{\nu',\varepsilon}(y(t)) & t \in [0,T] \\ \frac{1}{2}|\dot{y}(t)|^2 - \Phi_{\nu',\varepsilon}(y(t)) = -1 & t \in [0,T] \\ |y(t)| > R & t \in (0,T) \\ y(0) = p_0 \qquad y(T) = p_1 \end{cases} \tag{3.12}$$

when the distance between $p_0, p_1 \in \partial B_R(0)$ is sufficiently small; T has to be determined. To be precise we prove the following proposition.

Proposition 3.3.1. *There exist $\delta > 0$, $\varepsilon_2 > 0$ and $\nu'_1 > 0$ such that for every $(\varepsilon, \nu') \in (0,\varepsilon_2)\times(-\nu'_1, \nu'_1)$, for every $p_0, p_1 \in \partial B_R(0) : |p_1 - p_0| < 2\delta$, there exist a unique solution $y_{ext}(\cdot\,; p_0, p_1; \varepsilon, \nu')$ of (3.12) with $T = T_{ext}(p_0, p_1; \varepsilon, \nu') > 0$. This solution depends in a $\mathcal{C}^1$ way on the endpoints p_0 and p_1, and*

$$\begin{aligned} \max_{t\in[0,T_{ext}(p_0,p_1;\varepsilon,\nu')]} |y_{ext}(t; p_0, p_1; \varepsilon, \nu')| &\leq 2\left(\frac{M}{\alpha}\right)^{\frac{1}{\alpha}} \\ \max_{t\in[0,T_{ext}]} |\dot{y}_{ext}(t; p_0, p_1; \varepsilon, \nu')| &\leq 2\sqrt{2\left(-1 + \frac{M}{\alpha R^\alpha}\right)} \end{aligned} \tag{3.13}$$

for every $(p_0, p_1) \in \{(p_0, p_1) \in (\partial B_R(0))^2 : |p_0 - p_1| < 2\delta\}$, $\varepsilon \in (0, \varepsilon_2)$ and $\nu' \in (-\nu'_1, \nu'_1)$.

We follow the same line of reasoning of the proof of Theorem 2.3.1, with the only difference that here we add the parameter ν'. For the reader's convenience, we review the main steps. For every $p_0 = R\exp\{i\theta_0\} \in \partial B_R(0)$, the unperturbed problem ($\varepsilon = 0$ and $\nu' = 0$) is

$$\begin{cases} \ddot{y}(t) = -M\frac{y(t)}{|y(t)|^{\alpha+2}} & t \in [0,T] \\ \frac{1}{2}|\dot{y}(t)|^2 - \frac{M}{\alpha|y(t)|^\alpha} = -1 & t \in [0,T] \\ |y(t)| > R & t \in (0,T) \\ y(0) = p_0, \qquad y(T) = p_0. \end{cases}$$

Let us solve the Cauchy problem

$$\begin{cases} \ddot{y}(t) = -M\frac{y(t)}{|y(t)|^{\alpha+2}} \\ y(0) = p_0, \qquad \dot{y}(0) = \sqrt{2\left(-1 + \frac{M}{\alpha R^\alpha}\right)}\left(\frac{p_0}{R}\right). \end{cases}$$

The solution returns at the point p_0 after a certain time $\bar{T} > 0$, having swept the portion of the rectilinear brake orbit of energy -1 starting from p_0 and lying in $\mathbb{R}^2 \setminus B_R(0)$.

Our aim is to catch the behaviour of the solutions under small variations of the initial conditions. We consider

$$\begin{cases} \ddot{y}(t) = -M\frac{y(t)}{|y(t)|^{\alpha+2}} \\ y(0) = p_0, \qquad \dot{y}(0) = \dot{r}_0 e^{i\theta_0} + R\dot{\theta}_0 i e^{i\theta_0}, \end{cases} \tag{3.14}$$

where $\dot{r}_0$ is assigned as function of $\dot{\theta}_0$ by means of the energy integral. We denote as $y(\cdot\,;\theta_0,\dot{\theta}_0)$ the solution of (3.14). For the brake orbit $y\,(\cdot\,;\theta_0,0)$, it results

$$\theta(t;\theta_0,0) \equiv \theta_0 \qquad \forall t \in [0,\bar{T}].$$

We introduce $\psi : \Theta \times I \to \mathbb{R}^2$ as

$$\psi(\dot{\theta}_0, T) := y(T;\theta_0,\dot{\theta}_0),$$

where $\Theta \times I \subset S^1 \times \mathbb{R}$ is a neighbourhood of $(0,\bar{T})$ on which ψ is well defined. In Lemma 2.3.2 we showed that the Jacobian of ψ in $(0,\bar{T})$ is invertible.

Now we introduce the parameters ε and ν': let us define

$$\begin{aligned} \Psi :&\Theta \times I \times \partial B_R(0) \times \left[0, \frac{\varepsilon_1}{2}\right) \times \mathbb{R} \to \mathbb{R}^2 \\ &(\dot{\theta}_0, T, p_1, \varepsilon, \nu') \mapsto y(T;\theta_0,\dot{\theta}_0;\varepsilon,\nu') - p_1, \end{aligned}$$

where $y(\cdot\,;\theta_0,\dot{\theta}_0;\varepsilon,\nu')$ is the solution of

$$\begin{cases} \ddot{y}(t) + 2\nu' i\dot{y}(t) = \nabla\Phi_{\nu',\varepsilon}(y(t)) \\ y(0) = p_0, \qquad \dot{y}(0) = \dot{r}_{\nu',\varepsilon} e^{i\theta_0} + R\dot{\theta}_0 i e^{i\theta_0}, \end{cases} \tag{3.15}$$

and $\dot{r}_{\nu',\varepsilon}$ is assigned as function of $\dot{\theta}_0, \varepsilon, \nu'$ by means of the Jacobi constant. The proof of the following statement is a straightforward generalization of the proof of Lemma 2.3.3.

Lemma 3.3.2. *There exist $\delta > 0$, $0 < \varepsilon_2 < \varepsilon_1/2$ and $\nu_1' > 0$ such that for every $(\varepsilon,\nu') \in (0,\varepsilon_2) \times (-\nu_1',\nu_1')$, for every $p_1 \in \partial B_R(0) : |p_1 - p_0| < 2\delta$, there exists a unique solution $y(\cdot\,;\theta_0,\dot{\theta}_0;\varepsilon,\nu')$ of* (3.15) *defined in $[0,T]$ for a certain $T > 0$, and satisfying also* (3.12)*. Moreover, it is possible to choose δ, ε_2 and ν_1' independent on $p_0 \in \partial B_R(0)$.*

Proposition 3.3.1 follows. The solutions obtained are uniquely determined and depends in a smooth way on the ends p_0 and p_1, and on the parameters ε and ν' (by the implicit function theorem). Since a brake solution $y_{\mathrm{br}}(\cdot) = y(\cdot\,;p_0,p_0;0,0)$ of the Kepler problem is such that

$$\max_{t\in[0,\bar{T}]} |y_{\mathrm{br}}(t)| = \left(\frac{M}{\alpha}\right)^{\frac{1}{\alpha}} \quad \text{and} \quad \max_{t\in[0,\bar{T}]} |\dot{y}_{\mathrm{br}}(t)| = \sqrt{2\left(-1+\frac{M}{\alpha R^\alpha}\right)},$$

it is possible, if necessary, to replace ε_2 and ν_1' with smaller quantities in such a way that (3.13) is satisfied.

Definition 3.3.3. For any $\varepsilon \in (0, \varepsilon_2)$ we pose

$$\mathcal{OS}_\varepsilon := \{y_{\text{ext}}(\cdot\,; p_0, p_1; \varepsilon, \nu') : \; p_0, p_1 \in \partial B_R(0), \; |\nu'| < \nu_1'\},$$

i.e. $\mathcal{OS}_\varepsilon$ is the set of the *outer solutions* corresponding to a fixed value of ε.

Lemma 3.3.4. *For every $\varepsilon \in (0, \varepsilon_2)$ there exist $C_1, C_2 > 0$ such that*

$$C_1 \le T_{ext}(p_0, p_1; \varepsilon, \nu') \le C_2 \qquad \forall (p_0, p_1, \nu') \in (\partial B_R(0))^2 \times (-\widetilde{\nu}', \widetilde{\nu}').$$

Also, there exists $C_3 > 0$ such that

$$\|y_{ext}(\cdot\,; p_0, p_1; \varepsilon, \nu')\|_{H^1(0, T_{ext}(p_0, p_1; \varepsilon, \nu'))} \le C_3$$

for every $(p_0, p_1, \nu') \in (\partial B_R(0))^2 \times (-\widetilde{\nu}', \widetilde{\nu}')$.

Proof. The boundedness of $T_{\text{ext}}(p_0, p_1; \varepsilon, \nu')$ is a consequence of the continuous dependence of the solutions with respect to variations of initial data. As far as the bound in the H^1 norm is concerned, we can use (3.13) and the first part. □

Remark 3.3.5. We could make the boundedness properties described above uniform in ε. But we will use this lemma in Sections 3.5, 3.6 and 3.7, where ε will be fixed.

3.4 Inner dynamics

In contrast with the previous one, this section is not a direct generalization of Section 2.4; however, it is convenient to summarize the main ideas that we developed therein. Our goal was to find solutions of

$$\begin{cases} \ddot{y}(t) = \nabla V_\varepsilon(y(t)) & t \in [0, T] \\ \frac{1}{2}|\dot{y}|^2 - V_\varepsilon(y(t)) = -1 & t \in [0, T] \\ |y(t)| < R & t \in (0, T) \\ y(0) = p_1, \qquad y(T) = p_2. \end{cases} \tag{3.16}$$

satisfying particular topological requirements; T was not determined a priori, while the energy was fixed to -1; hence, in order to give a variational formulation of (3.16), it was convenient to adopt the Maupertuis principle rather then the minimal action principle. Let $[a, b] \subset \mathbb{R}$ and $p_1, p_2 \in \partial B_R(0)$, $p_1 = R \exp\{i\theta_1\}$, $p_2 = R \exp\{i\theta_2\}$ (the case $p_1 = p_2$ is admissible). We introduced the set of collision-free H^1 paths

$$\widehat{H}_{p_1 p_2}([a, b]) := \left\{ u \in H^1\left([a, b], \mathbb{R}^2\right) \;\middle|\; \begin{array}{l} u(a) = p_1, \; u(b) = p_2, \\ u(t) \neq c_j \; \forall t \in [a, b], \; \forall j \end{array} \right\},$$

the set of colliding H^1 functions

$$\mathfrak{Coll}_{p_1p_2}([a,b]) := \left\{ u \in H^1\left([a,b],\mathbb{R}^2\right) \;\middle|\; \begin{array}{l} u(a)=p_1,\ u(b)=p_2,\ u(t)=c_j \\ \text{for some } t\in[a,b] \text{ and } j\in\{1,\dots,N\} \end{array} \right\},$$

and their union

$$H_{p_1p_2}([a,b]) = \widehat{H}_{p_1p_2}([a,b]) \cup \mathfrak{Coll}_{p_1p_2}([a,b]).$$

Briefly, we write $\widehat{H}$, $\mathfrak{Coll}$ and H when there is not be possibility of misunderstanding. Note that H is the closure of $\widehat{H}$ in the weak topology of H^1. A path $u \in \widehat{H}$ can be characterized according to its winding number with respect to each centre. This number can be computed by artificially closing the path itself, in the following way: for any $u \in \widehat{H}$, let

$$\Gamma_u(t) := \begin{cases} \begin{cases} u(t) & t\in[a,b] \\ Re^{i(t-b+\theta_2)} & t\in(b, b+\theta_1+2\pi-\theta_2) \end{cases} & \text{if } \theta_1<\theta_2 \\ u(t) \qquad t\in[a,b] & \text{if } \theta_1=\theta_2 \\ \begin{cases} u(t) & t\in[a,b] \\ Re^{i(t-b+\theta_2)} & t\in(b, b+\theta_1-\theta_2) \end{cases} & \text{if } \theta_1>\theta_2, \end{cases}$$

i.e. if $p_1 \neq p_2$ we close the path u with the arc of $\partial B_R(0)$ connecting p_2 and p_1 in counterclockwise sense. Then it is well defined the usual winding number $\mathrm{Ind}\,(u([a,b]),c_j)$. Given $l=(l_1,\dots,l_N)\in\mathbb{Z}^N$, a connected component of $\widehat{H}$ is of the form

$$\widehat{\mathfrak{H}}_l^{p_1p_2}([a,b]) := \left\{ u \in \widehat{H}_{p_1p_2}([a,b]) : \mathrm{Ind}\,(u([a,b]),c_j) = l_j \quad \forall j=1,\dots,N \right\}.$$

We needed classes containing self-intersections-free paths, so that we considered $l\in\mathbb{Z}_2^N$ instead of $l\in\mathbb{Z}^N$, and set

$$\begin{aligned} \widehat{H}_l = \widehat{H}_l^{p_1p_2}([a,b]) & \\ := \left\{ u \in \widehat{H}_{p_1p_2}([a,b]) : \mathrm{Ind}\,(u([a,b]),c_j) \equiv l_j \mod 2 \quad \forall j=1,\dots,N \right\}; \end{aligned}$$

namely we collected together the components with winding numbers having the same parity with respect to each centre. We also made the following assumption (see (2.26)):

$$\exists j,k \in \{1,\dots,N\},\ j\neq k, \text{ such that } l_j \neq l_k \mod 2.$$

In this way, each $u \in \widehat{H}_l$ has to pass through the ball $B_\varepsilon(0)$, and cannot be constant even if $p_1=p_2$. Actually, the functions in $\widehat{H}_l$ are uniformly non-constant, in the sense that there exists $C>0$ such that

$$\|\dot{u}\|_2 \geq C \qquad \forall u \in \widehat{H}_l.$$

This can be easily checked repeating the argument developed in Lemma 2.4.18 to show that equation (2.28) holds true. Furthermore, the constant C can be chosen independently on p_1 and p_2 (see Lemma 2.4.49) and also on l (the proof is the same). We said that $l \in \mathbb{Z}_2^N$ is a *winding vector*, and we term $\mathfrak{I}^N := \{l \in \mathbb{Z}_2^N : \text{l satisfies (2.26)}\}$. In order to apply variational methods, we needed to consider $H_l = H_l^{p_1 p_2}([a,b])$, the closure of $\widehat{H}_l$ with respect to the weak topology of H^1; of course, in H_l there are collision-function. Since we searched functions whose images are in $B_R(0)$, we considered the subsets

$$\begin{aligned}\widehat{K}_l = \widehat{K}_l^{p_1 p_2}([a,b]) &:= \{u \in \widehat{H}_l : |u(t)| \le R \ \forall t \in [a,b]\} \\ K_l = K_l^{p_1 p_2}([a,b]) &:= \{u \in H_l : |u(t)| \le R \ \forall t \in [a,b]\}\,.\end{aligned}$$

The set K_l is weakly closed in H^1.

Recall the definition of the Maupertuis functional associated to problem (3.16):

$$M_{-1}(u) = M_{-1}([a,b];u) := \frac{1}{2}\int_a^b |\dot{u}|^2 \int_a^b (V_\varepsilon(u) - 1)\,; \tag{3.17}$$

Solutions of the fixed energy problem given by the first two equations in (3.16) are obtained as re-parametrizations of critical points of M_{-1} at positive level in the space $\widehat{H}$. It is also possible to consider re-parametrizations of critical points of the functional

$$L_{-1}(u) = L_{-1}([a,b];u) := \int_a^b \sqrt{(V_\varepsilon(u) - 1)\,|\dot{u}|^2},$$

which is defined in the closure with respect to the weak topology of H^1 of

$$H_{-1} = H_{-1}^{p_1 p_2}([a,b]) := \{u \in H_{p_1 p_2}([a,b]) : V(u(t)) > 1, |\dot{u}(t)| > 0 \text{ a.e. in } [a,b]\}\,.$$

Actually local minimizers of M_{-1} are local minimizers of L_{-1}, and the converse is true up to a re-parametrization. The functional L_{-1} has a useful geometric meaning, since for $u \in H_{-1}$ the value $L_{-1}(u)$ is the length of the curve parametrized by u with respect to the Jacobi metric $g_{ij}(y) = (V_\varepsilon(y) - 1)\,\delta_{ij}$, where δ_{ij} is the Kronecker delta; this metric makes the set $\{V_\varepsilon(u) > 1\}$ a Riemannian manifold.

Let us look at Theorem 2.4.14. We proved that there exists $\varepsilon_3 > 0$ such that for every $\varepsilon \in (0, \varepsilon_3)$, $p_1, p_2 \in \partial B_R(0)$ and $l \in \mathfrak{I}^N$, problem (3.16) has a solution $y_l(\cdot\,; p_1, p_2; \varepsilon, 0) \in K_l^{p_1 p_2}([0,T])$ ($T = T(p_1, p_2; \varepsilon; l)$) which is a re-parametrization of a local minimizer of the Maupertuis functional M_{-1} in $K_l^{p_1 p_2}([0,1])$, for some $T > 0$. If $p_1 = p_2$ and

$$l_1 = \cdots = l_{j-1} = l_{j+1} = \cdots = l_N \neq l_j \mod 2,$$

then this solution can be an ejection-collision solution with a unique collision in c_j, otherwise it has to be self-intersection-free and collision-free. The successive step consisted in the translation of this result in the language of partitions. This is possible since if $u \in \widehat{K}_l$ is self-intersection-free then it separates the centres in two different groups, which are determined by the particular choice of $l \in \mathfrak{I}^N$; namely, a self-intersection-free path in a class $\widehat{K}_l$ induces a partition of the centres in two non-empty sets. Hence we could define the application $\mathcal{A} : \mathfrak{I}^N \to \mathcal{P}$ which associates to a winding vector

$$l = (l_1, \dots, l_N) \text{ with } \begin{cases} l_k \equiv 0 \mod 2 & k \in A_0 \subset \{1, \dots, N\} \\ l_k \equiv 1 \mod 2 & k \in A_1 \subset \{1, \dots, N\} \end{cases}$$

the partition

$$\mathcal{A}(l) := \{\{c_k : l_k \in A_0\}, \{c_k : l_k \in A_1\}\}.$$

It is then natural to set

$$\widehat{K}_{P_j} = \widehat{K}_{P_j}^{p_1 p_2}([a,b]) := \left\{ u \in \widehat{K}_l^{p_1 p_2}([a,b]) : l \in \mathcal{A}^{-1}(P_j) \right\},$$
$$K_{P_j} = K_{P_j}^{p_1 p_2}([a,b]) := \left\{ u \in K_l^{p_1 p_2}([a,b]) : l \in \mathcal{A}^{-1}(P_j) \right\}.$$

From Theorem 2.4.14, noting that for each $P_j \in \mathcal{P}$ there are two $l \in \mathfrak{I}^N$ such that $A(l) = P_j$, we obtained, for every $\varepsilon \in (0, \varepsilon_3)$, $p_1, p_2 \in \partial B_R(0)$ and $P_j \in \mathcal{P}$, the existence of two solutions y_1 and y_2 of problem (3.16), which are respectively re-parametrizations of a global and a local minimizer of the Maupertuis functional M_{-1} in $K_{P_j}^{p_1 p_2}([0,1])$. If $p_1 = p_2$ and $P_j \in \mathcal{P}_1$ then they can be ejection-collision solutions with a unique collision in c_i, otherwise they are always collision-free; also, if one of y_1 and y_2 is collision-free, then they are two different solutions.

Let's come back to our "fixed Jacobi constant problem"

$$\begin{cases} \ddot{y}(t) + 2\nu' i \dot{y}(t) = \nabla \Phi_{\nu',\varepsilon}(y(t)) & t \in [0,T] \\ \frac{1}{2}|\dot{y}(t)|^2 - \Phi_{\nu',\varepsilon}(y(t)) = -1 & t \in [0,T] \\ |y(t)| < R & t \in [0,T] \\ y(0) = p_1 \qquad y(T) = p_2. \end{cases} \tag{3.18}$$

The variational formulation of (3.18) will be the object of Subsection 3.4.1. We will state the main result of this section in Subsection 3.4.2.

3.4.1 The variational formulation

Let us consider a general problem of type

$$\begin{cases} \ddot{z}(t) + 2\nu i \dot{z}(t) = \nabla \Phi_{\nu}(z(t)) & t \in [0,T] \\ \frac{1}{2}|\dot{z}(t)|^2 - \Phi_{\nu}(z(t)) = h & t \in [0,T] \\ z(0) = p_1 \qquad z(T) = p_2. \end{cases} \tag{3.19}$$

with $T > 0$ to be determined and $p_1, p_2 \in \mathbb{R}^2$. In order to solve it, we cannot use the Maupertuis functional because it is suited for fixed energy problems. However, exploiting the existence of the Jacobi constant, we can study the Maupertuis-type functional

$$M_{h,\nu}([a,b];u) := \sqrt{2}\left(\int_a^b |\dot{u}|^2\right)^{\frac{1}{2}}\left(\int_a^b \Phi_\nu(u)+h\right)^{\frac{1}{2}} + \nu\int_a^b \langle iu, \dot{u}\rangle.$$

We briefly write $M_{h,\nu}$ instead of $M_{h,\nu}([a,b];\cdot)$ when there is no possibility of misunderstanding. The domain of $M_{h,\nu}$ contains the closure in the weak topology of H^1 of

$$H_{h,\nu}^{p_1p_2}([a,b]) := \{u \in H_{p_1p_2}([a,b]) : \Phi_\nu(u(t)) > -h, |\dot{u}(t)| > 0 \text{ for a.e. } t \in [a,b]\}.$$

If

$$\sqrt{2}\left(\int_a^b |\dot{u}|^2\right)^{\frac{1}{2}}\left(\int_a^b \Phi_\nu(u)+h\right)^{\frac{1}{2}} > 0, \tag{3.20}$$

we can set

$$\omega^2 := \frac{\int_a^b \Phi_\nu(u)+h}{\frac{1}{2}\int_a^b |\dot{u}|^2} > 0 \tag{3.21}$$

and it makes sense to consider the re-parametrization $z(t) = u(\omega t)$, defined in $[a/\omega, b/\omega]$. The functional $M_{h,\nu}$ is differentiable in $\widehat{H} \cap H_{h,\nu}$ (seen as an affine space on H_0^1). We will consider $[a,b] = [0,1]$ for the sake of simplicity.

Theorem 3.4.1. *Let $u \in \widehat{H}_{p_1p_2}([0,1]) \cap H_{h,\nu}^{p_1p_2}([0,1])$ be a critical point of $M_{h,\nu}$, i.e. $dM_{h,\nu}(u)[v] = 0$ for every $v \in H_0^1([0,1])$, and assume that* (3.20) *is satisfied. Let ω be defined by* (3.21). *Then $z(t) := u(\omega t)$ is a classical solution of* (3.19) *with $T = 1/\omega$, while u itself is a classical solution of*

$$\begin{cases} \omega^2\ddot{u}(t) + 2\nu\omega i\dot{u}(t) = \nabla\Phi_\nu(u(t)) & t \in [0,1], \\ \frac{1}{2}|\dot{u}(t)|^2 - \frac{\Phi(u(t))}{\omega^2} = \frac{h}{\omega^2} & t \in [0,1], \\ u(0) = p_1, \quad u(1) = p_2. \end{cases}$$

Proof. For every $v \in H_0^1([0,1])$

$$\begin{aligned} dM_{h,\nu}(u)[v] &= \omega\int_0^1 \langle\dot{u},\dot{v}\rangle + \frac{1}{\omega}\int_0^1 \langle\nabla\Phi_\nu(u), v\rangle + \nu\int_0^1 (\langle iv,\dot{u}\rangle + \langle iu,\dot{v}\rangle) \\ &= \omega\int_0^1 \langle\dot{u},\dot{v}\rangle + \frac{1}{\omega}\int_0^1 \langle\nabla\Phi_\nu(u), v\rangle + \nu\int_0^1 (\langle iv,\dot{u}\rangle - \langle i\dot{u},v\rangle). \end{aligned}$$

We point out that the notation $\langle\cdot,\cdot\rangle$ is used for the usual scalar product in $\mathbb{R}^2$, so that

$$\langle x_1+iy_1, x_2+iy_2\rangle = x_1x_2 + y_1y_2.$$

Since $\langle iv, \dot u\rangle = -\langle i\dot u, v\rangle$, we have

$$dM_{h,\nu}(u)[v] = \omega \int_0^1 \langle \dot u, \dot v\rangle + \int_0^1 \langle -2\nu i \dot u + \frac{1}{\omega}\nabla\Phi_\nu(u), v\rangle.$$

This is zero for every $v \in H_0^1(0,1)$ if and only if u is a (weak, and by regularity strong) solution of

$$\omega^2 \ddot u(t) + 2\nu\omega i \dot u(t) = \nabla\Phi_\nu(u(t)) \qquad t \in [0,1].$$

Then $z(t) = u(\omega t)$ solves the first equation in (3.19). Note that the Jacobi constant for z reads

$$\frac{1}{2}|\dot z(t)|^2 - \Phi_\nu(z(t)) = k \qquad \forall t \in [0, 1/\omega],$$

with $k \in \mathbb{R}$. Equivalently,

$$\frac{\omega^2}{2}|\dot u(s)|^2 - \Phi_\nu(u(s)) = k \qquad \forall s \in [0,1],$$

from which we deduce

$$\omega^2 = \frac{\int_0^1 \Phi_\nu(u) + k}{\frac{1}{2}\int_0^1 |\dot u|^2};$$

comparing with (3.21), we obtain $k = h$. □

The previous statement says that the functional $M_{h,\nu}$ plays, for problem (3.19), the role that the classical Maupertuis functional M_h plays for a fixed energy problem of type (3.16), cf. with Theorem 2.4.1. In order to apply variational methods it is worthwhile working in H rather then in $\widehat{H}$, since $\widehat{H}$ is not weakly closed. As a consequence, it is not possible to rule out the occurrence of collisions from the beginning. This leads to the concept of weak solution for the problem (3.19).

Definition 3.4.2. Let u be a local minimizer of $M_{h,\nu}$ in $H_{h,\nu}^{p_1,p_2}([0,1])$ such that (3.20) holds true, and let ω be defined by (3.21). We say that $z(t) = u(\omega t)$ is a *weak solution* of (3.19) in the time interval $[0, 1/\omega]$.

If z is a weak solution, we can define the collision set as:

$$T_c(z) := \left\{ t \in \left[0, \frac{1}{\omega}\right] : z(t) = c_j \text{ for some } j = 1, \ldots, N \right\}.$$

It is not difficult to check that if z is a weak solution and $(a,b) \subset [0,1] \setminus T_c(z)$, then z is a classical solution of the restricted problem in (a,b), with Jacobi constant h: indeed for every $\varphi \in \mathcal{C}_c^\infty(a,b)$ it results

$$\frac{d}{d\lambda} M_{h,\nu}(u + \lambda\varphi)\bigg|_{\lambda=0} = 0. \tag{3.22}$$

One can verify that the set $T_c(z)$ is discrete and finite, so that z is a classical solution almost everywhere in $[0, 1/\omega]$. On the other hand, a local minimizer in K_l of $M_{h,\nu}$ does not satisfy the motion equation in every time interval $[c, d]$ such that $|u(t)| = R$ for every $t \in [c, d]$; indeed, in such a situation it is no more true that (3.22) holds true for every variation $\varphi \in \mathcal{C}_c^\infty([c, d])$. Nevertheless, the conservation of the Jacobi constant still holds true. This result is the counterpart of Theorem 2.4.3.

Proposition 3.4.3. *If* $u \in \overline{(H^{p_1 p_2}_{h,\nu}([0,1])}^{\sigma(H^1,(H^1)^*)}$ *is a local minimizer of* $M_{h,\nu}$*, then*

$$\frac{1}{2}|\dot{u}(t)|^2 - \frac{\Phi_\nu(u(t))}{\omega^2} = \frac{h}{\omega^2} \qquad \text{for a.e. } t \in [0,1]$$

Proof. It is a consequence of the extremality of u with respect to time re-parametrization keeping the ends fixed. For every $\varphi \in \mathcal{C}_c^\infty((0,1), \mathbb{R})$, let us consider $u_\lambda(t) := u(t+\lambda\varphi(t))$. For λ sufficiently small the function $t \mapsto t + \lambda\varphi(t)$ is increasing in $[0, 1]$, so that in particular it is invertible; the minimality of u implies

$$\left.\frac{d}{d\lambda} M_{h,\nu}(u_\lambda)\right|_{\lambda=0} = 0.$$

Now,

$$\begin{aligned}
M_{h,\nu}(u_\lambda) = \sqrt{2}\left(\int_0^1 |\dot{u}\left(t+\lambda\varphi(t)\right)|^2 \left(1+\lambda\dot{\varphi}(t)\right)^2 \, dt\right)^{\frac{1}{2}} \\
\cdot \left(\int_0^1 \left[\Phi_\nu\left(u\left(t+\lambda\varphi(t)\right)\right)+h\right] dt\right)^{\frac{1}{2}} \\
+ \nu \int_0^1 \langle iu\left(t+\lambda\varphi(t)\right), \dot{u}\left(t+\lambda\varphi(t)\right)\rangle \left(1+\lambda\dot{\varphi}(t)\right) dt \\
= \sqrt{2}\left(\int_0^1 |\dot{u}(s)|^2 \left(1+\lambda\dot{\varphi}(t(s))\right) ds\right)^{\frac{1}{2}} \left(\int_0^1 \frac{\Phi_\nu\left(u\left(s\right)\right)+h}{1+\lambda\dot{\varphi}(t(s))} \, ds\right)^{\frac{1}{2}} \\
+ \nu \int_0^1 \langle iu\left(s\right), \dot{u}\left(s\right)\rangle \, ds.
\end{aligned}$$

Observe that letting $\lambda \to 0$ the family of functions $t_\lambda(s) = s - \lambda\varphi(t(s))$ uniformly converges to s in $[0, 1]$:

$$|t_\lambda(s) - s| = |s - \lambda\varphi(t(s)) - s| = |\lambda| \|\varphi\|_\infty \qquad \forall s \in [0,1].$$

Hence

$$\frac{d}{d\lambda} M_h(u_\lambda)\bigg|_{\lambda=0} = \frac{\omega}{2}\int_0^1 |\dot{u}(s)|^2\,\dot{\varphi}(s)\,ds - \frac{1}{\omega}\int_0^1 \left[\Phi_\nu\left(u\left(s\right)\right) + h\right]\dot{\varphi}(s)\,ds$$
$$= \int_0^1 \left[\frac{\omega}{2}|\dot{u}(s)|^2 - \frac{1}{\omega}\left(\Phi_\nu(u(s)) + h\right)\right]\dot{\varphi}(s)\,ds.$$

Since this quantity vanishes for every $\varphi \in \mathcal{C}_c^\infty$ it results (see for instance [17])

$$\frac{\omega^2}{2}|\dot{u}(s)|^2 - \Phi_\nu(u(s)) - h = k \qquad \text{a.e. } s \in [0,1],$$

for some $k \in \mathbb{R}$. Integrating over $[0,1]$ we get

$$\omega^2 = \frac{\int_0^1 \left(\Phi_\nu(u) + h + k\right)}{\frac{1}{2}\int_0^1 |\dot{u}|^2}.$$

A comparison with definition (3.21) gives $k = 0$. □

Remark 3.4.4. Note that, if $\nu = 0$, the functional $M_{h,\nu}$ reduces to

$$M_{h,0}(u) := \sqrt{2}\left(\int_a^b |\dot{u}|^2\right)^{\frac{1}{2}}\left(\int_a^b (V(u) + h)\right)^{\frac{1}{2}} = 2\sqrt{M_h(u)},$$

where M_h is the classical Maupertuis functional of type (3.17). This reflects the perturbed nature of problem (3.18). Actually, due to the monotonicity of the square root for positive values of its argument it is immediate to deduce that u is a (local) minimizer of M_h at a positive level if and only if it is a (local) minimizer of $M_{h,0}$ such that (3.20) is satisfied. Therefore, if we work in a set in which M_h is bounded below by a positive constant, it is equivalent to minimize M_h or $M_{h,0}$. In particular, since in Lemma 2.4.18 we proved that for every $p_1, p_2 \in \partial B_R(0)$ and for every $l \in \mathfrak{I}^N$ there exists $C > 0$ such that

$$M_{-1}(u) \geq C > 0 \qquad \forall u \in K_l^{p_1 p_2}([0,1]),$$

the characterization of the minimizers of M_{-1} in K_l (and consequently also in K_{P_j}) described in Theorem 2.4.14 (or Corollary 2.4.16) applies for the minimizers of $M_{-1,0}$; this will be crucial in Section 3.6.

As announced in the introduction, there is an analogue counterpart for the functional L_h. We introduce $L_{h,\nu}([a,b];\cdot) : \overline{H_{h,\nu}}^{\sigma(H^1,(H^1)^*)} \to \mathbb{R} \cup \{+\infty\}$ as

$$L_{h,\nu}([a,b];u) := \int_a^b \sqrt{(\Phi_\nu(u) + h)}|\dot{u}| + \frac{1}{\sqrt{2}}\nu\int_a^b \langle iu, \dot{u}\rangle.$$

For $u \in H^1([a,b])$, we consider the following class of orientation preserving parametrizations

$$\Gamma_u := \left\{ \left([a',b'], f\right) \,\middle|\, \begin{array}{l} f:[a',b'] \to [a,b], f \in \mathcal{C}^1([a',b']) \text{ and increasing,} \\ \text{such that } u \circ f \in H^1(a',b') \end{array} \right\}.$$

It is not difficult to check that $L_{h,\nu}$ is invariant under re-parametrizations of Γ_u. We point out that this is false if we consider re-parametrizations which do not preserve the orientation. In particular, differently from L_h, $L_{h,\nu}$ is not a length. It is possible to check that if $|\nu|$ is sufficiently small then

$$\sqrt{\Phi_\nu(z)+h}\,|\dot z| + \nu\langle iu, \dot u\rangle$$

is a Finsler function which makes the "Hill region" $\{\Phi_\nu(z) > -h\}$ a Finsler manifold.

Theorem 3.4.5. *Let $u \in H^{p_1p_2}_{h,\nu}([0,1]) \cap \widehat{H}_{p_1p_2}([0,1])$ be a non-constant critical point of $L_{h,\nu}$. Then there exist a re-parametrization z of u which is a classical solution of* (3.19) *for some $T>0$.*

Proof. We can adapt the proof of Theorem 2.4.5 with minor changes. For every $v \in H^1_0(0,1)$ we have

$$\begin{aligned} dL_{h,\nu}(u)[v] &= \int_0^1 \frac{\sqrt{\Phi_\nu(u)+h}}{|\dot u|}\langle \dot u, \dot v\rangle + \int_0^1 \frac{|\dot u|}{2\sqrt{\Phi_\nu(u)+h}}\langle \nabla\Phi_\nu(u), v\rangle \\ &\quad + \frac{1}{\sqrt 2}\nu\int_0^1 \left(\langle iv, \dot u\rangle + \langle iu, \dot v\rangle\right) \\ &= \int_0^1 \langle \frac{\sqrt{\Phi_\nu(u)+h}}{|\dot u|}\dot u, \dot v\rangle + \int_0^1 \langle \frac{|\dot u|}{2\sqrt{\Phi_\nu(u)+h}}\nabla\Phi_\nu(u) - \sqrt 2\nu i\dot u, v\rangle, \end{aligned}$$

where we used again the fact that $\langle iv, \dot u\rangle = -\langle i\dot u, v\rangle$. This expression vanishes for every $v \in H^1_0(0,1)$ if and only if u is a (weak, and by regularity strong) solution of

$$-\frac{d}{dt}\left(\frac{\sqrt{\Phi_\nu(u(t))+h}}{|\dot u(t)|}\dot u(t)\right) + \frac{|\dot u(t)|}{2\sqrt{\Phi_\nu(u(t))+h}}\nabla\Phi_\nu(u(t)) - \sqrt 2\nu i\dot u(t) = 0 \qquad (3.23)$$

for every $t \in [0,1]$. Let us define, for $t \in [0,1]$,

$$s(t) := \int_0^t \frac{|\dot u(\tau)|^2}{\sqrt{\Phi_\nu(u(\tau))+h}}\,d\tau,$$

and set $\widetilde T = s(1)$. It results $([0,\widetilde T], s) \in \Gamma_u$, and for every $s \in [0,\widetilde T]$ (we denote with " $'$ " the differentiation with respect to the new parameter s)

$$\frac{dt}{ds}(s) = \left(\frac{ds}{dt}(t)\right)^{-1}\Bigg|_{t=t(s)} = \frac{\sqrt{\Phi_\nu(u(t(s)))+h}}{|\dot u(t(s))|}.$$

With this change of variable, setting $w(s) = u(t(s))$, equation (3.23) becomes

$$-\frac{1}{t'(s)}\frac{d}{ds}\left(\frac{w'(s)}{t'(s)}t'(s)\right) - \frac{1}{2t'(s)}\nabla\Phi_\nu(w(s)) - \sqrt{2}\nu i\frac{w'(s)}{t'(s)} = 0,$$

i.e.

$$w''(s) + \sqrt{2}\nu i w'(s) = \frac{1}{2}\nabla\Phi_\nu(w(s)) \qquad t \in [0, \widetilde{T}].$$

Setting $z(s) := w(\sqrt{2}s)$, it is straightforward to check that z is a solution of the first equation in (3.19) over $[0, T]$, where $T := \widetilde{T}/\sqrt{2}$. As far as the second equation is concerned

$$|w'(s)|^2 = |\dot{u}(t(s))t'(s)|^2 = \Phi_\nu(w(s)) + h \quad \Longrightarrow \quad \frac{1}{2}|z'(s)|^2 = \Phi_\nu(z(s)) + h$$

for every $s \in [0, T]$.

□

The relationship between minimizers of $M_{h,\nu}$ and $L_{h,\nu}$ is given by the following statement.

Proposition 3.4.6. *Let $u \in H_{h,\nu} \cap H$ be a non-constant (local) minimizer of $M_{h,\nu}$ such that* (3.20) *holds true. Then u is a (local) minimizer of $L_{h,\nu}$ in $H_{h,\nu} \cap H$.*
On the other hand, let $u \in H_{h,\nu} \cap H$ be a non-constant (local) minimizer of $L_{h,\nu}$. Then, up to a re-parametrization, u is a (local) minimizer of $M_{h,\nu}$ in $H_{h,\nu} \cap H$ such that (3.20) *holds true.*

Proof. Due to the Hölder inequality we have

$$\sqrt{2}L_{h,\nu}(u) \le M_{h,\nu}(u) \qquad \forall u \in H_{h,\nu} \cap \widehat{H},$$

with equality if and only if there exists $C > 0$ such that

$$|\dot{u}(t)|^2 = C\left(\Phi_\nu(u(t)) - 1\right) \qquad \forall t \in [0, 1].$$

Now we can follow step by step the proofs of Propositions 2.4.7 and 2.4.8. □

3.4.2 Existence of inner solutions

The following result is a partial counterpart of Theorem 2.4.14.

Proposition 3.4.7. *There exist $\varepsilon_4 > 0$ and $\nu_2' > 0$ such that for every $(p_1, p_2, \varepsilon, \nu', l) \in (\partial B_R(0))^2 \times (0, \varepsilon_4) \times (-\nu_2', \nu_2') \times \mathfrak{I}^N$, problem* (3.18) *has a weak solution $y_l(\cdot\,; p_1, p_2; \varepsilon, \nu') \in K_l^{p_1 p_2}([0, T])$ which is a re-parametrization of a local minimizer $u_l(\cdot; p_1, p_2; \varepsilon, \nu')$ of $M_{-1,\nu'}$ in $K_l^{p_1 p_2}([0, 1])$.*

Before proceeding with the proof of Theorem 2.4.14, we state the translation of this result in terms of partitions.

Corollary 3.4.8. *For every* $(p_1, p_2, \varepsilon, \nu', P_j) \in (\partial B_R(0))^2 \times (0, \varepsilon_4) \times (-\nu_2', \nu_2') \times \mathcal{P}$, *problem* (3.18) *has two weak solutions* $y_1 \in K_{P_j}^{p_1 p_2}([0, T_1])$ *and* $y_2^{P_j}(\cdot\,; p_1, p_2; \varepsilon, \nu') \in K_{P_j}^{p_1 p_2}([0, T_2])$*;* y_1 *(resp.* y_2*) is a re-parametrization of a global (resp. a local) minimizer* u_1 *(resp.* u_2*) of the Maupertuis-type functional* $M_{-1,\nu'}$ *in* $K_{P_j}^{p_1 p_2}([0, 1])$*. If one of them is collision-free, then they are different solutions.*

We fix $[a, b] = [0, 1]$ and the Jacobi constant to -1, so we write $M_{\nu'}$ instead of $M_{-1,\nu'}$. Also, we fix $p_1, p_2 \in \partial B_R(0)$ and $l \in \mathfrak{I}^N$.

Remark 3.4.9. In the statement of Theorem 2.4.14 the values ε_4 and ν_2' depend neither on $p_1, p_2 \in \partial B_R(0)$, nor on $l \in \mathfrak{I}^N$. But here we fixed p_1, p_2 and l before finding ε_4 and ν_2'. Actually, once we will find ε_4 and ν_2', we will see that they are independent on the previous quantities.

We aim at applying the direct methods of the calculus of variations in order to find a minimizer of $M_{\nu'}$ in K_l. Assuming that we can find such a minimizer $u_l(\cdot\,; p_1, p_2; \varepsilon, \nu')$, in order to obtain a weak solution of (3.18) we have to show that

1) $u_l(\cdot\,; p_1, p_2; \varepsilon, \nu')$ satisfies (3.20), 2) $|u_l(t; p_1, p_2; \varepsilon, \nu')| < R \quad \forall t \in (0, 1)$.

Note that the first requirement is satisfied, as shown in Lemma 2.4.18 (the definition of the sets K_l does not depend by the presence of ν'). We will discuss about the second condition after the minimization.

Lemma 3.4.10. *The functional* $M_{\nu'}$ *is coercive in* K_l*.*

Proof. Let $(u_n) \subset K_l$ such that $\|\dot{u}_n\|_{H^1} \to \infty$ for $n \to \infty$. Since $\|u_n\|_2 \leq R$, necessarily $\|\dot{u}_n\|_2 \to +\infty$ as $n \to \infty$. As $V_\varepsilon(y) - 1 \geq M_1 > 0$ in $B_R(0)$,

$$\begin{aligned} M_{\nu'}(u_n) &\geq \sqrt{2}\|\dot{u}_n\|_2 \left(M_1 + \frac{(\nu')^2}{2} \int_0^1 |u_n|^2 \right)^{\frac{1}{2}} - |\nu'| \int_0^1 |u_n||\dot{u}_n| \\ &= \sqrt{2}\|\dot{u}_n\|_2 \left(\frac{|\nu'|}{\sqrt{2}} \|u_n\|_2 + \lambda \right) - |\nu'| \|u_n\|_2 \|\dot{u}_n\|_2 \end{aligned}$$

for some $\lambda > 0$. Hence $M_{\nu'}(u_n) \geq \sqrt{2}\lambda\|\dot{u}_n\|_2$. □

Lemma 3.4.11. *The functional* $M_{\nu'}$ *is weakly lower semi-continuous in* K_l*.*

Proof. Let $(u_n) \subset K_l$ such that $u_n \rightharpoonup u$ weakly in H^1. Arguing as in the proof of Lemma 2.4.19, we obtain

$$\left(\int_0^1 |\dot{u}|\right)^{\frac{1}{2}} \left(\int_0^1 \Phi_{\nu',\varepsilon}(u) - 1\right)^{\frac{1}{2}} \leq \liminf_{n\to\infty} \left(\int_0^1 |\dot{u}_n|\right)^{\frac{1}{2}} \left(\int_0^1 \Phi_{\nu',\varepsilon}(u_n) - 1\right)^{\frac{1}{2}}.$$

It remains to show that

$$\nu' \int_0^1 \langle iu, \dot{u}\rangle \leq \liminf_{n\to\infty} \nu' \int_0^1 \langle iu_n, \dot{u}_n\rangle. \tag{3.24}$$

The weak convergence of u_n to u implies that $u_n \to u$ uniformly in $[0,1]$ and $\dot{u}_n \rightharpoonup \dot{u}$ weakly in L^2, as $n \to \infty$. We have

$$\nu' \int_0^1 \langle iu_n, \dot{u}_n\rangle = \nu' \int_0^1 \langle i(u_n - u), \dot{u}_n\rangle + \nu' \int_0^1 \langle iu, \dot{u}_n\rangle.$$

The first term tends to 0 and the second term tends to $\nu' \int_0^1 \langle iu, \dot{u}\rangle$ as $n \to \infty$; inequality (3.24) follows. □

Remark 3.4.12. The term $\nu \int_0^1 \langle iu, \dot{u}\rangle$ is not only weakly lower semi-continuous in H^1, but also continuous in the weak topology of H^1.

Due to the coercivity and the weak lower semi-continuity of $M_{\nu'}$, we can apply the direct methods of the calculus of variations on the functional $M_{\nu'}$ in the weakly closed set K_l. For every $(\varepsilon, \nu') \in (0, \varepsilon_1/2) \times \mathbb{R}$, we obtain a minimizer $u_l(\cdot; p_1, p_2; \varepsilon, \nu')$ for which (3.20) is satisfied. The following result concludes the proof of Proposition 3.4.7.

Lemma 3.4.13. *There are $\varepsilon_4, \nu'_2 > 0$ such that for every $(p_1, p_2, \varepsilon, \nu', l) \in (\partial B_R(0))^2 \times (0, \varepsilon_4) \times (-\nu'_2, \nu'_2) \times \mathfrak{I}^N$ the minimizer $u_l(\cdot\,; p_1, p_2; \varepsilon, \nu')$ is such that*

$$|u_l(\cdot\,; p_1, p_2; \varepsilon, \nu')| < R \qquad \forall t \in (0,1).$$

Proof. We can follow the same line of reasoning which was used in Chapter 2 in order to prove Proposition 2.4.26. For the reader's convenience, we report here the ingredients of the proof. Let us term

$$T_R(u) := \{t \in [0,1] : |u(t)| = R\}, \quad T^+_{R/2}(u) := \left\{t \in [0,1] : |u(t)| > \frac{R}{2}\right\}$$

A connected component of $T_R(u)$ is an interval (possibly a single point) $[t_1, t_2]$ with $t_1 \leq t_2$. It is possible to show that $u \in \mathcal{C}^1((0,1))$, and if (a,b) is a connected component of $T^+_{R/2}(u) \setminus T_R(u)$, then $u|_{(a,b)}$ is of class $\mathcal{C}^2$ and is a solution of

$$\omega^2 \ddot{u}(t) + 2\nu' \omega i \dot{u}(t) = \nabla \Phi_{\nu',\varepsilon}(u(t)), \quad \text{where} \quad \omega^2 := \frac{\int_0^1 (\Phi_{\nu'\varepsilon}(u) - 1)}{\frac{1}{2}\int_0^1 |\dot{u}|^2}.$$

Moreover, there are $\varepsilon_4, \nu_2', \tau > 0$ such that, if $(\varepsilon, \nu') \in (0, \varepsilon_4) \times (-\nu_2', \nu_2')$, then for every t_3, t_4 such that

$$|u(t_3)| = R, \quad |u(t_4)| = \frac{R}{2}, \quad \frac{R}{2} < |u(t)| < R \quad \forall t \in \begin{cases} (t_3, t_4) & \text{if } t_3 < t_4 \\ (t_4, t_3) & \text{if } t_3 > t_4 \end{cases},$$

there holds $|t_4 - t_3| \leq \tau$. Neither ε_4 nor ν_2' depend on p_1, p_2 or l. Let $[t_1, t_2]$ be a connected component of $T_R(u)$, let (a, b) be a connected component of $T^+_{R/2}$ such that $[t_1, t_2] \subset (a, b)$. Let us consider $y(t) := u(\omega t)$. Since $y \in \mathcal{C}^1((a/\omega, b/\omega))$, it can lean against the circle $\{y \in \mathbb{R}^2 : |y| = R\}$ with tangential velocity, and for every $\lambda > 0$ there exists $t_5 > t_2$ (or $t_5 < t_1$, and in this case the following inequality has to be changed in obvious way) such that

$$\left| y\left(\frac{t_5}{\omega}\right) - Re^{i\theta(t_2/\omega)} \right| + \left| \dot{y}\left(\frac{t_5}{\omega}\right) - R\dot{\theta}\left(\frac{t_2}{\omega}\right) ie^{i\theta(t_2/\omega)} \right| < \lambda.$$

Thus, recalling that R is the radius of the circular solution of energy -1 for the α-Kepler problem, the theorem of continuous dependence of the solutions with respect to the vector field and the initial data implies that y cannot enter (or exit from) the ball $B_{R/2}(0)$ in time τ, provided ε_4 and ν_2' are sufficiently small (if this was not true, we can replace them with smaller quantities); this is in contradiction with the choice of l. □

In order to exploit the description of the behaviour of the solution which we obtained for the N-centre problem in Theorem 2.4.14, we will replace ε_4 with $\min\{\varepsilon_3, \varepsilon_4\}$, where ε_3 has been introduced in the quoted statement.

Definition 3.4.14. Let us fix **arbitrarily** $\nu_3' \in \left(0, \min\{\nu_2', \sqrt{2M_1}/R\}\right)$. For every $\varepsilon \in (0, \varepsilon_4)$ we term

$$\mathcal{IM}_\varepsilon := \{u_l(\cdot; p_1, p_2; \varepsilon, \nu') : \ p_1, p_2 \in \partial B_R(0), \ l \in \mathbb{Z}_2^N, \ |\nu'| < \nu_3'\},$$

the set of the *inner minimizers* of $\{M_{\nu'}\}_{|\nu'|<\nu_3'}$ for a fixed value of ε, and

$$\mathcal{IS}_\varepsilon := \{y_l(\cdot; p_1, p_2; \varepsilon, \nu') : \ p_1, p_2 \in \partial B_R(0), \ l \in \mathbb{Z}_2^N, \ |\nu'| < \nu_3'\},$$

the set of the corresponding *inner solutions* for a fixed value of ε.

We conclude this section with a collection of boundedness properties for the functions of $\mathcal{IM}_\varepsilon$.

Proposition 3.4.15. *Let $\varepsilon \in (0, \varepsilon_4)$. There are $C_1, C_2, C_3, C_4, C_5 > 0$ such that*

$$C_1 \leq \inf_{u \in \mathcal{IM}_\varepsilon} \|\dot{u}\|_2 \leq \sup_{u \in \mathcal{IM}_\varepsilon} \|\dot{u}\|_2 \leq C_2,$$

$$C_3 \leq \inf_{u=u_l(\cdot\,;p_1,p_2,\varepsilon,\nu')\in\mathcal{IM}_\varepsilon} \int_0^1 \Phi_{\nu',\varepsilon}(u) - 1$$
$$\leq \sup_{u=u_l(\cdot\,;p_1,p_2,\varepsilon,\nu')\in\mathcal{IM}_\varepsilon} \int_0^1 \Phi_{\nu',\varepsilon}(u) - 1 \leq C_4,$$

and

$$\sup_{u=u_l(\cdot\,;p_1,p_2,\varepsilon,\nu')} M_{\nu'}(u) \leq C_5.$$

Remark 3.4.16. Since $\sup\{\|u\|_2 : u \in \mathcal{IM}_\varepsilon \leq R\}$, the set $\mathcal{IM}_\varepsilon$ is bounded in the H^1 norm.

Proof. It is a slightly modification of the proof of Lemma 2.4.49. Every $u \in \mathcal{IM}_\varepsilon$ is of type $u_l(\cdot\,;p_1,p_2;\varepsilon,\nu')$ for some $p_1,p_2 \in \partial B_R(0)$, $l \in \mathfrak{I}^N$, $\nu' \in (\nu_3', \nu_3')$. Since $\mathfrak{I}^N$ is discrete and finite, we can prove the statement for a fixed l. We have already pointed out that the functions of $\bigcup_{p_1,p_2\in\partial B_R(0)} K_l^{p_1p_2}([0,1])$ are uniformly non-constant, which ensures the existence of C_1. Furthermore, as an immediate consequence of the estimate in Remark 3.2.5, we obtain $C_3 = M_1$. Now let us fix $\widetilde{p}_1, \widetilde{p}_2 \in \partial B_R(0)$; there exists $\widetilde{u} \in K_l^{\widetilde{p}_1\widetilde{p}_2}([0,1])$ such that, for some $C_6 > 0$ and $\mu \in (0,\varepsilon)$, it results

$$|\dot{\widetilde{u}}(t)| = C_6, \quad |\widetilde{u}(t) - c_j| \geq \mu \quad \forall t \in [0,1], \forall j \in \{1,\dots,N\}.$$

For every $\nu' \in (-\nu_3', \nu_3')$ we have

$$\int_0^1 \Phi_{\nu',\varepsilon}(\widetilde{u}) = \int_0^1 \left(V_\varepsilon(\widetilde{u}) + \frac{(\nu')^2}{2}|\widetilde{u}|^2 \right) \leq \frac{M}{\alpha\mu^\alpha} + \frac{(\nu_3')^2}{2}R^2 =: C_7.$$

Starting from this bound it is possible to obtain a uniform bound with respect to p_1, p_2, ν' for the level of the minimizers of $M_{\nu'}$. If $(p_1,p_2) \neq (\widetilde{p}_1,\widetilde{p}_2)$, we consider the path

$$\widehat{u}(t) := \begin{cases} \zeta_R(3t; p_1, \widetilde{p}_1) & t \in [0,1/3] \\ \widetilde{u}(3t-1) & t \in (1/3,2/3] \\ \zeta_R\left(3t-2; \widetilde{p}_2, p_2\right) & t \in (2/3,1], \end{cases}$$

where, for $p_*, p_{**} \in \partial B_R(0)$, $\zeta_R(\cdot\,;p_*,p_{**}) : [0,1] \to \mathbb{R}^2$ parametrizes the shorter (in the Euclidean metric) arc of $\partial B_R(0)$ connecting p_* and p_{**} with constant velocity. As far as the velocity of $\zeta_R(\cdot\,;p_*,p_{**})$ is concerned, it is easy to see that it is uniformly bounded with respect to p_*, p_{**}. This, together with the assumptions on $\widetilde{u}$, implies that also the velocity of $\widehat{u}$ is bounded in $[0,1]$, and

$$M_{\nu'}(\widehat{u}) \leq C\left(\int_0^1 \Phi_{\nu',\varepsilon}(\widetilde{u}) - 1 + C\right)^{\frac{1}{2}} + |\nu'|RC \leq C_5.$$

The constant C_5 does not depend on the ends p_1 and p_2 or on the parameter ν'. Consequently, for the family of the minimizers there holds

$$M'_\nu(u_l(\cdot\,;p_1,p_2;\varepsilon,\nu')) \leq C_5 \qquad \forall p_1,p_2 \in \partial B_R(0),\ |\nu'| < \nu'_3. \tag{3.25}$$

Using the estimate (3.11), we obtain

$$\begin{aligned}\|\dot u_l(\cdot\,;p_1,p_2;\varepsilon,\nu')\|_2 &\leq \frac{C_5 - \nu'\int_0^1 \langle iu_l(\cdot\,;p_1,p_2;\varepsilon,\nu'),\dot u_l(\cdot\,;p_1,p_2;\varepsilon,\nu')\rangle}{\sqrt{2M_1}}\\ &\leq \frac{C_5 + |\nu'|R\|\dot u_l(\cdot\,;p_1,p_2;\varepsilon,\nu')\|_2}{\sqrt{2M_1}},\end{aligned}$$

for every $p_1,p_2 \in \partial B_R(0)$ and $|\nu'| < \nu'_3$. Now

$$\left(1 - \frac{|\nu'|R}{\sqrt{2M_1}}\right)\|\dot u_l(\cdot\,;p_1,p_2;\varepsilon,\nu')\|_2 \leq \frac{C_5}{\sqrt{2M_1}}.$$

Since $|\nu'| < \nu'_3 < \sqrt{2M_1}/R$, the coefficient on the left hand side is bounded below by a positive constant; therefore

$$\|\dot u_l(\cdot\,;p_1,p_2;\varepsilon,\nu')\|_2 \leq \frac{C_5}{\sqrt{2M_1}}\left(1 - \frac{|\nu'_3|R}{\sqrt{2M_1}}\right)^{-1} =: C_2 \qquad \forall (p_1,p_2,\nu') \in (\partial B_R(0))^2 \times (-\nu'_3,\nu'_3).$$

It remains to find C_4; from (3.25), using the existence of C_1, it follows

$$\begin{aligned}\left(\int_0^1 \Phi_{\nu',\varepsilon}(u_l(\cdot\,;p_1,p_2;\varepsilon,\nu')) - 1\right)^{\frac12} &\leq \frac{C_5 + |\nu'|R\|\dot u_l(\cdot\,;p_1,p_2;\varepsilon,\nu')\|_{L_2}}{\sqrt{2}\|\dot u_l(\cdot\,;p_1,p_2;\varepsilon,\nu')\|_2}\\ &\leq \frac{C_5}{\sqrt{2}C_1} + \frac{\nu'_3 R}{\sqrt{2}} =: C_4^{\frac12}. \quad \square\end{aligned}$$

Remark 3.4.17. Some constants depend on ε; this reflects the fact that the more the Jacobi constant is small, the more the admissible values of the angular velocity are small, see Remark 3.2.3, and this is why we keep ε fixed, letting ν' vary, instead of considering both ε and ν' as parameters.

We use the notation $[0, T_l(p_1,p_2;\varepsilon,\nu')]$ for the time interval of $y_l(\cdot\,;p_1,p_2;\varepsilon,\nu') \in \mathcal{IS}_\varepsilon$. It results

$$T_l(p_1,p_2;\varepsilon,\nu') = \frac{1}{\omega_l(p_1,p_2;\varepsilon,\nu')}, \quad \text{where} \quad \omega_l(p_1,p_2;\varepsilon,\nu') = \frac{\int_0^1 \Phi_{\nu',\varepsilon}(u_l(\cdot\,;p_1,p_2;\varepsilon,\nu')) - 1}{\frac12\|\dot u_l(\cdot\,;p_1,p_2;\varepsilon,\nu')\|^2}.$$

Corollary 3.4.18. *Let $\varepsilon \in (0, \varepsilon_4)$. There exist $C_1, C_2, C_3 > 0$ such that*

$$C_1 \le T_l(p_1, p_2; \varepsilon, \nu') \le C_2$$
$$\|y_l(\cdot\,; p_1, p_2; \varepsilon, \nu')\|_{H^1(0, T_l(p_1, p_2; \varepsilon, \nu'))} \le C_3$$

for every $(p_0, p_1, \nu', l) \in (\partial B_R(0))^2 \times (-\nu'_3, \nu'_3) \times \mathfrak{I}^N$.

3.4.3 Forward normal neighbourhoods

In Section 2.5 we exploited the geometric interpretation of L: it is the length in the Riemannian manifold $\{V_\varepsilon(y) > -1\}$ endowed with the Jacobi metric. In particular we used classical results concerning the existence of totally normal and strongly convex neighbourhoods. Now we are not dealing with a length anymore, but with a Finsler function; so, something similar can be proven. The following is a known result, but since we cannot find a proper reference we give a sketch of the proof for completeness.

Proposition 3.4.19. *Let $\rho > 0$ be small enough, chosen in such a way that*

$$B_\varepsilon(0) \subset \overline{B_{R/2-\rho}(0)} \subset \overline{B_{R+\rho}(0)} \subset \{\Phi_{\nu',\varepsilon}(y) > 1\}.$$

There exist $\varepsilon_5 \in (0, \varepsilon_4]$, $\nu'_4 \in (0, \nu'_3]$ and $\bar{r} \in (0, 2\rho)$ such that if $\varepsilon \in (0, \varepsilon_5)$, $|\nu'| < \nu'_4$, $p_1, p_2 \in \overline{B_R(0) \setminus B_{R/2}(0)}$ and $|p_1 - p_2| \le \bar{r}$ then there is a unique minimizer $u_{min}(\cdot\,; p_1, p_2; \varepsilon, \nu')$ of $M_{\nu'}$ in the set

$$\left\{u \in H_{p_1 p_2}([0,1]) : u(t) \in B_{R+\rho}(0) \setminus B_{R/2-\rho}(0)\ \forall t\right\}.$$

Moreover, it depends in a $\mathcal{C}^1$ way on its ends and on the parameters ε and ν', and is the unique global minimizer of $M_{\nu'}$ in $H_{p_1 p_2}([0,1])$.

Definition 3.4.20. Let $\varepsilon \in (0, \varepsilon_5)$, $|\nu'| < \nu'_4$, and let us take $\rho > 0$ as above; let $p \in \overline{B_R(0) \setminus B_{R/2}(0)}$. For every pair $p_1, p_2 \in \overline{B_{\bar{r}}}(p)$ there is a unique (up to a re-parametrization) local minimizer of $L_{\nu'}$ which starts from p_1 and arrives at p_2, depending smoothly on the ends. We will say that $B_{\bar{r}}(p)$ is a *forward normal neighbourhood* of p.

Proposition 3.4.19 says that every point of $\overline{B_R(0) \setminus B_{R/2}(0)}$ has a forward normal neighbourhood; moreover, the set $B_{R+\rho}(0) \setminus B_{R/2-\rho}(0)$ is "convex", in the sense that the minimizers $u_{\min}(\cdot\,; p_1, p_2; \varepsilon, \nu')$ stay in it.
Forward normal neighbourhoods plays the role of totally normal ones of a Riemaniann manifold, with the difference that, since our functional $L_{\nu'}$ is not invariant under orientation-reversing re-parametrizations, a minimizer of $L_{\nu'}$ in $H_{p_1 p_2}([0,1])$ could not be a minimizer of $L_{\nu'}$ in $H_{p_2 p_1}([0,1])$.
Actually for every $p \in \{\Phi_{\nu',\varepsilon}(y) > 1\}$ it is possible to prove the existence of a forward

normal neighbourhood, but due to the degeneracy of our Finsler function, which can become even negative if we are close to the boundary of the "Hill region", the radius of these neighbourhood becomes smaller and smaller and tends to 0 as p approaches $\{\Phi_{\nu',\varepsilon}(y) = 1\}$.

Proof. Let $p_1, p_2 \in \overline{B_R(0) \setminus B_{R/2}(0)}$, $\varepsilon \in (0, \varepsilon_4)$, $\nu' \in (-\nu'_4, \nu'_4)$. The existence can be proved applying the direct methods of the calculus of variations. If $p_1 = p_2$, observe that the minimizer is simply the constant function p_1.
Let $u_{\min}(\cdot\,; p_1, p_2; \varepsilon, \nu')$ be a minimizer in $H_{p_1p_2}([0,1])$; there exists $\bar{r} > 0$ such that if $|p_1 - p_2| \leq \bar{r}$, then $u_{\min}(\cdot\,; p_1, p_2; \varepsilon, \nu')$ is contained in $B_{R+\rho}(0) \setminus B_{R/2-\rho}(0)$: if not, there are sequences $(r_n) \subset \mathbb{R}^+$ and $((p_1^n, p_2^n)) \subset \overline{B_R(0) \setminus B_{R/2}(0)}$ such that $|p_1^n - p_2^n| \leq r_n$ and $u_{\min}(\cdot\,; p_1^n, p_2^n; \varepsilon, \nu')$ touches $\partial\left(B_{R+\rho}(0) \setminus B_{R/2-\rho}(0)\right)$. But this is absurd, because if $r_n \to 0$ the minimizers tends to be constant functions in $\overline{B_R(0) \setminus B_{R/2}(0)}$. The value ρ is independent on $\varepsilon \in (0, \varepsilon_4)$ and $|\nu'| < \nu'_4$. For the uniqueness and the $\mathcal{C}^1$ dependence, we consider the map

$$\left(\overline{B_R(0) \setminus B_{R/2}(0)}\right)^2 \times (0, \varepsilon_4) \times (-\nu'_4, \nu'_4) \times H_{p_1p_2}([0,1]) \to \left(H_{p_1p_2}([0,1])\right)^*$$
$$(p_1, p_2, \varepsilon, \nu', u) \mapsto dM_{\nu'}(u).$$

Let $\bar{u}$ be a minimizer of $M_{\nu'}$ in $H_{p_1p_2}([0,1])$, whose image is contained in $B_{R+\rho}(0) \setminus B_{R/2-\rho}(0)$; an explicit computation shows that, if $|p_1 - p_2|$ and ν' are sufficiently small, the second differential $d^2M_{\nu'}(u)$ is positive definite, so that it is invertible. Thus, the implicit function theorem applies to give uniqueness and smooth dependence. □

Remark 3.4.21. In Section 3.3 we prove that, if $p_1, p_2 \in \partial B_R(0)$ are sufficiently close together, we can find a "close to brake" solution of problem (3.12) which, of course, passes close to the boundary of the "Hill region" $\{\Phi_{\nu',\varepsilon}(y) > 1\}$. This is not in contradiction with the previous result, since an outer solution parametrizes a non-minimal critical point of $L_{\nu'}$.

3.5 A finite-dimensional reduction

In this section we glue the fixed ends trajectories previously obtained, alternating outer and inner arcs in order to construct periodic orbits of the restricted problem (3.3) in the whole plane. Since in this procedure we need smooth junctions, we are going to use the same variational argument which was introduced in Section 2.5. Let us set $\widetilde{\varepsilon} := \min\{\varepsilon_2, \varepsilon_5\}$, $\widetilde{\nu}' := \min\{\nu'_1, \nu'_4\}$. The quantities ε_2 and ν'_1 have been introduced in Proposition 3.3.1 (recall also the definition of δ therein), while ε_5 and ν'_4 have been introduced in Proposition 3.4.19.

Proposition 3.5.1. *There exist $\bar{\varepsilon}, \bar{\nu}' > 0$ such that, for every $(\varepsilon, \nu') \in (0, \bar{\varepsilon}) \times (-\bar{\nu}', \bar{\nu}')$, $n \in \mathbb{N}$ and $(P_{j_1}, \dots, P_{j_n}) \in \mathcal{P}^n$, there exists a periodic weak solution $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}$ of problem (3.6), which depends on $(P_{j_1}, \dots, P_{j_n})$ in the following way: the image of $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}$ crosses $2n$ times within one period the circle $\partial B_R(0)$, at times $(t_k)_{k=0,\dots,2n-1}$, and*

- *in (t_{2k}, t_{2k+1}) the solution stays outside $B_R(0)$ and*
$$|\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}(t_{2k}) - \gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}(t_{2k+1})| < \delta;$$
- *in (t_{2k+1}, t_{2k+2}) the solution lies inside $B_R(0)$, and, if it does not collide against any centre, then it separates them according to the partition P_{j_k}.*

Let us fix $\varepsilon \in (0, \widetilde{\varepsilon})$, $|\nu'| < \widetilde{\nu}'$, $n \in \mathbb{N}$, $(P_{k_1}, P_{k_2}, \dots, P_{k_n}) \in \mathcal{P}^n$. We recall the definition of D (cf. Section 2.5, Chapter 2):

$$D = \left\{ (p_0, \dots, p_{2n}) \in (\partial B_R(0))^{2n+1} \;\middle|\; \begin{array}{l} |p_{2j+1} - p_{2j}| \le \delta \\ \text{for } j = 0, \dots, n-1, \\ p_{2n} = p_0 \end{array} \right\},$$

Let $(p_0, \dots, p_{2n}) \in D$. For every $j \in \{0, \dots, n-1\}$, we can apply Proposition 2.3.1 to obtain an outer solution $y_{2j}(t) := y_{\text{ext}}(t; p_{2j}, p_{2j+1}; \varepsilon, \nu')$ defined in $[0, T_{2j}]$, where $T_{2j} := T_{\text{ext}}(p_{2j}, p_{2j+1}; \varepsilon, \nu')$. We recall that y_{2j} depends on p_{2j} and p_{2j+1} in a $\mathcal{C}^1$ manner. Also, from Corollary 2.4.16 we obtain an inner weak solution $y_{2j+1}(t) := y_{P_{k_{j+1}}}(t; p_{2j+1}, p_{2j+2}; \varepsilon, \nu')$ defined in $[0, T_{2j+1}]$, where $T_{2j+1} := T_{P_{k_{j+1}}}(p_{2j+1}, p_{2j+2}; \varepsilon, \nu')$ (recall that $\nu'_4 < \nu'_3$). Being $L_{\nu'}$ invariant under re-parametrizations which preserve the orientation, y_{2j+1} is a local minimizer of the functional $L_{\nu'}([0, T_{2j+1}]; \cdot)$. We point out that y_{2j+1} could not be unique; however, if there is more then one minimizer of $L_{\nu'}$ in K_{P_j}, we can arbitrarily choose one of them.
We set $\mathfrak{T}_k := \sum_{j=0}^{k} T_j$, $k = 0, \dots, 2n-1$, and

$$\gamma^{((P_{k_1},\dots,P_{k_n}),\varepsilon,\nu')}_{(p_0,\dots,p_{2n})}(s) := \begin{cases} y_0(s) & s \in [0, \mathfrak{T}_0] \\ y_1(s - \mathfrak{T}_0) & s \in [\mathfrak{T}_0, \mathfrak{T}_1] \\ \vdots & \\ y_{2n-2}(s - \mathfrak{T}_{2n-3}) & s \in [\mathfrak{T}_{2n-3}, \mathfrak{T}_{2n-2}] \\ y_{2n-1}(s - \mathfrak{T}_{2n-2}) & s \in [\mathfrak{T}_{2n-2}, \mathfrak{T}_{2n-1}]. \end{cases}$$

The function $\gamma^{((P_{k_1},\dots,P_{k_n}),\varepsilon,\nu')}_{(p_0,\dots,p_{2n})}$ is a piecewise differentiable $\mathfrak{T}_{2n-1}$-periodic function. It is a weak solution of the restricted problem (3.3) with Jacobi constant -1 in $[0, \mathfrak{T}_{2n-1}] \setminus$

$\{0, \mathfrak{T}_0, \ldots, \mathfrak{T}_{2n-1}\}$, but in general is not $\mathcal{C}^1$ in $\{0, \mathfrak{T}_0, \ldots, \mathfrak{T}_{2n-1}\}$; however, the right and left limits of the derivatives in these points are finite, so that it is in H^1. It is also possible that $\gamma_{(p_0,\ldots,p_{2n})}^{((P_{k_1},\ldots,P_{k_n}),\varepsilon,\nu')}$ has collisions. Thanks to Lemma 2.3.4 and Corollary 3.4.18, we are sure that the time interval of $\gamma_{(p_0,\ldots,p_{2n})}^{((P_{k_1},\ldots,P_{k_n}),\varepsilon,\nu')}$ is bounded above and bounded below, uniformly with respect to $(p_0, \ldots, p_{2n}) \in D$, by positive constants; therefore for every $(p_0, \ldots, p_{2n}) \in D$ the period of the associated function is neither trivial, nor infinite.

We introduce a function $F = F_{((P_{k_1},\ldots,P_{k_n}),\varepsilon,\nu')} : D \to \mathbb{R}$ defined by

$$F(p_0, \ldots, p_{2n}) := L_{\nu'}\left([0, \mathfrak{T}_{2n-1}]; \gamma_{(p_0,\ldots,p_{2n})}^{((P_{k_1},\ldots,P_{k_n}),\varepsilon,\nu')}\right) = \sum_{j=0}^{2n-1} L_{\nu'}\left([0, T_j]; y_j\right).$$

Proposition 3.5.2. *There exists $(\bar{p}_0, \ldots, \bar{p}_{2n}) \in D$ which minimizes F. There exist $\bar{\varepsilon}, \bar{\nu}' > 0$ such that, for every $(\varepsilon, \nu') \in (0, \bar{\varepsilon}) \times (-\bar{\nu}', \bar{\nu}')$, the associated function $\gamma_{(p_0,\ldots,p_{2n})}^{((P_{k_1},\ldots,P_{k_n}),\varepsilon,\nu')}$ is a periodic weak solution of the restricted problem* (3.6)*. The values $\bar{\varepsilon}$ and $\bar{\nu}'$ depends neither on n, nor on $(P_{k_1}, \ldots, P_{k_n}) \in \mathcal{P}^n$.*

Remark 3.5.3. Proposition 3.5.1 is an immediate consequence of this statement.

From now on, we will write $\gamma^{((P_{k_1},\ldots,P_{k_n}),\varepsilon,\nu')}$ to denote the periodic weak solution associated to an arbitrarily chosen minimizer of $F_{((P_{k_1},\ldots,P_{k_n}),\varepsilon,\nu')}$.

We reach the result through a series of lemmas, following the sketch of the proof of Proposition 2.5.1.

Lemma 3.5.4. *The function F is continuous, so that there exists a minimizer of F in the compact set D.*

Proof. It is not difficult to adapt the proof of Lemma 2.5.3. □

Let $(\bar{p}_0, \ldots, \bar{p}_{2n})$ be a minimizer of F. We aim at showing that the minimality of $(\bar{p}_0, \ldots, \bar{p}_{2n})$ implies smoothness in the junction times for the associated periodic function $\gamma_{(\bar{p}_0,\ldots,\bar{p}_{2n})}^{((P_{k_1},\ldots,P_{k_n}),\varepsilon,\nu')}$. In order to prove it, we would like to write explicitly the equation $\nabla F(\bar{p}_0, \ldots, \bar{p}_{2n}) = 0$. As noticed in Section 2.5, it is not evident that this can be done, because of the lack of uniqueness of inner minimizers of $M_{\nu'}$ in K_{P_j}: for this reason it is not immediate that an inner solution depends smoothly on its ends. In order to overcome the problem, we can use Proposition 3.4.19 and argue as in Section 2.5: for any $j \in \{0, \ldots, n-1\}$, we consider a forward normal neighbourhood U_{2j+1} of the point $\bar{p}_{2j+1}$. Let us choose $t_* \in (0, T_{2j+1})$ such that

$$\widetilde{p}_{2j+1} := y_{2j+1}(t_*) \in U_{2j+1}, \quad |\widetilde{p}_{2j+1}| < R, \quad y\left([0, t_*]\right) \subset \left(B_R(0) \setminus B_{R/2}(0)\right);$$

There exists a unique minimizer $\widehat{y}(\cdot;\bar{p}_{2j+1},\widetilde{p}_{2j+1};\varepsilon,\nu')$ of $M_{\nu'}$, and hence also of $L_{\nu'}$ (up to a re-parametrization), which connects p_{2j+1} and $\widetilde{p}_{2j+1}$ in time 1, and depends smoothly on its ends. For the uniqueness, $\widehat{y}$ has to be a re-parametrization of y_{2j+1}. Note that if $p_{2j+1}\in\overline{U_{2j+1}\cap B_R(0)}$, then there is a unique minimizer $\widehat{y}(\cdot;p_{2j+1},\widetilde{p}_{2j+1};\varepsilon,\nu')$ of $M_{\nu'}$ which connects p_{2j+1} and $\widetilde{p}_{2j+1}$. We consider its re-parametrization $\widetilde{y}(\cdot\,;p_{2j+1},\widetilde{p}_{2j+1};\varepsilon)$ such that

$$\begin{cases}\ddot{\widetilde{y}}(t)+2\nu' i\dot{\widetilde{y}}(t)=\nabla\Phi_{\nu',\varepsilon}(\widetilde{y}(t))\\ \frac{1}{2}|\dot{\widetilde{y}}(t)|^2-\Phi_{\nu',\varepsilon}(\widetilde{y}(t))=-1,\end{cases}$$

denoting by $[0,T(p_{2j+1},\widetilde{p}_{2j+1})]$ its domain. Due to the minimality of $\widehat{y}(\cdot\,;p_{2j+1},\widetilde{p}_{2j+1};\varepsilon,\nu')$ for $L_{\nu'}$, such a re-parametrization exists, see Theorem 3.4.5. In this way

$$\widetilde{y}(\cdot\,;\bar{p}_{2j+1},\widetilde{p}_{2j+1};\varepsilon,\nu')\equiv y_{P_{k_{j+1}}}(\cdot\,;\bar{p}_{2j+1},\widetilde{p}_{2j+1};\varepsilon,\nu')|_{[0,T(\bar{p}_{2j+1},\widetilde{p}_{2j+1})]}.$$

Let $D_{2j+1}:=\{p_{2j+1}\in\left(\partial B_R(0)\cap\bar{U}_{2j+1}\right):|\bar{p}_{2j}-p_{2j+1}|\le\delta\}$. We define $G_{2j+1}:D_{2j+1}\to\mathbb{R}$ by

$$\begin{aligned}G_{2j+1}(p_{2j+1}):=L\left([0,T(p_{2j+1})];y_{\text{ext}}(\cdot\,;\bar{p}_{2j},p_{2j+1};\varepsilon,\nu')\right)\\ +L\left([0,T(p_{2j+1},\widetilde{p}_{2j+1})];\widetilde{y}(\cdot\,;p_{2j+1},\widetilde{p}_{2j+1};\varepsilon,\nu')\right),\end{aligned}$$

where $T(p_{2j+1})$ denotes $T_{\text{ext}}(\bar{p}_{2j},p_{2j+1};\varepsilon,\nu')$ (we will adopt this notation in this section). Of course, with minor changes we can also define a function G_{2j}, for every $j\in\{0,\dots,n\}$. Note that G_k is continuous (for every k), since it is a sum of terms which are both continuous with respect to p_k. As a consequence, G_k has a minimum.

Lemma 3.5.5. *If $(\bar{p}_0,\dots,\bar{p}_{2n})$ is a minimizer for F, then $\bar{p}_k$ is a minimizer for G_k.*

Proof. The proof is the same of Lemma 2.5.5. □

The main reason to pass from the study of F to the study of the functions G_k is that, in contrast with F, G_k is evidently differentiable for every k: only to fix our minds, we focus on the case $k=2j+1$; $L\left([0,T(p_{2j+1})];y_{\text{ext}}(\cdot\,;\bar{p}_{2j},p_{2j+1};\varepsilon,\nu')\right)$ depends smoothly on p_{2j+1} for the differentiable dependence of outer solutions with respect to the ends, and $L\left([0,T(p_{2j+1},\widetilde{p}_{2j})];\widetilde{y}(\cdot\,;p_{2j+1},\widetilde{p};\varepsilon,\nu')\right)$ depends smoothly on p_{2j+1} for Proposition 3.4.19. Therefore, the minimality of $\bar{p}_{2j+1}$ implies that

$$\text{if } \bar{p}_{2j+1}\in D^\circ_{2j+1}\quad\Longrightarrow\quad\frac{\partial G_{2j+1}}{\partial p_{2j+1}}(\bar{p}_{2j+1})=0$$

(the notation D°_{2j+1} denotes the inner of D_{2j+1}). This partial derivative is a linear operator from the tangent space $T_{\bar{p}_{2j+1}}(\partial B_R(0))$ into $\mathbb{R}$. In what follows we show that, if ε and ν' are small enough, $\bar{p}_k \in D_k^\circ$ for every k, and that the stationarity conditions are nothing but regularity conditions for the functions

$$\zeta_{2j}(t) := \begin{cases} y_{P_{k_{j-1}}}(t + T_{2j-1} - T(\widetilde{p}_{2j}, \bar{p}_{2j}); \bar{p}_{2j-1}, \bar{p}_{2j}; \varepsilon, \nu') & \text{if } t \in [0, T(\widetilde{p}_{2j}, \bar{p}_{2j})] \\ y_{\text{ext}}(t - T(\widetilde{p}_{2j}, \bar{p}_{2j}); \bar{p}_{2j}, \bar{p}_{2j+1}; \varepsilon, \nu') \\ \qquad \text{if } t \in [T(\widetilde{p}_{2j}, \bar{p}_{2j}), T(\widetilde{p}_{2j}, \bar{p}_{2j}) + T(\bar{p}_{2j+1})] \end{cases}$$

and

$$\zeta_{2j+1}(t) := \begin{cases} y_{\text{ext}}(t; \bar{p}_{2j}, \bar{p}_{2j+1}; \varepsilon, \nu') & \text{if } t \in [0, T(\bar{p}_{2j+1})] \\ y_{P_{k_{j+1}}}(t - T(\bar{p}_{2j+1}); \bar{p}_{2j}, \bar{p}_{2j+1}; \varepsilon, \nu') \\ \qquad \text{if } t \in [T(\bar{p}_{2j+1}), T(\bar{p}_{2j+1}) + T(\bar{p}_{2j+1}, \widetilde{p}_{2j+1})]. \end{cases}$$

Taking into account that ζ_k is (up to a time translation) the restriction of $\gamma^{((P_{k_1},\ldots,P_{k_n}),\varepsilon,\nu')}$ in a neighbourhood of the junction time $\mathfrak{T}_{k-1}$, we obtain $\mathcal{C}^1$ regularity for $\gamma^{((P_{k_1},\ldots,P_{k_n}),\varepsilon,\nu')}$ itself.

Lemma 3.5.6. *For every $j = 0, \ldots, n-1$, $p_{2j} \in D_{2j}$, and for every $\varphi \in T_{p_{2j}}(B_R(0))$, we have*

$$\frac{\partial G_{2j}}{\partial p_{2j}}(p_{2j})[\varphi] = \frac{1}{\sqrt{2}} \langle \dot{\widetilde{y}}(T(\widetilde{p}_{2j}, p_{2j}); \widetilde{p}_{2j}, p_{2j}; \varepsilon, \nu') - \dot{y}_{ext}(0; p_{2j}, \bar{p}_{2j+1}; \varepsilon, \nu'), \varphi \rangle.$$

For every $j = 0, \ldots, n-1$, $p_{2j+1} \in D_{2j+1}$, and for every $\varphi \in T_{p_{2j+1}}(B_R(0))$, we have

$$\frac{\partial G_{2j+1}}{\partial p_{2j+1}}(p_{2j+1})[\varphi] = \frac{1}{\sqrt{2}} \langle \dot{y}_{ext}(T(p_{2j+1}); \bar{p}_{2j}, p_{2j+1}; \varepsilon, \nu') - \dot{\widetilde{y}}(0; p_{2j+1}, \widetilde{p}_{2j+1}; \varepsilon, \nu'), \varphi \rangle.$$

Proof. It is not restrictive to consider the derivative of G_1 to ease the notation. The same calculations work for the other cases. There holds

$$\begin{aligned} \frac{\partial G_1}{\partial p_1}(p_1) = \frac{\partial}{\partial p_1} L_{\nu'}\left([0, T(p_1)]; y_{\text{ext}}(\,\cdot\,; \bar{p}_0, p_1; \varepsilon, \nu')\right) \\ + \frac{\partial}{\partial p_1} L_{\nu'}\left([0, T(p_1, \widetilde{p}_1)]; \widetilde{y}(\,\cdot\,; p_1, \widetilde{p}_1; \varepsilon, \nu')\right). \end{aligned} \tag{3.26}$$

Let us consider the first term in the right side, writing simply y_0 instead of

$y_{\text{ext}}(\cdot\,; \bar{p}_0, p_1; \varepsilon, \nu')$; we consider $u_0(t) = y_0(T_0 t)$, defined in $[0,1]$. It results

$$\begin{aligned}
\frac{\partial}{\partial p_1} L_{\nu'}\left([0, T(p_1)]; y_0\right) &= \frac{\partial}{\partial p_1} L_{\nu'}\left([0,1]; u_0\right) \\
&= \frac{1}{\sqrt{2}} \int_0^1 \left[\langle \frac{\dot{u}_0}{T_0}, \frac{d}{dt}\frac{\partial u_0}{\partial p_1} \rangle + \langle T_0 \nabla \Phi_{\nu',\varepsilon}(u_0), \frac{\partial u_0}{\partial p_1} \rangle \right] \\
&\quad + \frac{1}{\sqrt{2}} \nu' \int_0^1 \left(\langle i \frac{\partial u_0}{\partial p_1}, \dot{u}_0 \rangle + \langle i u_0, \frac{d}{dt}\frac{\partial u_0}{\partial p_1} \right) \\
&= \frac{1}{\sqrt{2}} \int_0^1 \langle -\frac{\ddot{u}_0}{T_0} - 2\nu' i \dot{u}_0 + T_0 \nabla \Phi_{\nu',\varepsilon}(u_0), \frac{\partial u_0}{\partial p_1} \rangle \\
&\quad + \frac{1}{\sqrt{2}} \left[\langle \frac{\dot{u}_0(t)}{T_0} + \nu' i u_0(t), \frac{\partial u_0}{\partial p_1}(t) \rangle \right]_0^1 \\
&= \frac{1}{\sqrt{2}} \left[\langle \dot{y}_0(t) + \nu' y_0(t), \frac{\partial y_0}{\partial p_1}(t) \rangle \right]_0^{T(p_1)} .
\end{aligned}$$

In the second equality we use the Jacobi constant for y_0, in the last one we use the fact that y_0 is a classical solution of the motion equation. As in Lemma 2.5.6, we can compute

$$\frac{\partial}{\partial p_1} y_0(0) = 0 \qquad \frac{\partial}{\partial p_1} y_0(T(p_1)) = Id_{T_{p_1}(\partial B_R(0))}.$$

Hence

$$\frac{\partial}{\partial p_1} L_{\nu'}\left([0, T(p_1)]; y_0\right)[\varphi] = \frac{1}{\sqrt{2}} \left(\langle \dot{y}_0(T(p_1)), \varphi \rangle + \nu' \langle i p_1, \varphi \rangle \right).$$

We can repeat the same computations for the second term in the right side of the (3.26), with minor changes: terming $\widetilde{y}_1 = \widetilde{y}(\cdot\,; p_1, \widetilde{p}_1; \varepsilon, \nu')$, we obtain

$$\frac{\partial}{\partial p_1} L_{\nu'}\left([0, T(p_1, \widetilde{p})_1]; \widetilde{y}_1\right)[\varphi] = -\frac{1}{\sqrt{2}} \left(\langle \dot{\widetilde{y}}_1(0), \varphi \rangle + \nu' \langle i p_1, \varphi \rangle \right).$$ □

Lemma 3.5.7. *There exist $\bar{\varepsilon} > 0$ and $\bar{\nu}' > 0$ such that if $\varepsilon \in (0, \bar{\varepsilon})$ and $|\nu'| < \bar{\nu}'$ then*

$$\bar{p}_k \text{ minimizes } G_k \quad \Longrightarrow \quad \bar{p}_k \in D_k^\circ \qquad \forall k.$$

The values $\bar{\varepsilon}$ and $\bar{\nu}'$ are independent on $(P_{k_1}, \ldots, P_{k_n}) \in \mathcal{P}^n$.

Proof. It is possible to adapt the proof of Lemma 2.5.7. □

Conclusion of the proof of Proposition 3.5.2. We can follow the proof of Lemma 2.5.8 in order to check that each ζ_k is smooth. Recalling the construction of $\gamma^{((P_{k_1},\ldots,P_{k_n}),\varepsilon,\nu')}$, the proof is complete. □

3.6 Collision-free weak solutions

We will work with $\varepsilon \in (0, \bar{\varepsilon})$ which is fixed. The aim is to find a threshold $\bar{\nu}'_{th}(\varepsilon)$ such that, if $|\nu'| < \bar{\nu}'_{th}(\varepsilon)$, then $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}$ is collision-free. It is necessary to distinguish among:

$$1)\ \alpha = 1 \text{ and } N \geq 4, \qquad 2)\ \alpha = 1 \text{ and } N = 3, \qquad 3)\ \alpha \in (1, 2).$$

1) $\alpha = 1$ and $N \geq 4$. We start by looking at Theorem 2.5.1. Since $N \geq 4$, we have a simple way to choose $(P_{j_1}, \dots, P_{j_n})$ so that the weak solution $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,0)}$ is a collision-free solution of the N-centre problem $\ddot{y} = \nabla V_\varepsilon(y)$, with energy -1: it is sufficient to take $P_{j_k} \in \mathcal{P} \setminus \mathcal{P}_1$ for every $k = 1, \dots, n$. Indeed in such a situation the conditions (ii)-(b) or (ii)-(c) of the quoted statement cannot be satisfied. Note that if $N = 3$ the set $\mathcal{P} \setminus \mathcal{P}_1$ is empty, and this is way that case deserves a different discussion. Now, let $\varepsilon \in (0, \bar{\varepsilon})$, $\nu' \in (-\bar{\nu}', \bar{\nu}')$, $n \in \mathbb{N}$ and $(P_{j_1}, \dots, P_{j_n}) \in (\mathcal{P} \setminus \mathcal{P}_1)^n$; let $(\bar{p}_0, \dots, \bar{p}_{2n})$ be a minimizer of $F_{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}$ found in Proposition 3.5.2, and let $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}$ be the corresponding periodic weak solution of (3.6). Is it true that, for ν' sufficiently small, such a solution is still collision-free? The answer is affirmative: the idea is that if $\nu' \to 0$ the "minimizers" $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}$ are weakly convergent in H^1 to $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,0)}$, which is collision-free. This is true in a local sense, and can be considered as a kind of Gamma-convergence argument.

Continuity Lemma 3.6.1. *Let $\varepsilon \in (0, \bar{\varepsilon})$, $P_j \in \mathcal{P}$, $((p_1^m, p_2^m)) \subset (\partial B_R(0))^2$ and $(\nu'_m) \subset (-\bar{\nu}', \bar{\nu}')$. Let $u_m = u_{P_j}(\cdot\,; p_1^m, p_2^m; \varepsilon, \nu'_m)$ be a minimizer for the following variational problem:*

$$\min \left\{ M_{\nu'_m}(u) : u \in K_{P_j}^{p_1^m p_2^m}([0,1]) \right\}.$$

Assume that $(p_1^m, p_2^m) \to (\widetilde{p}_1, \widetilde{p}_2)$, $\nu'_m \to 0$, and $u_m \rightharpoonup \widetilde{u}$ weakly in H^1. Then $\widetilde{u}$ is a minimizer for

$$\min \left\{ M_0(u) : u \in K_{P_j}^{\widetilde{p}_1 \widetilde{p}_2}([0,1]) \right\}.$$

We postpone the proof of this lemma in the next section; now, as announced, we use it in order to prove the following proposition, which is the last step in the proof of Theorem 3.1.1 (recalling Proposition 3.2.1 and Remark 3.2.3).

Proposition 3.6.2. *Let $\alpha = 1$ and $N \geq 4$. Let $\varepsilon \in (0, \bar{\varepsilon})$. There exists $\bar{\nu}'_1(\varepsilon)$ such that for every $\nu' \in (-\bar{\nu}'_1(\varepsilon), \bar{\nu}'_1(\varepsilon))$, $n \in \mathbb{N}$ and $(P_{j_1}, \dots, P_{j_n}) \in (\mathcal{P} \setminus \mathcal{P}_1)^n$, the function $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}$ is collision-free.*

Proof. Let $(P_{j_1}, \dots, P_{j_n}) \in (\mathcal{P} \setminus \mathcal{P}_1)^n$ and $\nu' \in (-\bar{\nu}', \bar{\nu}')$. The key observation is the following: when $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}$ stays inside $B_R(0)$, it coincides with a re-parametrization of an inner minimizer $u_{P_j}(\cdot\,;p_1,p_2;\varepsilon,\nu')$, for some p_1,p_2 and P_j. Therefore the thesis follows if we show that there exist $\bar\nu_1'=\bar\nu_1'(\varepsilon),\beta_1=\beta_1(\varepsilon)>0$ such that

$$\min_{k\in\{1,\dots,N\}}\left(\min_{t\in[0,1]}|u_{P_j}(t;p_1,p_2;\varepsilon,\nu')-c_k|\right)\geq\beta_1$$

for every $(p_1,p_2,P_j,\nu')\in(\partial B_R(0))^2\times(\mathcal{P}\setminus\mathcal{P}_1)\times(-\bar\nu_1',\bar\nu_1')$.
Assume by contradiction that this claim is not true. Then there are $(\beta_m)\subset\mathbb{R}^+$, $(\nu_m')\subset(-\bar\nu',\bar\nu')$, $((p_1^m,p_2^m))\subset(\partial B_R(0))^2$, $(P_j^m)\subset(\mathcal{P}\setminus\mathcal{P}_1)$ and $(k_m)\subset\{1,\dots,N\}$ such that $\beta_m\to 0$, $\nu_m'\to 0$ for $m\to\infty$, and

$$\min_{t\in[0,1]}|u_{P_j^m}(t;p_1^m,p_2^m;\varepsilon,\nu_m')-c_{k_m}|=\beta_m\qquad\forall m.$$

Since $\{1,\dots,N\}$ and $\mathcal{P}\setminus\mathcal{P}_1$ are discrete and finite, we can assume $k_m=k$ and $P_j^m=P_j$ for every m. Also, since $\partial B_R(0)$ is compact, up to a subsequence $(p_1^m,p_2^m)\to(\widetilde p_1,\widetilde p_2)\in\partial B_R(0)$. We term $u_m=u_{P_j}(\cdot\,;p_1^m,p_2^m;\varepsilon,\nu_m')$. As shown in Proposition 3.4.15, the set of the minimizers $\mathcal{IM}_\varepsilon$ is bounded in the H^1 norm, therefore up to a subsequence $u_m\rightharpoonup\widetilde u\in K_{P_j}^{\widetilde p_1\widetilde p_2}([0,1])$ weakly in H^1 (and hence uniformly). In particular, the function $\widetilde u$ has at least one collision. Thus, Lemma 3.6.1 implies that $\widetilde u$ is a collision minimizer of M_0 in $K_{P_j}^{\widetilde p_1\widetilde p_2}([0,1])$; this is in contradiction with Theorem 2.4.14, since $P_j\notin\mathcal{P}_1$ (recall Remark 3.4.4). □

Remark 3.6.3. The continuity lemma permits to restrict the attention to a unique passage inside $B_R(0)$; in particular the argument is independent on n, which can be arbitrarily large.

2) $\alpha=1$ and $N=3$. This is the hardest part, since if we look at Theorem 2.5.1 we realize that it is not immediate to give conditions on $(P_{j_1},\dots,P_{j_n})$ to obtain a collision-free periodic solution $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,0)}$ for the fixed energy N-centre problem

$$\begin{cases}\ddot y(t)=\nabla V_\varepsilon(y(t))\\ \frac12|\dot y(t)|^2-V_\varepsilon(y(t))=-1.\end{cases}$$

In order to work with a set of symbols such that the corresponding solutions are collision-free, we introduced a particular set of symbols

$$\mathcal{G}=\{P_1P_1P_2P_3,P_2P_2P_3P_1\}\,;$$

for every n and for every $(P_{j_1},\dots,P_{j_{4n}})\in\mathcal{G}^n$, the weak solution $\gamma^{((P_{j_1},\dots,P_{j_{4n}}),\varepsilon,0)}$ of the N-centre problem is actually a classical solution, because no composed sequence of elements of $\mathcal{G}$ has the reflection symmetry which characterizes a collision trajectory (see Remark 2.6.4). For $\varepsilon \in (0,\bar{\varepsilon})$, we aim at showing that, if $|\nu'|$ is sufficiently small, for every $n \in \mathbb{N}$ and $(P_{j_1},\dots,P_{j_{4n}}) \in \mathcal{G}^n$ the function $\gamma^{((P_{j_1},\dots,P_{j_{4n}}),\varepsilon,\nu')}$ is still collision-free. The idea for the proof is exactly the same which we have already used in point 1. Unfortunately, while therein we can simply restrict our attention to the behaviour of any inner minimizer (that is a local argument), here this approach does not work. Indeed, for every $P_j \in \mathcal{P}$ and $p_1 \in \partial B_R(0)$ it is possible that a minimizer of M_0 in $K^{p_1p_1}_{P_j}([0,1])$ has collisions. Therefore we have to use an argument which is local, "but not too much"; it is important to keep in mind the procedure explained in Remark 2.6.4.

We collect the possible groups of 5 consecutive partitions in (2.75) in a set $\widetilde{\mathcal{P}}^5 \subset \mathcal{P}$. Let us fix $\varepsilon \in (0,\bar{\varepsilon})$, $p_1, p_{10} \in \partial B_R(0)$, $(P_{k_1},\dots,P_{k_5}) \in \widetilde{\mathcal{P}}^5$, $\nu' \in (-\bar{\nu},\bar{\nu})$. Let

$$B := \{(p_2,\dots,p_9) \in (\partial B_R(0))^8 : |p_{2j}-p_{2j+1}| \le \delta,\ j=1,\dots,4\}.$$

As we associated to each point of D a periodic function, to each point of B we can associate a (non-periodic) function in the following way. For each $j=1,\dots,4$ we can connect p_{2j} and p_{2j+1} with an outer solution $y_{2j} = y_{\text{ext}}(\cdot\,;p_{2j},p_{2j+1};\varepsilon,\nu')$ of (3.12); for each $j=0,\dots,4$ we can connect p_{2j+1} and p_{2j+2} with an inner solution $y_{2j+1} = y_{P_{k_{j+1}}}(\cdot\,;p_{2j+1},p_{2j+2};\varepsilon,\nu')$ of (3.18). We set $t_1 := 0$, $t_k := \sum_{j=1}^{k-1} T_j$ for $k=2,\dots,10$, where $[0,T_j]$ is the time interval of y_j. We define

$$\sigma^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}_{(p_2,\dots,p_9)}(t) := \begin{cases} y_1(t) & t\in[t_1,t_2] \\ y_2(t-t_2) & t\in[t_2,t_3] \\ \vdots & \\ y_9(t-t_9) & t\in[t_9,t_{10}]. \end{cases} \tag{3.27}$$

By the definition $\sigma^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}_{(p_2,\dots,p_9)}(t_k) = p_k$. We introduce a function $\mathfrak{F}_{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')} : B \to \mathbb{R}$ as

$$\mathfrak{F}_{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}(p_2,\dots,p_9) := L_{\nu'}\left([0,t_{10}];\sigma^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}_{(p_2,\dots,p_9)}\right).$$

Note the analogy between the definition of $\mathfrak{F} = \mathfrak{F}_{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}$ and of $F = F_{((P_{k_1},\dots,P_{k_n}),\varepsilon,\nu')}$. The function $\mathfrak{F}$ is continuous on the compact set B (apply the same proof already used for the continuity of F), therefore it has a minimum. We denote by $\sigma^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}$ the glued function associated to an arbitrarily chosen minimizer. Let $(P_{k_1},\dots,P_{k_{4n}}) \in \mathcal{G}^n$. The following Lemma relates the minimality properties of F and of $\mathfrak{F}$; in what follows the indexes have to be considered by periodicity: for instance writing $2j+5$ we mean $2j+5 \mod 8n$.

Lemma 3.6.4. *Let* $(\bar{p}_0, \ldots, \bar{p}_{8n}) \in D$ *be a minimizer of* $F_{((P_{k_1},\ldots,P_{k_{4n}}),\varepsilon,\nu')}$. *Then, for every* $j = 0, \ldots, 4n-1$, *the point* $(\bar{p}_{2j+2}, \ldots, \bar{p}_{2j+9}) \in B$ *is a minimizer of* $\mathfrak{F}_{((\bar{p}_{2j+1},\bar{p}_{2j+10}),(P_{k_{j+1}},\ldots,P_{k_{j+5}}),\varepsilon,\nu')}$. *In particular*

$$\gamma_{((P_{k_{j+1}},\ldots,P_{k_{j+5}}),\varepsilon,\nu')}|_{[\mathfrak{T}_{2j},\mathfrak{T}_{2j+10}]} \equiv \sigma^{((\bar{p}_{2j+1},\bar{p}_{2j+10}),(P_{k_{j+1}},\ldots,P_{k_{j+5}}),\varepsilon,\nu')}.$$

Proof. It is an immediate consequence of the additivity of the functional $L_{\nu'}$. □

As a consequence, the following statement can be proved applying the same argument already explained in Remark 2.6.4.

Lemma 3.6.5. *Let* $\varepsilon \in (0, \bar{\varepsilon})$. *For every* $((p_1, p_{10}), (P_{k_1}, \ldots, P_{k_5})) \in (\partial B_R(0))^2 \times \widetilde{\mathcal{P}}^5$ *the function* $\sigma^{((p_1,p_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,0)}$ *is collision-free during its third passage inside the ball* $B_R(0)$.

We denote with $T(\sigma)$ or $T^{((p_1,p_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,\nu')}_{(p_2,\ldots,p_9)}$ the maximum of the time interval of $\sigma = \sigma^{((p_1,p_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,\nu')}_{(p_2,\ldots,p_9)}$. We collect the boundedness properties of outer and inner solutions, see Lemma 3.3.4 and Corollary 3.4.18.

Lemma 3.6.6. *Let* $\varepsilon \in (0, \bar{\varepsilon})$. *There are* $C_1, C_2, C_3 > 0$ *such that*

$$C_1 \le T^{((p_1,p_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,\nu')}_{(p_2,\ldots,p_9)} \le C_2$$
$$\|\sigma^{((p_1,p_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,\nu')}_{(p_2,\ldots,p_9)}\|_{H^1([0,T(\sigma)])} \le C_3$$

for every $((p_2, \ldots, p_9), (p_1, p_{10}), (P_{k_1}, \ldots, P_{k_5}), \nu') \in B \times (\partial B_R(0))^2 \times \widetilde{\mathcal{P}}^5 \times (-\bar{\nu}', \bar{\nu}')$.

It is preferable to deal with functions defined in the same time interval. Therefore, for every $\sigma = \sigma^{((p_1,p_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,\nu')}_{(p_2,\ldots,p_9)}$ we introduce the re-parametrization

$$v(t) := v^{((p_1,p_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,\nu')}_{(p_2,\ldots,p_9)}(t) = \sigma^{((p_1,p_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,\nu')}_{(p_2,\ldots,p_9)}(T(\sigma)t),$$

defined for $t \in [0, 1]$.

Definition 3.6.7. We collect the "glued function" v in

$$\mathcal{GF}_\varepsilon := \left\{ v = v^{((p_1,p_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,\nu')}_{(p_2,\ldots,p_9)} \;\middle|\; \begin{array}{l} (p_2, \ldots, p_9) \in B,\ (p_1, p_{10}) \in (\partial B_R(0))^2, \\ (P_{k_1}, \ldots, P_{k_5}) \in \widetilde{\mathcal{P}}^5,\ |\nu'| < \bar{\nu}' \end{array} \right\}.$$

For each $v \in \mathcal{GF}_\varepsilon$ we term

$$\omega(v)^2 := \frac{\int_0^1 \Phi_{\nu',\varepsilon}(v) - 1}{\frac{1}{2}\int_0^1 |\dot{v}|^2}.$$

Note that, if $v(t) = \sigma(T(\sigma)t)$, then $\omega(v) = 1/T(\sigma)$. Note also that for every $\varepsilon \in (0, \bar{\varepsilon})$ there exists $C > 0$ such that $\|v\|_{H^1} \le C$ for every $v \in \mathcal{GF}_\varepsilon$. Indeed, this follows from Lemma 3.6.6, taking into account the boundedness properties for the time intervals of inner and outer solutions. In order to work with sequences of functions in $\mathcal{GF}_\varepsilon$, it is convenient to introduce some notation. Fixed $(P_{k_1}, \dots, P_{k_5}) \in \mathcal{P}^5$ and $\varepsilon \in (0, \bar{\varepsilon})$, assume that we have $((p_2^m, \dots, p_9^m))_m \subset B$, $((p_1^m, p_{10}^m))_m \subset (\partial B_R(0))^2$, $(\nu'_m) \subset (-\bar{\nu}', \bar{\nu}')$ such that

$$(p_2^m, \dots, p_9^m) \to (\widehat{p}_2, \dots, \widehat{p}_9) \qquad (p_1^m, p_{10}^m) \to (\widehat{p}_1, \widehat{p}_{10}) \qquad \nu'_m \to 0.$$

We will use the following notations

$$\begin{aligned} v_m &:= v_{(p_2^m,\dots,p_9^m)}^{((p_1^m,p_{10}^m),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu'_m)} \qquad && \omega_m := \omega(v_m) \\ \sigma_m &:= \sigma_{(p_2^m,\dots,p_9^m)}^{((p_1^m,p_{10}^m),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu'_m)} \qquad && T_m := T(\sigma_m); \end{aligned} \tag{3.28}$$

Subscripts will be replaced by the accent $\widehat{\cdot}$ for the function corresponding to the limit points. Recall that σ_m has been obtained by the juxtaposition of

$$y_{P_{k_{j+1}}}(\cdot\,; p_{2j+1}^m, p_{2j+2}^m; \varepsilon, \nu'_m) =: y_{2j+1}^m \quad \text{and} \quad y_{\text{ext}}(\cdot\,; p_{2j}^m, p_{2j+1}^m; \varepsilon, \nu'_m) =: y_{2j}^m.$$

Each y_j^m is defined over a time interval $[0, T_j^m]$. There are $0 = t_1^m < t_2^m < \dots t_9^m < t_{10}^m = T(\sigma_m)$ such that $\sigma_m(t_k^m) = p_k^m$ for every $k = 1, \dots, 10$. We have $T_j^m = t_{j+1}^m - t_j^m$. For $j = 0, \dots, 4$, recall that

$$y_{P_{k_{j+1}}}(\cdot\,; p_{2j+1}^m, p_{2j+2}^m; \varepsilon, \nu'_m) = u_{P_{k_{j+1}}}\left(\frac{\cdot}{T_{2j+1}^m}\cdot; p_{2j+1}^m, p_{2j+2}^m; \varepsilon, \nu'_m\right) =: u_{2j+1}^m.$$

Lemma 3.6.8. *Let $\varepsilon \in (0, \bar{\varepsilon})$, $(P_{k_1}, \dots, P_{k_5}) \in \mathcal{P}^5$. Assume that we have sequences $((p_2^m, \dots, p_9^m))_m \subset B$, $((p_1^m, p_{10}^m))_m \subset (\partial B_R(0))^2$, $(\nu'_m) \subset (-\bar{\nu}', \bar{\nu}')$ such that*

$$(p_2^m, \dots, p_9^m) \to (\widehat{p}_2, \dots, \widehat{p}_9) \qquad (p_1^m, p_{10}^m) \to (\widehat{p}_1, \widehat{p}_{10}) \qquad \nu'_m \to 0.$$

Using the notations previously introduced, assume that exists $v \in H^1([0,1])$ such that $v_m \rightharpoonup v$ weakly in H^1. Then

$$v = v_{(\widehat{p}_2,\dots,\widehat{p}_9)}^{((\widehat{p}_1,\widehat{p}_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,0)}.$$

Proof. It is sufficient to perform a series of successive re-parametrizations: indeed, we know that under the convergence of the ends and of ν'_m, inner and outer solutions y_k^m are weakly convergent to inner and outer solutions $\widehat{y}_k$ (see Propositions 3.3.1 and the Continuity Lemma 3.6.1); therefore it turns out that v is nothing but the function obtained by the juxtaposition of the limit arcs.

□

To each $((p_1,p_{10}),(P_{k_1},\ldots,P_{k_5}),\nu')\in(\partial B_R(0))^2\times\widetilde{\mathcal{P}}^5\times(-\bar\nu_1',\bar\nu_1')$ we can associate an element of $\mathcal{GF}_\varepsilon$ in the following way: it is well defined the function $\mathfrak{F}_{((p_1,p_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,\nu')}$, and we know that it has a minimum. To a minimum we associated the function $\sigma^{((p_1,p_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,\nu')}$, which can be re-parametrized obtaining $v^{((p_1,p_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,\nu')}$. We are ready to state the counterpart of the Continuity Lemma 3.6.1.

Continuity Lemma 3.6.9. *Let $\varepsilon\in(0,\bar\varepsilon)$, $(P_{k_1},\ldots,P_{k_5})\in\mathcal{P}^5$, $((p_1^m,p_{10}^m))\subset(\partial B_R(0))^2$ and $(\nu_m')\subset(-\bar\nu',\bar\nu')$. Let $v_m=v^{((p_1^m,p_{10}^m),(P_{k_1},\ldots,P_{k_5}),\varepsilon,\nu_m')}$ be a function of $\mathcal{GF}_\varepsilon$ associated to a minimizer of the following variational problem:*

$$\min\left\{\mathfrak{F}_{((p_1^m,p_{10}^m),(P_{k_1},\ldots,P_{k_5}),\varepsilon,\nu_m')}(p_2,\ldots,p_9):(p_2,\ldots,p_9)\in B\right\}.$$

Assume $(p_1^m,p_{10}^m)\to(\widetilde p_1,\widetilde p_{10})$, $\nu_m'\to0$, and $v_m\rightharpoonup\widetilde v$ weakly in H^1. Then $\widetilde v$ is the function associated to a minimizer for

$$\min\left\{\mathfrak{F}_{((\widetilde p_1,\widetilde p_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,0)}(p_2,\ldots,p_9):(p_2,\ldots,p_9)\in B\right\}.$$

This result permits to prove the following proposition, which is the last step in the proof of Theorem 3.1.2.

Proposition 3.6.10. *Let $\alpha=1$ and $N=3$. Let $\varepsilon\in(0,\bar\varepsilon)$. There exists $\bar\nu_2'(\varepsilon)$ such that for every $\nu'\in(-\bar\nu_2'(\varepsilon),\bar\nu_2'(\varepsilon))$, $n\in\mathbb{N}$ and $(P_{j_1},\ldots,P_{j_{4n}})\in\mathcal{G}^n$, the function $\gamma^{((P_{j_1},\ldots,P_{j_{4n}}),\varepsilon,\nu')}$ is collision-free.*

Proof. Let $(P_{j_1},\ldots,P_{j_{4n}})\in\mathcal{G}^n$ and $\nu'\in(-\bar\nu',\bar\nu')$. Let us consider the restriction of $\gamma=\gamma^{((P_{j_1},\ldots,P_{j_{4n}}),\varepsilon,\nu')}$ in a time interval $[s_1,s_2]$, chosen in such a way that $\gamma|_{[s_1,s_2]}$ describes one passage of γ inside $B_R(0)$. The goal is to show that $\gamma|_{[s_1,s_2]}$ is collision-free. There are

- $t_k\in\mathbb{R}$ and $p_k\in\partial B_R(0)$ such that $\gamma(t_k)=p_k$, for every $k=1,\ldots,10$.
- $(P_{k_1},\ldots,P_{k_5})\in\mathcal{P}^5$,

such that $\gamma|_{[t_1,t_{10}]}=\sigma^{((p_1,p_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,\nu')}=\sigma$ and $\gamma|_{[s_1,s_2]}=\sigma|_{[t_5,t_6]}$, where t_5 and t_6 have been defined in (3.27). This means that each passage of γ inside $\partial B_R(0)$ is the third passage inside $\partial B_R(0)$ of a function $\sigma^{((p_1,p_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,\nu')}$, for some $p_1,p_{10}\in\partial B_R(0)$ and $(P_{k_1},\ldots,P_{k_5})\in\mathcal{P}^5$. This observation is the key point of the proof: it implies that our thesis follows if we show that there are $\bar\nu_2',\beta_2>0$ such that

$$\min_{k\in\{1,\ldots,N\}}\left(\min_{t\in\left[\frac{t_5}{T(\sigma)},\frac{t_6}{T(\sigma)}\right]}\left|v^{((p_1,p_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,\nu')}(t)-c_{k_3}\right|\right)\geq\beta_2 \tag{3.29}$$

for every $((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\nu') \in (\partial B_R(0))^2 \times \widetilde{\mathcal{P}}^5 \times (-\bar\nu_2',\bar\nu_2')$; this implies that $v^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}$ (and hence $\sigma^{((p_1,p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu')}$) cannot have a collision in its third passage inside $B_R(0)$, independently on (p_1,p_{10}) and $(P_{k_1},\dots,P_{k_5})$.
Assume by contradiction that (3.29) is not true. Then there are $(\beta_m)\subset\mathbb{R}^+$, $(\nu_m')\subset(-\bar\nu',\bar\nu')$, $((p_1^m,p_2^m))\subset(\partial B_R(0))^2$, $((P_{k_1},\dots,P_{k_5})^m)\subset\widetilde{\mathcal{P}}^5$ such that $\beta_m\to0$, $\nu_m'\to0$ for $m\to\infty$, and

$$\min_{t\in\left[\frac{t_5^m}{T(\sigma_m)},\frac{t_6^m}{T(\sigma_m)}\right]} |v^{((p_1^m,p_{10}^m),(P_{k_1},\dots,P_{k_5})^m,\varepsilon,\nu_m')}(t)-c_{k_3^m}|=\beta_m \qquad \forall m.$$

Since $\widetilde{\mathcal{P}}^5$ is discrete and finite, we can assume $(P_{k_1},\dots,P_{k_5})^m=(P_{k_1},\dots,P_{k_5})$ for every m. Also, since $\partial B_R(0)$ is compact, up to a subsequence $(p_1^m,p_2^m)\to(\widehat p_1,\widehat p_2)\in\partial B_R(0)$. We term $v_m=v^{((p_1^m,p_{10}^m),(P_{k_1},\dots,P_{k_5})^m,\varepsilon,\nu_m')}$. The image of v_m intersects the circle $\partial B_R(0)$ in 8 points $(p_2^m,\dots,p_9^m)\in B$ in succession. Up to a subsequence $(p_2^m,\dots,p_9^m)\to(\widehat p_2,\dots,\widehat p_9)$. We observed that the set $\mathcal{GF}_\varepsilon$ is bounded in the H^1 norm, therefore up to a subsequence $v_m\rightharpoonup\widehat v\in H^1([0,1])$ weakly in H^1 (and hence uniformly). The image of $\widehat v$ intersects the circle in the 8 points $(\widehat p_2,\dots,\widehat p_9)$ in succession. To be precise

$$\widehat v = v_{(\widehat p_2,\dots,\widehat p_9)}^{((\widehat p_1,\widehat p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,0)}\in\mathcal{GF}_\varepsilon,$$

see Lemma 3.6.8. By the Continuity Lemma 3.6.9, the point $(\widehat p_2,\dots,\widehat p_9)$ minimizes $\mathfrak{F}_{((\widehat p_1,\widehat p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,0)}$ in B. But the uniform convergence implies that $\widehat v$ has a collision in its third passage inside $B_R(0)$, and this is in contradiction with Lemma 3.6.5. □

3) $\alpha\in(1,2)$**.** This is the easiest case, since for every $\varepsilon\in(0,\bar\varepsilon)$, $n\in\mathbb{N}$, $(P_{j_1},\dots,P_{j_n})\in\mathcal{P}^n$ the weak solution $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,0)}$ is collision-free (Theorem 2.5.1). Thus, we can simply follows the sketch already developed for point 1) with minor changes.

Proposition 3.6.11. *Let $\alpha\in(1,2)$. Let $\varepsilon\in(0,\bar\varepsilon)$. There exists $\bar\nu_3'(\varepsilon)$ such that for every $\nu'\in(-\bar\nu_3'(\varepsilon),\bar\nu_3'(\varepsilon))$, $n\in\mathbb{N}$ and $(P_{j_1},\dots,P_{j_n})\in\mathcal{P}^n$, the function $\gamma^{((P_{j_1},\dots,P_{j_n}),\varepsilon,\nu')}$ is collision-free.*

3.7 Proofs of the continuity lemmas

3.7.1 Proof of Continuity Lemma 3.6.1

Let u_0 be a minimizer of M_0 in $K_{P_j}^{\widetilde p_1\widetilde p_2}([0,1])$. We aim at proving that $M_0(\widetilde u)=M_0(u_0)$. We will briefly write L_m for $L_{\nu_m'}$ and M_m for $M_{\nu_m'}$. The following statement is a continuity property for the functionals $\{M_m\}$ in the set of the minimizers $\{u_m\}$.

Lemma 3.7.1. *The family* $\{M_m\}_m$ *tends to* M_0 *as* $m \to \infty$*, uniformly in the set* $\{u_m : m \in \mathbb{N}\}$*. This means that for every* $\lambda > 0$ *exists* $m_1 \in \mathbb{N}$ *such that*

$$m > m_1 \quad \Longrightarrow \quad |M_m(u_{\bar{m}}) - M_0(u_{\bar{m}})| \leq \lambda \quad \forall \bar{m} \in \mathbb{N}.$$

Proof. Let $\bar{m} \in \mathbb{N}$. For every m we have

$$|M_m(u_{\bar{m}}) - M_0(u_{\bar{m}})| \leq |\nu'_m| \int_0^1 |u_{\bar{m}}||\dot{u}_{\bar{m}}| \\ + \sqrt{2}\left(\int_0^1 |\dot{u}_{\bar{m}}|\right)^{\frac{1}{2}} \left| \left(\int_0^1 V_\varepsilon(u_{\bar{m}}) - 1 + \frac{(\nu'_m)^2}{2}|u_{\bar{m}}|^2\right)^{\frac{1}{2}} - \left(\int_0^1 V_\varepsilon(u_{\bar{m}}) - 1\right)^{\frac{1}{2}} \right|$$

Let $\varphi_{\bar{m}}(\nu) := \left(\int_0^1 V_\varepsilon(u_{\bar{m}}) - 1 + \frac{(\nu^2)}{2}|u_{\bar{m}}|^2\right)^{1/2}$. It results

$$|\varphi_{\bar{m}}(\nu'_m) - \varphi_{\bar{m}}(0)| \leq \frac{1}{2}\left(\int_0^1 V_\varepsilon(u_{\bar{m}}) - 1\right)^{-\frac{1}{2}} \int_0^1 |u_{\bar{m}}|^2 (\nu'_m)^2 \leq \frac{R^2}{2\sqrt{M_1}}(\nu'_m)^2,$$

so that

$$|M_m(u_{\bar{m}}) - M_0(u_{\bar{m}})| \leq R\|\dot{u}_{\bar{m}}\|_2 |\nu'_m| + \frac{R^2}{\sqrt{2M_1}}\|\dot{u}_{\bar{m}}\|_2 (\nu'_m)^2 \leq C(|\nu'_m| + (\nu'_m)^2),$$

where C is a constant independent on $\bar{m}$ (see Proposition 3.4.15). □

We want to compare $M_m(u_m)$ with $M_m(u_0)$. Because of the minimality property of u_m it seems reasonable to think that $M_m(u_m) \leq M_m(u_0)$. This is not immediate, and not necessarily true, since u_m is a minimizer of M_m for the fixed ends problem $\min\{M_m(u) : u \in K_{P_j}^{p_1^m p_2^m}([0,1])\}$, while u_0 connects $\widetilde{p}_1$ and $\widetilde{p}_2$. However, the fact that $p_1^m \to \widetilde{p}_1$ and $p_2^m \to \widetilde{p}_2$ suggests that maybe we can prove something similar (which in fact will be equation (3.32)). For every $p_*, p_{**} \in \partial B_R(0)$ we consider again the function $\zeta_R(\cdot\,; p_*, p_{**})$ which parametrizes the shorter arc of $\partial B_R(0)$ connecting p_* and p_{**} in time 1 with constant angular velocity. It is easy to check that

$$\forall \lambda > 0\ \exists \rho > 0 :\ |p_* - p_{**}| < \rho \quad \Longrightarrow \quad M_0(\zeta_R(\cdot\,; p_*, p_{**})) < \lambda,$$

so that

$$\forall \lambda > 0\ \exists m_2 \in \mathbb{N} :\ m > m_2 \quad \Longrightarrow \quad \begin{cases} M_0(\zeta_R(t; p_1^m, \widetilde{p}_1)) < \lambda \\ M_0(\zeta_R(t; \widetilde{p}_2, p_2^m)) < \lambda. \end{cases} \tag{3.30}$$

Furthermore, the following continuity property holds true.

Lemma 3.7.2. *The family $\{M_m\}_m$ tends to M_0 as $m \to \infty$, uniformly in the set $\{\zeta_R(\cdot\,; p_*, p_{**}) :\ p_*, p_{**} \in \partial B_R(0)\}$. This means that for every $\lambda > 0$ exists $m_3 \in \mathbb{N}$ such that*

$$m > m_3 \quad \Longrightarrow \quad |M_m(\zeta_R(\cdot\,; p_*, p_{**})) - M_0(\zeta_R(\cdot\,; p_*, p_{**}))| \leq \lambda \quad \forall p_*, p_{**} \in \partial B_R(0).$$

Proof. We can adapt the proof of Lemma 3.7.1 with minor changes. □

Conclusion of the proof of the Continuity Lemma 3.6.1. Because of the minimality of u_0 and the weak lower semi-continuity of M_0 it results

$$M_0(u_0) \leq M_0(\widetilde{u}) \leq \liminf_{m\to\infty} M_0(u_m). \tag{3.31}$$

For every $m \in \mathbb{N} \cup \{0\}$ we have

$$\frac{\omega_m^2}{2}|\dot{u}_m|^2 - \Phi_{\nu'_m,\varepsilon}(u_m) = -1 \quad \text{a.e. in } [0,1] \qquad \Longrightarrow \qquad \sqrt{2}L_m(u_m) = M_m(u_m),$$

where $\omega_m = \omega_{P_j}(p_1^m, p_2^m; \varepsilon, \nu'_m)$. The variational characterization of u_m implies that

$$\begin{aligned} M_m(u_m) = \sqrt{2}L_m(u_m) &\leq \sqrt{2}L_m(\zeta_R(\cdot\,; p_1^m, \widetilde{p}_1)) + \sqrt{2}L_m(u_0) + \sqrt{2}L_m(\zeta_R(\cdot\,; \widetilde{p}_2, p_2^m)) \\ &\leq M_m(\zeta_R(\cdot\,; p_1^m, \widetilde{p}_1)) + M_m(u_0) + M_m(\zeta_R(\cdot\,; \widetilde{p}_2, p_2^m)). \end{aligned} \tag{3.32}$$

We passed to the functional L_m in order to exploit its additivity property, which does not hold for M_m. Lemmas 3.7.1, 3.7.2 and equation (3.30) imply that for every $\lambda > 0$ if $m > \max\{m_1, m_2, m_3\}$ then

$$\begin{cases} M_m(u_m) > M_0(u_m) - \lambda \\ M_m(\zeta_R(\cdot\,; p_1^m, \widetilde{p}_1)) < M_0(\zeta_R(\cdot\,; p_1^m, \widetilde{p}_1)) + \lambda < 2\lambda \\ M_m(\zeta_R(\cdot\,; \widetilde{p}_2, p_2^m)) < M_0(\zeta_R(\cdot\,; \widetilde{p}_2, p_2^m)) + \lambda < 2\lambda \\ M_m(u_0) < M_0(u_0) + \lambda. \end{cases}$$

Hence, from equation (3.32) we deduce that, given $\lambda > 0$, if $m > \max\{m_1, m_2, m_3\}$, then

$$M_0(u_m) - \lambda \leq M_0(u_0) + 5\lambda \quad \Longrightarrow \quad \limsup_{m\to\infty} M_0(u_m) \leq M_0(u_0).$$

This, together with (3.31), says that the sequence $(M_0(u_m))_m$ has a limit and $M_0(u_0) = M_0(\widetilde{u}) = \lim_m M_0(u_m)$; in particular $\widetilde{u}$ is a minimizer of M_0 in $K_{P_j}^{\widetilde{p}_1\widetilde{p}_2}([0,1])$. □

3.7.2 Proof of Continuity Lemma 3.6.9

Let $\sigma_0 = \sigma^{((\widetilde{p}_1,\widetilde{p}_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,0)} = \sigma^{((\widetilde{p}_1,\widetilde{p}_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,0)}_{(\widehat{p}_2,\ldots,\widehat{p}_9)}$, where $(\widehat{p}_2,\ldots,\widehat{p}_9)$ is a minimizer of $\mathfrak{F}_{((\widetilde{p}_1,\widetilde{p}_{10}),(P_{k_1},\ldots,P_{k_5}),\varepsilon,0)}$, and let $v_0(t) = \sigma_0(T(\sigma_0 t))$. We aim at proving that $M_0(\widetilde{v}) = M_0(v_0)$. We need two intermediate results. The first one is a generalization of Lemma 3.7.1 for the glued functions.

Lemma 3.7.3. *Let $(v_m) \subset \mathcal{GF}_\varepsilon$, where each v_m is a glued function defined by* (3.28)*. The family $\{M_m\}_m$ tends to M_0 for $m \to \infty$, uniformly in $\{v_m\}_m$. This means that for every $\lambda > 0$ exists $m_1 \in \mathbb{N}$ such that*

$$m > m_1 \quad \Longrightarrow \quad |M_m(v_{\bar m}) - M_0(v_{\bar m})| < \lambda \qquad \forall \bar m.$$

Proof. We can adapt the proof of Lemma 3.7.3; the only difference is that we used the uniform bounds

$$\|u\|_2 \le R \qquad \|\dot u\|_2 \le C \qquad \int_0^1 V_\varepsilon(u) - 1 \ge M_1 \qquad \forall u \in \mathcal{IM}_\varepsilon.$$

Now we are considering glued functions, so we need similar properties for the function of $\mathcal{GF}_\varepsilon$. We have already noticed that there is $C > 0$ such that $\|\dot v_{\bar m}\|_{H^1} \le C$ for every $\bar m$; furthermore,

$$\int_0^1 V_\varepsilon(v_{\bar m}) - 1 \ge \frac{1}{T(\sigma_{\bar m})} \sum_{j=1}^{4} \int_0^{T_{2j+1}} (V_\varepsilon(y_{2j+1}) - 1) \ge \frac{4M_1}{C}. \qquad \square$$

In what follows we will show that the "length" $L_{\nu'}$ of an outer solution is a continuous function of the parameter ν' in $\nu' = 0$.

Lemma 3.7.4. *Let $p_{2j}, p_{2j+1} \in \partial B_R(0)$ be such that $|p_{2j} - p_{2j+1}| \le \delta$, let $(\nu'_m) \subset (-\bar\nu', \bar\nu')$ be such that $\nu'_m \to 0$ as $m \to \infty$. For every $\lambda > 0$ there is $m_4 = m_4(p_{2j}, p_{2j+1}) \in \mathbb{N}$ such that*

$$|L_{\nu'_{\bar m}}(y_{ext}(\cdot\,; p_{2j}, p_{2j+1}; \varepsilon, \nu'_m)) - L_{\nu'_{\bar m}}(y_{ext}(\cdot\,; p_{2j}, p_{2j+1}; \varepsilon, 0))| < \lambda$$

for every $\bar m' \in \mathbb{N}$.

Proof. We will write y_m instead of $y_{\text{ext}}(\cdot\,; p_{2j}, p_{2j+1}; \varepsilon, \nu'_m)$ and $L_{\bar m}$ instead of $L_{\nu'_{\bar m}}$ to ease the notation. Let T_m be such that $y_m(T_m) = p_{2j+1}$.

$$\begin{aligned} &|L_{\bar m}(y_m) - L_{\bar m}(y_0)| \\ &\quad \le \left| \int_0^{T_m} \sqrt{\Phi_{\nu'_{\bar m},\varepsilon}(y_m(t)) - 1}\,|\dot y_m(t)|\,dt - \int_0^{T_0} \sqrt{\Phi_{\nu'_{\bar m},\varepsilon}(y_0(t)) - 1}\,|\dot y_0(t)|\,dt \right| \\ &\qquad + \left| \int_0^{T_m} \langle i y_m(t), \dot y_m(t)\rangle\,dt - \int_0^{T_0} \langle i y_0(t), \dot y_0(t)\rangle\,dt \right|. \end{aligned} \tag{3.33}$$

We have already observed (Remark 3.4.12) that $\int_0^1 \langle iu, \dot{u}\rangle$ is continuous in the weak topology of H^1. We know that $y_m \to y_0$ $\mathcal{C}^1$-uniformly; it is not difficult to check that consequently

$$y_m(T_m t) \to y_0(T_0 t) \quad \mathcal{C}^1\text{-uniformly in } [0,1], \tag{3.34}$$

so that the second term in the right hand side of (3.33) tends to 0 as $m \to \infty$ (independently on $\bar{m}$). As far as the first term in the right hand side of (3.33) is concerned, it results

$$\begin{aligned}
&\left| \int_0^{T_m} \sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_m(t)) - 1}|\dot{y}_m(t)|\,dt - \int_0^{T_0} \sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_0(t)) - 1}|\dot{y}_0(t)|\,dt \right| \\
&\quad = \left| \int_0^1 \left(\sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_m(T_m t)) - 1}|\dot{y}_m(T_m t)| - \sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_0(T_0 t)) - 1}|\dot{y}_0(T_0 t)| \right) dt \right| \\
&\qquad \le \int_0^1 \left| \sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_m(T_m t)) - 1} - \sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_0(T_0 t)) - 1} \right| |\dot{y}_m(T_m t)|\,dt \\
&\qquad\qquad + \int_0^1 \sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_0(T_0 t)) - 1}\,\big||\dot{y}_0(T_0 t)| - |\dot{y}_m(T_m t)|\big|\,dt.
\end{aligned} \tag{3.35}$$

It is well known that the function $\sqrt{\cdot}$ is 1/2-Hölder continuous, so that for every $\bar{m}$

$$\begin{aligned}
&\int_0^1 |\sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_m(T_m t)) - 1} - \sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_0(T_0 t)) - 1}||\dot{y}_m(T_m t)|\,dt \\
&\quad \le \left(\int_0^1 \left| \sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_m(T_m t)) - 1} - \sqrt{\Phi_{\nu'_{\bar{m}},\varepsilon}(y_0(T_0 t)) - 1} \right|^2 dt \right)^{\frac{1}{2}} \|\dot{y}_m(T_m \cdot)\|_2 \\
&\qquad \le C \left(\int_0^1 |\Phi_{\nu'_{\bar{m}},\varepsilon}(y_m(T_m t)) - \Phi_{\nu'_{\bar{m}},\varepsilon}(y_0(T_0 t))|\,dt \right)^{\frac{1}{2}};
\end{aligned} \tag{3.36}$$

In the last inequality, we took advantage of the uniform bound for the L^2 norm of outer solutions, as usual. Both y_m and y_0 are outer solutions, therefore we can exploit the fact that V_ε is $\mathcal{C}^\infty$ with bounded derivatives outside $\partial B_R(0)$; using also (3.34) and the first estimate (3.13), we obtain

$$\begin{aligned}
&\sup_{t\in[0,1]} |\Phi_{\nu'_{\bar{m}},\varepsilon}(y_m(T_m t)) - \Phi_{\nu'_{\bar{m}},\varepsilon}(y_0(T_0 t))| \\
&\qquad \le (C + |\nu'_{\bar{m}}|^2) \sup_{t\in[0,1]} |y_m(T_m t) - y_0(T_0 t)| \to 0
\end{aligned} \tag{3.37}$$

as $m \to \infty$, independently on $\bar{m}$ (recall that $|\nu'_{\bar{m}}| \le \bar{\nu}'$).

Furthermore, using again (3.34)

$$\int_0^1 \sqrt{\Phi_{\nu'_{\bar m},\varepsilon}(y_0(T_0t))-1}\,||\dot y_0(T_0t)|-|\dot y_m(T_mt)||\;dt$$

$$\le \||\dot y_m(T_m\cdot)|-|\dot y_0(T_0\cdot)|\|_2\left(\int_0^1 \Phi_{\nu'_{\bar m},\varepsilon}(y_o(T_0t))-1\,dt\right)^{\frac12}$$

$$=\left(C+2|\nu'_{\bar m}|^2R^2\right)^{\frac12}\|\dot y_m(T.)-\dot y_0(T_0\cdot)\|_{L^\infty}\to 0, \quad (3.38)$$

as $m\to\infty$, independently on $\bar m$. Collecting (3.36), (3.37), (3.38) and comparing with (3.35) we deduce

$$\lim_{m\to\infty}\left|\int_0^{T_m}\sqrt{\Phi_{\nu'_{\bar m},\varepsilon}(y_m(t))-1}|\dot y_m(t)|\,dt-\int_0^{T_0}\sqrt{\Phi_{\nu'_{\bar m},\varepsilon}(y_0(t))-1}|\dot y_0(t)|\,dt\right|=0$$

uniformly in $\bar m$, which gives the thesis. □

Conclusion of the proof of the Continuity Lemma 3.6.9. The conservation of the Jacobi constant holds true both for v_0 and $\widetilde v$ (recall that $\widetilde v\in\mathcal{GF}_\varepsilon$, as showed in Lemma 3.6.8); using this conservation property, the minimality of σ_0 and the weak lower semi-continuity of M_0, we have

$$M_0(v_0)=L_0(v_0)\le L_0(\widetilde v)=M_0(\widetilde v)\le\liminf_{m\to\infty}M_0(v_m). \quad (3.39)$$

In what follows we briefly write $M_m(\sigma_m)=M_m([0,T(\sigma_m)];\sigma_m)$, and pose $\widehat p_1:=\widetilde p_1$ and $\widehat p_{10}:=\widetilde p_{10}$. The minimality of $(p_2^m,\dots,p_9^m)$ for $\mathfrak{F}_{((p_1^m,p_{10}^m),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu'_m)}$ implies that

$$\begin{aligned}
M_m(\sigma_m)&=\sqrt2L_m(\sigma_m)\le\sqrt2L_m(\sigma^{((p_1^m,p_{10}^m),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu'_m)}_{(\widehat p_2,\dots,\widehat p_9)})\\
&\le\sqrt2\left(L_m(\sigma^{((\widehat p_1,\widehat p_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,\nu'_m)}_{(\widehat p_2,\dots,\widehat p_9)})+L_m(\zeta_R(\cdot\,;p_1^m,\widehat p_1))+L_m(\zeta_R(\cdot\,;\widehat p_{10},p_{10}^m))\right)\\
&\le\sqrt2\left(\sum_{j=0}^4L_m(y_{P_{k_{j+1}}}(\cdot\,;\widehat p_{2j+1},\widehat p_{2j+2};\varepsilon,\nu'_m))+\sum_{j=1}^4L_m(y_{\text{ext}}(\cdot\,;\widehat p_{2j},\widehat p_{2j+1};\varepsilon,\nu'_m))\right.\\
&\qquad\left.+L_m(\zeta_R(\cdot\,;p_1^m,\widehat p_1))+L_m(\zeta_R(\cdot\,;\widehat p_{10},p_{10}^m))\right)\\
&\le\sqrt2\left(\sum_{j=0}^4L_m(y_{P_{k_{j+1}}}(\cdot\,;\widehat p_{2j+1},\widehat p_{2j+2};\varepsilon,0))+\sum_{j=1}^4L_m(y_{\text{ext}}(\cdot\,;\widehat p_{2j},\widehat p_{2j+1};\varepsilon,\nu'_m))\right.\\
&\qquad\left.+L_m(\zeta_R(\cdot\,;p_1^m,\widehat p_1))+L_m(\zeta_R(\cdot\,;\widehat p_{10},p_{10}^m))\right). \quad (3.40)
\end{aligned}$$

In the last inequality we used the minimality of $y_{P_{k_{j+1}}}(\cdot\,;\widehat{p}_{2j},\widehat{p}_{2j+1};\varepsilon,\nu'_m)$. Now we collect the uniform estimates of equation (3.30), Lemmas 3.7.2, 3.7.3 and 3.7.4: for every $\lambda>0$ exists $m_5:=\max\{m_1,\dots,\max\{m_4(\widehat{p}_{2j},\widehat{p}_{2j+1}) : j=1,\dots,4\}\}$ such that

$$\begin{cases} M_m(v_m) > M_0(v_m)-\lambda \\ \sqrt{2}L_m(\zeta_R(\cdot\,;p_1^m,\widehat{p}_1)) \le M_m(\zeta_R(\cdot\,;p_1^m,\widehat{p}_1)) < 2\lambda \\ \sqrt{2}L_m(\zeta_R(\cdot\,;\widehat{p}_{10},p_{10}^m)) \le M_m(\zeta_R(\cdot\,;\widehat{p}_{10},p_{10}^m)) < 2\lambda \\ L_m(y_{\text{ext}}(\cdot\,;\widehat{p}_{2j},\widehat{p}_{2j+1};\varepsilon,\nu'_m)) < L_m(y_{\text{ext}}(\cdot\,;\widehat{p}_{2j},\widehat{p}_{2j+1};\varepsilon,0))+\lambda \\ M_m(v_0) < M_0(v_0)+\lambda \end{cases}$$

for every $m>m_5$. Therefore, for every $\lambda>0$ the chain of inequalities (3.40) gives

$$\begin{aligned} M_0(\sigma_m)-\lambda \le \sqrt{2}\Bigg(&\sum_{j=0}^{4} L_m(y_{P_{k_{j+1}}}(\cdot\,;\widehat{p}_{2j+1},\widehat{p}_{2j+2};\varepsilon,0)) \\ &+\sum_{j=1}^{4} L_m(y_{\text{ext}}(\cdot\,;\widehat{p}_{2j},\widehat{p}_{2j+1};\varepsilon,0))\Bigg)+(1+\sqrt{2})4\lambda \\ &=\sqrt{2}L_m(\sigma_{(\widehat{p}_2,\dots,\widehat{p}_9)}^{((\widehat{p}_1,\widehat{p}_{10}),(P_{k_1},\dots,P_{k_5}),\varepsilon,0)})+(1+\sqrt{2})4\lambda \\ &\le M_m(\sigma_0)+(1+\sqrt{2})4\lambda \end{aligned}$$

if $m>m_5$. With a change of variable, the previous inequality is equivalent to

$$M_0(v_m)-\lambda \le M_m(v_0)+C\lambda \quad\Longrightarrow\quad M_0(v_m)\le M_0(v_0)+(C+1)\lambda$$

if $m>m_5$; since λ has been arbitrarily chosen, it results $\limsup_m M_0(v_m)\le M_0(v_0)$; comparing with (3.39) we deduce that $M_0 v_0)=M_0(\widetilde{v})$, and the proof is complete. □

Chapter 4

Entire solutions with exponential growth for a nonlinear elliptic system modelling phase separation

4.1 Introduction and main results

In this chapter, which collects the results of [81], we investigate the existence of entire solutions with exponential growth for the following semi-linear elliptic system:

$$\begin{cases} -\Delta u = -uv^2 & \text{in } \mathbb{R}^N \\ -\Delta v = -u^2 v & \text{in } \mathbb{R}^N \\ u, v > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{4.1}$$

where $N \geq 2$.

Therein, (4.1) appears in the analysis of phase-separation phenomena for Bose-Einstein condensates with multiple states. In what follows, we give a brief overview concerning the physical background, and we refer to [5, 48, 50, 65, 72, 73, 74, 75, 86] and to the references therein for a complete discussion.

Wave-particle duality in quantum mechanics. At the end of the 19th century, light was thought to consist of waves of electromagnetic fields which propagated according to Maxwell equations, while matter was thought to consist of localized particles. This division was challenged when, in his 1905 paper on the photoelectric effect, Albert Einstein postulated that light was emitted and absorbed as localized packets, or "quanta".

Einstein postulate was confirmed experimentally by Robert Millikan and Arthur Compton over the next two decades; thus, it became apparent that light has both wave-like and particle-like properties. Louis De Broglie, in his 1924 PhD thesis, sought to expand this wave-particle duality to all particles: this idea has been formalized with the introduction of a wave-function $\psi = \psi(t, x)$ which rules the evolution of a particle along the time, in such a way that $|\psi(t, x)|^2$ gives the probability to find the considered particle at a given place x in a given time t. In 1926, Erwin Schrödinger published an equation describing how this wave-function should evolve - the matter wave equivalent of Maxwell equations - and used it to derive the energy spectrum of hydrogen.

Bose-Einstein condensates. Bosons constitute a family of particles which do not obey the Pauli exclusion principle, so that it is possible that many bosons are in the same quantum state. The Bose-Einstein condensation occurs when most of the particles of a gas of bosons occupy the lowest energy quantum state; this phenomenon, predicted by Bose and Einstein in the '20s, has been experimentally observed only at the end of the 20th century by cooling a dilute gas of bosons to temperatures very close to the absolute zero [5].

To describe the state of a Bose-Einstein condensate, Gross and Pitaevskii proposed to employ the *Hartree-Fock approximation* and the *pseudopotential interaction model.*

In the Hartree-Fock approximation the total wave function Ψ of the system of n bosons is taken as a product of single particle functions ψ:

$$\Psi(t, x_1, \dots, x_n) = \prod_{i=1}^{n} \psi(t, x_i),$$

where x_i is the coordinate of the i-th boson.

The pseudopotential model Hamiltonian of the system is

$$H(x) = \left(-\frac{\hbar^2}{2m}\Delta + V(x)\right) + \sum_{1 \le i < j \le n} \frac{4\pi\hbar^2}{m}\omega\delta(x_i - x_j),$$

where $\hbar$ is the reduced Plank constant, m is the atom mass of the boson, V is the external potential, δ denotes the Dirac delta distribution, and ω is the scattering length relating different particles; if $\omega > 0$, then the interaction between two different bosons is attractive, while if $\omega < 0$, it is repulsive.

It turns out that if the single particle wave-function ψ solves the so-called Gross-Pitaevskii equation (which we report up to multiplicative positive constants)

$$i\hbar\partial_t\psi = \left(-\Delta + V(x) - \omega|\psi|^2\right)\psi,$$

and satisfies a suitable normalization, then the total wave-function Ψ minimizes the expectation value of the model Hamiltonian.

Recently, the condensation has been observed also in mixture of gas [65, 75], and in this case the model has to be modified in the following system of Schrödinger equations (again, up to multiplicative positive constants):

$$\begin{cases} i\partial_t \psi_i = \left(-\Delta + V(x) - \omega_i |\psi_i|^2 - \sum_{j\neq i} \beta_{ij} |\psi_j|^2\right)\psi_i \\ \psi_i \in H^1_0(\Omega;\mathbb{C}) \text{ for every } t>0,\, i=1,\dots,k. \end{cases}$$

Here ω_i and β_{ij} are the intraspecies and interspecies scattering length, respectively. The sign of ω_i describes the interaction between particles of the same condensate: $\omega_i > 0$ means attractive interaction, while $\omega_i < 0$ means repulsive interaction. Analogously, β_{ij} describes the interaction between particles of two different condensates. Concerning the boundary conditions, it is coherent with the model.

In what follows, we are interested in the so-called focusing case $\omega_i > 0$ (we point out that also the defocusing one $\omega_i < 0$ has been studied, we refer e.g. to [20]), and, only for the sake of simplicity, we pose $k = 2$. When looking for solutions of the form $\psi_1(x,t) = e^{-i\lambda_1 t} u_\beta(x)$, $\psi_2(x,t) = e^{-i\lambda_2 t} v_\beta(x)$, to which one usually refers as *solitons*, one finds the following equations for the densities u_β, v_β:

$$\begin{cases} -\Delta u_\beta + (\lambda_1 + V(x))u_\beta = \omega_1 u_\beta^2 u_\beta + \beta u_\beta v_\beta^2 \\ -\Delta v_\beta + (\lambda_2 + V(x))v_\beta = \omega_2 v_\beta^2 v_\beta + \beta u_\beta^2 v_\beta \\ u_\beta, v_\beta \in H^1_0(\Omega). \end{cases} \tag{4.2}$$

Starting from the pioneering paper [57], this system has been intensively studied, and by now several results concerning existence, multiplicity, and qualitative properties of the solutions are available (see e.g. [2, 6, 7, 58, 77, 85] and the references therein). In what follows, we only make use of the fact that for every $\beta < 0$ there exists a solution (u_β, v_β) which is positive, in the sense that both $u_\beta > 0$ and $v_\beta > 0$ in Ω.

Phase-separation for Bose-Einstein condensates with multiple states. In recent experiments regarding multiple condensates, it has been observed the occurrence of phase-separation phenomena [50, 72]; that is, it has been observed that when the interspecies scattering length β is negative and becomes larger and larger in absolute value (this condition can be realized from a physical point of view, as explained in [72]), the wave-functions of different condensates tend to assume disjoint supports: $u_\beta v_\beta \equiv 0$ as $\beta \to -\infty$ in Ω. From a mathematical point of view, the idea is that, as β is negative and $|\beta|$ becomes larger and larger, the competition between u_β and v_β becomes stronger and stronger, so that it seems reasonable to think that the densities concentrate in different sub-domains of Ω.

The occurrence of phase-separation has been rigorously proved in [24, 25, 67, 91]: under some additional assumptions, if $\{(u_\beta, v_\beta) : \beta \ll -1\}$ is a family of *positive solu-*

tions to (4.2), then up to a subsequence it is convergent to some limiting profile (u, v), which is a solution to

$$\begin{cases} -\Delta u + \lambda_1 u = \omega_1 u^3 & \text{in } \Omega_u \\ -\Delta v + \lambda_2 v = \omega_2 v^3 & \text{in } \Omega_v \\ uv \equiv 0 & \text{in } \Omega, \end{cases}$$

where $\Omega_u := \{x \in \Omega : u > 0\}$ and $\Omega_v := \{x \in \Omega : v > 0\}$ are positivity domains composed of a finite number of disjoint connected components with positive Lebesgue measure. At this point several questions naturally arise, such as the regularity of the limiting profile and of the free boundary, as well as uniform bounds in suitable functional spaces for the family $\{(u_\beta, v_\beta) : \beta \ll -1\}$. To answer these questions, one is induced to perform a blow-up analysis on the interface between u_β and v_β. Let us consider points

$$x_\beta \in \Omega \text{ such that } u_\beta(x_\beta) = v_\beta(x_\beta) =: m_\beta;$$

we wish to scale the equations for u_β and v_β around such points in order to deduce some information about the limit configuration. At least in dimension $N = 1$, Berestycki, Lin, Wei and Zhao in [12] showed that

$$m_\beta^4 \beta \to C \in (0, \infty) \qquad \text{as } \beta \to \infty;$$

the knowledge of the asymptotic of m_β permits to understand the correct scaling rate of the equation: letting

$$\hat{u}_\beta(x) := \frac{1}{m_\beta} u_\beta(m_\beta x + x_\beta), \quad \hat{v}_\beta(x) := \frac{1}{m_\beta} v_\beta(m_\beta x + x_\beta)$$

there is accumulation of the sequence, in $\mathcal{C}^2_{\text{loc}}(\mathbb{R})$, to a positive solution of

$$\begin{cases} u'' = uv^2 & \text{in } \mathbb{R} \\ v'' = u^2 v & \text{in } \mathbb{R}. \end{cases}$$

This is exactly system (4.1) in case $N = 1$.

For higher dimensions, the behaviour of m_β as $\beta \to +\infty$ has not been rigorously determined yet, but it is conjectured by the authors of [12] that the same asymptotic for m_β should hold. Under this assumption, it is possible to show that limits of the same scaling converge to a solution of (4.1). Therefore, to understand the geometry of the solutions of this system is clarifying about the behaviour of the segregation.

As a final remark, we observe that from a physical point of view also sign-changing solutions of (4.2) are interested, but, so far, the analysis of the phase-separation has been leaded only for positive solutions. This is the motivation which induced us to concentrate on $u, v > 0$ in (4.1).

Entire solutions of (4.1)**.** In order to better motivate our interest in the existence of solutions with super-algebraic growth, and to understand the main difficulties that one has to face when dealing with (4.1), we review the known results concerning the existence of positive solutions of the considered problem.

In [67], Noris, Tavares, Terracini and Verzini proved that if (u, v) is a nonnegative solution of (4.1), and both u and v are globally α-Hölder continuous (for some $\alpha \in (0,1)$), then one between u and v has to be identically 0, and the other has to be constant. As a consequence, it is possible to show that there are no solution to (4.1) (thus, strictly positive in each component) satisfying the sub-linear growth condition

$$u(x) + v(x) \leq C(1 + |x|^\alpha) \qquad \forall x \in \mathbb{R}^N,$$

for some $\alpha \in (0,1)$.

On the other hand, Berestycki, Lin, Wei and Zhao [12] proved the existence of a positive solution for system (4.1) with linear growth, when $N = 1$: there exists $C > 0$ such that

$$u(t) + v(t) \leq C(1 + |t|) \qquad \forall t \in \mathbb{R};$$

this solution is reflectionally symmetric with respect to a certain $t_0 \in \mathbb{R}$, in the sense that

$$u(t_0 + t) = v(t_0 - t) \qquad \forall t \in \mathbb{R},$$

and the following monotonicity condition holds true:

$$\text{either } u' > 0 \text{ and } v' < 0 \text{ in } \mathbb{R}, \text{ or } u' < 0 \text{ and } v' > 0 \text{ in } \mathbb{R}.$$

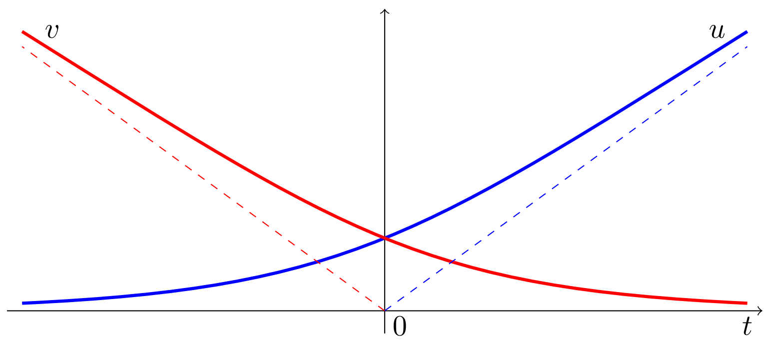
The monotonicity insures that there exist the limits of u and v as $t \to \pm\infty$. Assuming that $u' > 0$ and $v' < 0$ in $\mathbb{R}$, it results

$$\lim_{t \to -\infty} u(t) = 0 \quad \text{and} \quad \lim_{t \to +\infty} u(t) = +\infty$$
$$\lim_{t \to -\infty} v(t) = +\infty \quad \text{and} \quad \lim_{t \to +\infty} v(t) = 0.$$

In a successive paper, Berestycki, Terracini, Wang and Wei [13] proved the uniqueness of the positive solution for the 1-dimensional problem, up to rotations, translations, scaling of type

$$(u(x), v(x)) \mapsto (\lambda u(\lambda x), \lambda v(\lambda x)),$$

and exchange of the components (we explicitly remark that system (4.1) is invariant under these transformations). The following picture represents the qualitative graph of the 1-dimensional profile which is symmetric with respect to $t_0 = 0$.



Despite the lack of solutions with sub-linear growth, and the uniqueness of the solution with linear growth, system (4.1) has a large number of "geometrically distinct" entire solutions; saying that two solutions (u_1, v_1) and (u_2, v_2) are geometrically distinct, we mean that one cannot obtain (u_1, v_1) by (u_2, v_2) through a rotation, a translation, a scaling or an exchange of the components. Concerning the terminology, we also say that a solution (u, v) to (4.1) has algebraic growth if there exist $p \geq 1$ and $C > 0$ such that

$$u(x) + v(x) \leq C(1 + |x|^p) \qquad \forall x \in \mathbb{R}^N.$$

In [13], the authors constructed entire solutions with arbitrary integer algebraic growth in the plane $\mathbb{R}^2$, which are not 1-dimensional. To be precise, for any $d \in \mathbb{N} \setminus \{0\}$, they proved the existence of a solution (u, v) such that

$$\lim_{r \to \infty} \frac{1}{r^{2d+N-1}} \int_{\partial B_r(0)} u^2 + v^2 \in (0, +\infty), \tag{4.3}$$

where $N = 2$. The quantity

$$\frac{1}{r^{N-1}} \int_{\partial B_r(0)} u^2 + v^2$$

is the square of the quadratic mean of the pair (u, v) on the sphere $\partial B_r(0)$; hence, equation (4.3) says that the solution (u, v) grows (in quadratic mean) like $|x|^d$, as $|x| \to +\infty$.

We point out that all these results concern algebraically growing solutions, and this growth condition plays a crucial role in their proofs.

As a second remark, we underline that all the above results admit a counterpart for the k component system

$$\begin{cases} -\Delta u_i = -u_i \sum_{j \neq i} u_j^2 & \text{in } \mathbb{R}^N \\ u_i > 0 & \text{in } \mathbb{R}^N; \end{cases}$$

we refer again to [13] (see the forthcoming Theorem 4.5.1 for the statement of the main result).

Finally, to complete this bibliographic introduction, we mention the fact that, motivated by some considerations in [12], great efforts have been devoted in proving some De Giorgi-type conjectures for solutions of (4.1); saying "De Giorgi-type conjectures", we mean the research of "reasonable" assumptions which ensure the 1-dimensional symmetry of a positive solution of system (4.1) in $\mathbb{R}^N$. We refer to Chapter 6 and to the references therein for more details.

The aforementioned results lead quite naturally to the following question: do there exist solutions of (4.1) having super-algebraic growth? We can answer positively to this question, proving the existence of different types of solutions with exponential growth for the considered system in $\mathbb{R}^2$ (thus in $\mathbb{R}^N$ for every $N \geq 2$).

In our construction we adapt the same line of reasoning used in the proof of Theorem 1.3 of [13], which we briefly describe. Firstly, for a fixed $d \in \mathbb{N}$ and for any $R \geq 1$, the authors proved the existence of a solution (u_R, v_R) to (4.1) in the ball $B_R(0) \subset \mathbb{R}^2$, with boundary conditions

$$u = (\mathfrak{Re}(z^d))^+ \quad v = (\mathfrak{Re}(z^d))^- \qquad \text{on } \partial B_R(0),$$

and with the same symmetries of the pair $(\mathfrak{Re}(z^d)^+, \mathfrak{Re}(z^d)^-)$. In a second time, they passed to the limit as $R \to +\infty$; by means of some monotonicity formulae, and strongly exploiting the symmetries of (u_R, v_R), they showed that $\{(u_R, v_R)\}$ converges (up to a subsequence) to a nontrivial entire solution (u, v), which inherits by $(\mathfrak{Re}(z^d)^+, \mathfrak{Re}(z^d)^-)$ the symmetries and the asymptotic rate of growth. In light of this constructive method, we say that (u, v) *is modelled on* the harmonic function $\mathfrak{Re}(z^d)$.

We explicitly remark that the choice of a harmonic boundary condition is not accidental: indeed, there is a deep relationship between solutions of systems of type (4.1) and harmonic functions. Roughly speaking, if we consider a sequence $\{(u_\beta, v_\beta)\}$, where

$$\begin{cases} -\Delta u_\beta = -\beta u_\beta v_\beta^2 \\ -\Delta v_\beta = -\beta u_\beta^2 v_\beta, \end{cases}$$

and we pass to the limit as $\beta \to +\infty$, then the pair (u_β, v_β) converges (under suitable additional assumptions) to a segregated profile (u_∞, v_∞) such that $u_\infty - v_\infty$ is harmonic (cf. the forthcoming Theorem 6.2.3). It is not difficult to see that a sequence $\{(u_\beta, v_\beta)\}$ of the previous type can be obtained after a suitable scaling starting from a solution of (4.1) (cf. Theorem 6.2.13). In light of the procedure described above, the existence theorem of algebraically growing solutions in [13] can be seen as a counterpart of these convergence results.

Here, having in mind the construction of solutions with exponential growth, we start by considering the harmonic function

$$\Phi(x, y) := \cosh x \sin y.$$

The first of our main results is the following.

Theorem 4.1.1. *There exists an entire solution* $(u,v) \in (\mathcal{C}^\infty(\mathbb{R}^2))^2$ *to system* (4.1) *such that*

1) $u(x,y+2\pi) = u(x,y)$ *and* $v(x,y+2\pi) = v(x,y)$*;*

2) $u(-x,y) = u(x,y)$ *and* $v(-x,y) = v(x,y)$*;*

3) the symmetries

$$v(x,y) = u(x,y-\pi) \qquad u(x,\pi-y) = v(x,\pi+y)$$
$$u\left(x,\frac{\pi}{2}+y\right) = u\left(x,\frac{\pi}{2}-y\right) \qquad v\left(x,\frac{3}{2}\pi+y\right) = v\left(x,\frac{3}{2}\pi-y\right)$$

hold;

4) $u - v > 0$ *in* $\{\Phi > 0\}$ *and* $v - u > 0$ *in* $\{\Phi < 0\}$*;*

5) $u > \Phi^+$ *and* $v > \Phi^-$ *in* $\mathbb{R}^2$*;*

*6) the function (*Almgren quotient*)*

$$r \mapsto \frac{\int_{(0,r)\times(0,2\pi)} |\nabla u|^2 + |\nabla v|^2 + 2u^2v^2}{\int_{\{r\}\times[0,2\pi]} u^2 + v^2}$$

is nondecreasing, and

$$\lim_{r\to+\infty} \frac{\int_{(0,r)\times(0,2\pi)} |\nabla u|^2 + |\nabla v|^2 + 2u^2v^2}{\int_{\{r\}\times[0,2\pi]} u^2 + v^2} = 1;$$

7) there exists the limit

$$\lim_{r\to+\infty} \frac{1}{e^{2r}} \int_{\{r\}\times[0,2\pi]} u^2 + v^2 =: \alpha \in (0,+\infty).$$

Remark 4.1.2. This solution *is modelled on* the harmonic function Φ, in the sense that it inherits the symmetries of (Φ^+, Φ^-) and has the same rate of growth of Φ.

Remark 4.1.3. Point 7) of the Theorem gives a lower and a upper bound on the rate of growth of the quadratic mean of (u,v) on $\{r\} \times [0,2\pi]$ when r varies:

$$\left(\int_{\{r\}\times[0,2\pi]} u^2 + v^2\right)^{\frac{1}{2}} = O(e^r) \qquad \text{as } r \to +\infty.$$

The domain of integration takes into account the periodicity of (u, v): the quadratic mean of (u, v) on $\{r\} \times [0, 2\pi]$ has exponential growth, and the rate of growth is the same of the function e^r, which in turn has the same rate of growth of Φ. Note that the coefficient 1 in the exponent of e^r coincides with the limit as $r \to +\infty$ of the Almgren quotient defined in point 6).

Remark 4.1.4. With a scaling argument, it is not difficult to prove the existence of entire solutions with exponential growth of order λ for every $\lambda > 0$ (in the previous sense). To see this, let

$$(u_\lambda(x, y), v_\lambda(x, y)) = (\lambda u(\lambda x, \lambda y), \lambda v(\lambda x, \lambda y))\,.$$

It is straightforward to check that (u_λ, v_λ) is still a solution to (4.1) in the plane, is $2\pi/\lambda$-periodic in y and is such that

$$u_\lambda(x, y) \geq \lambda\,(\cosh(\lambda x)\sin(\lambda y))^+ \quad \text{and} \quad v_\lambda(x, y) \geq \lambda\,(\cosh(\lambda x)\sin(\lambda y))^-\,.$$

Moreover,

$$\lim_{r\to+\infty} \frac{\int_{(0,r)\times\left(0,\frac{2\pi}{\lambda}\right)} |\nabla u_\lambda|^2 + |\nabla v_\lambda|^2 + 2u_\lambda^2 v_\lambda^2}{\int_{\{r\}\times\left[0,\frac{2\pi}{\lambda}\right]} u_\lambda^2 + v_\lambda^2} = \lambda, \tag{4.4}$$

and

$$\lim_{r\to+\infty} \frac{1}{e^{2\lambda r}} \int_{\{r\}\times\left[0,\frac{2\pi}{\lambda}\right]} u_\lambda^2 + v_\lambda^2 = \lambda\alpha.$$

The solution (u_λ, v_λ) is modelled on the harmonic function $\cosh(\lambda x)\sin(\lambda y)$. This reveals that there exists a correspondence

$$\{(u_\lambda, v_\lambda) : \lambda > 0\} \leftrightarrow \{\sin(\lambda x)\cosh(\lambda y) : \lambda > 0\}.$$

Due to the invariance under translations and rotations of problem (4.1), the family $\{(u_\lambda, v_\lambda) : \lambda > 0\}$ can equivalently be related with the families of harmonic functions $\{\cosh(\lambda x)\,[C_1\cos(\lambda y) + C_2\sin(\lambda y)]\}$ or $\{[C_3\cos(\lambda x) + C_4\sin(\lambda x)]\cosh(\lambda y) : \lambda > 0\}$, where $C_1, C_2, C_3, C_4 \in \mathbb{R}$.

As observed in Remark 4.1.3, the limit of the Almgren quotient in (4.4) describes the rate of the growth of the quadratic mean of (u_λ, v_λ) computed on an interval of periodicity in the y variable. The previous computation reveals that for every $\lambda > 0$ we can construct a solution having rate of growth equal to λ. This marks a relevant difference between entire solutions with polynomial growth and entire solutions with exponential growth: while in the former case the admissible rates of growth are quantized (Theorem 1.4 of [13], which we reported in Chapter 6, Theorem 6.2.13), in the latter one we can prescribe any positive real value as rate of growth.

Remark 4.1.4 reveals that, starting from the solution found in Theorem 4.1.1, we can build infinitely-many entire solutions with different exponential growth. However, noting that system (4.1) is invariant under rotations, translations and scaling, intuitively speaking they are all the same solution. We wonder if there exists an entire solution of (4.1) having exponential growth which cannot be obtained by the one found in Theorem 4.1.1 through a rotation, a translation or a scaling; the answer is affirmative. We denote

$$\Gamma(x,y) := e^x \sin y.$$

Theorem 4.1.5. *There exists an entire solution* $(u,v) \in (\mathcal{C}^\infty(\mathbb{R}^2))^2$ *to system* (4.1) *which satisfies points 1) and 3) of Theorem 4.1.1; moreover*

2) for every $r \in \mathbb{R}$

$$\int_{(-\infty,r)\times(0,2\pi)} |\nabla u|^2 + |\nabla v|^2 + u^2 v^2 < +\infty;$$

4) $u - v > 0$ *in* $\{\Gamma > 0\}$ *and* $v - u > 0$ *in* $\{\Gamma < 0\}$*;*

5) $u > \Gamma^+$ *and* $v > \Gamma^-$ *in* $\mathbb{R}^2$*;*

*6) the function (*Almgren quotient*)*

$$r \mapsto \frac{\int_{(-\infty,r)\times(0,2\pi)} |\nabla u|^2 + |\nabla v|^2 + 2u^2v^2}{\int_{\{r\}\times(0,2\pi)} u^2 + v^2}$$

is nondecreasing, and

$$\lim_{r\to+\infty} \frac{\int_{(-\infty,r)\times(0,2\pi)} |\nabla u|^2 + |\nabla v|^2 + 2u^2v^2}{\int_{\{r\}\times(0,2\pi)} u^2 + v^2} = 1;$$

7) there exist the limits

$$\lim_{r\to+\infty} \frac{1}{e^{2r}} \int_{\{r\}\times[0,2\pi]} u^2 + v^2 =: \beta \in (0,+\infty) \quad \textit{and} \quad \lim_{r\to-\infty} \int_{\{r\}\times[0,2\pi]} u^2 + v^2 = 0.$$

Remark 4.1.6. This solution is modelled on the harmonic function Γ. As explained in Remark 4.1.3, it is possible to obtain a family of entire solutions which is in correspondence with a family of harmonic functions.

Remark 4.1.7. Note that the Almgren quotients that we defined in Theorem 4.1.1 and 4.1.5 are different. They are both different to the Almgren quotient which has been defined in [13], and which will be considered in Chapter 6. This depends on the different type of geometry of solutions with algebraic growth, which are asymptotic to some *homogeneous* harmonic polynomial (in the sense specified by Theorem 6.2.13), and of the two types of solutions with exponential growth we constructed, which are completely non-homogeneous: indeed, in both the cases there exists one variable which carries the periodicity of the solution, and one variable which carries the information on the growth of the solution.

Remark 4.1.8. In [13], the authors formulated the following question (see Open Problem 4 in the quoted paper): "are there solutions to (4.1) such that the set $\{u = v\}$ contains disjoint multiple curves?" The solutions constructed in Theorems 4.1.1 and 4.1.5 permits to answer affirmatively.

We can generalize our existence result to the case of systems with many components. To be precise, given an integer k, we construct a solution $(u_1, \dots, u_k)$ of

$$\begin{cases} -\Delta u_i = -u_i \sum_{j \neq i} u_j^2 \\ u_i > 0, \end{cases} \qquad i = 1, \dots, k, \tag{4.5}$$

in the whole plane $\mathbb{R}^2$ having the same growth and the same symmetries of Γ, in the sense specified by the following statement. Here and in the rest of the chapter we consider the indexes $\mod k$.

Theorem 4.1.9. *There exists an entire solution $(u_1, \dots, u_k) \in (\mathcal{C}^\infty(\mathbb{R}^2))^k$ to system* (4.5) *such that, for every $i = 1, \dots, k$,*

1) $u_i(x, y + k\pi) = u_i(x, y)$;

2) the symmetries

$$u_{i+1}(x, y) = u_i(x, y - \pi) \qquad u_1\left(x, \frac{\pi}{2} + y\right) = u_1\left(x, \frac{\pi}{2} - y\right)$$

hold;

3) for every $r \in \mathbb{R}$

$$\int_{(-\infty, r) \times (0, k\pi)} \sum_{i=1}^k |\nabla u_i|^2 + \sum_{1 \leq i < j \leq k} u_i^2 u_j^2 < +\infty;$$

*4) the function (*Almgren quotient*)*

$$r \mapsto \frac{\int_{(-\infty,r)\times(0,k\pi)} \sum_{i=1}^k |\nabla u_i|^2 + 2\sum_{1\le i<j\le k} u_i^2 u_j^2}{\int_{\{r\}\times[0,k\pi]} \sum_{i=1}^k u_i^2}$$

is nondecreasing, and

$$\lim_{r\to+\infty} \frac{\int_{(-\infty,r)\times(0,k\pi)} \sum_{i=1}^k |\nabla u_i|^2 + 2\sum_{1\le i<j\le k} u_i^2 u_j^2}{\int_{\{r\}\times[0,k\pi]} \sum_{i=1}^k u_i^2} = 1;$$

5) there exist the limits

$$\lim_{r\to+\infty} \frac{1}{e^{2r}} \int_{\{r\}\times[0,k\pi]} \sum_{i=1}^k u_i^2 =: \gamma \in (0,+\infty) \quad \textit{and} \quad \lim_{r\to-\infty} \int_{\{r\}\times[0,k\pi]} \sum_{i=1}^k u_i^2 = 0.$$

This solution is modelled on Γ. It is also possible to obtain a solution of the component system (4.5) modelled on Φ; we describe the construction of such a solution in Remark 4.5.15.

Our last main result is the counterpart of Theorem 1.4 of [13] in our setting. This can be quite surprising because, as we already observed, we cannot expect a quantization of the admissible rates of growth dealing with solutions with exponential growth, see Remark 4.1.4. Nevertheless, if we consider solutions which are periodic in one direction, prescribing a common period such a quantization can be recovered.

Theorem 4.1.10. *Let (u,v) be a nontrivial solution of (4.1) in $\mathbb{R}^2$ which is 2π-periodic in y, and such that one of the following situation occurs:*

(i) it holds

$$\lim_{r\to-\infty} \int_{\{r\}\times[0,2\pi]} u^2+v^2 = 0,$$

and

$$d := \lim_{r\to+\infty} \frac{\int_{(-\infty,r)\times(0,2\pi)} |\nabla u|^2 + |\nabla v|^2 + u^2v^2}{\int_{\{r\}\times[0,2\pi]} u^2+v^2} < +\infty;$$

(ii) $\partial_x u = 0 = \partial_x v$ on $\{a\}\times[0,2\pi]$ for some $a\in\mathbb{R}$, and

$$d := \lim_{r\to+\infty} \frac{\int_{(a,r)\times(0,2\pi)} |\nabla u|^2 + |\nabla v|^2 + u^2v^2}{\int_{\{r\}\times[0,2\pi]} u^2+v^2} < +\infty;$$

then d is a positive integer, and

$$\left(\int_{\{r\}\times[0,2\pi]} u^2+v^2\right)^{\frac{1}{2}} = O(e^{dr}) \qquad \text{as } r\to+\infty;$$

moreover, up to a subsequence, the family $\{(u_R,v_R): R>0\}$, defined by

$$(u_R(x,y),v_R(x,y)) := \frac{1}{\left(\int_{\{R\}\times[0,2\pi]} u^2+v^2\right)^{\frac{1}{2}}}\left(u(x+R,y),v(x+R,y)\right),$$

converges in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^2)$ and in $H^1_{\mathrm{loc}}(\mathbb{R}^2)$, as $R\to+\infty$, to (Ψ^+,Ψ^-), where

$$\Psi(x,y) = e^{dx}\left(C_1\cos(dy)+C_2\sin(dy)\right)$$

for some $C_1, C_2 \in \mathbb{R}$.

Notation. We deal with functions defined in domains of type $(a,b)\times\mathbb{R}$, where $a<b$ are extended real numbers ($a=-\infty$ and $b=+\infty$ are admissible). We often assume that $(u_1,\dots,u_k)$ is $k\pi$-periodic in y; therefore, we can think to $(u_1,\dots,u_k)$ as defined on the cylinder

$$C_{(a,b)} := (a,b)\times\mathbb{S}_k \quad \text{where} \quad \mathbb{S}_k = \mathbb{R}/(k\pi\mathbb{Z}).$$

We also denote $\Sigma_r := \{r\}\times\mathbb{S}_k$. In case $b>0$, $a=-b$, we simply write C_b instead of $C_{(-b,b)}$, to simplify the notation.

Structure of the chapter. In Section 4.2, we prove some monotonicity formulae which we use in the rest of the chapter. We can deal with two types of solutions: solutions satisfying a homogeneous Neumann condition, defined in a cylinder $C_{(a,b)}$ with $a>-\infty$, or solutions defined in a semi-infinite cylinder of type $C_{(-\infty,b)}$ and decaying at $x\to-\infty$. For the sake of completeness, and having in mind to use some monotonicity formulae in the proof of Theorem 4.1.9, we always consider the case of systems with k components.

The proof of Theorem 4.1.1 is the object of Section 4.3. It follows the same sketch of the proof than Theorem 1.3 in [13]: we start by showing that for any $R>0$ there exists a solution (u_R,v_R) to (4.1) in the cylinder C_R, with Dirichlet boundary condition

$$u_R=\Phi^+ \quad \text{and} \quad v_R=\Phi^- \quad \text{on } \{-R,R\}\times[0,2\pi],$$

and exhibiting the same symmetries of (Φ^+,Φ^-). In order to obtain a solution defined in the whole C_∞, we wish to prove the $\mathcal{C}^2_{\mathrm{loc}}(C_\infty)$ convergence of the family $\{(u_R,v_R): R>1\}$, as $R\to+\infty$. To show that this convergence occurs, we exploit the monotonicity formulae proved in Subsection 4.2.1. With respect to Theorem 1.3 of [13], major difficulties arise in the precise characterization of the growth of (u, v), points 6) and 7) of Theorem 4.1.1.

In Section 4.4, we prove Theorem 4.1.5. One could be tempted to try to adapt the proof of Theorem 4.1.1 replacing Φ with Γ. Unfortunately, in such a situation we could not exploit the results of subsection 4.2.1; this is related to the lack of the even symmetry in the x variable of the function Γ (note that the function Φ enjoys this symmetry). A possible way to overcome this problem is to work in semi-infinite cylinders $C_{(-\infty,R)}$ and use the monotonicity formulae proved in subsection 4.2.2. But to work in an unbounded set introduces further complications: for instance, the compactness of the Sobolev embedding and of some trace operators, a property that we use many times in Section 4.3, does not hold in $C_{(-\infty,R)}$. Although we believe that this kind of obstacle can be overcome, we propose a different approach for the construction of solutions modelled on Γ, which is based on the elementary limit

$$\lim_{R\to+\infty} \Phi_R(x, y) = \Gamma(x, y) \qquad \forall (x, y) \in \mathbb{R}^2,$$

where $\Phi_R(x, y) = 2e^{-R} \cosh(x + R) \sin y$. We prove the existence of a solution (u_R, v_R) of (4.1) in $C_{(-3R,R)}$, with Dirichlet boundary condition

$$u_R = \Phi_R^+ \quad \text{and} \quad v_R = \Phi_R^- \quad \text{on } \{-3R, R\} \times [0, 2\pi],$$

and exhibiting the same symmetries of (Φ_R^+, Φ_R^-). Then, using again the results of Section 4.2, we pass to the limit as $R \to +\infty$, proving the compactness of $\{(u_R, v_R)\}$.

Section 4.5 is devoted to the study of systems with many components. As in [13] the authors could prove in one shot an existence theorem for 2 or k components (there are no substantial changes in the proofs), it is natural to wonder if here we can simply adapt step by step the construction carried on in Sections 4.3 and 4.4, or not. Unfortunately, the answer is negative: following the sketch of the proof of Theorem 4.1.1, we can adapt most the results of Sections 4.3 and 4.4 with minor changes, but with respect to the results of Subsections 4.3.2 and 4.4.2, we cannot show that the limit of the sequence $(u_{1,R}, \ldots, u_{k,R})$ does not vanish (this fact follows from a subtle technical point). This is why we have to use a completely different argument which is not based on the existence of solutions for the system of k components in bounded cylinders (or in semi-infinite cylinders), but rests on Theorem 1.6 of [13]. Roughly speaking, we will obtain the existence of a solution of (4.5) with exponential growth as a limit of solutions of the same system having algebraic growth. Roughly speaking, we translate the limit

$$\lim_{d\to+\infty} \mathfrak{Im} \left[\left(1 + \frac{z}{d}\right)^d \right] = e^x \sin y$$

in terms of solutions to (4.1): we consider a sequence of solutions to (4.1) with polynomial growth of order d, and, after suitable scaling, we show that it converges to a solution of (4.1) having exponential growth, that is, $\int_{\{r\}\times[0,k\pi]} \sum_{i=1}^k u_i^2 = O(e^{2r})$ as $r \to +\infty$.

The proof of Theorem 4.1.10 is the object of Section 4.6.

4.2 Almgren-type monotonicity formulae

Let $k \geq 2$ be a fixed integer. In this section we are going to prove some monotonicity formulae for solutions of

$$\begin{cases} -\Delta u_i = -u_i \sum_{j\neq i} u_j^2 \\ u_i > 0 \end{cases} \tag{4.6}$$

defined in a cylinder $C_{(a,b)}$ (this means that we assume from the beginning that the function $(u_1, \ldots, u_k)$ is $k\pi$-periodic in y).

In this section we use many times the following general result:

Lemma 4.2.1. *Let $(u_1, \ldots, u_k)$ be a solution of (4.5) in $C_{(a,b)}$. Then the function*

$$r \mapsto \int_{\Sigma_r} \sum_{i=1}^{k} |\nabla u_i|^2 + \sum_{1\leq i<j\leq k} u_i^2 u_j^2 - 2\int_{\Sigma_r} \sum_{i=1}^{k} (\partial_x u_i)^2$$

is constant in (a,b).

Proof. Let $a < r_1 < r_2 < b$. We test equation (4.6) with $(\partial_x u_1, \ldots, \partial_x u_k)$ in $C_{(r_1,r_2)}$: for every i it results

$$\int_{C_{(r_1,r_2)}} \frac{1}{2}\partial_x \left(|\nabla u_i|^2\right) + \left(\sum_{j\neq i} u_j^2\right) u_i \partial_x u_i = \int_{\Sigma_{r_2}} (\partial_x u_i)^2 - \int_{\Sigma_{r_1}} (\partial_x u_i)^2.$$

Summing for $i = 1, \ldots, k$ we obtain

$$\int_{C_{(r_1,r_2)}} \partial_x \left(\sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2\right) = 2\int_{\Sigma_{r_2}} \sum_i (\partial_x u_i)^2 - 2\int_{\Sigma_{r_1}} \sum_i (\partial_x u_i)^2,$$

which gives the thesis. □

4.2.1 Solutions with Neumann boundary conditions

In this subsection we are interested in solutions to (4.6) defined in $C_{(a,b)}$ (thus $k\pi$-periodic in y), with $a > -\infty$ and $b \in (a, +\infty]$, and satisfying a homogeneous Neumann boundary condition on Σ_a, that is,

$$\partial_x u_i = 0 \qquad \text{on } \Sigma_a, \text{ for every } i = 1, \ldots, k. \tag{4.7}$$

Firstly, we observed that under this assumption Lemma 4.2.1 implies

Lemma 4.2.2. *Let* $(u_1, \dots, u_k)$ *be a solution of* (4.6) *in* $C_{(a,b)}$, *such that* (4.7) *holds true. For every* $r \in (a, b)$ *the following identity holds:*

$$\int_{\Sigma_r} \sum_{i=1}^{k} |\nabla u_i|^2 + \sum_{1 \le i < j \le k} u_i^2 u_j^2 = 2 \int_{\Sigma_r} \sum_{i=1}^{k} (\partial_x u_i)^2 + \int_{\Sigma_a} \sum_{i=1}^{k} (\partial_y u_i)^2 + \sum_{1 \le i < j \le k} u_i^2 u_j^2.$$

For a solution $(u_1, \dots, u_k)$ of (4.6) in $C_{(a,b)}$ satisfying (4.7), we define

$$E^{sym}(r) := \int_{C_{(a,r)}} \sum_{i=1}^{k} |\nabla u_i|^2 + 2 \sum_{1 \le i < j \le k} u_i^2 u_j^2,$$

$$\mathcal{E}^{sym}(r) := \int_{C_{(a,r)}} \sum_{i=1}^{k} |\nabla u_i|^2 + \sum_{1 \le i < j \le k} u_i^2 u_j^2,$$

$$H(r) := \int_{\Sigma_r} \sum_{i=1}^{k} u_i^2$$

Remark 4.2.3. The index *sym* denotes the fact that, as we will see, the quantities E^{sym} and $\mathcal{E}^{sym}$ are well suited to describe the growth of the solution $(u_1, \dots, u_k)$ only if $(u_1, \dots, u_k)$ satisfies the (4.7), which can be considered as a symmetry condition. Indeed, under (4.7) one can extend $(u_1, \dots, u_k)$ on $C_{(2a-b,b)}$ by even symmetry in the x variable.

By regularity, E, $\mathcal{E}$ and H are smooth. A direct computation shows that they are nondecreasing functions: in particular

$$H'(r) = 2 \int_{\Sigma_r} \sum_i u_i \partial_\nu u_i = 2E(r), \tag{4.8}$$

where the last identity follows from the divergence theorem and the boundary conditions of $(u_1, \dots, u_k)$. Our next result consist in showing that also the ratio between E (or $\mathcal{E}$) and H is nondecreasing.

Proposition 4.2.4. *Let* $(u_1, \dots, u_k)$ *be a solution of* (4.6) *in* $C_{(a,b)}$ *such that* (4.7) *holds true. The* Almgren quotient

$$N^{sym}(r) := \frac{E^{sym}(r)}{H(r)}$$

is well defined and nondecreasing in (a, b). *Moreover*

$$\int_a^r \frac{\int_{\Sigma_s} \sum_{i<j} u_i^2 u_j^2}{H(s)} \, ds \le N(r).$$

Analogously, the function (which we will call Almgren quotient*, too)*

$$\mathfrak{N}^{sym}(r) := \frac{E^{sym}(r)}{H(r)}$$

is well defined and nondecreasing in (a,b)*, and*

$$\mathfrak{N}'(r) \geq 2\mathfrak{N}(r)\frac{\int_{C_{(a,r)}} \sum_{i<j} u_i^2 u_j^2}{H(r)} + 2\left(\frac{\int_{C_{(a,r)}} \sum_{i<j} u_i^2 u_j^2}{H(r)}\right)^2.$$

In the rest of this subsection we briefly write E, $\mathcal{E}$, N and $\mathfrak{N}$ instead of E^{sym}, $\mathcal{E}^{sym}$, N^{sym} and $\mathfrak{N}^{sym}$ to ease the notation.

Proof. Since $(u,v) \in H^1_{\text{loc}}(C_{(a,b)})$ is nontrivial, E and H are positive in (a,b) and bounded for r bounded. We compute, by means of Lemma 4.2.2

$$\begin{aligned} E'(r) &= \int_{\Sigma_r} \sum_i |\nabla u_i|^2 + 2\sum_{i<j} u_i^2 u_j^2 \\ &= \int_{\Sigma_r} 2\sum_i (\partial_x u_i)^2 + \sum_{i<j} u_i^2 u_j^2 + \int_{\Sigma_a} \sum_i (\partial_y u_i)^2 + \sum_{i<j} u_i^2 u_j^2. \end{aligned}$$

Note that $\partial_x u_i = \partial_\nu u_i$ on Σ_r. Using the previous identity and the (4.8) we are in position to compute the logarithmic derivative of N:

$$\begin{aligned} \frac{N'(r)}{N(r)} &= \frac{E'(r)}{E(r)} - \frac{H'(r)}{H(r)} \\ &= 2\frac{\int_{\Sigma_r} \sum_i (\partial_\nu u_i)^2}{\int_{\Sigma_r} \sum_i u\partial_\nu u_i} + \frac{\int_{\Sigma_a} \sum_i (\partial_y u_i)^2 + \sum_{i<j} u_i^2 u_j^2 + \int_{\Sigma_r} \sum_{i<j} u_i^2 u_j^2}{E(r)} \\ &\quad - 2\frac{\int_{\Sigma_r} \sum_i u\partial_\nu u_i}{\int_{\Sigma_r} \sum_i u_i^2} \\ &\geq 2\left(\frac{\int_{\Sigma_r} \sum_i (\partial_\nu u_i)^2}{\int_{\Sigma_r} \sum_i u\partial_\nu u_i} - \frac{\int_{\Sigma_r} \sum_i u\partial_\nu u_i}{\int_{\Sigma_r} \sum_i u_i^2}\right) + \frac{\int_{\Sigma_r} \sum_{i<j} u_i^2 u_j^2}{E(r)} \\ &\geq \frac{\int_{\Sigma_r} \sum_{i<j} u_i^2 u_j^2}{E(r)} \geq 0, \end{aligned}$$

where we used the Cauchy-Schwarz and the Young inequalities. As a consequence, N is nondecreasing in (a,b). Note also that

$$N'(r) \geq \frac{\int_{\Sigma_r} \sum_{i<j} u_i^2 u_j^2}{H(r)} \quad \Longrightarrow \quad N(r) \geq \int_a^r \frac{\int_{\Sigma_s} \sum_{i<j} u_i^2 u_j^2}{H(s)}\, ds$$

for every $r > a$. The same argument can be adapted with minor changes to prove the monotonicity of $\mathfrak{N}$. $\square$

As a first consequence, we have the following

Corollary 4.2.5. *Let $(u_1, \ldots, u_k)$ be a solution of (4.6) in $C_{(a,b)}$ such that (4.7) holds.*

(i) If $N(r) \geq \underline{d}$ for $r \geq s > a$, then

$$\frac{H(r_1)}{e^{2\underline{d}r_1}} \leq \frac{H(r_2)}{e^{2\underline{d}r_2}} \qquad \forall\ s \leq r_1 < r_2 < b,$$

ii) If $N(r) \leq \overline{d}$ for $r \leq t < b$, then

$$\frac{H(r_1)}{e^{2\overline{d}r_1}} \geq \frac{H(r_2)}{e^{2\overline{d}r_2}} \qquad \forall\ a < r_1 < r_2 \leq t.$$

Proof. We prove only (ii). Recalling that $H'(r) = 2E(r)$ (see (4.8)), we have

$$\frac{d}{dr} \log H(r) = 2N(r) \leq 2\overline{d} \qquad \forall r \in (a, t].$$

By integrating, the thesis follows. $\square$

The next step is to prove a similar monotonicity property for the function E. Our result rests on Theorem 5.6 of [13] (see also [12]), which we state here for the reader's convenience.

Theorem 4.2.6. *Let k be a fixed integer and let $\Lambda > 1$. Let*

$$\mathcal{L}(k, \Lambda) :=$$
$$\min \left\{ \int_0^{2\pi} \sum_i (f_i')^2 + \Lambda \sum_{i<j} f_i^2 f_j^2 \;\middle|\; \begin{array}{l} f_1, \ldots, f_k \in H^1([0, 2\pi]),\ \int_0^{2\pi} \sum_i f_i^2 = 1 \\ f_{i+1}(t) = f_i\left(t - \frac{2\pi}{k}\right),\ f_1(\pi + t) = f_1(\pi - t) \end{array} \right\},$$

where the indexes are counted $\mod k$. *There exists $C > 0$ such that*

$$\left(\frac{k}{2}\right)^2 - C\Lambda^{-1/4} \leq \mathcal{L}(k, \Lambda) \leq \left(\frac{k}{2}\right)^2.$$

Remark 4.2.7. Having in mind to apply Theorem 4.2.6 on 2π-periodic functions, we note that the condition $f_1(\pi + t) = f_1(\pi - t)$ can be replaced by $f_1(t + \tau) = f_1(\tau - t)$ for any $\tau \in [0, 2\pi)$.

For a fixed $r_0 \in (a,b)$, let us introduce

$$\varphi(r;r_0) := \int_{r_0}^{r} \frac{ds}{H(s)^{1/4}}.$$

The function φ is positive and increasing in r; thanks to point (i) of Corollary 4.2.5 and to the monotonicity of N, whenever (u,v) is nontrivial φ is bounded by a quantity depending only on $H(r_0)$ and $N(r_0)$. To be precise:

$$\varphi(r;r_0) \leq 2\frac{e^{\frac{1}{2}N(r_0)r_0}}{H(r_0)^{\frac{1}{4}}N(r_0)}\left[e^{-\frac{1}{2}N(r_0)r_0} - e^{-\frac{1}{2}N(r_0)r}\right]. \tag{4.9}$$

This, together with the monotonicity of $\varphi(\cdot;r_0)$, implies that if $b=+\infty$ then there exists the limit

$$\lim_{r\to+\infty} \varphi(r;r_0) < +\infty. \tag{4.10}$$

Lemma 4.2.8. *Let* $(u_1,\dots,u_k)$ *be a solution of* (4.1) *in* $C_{(a,b)}$ *such that* (4.7) *holds. Let* $r_0 \in (a,b)$*, and assume that*

$$u_{i+1}(x,y) = u_i(x,y-\pi) \quad \textit{and} \quad u_1(x,\tau+y) = u_1(x,\tau-y) \tag{4.11}$$

where $\tau \in [0,k\pi)$*. Then there exists* $C>0$ *such that the function* $r \mapsto \dfrac{E(r)}{e^{2r}}e^{C\varphi(r;r_0)}$ *is nondecreasing in* r *for* $r > r_0$*.*

Proof. Recalling the (4.8), we compute the logarithmic derivative

$$\frac{d}{dr}\log\left(\frac{E(r)}{e^{2r}}\right) = -2 + \frac{\int_{\Sigma_r}\sum_i(\partial_\nu u_i)^2 + \int_{\Sigma_r}\sum_i(\partial_y u_i)^2 + 2\sum_{i<j}u_i^2u_j^2}{\int_{\Sigma_r}\sum_i u_i\partial_\nu u_i} \tag{4.12}$$

To apply Theorem 4.2.6, we observe that $\Sigma_r = \{r\}\times[0,k\pi]$, so that

$$\begin{aligned}\int_{\Sigma_r}\sum_i(\partial_y u_i)^2 + 2\sum_{i<j}u_i^2u_j^2 &= \int_0^{k\pi}\sum_i(\partial_y u_i(r,y))^2 + 2\sum_{i<j}u_i(r,y)^2u_j(r,y)^2\,dy \\ &= \frac{2}{k}\int_0^{2\pi}\sum_i(\partial_y\tilde{u}_i(r,y))^2 + 2\left(\frac{k}{2}\right)^2\sum_{i<j}\tilde{u}_i(r,y)^2\tilde{u}_j(r,y)^2\,dy, \end{aligned}\tag{4.13}$$

where $\tilde{u}_i(r,y) = u_i(r,ky/2)$. By a scaling argument, thanks to assumption (4.11) (see also Remark 4.2.7) we can say that for every $\Lambda > 1/2$ it holds

$$\begin{aligned}\int_0^{2\pi}\sum_i(\partial_y\tilde{u}_i(r,y))^2 + \left(\frac{k}{2}\right)^2\frac{2\Lambda}{\int_0^{2\pi}\sum_i\tilde{u}_i(r,y)^2\,dy}\sum_{i<j}\tilde{u}_i(r,y)^2\tilde{u}_j(r,y)^2\,dy \\ \geq \mathcal{L}\left(k,2\Lambda\left(\frac{k}{2}\right)^2\right)\int_0^{2\pi}\sum_i\tilde{u}_i(r,y)^2\,dy = \frac{2}{k}\mathcal{L}\left(k,2\Lambda\left(\frac{k}{2}\right)^2\right)\int_{\Sigma_r}\sum_i u_i^2\end{aligned}$$

The choice

$$\Lambda = \int_0^{2\pi} \sum_i \tilde{u}_i(r,y)^2 \, dy = \frac{2}{k} H(r)$$

yields

$$\int_0^{2\pi} \sum_i (\partial_y \tilde{u}_i(r,y))^2 + 2\left(\frac{k}{2}\right)^2 \sum_{i<j} \tilde{u}_i(r,y)^2 \tilde{u}_j(r,y)^2 \, dy \geq \frac{2}{k} \mathcal{L}\left(k, kH(r)\right) \int_{\Sigma_r} \sum_i u_i^2,$$

and coming back to (4.13) we obtain

$$\int_{\Sigma_r} \sum_i (\partial_y u_i)^2 + 2 \sum_{i<j} u_i^2 u_j^2 \geq \left(\frac{2}{k}\right)^2 \mathcal{L}\left(k, kH(r)\right) \int_{\Sigma_r} \sum_i u_i^2.$$

Plugging this estimate into the (4.12) we see that

$$\begin{aligned} \frac{d}{dr} \log\left(\frac{E(r)}{e^{2r}}\right) &\geq -2 + \frac{\int_{\Sigma_r} \sum_i (\partial_\nu u_i)^2 + \left(\frac{2}{k}\right)^2 \mathcal{L}\left(k, kH(r)\right) \int_{\Sigma_r} \sum_i u_i^2}{\int_{\Sigma_r} \sum_i u_i \partial_\nu u_i} \\ &\geq -2 + 2\frac{2}{k}\sqrt{\mathcal{L}\left(k, kH(r)\right)} \geq -\frac{C}{H(r)^{1/4}} \end{aligned}$$

where we used Theorem 4.2.6. An integration gives the desired result. □

Lemma 4.2.9. *Let $(u_1, \ldots, u_k)$ be a nontrivial solution of (4.6) in $C_{(a,+\infty)}$, and assume that (4.7) and (4.11) hold. If $d := \lim_{r\to+\infty} N(r) < +\infty$, then $d \geq 1$ and*

$$\lim_{r\to+\infty} \frac{E(r)}{e^{2r}} > 0.$$

Proof. Let us fix $r_0 > a$. Firstly, from the previous lemma and the (4.10), we deduce that there exists the limit

$$l := \lim_{r\to+\infty} \frac{E(r)}{e^{2r}} \geq 0.$$

Recalling that $\varphi(r; r_0)$ is bounded, it results

$$\frac{E(r)}{e^{2r}} \geq e^{-C\varphi(r;r_0)} \frac{E(r_0)}{e^{2r_0}} \geq C > 0 \qquad \forall r > r_0,$$

so that the value l is strictly greater then 0. Now, assume by contradiction that $d = \lim_{r\to+\infty} N(r) < 1$. The monotonicity of N implies $N(r) \leq d$ for every $r > 0$. Hence, from Corollary 4.2.5 we deduce

$$\frac{H(r)}{e^{2dr}} \leq \frac{H(r_0)}{e^{2dr_0}} \quad \forall r > r_0 \quad \Longrightarrow \quad \limsup_{r\to+\infty} \frac{H(r)}{e^{2dr}} < +\infty \quad \Longrightarrow \quad \lim_{r\to+\infty} \frac{H(r)}{e^{2r}} = 0,$$

which in turns gives

$$0 < l = \lim_{r\to+\infty} \frac{E(r)}{e^{2r}} = \lim_{r\to+\infty} N(r) \lim_{r\to+\infty} \frac{H(r)}{e^{2r}} = 0,$$

a contradiction. □

4.2.2 Solutions with finite energy in unbounded cylinders

In what follows we consider a solution $(u_1,\dots,u_k)$ of (4.6) defined in an unbounded cylinder $C_{(-\infty,b)}$, with $b \in \mathbb{R}$ (the choice $b = +\infty$ is admissible). In this setting we assume that $(u_1,\dots,u_k)$ has a sufficiently fast decay as $x \to -\infty$, in the sense that

$$H(r) := \int_{\Sigma_r} \sum_{i=1}^{k} u_i^2 \to 0 \quad \text{as } r \to -\infty. \tag{4.14}$$

First of all, we can show that under assumption (4.14) the function $(u_1,\dots,u_k)$ has finite energy in $C_{(-\infty,b)}$.

Lemma 4.2.10. *Let $(u_1,\dots,u_k)$ be a solution of (4.5) in $C_{(-\infty,b)}$, such that (4.14) holds. Then*

$$\mathcal{E}^{unb}(r) := \int_{C_{(-\infty,r)}} \sum_{i=1}^{k} |\nabla u_i|^2 + \sum_{1\le i<j\le k} u_i^2 u_j^2 < +\infty \qquad \forall r < b.$$

The index *unb* stands for the fact that the energy is evaluated in an unbounded cylinder, and will be omitted in the rest of the subsection.

Proof. Firstly, being a solution in $C_{(-\infty,b)}$, it results $(u_1,\dots,u_k) \in H^1_{\mathrm{loc}}(C_{(-\infty,b)})$. Thus, under assumption (4.14), there exists $C > 0$ such that $H(r) \le C$ for every $r < b$.

Let $r_0 < b$. Let us introduce, for $r > 0$, the functional

$$e(r) := \int_{C_{(-r+r_0,r_0)}} \sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2.$$

For the sake of simplicity, in the rest of the proof we assume $r_0 = 0$ (thus $b > 0$). By direct computation and an application of Lemma 4.2.1, we find

$$\begin{aligned} e'(r) &= \int_{\Sigma_{-r}} \sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 \\ &= 2\int_{\Sigma_{-r}} \sum_i (\partial_x u_i)^2 + \int_{\Sigma_0} \sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 - 2\int_{\Sigma_0} \sum_i (\partial_x u_i)^2, \end{aligned}$$

that is

$$\int_{\Sigma_{-r}} \sum_i (\partial_x u_i)^2 = \frac{1}{2} e'(r) + C_0.$$

On the other hand, testing equation (4.5) in $C_{(-r,0)}$ by $(u_1, \ldots, u_k)$ and summing for $i = 1, \ldots, k$, we find

$$\begin{aligned} e(r) &\le \int_{C_{(-r,0)}} \sum_i |\nabla u_i|^2 + 2 \sum_{i<j} u_i^2 u_j^2 = \int_{\Sigma_0} \sum_i u_i \partial_x u_i - \int_{\Sigma_{-r}} \sum_i u_i \partial_x u_i \\ &\le \int_{\Sigma_0} \sum_i u_i \partial_x u_i + \left(\int_{\Sigma_{-r}} \sum_i (\partial_x u_i)^2 \right)^{\frac{1}{2}} \left(\int_{\Sigma_{-r}} \sum_i u_i^2 \right)^{\frac{1}{2}} \end{aligned}$$

Let us assume by contradiction that $e(r) \to +\infty$ as $r \to +\infty$. Taking the square of the previous inequality, using the boundedness of H and assumption (4.14), we have

$$\begin{cases} \frac{1}{C^2}(e(r) + C_1)^2 - 2C_0 \le e'(r) & \text{for } r > \bar{r} \\ e(\bar{r}) > 0, \end{cases}$$

for some $C_0, C_1 > 0$ and $\bar{r}$ sufficiently large. Any solution to the previous differential inequality blows up in finite time, in contradiction with the fact that $(u_1, \ldots, u_k) \in H^1_{\mathrm{loc}}(C_{(-\infty,b)})$. As a consequence e is bounded and, by regularity,

$$\int_{C_{(-\infty,r)}} \sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 < +\infty \qquad \forall r < b. \qquad \square$$

Remark 4.2.11. As a byproduct of the previous lemma, if $(u_1, \ldots, u_k)$ solves the (4.5) in $C_{(-\infty,b)}$ and (4.14) holds, then

$$\lim_{r \to -\infty} \mathcal{E}(r) = 0.$$

Having in mind to recover the monotonicity formulae of the previous subsection in the present situation, we cannot adapt the proof of Lemma 4.2.2, where assumption (4.7) played an important role. However, we can obtain a similar result with a different proof.

Lemma 4.2.12. *Let $(u_1, \ldots, u_k)$ be a solution to (4.1) in $C_{(-\infty,b)}$, such that (4.14) holds. Then*

$$\int_{\Sigma_r} \sum_{i=1}^k |\nabla u_i|^2 + \sum_{1 \le i<j \le k} u_i^2 u_j^2 = 2 \int_{\Sigma_r} \sum_{i=1}^k (\partial_x u_i)^2$$

for every $r < b$.

Proof. We use the method of the variations of the domains: for $\psi \in \mathcal{C}^1_c(-\infty, r)$, we consider

$$u_{i,\varepsilon}(x,y) = u_i(x + \varepsilon\psi(x), y) \qquad i = 1, \dots, k.$$

It is possible to see $(u_{1,\varepsilon}, \dots, u_{k,\varepsilon})$ as a smooth variations of $(u_1, \dots, u_k)$ with compact support in $C_{(-\infty,r)}$: indeed

$$u_i(x + \varepsilon\psi(x), y) - u_i(x,y) = \varepsilon\partial_x u(\xi_x, y)\psi(x),$$

where $\xi_x \in (x, x + \varepsilon\psi(x))$. To proceed, we explicitly remark that any solution to (4.5) is critical for the energy functional

$$J(v_1, \dots, v_k) := \int_{C_{(-\infty,b)}} \sum_{i=1}^k |\nabla v_i|^2 + \sum_{1 \le i < j \le k} v_i^2 v_j^2$$

with respect to variations with compact support in $\mathcal{C}^\infty_c(C_{(-\infty,b)})$. We observe that $J(u_1, \dots, u_k) = \mathcal{E}(b)$. As $(u_1, \dots, u_k)$ is a smooth solution of (4.5) with finite energy $\mathcal{E}(r)$, it follows that

$$\begin{aligned}
0 &= \lim_{\varepsilon \to 0} \frac{\int_{C_{(-\infty,r)}} \sum_i |\nabla u_{i,\varepsilon}|^2 + \sum_{i<j} u_{i,\varepsilon}^2 u_{j,\varepsilon}^2 - \mathcal{E}(r)}{\varepsilon} \\
&= \int_{C_{(-\infty,r)}} \frac{\partial}{\partial \varepsilon} \left(\sum_i |\nabla u_i(x + \varepsilon\psi(x), y)|^2 \right. \\
&\qquad \left. + \sum_{i<j} u_i^2(x + \varepsilon\psi(x), y) u_j^2(x + \varepsilon\psi(x), y) \right) \Bigg|_{\varepsilon=0} dxdy \qquad (4.15) \\
&\qquad + 2 \lim_{\varepsilon \to 0} \int_{C_{(-\infty,r)}} \psi'(x) \sum_i (\partial_x u_i)^2 (x + \varepsilon\psi(x))\, dxdy \\
&= \int_{C_{(-\infty,r)}} \left(2 \sum_i (\partial_x u_i)^2 - \left(\sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 \right) \right) \psi'
\end{aligned}$$

for every $\psi \in \mathcal{C}^1_c(-\infty, r)$. Since $\mathcal{E}(r) < +\infty$, for every $\varepsilon > 0$ there exists a compact $K_\varepsilon \subset C_{(-\infty,r)}$ such that

$$\int_{C_{(-\infty,r)} \setminus K_\varepsilon} \sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 < \varepsilon.$$

Let now $\psi \in \mathcal{C}^1(-\infty, r)$ be such that $\|\psi\|_{\mathcal{C}^1(-\infty,r)} < +\infty$ and $\psi = 0$ in a neighbourhood of r. It is possible to write $\psi = \psi_1 + \psi_2$ where $\psi_1 \in \mathcal{C}^1_c(-\infty, r)$ and $\operatorname{supp} \psi_2 \times (\mathbb{R}/k\pi\mathbb{Z}) \subset$

$(C_{(-\infty,r)} \setminus K_\varepsilon)$. Therefore, from (4.15) it follows

$$\int_{C_{(-\infty,r)}} \left(2\sum_i (\partial_x u_i)^2 - \left(\sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2\right)\right)\psi'$$
$$= \int_{C_{(-\infty,r)}\setminus K_\varepsilon} \left(2\sum_i (\partial_x u_i)^2 - \left(\sum_i |\nabla u|^2 + \sum_{i<j} u_i^2 u_j^2\right)\right)\psi_2'$$
$$\leq 3\|\psi\|_{\mathcal{C}^1(-\infty,r)} \int_{C_{(-\infty,r)}\setminus K_\varepsilon} \left(\sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2\right) < C\varepsilon.$$

Since ε has been arbitrarily chosen, we obtain

$$\int_{C_{(-\infty,r)}} \left(2\sum_i (\partial_x u_i)^2 - \left(\sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2\right)\right)\psi' = 0 \tag{4.16}$$

for every $\psi \in \mathcal{C}^1(-\infty,r)$ such that $\|\psi\|_{\mathcal{C}^1(-\infty,r)} < +\infty$ and $\psi = 0$ in a neighbourhood of r.
Now, let $\psi \in \mathcal{C}^1((-\infty,r])$ be such that $\|\psi\|_{\mathcal{C}^1((-\infty,r])} < +\infty$. For a given $\varepsilon > 0$, we introduce a cut-off function $\eta \in \mathcal{C}^\infty(\mathbb{R})$ such that

$$\eta(s) = \begin{cases} 1 & \text{if } s \leq r-\varepsilon \\ 0 & \text{if } s \geq r-\varepsilon/2. \end{cases}$$

Since $\eta\psi \in \mathcal{C}^1(-\infty,r)$, $\|\eta\psi\|_{\mathcal{C}^1(-\infty,r)} < +\infty$ and $\eta\psi = 0$ in a neighbourhood of r, from (4.16) we deduce

$$\int_{C_{(-\infty,r)}} \left(2\sum_i (\partial_x u_i)^2 - \left(\sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2\right)\right)\eta\psi'$$
$$= \int_{C_{(-\infty,r)}} \left(\sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 - 2\sum_i (\partial_x u_i)^2\right)\eta'\psi. \tag{4.17}$$

Denoting by

$$\gamma = \left(\sum_i |\nabla u_i|^2 + \sum_{i<j} u_i^2 u_j^2 - 2\sum_i (\partial_x u_i)^2\right)\psi,$$

the right hand side is

$$\int_0^{k\pi}\left(\int_{r-\varepsilon}^{r}\eta'(x)\gamma(x,y)\,dx\right)dy = -\int_0^{k\pi}\gamma(r-\varepsilon,y)\,dy$$
$$-\int_0^{k\pi}\left(\int_{r-\varepsilon}^{r}\eta(x)\partial_x\gamma(x,y)\,dx\right)dy$$
$$=\int_{\Sigma_r}\left(2\sum_i(\partial_x u_i)^2-\left(\sum_i|\nabla u_i|^2+\sum_{i<j}u_i^2u_j^2\right)\right)\psi+o(1)$$

as $\varepsilon\to 0$, where the last identity follows from the regularity of $(u_1,\dots,u_k)$ and from the $\mathcal{C}^1$-boundedness of ψ and η. Passing to the limit as $\varepsilon\to 0$ in the (4.17), we deduce that for every $\psi\in\mathcal{C}^1((-\infty,r])$ such that $\|\psi\|_{\mathcal{C}^1((-\infty,r])}<+\infty$ it results

$$\int_{C_{(-\infty,r)}}\left(2\sum_i(\partial_x u_i)^2-\left(\sum_i|\nabla u_i|^2+\sum_{i<j}u_i^2u_j^2\right)\right)\psi'$$
$$=\int_{\Sigma_r}\left(2\sum_i(\partial_x u_i)^2-\left(\sum_i|\nabla u_i|^2+\sum_{i<j}u_i^2u_j^2\right)\right)\psi.$$

Choosing $\psi=1$ we obtain the thesis. □

This result permits to prove an Almgren monotonicity formula for a solution $(u_1,\dots,u_k)$ of (4.5) in $C_{(-\infty,b)}$ such that (4.14) holds. Let us set

$$E^{unb}(r):=\int_{C_{(-\infty,r)}}\sum_{i=1}^k|\nabla u_i|^2+2\sum_{1\le i<j\le k}u_i^2u_j^2,$$

We briefly write E in the rest of the subsection. Clearly, Lemma 4.2.10 and the fact that $\mathcal{E}(r)\to 0$ as $r\to-\infty$ (see Remark 4.2.11) implies that

$$E(r)<+\infty\quad\forall r<b\quad\text{and}\quad\lim_{r\to-\infty}E(r)=0. \tag{4.18}$$

By regularity, $E,\mathcal{E}$ and H are smooth. A direct computation shows that E and $\mathcal{E}$ are increasing in r. As far as H is concerned, with respect to the previous subsection we cannot deduce the identity (4.8) by means of a simple integration by parts, since we are working in an unbounded domain. However,

Lemma 4.2.13. *Let $(u_1, \dots, u_k)$ be a solution to (4.5) in $C_{(-\infty,b)}$, such that (4.14) holds. Then*

$$H'(r) = 2\int_{\Sigma_r} \sum_{i=1}^k u_i \partial_\nu u_i = 2E(r)$$

for every $r < b$. In particular, H is nondecreasing.

Proof. For every $s < r < b$, the divergence theorem and the periodicity of $(u_1, \dots, u_k)$ imply that

$$\begin{aligned} E(r) &= E(s) + \int_{C_{(s,r)}} \sum_i |\nabla u_i|^2 + 2\sum_{i<j} u_i^2 u_j^2 \\ &= E(s) - \int_{\Sigma_s} \sum_i u_i \partial_x u_i + \int_{\Sigma_r} \sum_i u_i \partial_\nu u_i. \end{aligned} \tag{4.19}$$

We consider the second term on the right hand side. Let $\eta \in C_c^\infty(-1,1)$ be a non negative cut-off function, even with respect to $r = 0$, such that $\eta(0) = 1$ and $\eta \le 1$ in $(-1,1)$. Let $\eta_s(x) = \eta(x - s)$; testing equation (4.6) with $u_i\eta_s$ in $C_{(s-1,s)}$, we find

$$\int_{C_{(s-1,s)}} \nabla u_i \cdot \nabla(u_i \eta_s) + u_i^2 \sum_{i \neq j} u_j^2 \eta_s = \int_{\Sigma_s} u_i \partial_x u_i$$

Summing for $i = 1, \dots, k$, we obtain

$$\begin{aligned} \int_{\Sigma_s} \sum_i u_i \partial_x u_i &= \int_{C_{(s-1,s)}} \sum_i \left(u_i \partial_x u_i \eta_s' + |\nabla u_i|^2 \eta_s \right) + 2\sum_{i<j} u_i^2 u_j^2 \eta_s \\ &\le C(\eta') \sum_i \|u_i\|^2_{H^1(C_{(s-1,s)})} + E(s), \end{aligned} \tag{4.20}$$

where the last estimate follows from the Hölder inequality. We claim that

$$\sum_i \|u_i\|_{H^1(C_{(s-1,s)})} \to 0 \qquad \text{as } s \to -\infty.$$

This is a consequence of the Poincaré inequality

$$\int_{C_{(s-1,s)}} u^2 \le C\left(\int_{\Sigma_s} u^2 + \int_{C_{(s-1,s)}} |\nabla u|^2 \right) \qquad \forall u \in H^1(C_{(s-1,s)})$$

together with assumption (4.14) and the fact that $E(s) \to 0$ as $s \to -\infty$ (see (4.18)). Thus, from the (4.20) we deduce that

$$\lim_{s \to -\infty} \int_{\Sigma_s} \sum_i u_i \partial_x u_i = 0,$$

which in turn can be used in the (4.19) to obtain the thesis:

$$E(r)=\lim_{s\to-\infty}\left(E(s)-\int_{\Sigma_s}\sum_i u_i\partial_x u_i+\int_{\Sigma_x}\sum_i u_i\partial_\nu u_i\right)=\int_{\Sigma_x}\sum_i u_i\partial_\nu u_i. \qquad\square$$

In light of the previous results, the proof of the following statements are straightforward modification of the proofs of Proposition 4.2.4, Corollary 4.2.5 and Lemmas 4.2.8 and 4.2.9.

Proposition 4.2.14. *Let $(u_1,\dots,u_k)$ be a solution of* (4.6) *in $C_{(-\infty,b)}$ such that* (4.14) *holds. The* Almgren quotient

$$N^{unb}(r):=\frac{E^{unb}(r)}{H(r)}$$

is well defined in $(-\infty,b)$ and nondecreasing. Moreover,

$$\int_{-\infty}^{r}\frac{\int_{\Sigma_s}\sum_{i<j}u_i^2u_j^2}{H(s)}\,ds\le N(r).$$

Analogously, the function $\mathfrak{N}^{unb}(r):=\dfrac{\mathcal{E}^{unb}(r)}{H(r)}$ is well defined in $(-\infty,b)$ and nondecreasing.

We will briefly write N and $\mathfrak{N}$ instead of N^{unb} and $\mathfrak{N}^{unb}$ in the rest of this subsection.

Corollary 4.2.15. *Let $(u_1,\dots,u_k)$ be a solution of* (4.6) *in $C_{(-\infty,b)}$ such that* (4.14) *holds.*

(i) If $N(r)\ge\underline{d}$ for $r\ge s$, then

$$\frac{H(r_1)}{e^{2\underline{d}r_1}}\le\frac{H(r_2)}{e^{2\underline{d}r_2}}\qquad\forall\ s\le r_1<r_2<b,$$

ii) If $N(r)\le\overline{d}$ for $r\le t<b$, then

$$\frac{H(r_1)}{e^{2\overline{d}r_1}}\ge\frac{H(r_2)}{e^{2\overline{d}r_2}}\qquad\forall\ r_1<r_2\le t.$$

For a fixed $r_0<b$, let us introduce

$$\varphi(r;r_0):=\int_{r_0}^{r}\frac{ds}{H(s)^{1/4}}.$$

The function φ is positive and increasing in $\mathbb{R}^+$; thanks to point (i) of Corollary 4.2.15 and to the monotonicity of N, whenever (u, v) is nontrivial φ is bounded by a quantity depending only $H(r_0)$ and $N(r_0)$:

$$\varphi(r; r_0) \leq 2\frac{e^{\frac{1}{2}N(r_0)r_0}}{H(r_0)^{\frac{1}{4}}N(r_0)}\left[e^{-\frac{1}{2}N(r_0)r_0} - e^{-\frac{1}{2}N(r_0)r}\right].$$

This, together with the monotonicity of $\varphi(\cdot; r_0)$, implies that if $b = +\infty$ then there exists the limit

$$\lim_{r\to+\infty} \varphi(r; r_0) < +\infty.$$

Lemma 4.2.16. *Let* $(u_1, \ldots, u_k)$ *be a solution of* (4.1) *in* $C_{(-\infty,b)}$ *such that* (4.14) *holds. Let* $r_0 \in (-\infty, b)$*, and assume that*

$$u_{i+1}(x, y) = u_i(x, y - \pi) \quad \textit{and} \quad u_1(x, \tau + y) = u_1(x, \tau - y) \tag{4.21}$$

where $\tau \in [0, k\pi)$*. There exists* $C > 0$ *such that the function* $r \mapsto \dfrac{E(r)}{e^{2r}}e^{C\varphi(r;r_0)}$ *is nondecreasing in* r *for* $r > r_0$*.*

Lemma 4.2.17. *Let* $(u_1, \ldots, u_k)$ *be a nontrivial solution of* (4.6) *in* C_∞*, and assume that* (4.14) *and* (4.21) *hold. If* $d := \lim_{r\to+\infty} N(r) < +\infty$*, then* $d \geq 1$ *and*

$$\lim_{r\to+\infty} \frac{E(r)}{e^{2r}} > 0.$$

Remark 4.2.18. The achievements of this section hold true for solutions to

$$\begin{cases} -\Delta u_i = -\beta u_i \sum_{j\neq i} u_j^2 \\ u_i > 0 \end{cases}$$

with the energy density

$$\sum_i |\nabla u_i|^2 + 2\sum_{i<j} u_i^2 u_j^2 \quad \text{replaced by} \quad \sum_i |\nabla u_i|^2 + 2\beta\sum_{i<j} u_i^2 u_j^2.$$

4.2.3 Monotonicity formulae for harmonic functions

Here we prove some monotonicity formulae for harmonic functions of the plane which are 2π periodic in one variable. In what follows, in the definition of $C_{(a,b)}$ and Σ_r we mean $k = 2$. The following results will be useful in Section 4.6.

Firstly, it is not difficult to obtain the counterpart of Lemma 4.2.1.

Lemma 4.2.19. *Let* Ψ *be an entire harmonic function in* $C_{(a,b)}$*. Then*

$$r \mapsto \int_{\Sigma_r} |\nabla\Psi|^2 - 2\Psi_x^2$$

is constant.

Proof. We proceed as in the proof of Lemma 4.2.1: for $a < r_1 < r_2 < b$, we test the equation $-\Delta\Psi = 0$ with Ψ_x in $C_{(r_1,r_2)}$ and integrate by parts. □

In what follows we consider a harmonic function Ψ defined in an unbounded cylinder $C_{(-\infty,b)}$, with $b \in \mathbb{R}$ or $b = +\infty$. We assume that

$$H(r;\Psi) := \int_{\Sigma_r} \Psi^2 \to 0 \quad \text{as } r \to -\infty. \tag{4.22}$$

Lemma 4.2.20. *Let* Ψ *be a harmonic function in* $C_{(-\infty,b)}$ *such that* (4.22) *holds true. Then*

(*i*) *for every* $r \in \mathbb{R}$ *it results* $E^{unb}(r;\Psi) := \int_{C_{(-\infty,r)}} |\nabla\Psi|^2 < +\infty$

(*ii*) *it results*

$$\int_{\Sigma_r} |\nabla\Psi|^2 = 2\int_{\Sigma_r} (\partial_x\Psi)^2 \tag{4.23}$$

Proof. In light of Lemma 4.2.19, it is not difficult to adapt the proof of Lemma 4.2.10 and obtain (*i*). As far as (*ii*), we can proceed as in the proof of Lemma 4.2.12. □

Proposition 4.2.21. *Let* Ψ *be a nontrivial harmonic function in* $C_{(-\infty,b)}$*, such that* (4.22) *holds true. The* Almgren quotient

$$N^{unb}(r;\Psi) := \frac{\int_{C_{(-\infty,r)}} |\nabla\Psi|^2}{\int_{\Sigma_r} \Psi^2}$$

is nondecreasing in r*. If* $N(\cdot;\Psi)$ *is constant for* r *in some non empty open interval* (r_1, r_2)*, then* $N(r;\Psi)$ *is constant for all* $r \in \mathbb{R}$ *and there exists a positive integer* $d \in \mathbb{N}$ *such that* $N(r;\Psi) = d$*; furthermore,*

$$\Psi(x,y) = [C_1 \cos(dy) + C_2 \sin(dy)]\, e^{dx}$$

for some $C_1, C_2 \in \mathbb{R}$*.*

Proof. The Almgren quotient is well defined, thanks to Lemma 4.2.20. To prove its monotonicity, we compute the logarithmic derivative by means of the Pohozaev identity (4.23) and the fact that $H'(r;\Psi) = 2E^{unb}(r;\Psi)$ (this follows from (4.22), and can be proved using the argument in Lemma 4.2.13):

$$\begin{aligned}\frac{(N^{unb})'(r;\Psi)}{N^{unb}(r;\Psi)} &= \frac{\int_{\Sigma_r} |\nabla\Psi|^2}{\int_{C_{(-\infty,r)}} |\nabla\Psi|^2} - 2\frac{\int_{\Sigma_r} \Psi\partial_x\Psi}{\int_{\Sigma_r} \Psi^2} \\ &= 2\frac{\int_{\Sigma_r} |\partial_x\Psi|^2}{\int_{\Sigma_r} \Psi\partial_x\Psi} - 2\frac{\int_{\Sigma_r} \Psi\partial_x\Psi}{\int_{\Sigma_r} \Psi^2} \geq 0,\end{aligned}$$

where in the last step we used the Cauchy-Schwarz inequality.

Let us assume now that $N^{unb}(r;\Psi)$ is constant for $r \in (r_1, r_2)$. By the previous computations it follows that necessarily

$$\int_{\Sigma_r} |\partial_x\Psi|^2 \int_{\Sigma_r} \Psi^2 = \left(\int_{\Sigma_r} \Psi\partial_x\Psi\right)^2$$

for every $r \in (r_1, r_2)$. Again from the Cauchy-Schwarz inequality, we evince that it must be

$$\partial_x\Psi = \lambda\Psi \qquad \text{on } \Sigma_r$$

for some constant $\lambda \in \mathbb{R}$ and for every $r \in (r_1, r_2)$. Solving the differential equation, we find that Ψ is of the form

$$\Psi(x,y) = \psi(y)e^{\lambda x}.$$

This, together with the equation $\Delta\Psi = 0$, yields

$$\psi'' + \lambda^2\psi = 0 \quad \Longrightarrow \quad \Psi(x,y) = [C_1\cos(\lambda y) + C_2\sin(\lambda y)]\, e^{\lambda x}$$

for every $(x,y) \in (r_1, r_2) \times \mathbb{R}$, and Ψ can be uniquely extended to $\mathbb{R}^2$ by the unique continuation principle for harmonic functions. Since Ψ satisfies the condition (4.22) and is nontrivial, it follows that $\lambda > 0$. The proof is complete, recalling the periodicity in y of the function Ψ and computing its Almgren quotient. □

4.3 Proof of Theorem 4.1.1

In this section we construct a solution to (4.1) modelled on the harmonic function $\Phi(x,y) = \cosh x \sin y$. Before proceeding with the proof, we report some results which will be useful in this and in the next sections, for which we cannot find proper references.

We start with the following version of the parabolic minimum principle.

Lemma 4.3.1. *Let $N \geq 2$, let $\Omega = (a,b) \times \Omega' \subset \mathbb{R}^N$ be open and connected, let $c \in L^\infty(\Omega)$ and let $w \in H^1(\Omega)$ be such that*

$$\begin{cases} w_t - \Delta w \geq c(x)w & \text{in } [0,T] \times \Omega \\ w \geq 0 & \text{on } \{0\} \times \overline{\Omega} \\ w \geq 0 & \text{on } (0,T) \times (a,b) \times \partial\Omega', \end{cases}$$

and w has $(b-a)$-periodic boundary condition on $\{a,b\} \times \Omega'$. Then $w \geq 0$.

Proof. Let $J(t) := \frac{1}{2}\int_\Omega (w^-)^2$. A direct computation shows that $J'(t) \leq 2\|c\|_{L^\infty(\Omega)} J(t)$, where we used the boundary conditions. Consequently,

$$J(t) \leq J(0) e^{2\|c\|_{L^\infty(\Omega)} t} = 0 \qquad \forall t \in [0,T]$$

where the last identity follows by the initial condition. □

Remark 4.3.2. Note that we do not require anything about the sign of c.

We will exploit many times the following properties of the trace operators.

Theorem 4.3.3. *For $a < b$ real numbers, let $C_{(a,b)} = (a,b) \times \mathbb{S}_k$ be a bounded cylinder. The trace operator*

$$Tr_{C_{(a,b)}} : u \in H^1(C_{(a,b)}) \mapsto u|_{\Sigma_a \cup \Sigma_b} \in L^2(\Sigma_a \cup \Sigma_b)$$

is compact.

Proof. For the sake of simplicity we consider the case $a = 0$ and $b = 1$. Let $(u_n) \subset H^1(C_{(0,1)})$ be such that $u_n \rightharpoonup 0$. We show that $u_n|_{\Sigma_0 \cup \Sigma_1} \to 0$ in $L^2(\Sigma_0 \cup \Sigma_1)$. Let $w(x,y) := x(x-1)$. We note that $\partial_\nu w = 1$ on $\Sigma_0 \cup \Sigma_1$. Let

$$F(x,y) = \nabla w(x,y) = (2x-1, 0) \quad \text{and} \quad g(x,y) = \Delta w(x,y) = 2.$$

By the divergence theorem

$$2\int_{C_{(0,1)}} u_n^2 = \int_{C_{(0,1)}} (\mathrm{div} F) u_n^2 = -\int_{C_{(0,1)}} 2u_n F \cdot \nabla u_n + \int_{\Sigma_0 \cup \Sigma_1} u_n^2,$$

so that

$$\int_{\Sigma_0 \cup \Sigma_1} u_n^2 \leq 2\|u_n\|^2_{L^2(C_{(0,1)})} + 2\|u_n\|_{L^2(C_{(0,1)})} \|\nabla u_n\|_{L^2(C_{(0,1)})} \to 0$$

as $n \to \infty$, by the compactness of the Sobolev embedding $H^1(C_{(0,1)}) \hookrightarrow L^2(C_{(0,1)})$. □

Corollary 4.3.4. *For $a < b$ real numbers, let $C_{(a,b)} = (a,b) \times \mathbb{S}_k$ be a bounded cylinder. The local trace operator*

$$T_{\Sigma_b} : u \in H^1(C_{(a,b)}) \mapsto u|_{\Sigma_b} \in L^2(\Sigma_b)$$

is compact.

Proof. It is an easy consequence of Theorem 4.3.3 and of the fact that the linear operator $L_f : \varphi \in L^2(\Sigma_a \cup \Sigma_b) \mapsto f\varphi \in L^2(\Sigma_a \cup \Sigma_b)$ is continuous for every $f \in L^\infty(\Sigma_a \cup \Sigma_b)$. As $T_{\Sigma_b} = L_{\chi_{\Sigma_b}} \circ Tr_{C_{(a,b)}}$, where χ_{Σ_b} is the characteristic function of Σ_b, T_{Σ_b} is compact. □

4.3.1 Existence in bounded cylinders

For every $R > 0$ we construct a solution (u_R, v_R) to

$$\begin{cases} -\Delta u = -uv^2 & \text{in } C_R \\ -\Delta v = -u^2 v & \text{in } C_R \\ u, v > 0 \end{cases} \tag{4.24a}$$

(equivalently, we can consider the problem in $(-R, R) \times (0, 2\pi)$ with periodic boundary condition on the sides $[-R, R] \times \{0, 2\pi\}$), with Dirichlet boundary condition

$$u = \Phi^+, \quad v = \Phi^- \quad \text{on } \Sigma_R \cup \Sigma_{-R}, \tag{4.24b}$$

and exhibiting the same symmetries of (Φ^+, Φ^-). To be precise:

Proposition 4.3.5. *There exists a solution (u_R, v_R) to problem (4.24a) with the prescribed boundary conditions (4.24b), such that*

1) $u_R(-x, y) = u_R(x, y)$ and $v_R(-x, y) = v_R(x, y)$;

2) the symmetries

$$v_R(x, y) = u_R(x, y - \pi) \qquad u_R(x, \pi - y) = v_R(x, \pi + y)$$
$$u_R\left(x, \frac{\pi}{2} + y\right) = u_R\left(x, \frac{\pi}{2} - y\right) \qquad v_R\left(x, \frac{3}{2}\pi + y\right) = v_R\left(x, \frac{3}{2}\pi - y\right)$$

hold;

3) $u_R - v_R > 0$ in $\{\Phi > 0\}$ and $v_R - u_R > 0$ in $\{\Phi < 0\}$;

4) $u_R > \Phi^+$ and $v_R > \Phi^-$.

Remark 4.3.6. In light of the evenness of (u_R, v_R) in x, it results

$$\partial_x u = 0 = \partial_x v \qquad \text{on } \Sigma_0.$$

As a consequence, the monotonicity formulae proved in Subsection 4.2.1 hold true for (u_R, v_R) in the semi-cylinder $C_{(0,R)}$.

In order to keep the notation as simple as possible, in what follows we refer to a solution of (4.24a)-(4.24b) as to a solution of (4.24).

Proof. Let

$$\mathcal{U}_R := \left\{ (u,v) \in (H^1(C_R))^2 \;\middle|\; \begin{array}{l} u = \Phi^+,\ v = \Phi^- \text{ on } \Sigma_R \cup \Sigma_{-R},\ u \geq 0, \\ u - v \geq 0 \text{ in } \{\Phi \geq 0\}, \\ v(x,y) = u(x, y-\pi),\ u(-x,y) = u(x,y), \\ u(x, \pi - y) = v(x, \pi + y),\ u\left(x, \frac{\pi}{2} + y\right) = u\left(x, \frac{\pi}{2} - y\right) \end{array} \right\}.$$

Note that if $(u,v) \in \mathcal{U}_R$ then v is nonnegative, even in x and symmetric in y with respect to $3\pi/2$; moreover, $u - v \leq 0$ in $\{\Phi < 0\}$. It is immediate to check that $\mathcal{U}_R$ is weakly closed with respect to the H^1 topology. We seek solutions of (4.24) as minimizers of the energy functional

$$J(u,v) := \int_{C_R} |\nabla u|^2 + |\nabla v|^2 + u^2 v^2$$

in $\mathcal{U}_R$. The existence of at least one minimizer is given by the direct method of the calculus of variations; for the coercivity of the functional J, we use the following Poincaré inequality:

$$\int_{C_R} u^2 \leq C \left(\int_{\Sigma_{-R}} u^2 + \int_{C_R} |\nabla u|^2 \right) \qquad \forall u \in H^1(C_R), \tag{4.25}$$

where C depends only on R. To show that a minimizer satisfies equation (4.24), we consider the parabolic problem

$$\begin{cases} U_t - \Delta U = -UV^2 & \text{in } (0,+\infty) \times C_R \\ V_t - \Delta V = -U^2 V & \text{in } (0,+\infty) \times C_R \\ U = \Phi^+,\ V = \Phi^- & \text{on } (0,+\infty) \times (\Sigma_R \cup \Sigma_{-R}) \end{cases} \tag{4.26}$$

with initial condition in $\mathcal{U}_R$. There exists a unique local solution (U,V); by Lemma 4.3.1 if follows $U, V \geq 0$; hence, the maximum principle gives

$$0 \leq U \leq \sup_{C_R} \Phi^+ \quad \text{and} \quad 0 \leq V \leq \sup_{C_R} \Phi^-.$$

This control reveals that (U, V) can be uniquely extended in the whole $(0, +\infty)$. Since

$$\frac{d}{dt} J(U(t,\cdot), V(t,\cdot)) = -2 \int_{C_R} \left(U_t^2 + V_t^2\right) \leq 0, \tag{4.27}$$

that is, the energy is a Lyapunov functional, from the parabolic theory it follows that for every sequence $t_i \to +\infty$ there exists a subsequence (t_j) such that $(U(t_j,\cdot), V(t_j,\cdot))$ converges to a solution (u, v) of (4.24). Therefore, in order to prove that (u_R, v_R) solves (4.24), it is sufficient to show that there exists an initial condition in $\mathcal{U}_R$ such that the limiting profile (u, v) coincides with (u_R, v_R). We use the fact that

$$\mathcal{U}_R \text{ is positively invariant under the parabolic flow.} \tag{4.28}$$

To prove this claim, we firstly note that by the symmetry of initial and boundary conditions and by the uniqueness of the solution to problem (4.26), we have

$$\begin{aligned} V(t,x,y) &= U(t,x,y-\pi), \qquad & U(t,-x,y) &= U(t,x,y), \\ V(t,x,\pi+y) &= U(t,x,\pi-y), \qquad & U\left(t,x,\frac{\pi}{2}+y\right) &= U\left(t,x,\frac{\pi}{2}-y\right). \end{aligned} \tag{4.29}$$

This implies

$$U(t,x,\pi) - V(t,x,\pi) = 0 \qquad \forall (t,x) \in (0,+\infty) \times [-R,R].$$

Furthermore, using the (4.29) and the periodicity of (U, V),

$$\begin{aligned} U(t,x,0) - V(t,x,0) = U(t,x,0) - V(t,x,2\pi) = 0 \qquad & \forall (t,x) \in (0,+\infty) \times [-R,R] \\ U(t,x,2\pi) - V(t,x,2\pi) = U(t,x,2\pi) - V(t,x,0) = 0 \qquad & \forall (t,x) \in (0,+\infty) \times [-R,R]. \end{aligned}$$

This means that $U - V = 0$ on $\{\Phi = 0\}$. Let us introduce $D_R := \{\Phi > 0\} \cap C_R$. For each initial datum in $\mathcal{U}_R$, we have

$$\begin{cases} (U-V)_t - \Delta(U-V) = UV(U-V) & \text{in } (0,+\infty) \times D_R \\ U - V \geq 0 & \text{on } \{0\} \times D_R \\ U - V \geq 0 & \text{on } [0,+\infty) \times \partial D_R. \end{cases}$$

Lemma 4.3.1 implies $U - V \geq 0$ in $(0, +\infty) \times D_R$. This completes the proof of the claim.

Let us consider equation (4.26) with the initial conditions $U(0, x, y) = u_R(x, y)$, $V(0, x, y) = v_R(x, y)$; let us denote (U^R, V^R) the corresponding solution. On one side, by minimality,

$$J(u_R, v_R) \leq J(U^R(t,\cdot), V^R(t,\cdot)) \qquad \forall t \in (0, +\infty);$$

we point out that this comparison is possible because of (4.28). On the other side, by the (4.27),

$$J(U^R(t,\cdot),V^R(t,\cdot)) \leq J(u_R,v_R) \qquad \forall t \in (0,+\infty).$$

We deduce that $J(U^R,V^R)$ is constant, which in turns implies (we can use again the (4.27)),

$$U_t^R(t,x,y) = V_t^R(t,x,y) \equiv 0 \quad \Longrightarrow \quad U^R(t,x,y) = u_R(x,y), \quad V^R(t,x,y) = v_R(x,y).$$

By the above argument, as (u_R,v_R) coincides with the asymptotic profile of a solution of the parabolic problem (4.26), it solves (4.24). Points 1)-3) of the thesis are satisfied due to the positive invariance of $\mathcal{U}_R$. The strong maximum principle yields $u_R > 0$ and $v_R > 0$. Moreover,

$$\begin{cases} -\Delta(u_R - v_R - \Phi) = u_R v_R (u_R - v_R) \geq 0 & \text{in } D_R \\ u_R - v_R - \Phi = 0 & \text{on } \partial D_R \end{cases}$$

which implies $u_R - v_R - \Phi \geq 0$ in D_R. By the strong maximum principle and the fact that $u_R, v_R > 0$ we deduce $u_R > \Phi^+$. Analogously, $v_R > \Phi^-$. □

Remark 4.3.7. The existence of a positive solution of (4.24) satisfying the conditions 1)-2) of the Proposition 4.3.5 can be easily proved by means of the celebrated Palais Principle of Symmetric Criticality (see [71]):

Theorem 4.3.8. *Let G be a group of isometries of a Riemaniann manifold M, and let $f : M \to \mathbb{R}$ be a $\mathcal{C}^1$ function invariant under G; let Σ be the set of the stationary points of M under the action of G. Under these assumptions, if p is a critical point of f restricted to Σ, then it is a free critical point of f on M.*

Now, let us consider the weakly closed set

$$\mathcal{S} := \left\{ (u,v) \in (H^1(C_R))^2 \;\middle|\; \begin{array}{l} u = \Phi^+,\, v = \Phi^- \text{ on } \Sigma_R \cup \Sigma_{-R}, \\ v(x,y) = u(x,y-\pi),\, u(-x,y) = u(x,y), \\ u(x,\pi - y) = v(x,\pi+y),\, u\left(x,\frac{\pi}{2}+y\right) = u\left(x,\frac{\pi}{2}-y\right) \end{array} \right\},$$

Arguing as in the first part of the proof of Proposition 4.3.5, one can easily show that there exists a minimizer of J in $\mathcal{S}$; moreover, it is immediate to check that J is invariant under the symmetries defining the set $\mathcal{S}$, so that the the Palais Principle implies that the constrained minimizer is, in fact, a free critical point of J, and then it is a positive solution of (4.24) (for the positivity we can apply the maximum principle). We chose a more complicated proof since we will strongly use the pointwise estimates given by point 4).

4.3.2 Compactness of the family $\{(u_R, v_R)\}$

In this section we aim at proving that, up to a subsequence, the family $\{(u_R, v_R) : R > 1\}$ obtained in Proposition 4.3.5 converges, as $R \to +\infty$, to a solution (u, v) of (4.1) defined in the whole C_∞. Then, by looking at (u, v) as defined in $\mathbb{R}^2$ (this is possible thanks to the periodicity), we obtain a solution of (4.1) satisfying the conditions 1)-5) of Theorem 4.1.1. At a later stage, we will also obtain the estimates of points 6) and 7).

We denote $E_R, \mathcal{E}_R, H_R, N_R$ and $\mathfrak{N}_R$ the functions $E^{sym}, H, \mathcal{E}^{sym}, N^{sym}$ and $\mathfrak{N}^{sym}$ (which have been defined in Subsection 4.2.1) when referred to (u_R, v_R). As observed in Remark 4.3.6, for these quantities the results of Subsection 4.2.1 apply.

We will obtain compactness of the sequence (u_R, v_R) using some uniform-in-R control on N_R and H_R. We start with a uniform (in both r and R) upper bound for the Almgren quotients $N_R(r)$.

Lemma 4.3.9. *It results that $N_R(r) \le 2$, for every $R > 0$ and $r \in (0, R)$.*

Proof. It is an easy consequence of the monotonicity of N_R and of the minimality of (u_R, v_R) for the functional J in $\mathcal{U}_R$: noting that $J(u_R, v_R) = \mathcal{E}_R(R)$, we compute

$$N_R(r) \le N_R(R) \le \frac{2\mathcal{E}_R(R)}{H_R(R)} \le \frac{2}{\int_{\Sigma_R} \Phi^2} \int_{C_{(0,R)}} |\nabla \Phi|^2 = 2 \tanh R.$$

We used the fact that the restriction of (Φ^+, Φ^-) in C_R is an element of $\mathcal{U}_R$ for every R, and the boundary condition of (u_R, v_R) on Σ_R. □

In the proof of the following lemma we exploit the compactness of the local trace operator $T_{\Sigma_1} : u \in H^1(C_{(0,1)}) \mapsto u|_{\Sigma_1} \in L^2(\Sigma_1)$, see Corollary 4.3.4.

Lemma 4.3.10. *There exists $C > 0$ such that $H_R(1) \le C$ for every $R > 1$.*

Proof. By contradiction, assume that $H_{R_n}(1) \to +\infty$ for a sequence $R_n \to +\infty$. Let us introduce the sequence of scaled functions

$$(\hat{u}_n(x, y), \hat{v}_n(x, y)) := \frac{1}{\sqrt{H_{R_n}(1)}} \left(u_{R_n}(x, y), v_{R_n}(x, y)\right).$$

We wish to prove a convergence result for such a sequence, in order to obtain a uniform lower bound for $N_{R_n}(1)$. In a natural way, the scaling leads us to consider, for $r \in (0, 1)$, the quantities

$$\hat{E}_n(r) := \int_{C_{(0,r)}} |\nabla \hat{u}_n|^2 + |\nabla \hat{v}_n|^2 + 2H_{R_n}(1)\hat{u}_n^2 \hat{v}_n^2,$$

$$\hat{H}_n(r) := \int_{\Sigma_r} \hat{u}_n^2 + \hat{v}_n^2, \quad \hat{N}_n(r) := \frac{\hat{E}_n(r)}{\hat{H}_n(r)}.$$

By construction, it holds $\hat{H}_n(1) = 1$ and $\hat{N}_n(r) = N_{R_n}(r) \leq 2$, thanks to Lemma 4.3.9. Now,

$$\int_{C_{(0,1)}} |\nabla \hat{u}_n|^2 + |\nabla \hat{v}_n|^2 \leq \hat{E}_n(1) = \hat{N}_n(1)\hat{H}_n(1) \leq 2, \tag{4.30}$$

which gives a uniform bound in the $H^1(C_{(0,1)})$ norm of the sequence $(\hat{u}_n, \hat{v}_n)$ (we can use a Poincaré inequality of type (4.25)). Then, we can extract a subsequence which converges weakly in $H^1(C_{(0,1)})$ to some limiting profile $(\hat{u}, \hat{v})$, which is nontrivial in light of the compactness of the local trace operator T_{Σ_1} and of the fact that $\hat{H}_n(1) = 1$. Since

$$\mathcal{V} := \left\{ (u,v) \in \left(H^1(C_{(0,1)})\right)^2 \;\middle|\; \begin{array}{l} u - v \geq 0 \text{ in } \{\Phi \geq 0\},\, v(x,y) = u(x, y-\pi), \\ u(x, \pi - y) = v(x, \pi + y),\, u\left(x, \frac{\pi}{2} + y\right) = u\left(x, \frac{\pi}{2} - y\right) \end{array} \right\},$$

is closed in the weak $H^1(C_{(0,1)})$ topology, and $(\hat{u}_n|_{C_{(0,1)}}, \hat{v}_n|_{C_{(0,1)}}) \in \mathcal{V}$ for every n, $\hat{u}$ and $\hat{v}$ are nonnegative functions with the same symmetries of (u_R, v_R); moreover we can show that $(\hat{u}, \hat{v})$ satisfies the segregation condition $\hat{u}\hat{v} = 0$ a.e. in $C_{(0,1)}$. Indeed, by the compactness of the Sobolev embedding $H^1(C_{(0,1)}) \hookrightarrow L^4(C_{(0,1)})$ we deduce that the interaction term

$$I(u,v) := \int_{C_{(0,1)}} u^2 v^2$$

is continuous in the weak topology of $(H^1(C_{(0,1)}))^2$. From the estimate (4.30), we infer

$$2H_{R_n}(1) I(\hat{u}_n, \hat{v}_n) \leq \hat{E}_n(1) \leq 2;$$

passing to the limit as $n \to +\infty$, we conclude

$$I(\hat{u}, \hat{v}) = \lim_{n\to\infty} I(\hat{u}_n, \hat{v}_n) = 0 \quad \Longrightarrow \quad \hat{u}\hat{v} = 0 \text{ a.e. in } C_{(0,1)}.$$

Moreover, from the compactness of the local trace operator T_{Σ_1}, we also deduce $\int_{\Sigma_1} \hat{u}^2 + \hat{v}^2 = 1$. Let us consider the functional

$$J^\infty(u,v) := \int_{C_{(0,1)}} |\nabla u|^2 + |\nabla v|^2,$$

defined in the set

$$\mathcal{M} := \left\{ (u,v) \in (H^1(C_{(0,1)}))^2 \;\middle|\; \begin{array}{l} \int_{\Sigma_1} u^2 + v^2 = 1, \\ v(x,y) = u(x, y - \pi),\ uv = 0 \text{ a.e. in } C_1 \end{array} \right\}.$$

Due to the compactness of the trace operator, one can check that $\mathcal{M}$ is closed in the weak $(H^1(C_{(0,1)}))^2$ topology. It is clear that $(\hat{u}, \hat{v}) \in \mathcal{M}$. We claim that

$$\inf_{(u,v)\in\mathcal{M}} J^\infty(u,v) =: m > 0.$$

Indeed, let us assume by contradiction that the infimum is 0: since the set $\mathcal{M}$ is weakly closed and J^∞ is weakly lower semi-continuous and coercive, there exists $(\bar{u}, \bar{v})$ such that $J^\infty(\bar{u}, \bar{v}) = 0$. It follows that $(\bar{u}, \bar{v})$ is a vector of constant functions; the symmetry and the segregation condition imply that $(\bar{u}, \bar{v}) \equiv (0, 0)$, but this is in contrast with the fact that $(\bar{u}, \bar{v}) \in \mathcal{M}$. Thus, the weak convergence of the sequence $(\hat{u}_n, \hat{v}_n)$ entails

$$\liminf_{n\to\infty} \hat{N}_n(1) \geq \liminf_{n\to\infty} \int_{C_{(0,1)}} |\nabla \hat{u}_n|^2 + |\nabla \hat{v}_n|^2 \geq m > 0,$$

so that whenever n is sufficiently large

$$N_{R_n}(1) = \hat{N}_n(1) \geq \frac{1}{2} m. \tag{4.31}$$

Thanks to Lemma 4.3.9 we know that $m/2 \leq N_{R_n}(1) \leq 2$, and from the assumption $H_{R_n}(1) \to +\infty$ we deduce that (recall the (4.9))

$$\begin{aligned}\varphi_{R_n}(r;1) :&= \int_1^r \frac{ds}{H_{R_n}(s)^{1/4}} \\ &\leq 2 \frac{e^{\frac{1}{2}N_{R_n}(1)}}{H_{R_n}(1)^{\frac{1}{4}} N_{R_n}(1)} \left[e^{-\frac{1}{2}N_{R_n}(1)} - e^{-\frac{1}{2}N_{R_n}(1)r} \right] \to 0\end{aligned}$$

as $n \to \infty$, for every $r > 1$. In particular, there exists $C > 0$ such that

$$\varphi_{R_n}(r;1) \leq C \qquad \forall 1 \leq r \leq R_n, \ \forall n. \tag{4.32}$$

This implies that the sequence $(E_{R_n}(1))_n$ is bounded. To see this, we firstly note that (u_{R_n}, v_{R_n}) satisfies the symmetry condition (4.11) which is necessary to apply Lemma 4.2.8; consequently, the variational characterization of (u_{R_n}, v_{R_n}) (see also the proof of Lemma 4.3.9 and the (4.32)) implies that

$$\begin{aligned}\frac{E_{R_n}(1)}{e^2} &\leq e^{C\varphi_{R_n}(R_n;1)} \frac{E_{R_n}(R_n)}{e^{2R_n}} \leq 2C \frac{\mathcal{E}_{R_n}(R_n)}{e^{2R_n}} \\ &\leq C \frac{\int_{C_{(0,R_n)}} |\nabla \Phi|^2}{e^{2R_n}} = C \frac{\sinh R_n \cosh R_n}{e^{2R_n}} \leq C,\end{aligned}$$

where C does not depend on n. Since $(E_{R_n}(1))_n$ is bounded and $(H_{R_n}(1))_n$ tends to infinity, we obtain

$$\lim_{n\to\infty} N_{R_n}(1) = \lim_{n\to\infty} \frac{E_{R_n}(1)}{H_{R_n}(1)} = 0,$$

in contradiction with (4.31). □

Lemma 4.3.11. *Up to a subsequence,* $\{(u_R, v_R)\}$ *converges in* $\mathcal{C}^2_{\text{loc}}(C_\infty)$*, as* $R \to +\infty$*, to a solution* (u, v) *of* (4.1) *in the whole* C_∞*. This solution satisfies point 2)-5) of Theorem 4.1.1, and its Almgren quotient* N *is such that*

$$N(r) \leq 2 \quad \forall r > 0 \quad \textit{and} \quad \lim_{r\to+\infty} N(r) \geq 1.$$

Proof. As $H_R(1)$ is bounded in R and $N_R(1) \leq 2$, $E_R(1)$ is also bounded in R. By means of a Poincaré inequality of type (4.25), this induces a uniform-in-R bound for the $H^1(C_{(0,1)})$ norm of (u_R, v_R), which in turns, by the compactness of the trace operator, gives a uniform-in-R bound for the $L^2(\partial C_{(0,1)})$ norm. Due to the subharmonicity of (u_R, v_R), the $L^2(\partial C_{(0,1)})$ bound provides a uniform-in-R bound for the L^∞ norm of (u_R, v_R) in every compact subset of $C_{(0,1)}$; the regularity theory for elliptic equations (see [45]) ensures that, up to a subsequence, (u_R, v_R) converges in $\mathcal{C}^2_{\text{loc}}(C_{(0,1)})$, as $R \to +\infty$, to a solution (u^1, v^1) of (4.1) in $C_{(0,1)}$. As each (u_R, v_R) is even in x, this solution can be extended by even symmetry in x to C_1, and here satisfies the conditions 1)-4) of Proposition 4.3.5 (hence both u^1 and v^1 are nontrivial). The previous argument can be iterated: indeed, by Corollary 4.2.5 and Lemma 4.3.9, we deduce

$$H_R(r) \leq \frac{H_R(1)}{e^4} e^{4r} \leq C e^{4r} \qquad \forall r > 1;$$

that is, a uniform-in-R bound for $H_R(1)$ induces a uniform-in-R bound for $H_R(r)$ for every $r > 1$. As a consequence we obtain, for every $r > 1$, a solution (u^r, v^r) to equation (4.1) in C_r. A diagonal selection gives the existence of a solution (u, v) to (4.1) in the whole C_∞. This solution inherits by (u^r, v^r) the conditions 1)-4) of Proposition 4.3.5, and thanks to the $\mathcal{C}^2_{\text{loc}}(C_\infty)$ convergence and Lemma 4.3.9 it holds

$$N(r) = \frac{\int_{C_{(0,r)}} |\nabla u|^2 + |\nabla v|^2 + 2u^2v^2}{\int_{\Sigma_r} u^2 + v^2} \leq 2 \qquad \forall r > 0.$$

From Lemma 4.2.9, which we can apply in light of the symmetries of (u, v), we conclude

$$\lim_{r\to+\infty} N(r) \geq 1. \qquad \square$$

The following Lemma completes the proof of point 6) of Theorem 4.1.1. After that, by means of the pointwise estimates $u > \Phi^+$ and $v > \Phi^-$ and Corollary 4.2.5, it is straightforward to obtain also point 7).

Lemma 4.3.12. *It holds* $d := \lim_{r\to\infty} N(r) = 1$.

Proof. In light of the fact that $d \geq 1$, it is sufficient to show that $d \leq 1$. Let (u_{R_n}, v_{R_n}) be the convergent subsequence found in Lemma 4.3.11, which we will simply denote $\{(u_n, v_n)\}$. For $r > 0$ we let

$$f_n(r) := \frac{\int_{C_{(0,r)}} u_n^2 v_n^2}{H_{R_n}(r)}, \quad g_n(r) := \frac{\int_{\Sigma_r} u_n^2 v_n^2}{H_{R_n}(r)}.$$

With f and g we identify the same quantities computed for the limiting profile (u, v). Observe that f_n, g_n, f and g are continuous and nonnegative. By definition,

$$f_n(r) \leq \frac{1}{2} N_{R_n}(r) \leq 1 \qquad \forall r > 0, \tag{4.33}$$

where we used Lemma 4.3.9. The uniform convergence of (u_n, v_n) implies that $f_n \to f$ and $g_n \to g$ uniformly on compact intervals, while by Proposition 4.2.4 we have

$$\int_0^r g_n(s)\,\mathrm{d}s \leq N_{R_n}(r) \quad \text{and} \quad \int_0^r g(s)\,\mathrm{d}s \leq N(r),$$

so that in particular $g_n \in L^1(0, R)$ and $g \in L^1(\mathbb{R}^+)$. By means of the monotonicity formula for the Almgren quotient $\mathfrak{N}$, Proposition 4.2.4, it is possible to refine the computation in Lemma 4.3.9:

$$N_{R_n}(r) = \mathfrak{N}_{R_n}(r) + f_n(r) \leq \mathfrak{N}_{R_n}(R_n) + f_n(r) \leq 1 + f_n(r).$$

In light of the strong $H^1_{\mathrm{loc}}(C_\infty)$ convergence of (u_n, v_n) to (u, v), we deduce

$$N(r) \leq 1 + \lim_{n \to +\infty} f_n(r) = 1 + f(r).$$

We have to show that $f(r) \to 0$ as $r \to +\infty$. To prove this, we begin by computing the logarithmic derivative of f_n:

$$\frac{f_n'(r)}{f_n(r)} = \frac{\int_{\Sigma_r} u_n^2 v_n^2}{\int_{C_{(0,r)}} u_n^2 v_n^2} - 2\frac{E_{R_n}(r)}{H_{R_n}(r)} = \frac{g_n(r)}{f_n(r)} - 2N_{R_n}(r),$$

where we used the fact that $H_{R_n}'(r) = 2E_{R_n}(r)$, see equation (4.8). Exploiting the strong H^1 convergence of the sequence $\{(u_n, v_n)\}$ and the fact that $\lim_{r \to +\infty} N(r) \geq 1$, we deduce that there exist $r_0, \delta > 0$ such that $N_{R_n}(r_0) > \delta$ for every n sufficiently large. Consequently, f_n satisfies the inequality

$$f_n'(r) + 2\delta f_n(r) \leq g_n(r) \qquad \text{for } r \in (r_0, R_n).$$

Multiplying by $e^{2\delta r}$ and integrating in (r_1, r_2) for $r_0 < r_1 < r_2 < R_n$, we obtain

$$f_n(r_2) \leq e^{2\delta(r_1-r_2)} f_n(r_1) + \int_{r_1}^{r_2} g_n(s) e^{2\delta(s-r_2)}\,\mathrm{d}s \leq e^{2\delta(r_1-r_2)} + \int_{r_1}^{r_2} g_n(s)\,\mathrm{d}s,$$

where we used the estimate (4.33). This implies

$$f(r_2) \leq e^{2\delta(r_1-r_2)} + \int_{r_1}^{r_2} g(s)\,\mathrm{d}s \qquad \text{for } r_0 < r_1 < r_2.$$

Since $g \in L^1(\mathbb{R}^+)$ and $f \geq 0$, choosing $r_1 = r_2/2$ we find

$$\limsup_{r\to+\infty} f(r) = 0 = \lim_{r\to+\infty} f(r). \qquad \square$$

4.4 Proof of Theorem 4.1.5

In this section we construct a solution to (4.1) modelled on the harmonic function $\Gamma(x,y) = e^x \sin y$. Our construction is based on the trivial observation that

$$\Phi_R(x,y) := 2\cosh(x+R)e^{-R}\sin y \to \Gamma(x,y) \quad \text{ as } R \to +\infty.$$

4.4.1 Existence in bounded cylinders

As a first step, using the same line of reasoning developed in Proposition 4.3.5, it is possible to show the existence of solution to the system

$$\begin{cases} -\Delta u = -uv^2 & \text{in } C_{(-3R,R)} \\ -\Delta v = -u^2 v & \text{in } C_{(-3R,R)} \\ u, v > 0 \end{cases} \tag{4.34a}$$

(equivalently, we can consider the problem in the rectangle $(-3R,R) \times (0,2\pi)$ with periodic boundary condition on the sides $[-3R,R] \times \{0,2\pi\}$) and such that

$$u_R = \Phi_R^+, \quad v_R = \Phi_R^- \quad \text{on } \Sigma_R \cup \Sigma_{-3R}. \tag{4.34b}$$

More precisely:

Proposition 4.4.1. *There exists a solution (u_R, v_R) to problem (4.34a) with the prescribed boundary conditions (4.34b), such that*

1) $u_R(-R-x,y) = u_R(-R+x,y)$ and $v_R(-R-x,y) = v_R(-R+x,y)$, *2) the symmetries*

$$v_R(x,y) = u_R(x, y-\pi) \qquad u_R(x, \pi - y) = v_R(x, \pi + y)$$
$$u_R\left(x, \frac{\pi}{2} + y\right) = u_R\left(x, \frac{\pi}{2} - y\right) \qquad v_R\left(x, \frac{3}{2}\pi + y\right) = v_R\left(x, \frac{3}{2}\pi - y\right)$$

hold,

3) $u_R - v_R > 0$ *in* $\{\Phi_R > 0\}$ *and* $v_R - u_R > 0$ *in* $\{\Phi_R < 0\}$,

4) $u_R > (\Phi_R)^+$ *and* $v_R > (\Phi_R)^-$.

Sketch of proof. One can recast the proof of Proposition 4.3.5 in this setting. □

Remark 4.4.2. In light of point 1) of the Proposition, it results

$$\partial_x u_R = 0 = \partial_x v_R \qquad \text{on } \Sigma_{-R}.$$

Therefore, the monotonicity formulae proved in Subsection 4.2.1 hold true for (u_R, v_R) in the semi-cylinder C_R.

4.4.2 Compactness of the family $\{(u_R, v_R)\}$

As in the previous section, we denote as E_R, $\mathcal{E}_R$, N_R and $\mathfrak{N}_R$ the functions E^{sym}, $\mathcal{E}^{sym}$, N^{sym} and $\mathfrak{N}^{sym}$ defined in Subsection 4.2.1 when referred to (u_R, v_R). We follow here the same line of reasoning adopted in Subsection 4.3.2. Firstly, it is not difficult to modify the proof of Lemmas 4.3.9 and 4.3.10 obtaining the following estimates:

Lemma 4.4.3. *There holds* $N_R(r) \le 2$, *for every* $R > 0$ *and* $r \in (-R, R)$.

Lemma 4.4.4. *There exists* $C > 0$ *such that* $H_R(1) \le C$ *for every* $R > 1$.

We are in position to show that the family $\{(u_R, v_R)\}$ is compact, in the following sense.

Lemma 4.4.5. *Up to a subsequence,* $\{(u_R, v_R)\}$ *converges in* $\mathcal{C}^2_{\text{loc}}(C_\infty)$, *as* $R \to +\infty$, *to a solution* (u, v) *of* (4.1) *in the whole* C_∞. *This solution has the properties 2)-4) of Proposition 4.4.1.*

Proof. As $H_R(1)$ is bounded in R and $N_R(1) \le 2$, also $E_R(1)$ is bounded in R, and a fortiori

$$\int_{C_1} |\nabla u_R|^2 + |\nabla v_R|^2 \le C \qquad \forall R > 1.$$

This estimate, the boundedness of $H_R(1)$ and a Poincaré inequality of type (4.25) imply that $\{(u_R, v_R)\}$ is bounded in $H^1(C_1)$. Consequently, it is possible to argue as in the proof of Lemma 4.3.11 and obtain the existence of a subsequence of $\{(u_R, v_R)\}$ which converges in $\mathcal{C}^2_{\mathrm{loc}}(C_1)$ to a solution (u^1, v^1) of (4.1) in C_1, which inherits by $\{(u_R, v_R)\}$ the properties 2)-4) of Proposition 4.4.1. In light of Corollary 4.2.5 and Lemma 4.4.3, this procedure can be iterated: indeed

$$H_R(r) \leq \frac{H_R(1)}{e^4} e^{4r} \leq C e^{4r} \qquad \forall r > 1,$$

so that applying the previous argument we obtain a subsequence of $\{(u_R, v_R)\}$ which converges in $\mathcal{C}^2_{\mathrm{loc}}(C_r)$ to a solution (u^r, v^r) of (4.1) in C_r, and inherits by $\{(u_R, v_R)\}$ the properties 2)-4) of Proposition 4.4.1. A diagonal selection gives the existence of a solution (u, v) of (4.1) in the whole C_∞, and this solution enjoys the properties 2)-4) of Proposition 4.4.1. □

Remark 4.4.6. The monotonicity formulae proved in Subsection 4.2.1 do not apply on (u, v), because passing to the limit we lose the Neumann condition $\partial_x u_R = 0 = \partial_x v_R$ on Σ_{-R}.

In the next lemma, we show that (u, v) is a solution with finite energy, so that the achievements proved in Subsection 4.2.2 applies.

Lemma 4.4.7. *Let (u, v) be the solution found in Lemma 4.4.5. It results*

$$\mathcal{E}^{unb}(r) := \int_{C_{(-\infty,r)}} |\nabla u|^2 + |\nabla v|^2 + u^2 v^2 < +\infty \qquad \forall r \in \mathbb{R} \tag{4.35}$$

and

$$\lim_{r \to -\infty} H(r) = \lim_{r \to -\infty} \int_{\Sigma_r} u^2 + v^2 = 0.$$

Recall that $\mathcal{E}^{unb}$ has been defined in subsection 4.2.2.

Proof. Let $\{(u_{R_n}, v_{R_n})\}$ be the converging subsequence found in Lemma 4.4.5, which we simply denote $\{(u_n, v_n)\}$. Since $\{(u_n, v_n)\}$ converges to (u, v) in $\mathcal{C}^2_{\mathrm{loc}}(C_\infty)$, it follows that as $n \to \infty$

$$\left(|\nabla u_n|^2 + |\nabla v_n|^2 + u_n^2 v_n^2\right) \chi_{C_{(-R_n,r)}} \to \left(|\nabla u|^2 + |\nabla v|^2 + u^2 v^2\right) \chi_{C_{(-\infty,r)}}$$

almost everywhere in $C_{(-\infty,r)}$, for every $r > 1$. Therefore, applying Corollary 4.2.5 on (u_n, v_n), Lemma 4.4.4 and the Fatou lemma, we deduce

$$\begin{aligned}\mathcal{E}^{unb}(r) &\leq \liminf_{n \to \infty} \int_{C_{(-\infty,r)}} \left(|\nabla u_n|^2 + |\nabla v_n|^2 + u_n^2 v_n^2\right) \chi_{C_{(-R_n,r)}} \leq \liminf_{n \to \infty} E_{R_n}(r) \\ &= \liminf_{n \to \infty} N_{R_n}(r) H_{R_n}(r) \leq \liminf_{n \to \infty} 2 \frac{H_{R_n}(1)}{e^4} e^{4r} \leq C e^{4r},\end{aligned}$$

which proves the (4.35). To complete the proof, we firstly note that necessarily $\mathcal{E}^{unb}(r) \to 0$ as $r \to -\infty$, and hence the same holds for E^{unb} (which has been defined in Subsection 4.2.2). Assume by contradiction that for a sequence $r_n \to -\infty$ it results $H(r_n) \geq C > 0$. We define

$$(\hat{u}_n(x,y), \hat{v}_n(x,y)) := \frac{1}{\sqrt{H(r_n)}} \left(u(x+r_n, y), v(x+r_n, y) \right).$$

A direct computation shows that

$$\begin{aligned} \int_{C_{(-\infty,0)}} |\nabla \hat{u}_n|^2 + |\nabla \hat{v}_n|^2 &\leq \int_{C_{(-\infty,0)}} |\nabla \hat{u}_n|^2 + |\nabla \hat{v}_n|^2 + 2H(r_n)\hat{u}_n^2\hat{v}_n^2 \\ &= \frac{1}{H(r_n)} E^{unb}(r_n) \to 0 \end{aligned}$$

as $n \to \infty$. Consequently, $(\hat{u}_n, \hat{v}_n)$ tend to be a pair of constant functions of type $(\hat{u}, \hat{v})$ with $\hat{u} = \hat{v}$ (this follows from the symmetries of (u,v)). As

$$C \int_{C_{(-\infty,0)}} \hat{u}_n^2 \hat{v}_n^2 \leq H(r_n) \int_{C_{(-\infty,0)}} \hat{u}_n^2 \hat{v}_n^2 \to 0,$$

necessarily $(\hat{u}_n, \hat{v}_n) \to (0,0)$ almost everywhere in $C_{(-\infty,0)}$. This is in contradiction with the fact that $\int_{\Sigma_0} \hat{u}_n^2 + \hat{v}_n^2 = H(r_n) \geq C$. □

So far we proved that the solution (u,v), found in Lemma 4.4.5, enjoys properties 1)-5) of Theorem 4.1.5, and is such that $H(r) \to 0$ as $r \to -\infty$. The previous lemma enables us to apply the achievements of Subsection 4.2.2 for E^{unb}, H, N^{unb} and $\mathfrak{N}^{unb}$ (which we consider referred to the solution (u,v) found in Lemma 4.4.5), and permits to complete the description of the growth of (u,v), points 6)-7) of Theorem 4.1.5.

Lemma 4.4.8. *Let (u,v) be the solution found in Lemma 4.4.5. It results*

$$\lim_{r \to +\infty} N^{unb}(r) = 1.$$

Proof. Let $\{(u_{R_n}, v_{R_n})\}$be the converging subsequence found in Lemma 4.4.5, which we simply denote $\{(u_n, v_n)\}$. Firstly, arguing as in the proof of the previous lemma, we note that by the $\mathcal{C}^2_{\text{loc}}(C_\infty)$ convergence of (u_n, v_n) to (u,v) it follows that

$$N^{unb}(r) \leq \liminf_{n \to \infty} N_{R_n}(r) \leq 2 \qquad \forall r \in \mathbb{R},$$

thanks to the Fatou lemma. This, together with the symmetries of (u,v), permits to use Lemma 4.2.17, which gives $\lim_{r \to +\infty} N^{unb}(r) \geq 1$. To complete the proof, it is sufficient to show that $\lim_{r \to +\infty} N^{unb}(r) \leq 1$. For any $r > 0$, let

$$f_n(r) := \frac{\int_{C_r} u_n^2 v_n^2}{H_{R_n}(r)}, \qquad g_n(r) := \frac{\int_{\Sigma_r \cup \Sigma_{-r}} u_n^2 v_n^2}{H_{R_n}(r)},$$

and let f and g the same quantities referred to the solution (u, v). Observe that f_n, g_n, f and g are continuous and nonnegative. The uniform convergence of (u_n, v_n) to (u, v) implies that $f_n \to f$ and $g_n \to g$, as $n \to \infty$, uniformly on compact intervals. By definition,

$$f_n(r) \leq \frac{1}{2} N_{R_n}(r) \leq 1 \qquad \forall r > 0.$$

whenever $R_n \geq r$. We claim that $g \in L^1(\mathbb{R}^+)$. Indeed, by the monotonicity of H and Proposition 4.2.14, it follows that

$$\begin{aligned}\int_0^r g(s)\, ds &= \int_0^r \frac{\int_{\Sigma_s} u^2 v^2}{H(s)}\, ds + \int_{-r}^0 \frac{\int_{\Sigma_s} u^2 v^2}{H(-s)}\, ds \\ &\leq \int_{-r}^r \frac{\int_{\Sigma_s} u^2 v^2}{H(s)}\, ds \leq \int_{-\infty}^r \frac{\int_{\Sigma_s} u^2 v^2}{H(s)}\, ds \leq N^{unb}(r),\end{aligned}$$

for every $r > 0$. Let $r > 0$; it is possible to refine the computation on Lemma 4.3.9 to obtain

$$N_{R_n}(r) \leq 1 + f_n(r) + \frac{\int_{C_{(-R_n,-r)}} u_n^2 v_n^2}{H_{R_n}(r)} \leq 1 + f_n(r) + \frac{E_{R_n}(-r)}{H_{R_n}(r)}$$

Therefore, using again the Fatou lemma we deduce

$$N^{unb}(r) \leq \liminf_{n\to\infty} N_{R_n}(r) \leq 1 + f(r) + \liminf_{n\to\infty} \frac{E_{R_n}(-r)}{H_{R_n}(r)},$$

and to complete the proof we show that

$$\lim_{r\to+\infty} \left(f(r) + \liminf_{n\to\infty} \frac{E_{R_n}(-r)}{H_{R_n}(r)} \right) = 0. \tag{4.36}$$

Firstly, we note that

$$\liminf_{n\to\infty} \frac{E_{R_n}(-r)}{H_{R_n}(r)} = \liminf_{n\to\infty} \frac{N_{R_n}(-r) H_{R_n}(-r)}{H_{R_n}(r)} \leq 2 \liminf_{n\to\infty} \frac{H_{R_n}(-r)}{H_{R_n}(r)}.$$

From the $\mathcal{C}^2_{\mathrm{loc}}(C_\infty)$ convergence of (u_n, v_n) to (u, v) it follows

$$2 \liminf_{n\to\infty} \frac{H_{R_n}(-r)}{H_{R_n}(r)} = 2 \frac{H(-r)}{H(r)} \to 0 \qquad \text{as } r \to +\infty$$

where we used Lemma 4.4.7 and the fact that $H(r) > H(0) > 0$ for every $r > 0$. For the (4.36) it remains to prove that $f(r) \to 0$ as $r \to +\infty$. Having observed that $\lim_{r\to+\infty} N(r) \geq 1$ and that $g \in L^1(\mathbb{R}^+)$, it is not difficult to adapt the conclusion of the proof of Lemma 4.3.12. □

4.5 Systems with many components

In this section we are going to prove the existence of entire solutions with exponential growth for the k component system (4.5). Our construction is based on the elementary limit

$$\lim_{d\to+\infty} \mathfrak{Im}\left[\left(1+\frac{z}{d}\right)^d\right] = e^x \sin y,$$

which shows that the harmonic function $e^x \sin y$ can be obtained as limit of homogeneous harmonic polynomial. We wish to prove that the same idea applies to solutions of the system (4.5): there exists an entire solution to (4.5) having exponential growth which can be obtained as limit of entire solutions having algebraic growth.

4.5.1 Preliminary results

We recall some results contained in [13]. For $d \in \mathbb{N}/2$, let G_d be the rotation of angle π/d in counterclockwise sense.

Theorem 4.5.1 (Theorem 1.6 of [13]). *Let $k \geq 2$ be a positive integer, let $d \in \mathbb{N}/2$ be such that*

$$2d = hk \qquad \textit{for some } h \in \mathbb{N}.$$

There exists a solution $(u_1^d, \dots, u_k^d)$ to the system (4.5) which enjoys the following symmetries:

$$\begin{aligned} u_i^d(x,y) &= u_i^d(G_d^k(x,y)) \\ u_i^d(x,y) &= u_{i+1}^d(G_d(x,y)) \\ u_{k+1-i}^d(x,y) &= u_i^d(x,-y), \end{aligned} \tag{4.37}$$

where we recall that indexes are meant $\mod k$. *Moreover,*

$$\lim_{r\to+\infty} \frac{1}{r^{1+2d}} \int_{\partial B_r} \sum_{i=1}^k \left(u_i^d\right)^2 = b \in (0,+\infty), \tag{4.38}$$

and

$$\lim_{r\to+\infty} \frac{r\int_{B_r} \sum_{i=1}^k |\nabla u_i^d|^2 + \sum_{1\leq i<j\leq k} \left(u_i^d u_j^d\right)^2}{\int_{\partial B_r} \sum_{i=1}^k \left(u_i^d\right)^2} = d, \tag{4.39}$$

where B_r denotes the ball of center 0 and radius r.

The solution $(u_1^d, \dots, u_k^d)$ is modelled on the harmonic function $\mathfrak{Im}(z^d)$, as specified by the symmetries (4.37) and by the growth condition (4.38). In the quoted statement, the authors modelled their construction on the functions $\mathfrak{Re}(z^d)$: it is straightforward to obtain an analogous result replacing the real part with the imaginary one.

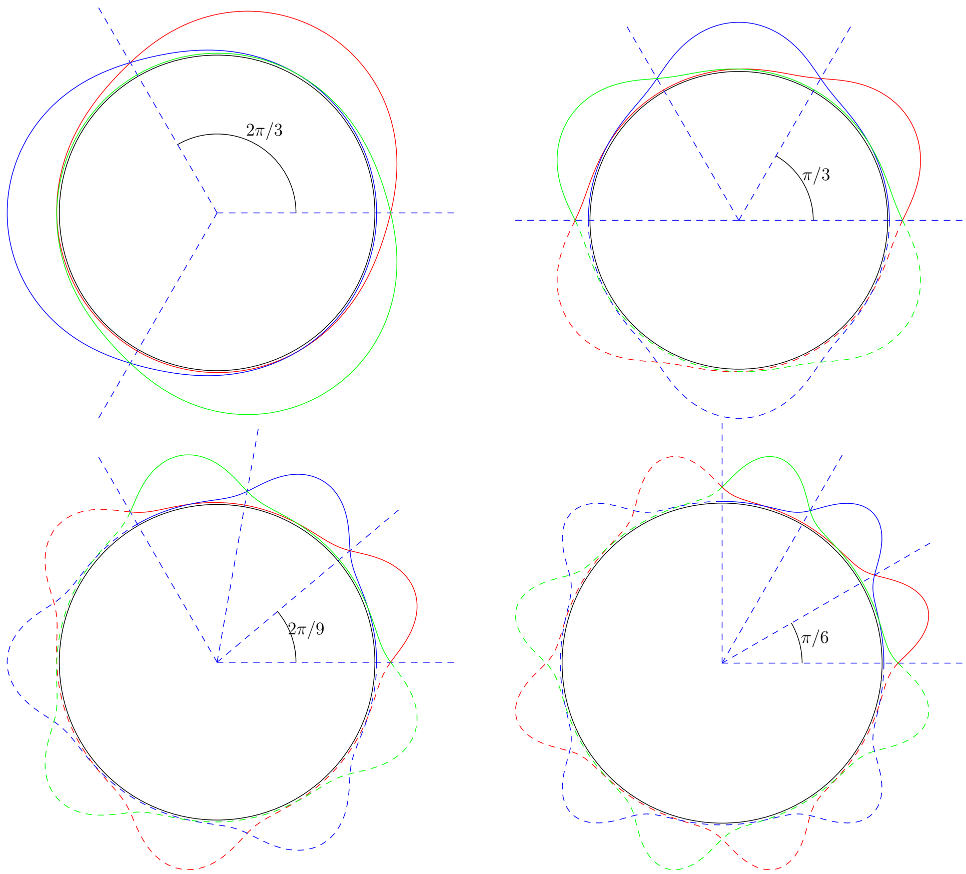


Figure 4.1: In the figure we represent some of the solutions obtained in Theorem 4.5.1. Here the number of components is set as $k = 3$: each component is drawn with a different color. On the other hand the periodicity (that is, how many times the patch of 3-components is replicated in the circle) is given by $h = 1$ (up left), $h = 2$ (up right), $h = 3$ (down left) and $h = 4$ (down right), respectively. As a consequence, the growth rate d varies as $d = 3/2, 3, 9/2, 6$, following the same order.

Remark 4.5.2. We point out that the symmetries (4.37) implies that u_1^d is symmetric with respect to the reflection with the axis $y = \tan(\pi/(2d))\, x$.

For a solution $(u_1, \dots, u_k)$ of system (4.5) in $\mathbb{R}^2$, we introduce the functionals

$$\begin{aligned} E^{alg}(r;\Lambda) &:= \int_{B_r} \sum_{i=1}^{k} |\nabla u_i|^2 + \Lambda \sum_{1\le i<j\le k} (u_i u_j)^2 \\ H^{alg}(r) &:= \frac{1}{r}\int_{\partial B_r} \sum_{i=1}^{k} (u_i)^2 \end{aligned} \tag{4.40}$$

The index *alg* denotes the fact that these quantities are well suited to describe the growth of $(u_1, \dots, u_k)$ under the assumption that $(u_1, \dots, u_k)$ has algebraic growth. In particular, as proved in Lemma 2.1 of [34] and Corollary A.8 of [36] for the case $k = 2$, the Almgren quotient

$$N^{alg}(r;1) := \frac{E^{alg}(r;1)}{H^{alg}(r)}$$

is bounded in $r \in \mathbb{R}^+$ if and only if $(u_1, \dots, u_k)$ has algebraic growth.

It is not difficult to adapt the proof of Proposition 5.2 in [13] to obtain the following general result (in the sense that it holds true for an arbitrary solution of (4.5) in $\mathbb{R}^N$, for any dimension $N \ge 2$).

Proposition 4.5.3 (see Proposition 5.2 of [13])**.** *Let $N \ge 2$,*

$$\Lambda \in \begin{cases} \left[1, \frac{N}{N-2}\right] & \text{if } N > 2 \\ [1, +\infty) & \text{if } N = 2, \end{cases}$$

and let $(u_1, \dots, u_k)$ be a solution of (4.5) in $\mathbb{R}^N$; the Almgren quotient

$$N^{alg}(r;\Lambda) := \frac{E^{alg}(r;\Lambda)}{H^{alg}(r)} = \frac{r \int_{B_r} \sum_{i=1}^{k} |\nabla u_i|^2 + \Lambda \sum_{1\le i<j\le k} (u_i u_j)^2}{\int_{\partial B_r} \sum_{i=1}^{k} (u_i)^2}$$

is nondecreasing in r.

Proof. We observe that

$$\begin{aligned}\frac{d}{dr}E^{alg}(r;\Lambda) &= \frac{d}{dr}\left(\frac{1}{r^{N-2}}\int_{B_r}\sum_i |\nabla u_i|^2 + \sum_{i<j}(u_iu_j)^2\right)\\ &\quad + \frac{d}{dr}\left(\frac{\Lambda-1}{r^{N-2}}\int_{B_r}\sum_{i<j}(u_iu_j)^2\right)\\ &= \frac{2}{r^{N-2}}\int_{\partial B_r}\sum_i(\partial_\nu u_i)^2 + \frac{2}{r^{N-1}}\int_{B_r}\sum_{i<j}(u_iu_j)^2\\ &\quad + \frac{(2-N)(\Lambda-1)}{r^{N-1}}\int_{B_r}\sum_{i<j}u_i^2u_j^2 + \frac{\Lambda-1}{r^{N-2}}\int_{\partial B_r}\sum_{i<j}u_i^2u_j^2,\end{aligned}$$

where we used equation (5.3) in [13]. Proceeding as in the proof of Proposition 5.2 in [13], one obtains

$$\frac{d}{dr}N^{alg}(r;\Lambda) \geq (2+(\Lambda-1)(2-N))\frac{\int_{B_r}\sum_{i<j}u_i^2u_j^2}{r^{N-1}H^{alg}(r)} + \frac{(\Lambda-1)\int_{\partial B_r}\sum_{i<j}u_i^2u_j^2}{r^{N-2}H^{alg}(r)},$$

which is ≥ 0 by our assumption on Λ. □

Remark 4.5.4. In [13] the authors considered the case $\Lambda = 1$.

We work in the plane $\mathbb{R}^2$, so that it is possible to choose $\Lambda = 2$ in Proposition 4.5.3. We denote $E_d(\cdot;\Lambda)$ and H_d the quantities defined in (4.40) when referred to the functions $(u_1^d,\ldots,u_k^d)$ defined in Theorem 4.5.1; also, we denote

$$N_d(\cdot;\Lambda) := \frac{E_d(\cdot;\Lambda)}{H_d}.$$

In case $\Lambda = 2$, we simply write E_d and N_d to ease the notation.

Lemma 4.5.5. *Let $(u_1^d,\ldots,u_k^d)$ be defined in Theorem 4.5.1. It holds* $\lim_{r\to+\infty} N_d(r) = d$.

Proof. It is an easy consequence of the (4.39) and of Corollary 5.8 in [13], where it is proved that for the solution $(u_1^d,\ldots,u_k^d)$ it holds

$$\lim_{r\to+\infty}\frac{E_d(r;2)}{r^{2d}} = \lim_{r\to+\infty}\frac{E_d(r;1)}{r^{2d}}.$$

Therefore,

$$\lim_{r\to+\infty} N_d(r) = \lim_{r\to+\infty} \frac{E_d(r;2)}{H_d(r)} = \lim_{r\to+\infty} \frac{E_d(r;2)}{r^{2d}} \cdot \lim_{r\to+\infty} \frac{r^{2d}}{H_d(r)}$$
$$= \lim_{r\to+\infty} \frac{E_d(r;1)}{r^{2d}} \cdot \lim_{r\to+\infty} \frac{r^{2d}}{H_d(r)} = \lim_{r\to+\infty} N_d(r;1) = d. \qquad \square$$

As a consequence, the following doubling property holds true:

Proposition 4.5.6 (See Proposition 5.3 of [13])**.** *For any $0 < r_1 < r_2$ it holds*

$$\frac{H_d(r_2)}{r_2^{2d}} \leq \frac{H_d(r_1)}{r_1^{2d}}.$$

Proof. A direct computation shows that

$$\frac{d}{dr} \log \frac{H_d(r)}{r^{2d}} = \frac{2N_d(r)}{r} - \frac{2d}{r} \leq 0;$$

an integration gives the thesis. $\square$

Let us consider the scaling

$$(u^d_{1,R}, \ldots, u^d_{k,R}) := \left(\frac{2d}{kH_d(R)}\right)^{\frac{1}{2}} \left(u^d_1(Rx, Ry), \ldots, u^d_k(Rx, Ry)\right),$$

where R will be determined later as a function of d. We see that

$$\begin{cases} -\Delta u^d_{i,R} = -\beta^d_R\, u^d_{i,R} \sum_{j\neq i} \left(u^d_{j,R}\right)^2 & \text{in } \mathbb{R}^2 \\ \int_{\partial B_1} \sum_{i=1}^k \left(u^d_{i,R}\right)^2 = \frac{2d}{k} \end{cases} \tag{4.41}$$

where $\beta^d_R := kH_d(R)R^2/(2d)$.

Remark 4.5.7. As a function of R, β^d_R is continuous and such that $\beta^d_R \to 0$ if $R \to 0$ and $\beta^d_R \to \infty$ if $R \to \infty$.

Accordingly with our scaling, we introduce the new Almgren quotient

$$N_{d,R}(r) := \frac{E_{d,R}(r)}{H_R(r)} = \frac{r \int_{B_r} \sum_{i=1}^k |\nabla u^d_{i,R}|^2 + 2\beta^d_R \sum_{1\leq i<j\leq k} \left(u^d_{i,R}\, u^d_{j,R}\right)^2}{\int_{\partial B_r} \sum_{i=1}^k \left(u^d_{i,R}\right)^2}.$$

We point out that $N_{d,R}(r) = N_d(Rr)$, so that from Lemma 4.5.5 and the monotonicity of N_d we deduce

$$N_{d,R}(r) \leq d \qquad \forall r, R > 0, \tag{4.42}$$

for every d. By the symmetries, the solution $(u^d_{1,R}, \ldots, u^d_{k,R})$ is $k\pi/d$-periodic with respect to the angular component, thus it is convenient to restrict our attention to the cones

$$S^d_r := \left\{(\rho,\theta) : \rho \in (0,r), \theta \in \left(0, \frac{k\pi}{d}\right)\right\} \quad \text{and} \quad S^d := \left\{(\rho,\theta); \rho > 0, \theta \in \left(0, \frac{k\pi}{d}\right)\right\}.$$

The boundary ∂S^d_r can be decomposed as $\partial S^d_r = \partial_p S^d_r \cup \partial_r S^d_r$, where

$$\partial_p S^d_r := (0,r) \times \left\{0, \frac{k\pi}{d}\right\} \quad \text{and} \quad \partial_r S^d_r := \{r\} \times \left(0, \frac{k\pi}{d}\right).$$

Taking into account the periodicity of $(u^d_{1,R}, \ldots, u^d_{k,R})$, we note that $(u^d_{1,R}, \ldots, u^d_{k,R})$ has periodic boundary conditions on $\partial_p S^d_r$; furthermore

$$\begin{aligned}
E_{d,R}(r) &= \frac{2d}{k} \int_{S^d_r} \sum_i |\nabla u^d_{i,R}|^2 + 2\beta^d_R \sum_{i<j} \left(u^d_{i,R}\, u^d_{j,R}\right)^2 \\
H_{d,R}(r) &= \frac{2d}{kr} \int_{\partial_r S^d_r} \sum_i \left(u^d_{i,R}\right)^2 \\
N_{d,R}(r) &= \frac{r \int_{S^d_r} \sum_i |\nabla u^d_{i,R}|^2 + 2\beta^d_R \sum_{i<j} \left(u^d_{i,R}\, u^d_{j,R}\right)^2}{\int_{\partial S^d_r} \sum_i \left(u^d_{i,R}\right)^2}.
\end{aligned} \tag{4.43}$$

4.5.2 A blow-up in a neighbourhood of $(1,0)$

In order to pursue our strategy, we consider the further scaling

$$(\hat{u}^d_{1,R}(x,y), \ldots, \hat{u}^d_{k,R}(x,y)) = \frac{\sqrt{\beta^d_R}}{d} \left(u^d_{1,R}\left(1 + \frac{x}{d}, \frac{y}{d}\right), \ldots, u^d_{k,R}\left(1 + \frac{x}{d}, \frac{y}{d}\right)\right).$$

Accordingly, we consider the scaled domains

$$\hat{S}^d_r = d\left(S^d_r - (1,0)\right) \quad \text{and} \quad \hat{S}^d = d\left(S^d - (1,0)\right)$$

and the respective boundaries. Having in mind to let $d \to \infty$, we observe that this scaling is a blow-up centred in the point $(1,0)$. It is easy to verify that $(\hat{u}^d_{1,R}, \ldots, \hat{u}^d_{k,R})$

solves (see (4.41))

$$\begin{cases} -\Delta \hat{u}^d_{i,R} = -\hat{u}^d_{i,R} \sum_{j\neq i} \left(\hat{u}^d_{j,R}\right)^2 & \text{in } \hat{S}^d \\ \int_{\partial_r \hat{S}^d_1} \sum_{i=1}^k \left(\hat{u}^d_{i,R}\right)^2 = \frac{\beta^d_R}{d}, \end{cases} \tag{4.44}$$

with suitable periodic conditions on $\partial\hat{S}^d$. A direct computation shows that from (4.43) it follows

$$N_{d,R}(r) = d\frac{r\int_{\hat{S}^d_r} \sum_i |\nabla \hat{u}^d_{i,R}|^2 + 2\sum_{i<j}\left(\hat{u}^d_{i,R}\hat{u}^d_{j,R}\right)^2}{\int_{\partial_r \hat{S}^d_r} \sum_i \left(\hat{u}^d_{i,R}\right)^2},$$

where in the new coordinates

$$r = \sqrt{\left(1+\frac{x}{d}\right)^2 + \left(\frac{y}{d}\right)^2}. \tag{4.45}$$

Therefore, we are led to define a different Almgren quotient for the scaled functions $(\hat{u}^d_{1,R}, \dots, \hat{u}^d_{k,R})$:

$$\begin{aligned} \hat{E}_{d,R}(r) &:= \int_{\hat{S}^d_r} \sum_{i=1}^k |\nabla \hat{u}^d_{i,R}|^2 + 2\sum_{1\le i<j\le k} \left(\hat{u}^d_{i,R}\hat{u}^d_{j,R}\right)^2 \\ \hat{H}_{d,R}(r) &:= \frac{1}{r}\int_{\partial_r \hat{S}^d_r} \sum_{i=1}^k \left(\hat{u}^d_{i,R}\right)^2 \\ \hat{N}_{d,R}(r) &:= \frac{\hat{E}_{d,R}(r)}{\hat{H}_{d,R}(r)} = \frac{1}{d}N_{d,R}(r). \end{aligned}$$

From equation (4.42), we deduce

$$\hat{N}_{d,R}(r) \le 1 \qquad \forall r, R > 0,\ \forall d \in \frac{\mathbb{N}}{2}. \tag{4.46}$$

In order to understand the behaviour of $(\hat{u}^d_{1,R}, \dots, \hat{u}^d_{k,R})$ when $d \to \infty$, we fix $R = R(d)$ to get a non-degeneracy condition.

Lemma 4.5.8. *For every $d \in \mathbb{N}/2$ there exists $R_d > 0$ such that*

$$\hat{H}_{d,R_d}(1) = \int_{\partial_r \hat{S}^d_1} \sum_i \left(\hat{u}^d_{i,R_d}\right)^2 = 1.$$

Proof. By (4.44) we know that $\hat{H}_d(1) = \beta_R^d/d$, so that we have to find R_d such that $\beta_R^d = d$. As observed in Remark 4.5.7, this choice is possible. □

We denote $(\hat{u}_1^d, \dots, \hat{u}_k^d) := (\hat{u}_{1,R_d}^d, \dots, \hat{u}_{k,R_d}^d)$, $\hat{H}_d := \hat{H}_{d,R_d}$, $\hat{E}_d := \hat{E}_{d,R_d}$, $\hat{N}_d := \hat{N}_{d,R_d}$ and $\beta^d := \beta_{R_d}^d$. We aim at proving that, up to a subsequence, the family $\left\{(\hat{u}_1^d, \dots, \hat{u}_k^d) : \ d \in \mathbb{N}/2\right\}$ converges, as $d \to +\infty$, to a solution of (4.5). To this aim, major difficulties arise from the fact that $\hat{S}_r^d$ and $\hat{S}^d$ depend on d; in the next lemma we show that this problem can be overcome thanks to a convergence property of these domains.

Lemma 4.5.9. *For any $r > 1$, the sets $\hat{S}_r^d$ converge to $\mathbb{R} \times (0, k\pi)$ as $k \to +\infty$, in the sense that*

$$\mathbb{R} \times (0, k\pi) = \operatorname{Int}\left(\bigcap_{n \in \frac{\mathbb{N}}{2}} \bigcup_{d > n} \hat{S}_r^d\right),$$

where for $A \subset \mathbb{R}^2$ we mean that $\operatorname{Int}(A)$ *denotes the inner part A. Analogously,*

$$\mathbb{R} \times (0, k\pi) = \operatorname{Int}\left(\bigcap_{n \in \frac{\mathbb{N}}{2}} \bigcup_{d > n} \hat{S}^d\right) \quad \text{and} \quad (-\infty, 0) \times (0, k\pi) = \operatorname{Int}\left(\bigcap_{n \in \frac{\mathbb{N}}{2}} \bigcup_{d > n} \hat{S}_1^d\right),$$

and for every $\bar{x} \in \mathbb{R}$

$$(-\infty, \bar{x}) \times (0, k\pi) = \operatorname{Int}\left(\bigcap_{n \in \frac{\mathbb{N}}{2}} \bigcup_{d > n} \hat{S}_{1+\frac{\bar{x}}{d}}^d\right).$$

Proof. We prove only the first claim. Let $r > 1$.

Step 1) $\mathbb{R} \times (0, k\pi) \subset \bigcap_{n \in \frac{\mathbb{N}}{2}} \bigcup_{d > n} \hat{S}_r^d$.

Let $(x, y) \in \mathbb{R} \times (0, k\pi)$. We show that for every $d \in \mathbb{N}/2$ sufficiently large $(x, y) \in \hat{S}_r^d$, that is, $(1 + x/d, y/d) \in S_r^d$, which means

$$\sqrt{\left(1 + \frac{x}{d}\right)^2 + \left(\frac{y}{d}\right)^2} < r \quad \text{and} \quad \arctan\left(\frac{y}{x+d}\right) \in \left(0, \frac{k\pi}{d}\right).$$

For the first condition it is possible to choose d sufficiently large, as $r > 1$. To prove the second condition, we start by considering $d > -x$, so that $\arctan\left(y/(x+d)\right) > 0$. Now, provided d is sufficiently large

$$\arctan\left(\frac{y}{x+d}\right) < \frac{k\pi}{d} \quad \Longleftrightarrow \quad y < (x+d)\tan\left(\frac{k\pi}{d}\right).$$

Since $y < k\pi$, there exists $\varepsilon > 0$ such that $y \le k(1-\varepsilon)\pi$. Let $\bar d$ be sufficiently large so that

$$x + d > \left(1 - \frac{\varepsilon}{2}\right) d \quad \text{and} \quad \frac{d}{k\pi} \tan\left(\frac{k\pi}{d}\right) > 1 - \frac{\varepsilon}{2}$$

for every $d > \bar d$. Then

$$(x+d)\tan\left(\frac{k\pi}{d}\right) > \left(1 - \frac{\varepsilon}{2}\right)^2 k\pi > (1-\varepsilon)k\pi \ge y$$

whenever $d > \bar d$.

Step 2) $\displaystyle\bigcap_{n \in \frac{\mathbb{N}}{2}} \bigcup_{d>n} \hat S_r^d \subset \mathbb{R} \times [0, k\pi]$.

We show that

$$(\mathbb{R} \times [0,k\pi])^c \subset \left(\bigcap_{n \in \frac{\mathbb{N}}{2}} \bigcup_{d>n} \hat S_r^d \right)^c .$$

If $(x,y) \notin \mathbb{R} \times [0,k\pi]$, then $y > k\pi$ or $y < 0$. We consider only the case $y > k\pi$; in such a situation

$$y > k\pi = \lim_{d\to\infty} (x+d)\tan\left(\frac{k\pi}{d}\right),$$

so that $(x,y) \notin \hat S_r^d$ for every d sufficiently large. □

Remark 4.5.10. As a consequence of the previous result, we see that

$$\partial_r \hat S_1^d \to \{0\} \times [0,k\pi] \quad \text{and} \quad \partial_r \hat S^d_{1+\frac{\bar x}{d}} \to \{\bar x\} \times [0,k\pi]$$

for every $\bar x \in \mathbb{R}$.

Remark 4.5.11. Recall the expression of r in the new variable, given by (4.45). For every $r > 0$ and $d \in \mathbb{N}/2$ there exists $\xi(r,d)$ such that

$$r = 1 + \frac{\xi(r,d)}{d} \quad \Longleftrightarrow \quad \xi(r,d) = d(r-1).$$

Note that for every $(x,y) \in \partial_r \hat S_r^d$ it results $x < \xi(r,d)$. On the contrary, fixing $(x,y) \in \partial_r \hat S_r^d$ there exists $\zeta(d,x,y)$ such that

$$r = \sqrt{\left(1 + \frac{x}{d}\right)^2 + \left(\frac{y}{d}\right)^2} = 1 + \frac{x}{d} + \zeta(d,x,y).$$

In particular, if $y = 0$ we have $\zeta(d,x,0) = 0$, while if $y > 0$, $\zeta(d,x,y) \sim d^{-2}$.

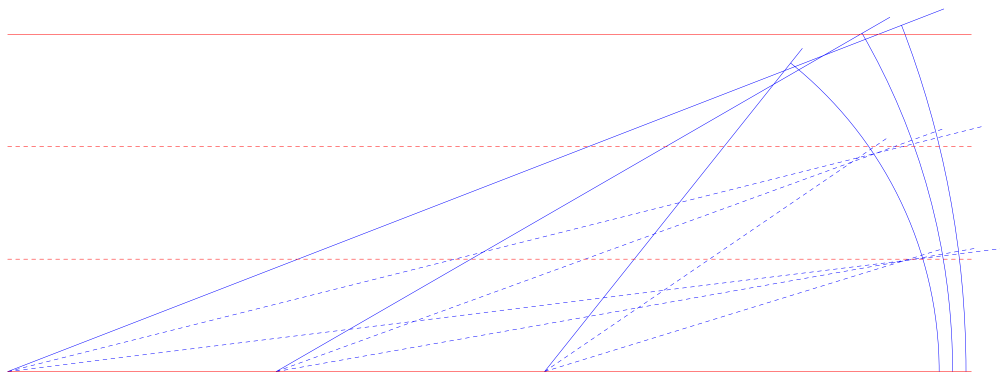


Figure 4.2: Visualization of the construction in Lemma 4.5.9. In red the limiting set $\mathbb{R} \times (0, k\pi)$. In blue some of the scaled domains $\hat{S}_r^d$, for $r > 1$.

We are ready to prove the convergence of $\{(\hat{u}_1^d, \ldots, \hat{u}_k^d)\}$ as $d \to \infty$.

Lemma 4.5.12. *Up to a subsequence, $\{(\hat{u}_1^d, \ldots, \hat{u}_k^d)\}$ converges in $\mathcal{C}^2_{\mathrm{loc}}(C_\infty)$, as $d \to \infty$, to a nontrivial solution $(\hat{u}_1, \ldots, \hat{u}_k)$ of (4.5). This solution, which is $k\pi$-periodic in y, enjoys the symmetries*

$$\hat{u}_{i+1}(x, y) = \hat{u}_i(x, y - \pi) \quad \textit{and} \quad \hat{u}_1\left(x, y + \frac{\pi}{2}\right) = \hat{u}_1\left(x, y - \frac{\pi}{2}\right)$$

Proof. From Proposition 4.5.6 and Lemma 4.5.8, we deduce that for any $r \geq 1$ and d the inequality

$$\frac{\hat{H}_d(r)}{r^{2d}} = \frac{k\beta^d H_d(r)}{2d^2 r^{2d}} \leq \frac{k\beta^d}{2d^2} H_d(1) = \hat{H}_d(1) = 1$$

holds. For every $x > 0$, let $r = 1 + x/d$; for every d sufficiently large, we have

$$\hat{H}_d\left(1 + \frac{x}{d}\right) \leq \left(1 + \frac{x}{d}\right)^{2d} \leq 2e^{2x} \tag{4.47}$$

Recalling the (4.46) (which we apply for $R = R_d$), we deduce

$$\hat{E}_d\left(1 + \frac{x}{d}\right) = \hat{N}_d\left(1 + \frac{x}{d}\right)\hat{H}_d\left(1 + \frac{x}{d}\right) \leq 2e^{2x} \tag{4.48}$$

for every d sufficiently large. Recall that $(\hat{u}_1^d, \ldots, \hat{u}_k^d)$ can be extended by angular periodicity in the whole plane $\mathbb{R}^2$. Let us introduce

$$T_r^d := \left\{(\rho, \theta) : \rho < r,\ \theta \in \left(-\frac{\pi}{d}, (k+1)\frac{\pi}{d}\right)\right\} \supset S_r^d,$$

and let $\hat{T}_r^d := d\left(T_r^d - (1,0)\right) \supset \hat{S}_r^d$. Suitably modifying the argument in Lemma 4.5.9, it is not difficult to see that

$$\mathrm{Int}\left(\bigcap_{n\in\frac{\mathbb{N}}{2}}\bigcup_{d>n}\hat{T}^d_{1+\frac{\bar{x}}{d}}\right) = (-\infty,\bar{x})\times(-\pi,(k+1)\pi)$$

for every $\bar{x}\in\mathbb{R}$. Hence, let B an open ball contained in $\mathbb{R}\times(-\pi,(k+1)\pi)$, and let $x_B := \sup\{x : (x,y)\in B\}$, so that $B\subset(-\infty,x_B+1)\times(-\pi,(k+1)\pi)$. Using the same argument in the proof of Lemma 4.5.9, it is possible to show that

$$B\subset \hat{T}^d_{1+\frac{x_B+1}{d}},$$

for every d sufficiently large, and by the (4.48) and the periodicity of $(\hat{u}_1,\dots,\hat{u}_k)$ we deduce

$$\int_B\sum_i|\nabla\hat{u}_i^d|^2 \le 3\hat{E}_d\left(1+\frac{x_B+1}{d}\right)\le 6e^{2(x_B+1)}$$

whenever d is sufficiently large. This, together with (4.47), implies that $\{(\hat{u}_1^d,\dots,\hat{u}_k^d)\}$ is uniformly bounded in $H^1(B)$, for every $B\subset\mathbb{R}\times(-\pi,(k+1)\pi)$. By the boundedness of the trace operator, this bound provides a uniform-in-d bound on the $L^2(\partial K)$ norm for every compact $K\subset\subset\mathbb{R}\times(-\pi,(k+1)\pi)$, which in turns, due to the subharmonicity of u_i^d, gives a uniform-in-d bound on the $L^\infty(K)$ norm of $\{(\hat{u}_1^d,\dots,\hat{u}_k^d)\}$, for every compact set $K\subset\subset\mathbb{R}\times(-\pi,(k+1)\pi)$. The standard regularity theory for elliptic equations guarantees that when $d\to\infty$ then $\{(\hat{u}_1^d,\dots,\hat{u}_k^d)\}$ converges in $\mathcal{C}^2_{\mathrm{loc}}(\mathbb{R}\times(-\pi,(k+1)\pi))$, up to a subsequence, to a function $(\hat{u}_1,\dots,\hat{u}_k)$ which is a solution to (4.5). By the convergence and by the normalization required in Lemma 4.5.8, we deduce that (recall also the convergence of the boundaries $\partial\hat{S}_1^d$, Remark 4.5.10)

$$\int_0^{k\pi}\sum_i\hat{u}_i(0,y)^2\,dy = 1;$$

in particular, $(\hat{u}_1,\dots,\hat{u}_k)$ is nontrivial. The $k\pi$-periodicity in y follows directly form the convergence of the domains, Lemma 4.5.9. By the pointwise convergence of $(\hat{u}_1^d,\dots,\hat{u}_k^d)$ to $(\hat{u}_1,\dots,\hat{u}_k)$ and by the symmetries of each function $(\hat{u}_1^d,\dots,\hat{u}_k^d)$ (see equation (4.37) and Remark 4.5.2) we deduce also that

$$\hat{u}_{i+1}(x,y) = \hat{u}_i\left(x,y-\pi\right)\quad\text{and}\quad\hat{u}_1\left(x,y+\frac{\pi}{2}\right) = \hat{u}_1\left(x,y-\frac{\pi}{2}\right). \qquad \square$$

4.5.3 Characterization of the growth of $(\hat{u}_1, \dots, \hat{u}_k)$

So far we proved the existence of a solution $(\hat{u}_1, \dots, \hat{u}_k)$ of (4.5) which enjoys the properties 1) and 2) of Theorem 4.1.9. In this subsection, we are going to complete the proof of the quoted statement, showing that $(\hat{u}_1, \dots, \hat{u}_k)$ enjoys also the properties 3)-5). We denote by $\hat{\mathcal{E}}, \hat{E}, \hat{H}$ and $\hat{N}$ the quantities $\mathcal{E}^{unb}, E^{unb}, H$ and N^{unb} introduced in Subsection 4.2.2 when referred to the function $(\hat{u}_1, \dots, \hat{u}_k)$. Firstly, we show that $(\hat{u}_1, \dots, \hat{u}_k)$ has finite energy, point 3) of Theorem 4.1.9, and that $\hat{H}(x) \to 0$ as $x \to -\infty$.

Lemma 4.5.13. *For every $x \in \mathbb{R}$ there holds $\hat{\mathcal{E}}(x) < +\infty$. In particular*

$$\hat{\mathcal{E}}(x) \leq \liminf_{d\to\infty} \hat{\mathcal{E}}_d\left(1+\frac{x}{d}\right) \quad \text{and} \quad \hat{E}(x) \leq \liminf_{d\to\infty} \hat{E}_d\left(1+\frac{x}{d}\right).$$

Furthermore, $\lim_{x\to-\infty} \hat{H}(x) = 0$.

Proof. By the $\mathcal{C}^2_{\text{loc}}(\mathbb{R}^2)$ convergence of $(\hat{u}_1^d, \dots, \hat{u}_k^d)$ to $(\hat{u}_1, \dots, \hat{u}_k)$ and by the convergence properties of the domains $\hat{S}^d_{1+\frac{x}{d}}$, Lemma 4.5.9, we deduce that as $n \to \infty$

$$\left(\sum_i |\nabla \hat{u}_i^d|^2 + \sum_{i<j}\left(\hat{u}_i^d \hat{u}_j^d\right)^2\right)\chi_{\hat{S}^d_{1+\frac{x}{d}}} \to \left(\sum_i |\nabla \hat{u}_i|^2 + \sum_{i<j}(\hat{u}_i\hat{u}_j)^2\right)\chi_{C_{(-\infty,x)}},$$

almost everywhere in C_∞, for every $x \in \mathbb{R}$. As a consequence, we can apply the Fatou lemma obtaining

$$\hat{\mathcal{E}}(x) \leq \liminf_{d\to\infty} \hat{\mathcal{E}}_d\left(1+\frac{x}{d}\right) \leq 2e^{2x},$$

where the uniform boundedness of $\hat{\mathcal{E}}_d(1+x/d)$ comes from (4.48). To prove that $\hat{H}(x) \to 0$ as $x \to -\infty$, we can proceed with the same argument developed in Lemma 4.4.7. □

In light of the previous lemma, the monotonicity formulae proved in Subsection 4.2.2 applies for $\hat{\mathcal{E}}, \hat{E}, \hat{H}$ and $\hat{N}$.

Lemma 4.5.14. *It holds*

$$\lim_{x\to+\infty} \hat{N}(x) = 1.$$

Proof. By Proposition 4.2.14, we know that $\hat{N}$ is nondecreasing in x, and thanks to the symmetries of $(\hat{u}_1, \dots, \hat{u}_k)$, see Lemma 4.5.12, Lemma 4.2.17 implies that

$$\lim_{x\to+\infty} \hat{N}(x) \geq 1.$$

It remains to show that this limit is smaller than 1. This follows from the estimates of Lemma 4.5.13 and from the strong convergence of $(\hat{u}_1^d, \ldots, \hat{u}_k^d) \to (\hat{u}_1, \ldots, \hat{u}_k)$, which implies that $\hat{H}_d(1 + x/d) \to \hat{H}(x)$ as $d \to \infty$: therefore, for every $x \in \mathbb{R}$

$$\hat{N}(x) = \frac{\hat{E}(x)}{\hat{H}(x)} \leq \frac{\liminf_{d\to\infty} \hat{E}_d(x)}{\lim_{d\to\infty} \hat{H}_d(x)} = \liminf_{d\to\infty} \hat{N}_d(x) \leq 1,$$

where we used the (4.46). □

In light of this achievement, we can apply Corollary 4.2.15 to complete the proof of point 5) of Theorem 4.1.9. The fact that $\gamma > 0$ follows by Lemmas 4.5.14 and 4.2.17:

$$\lim_{r\to+\infty} \frac{\hat{H}(r)}{e^{2r}} = \lim_{r\to+\infty} \frac{\hat{E}(r)}{e^{2r}} \cdot \lim_{r\to+\infty} \frac{1}{\hat{N}(r)} > 0.$$

Remark 4.5.15. With a similar construction, it is possible to obtain the existence of solutions to (4.5) in $\mathbb{R}^2$ modelled on $\cosh x \sin y$. To do this, we can first construct solutions of (4.5) having algebraic growth defined outside the ball of radius 1, with homogeneous Neumann boundary conditions on ∂B_1. This can be done suitably modifying the proof of Theorem 1.6 in [13]. Then, performing a new blow-up in a neighbourhood of $(1,0)$, we can obtain a solution of (4.5) defined in $\mathbb{R}^2_+$, with homogeneous Neumann condition on $\{x = 0\}$; this solution can be extended by even-symmetry in x in the whole $\mathbb{R}^2$.

4.6 Asymptotics of solutions which are periodic in one variable

In this section we prove Theorem 4.1.10.

Proof of Theorem 4.1.10. Let us start with case (i). Since the solution (u, v) is nontrivial, $N(0) > 0$: in particular, from point (i) of Corollary 4.2.15 it follows that $H(r) \to +\infty$ as $r \to +\infty$. Let us consider the shifted functions

$$(u_R(x,y), v_R(x,y)) := \frac{1}{\sqrt{H(R)}}(u(x+R,y), v(x+R,y))$$

which solve the system

$$\begin{cases} -\Delta u_R = -H(R) u_R v_R^2 & \text{in } C_\infty \\ -\Delta v_R = -H(R) u_R^2 v_R & \text{in } C_\infty \\ \int_{\Sigma_0} u_R^2 + v_R^2 = 1 \end{cases}$$

and share the same periodicity of (u, v). We introduce

$$E_R(r) := \int_{C_{(-\infty,r)}} |\nabla u_R|^2 + |\nabla v_R|^2 + 2H(R)u_R^2 v_R^2,$$

$$H_R(r) := \int_{\Sigma_r} u_R^2 + v_R^2 \quad \text{and} \quad N_R(r) := \frac{E_R(r)}{H_R(r)}.$$

It is easy to see that

$$\begin{aligned} E_R(r) &= \frac{1}{H(R)} E^{unb}(r+R) \\ H_R(r) &= \frac{1}{H(R)} H(r+R) \end{aligned} \qquad \Longrightarrow \qquad N_R(r) = N^{unb}(r+R)$$

for any r (recall that E^{unb} and N^{unb} have been defined in Subsection 4.2.2). We point out that, by the monotonicity of N^{unb}, Proposition 4.2.14, we have

- $N_{R_1}(r) \leq N_{R_2}(r)$ for every $R_1 < R_2$;
- $N_R(r) \leq d = \lim_{r\to\infty} N(r)$ for every r, R, and $N_R(r) \to d$ as $R \to \infty$ for every $r \in \mathbb{R}$.

Therefore, N_R tends to the constant function d in $L^1_{\mathrm{loc}}(\mathbb{R})$, as $R \to +\infty$.

Thanks to the normalization condition $H_R(0) = 1$ and the uniform bound $N_R(r) \leq d$, applying Corollary 4.2.15 (see also Remark 4.2.18) we deduce that $H_R(r)$ is uniformly bounded in R for every $r > 0$. Consequently, also $E_R(r)$ is uniformly bounded in R for every $r > 0$. By means of a Poincaré inequality of type (4.25), we deduce that the sequence (u_R, v_R) is uniformly bounded in $H^1_{\mathrm{loc}}(C_\infty)$ and, by standard elliptic estimates, in $L^\infty_{\mathrm{loc}}(C_\infty)$. From Theorem 2.6 of [90] (it is a local version of Theorem 1.1 of [67]), we evince that the sequence (u_R, v_R) is uniformly bounded also in $\mathcal{C}^{0,\alpha}_{\mathrm{loc}}(C_\infty)$ for any $\alpha \in (0, 1)$. Consequently, up to a subsequence, (u_R, v_R) converges in $\mathcal{C}^0_{\mathrm{loc}}(C_\infty)$ and in $H^1_{\mathrm{loc}}(C_\infty)$ to a pair (Ψ^+, Ψ^-), where Ψ is a nontrivial harmonic function (see the forthcoming Theorem 6.2.3). By the convergence, Ψ has to be 2π-periodic in y.

Firstly, we prove that $H(r; \Psi) \to 0$ as $r \to -\infty$, so that the results of Subsection 4.2.3 hold true for Ψ. As already observed, $N_R(r) \geq N_{\bar{R}}(r)$ for every $r \in \mathbb{R}$, for every $R > \bar{R}$. By the expression of the logarithmic derivative of H_R, see Corollary 4.2.15 (see also Remark 4.2.18) we have

$$\frac{d}{dr} \log H_R(r) = 2N_R(r) \geq 2N_{\bar{R}}(r) = \frac{d}{dr} \log H_{\bar{R}}(r) \qquad \forall r.$$

As a consequence, taking into account that $H_R(0) = 1$ for every R, for every $r < 0$ it results

$$\frac{H_R(0)}{H_R(r)} \geq \frac{H_{\bar{R}}(0)}{H_{\bar{R}}(r)} \qquad \Longleftrightarrow \qquad H_{\bar{R}}(r) \geq H_R(r) \qquad \forall R > \bar{R}.$$

Passing to the limit as $R \to +\infty$, by the $\mathcal{C}^0_{\text{loc}}(\mathbb{R}^2)$ convergence of (u_R, v_R) to (Ψ^+, Ψ^-) it follows that $H_{\bar{R}}(r) \geq H(r;\Psi)$, which gives $H(r;\Psi) \to 0$ as $r \to -\infty$ in light of our assumption on (u,v).

Using again the expression of the logarithmic derivative of H_R and $H(\cdot;\Psi)$, we deduce

$$\log \frac{H_R(r_2)}{H_R(r_1)} = 2\int_{r_1}^{r_2} N_R(s)\,ds \quad \text{and} \quad \log \frac{H(r_2;\Psi)}{H(r_1;\Psi)} = 2\int_{r_1}^{r_2} N(s;\Psi)\,ds,$$

where $r_1 < r_2$. The left hand side of the first identity converges to the left hand side of the second identity; recalling that $N_R \to d$ in $L^1_{\text{loc}}(\mathbb{R})$, we deduce

$$\int_{r_1}^{r_2} N(s;\Psi)\,ds = \lim_{R\to+\infty} \int_{r_1}^{r_2} N_R(s)\,ds = d(r_2 - r_1) \\ \implies \quad \frac{1}{r_2 - r_1}\int_{r_1}^{r_2} N(s;\Psi)\,ds = d.$$

for every $r_1 < r_2$. It is well known that, being $N(\cdot;\Psi) \in L^1_{\text{loc}}(\mathbb{R})$, the limit as $r_2 \to r_1$ of the left hand side converges to $N(r_1;\Psi)$ for almost every $r_1 \in \mathbb{R}$. Hence, $N(r;\Psi) = d$ for every $r \in \mathbb{R}$. We are then in position to apply Proposition 4.2.21:

$$\lim_{R\to+\infty} N(R) = \lim_{R\to+\infty} N_R(0) = N(0;\Psi) = d \in \mathbb{N} \setminus \{0\},$$

and $\Psi(x,y) = [C_1 \cos(dy) + C_2 \sin(dy)]\, e^{dx}$ for some constant $C_1, C_2 \in \mathbb{R}$.

As far as case (ii) is concerned, for the sake of simplicity we assume $a = 0$. One can repeat the proof with minor changes replacing E^{unb} and N^{unb} with E^{sym} and N^{sym} (which have been defined in Subsection 4.2.1). The unique nontrivial step consists in proving that in this setting $H(r;\Psi) \to 0$ as $r \to -\infty$. To this aim, we note that, as before,

$$H_R(r) \leq H_{\bar{R}}(r) \qquad \forall R > \bar{R},$$

for every $r > -\bar{R}$. In particular, if $r \in (1 - \bar{R}, 0)$, by Proposition 4.2.4 and Corollary 4.2.5 we deduce

$$H_R(r) \leq H_{\bar{R}}(r) = \frac{H(r + \bar{R})}{H(\bar{R})} \leq \frac{e^{2N(1)(r+\bar{R})}}{e^{2N(1)\bar{R}}} = e^{2N(1)r} \qquad \forall R > \bar{R}.$$

Passing to the limit as $R \to +\infty$, by $\mathcal{C}^0_{\text{loc}}(\mathbb{R}^2)$ convergence we obtain

$$H(r;\Psi) \leq e^{2N(1)r} \qquad \forall r \in (-\infty, 0),$$

which yields $H(r;\Psi) \to 0$ as $r \to -\infty$. □

Part II

Qualitative properties for solutions of some elliptic problems in unbounded domains

Chapter 5

Symmetry and uniqueness for nonnegative solutions of some problems in the half-space

5.1 Introduction

This chapter is devoted to classification results for *nonnegative* solutions of

$$\begin{cases} -\mathrm{div}(A(x)\nabla u) = u - g(x) & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{5.1}$$

in low dimension. Here $\mathbb{R}^N_+$ is the half-space $\mathbb{R}^{N-1} \times \mathbb{R}^+$ and $\mathrm{div}(A(x)\nabla)$ is an elliptic operator. We consider different types of inhomogeneous terms g, and obtain different results according to the properties of such functions.

The interest in this kind of problems comes from Berestycki, Caffarelli and Niremberg: in [10] they proved that a *positive and bounded* solution to

$$\begin{cases} -\Delta u = u - 1 & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{5.2}$$

does not exist when $N \leq 3$; on the other hand, it is immediate to check that $u(x', x_N) = 1 - \cos x_N$ is a *nonnegative* solution. Their non-existence result fits in a wider study of 1-dimensional symmetry and monotonicity for *positive and bounded* solutions to general problems of type

$$\begin{cases} -\Delta u = f(u) & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases} \tag{5.3}$$

with f Lipschitz continuous. We review here the main results: if $N \geq 2$ and $f(0) \geq 0$, then a positive and bounded solution is strictly increasing in the x_N variable (see [10, 27]). Furthermore (we refer again to [10]), if $N \leq 3$, $f \in \mathcal{C}^1(\mathbb{R})$ and $f(0) \geq 0$, then a positive and bounded solution depends only on one variable (1-dimensional symmetry). Another contribution contained in [10] is that the monotonicity and the 1-D symmetry hold true for $N = 2$ without any restriction on the sign of $f(0)$. The proofs of the quoted results are based on the moving planes method and on a previous achievement contained in [11], where it is shown that if u is a positive and bounded solution of (5.3) and

$$f(M) \leq 0 \quad \text{where} \quad M = \sup_{x \in \mathbb{R}^N_+} u(x),$$

then u is symmetric and monotone, and $f(M) = 0$. When f is a power (thus $f(0) = 0$), similar results has been achieved in [44, 49]. We point out that our contribution is not included in the existing literature, because we are considering *nonnegative and not necessarily bounded* solutions, and because in general we are interested in the case $f(0) < 0$. In such a situation the moving planes method gives just partial results, as shown by Dancer [28]. We emphasize the fact that the difference between *positive* and *nonnegative* is substantial for $f(0) < 0$, since in this case natural solutions are nonnegative and non-monotone, and a positive solution does not necessarily exist; this is clearly the case of the model problem (5.2). For all these reasons, our approach is different, and it is based upon a combination of Fourier series and Liouville theorems.
To complete the essential bibliography for this kind of problems, we mention also the work [39], where symmetry and monotonicity are obtained under weaker regularity assumptions on f, and an extension in dimension 4 and 5 is given for a wide class of nonlinearities.

We already announced that our approach is based upon a combination of Fourier series and Liouville theorems. Hence, in Section 5.2 we state and prove some preliminary results, for the sake of completeness.

Our first contribution regards the model problem (5.2):

$$\begin{cases} -\Delta u = u - 1 & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases}$$

In Section 5.3, we prove that, if $N = 2$ or 3, under the additional assumption that u is bounded in any strip of type $\mathbb{R}^{N-1} \times [0, M]$, the unique nonnegative solution of (5.2) is $u(x', x_N) = 1 - \cos x_N$; we point out that u is not necessarily bounded, and no assumption about its growth in the x_N direction is required. This is a result of uniqueness and of 1-D symmetry, i.e. the (unique) nonnegative solution of (5.2) is a function depending only on x_N. The assumption "$N = 2$ or 3" is substantial for our proof. However, we can still say something for the model problem in higher dimension. This is the object of Subsection 5.3.1.

As far as the generalization towards problem (5.1) is concerned, we see in Section 5.4, Theorem 5.4.1, that the presence of div $(A(x)\nabla)$ instead of the Laplacian does not affect the previous result, under suitable assumptions on A.

A further natural generalization of problem (5.2) consists in introducing a g depending only on x_N instead of the constant function 1:

$$\begin{cases} -\mathrm{div}\,(A(x)\nabla u) = u - g(x_N) & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+ \end{cases} \tag{5.4}$$

In this setting, Theorem 5.5.8 is the counterpart of Theorem 5.3.1; roughly speaking, it says that if $N = 2$ or 3, under suitable and natural assumptions on A and on $g = g(x_N)$, if u solves (5.4) and is bounded in any strip of type $\mathbb{R}^{N-1} \times [0, M]$, then u is uniquely determined and depends only on x_N.

Finally, we show how to use the method developed in the previous sections in order to deal with a wider class of inhomogeneous terms (depending also on x'), obtaining sharp results for some particular cases; for instance, we prove that if $g = g(x')$ and there exists a solution u of (5.1) satisfying (5.11), then g has to be constant.

Notation. We will consider problems in the half-space $\mathbb{R}^N_+ := \mathbb{R}^{N-1} \times (0, +\infty)$. As usual, we write (x', x_N) to denote a point of $\mathbb{R}^N_+$. The symbols ∇', div$'$ or Δ' are used for the gradient, the divergence or the Laplacian in $\mathbb{R}^{N-1}$, respectively. The notation u_j indicates the partial derivative of u with respect to the x_j variable. For any $x \in \mathbb{R}^N$, for any $R > 0$, we set

$$B_R(x) := \left\{ y \in \mathbb{R}^N : |y - x| < R \right\};$$

if $x = 0$, we simply write B_R. For any $A \subset \mathbb{R}^N$, χ_A denotes the characteristic function of A. We use the notation $\langle \cdot, \cdot \rangle$ for the usual scalar product in any Euclidean space. Given a real valued function v, we denote its positive part as v^+.

5.2 Liouville theorems for subharmonic functions

One of the most celebrated theorems in complex analysis is the Liouville theorem for holomorphic function.

Theorem 5.2.1. *Let $f : \mathbb{C} \to \mathbb{C}$ be a holomorphic function. If f is bounded, then it is constant.*

A proof of this statement, usually based on the Cauchy integral formula, can be found in any textbook of complex analysis. Another way to prove it is to use a similar result concerning real-valued harmonic functions defined in the whole $\mathbb{R}^N$.

Theorem 5.2.2. *Let $N \geq 1$, and let $u \in \mathcal{C}^2(\mathbb{R}^N)$ be a bounded solution of $-\Delta u = 0$ in $\mathbb{R}^N$. If u is bounded, then it is constant.*

For the proof of Theorem 5.2.2 we refer, e.g., to [34]. Theorem 5.2.1 can be obtained by Theorem 5.2.2, recalling that the real and the imaginary parts of a holomorphic function satisfy the Cauchy-Riemann equation; therefore, if f is holomorphic and bounded, then $\mathfrak{Re}(u)$ and $\mathfrak{Im}(u)$ are bounded real-valued harmonic functions in $\mathbb{R}^2$.

Starting from these classical results, several propositions concerning the classification of solutions to elliptic differential equations defined in the entire Euclidean space $\mathbb{R}^N$ have been labelled under the name "Liouville-type theorems". We are particularly interested in the following extension of Theorem 5.2.2, which concerns subharmonic/superharmonic functions defined in the Euclidean plane $\mathbb{R}^2$, and which are bounded above/below.

Theorem 5.2.3. *Let $u \in \mathcal{C}^2(\mathbb{R}^2)$ be a subharmonic function: $-\Delta u \leq 0$ in $\mathbb{R}^2$. If u is bounded from above, then it is constant.*

Remark 5.2.4. 1) The statement does not hold if we replace the assumption that u is bounded from above by the assumption that u is bounded from below, as shown by the non-constant subharmonic function $u(x) = |x|^2$.
2) The theorem does not hold in higher dimension. Indeed, for any $N \geq 3$, let $u \in \mathcal{C}^2(\mathbb{R}^N)$ be a positive and radially symmetric solution of

$$-\Delta u = |u|^{2^*-2}u \qquad \text{in } \mathbb{R}^N,$$

that is,

$$u(x) = \frac{c}{(1+|x|^2)^{\frac{N-2}{2}}} \qquad \text{for some } c > 0.$$

Then $v := -u$ is subharmonic and negative (in particular it is bounded above), but it is not constant.

To prove Theorem 5.2.3 we use a variant of the so-called *Hadamard three-circles theorem* for subharmonic functions, for which we refer to [34] (Theorem 3.2) and to the references therein.

Theorem 5.2.5. *Let $\Omega \subset \mathbb{R}^2$ be a domain containing the closure of an annulus*

$$A_{r_1 r_2} := \left\{ x \in \mathbb{R}^2 : 0 < r_1 < |x| < r_2 \right\}.$$

Let $u \in \mathcal{C}^2(\Omega)$ be a subharmonic function: $-\Delta u \leq 0$ in Ω. If $M(r)$ denotes the maximum of u on any circle $|x| = r$, then

$$M(r) \leq M(r_1)\frac{\log(r_2/r)}{\log(r_2/r_1)} + M(r_2)\frac{\log(r/r_1)}{\log(r_2/r_1)} \tag{5.5}$$

for $r_1 < r < r_2$.

Proof of Theorem 5.2.5. For every $x \in \overline{A}_{r_1 r_2}$ we consider

$$w(x) := u(x) - M(r_1)\frac{\log(r_2/|x|)}{\log(r_2/r_1)} - M(r_2)\frac{\log(|x|/r_1)}{\log(r_2/r_1)}.$$

The thesis follows if we show that $w \leq 0$ in $A_{r_1 r_2}$. Clearly, $w \in \mathcal{C}^2(A_{r_1 r_2})$, and since

$$\Delta\left(\log\left(\frac{r_2}{|x|}\right)\right) = 0 \quad \text{and} \quad \Delta\left(\log\left(\frac{|x|}{r_1}\right)\right) = 0,$$

we have

$$\begin{cases} -\Delta w \leq 0 & \text{in } A_{r_1 r_2} \\ w \leq 0 & \text{on } \partial A_{r_1 r_2}. \end{cases}$$

An application of the maximum principle gives the thesis. □

Proof of Theorem 5.2.3. The function u is bounded above and satisfy the assumptions of Theorem 5.2.5. Hence, letting $r_2 \to +\infty$ in inequality (5.5), we have

$$M(r) \leq M(r_1) \qquad \forall (r_1, r):\ 0 < r_1 \leq r. \tag{5.6}$$

On the other hand, being u subharmonic in any disc B_r, the maximum principle implies that

$$M(r_1) \leq \sup_{x \in B_r} u(x) \leq \max_{x \in \partial B_r} u(x) = M(r) \qquad \forall (r_1, r):\ 0 \leq r_1 \leq r. \tag{5.7}$$

A comparison between (5.6) and (5.7) yields

$$M(r) = M(r_1) \quad \forall (r_1, r):\ 0 \leq r_1 \leq r \quad \Longrightarrow \quad M(r) = M(0) = u(0) \quad \forall r > 0.$$

As, by subharmonicity, u cannot have any interior maximum point, it is constant. □

Remark 5.2.6. The proof of the Hadamard circle theorem works for $N \geq 3$ if we replace inequality (5.5) with

$$M(r) \leq M(r_1)\frac{r_2^{2-N} - r^{2-N}}{r_2^{2-N} - r_1^{2-N}} + M(r_2)\frac{r^{2-N} - r_1^{2-N}}{r_2^{2-N} - r_1^{2-N}} \tag{5.8}$$

for $r_1 < r < r_2$. Note that

$$\log(\rho_1/\rho_2) = \Gamma_2(\rho_1) - \Gamma_2(\rho_2), \quad \text{and} \quad \rho_1^{2-N} - \rho_2^{2-N} = \Gamma_N(\rho_1) - \Gamma_N(\rho_2),$$

where Γ_N denotes, up to a constant depending on N, the fundamental solution of the Laplace equation in $\mathbb{R}^N$, for any $N \geq 3$.

On the other hand, the proof of Theorem 5.2.3 cannot be generalized for $N \geq 3$: indeed, if we consider the limit as $r_2 \to +\infty$ in (5.8), we obtain

$$M(r) \leq M(r_1)\frac{r^{2-N}}{r_1^{2-N}} + \lim_{r_2 \to +\infty} M(r_2)\left(1 - \frac{r^{2-N}}{r_1^{2-N}}\right),$$

which does not ensure that $M(r) \leq M(r_1)$ for every (r_1, r) such that $0 < r_1 \leq r$. This observation reveals that the duality $N = 2$ or $N \geq 3$ in Liouville type results for subharmonic functions reflects the different behaviour at infinity of the fundamental solution of the Laplace equation for $N = 2$ or $N \geq 3$: on one side $\log \rho \to +\infty$ as $\rho \to +\infty$, on the other side $\rho^{2-N} \to 0$ as $\rho \to +\infty$.

We conclude this section with a generalization of Theorem 5.2.3 for some elliptic operators. The following result is a particular case of Proposition 7.7 in [34], and provides a new method to prove Theorem 5.2.3.

Theorem 5.2.7. *Let $B(x) = (b_{ij}(x))$ be a symmetric real matrix, whose entries are $L^\infty(\mathbb{R}^2)$ functions satisfying:*

$$\text{for a.e. } x \in \mathbb{R}^2,\ \forall \xi \in \mathbb{R}^2 \setminus \{0\}: \quad \sum_{i,j=1}^{2} b_{ij}(x)\xi_i\xi_j > 0.$$

Let $v \in H^1_{\mathrm{loc}}(\mathbb{R}^2)$ be a distribution solution of

$$\begin{cases} -\operatorname{div}(B(x)\nabla v) \geq 0 & \text{in } \mathbb{R}^2 \\ v(x) \geq -C & \text{a.e. in } \mathbb{R}^2, \end{cases} \tag{5.9}$$

for some positive constant C. Then v is a constant function.

Proof. Without loss of generality, we can assume that $v \geq C > 0$. For any $R > 1$ and $x \in \mathbb{R}^2$, we set

$$\phi_R(x) := \begin{cases} 1 & \text{if } |x| \leq R \\ 1 - \frac{1}{\log R}\log\left(\frac{|x|}{R}\right) & \text{if } R < |x| \leq R^2 \\ 0 & \text{if } |x| > R^2. \end{cases}$$

Let us test the first equation in (5.9) with $\phi_R^2 v^{-1}$:

$$\begin{aligned} \int_{\mathbb{R}^2}\left(\frac{\phi_R}{v}\right)^2 \langle B(x)\nabla v, \nabla v\rangle &\leq 2\int_{\mathbb{R}^2}\frac{\phi_R}{v}\langle B(x)\nabla v, \nabla \phi_R\rangle \\ &\leq 2\left(\int_{\mathbb{R}^2}\left(\frac{\phi_R}{v}\right)^2\langle B(x)\nabla v, \nabla v\rangle\right)^{\frac{1}{2}}\left(\int_{\mathbb{R}^2}\langle B(x)\nabla\phi_R, \nabla\phi_R\rangle\right)^{\frac{1}{2}}; \end{aligned}$$

for the last inequality we observe that, by the assumptions on B, for almost every $x \in \mathbb{R}^2$ the function

$$(\xi_1, \xi_2) \in \left(\mathbb{R}^2\right)^2 \mapsto \langle B(x)\xi_1, \xi_2 \rangle \in \mathbb{R}$$

defines a bilinear symmetric positive definite form, so that the Cauchy-Schwarz inequality holds. Applying the Young inequality on the right hand side, we easily deduce that

$$\int_{\mathbb{R}^2} \left(\frac{\phi_R}{v}\right)^2 \langle B(x)\nabla v, \nabla v \rangle \le 16 \int_{\mathbb{R}^2} \langle B(x)\nabla \phi_R, \nabla \phi_R \rangle \le C \int_{\mathbb{R}^2} |\nabla \phi_R|^2 \le \frac{C}{\log R}$$

for some positive constant $C > 0$ independent on R. Letting $R \to +\infty$, we obtain the desired result. □

The result can be extended straightforwardly to the 1 dimensional case.

Corollary 5.2.8. *Let $b \in L^\infty(\mathbb{R})$ such that $b > 0$ a.e. in $\mathbb{R}$. Let $v \in H^1_{\mathrm{loc}}(\mathbb{R})$ be a distribution solution of*

$$\begin{cases} -\left(b(x)v'\right)' \ge 0 & \text{in } \mathbb{R} \\ v(x) \ge -C & \text{a.e. in } \mathbb{R}, \end{cases} \tag{5.10}$$

for some positive constant C. Then v is a constant function.

Proof. We set

$$B(x_1, x_2) := \begin{pmatrix} b(x_1) & 0 \\ 0 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{v}(x_1, x_2) := v(x_1).$$

In this way, B is a real diagonal matrix with positive diagonal entries, $\tilde{v} \ge -C$ a.e. in $\mathbb{R}^2$, and

$$-\mathrm{div}\left(B(x)\nabla \tilde{v}\right) = -\partial_{x_1}\left(B_{11}(x)\partial_{x_1}\tilde{v}\right) - \partial_{x_2 x_2}\tilde{v} = -\left(b(x_1)v'\right)' \ge 0,$$

thanks to the (5.10); then, an application of Theorem 5.2.7 gives the desired result. □

In order to keep the notation as simple as possible, in what follows we always refer to Theorem 5.2.7 and Corollary 5.2.8 simply as Theorem 5.2.7.

5.3 The model problem

In this section we consider problem (5.2):

$$\begin{cases} -\Delta u = u - 1 & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases}$$

We aim at proving the following statement.

Theorem 5.3.1. *Let $N = 2$ or 3. If $u \in \mathcal{C}^2\left(\overline{\mathbb{R}^N_+}\right)$ solves problem (5.2) and*

$$\forall M > 0 \quad \exists C(M) > 0 : \quad 0 \le u(x) \le C(M) \quad \forall x \in \mathbb{R}^{N-1} \times [0, M], \tag{5.11}$$

then

$$u(x', x_N) = 1 - \cos x_N.$$

Remark 5.3.2. Assumption (5.11) says that u is nonnegative in the whole $\mathbb{R}^N_+$ and bounded in every strip of type $\mathbb{R}^{N-1} \times [0, M]$ (but with arbitrary growth in the x_N-direction). Assumption (5.11) is obviously satisfied if u is nonnegative and bounded. Actually, it is sufficient to assume that u is nonnegative and ∇u is bounded, in order to ensure (5.11). Indeed for every $M > 0$ we have

$$\begin{aligned} |u(x', x_N)| = |u(x', x_N) - u(x', 0)| &\le \sup_{\xi \in [0, x_N]} |\nabla u(x', \xi)| x_N \\ &\le \|\nabla u\|_\infty M = C(M) \qquad \forall (x', x_N) \in \mathbb{R}^{N-1} \times [0, M]. \end{aligned}$$

In particular we recover the non-existence of positive solutions to (5.2) of Berestycki, Caffarelli and Nirenberg, mentioned in the introduction to this part (contained in [10]).

First we focus on problem (5.2) in the strip $\overline{\Sigma} = \mathbb{R}^{N-1} \times [0, 2\pi]$, $N \ge 2$. For every $x' \in \mathbb{R}^{N-1}$, let $\tilde{u}(x', \cdot)$ be the 2π-periodic extension of $x_N \mapsto u(x', x_N)$. In view of the smoothness of u, it follows that the Fourier expansion of $x_N \mapsto \tilde{u}(x', x_N)$, given by

$$\frac{a_0(x')}{2} + \sum_{m=1}^{+\infty} \big(a_m(x') \cos(m x_N) + b_m(x') \sin(m x_N)\big), \tag{5.12}$$

where

$$\begin{aligned} a_m(x') &:= \frac{1}{\pi} \int_0^{2\pi} u(x', x_N) \cos(m x_N)\, dx_N \qquad \forall m \ge 0, \\ b_m(x') &:= \frac{1}{\pi} \int_0^{2\pi} u(x', x_N) \sin(m x_N)\, dx_N \qquad \forall m \ge 1, \end{aligned} \tag{5.13}$$

is convergent.
Now we determine the equations satisfied by the coefficients above.

Lemma 5.3.3. *Let $N \ge 2$. For any $m \ge 1$ we have*

$$\Delta' a_m(x') = (m^2 - 1) a_m(x') + \frac{1}{\pi}\big(u_N(x', 0) - u_N(x', 2\pi)\big) \tag{5.14}$$

$$\Delta' b_m(x') = (m^2 - 1) b_m(x') + \frac{m}{\pi} u(x', 2\pi). \tag{5.15}$$

Also,

$$\Delta' a_0(x') = 2 - a_0(x') + \frac{1}{\pi}\big(u_N(x', 0) - u_N(x', 2\pi)\big).$$

Proof. For any $m \geq 1$ we have

$$\begin{aligned}
\Delta' a_m(x') &= \frac{1}{\pi}\int_0^{2\pi} \Delta' u(x', x_N)\cos(mx_N)\,dx_N \\
&= \frac{1}{\pi}\int_0^{2\pi} \big(1 - u(x', x_N) - u_{NN}(x', x_N)\big)\cos(mx_N)\,dx_N \\
&= -\frac{1}{\pi}\int_0^{2\pi} \big(u(x', x_N) + u_{NN}(x', x_N)\big)\cos(mx_N)\,dx_N.
\end{aligned}$$

Integrating by parts twice the last term we obtain

$$\begin{aligned}
\Delta' a_m(x') =& -\frac{1}{\pi}\int_0^{2\pi} \big(u(x', x_N)\cos(mx_N) + m u_N(x', x_N)\sin(mx_N)\big)\,dx_N \\
&- \frac{1}{\pi}\big[u_N(x', x_N)\cos(mx_N)\big]_{x_N=0}^{2\pi} \\
=& \frac{m^2-1}{\pi}\int_0^{2\pi} u(x', x_N)\cos(mx_N)\,dx_N \\
&- \frac{1}{\pi}\big(u_N(x', 2\pi) - u_N(x', 0)\big),
\end{aligned}$$

which is equation (5.14).
With the same procedure we can find equation (5.15): for any $m \geq 1$

$$\begin{aligned}
\Delta' b_m(x') =& \frac{1}{\pi}\int_0^{2\pi} \Delta' u(x', x_N)\sin(mx_N)\,dx_N \\
=& \frac{1}{\pi}\int_0^{2\pi} \big(1 - u(x', x_N) - u_{NN}(x', x_N)\big)\sin(mx_N)\,dx_N \\
=& \frac{1}{\pi}\int_0^{2\pi} \big(-u(x', x_N)\sin(mx_N) + m u_N(x', x_N)\cos(mx_N)\big)\,dx_N \\
=& \frac{m^2-1}{\pi}\int_0^{2\pi} u(x', x_N)\sin(mx_N)\,dx_N \\
&+ \frac{m}{\pi}\big[u(x', x_N)\cos(mx_N)\big]_{x_N=0}^{2\pi}.
\end{aligned}$$

As far as a_0 is concerned, we have

$$\begin{aligned}
\Delta' a_0(x') &= \frac{1}{\pi}\int_0^{2\pi} \Delta' u(x', x_N)\,dx_N \\
&= \frac{1}{\pi}\int_0^{2\pi} \big(1 - u(x', x_N) - u_{NN}(x', x_N)\big)\,dx_N \\
&= 2 - \frac{1}{\pi}\int_0^{2\pi} u(x', x_N)\,dx_N + \frac{1}{\pi}\big(u_N(x', 0) - u_N(x', 2\pi)\big).
\end{aligned}$$

□

Lemma 5.3.4. *Both b_1 and a_1 are constant; moreover,*

$$u(x',2\pi)=0, \quad u_N(x',0)=0 \quad \text{and} \quad u_N(x',2\pi)=0 \quad \forall x'\in\mathbb{R}^{N-1}.$$

Proof. Using (5.15) with $m=1$ we have

$$\Delta' b_1(x')=\frac{1}{\pi}u(x',2\pi)\geq 0.$$

Therefore, thanks to (5.11), b_1 is a subharmonic and bounded function in $\mathbb{R}^{N-1}$ with $N=2$ or 3; Theorem 5.2.3 implies that it is constant, so that in particular $\Delta' b_1\equiv 0$, i.e. $u(x',2\pi)=0$ for every $x'\in\mathbb{R}^{N-1}$.
Note that, since $u\geq 0$ and $u(x',2\pi)=0$, each $(x',2\pi)$ is a point of minimum for u; consequently $u_N(x',2\pi)=0$, and this makes possible to prove that also a_1 is constant: indeed

$$\Delta' a_1(x')=\frac{1}{\pi}u_N(x',0).$$

Since $u(x',0)=0$ and $u\geq 0$ in Σ, it follows that $u_N(x',0)\geq 0$ for every $x'\in\mathbb{R}^{N-1}$. Hence a_1 is a subharmonic and bounded function in $\mathbb{R}$ or $\mathbb{R}^2$, which has to be constant by Theorem 5.2.3. It follows in particular that $u_N(x',0)\equiv 0$ in $\mathbb{R}^{N-1}$. □

An important consequence of the previous Lemma is that the equations for a_m and b_m simplify as

$$\Delta' a_m(x')=(m^2-1)a_m(x') \qquad \forall m\geq 2 \tag{5.16}$$

$$\Delta' b_m(x')=(m^2-1)b_m(x') \qquad \forall m\geq 2. \tag{5.17}$$

Hence, for $m\geq 2$, a_m and b_m satisfy an equation of type

$$-\Delta' v(x')+\lambda v(x')=0 \qquad \text{in } \mathbb{R}^{N-1}, \tag{5.18}$$

with $\lambda>0$. We point out that both a_m and b_m are bounded in absolute value in Σ (this follows from assumption (5.11)).
Bounded solutions of (5.18) has to vanish identically. This is an immediate consequence of the following general result.

Lemma 5.3.5. *Assume $N\geq 2$ and let $v\in\mathcal{C}^2(\mathbb{R}^{N-1})$ be a subsolution of*

$$-\Delta' v(x')+c(x')v(x')\leq 0 \qquad \text{in } \mathbb{R}^{N-1}, \tag{5.19}$$

with $c(x')\geq\lambda>0$ in $\mathbb{R}^{N-1}$.
If v^+ has at most algebraic growth at infinity, then $v\leq 0$ in $\mathbb{R}^{N-1}$,

Here and in what follows we say that v has algebraic growth if there exist $C > 0$ and $p \geq 1$ such that

$$|v(x')| \leq C(1 + |x'|^p) \qquad \forall x' \in \mathbb{R}^{N-1}.$$

For the proof, it will be useful the following lemma.

Lemma 5.3.6. *Let $\vartheta > 0$, $\gamma > 0$, be such that $\vartheta < 2^{-\gamma}$. Let $R_0 > 0$, $C > 0$ and $I : (R_0, +\infty) \to [0, +\infty)$ be such that*

$$\begin{cases} I(R) \leq \vartheta I(2R) & \forall R > R_0 \\ I(R) \leq C R^\gamma & \forall R > R_0. \end{cases} \tag{5.20}$$

Then $I(R) = 0$ for every $R > R_0$.

Proof. Iterating the first one of (5.20) we obtain, for every $k \in \mathbb{N}$,

$$I(R) \leq \vartheta^k I(2^k R) \qquad \forall R > R_0.$$

Now the second one gives

$$I(R) \leq C \left(\vartheta 2^\gamma\right)^k R^\gamma \qquad \forall R > R_0,\ \forall k \in \mathbb{N}.$$

Since $0 < \vartheta 2^\gamma < 1$, letting $k \to \infty$ we obtain $I(R) \leq 0$ for $R > R_0$. □

Proof of Lemma 5.3.5. We introduce a $\mathcal{C}^\infty$ cut-off function $\varphi : [0, +\infty) \to \mathbb{R}$ such that

$$\begin{cases} \varphi(t) = 1 & t \in [0, 1] \\ \varphi(t) = 0 & t \in [2, +\infty) \\ 0 \leq \varphi(t) \leq 1 & t \in (1, 2). \end{cases}$$

We set, for every $R > 0$, $\varphi_R(x') := \varphi(|x'|/R)$, which is defined on $\mathbb{R}^{N-1}$. Hence

$$\nabla' \varphi_R(x') = \frac{x'}{R|x'|} \varphi' \left(\frac{|x'|}{R}\right).$$

In particular

$$\left|\nabla' \varphi_R(x')\right| \leq \frac{C}{R} \chi_{B_{2R}}(x') \tag{5.21}$$

where C is a constant independent of R.
Testing (5.19) with $v^+ \varphi_R^2$ we obtain

$$\begin{aligned} \int_{\mathbb{R}^{N-1}} \left(|\nabla' v^+|^2 + \lambda \left(v^+\right)^2\right) \varphi_R^2 &\leq \int_{\mathbb{R}^{N-1}} \left(|\nabla' v^+|^2 + c \left(v^+\right)^2\right) \varphi_R^2 \\ &\leq -2 \int_{\mathbb{R}^{N-1}} v^+ \varphi_R \left\langle \nabla' v^+, \nabla' \varphi_R \right\rangle \leq 2 \int_{\mathbb{R}^{N-1}} v^+ \varphi_R \left|\left\langle \nabla' v^+, \nabla' \varphi_R \right\rangle\right|. \end{aligned} \tag{5.22}$$

We can use the Cauchy-Schwarz and the Young inequalities: for every $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$
\begin{aligned}
2\int_{\mathbb{R}^{N-1}} v^+\varphi_R \left|\langle \nabla' v^+, \nabla'\varphi_R\rangle\right| &\le 2\int_{\mathbb{R}^{N-1}} v^+\varphi_R |\nabla' v^+||\nabla'\varphi_R| \\
&\le 2\varepsilon \int_{\mathbb{R}^{N-1}} \varphi_R^2 |\nabla' v^+|^2 + 2C_\varepsilon \int_{\mathbb{R}^{N-1}} \left(v^+\right)^2 \left|\nabla'\varphi_R\right|^2.
\end{aligned}
$$

Coming back to (5.22), we obtain

$$
\int_{\mathbb{R}^{N-1}} \left((1-2\varepsilon)|\nabla' v^+|^2 + \lambda\left(v^+\right)^2\right)\varphi_R^2 \le 2C_\varepsilon \int_{\mathbb{R}^{N-1}} \left(v^+\right)^2 |\nabla'\varphi_R|^2.
$$

Choosing $\varepsilon < 1/2$ and using the (5.21), we deduce

$$
\int_{B_R} \left(v^+\right)^2 \le \frac{C}{\lambda R^2}\int_{B_{2R}} \left(v^+\right)^2.
$$

Also, since v^+ has at most algebraic growth at infinity, we have for any $R > 1$

$$
\int_{B_R} (v^+)^2 \le C' R^{N+2k}
$$

for some $k \ge 0$, $C' > 0$ independent of R. We are in position to apply Lemma 5.3.6, with

$$
I(R) := \int_{B_R} (v^+)^2.
$$

Here $\gamma = N + 2k$; note that there exists $R_0 > 1$ such that $\frac{C'}{\lambda R^2} < 2^{-N-2k}$ for every $R \ge R_0$. We set $\vartheta = \frac{C'}{\lambda R_0^2}$ and we apply Lemma 5.3.6 to obtain

$$
\int_{B_R} (v^+)^2 = 0 \qquad \forall R > R_0 \quad \Longrightarrow \quad v^+ \equiv 0. \qquad \square
$$

Conclusion of the proof of Theorem 5.3.1. Applying Lemma 5.3.5 to equations (5.16) and (5.17), we have that the Fourier coefficients a_m and b_m are identically 0 for any $m \ge 2$. Hence, the Fourier series (5.12) is reduced to

$$
\frac{a_0(x')}{2} + a_1 \cos x_N + b_1 \sin x_N \tag{5.23}
$$

and, for every $x \in \overline{\Sigma}$, it is equal to $u(x)$.

The boundary condition $u(x',0)=0$ reads

$$\frac{a_0(x')}{2}+a_1=0 \quad \Longrightarrow \quad a_0 \text{ is constant, equal to } -2a_1.$$

We also proved that $u_N(x',0)=0$, which implies $b_1=0$.
Plugging the expression of u inside the equation $-\Delta u = u-1$, we obtain

$$-a_1\cos x_N + \frac{a_0}{2} + a_1\cos x_N - 1 = 0 \quad \Longrightarrow \quad a_0 = 2,$$

and hence $a_1=-1$.
We proved that if $u \in \mathcal{C}^2\left(\overline{\mathbb{R}^N_+}\right)$ is a solution of (5.2) satisfying (5.11), then $u(x',x_N) = 1-\cos x_N$ in $\overline{\Sigma} = \mathbb{R}^{N-1}\times[0,2\pi]$.
To extend the result in the whole $\mathbb{R}^N_+$ we set

$$v_1(x',x_N) := u(x',x_N+2\pi).$$

It is straightforward to check that v_1 is a nonnegative solution of (5.2) and satisfies (5.11), so that it has to coincide with $1-\cos x_N$ in $\overline{\Sigma}$; this means that $u(x',x_N)=1-\cos x_N$ for $(x',x_N)\in\mathbb{R}^{N-1}\times[0,4\pi]$. The thesis follows by iteration of this argument. □

5.3.1 The model problem in higher dimension

In our proof it was crucial the possibility of applying the Liouville theorem for subharmonic functions, which holds only in $\mathbb{R}$ and $\mathbb{R}^2$. Therefore, despite the fact that our statement seems to be natural in any dimension, we cannot prove it. However, it is still possible to collect some properties of any solution of problem (5.2) satisfying (5.11) for $N\geq 4$.
We can focus again on the problem in the strip $\overline{\Sigma}$, considering the formal Fourier series (with respect to x_N variable) of the 2π-periodic extension $\widetilde{u}$ of u in Σ. Note that Lemma 5.3.3 still holds true. Now, in our analysis the key properties of the solutions was

$$u(x',2\pi)\equiv 0 \quad \text{and} \quad u_N(x',0)\equiv 0 \quad \text{in } \mathbb{R}^{N-1}. \tag{5.24}$$

Equation (5.24) implies that $u_N(x',2\pi)\equiv 0$ in $\mathbb{R}^{N-1}$, so that equations (5.14) and (5.15) are considerably simplified, since all the boundary terms have to vanish identically. This allowed to prove Theorem 5.3.1.

Proposition 5.3.7. *Let $N\geq 2$. Let $u\in\mathcal{C}^2\left(\overline{\mathbb{R}^N_+}\right)$ be a solution of problem* (5.2) *which satisfies* (5.11)*. Let a_m and b_m its formal Fourier coefficients, defined by* (5.13)*. Assume that* (5.24) *holds true. Then*

$$u(x',x_N) = 1-\cos x_N.$$

Proof. Under assumption (5.24), equations (5.14) and (5.15) are reduced to

$$\Delta' a_m(x') = (m^2 - 1)a_m(x') \qquad \forall m \geq 1$$
$$\Delta' b_m(x') = (m^2 - 1)b_m(x') \qquad \forall m \geq 1$$

(note that $u_N(x', 2\pi) = 0$ since $u(x', 2\pi) = 0$ and $u \geq 0$). Hence, Lemma 5.3.5 implies that $a_m \equiv 0 \equiv b_m$ for every $m \geq 2$, while from the classical Liouville theorem for harmonic function it follows that a_1 and b_1 are constant, so that

$$u(x', x_N) = \frac{a_0(x')}{2} + a_1 \cos x_N + b_1 \sin x_N \qquad \text{in } \Sigma.$$

Now we can conclude as in the proof of Theorem 5.3.1. □

Also if we cannot prove (5.24), it is possible to deduce something for the formal Fourier coefficients.

Proposition 5.3.8. *Let $N \geq 2$. Let $u \in \mathcal{C}^2\left(\overline{\mathbb{R}^N_+}\right)$ be a solution of problem* (5.2) *which satisfies* (5.11)*. Let a_m and b_m its formal Fourier coefficients, defined by* (5.13)*. Then*

- *(i)* $b_m \leq 0$ *for every* $m \geq 2$.
- *(ii)* $\frac{b_n}{n} \geq \frac{b_m}{m}$ *for every* $n > m \geq 2$.
- *(iii) for every* $m \geq 2$, *either* $b_m < 0$ *or* $b_m \equiv 0$ *in* $\mathbb{R}^{N-1}$.

Proof. (i) For every $m \geq 2$ we have

$$-\Delta' b_m(x') + (m^2 - 1)b_m(x') = -\frac{m}{\pi}u(x', 2\pi) \leq 0.$$

From Lemma 5.3.5, *which holds true in any dimension*, we deduce that

$$b_m \leq 0 \qquad \forall m \geq 2.$$

(ii) For $m \geq 2$, let us divide equation (5.15) by m:

$$\frac{\Delta' b_m(x')}{m} = \frac{m^2 - 1}{m}b_m(x') + \frac{1}{\pi}u(x', 2\pi).$$

If $n > m \geq 2$

$$-\Delta'\left(\frac{b_m(x')}{m} - \frac{b_n(x')}{n}\right) + \left(n^2 - 1\right)\left(\frac{b_m(x')}{m} - \frac{b_n(x')}{n}\right) = \left(n^2 - m^2\right)\frac{b_m(x')}{m} \leq 0,$$

thanks to the fact that $b_m \leq 0$. Again, by means of Lemma 5.3.5, we obtain

$$\frac{b_n}{n} \geq \frac{b_m}{m} \qquad \forall\ n > m \geq 2.$$

(iii) if there exists $\bar{x}' \in \mathbb{R}^{N-1}$ such that $b_m(\bar{x}') = 0$, the strong maximum principle implies $b_m \equiv 0$. □

It is particularly interesting to observe that, if we knew that one b_m vanishes in one point of $\mathbb{R}^{N-1}$, then we could recover Theorem 5.3.1.

Corollary 5.3.9. *Let* $N \geq 2$. *Let* $u \in \mathcal{C}^2\left(\overline{\mathbb{R}^N_+}\right)$ *be a solution of problem* (5.2) *satisfying* (5.11). *Let* a_m *and* b_m *its formal Fourier coefficients, defined by* (5.13).
If there exist $\bar{m} \geq 2$ *and* $\bar{x}' \in \mathbb{R}^{N-1}$ *such that* $b_{\bar{m}}(\bar{x}') = 0$, *then* $u(x', x_N) = 1 - \cos x_N$.

Proof. By point (iii) of the previous Proposition we know that $b_{\bar{m}} \equiv 0$.
Hence, from (5.15) for $\bar{m}$ we deduce

$$u(x', 2\pi) \equiv 0 \qquad \text{in } \mathbb{R}^{N-1}.$$

As a consequence

$$-\Delta' b_m(x') + (m^2 - 1) b_m(x') = 0 \qquad \forall m \geq 1,$$

which implies through Lemma 5.3.5 that $b_m \equiv 0$ for every $m \geq 2$; also, b_1 turns out to be a bounded harmonic function on the whole $\mathbb{R}^{N-1}$, so that it has to be constant. Now we show that $b_1 = 0$. Note that

$$\widetilde{u}(x', x_N) - b_1 \sin x_N = \frac{a_0(x')}{2} + \sum_{m=0}^{+\infty} a_m(x') \cos(m x_N);$$

hence, $w(x', x_N) = \widetilde{u}(x', x_N) - b_1 \sin x_N$ is an even 2π-periodic function in the x_N variable. Since we are assuming $u \in \mathcal{C}^2\left(\overline{\mathbb{R}^N_+}\right)$ and $u(x', 2\pi) = 0$, the function w is continuous on the whole $\mathbb{R}$, and has continuous derivative with respect to x_N, except at most in $(x', 0 + 2k\pi)$, with $k \in \mathbb{Z}$. However, the right and left derivatives in these points exist, and in particular

$$w_N(x', 2\pi^-) = u_N(x', 2\pi^-) - b_1 = -b_1.$$

By periodicity and oddness of w_N it results

$$b_1 = w_N(x', 0^+) = u_N(x', 0^+) - b_1 = u_N(x', 0) - b_1 \quad \Longrightarrow \quad u_N(x', 0) = 2b_1.$$

Note that $u_N(x', 0)$ is constant. Now, plugging this expression in equation (5.14) with $m = 1$ we obtain

$$\Delta' a_1(x') = \frac{2b_1}{\pi} \quad \Longrightarrow \quad \Delta'\left(a_1(x') - \frac{b_1 x_1^2}{\pi}\right) = 0:$$

the function $a_1(x') - b_1 x_1^2/\pi$ is harmonic in the whole $\mathbb{R}^N$ and has at most algebraic growth with rate 2 (since a_1 is bounded): therefore, the classical Liouville theorem for harmonic functions with algebraic growth implies that

$$a_1(x') = \frac{b_1 x_1^2}{\pi} + P(x'),$$

where P is a harmonic polynomial. To sum up, a_1 is a bounded polynomial, thus it is constant, which in turns gives $\Delta' a_1 = 0$, i.e. $b_1 = 0$ and finally $u_N(x', 0) = 0$. The thesis follows now from Proposition 5.3.7. $\square$

5.4 More general operators

In this section we generalize the approach adopted for the model problem to a more general family of elliptic equations (not necessarily uniformly elliptic) obtained by substituting the Laplacian with a class of operators in divergence form. To be precise, let $A(x')$ be a $N \times N$ matrix of type

$$A(x') = \begin{pmatrix} \widehat{A}(x') & 0 \\ 0 & 1 \end{pmatrix}, \tag{5.25}$$

where $\widehat{A}(x')$ is a $(N-1) \times (N-1)$ symmetric and real matrix with entries $a_{ij} \in C^1\left(\mathbb{R}^{N-1}\right) \cap L^\infty\left(\mathbb{R}^{N-1}\right)$, such that

$$\forall x' \in \mathbb{R}^{N-1},\ \forall \xi \in \mathbb{R}^{N-1} \setminus \{0\} : \quad \sum_{i,j=1}^{N-1} a_{ij}(x')\xi_i\xi_j > 0. \tag{5.26}$$

Of course, if $N = 2$ then $\widehat{A}(x')$ is a scalar positive function. Thanks to Theorem 5.2.7, we can try to use the arguments of Section 5.3 for the study of

$$\begin{cases} -\operatorname{div}(A(x)\nabla u) = u - 1 & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{5.27}$$

Note that, due to the particular form of A (cf. equation (5.25)), it is reasonable to think that (5.27) inherits the structure of the model problem solved in the previous section. It is also immediate to check that the function $u(x', x_N) = 1 - \cos x_N$ is, again, a nonnegative solution of (5.27) satisfying (5.11). We wish to prove that it is also unique in this class for $N = 2$ and 3.

Theorem 5.4.1. *Let $N = 2$ or 3. Let $A(x')$ be a $N \times N$ matrix of type* (5.25)*, where $\widehat{A}(x')$ is a $(N-1) \times (N-1)$ symmetric and real matrix with entries $a_{ij} \in C^1\left(\mathbb{R}^{N-1}\right) \cap L^\infty\left(\mathbb{R}^{N-1}\right)$, and such that* (5.26) *holds true. If $u \in \mathcal{C}^2\left(\overline{\mathbb{R}^N_+}\right)$ solves* (5.27) *and satisfies* (5.11)*, then*

$$u(x', x_N) = 1 - \cos x_N.$$

We can follow the proof of Theorem 5.3.1. Again, $\Sigma = \mathbb{R}^{N-1} \times (0, 2\pi)$. For every $x' \in \mathbb{R}^{N-1}$, we denote by $\tilde{u}(x', \cdot)$ the 2π-periodic extension of $x_N \mapsto u(x', x_N)$. As for the model problem, from the smoothness of u it follows that the Fourier expansion of $x_N \mapsto \tilde{u}(x', x_N)$ is convergent:

$$\widetilde{u}(x', x_N) = \frac{a_0(x')}{2} + \sum_{m=1}^{+\infty} \big(a_m(x') \cos(m x_N) + b_m(x') \sin(m x_N)\big)\,,$$

where a_m and b_m are defined by (5.13).

With a slightly modification of the proof of Lemma 5.3.3, we obtain

Lemma 5.4.2. *Let $N \geq 2$. For any $m \geq 1$ we have*

$$\operatorname{div}'\left(\widehat{A}(x')\nabla' a_m(x')\right) = (m^2-1)a_m(x') + \frac{1}{\pi}\big(u_N(x',0) - u_N(x',2\pi)\big) \tag{5.28}$$

$$\operatorname{div}'\left(\widehat{A}(x')\nabla' b_m(x')\right) = (m^2-1)b_m(x') + \frac{m}{\pi}u(x',2\pi). \tag{5.29}$$

Also,

$$\operatorname{div}'\left(\widehat{A}(x')\nabla' a_0(x')\right) = 2 - a_0(x') + \frac{1}{\pi}\big(u_N(x',0) - u_N(x',2\pi)\big)\,. \tag{5.30}$$

Proof. For any $m \geq 1$ we have

$$\operatorname{div}'\left(\widehat{A}(x')\nabla' a_m(x')\right) = \frac{1}{\pi}\int_0^{2\pi} \operatorname{div}'\left(\widehat{A}(x')\nabla' u(x',x_N)\right)\cos(m x_N)\,dx_N. \tag{5.31}$$

Since $a_{iN} = a_{Nj} \equiv 0$ for any $i, j \neq N$, we have

$$\begin{aligned}
\operatorname{div}'\left(\widehat{A}(x')\nabla' u(x',x_N)\right) &= \sum_{i=1}^{N-1} \partial_i \left(\sum_{j=1}^{N-1} a_{ij}(x') u_j(x',x_N)\right) \\
&= \sum_{i=1}^{N} \partial_i \left(\sum_{j=1}^{N} a_{ij}(x') u_j(x',x_N)\right) - u_{NN}(x',x_N) \\
&= 1 - u(x',x_N) - u_{NN}(x',x_N).
\end{aligned}$$

Hence equation (5.31) becomes

$$\operatorname{div}'\left(\widehat{A}(x')\nabla' a_m(x')\right) = \frac{1}{\pi}\int_0^{2\pi} \big(1 - u(x',x_N) - u_{NN}(x',x_N)\big)\cos(m x_N)\,dx_N.$$

Now, as usual, we can integrate by parts twice the last term and pass to

$$\operatorname{div}'\left(\widehat{A}(x')\nabla' a_m(x')\right) = (m^2-1)a_m(x') + \frac{1}{\pi}\big(u_N(x',0) - u_N(x',2\pi)\big)\,,$$

which is (5.28). The same procedure gives (5.29) and (5.30). □

Lemma 5.4.3. *Both b_1 and a_1 are constant. Moreover,*

$$u(x', 2\pi) = 0, \quad u_N(x', 2\pi) = 0 \quad \textit{and} \quad u_N(x', 0) = 0 \quad \forall x' \in \mathbb{R}^{N-1}.$$

Proof. In light of the previous Lemma, we have

$$\operatorname{div}' \left(\widehat{A}(x') \nabla' b_1(x') \right) = \frac{1}{\pi} u(x', 2\pi) \geq 0;$$

the function b_1 is bounded (since u satisfies (5.11)), and since $N = 2$ or 3 we are in position to apply Theorem 5.2.7:

$$b_1 = const. \quad \Longrightarrow \quad u(x', 2\pi) = \pi \operatorname{div}' \left(\widehat{A}(x') \nabla' b_1(x') \right) = 0.$$

Note that now $u_N(x', 2\pi) = 0$, since $(x', 2\pi)$ is a point of minimum of u for every $x' \in \mathbb{R}^{N-1}$. Therefore, equation (5.28) becomes

$$\operatorname{div}' \left(\widehat{A}(x') \nabla' a_1(x') \right) = \frac{1}{\pi} u_N(x', 0) \geq 0;$$

this means that a_1 satisfies the assumptions of Theorem 5.2.7:

$$a_1 = const. \quad \Longrightarrow \quad u_N(x', 0) = \pi \operatorname{div}' \left(\widehat{A}(x') \nabla' a_1(x') \right) = 0. \qquad \square$$

As a consequence, the equations for a_m and b_m simplify:

$$\begin{aligned} \operatorname{div}' \left(\widehat{A}(x') \nabla' b_m(x') \right) &= (m^2 - 1) b_m(x') \qquad \forall m \geq 2 \\ \operatorname{div}' \left(\widehat{A}(x') \nabla' a_m(x') \right) &= (m^2 - 1) a_m(x') \qquad \forall m \geq 2. \end{aligned}$$

In this way, we proved that for any $m \geq 2$ both the coefficients a_m and b_m are bounded solution of an equation of type

$$-\operatorname{div}' \left(\widehat{A}(x') \nabla' v(x') \right) + \lambda v(x') = 0 \qquad \text{in } \mathbb{R}^{N-1}, \tag{5.32}$$

with $\lambda > 0$. In analogy with the model problem, we state the following result.

Lemma 5.4.4. *Assume $N \geq 2$ and let $v \in \mathcal{C}^2 \left(\mathbb{R}^{N-1} \right)$ a subsolution of*

$$-\operatorname{div}' \left(\widehat{A}(x') \nabla' v(x') \right) + c(x') v(x') \leq 0 \qquad \textit{in } \mathbb{R}^{N-1},$$

with $c(x') \geq \lambda > 0$. Here $\widehat{A}(x')$ is an $(N-1) \times (N-1)$ matrix with entries a_{ij} in $L^\infty(\mathbb{R}^{N-1})$ and such that (5.26) holds true.
If v^+ has at most algebraic growth at infinity, then $v \leq 0$.

Proof. For any $R > 0$, let φ_R be as in the proof of Lemma 5.3.5. Recall the (5.21):

$$\left|\nabla' \varphi_R(x')\right| \leq \frac{C}{R}\chi_{B_{2R}}(x').$$

Let us test equation (5.32) with $v^+\varphi_R^2$:

$$\int_{\mathbb{R}^{N-1}} \left(\langle \widehat{A}(x')\nabla' v^+, \nabla' v^+\rangle + c(x')\left(v^+\right)^2\right)\varphi_R^2 \\ = -2\int_{\mathbb{R}^{N-1}} v^+\varphi_R \langle \widehat{A}(x')\nabla' v, \nabla' \varphi_R\rangle. \quad (5.33)$$

Under our assumptions on $\widehat{A}$, for almost every $x' \in \mathbb{R}^{N-1}$ the function

$$(\xi_1, \xi_2) \in \mathbb{R}^{2(N-1)} \mapsto \langle \widehat{A}(x')\xi_1, \xi_2\rangle \in \mathbb{R}$$

defines a bilinear symmetric positive definite form, so that in particular the Cauchy-Schwarz inequality holds true. Hence, using also the Young inequality, we can control the right hand side: for any $\varepsilon > 0$ there exists $C_\varepsilon > 0$ such that

$$\begin{aligned} -2\int_{\mathbb{R}^{N-1}} v^+\varphi_R\langle \widehat{A}(x')\nabla' v^+, \nabla'\varphi_R\rangle &\leq 2\int_{\mathbb{R}^{N-1}} v^+\varphi_R\left|\langle \widehat{A}(x')\nabla' v^+, \nabla'\varphi_R\rangle\right| \\ &\leq 2\int_{\mathbb{R}^{N-1}} v^+\varphi_R\sqrt{\langle \widehat{A}(x')\nabla' v^+, \nabla' v^+\rangle}\sqrt{\langle \widehat{A}(x')\nabla'\varphi_R, \nabla'\varphi_R\rangle} \\ &\leq 2\varepsilon\int_{\mathbb{R}^{N-1}} \varphi_R^2\langle \widehat{A}(x')\nabla' v^+, \nabla' v^+\rangle + 2C_\varepsilon\int_{\mathbb{R}^{N-1}} \left(v^+\right)^2\langle \widehat{A}(x')\nabla'\varphi_R, \nabla'\varphi_R\rangle. \end{aligned}$$

Coming back to equation (5.33), using also the fact that $c(x') \geq \lambda$, we find

$$\int_{\mathbb{R}^{N-1}} \left((1-2\varepsilon)\langle \widehat{A}(x')\nabla' v^+, \nabla' v^+\rangle + \lambda\left(v^+\right)^2\right)\varphi_R^2 \\ \leq 2C_\varepsilon\int_{\mathbb{R}^{N-1}} \left(v^+\right)^2\langle \widehat{A}(x')\nabla'\varphi_R, \nabla'\varphi_R\rangle.$$

Choosing $\varepsilon < 1/2$, using the assumptions of $\widehat{A}$ and the estimate (5.21), we deduce

$$\int_{B_R} \left(v^+\right)^2 \leq \frac{C}{\lambda R^2}\int_{B_{2R}} \left(v^+\right)^2.$$

Also, since v^+ has at most algebraic growth, we have

$$\int_{B_R} \left(v^+\right)^2 \leq C' R^{N+2k}.$$

We can apply Lemma 5.3.6 again, to find

$$\int_{B_R} \left(v^+\right)^2 = 0 \qquad \forall R > R_0 \text{ sufficiently large,}$$

which implies $v^+ \equiv 0$. □

Conclusion of the proof of Theorem 5.4.1. The previous lemma implies that $a_m \equiv 0$ and $b_m \equiv 0$ for every $m \geq 2$. Therefore, a solution u of (5.27) which satisfies (5.11) has the following expansion in $\overline{\Sigma}$:

$$u(x', x_N) = \frac{a_0(x')}{2} + a_1 \cos x_N + b_1 \sin x_N,$$

which is the same of equation (5.23). Moreover, we showed that

$$u(x', 0) = 0 \quad \text{and} \quad u_N(x', 0) = 0 \quad \forall x' \in \mathbb{R}^{N-1},$$

hence we can repeat step by step the conclusion of the proof of Theorem 5.3.1. □

5.5 More general problems

In this section we apply the previous method to study and classify solutions of

$$\begin{cases} -\mathrm{div}\,(A(x')\nabla u) = u - g(x', x_N) & \text{in } \mathbb{R}^N_+ \\ u = 0 & \text{on } \partial\mathbb{R}^N_+, \end{cases}$$

satisfying (5.11), when $N = 2$ or 3. The situation here is much more involved than the one in the previous sections. Indeed, we have to face the occurrence of various phenomena such as: non-existence of solutions and/or the existence and the multiplicity of solutions. Moreover, a solution might not be a function of the x_N variable only (in fact, if g depends on x' such a result cannot be expected). The results we shall prove will strongly depend on the form of the function g.

In what follows, we always assume that the matrix A satisfies the assumptions already imposed in the previous section. Therefore, we will no more write explicitly these assumptions. Since we are interested in classical solutions, we assume that $g \in \mathcal{C}\left(\overline{\mathbb{R}^N_+}\right)$. We can consider the 2π-periodic extension of $x_N \in (0, 2\pi) \mapsto g(x', x_N)$: inside Σ we have the expansion

$$\widetilde{g}(x', x_N) = \frac{c_0(x')}{2} + \sum_{m=1}^{\infty} \left(c_m(x') \cos(m x_N) + d_m(x') \sin(m x_N)\right),$$

where

$$c_m(x') := \frac{1}{\pi}\int_0^{2\pi} g(x', x_N)\cos(mx_N)\,dx_N \qquad \forall m \geq 0$$
$$d_m(x') := \frac{1}{\pi}\int_0^{2\pi} g(x', x_N)\sin(mx_N)\,dx_N \qquad \forall m \geq 1. \tag{5.34}$$

Let us define again a_m and b_m by (5.13); these are the formal Fourier coefficients of u with respect to the x_N-variable in Σ. We start writing down the equations satisfied by a_m and b_m.

Lemma 5.5.1. *For any $m \geq 0$ it results*

$$\begin{aligned}\operatorname{div}'\left(\widehat{A}(x')\nabla' a_m(x')\right) &= \left(m^2-1\right)a_m(x') + c_m(x') \\ &\quad + \frac{1}{\pi}\left(u_N(x',0) - u_N(x',2\pi)\right);\end{aligned} \tag{5.35}$$

For any $m \geq 1$ it results

$$\operatorname{div}'\left(\widehat{A}(x')\nabla' b_m(x')\right) = \left(m^2-1\right)b_m(x') + d_m(x') + \frac{m}{\pi}u(x',2\pi). \tag{5.36}$$

Proof. For any $m \geq 1$:

$$\begin{aligned}\operatorname{div}'\left(\widehat{A}(x')\nabla' a_m(x')\right) &= \frac{1}{\pi}\int_0^{2\pi}\operatorname{div}'\left(\widehat{A}(x')\nabla' u(x',x_N)\right)\cos(mx_N)\,dx_N \\ &= \frac{1}{\pi}\int_0^{2\pi}\left(g(x',x_N) - u(x',x_N) - u_{NN}(x',x_N)\right)\cos(mx_N)\,dx_N.\end{aligned}$$

Now we can go on with the same computations already developed in Lemma 5.3.3, with the only difference that

$$\frac{1}{\pi}\int_0^{2\pi}\cos(mx_N)\,dx_N = 0 \quad \text{while} \quad \frac{1}{\pi}\int_0^{2\pi} g(x',x_N)\cos(mx_N)\,dx_N = c_m(x').$$

In the end, we obtain

$$\operatorname{div}'\left(\widehat{A}(x')\nabla' a_m(x')\right) = \left(m^2-1\right)a_m(x') + c_m(x') + \frac{1}{\pi}\left(u_N(x',0) - u_N(x',2\pi)\right).$$

The same procedure gives the equations for b_m and for a_0. □

For a quite general g the study of these equations does not give a complete classification for the possible solutions of (5.1). However, in some particular cases we can obtain sharp results. This is be the object of the following subsections.

5.5.1 Inhomogeneous terms independent of x_N

The first generalization concerns a constant g. It is straightforward to adapt the arguments of the previous sections, obtaining the following result.

Theorem 5.5.2. *Let $N = 2$ or 3. If $g(x', x_N) = \theta \in \mathbb{R}$, one of the following alternatives occurs:*

(i) if $\theta \geq 0$ there exists a unique solution of (5.1) satisfying (5.11). This solution is given by

$$u(x', x_N) = \theta(1 - \cos x_N).$$

(ii) if $\theta < 0$, problem (5.1) does not admit any solution satisfying (5.11).

The next step in the study is to treat the case $g = g(x')$. If we are interested in solutions satisfying (5.11) and g is not constant, we can show that we do not have such a kind of solution at all.

Theorem 5.5.3. *Let $N = 2$ or 3, let $g = g(x') \in \mathcal{C}(\mathbb{R}^{N-1})$. If g is not constant, problem (5.1) does not admit any solution satisfying (5.11).*
Equivalently, if there exists $u \in \mathcal{C}^2\left(\overline{\mathbb{R}^N_+}\right)$ which solves (5.1) and satisfies (5.11), then g is constant.

Proof. Assume that g is not constant; the formal Fourier coefficients of g are

$$c_0(x') = 2g(x') \quad c_m(x') \equiv 0 \quad d_m(x') \equiv 0 \quad \forall m \geq 1.$$

By contradiction, let u be a solution of (5.1) satisfying (5.11). Since $c_1 \equiv 0$ and $d_1 \equiv 0$, equations (5.35) and (5.36) for $m = 1$ are

$$\begin{aligned}
\operatorname{div}'\left(\widehat{A}(x')\nabla' a_1(x')\right) &= \frac{1}{\pi}\left(u_N(x', 0) - u_N(x', 2\pi)\right) \\
\operatorname{div}'\left(\widehat{A}(x')\nabla' b_1(x')\right) &= \frac{1}{\pi} u(x', 2\pi).
\end{aligned}$$

Hence we are in position to follow the proof of Lemma 5.4.3: a_1 and b_1 are constant, and $u(x', 2\pi), u_N(x', 0), u_N(x', 2\pi) \equiv 0$ in $\mathbb{R}^{N-1}$. As a consequence, equations (5.35) and (5.36) become

$$\begin{aligned}
\operatorname{div}'\left(\widehat{A}(x')\nabla' a_m(x')\right) &= \left(m^2 - 1\right) a_m(x') \\
\operatorname{div}'\left(\widehat{A}(x')\nabla' b_m(x')\right) &= \left(m^2 - 1\right) b_m(x').
\end{aligned}$$

Therefore Lemma 5.4.4 applies: $a_m = b_m \equiv 0$ for every $m \geq 2$, so that

$$u(x', x_N) = \frac{a_0(x')}{2} + a_1 \cos x_N + b_1 \sin x_N.$$

The boundary condition $u(x', 0) = 0$ implies that a_0 is constant, but (5.35) for $m = 0$ yields

$$0 = \operatorname{div}' \left(\widehat{A}(x') \nabla' a_0 \right) = 2g(x') - a_0,$$

a contradiction. □

5.5.2 A 1-D inhomogeneous term

In this subsection we deal with $g = g(x_N)$. In this situation various phenomena may occur. Let us start with :

Non-existence. If $g(x_N) = \sin x_N$, problem (5.1) does not admit any solution satisfying (5.11). This follows from the following general result.

Proposition 5.5.4. *Let $N = 2$ or 3, let $g \in \mathcal{C}\left(\overline{\mathbb{R}^N_+}\right)$ and assume that $d_1 \geq 0$. If there exists a solution u of* (5.1) *such that* (5.11) *holds true, then $d_1 = 0$, b_1 is constant,*

$$u(x', 2\pi) = 0 \quad \text{and} \quad u_N(x', 2\pi) = 0 \quad \forall x' \in \mathbb{R}^{N-1}.$$

Proof. Let us consider equation (5.36) for $m = 1$: since $d_1 \geq 0$ and $u \geq 0$, we have

$$\operatorname{div}' \left(\widehat{A}(x') \nabla' b_1(x') \right) = d_1 + \frac{1}{\pi} u(x', 2\pi) \geq 0.$$

Due to the boundedness of u in the strip Σ, b_1 is bounded in absolute value. Since $N = 2$ or 3, we can apply Theorem 5.2.7, obtaining that b_1 is constant, which in turns gives

$$\frac{1}{\pi} u(x', 2\pi) = -d_1 \quad \Longrightarrow \quad u(x', 2\pi) = 0 = d_1 \qquad \forall x' \in \mathbb{R}^{N-1},$$

because u is nonnegative. Note that necessarily $u_N(x', 2\pi) = 0$. □

Remark 5.5.5. 1) Note that since g depends only on x_N, both c_m and d_m are constant.
2) The previous proposition applies not only if $g = g(x_N)$. For instance, it gives analogous non-existence results when

- $g(x', x_N)|_\Sigma$ is decreasing in the x_N direction ($g \neq$ const.) .
- $g(x', x_N) \geq g(x', 2\pi - x_N)$ for every $(x', x_N) \in \mathbb{R}^{N-1} \times (0, \pi)$, with strict inequality in one point.

We have a counterpart of the previous statement which rules out the existence of solutions of (5.1) satisfying (5.11) when $g(x_N) = \cos x_N$.

Proposition 5.5.6. *Let $N = 2$ or 3, let $g \in \mathcal{C}\left(\overline{\mathbb{R}^N_+}\right)$ and assume that $d_1 = 0$, $c_1 \geq 0$. If there exists a solution u of* (5.1) *such that* (5.11) *holds true, then $c_1 \equiv 0$, a_1 is constant, and*

$$u_N(x', 0) = 0 \qquad \forall x' \in \mathbb{R}^{N-1}.$$

Proof. In light of Proposition 5.5.4, we know that $u_N(x', 2\pi) = 0$. Moreover, as already observed, from $u(x', 0) = 0$ and $u \geq 0$ it follows $u_N(x', 0) \geq 0$. Thus, considering equation (5.35) for $m = 1$, we obtain

$$\operatorname{div}'\left(\widehat{A}(x')\nabla' a_1(x')\right) = c_1 + \frac{1}{\pi}u_N(x', 0) \geq 0,$$

since $c_1 \geq 0$. The function a_1 is bounded in absolute value, hence by Theorem 5.2.7 it is constant. Therefore

$$\frac{1}{\pi}u_N(x', 0) = -c_1 \quad \Longrightarrow \quad u_N(x', 0) = 0 = c_1 \qquad \forall x' \in \mathbb{R}^{N-1}. \qquad \square$$

Existence and multiplicity. For every $N \geq 2$,

$$u_A(x', x_N) = x_N + A \sin x_N, \qquad A \in [-1, 1],$$

is a one-parameter family of solutions of (5.1) with $g(x_N) = x_N$; each u_A satisfies (5.11). Note that in this case $c_1 = 0$ while $d_1 < 0$, so that the previous propositions do not apply. Note also that u_A is unbounded in $\mathbb{R}^N_+$ for every $A \in [-1, 1]$.

Existence, uniqueness and 1-D symmetry. For $m \geq 2$, the function $u(x', x_N) = (1 - \cos(mx_N))/(m^2 - 1)$ is the *unique* solution, satisfying (5.11), of problem (5.1) for $g(x_N) = 1/(m^2 - 1) + \cos(mx_N)$. Furthermore, u has 1-D symmetry. The uniqueness result is a consequence of the following general result.

Theorem 5.5.7. *Let $N = 2$ or 3, let $g \in \mathcal{C}(\mathbb{R})$ be such that*

$$c_1 \geq 0 \quad \text{and} \quad d_1 \geq 0, \tag{5.37}$$

where c_m and d_m are the Fourier coefficients of the function g in $(0, 2\pi)$, defined by (5.34). *If there exists $u \in \mathcal{C}^2\left(\overline{\mathbb{R}^N_+}\right)$ which solves problem* (5.1) *and satisfies* (5.11), *then* *necessarily* $c_1 = d_1 = 0$. *In this case, the restriction of* u *to* $\overline{\Sigma}$ *is* 1*-dimensional and is uniquely determined as the solution of*

$$\begin{cases} -u''(x_N) = u(x_N) - g(x_N) & \text{in } (0, 2\pi) \\ u(0) = u(2\pi) = 0 \\ u'(0) = u'(2\pi) = 0. \end{cases} \tag{5.38}$$

In particular, in $\overline{\Sigma}$ *we have*

$$\begin{aligned} u(x', x_N) = \frac{c_0}{2} + \left(-\frac{c_0}{2} + \sum_{m=2}^{+\infty} \frac{c_m}{m^2-1} \right) \cos x_N + \left(\sum_{m=2}^{+\infty} \frac{m\, d_m}{m^2-1} \right) \sin x_N \\ - \sum_{m=2}^{+\infty} \left(\frac{c_m}{m^2-1} \cos(m x_N) + \frac{d_m}{m^2-1} \sin(m x_N) \right). \end{aligned} \tag{5.39}$$

Proof. From Propositions 5.5.4 and 5.5.6 we know that, if u exists, then c_1 and d_1 has to be 0; in this case a_1 and b_1 are constant, and $u(x', 2\pi)$, $u_N(x', 0)$, $u_N(x', 2\pi) = 0$ in $\mathbb{R}^{N-1}$. Therefore, equations (5.35) and (5.36) for $m \geq 2$ simplify as

$$\begin{aligned} \operatorname{div}' \left(\widehat{A}(x') \nabla' a_m(x') \right) &= (m^2-1) a_m(x') + c_m \\ \operatorname{div}' \left(\widehat{A}(x') \nabla' b_m(x') \right) &= (m^2-1) b_m(x') + d_m, \end{aligned}$$

i.e.

$$\begin{aligned} -\operatorname{div}' \left(\widehat{A}(x') \nabla' \left(a_m(x') + \frac{c_m}{m^2-1} \right) \right) + (m^2-1) \left(a_m(x') + \frac{c_m}{m^2-1} \right) = 0 \\ -\operatorname{div}' \left(\widehat{A}(x') \nabla' \left(b_m(x') + \frac{d_m}{m^2-1} \right) \right) + (m^2-1) \left(b_m(x') + \frac{d_m}{m^2-1} \right) = 0. \end{aligned}$$

We can apply Lemma 5.4.4, obtaining

$$\begin{aligned} a_m(x') = a_m = -\frac{c_m}{m^2-1} \qquad \forall m \geq 2 \\ b_m(x') = b_m = -\frac{d_m}{m^2-1} \qquad \forall m \geq 2. \end{aligned}$$

Now, let us consider in Σ

$$\frac{a_0(x')}{2} + a_1 \cos x_N + b_1 \sin x_N - \sum_{m=2}^{+\infty} \left(\frac{c_m}{m^2-1} \cos(m x_N) + \frac{d_m}{m^2-1} \sin(m x_N) \right).$$

It is a series of $\mathcal{C}^\infty$ functions, which is convergent together with the series of the derivatives w.r.t. x_N, since the sequences $\{c_m\}$ and $\{d_m\}$ belong to l^2. In Σ the series is equal to u, and the equality can be extended up to the boundary since both the series itself and u are $\mathcal{C}^1(\overline{\Sigma})$. We also know that $u(x',0)=0$ and $u_N(x',0)=0$. Using the Dirichlet boundary condition we deduce that a_0 is constant too, and in particular equation (5.35) for $m=0$ implies $a_0=c_0$. Now, the "initial" conditions give the expressions of a_1 and b_1. To sum up, we proved that $u|_\Sigma$ is 1-D, thus a solution of

$$-u''(x_N)=u(x_N)-g(x_N) \qquad \text{for } x_N\in(0,2\pi)$$

with the boundary conditions stated in (5.38). □

As an immediate consequence, we obtain

Theorem 5.5.8. *Let $N=2$ or 3, let $g\in\mathcal{C}(\mathbb{R})$ be a 2π-periodic function satisfying* (5.37)*, where c_m and d_m are the Fourier coefficients of the function g. If there exists $u\in\mathcal{C}^2\left(\overline{\mathbb{R}^N_+}\right)$ which solves problem* (5.1) *and satisfies* (5.11)*, then necessarily $c_1=d_1=0$. In this case, u is 1-dimensional, 2π-periodic and it is uniquely determined as the solution of*

$$\begin{cases} -u''(x_N)=u(x_N)-g(x_N) & \text{in } (0,+\infty)\\ u(0)=u(2\pi)=0 \\ u'(0)=u'(2\pi)=0.\end{cases}$$

The expression of u in Fourier series is given by (5.39)*.*

In view of the example with $g(x_N)=x_N$ ($c_1=0$, $d_1<0$), we see that the assumptions $c_1\geq 0$ and $d_1\geq 0$ are necessary for Theorem 5.5.7 and Theorem 5.5.8. We also remark that the non-negativity of both c_1 and d_1 is not sufficient to guarantee the existence of a solution of (5.1) which satisfies (5.11). Indeed, as an immediate consequence of Proposition 5.5.13 (proved in the next subsection), we have non-existence of solutions of (5.1) satisfying (5.11) in case

$$g(x_N)=C_1\sin(mx_N) \quad \text{or} \quad g(x_N)=C_2\cos(mx_N) \quad m\geq 2, C_1,C_2\in\mathbb{R}.$$

Note that $c_1=d_1=0$ in the above examples.

Another class of functions g for which there is non-existence is considered in the next result, of independent interest,

Proposition 5.5.9. *Let $N\geq 2$. If $g\leq 0$ and it is non-constant, then a nonnegative solution of* (5.1) *has to be positive. In particular, if $N=2,3$ and $d_1\geq 0$, then problem* (5.1) *does not admit any solution satisfying* (5.11)*.*

Proof. By the strong maximum principle u must be positive in $\mathbb{R}^N_+$. Since $d_1 \geq 0$, if a solution u existed, from Proposition 5.5.4 it should satisfy $u(x', 2\pi) = 0$ for every $x' \in \mathbb{R}^{N-1}$. A contradiction. □

A typical example is given by the function $g(x_N) = -\theta - \cos x_N$, with $\theta \geq 1$. Note that $c_1 < 0$ and $d_1 = 0$ in this example.

5.5.3 General inhomogeneous terms

In this subsection we consider some g depending on both x' and x_N. As before, we will denote by c_m and d_m the Fourier coefficient of the 2π-periodic extension of $x_N \in (0, 2\pi) \mapsto g(x', x_N)$.

We have always began our analysis trying to prove that

$$u(x', 2\pi) \equiv 0 \quad \text{and} \quad u_N(x', 0) \equiv 0 \quad \text{in } \mathbb{R}^{N-1}, \tag{5.40}$$

so that also $u_N(x', 2\pi) = 0$ in $\mathbb{R}^{N-1}$, and all the boundary terms in equations (5.35) and (5.36) vanish identically:

$$\operatorname{div}'\left(\widehat{A}(x')\nabla' a_m(x')\right) = \left(m^2 - 1\right) a_m(x') + c_m(x')$$
$$\operatorname{div}'\left(\widehat{A}(x')\nabla' b_m(x')\right) = \left(m^2 - 1\right) b_m(x') + d_m(x').$$

We have already observed that, if $N = 2$ or 3, sufficient conditions in order to obtain (5.40) are $c_1 \geq 0$ and $d_1 \geq 0$.
In general (for every $N \geq 2$), assume that (5.40) holds true. Assume also that there exists $\bar{m} \geq 2$ such that $c_{\bar{m}} \equiv 0$. Then

$$\operatorname{div}'\left(\widehat{A}(x')\nabla' a_{\bar{m}}(x')\right) = \left(\bar{m}^2 - 1\right) a_{\bar{m}}(x'),$$

which is of type (5.32) with $\lambda > 0$. From Lemma 5.4.4 it follows $a_{\bar{m}} \equiv 0$. The same holds true for every $b_{\bar{m}}$ such that $d_{\bar{m}} \equiv 0$. We point out that this is true even for $N > 3$.

Proposition 5.5.10. *Let $N \geq 2$, let $g \in \mathcal{C}\left(\overline{\mathbb{R}^N_+}\right)$, let u be a solution of* (5.1) *satisfying* (5.11)*, and let a_m and b_m be its formal Fourier coefficients in Σ defined by* (5.13)*; assume that* (5.40) *holds true. Then for every $m \geq 2$ such that $c_m \equiv 0$ it results $a_m \equiv 0$, and for every $m \geq 2$ such that $d_m \equiv 0$ it results $b_m \equiv 0$.*

As far as the coefficient a_0 is concerned, we have a similar result, but only in low dimension.

Proposition 5.5.11. *Let* $N = 2$ *or* 3*, let* $g \in \mathcal{C}\left(\overline{\mathbb{R}^N_+}\right)$*, let* u *be a solution of* (5.1) *satisfying* (5.11)*, and let* a_m *and* b_m *be its formal Fourier coefficients defined by* (5.13)*; assume that* (5.40) *holds true. If* $c_0 \le 0$*, then* $a_0 = c_0 \equiv 0$*.*

Proof. Since $u \ge 0$, equation (5.35) for $m = 0$ is

$$\operatorname{div}'\left(\widehat{A}(x')\nabla' a_0(x')\right) = -a_0(x') + c_0(x') \le 0 \qquad \text{in } \mathbb{R}^{N-1}.$$

Since a_0 is bounded and $N = 2$ or 3, for Theorem 5.2.7 a_0 is constant. But then

$$0 = \operatorname{div}'\left(\widehat{A}(x')\nabla' a_0(x')\right) = -a_0 + c_0.$$

Thus, $0 \le a_0 = c_0 \le 0$. □

In what follows we first consider

$$g(x', x_N) = f(x')\varphi(x_N) \in \mathcal{C}\left(\overline{\mathbb{R}^N_+}\right), \qquad g \not\equiv 0.$$

In the expansion of the 2π-periodic extension of $x_N \in (0, 2\pi) \mapsto g(x', x_N)$, the Fourier coefficients are

$$c_m(x') = f(x')\gamma_m \quad \forall m \ge 0, \qquad d_m(x') = f(x')\delta_m \quad \forall m \ge 1,$$

where γ_m and δ_m are the (constant) Fourier coefficients of the 2π-periodic extension of $x_N \in (0, 2\pi) \mapsto \varphi(x_N)$.

Remark 5.5.12. Let $N = 2$ or 3. In light of Propositions 5.5.4 and 5.5.6, we know that if $f(x')\delta_1 \ge 0$, $f(x')\delta_1 \ne 0$, then there are no solutions of (5.1) satisfying (5.11). The same holds true if $f(x')\gamma_1 \ge 0$, $f(x')\gamma_1 \ne 0$ and $\delta_1 = 0$.

If $N = 2$ or 3 and $\gamma_1 = \delta_1 = 0$, from Propositions 5.5.4 and 5.5.6 we know that a_1 and b_1 are constant and (5.40) holds true. Hence, equations (5.35) and (5.36) simplify as

$$\operatorname{div}'\left(\widehat{A}(x')\nabla' a_m\right) = (m^2 - 1)a_m + f(x')\gamma_m \qquad \forall m \ne 1,$$
$$\operatorname{div}'\left(\widehat{A}(x')\nabla' b_m\right) = (m^2 - 1)b_m + f(x')\delta_m \qquad \forall m \ge 2.$$

It is not difficult to obtain the following non-existence result.

Proposition 5.5.13. *Let $N = 2$ or 3, let $g \in \mathcal{C}\left(\overline{\mathbb{R}^N_+}\right)$ and assume that*

$$g(x', x_N) = f(x')\cos(mx_N) \quad \text{or} \quad g(x', x_N) = f(x')\sin(mx_N),$$

where $m \geq 2$ and f is not identically 0. Then there are no solutions of (5.1) satisfying (5.11).

Proof. Let us first consider the case $g(x', x_N) = f(x')\sin(mx_N)$. By contradiction, let u be a solution of (5.1) satisfying (5.11). Applying Propositions 5.5.4, 5.5.6, 5.5.10 and 5.5.11, we obtain the following particular form for u in Σ:

$$u(x', x_N) = a_1 \cos x_N + b_1 \sin x_N + b_m(x') \sin(mx_N).$$

Since $u_N(x', 0) \equiv 0$, we deduce that $b_m = -\frac{b_1}{m}$ is constant. On the other hand it is a solution of

$$0 = \operatorname{div}'\left(\widehat{A}(x')\nabla' b_m\right) = (m^2 - 1)b_m + f(x');$$

thus f must be constant, that is $f(x') = f \equiv \theta \in \mathbb{R} \setminus \{0\}$. But in this case, imposing the initial condition $u(x', 0) = 0$, we obtain $a_1 = 0$, and consequently

$$u(x', x_N) = \frac{\theta}{m^2 - 1}\left(m \sin x_N - \sin(mx_N)\right),$$

which does not satisfy (5.11) because it assumes negative values (it is odd, 2π-periodic and not identically zero).

When $g(x', x_N) = f(x')\cos(mx_N)$, we can argue as before to find that u has the form

$$u(x', x_N) = a_1 \cos x_N + b_1 \sin x_N + a_m(x') \cos(mx_N).$$

The boundary condition $u_N(x', 0) = 0$ implies $b_1 = 0$, while from $u(x', 0) = 0$ we deduce that $a_m(x') = -a_1 = \textit{const.}$. Hence $0 = (m^2 - 1)a_m + f(x')$ and so $f(x') = f \equiv \theta \in \mathbb{R} \setminus \{0\}$. Finally u has the form

$$u(x', x_N) = \frac{\theta}{m^2 - 1}\left(\cos(x_N) - \cos(mx_N)\right).$$

Observe that

$$0 \leq u\left(x', \frac{2\pi}{m}\right) = \frac{\theta}{m^2 - 1}\left(\cos\left(\frac{2\pi}{m}\right) - 1\right) \quad \Longrightarrow \theta \leq 0,$$

while

$$0 \leq u\left(x', \frac{\pi}{m}\right) = \frac{\theta}{m^2 - 1}\left(\cos\left(\frac{\pi}{m}\right) + 1\right) \quad \Longrightarrow \theta \geq 0,$$

a contradiction. □

More in general, the same proof yields

Proposition 5.5.14. *Let $N = 2$ or 3, let $g \in \mathcal{C}\left(\overline{\mathbb{R}^N_+}\right)$ and assume that*

$$g(x', x_N) = \frac{c_0(x')}{2} + \sum_{m \in I_1} c_m(x') \cos(m x_N) + d_{\bar n}(x') \sin(\bar n x_N)$$

$$\text{or} \quad g(x', x_N) = c_{\bar m}(x') \cos(\bar m x_N) + \sum_{n \in I_2} d_n(x') \sin(n x_N),$$

where $I_1, I_2 \subset (\mathbb{N} \setminus \{0,1\})$ are finite sets, $\bar n \geq 2$, $\bar m \in (\mathbb{N} \setminus \{1\})$, and $d_{\bar n}$, $c_{\bar m}$ are not identically constant. Then there are no solutions of (5.1) *satisfying* (5.11)*.*

In what follows we set $N = 2$ or 3 and we show that it is possible to use the method of the Fourier coefficients in order to obtain a complete classification when $c_1 = d_1 = 0$ and only a finite number of the Fourier coefficients of g are not identically zero.

Let

$$g(x', x_N) = \frac{c_0(x')}{2} + \sum_{m \in I_1} c_m(x') \cos(m x_N) + \sum_{n \in I_2} d_n(x') \sin(n x_N), \tag{5.41}$$

where $I_1 = \{m_1, \ldots, m_{k_1}\}, I_2 = \{n_1, \ldots, n_{k_2}\} \subset (\mathbb{N} \setminus \{0,1\})$. As far as c_0 is concerned, it can be identically 0 or not. Only to fix our minds, we assume $c_0(x') \neq 0$; furthermore, for the sake of simplicity, we suppose that $c_0, c_{m_j}, d_{n_j} \in \mathcal{C}^\infty(\mathbb{R}^{N-1})$.
We show that, *if there exists $u \in \mathcal{C}^2\left(\overline{\mathbb{R}^N_+}\right)$ which solves* (5.1) *for this particular g and satisfies* (5.11)*, then we can determine the explicit expression of u.*
Note that, since $c_1 = d_1 = 0$, Propositions 5.5.4 and 5.5.6 imply that a_1 and b_1 are constant, and (5.40) holds true; thus, by Proposition 5.5.10 we obtain

$$u(x', x_N) = \frac{a_0(x')}{2} + a_1 \cos x_N + b_1 \sin x_N + \sum_{j=1}^{k_1} a_{m_j}(x') \cos(m_j x_N) + \sum_{j=1}^{k_2} b_{n_j}(x') \sin(n_j x_N),$$

in Σ, where a_0, a_{m_j} and b_{n_j} are solutions of

$$\mathrm{div}'\left(\widehat{A}(x') \nabla' a_0(x')\right) = -a_0(x') + c_0(x') \tag{5.42}$$

$$\mathrm{div}'\left(\widehat{A}(x') \nabla' a_{m_j}(x')\right) = (m_j^2 - 1) a_{m_j}(x') + c_{m_j}(x') \tag{5.43}$$

$$\mathrm{div}'\left(\widehat{A}(x') \nabla' b_{n_j}(x')\right) = (n_j^2 - 1) b_{n_j}(x') + d_{n_j}(x').$$

Propositions 5.5.13 and 5.5.14 imply that, if there exists a unique $m \in \mathbb{N} \setminus \{1\}$ such that $c_m \neq 0$ and is not constant, or if there exists a unique $m \geq 2$ such that $d_m \neq 0$ and is not constant, then a solution of (5.1) satisfying (5.11) does not exist. If we are not in this situation and such a solution exists, this system of PDEs (or ODEs if $N = 2$), together with the boundary conditions $u(x', 0) = 0$ and $u_N(x', 0) = 0$ permits to determine the explicit expression of a_0, a_{m_j} and b_{n_j}. We start observing that the boundary condition $u(x', 0) = 0$ involves only a_0 and a_{m_j}, while $u_N(x', 0) = 0$ involves the b_{n_j}. Thus, we can consider the system of $k_1 + 2$ equations given by $u(x', 0) = 0$ together with (5.42) and (5.43); the unknowns are the functions a_0 and a_{m_j}, while we consider a_1 as a parameter; from $u(x', 0) = 0$ we deduce

$$a_0(x') = -2a_1 - 2\sum_{j=1}^{k_1} a_{m_j}(x'); \tag{5.44}$$

As a consequence

$$\operatorname{div}'\left(\widehat{A}(x')\nabla' a_0(x')\right) = -2\sum_{j=1}^{k_1} \operatorname{div}'\left(\widehat{A}(x')\nabla' a_{m_j}(x')\right),$$

and

$$-a_0(x') + c_0(x') = 2a_1 + 2\sum_{j=1}^{k_1} a_{m_j}(x') + c_0(x'),$$

so that equation (5.42) gives

$$\sum_{j=1}^{k_1} \operatorname{div}'\left(\widehat{A}(x')\nabla' a_{m_j}(x')\right) = -a_1 - \sum_{j=1}^{k_1} a_{m_j}(x') - \frac{c_0(x')}{2}.$$

We plug (5.43) for $j \geq 1$ on the left hand side:

$$\sum_{j=1}^{k_1} \left[(m_j^2 - 1)a_{m_j}(x') + c_{m_j}(x')\right] = -a_1 - \sum_{j=1}^{k_1} a_{m_j}(x') - \frac{c_0(x')}{2},$$

i.e.

$$a_{m_1}(x') = \frac{1}{m_1^2}\left[-a_1 - f(x') - \sum_{j=2}^{k_1} m_j^2 a_{m_j}(x')\right], \tag{5.45}$$

where $f(x') = c_0(x')/2 + \sum_{j=1}^{k_1} c_{m_j}(x')$. Note that now equation (5.45) together with (5.43) for $j \geq 2$ is a system of $k_1 + 1$ equations in the unknowns a_{m_j} but without a_0. If

we can solve it, we can recover a_0 using the (5.44).
We can iterate the same argument: from (5.45) we have

$$\operatorname{div}'\left(\widehat{A}(x')\nabla' a_{m_1}(x')\right) = \frac{1}{m_1^2}\left[-\operatorname{div}'\left(\widehat{A}(x')\nabla' f(x')\right) - \sum_{j=2}^{k_1} m_j^2 \operatorname{div}'\left(\widehat{A}(x')\nabla' a_{m_j}(x')\right)\right]$$

(this is why we required the c_{m_j} smooth) and

$$(m_1^2-1)a_{m_1}(x') + c_{m_1}(x') = \frac{m_1^2-1}{m_1^2}\left[-a_1 - f(x') - \sum_{j=2}^{k_1} m_j^2 a_{m_j}(x')\right] + c_{m_1}(x');$$

equation (5.43) for $j=1$ gives

$$-\operatorname{div}'\left(\widehat{A}(x')\nabla' f(x')\right) - \sum_{j=2}^{k_1} m_j^2 \operatorname{div}'\left(\widehat{A}(x')\nabla' a_{m_j}(x')\right) = -(m_1^2-1)a_1 - (m_1^2-1)f(x') - (m_1^2-1)\sum_{j=2}^{k_1} m_j^2 a_{m_j}(x') + m_1^2 c_{m_1}(x'),$$

i.e.

$$a_{m_2}(x') = \frac{1}{m_2^2(m_2^2-m_1^2)} \cdot \left[(m_1^2-1)a_1 - f_1(x') - \sum_{j=3}^{k_1} m_j^2(m_j^2-m_1^2)a_{m_j}(x')\right], \quad (5.46)$$

where

$$f_1(x') = -\operatorname{div}'\left(\widehat{A}(x')\nabla' f(x')\right) - \sum_{j=2}^{k_1} m_j^2 c_{m_j}(x') - c_{m_1}.$$

Equation (5.46) together with (5.43) for $j \geq 2$ is a system of k_1 equations in the unknowns a_{m_j} for $j \geq 2$, but without a_0 and a_{m_1}. If we can solve it, we can recover a_{m_1} using the (5.45), and then a_0 using the (5.44).
Iterating the procedure k_1+2 times (here we have to assume k_1 finite), we obtain $a_{m_{k_1}}$ as function of the Fourier coefficients of the g (note that the more k_1 is large the more we have to require the coefficients c_{m_j} smooth), and successively the others a_{m_j}. Note that a_0 and a_{m_j} are functions of a_1.
The same procedure works for the coefficients b_{n_j}-s, starting from $u_N(x',0)=0$. In the end we obtain the explicit expression of u in function of the two "parameters" a_1 and b_1.

At this point it is sufficient to impose that u solves the considered differential equation to determine a_1 and b_1.

Let us see the iterative procedure in action with an example: let $N = 2$ and

$$\begin{aligned} g(x,y) = &\left(\frac{2}{(1+x^2)^2} - 4\frac{x}{(1+x^2)^2}\arctan x + (\arctan x)^2\right) \\ &+ \left(-\frac{2}{(1+x^2)^2} + 4\frac{x}{(1+x^2)^2}\arctan x + 3(\arctan x)^2\right)\cos(2y) \\ &= \frac{c_0(x)}{2} + c_2(x)\cos(2y). \end{aligned}$$

Proposition 5.5.15. *There is a unique solution of*

$$\begin{cases} -\Delta u = u - g & \text{in } \mathbb{R}^N_+ \\ u(x,0) = 0 \\ u \text{ satisfies } (5.11), \end{cases} \tag{5.47}$$

whose explicit expression is

$$u(x,y) = (\arctan x)^2 \left(1 - \cos(2y)\right).$$

Proof. Due to the form of g we know that if u solves (5.47) then

$$u(x,y) = \frac{a_0(x)}{2} + a_1 \cos y + b_1 \sin y + a_2(x)\cos(2y), \tag{5.48}$$

with $u_y(x,0) = 0$ (Lemma 5.4.3 and Proposition 5.5.10). Thus $b_1 = 0$. As far as a_0 and a_2 is concerned, they solve

$$a_0''(x) = -a_0(x) + c_0(x) \tag{5.49}$$

$$a_2''(x) = 3a_2(x) + c_2(x). \tag{5.50}$$

From $u(x,0) = 0$ we deduce

$$a_0(x) = -2a_1 - 2a_2(x). \tag{5.51}$$

Hence (5.49) gives

$$a_2''(x) = -a_1 - a_2(x) - \frac{c_0(x)}{2};$$

we plug (5.50) on the left hand side, obtaining

$$a_2(x) = -\frac{c_0(x)}{8} - \frac{c_2(x)}{4} - \frac{a_1}{4} = -(\arctan x)^2 - \frac{a_1}{4}, \tag{5.52}$$

and consequently from (5.51)

$$a_0(x) = \frac{c_0(x)}{4} + \frac{c_2(x)}{2} - \frac{3}{2}a_1 = 2\left(\arctan x\right)^2 - \frac{3}{2}a_1. \tag{5.53}$$

Note that it is sufficient to substitute the explicit expressions of c_0 and c_2 (which are given by g) in order to get a_0 and a_2, and no integration is required.
So far, we proved that a solution of (5.47) is of type

$$\begin{aligned} u_{a_1}(x,y) &= (\arctan x)^2\left(1-\cos(2y)\right) - \frac{a_1}{4}\left(3 - 4\cos y + \cos(2y)\right) \\ &= (\arctan x)^2\left(1-\cos(2y)\right) - \frac{a_1}{2}\left(1-\cos y\right)^2, \end{aligned}$$

which is non negative if and only if $a_1 \le 0$. It is straightforward to check that u_{a_1} solves (5.47) only if $a_1 = 0$. □

Remark 5.5.16. For a generic g of the form (5.41), the iterative procedure we introduced above can be used as a test in order to check if (5.47) has at least one solution satisfying (5.11).

For instance it is immediate to check that (5.47) with

$$g(x,y) = \cos(2x) + \sin(3x)\cos(2y),$$

has not a solution satisfying (5.11). Indeed, if such a solution existed, then its explicit expression would be (5.48) with a_0 and a_2 given by

$$a_0(x) = \frac{c_0(x)}{4} + \frac{c_2(x)}{2} - \frac{3}{2}a_1 \quad a_2(x) = -\frac{c_0(x)}{8} - \frac{c_2(x)}{4} - \frac{a_1}{4},$$

(cf. (5.53) and (5.52)) where $c_0(x) = 2\cos(2x)$ and $c_2(x) = \sin(3x)$; but $a_0(x)/2 + a_1\cos(y) + a_2(x)\cos(2y)$ is not a solution of $-\Delta u = u - g$.

Last but not least, we also remark that if $\lambda_1, ..., \lambda_k$ are nonnegative real numbers and $u_1, ..., u_k$ are solutions of (5.47) with $g = g_j$, $j = 1, ..., k$, then the function $u = \sum_{j=1}^k \lambda_j u_j$ is a solution of (5.47) with $g = \sum_{j=1}^k \lambda_j g_j$. Thus, combining in a suitable way the examples considered before, we can construct many other functions g for which we have existence and uniqueness of the solution or existence and multiplicity of the solutions.

Chapter 6

Monotonicity and 1-dimensional symmetry for solutions of an elliptic system modelling phase separation

6.1 Introduction and main results

In this chapter we are interested in monotonicity and 1-dimensional symmetry for solutions of system

$$\begin{cases} -\Delta u = -uv^2 & \text{in } \mathbb{R}^N \\ -\Delta v = -u^2 v & \text{in } \mathbb{R}^N \\ u, v > 0 & \text{in } \mathbb{R}^N, \end{cases} \tag{6.1}$$

which we dealt with in Chapter 4.

The peculiar problem we consider has been proposed by Berestycki, Lin, Wei and Zhao in [12], where the authors established some similarities between system (6.1) and the celebrated Allen-Cahn equation

$$-\Delta u = u - u^3 \qquad \text{in } \mathbb{R}^N. \tag{6.2}$$

We roughly describe now the aforementioned relationship, referring to [12] for a more precise discussion. In a binary fluid such as a mixture of oil and water, the two components of the fluid may spontaneously separate and form two segregated domains, divided by an interface. This phenomenon is called phase separation. To understand from a mathematical point of view the law which governs the interface formation, we may assume that it is driven by a variational principle, in the sense that the pattern of separation is given as a minimizer of a suitable energy functional. The functional

$$J(u) = \int_\Omega \frac{\varepsilon}{2}|\nabla u|^2 + \frac{1}{4\varepsilon}(1-u^2)^2 \qquad u \in H^1(\Omega), \varepsilon > 0,$$

has been proposed to describe the process of phase-separation in this setting. Up to a space dilation, we may focus on the case $\varepsilon = 1$. Then, we find the Allen-Cahn equation as Euler-Lagrange equation of J.

As explained in the introduction to Chapter 4, phase separation occurs also in Bose-Einstein condensation. The idea which is explained in [12] is that, in this latter situation, system (6.1) plays the same role which the Allen-Cahn equation has in the previous model. In this sense, it seems reasonable to think that equations (6.1) and (6.2) share some common features, and, having this in mind, the authors of [12] took some interesting issues concerning solutions to (6.2) and translated in a convenient way to solutions of (6.1).

The classical De Giorgi conjecture and Gibbons conjecture. In 1978, De Giorgi formulated in [30] the following famous conjecture.

Conjecture. Let u be a bounded solution of (6.2) such that $\partial_N u > 0$ in $\mathbb{R}^N$. Then, at least when $N \leq 8$, the level sets $\{u = c\}$ are hyperplanes.

The thesis of the conjecture is equivalent to the fact that, up to a rotation, u depends only on one variable. One can easily check that in this case u has to be of type

$$u(x) = \tanh\left(\frac{\langle \nu, x\rangle - b}{\sqrt{2}}\right),$$

where $\nu \in \mathbb{S}^{N-1}$ with $\nu_N > 0$, and $b \in \mathbb{R}$.

A variant of the De Giorgi conjecture is the so-called Gibbons conjecture, see [18, 43].

Conjecture. Let $u \in \mathcal{C}^2(\mathbb{R}^N) \cap L^\infty(\mathbb{R}^N)$ be a solution of (6.2), and assume that

$$\lim_{x_N \to +\infty} u(x', x_N) = +1 \quad \text{and} \quad \lim_{x_N \to -\infty} u(x', x_N) = -1,$$

the limit being uniform in $x' \in \mathbb{R}^{N-1}$. Then the level sets $\{u = c\}$ are hyperplanes.

Note that in this last case there is no restriction on the dimension N.

The conjectures above have been intensively investigated in recent years and have been essentially settled by now (even though the De Giorgi conjecture is still open, in its complete generality, for $4 \leq N \leq 8$). We refer to [38] for a complete overview of the known results.

The counterparts for solutions to (6.1)**.** We recall that system (6.1) has a unique (up to translation, rotation, and scaling) positive solution $(\bar{u}, \bar{v})$ when $N = 1$. It has linear growth, that is, there exists $C > 0$ such that

$$u(x) + v(x) \leq C(1 + |x|) \qquad \forall x \in \mathbb{R}^N,$$

and satisfies the following monotonicity condition:

$$\bar{u}' > 0 \text{ and } \bar{v}' < 0 \text{ in } \mathbb{R}.$$

Furthermore,

$$\lim_{t \to -\infty} u(t) = 0 \quad \text{and} \quad \lim_{t \to +\infty} u(t) = +\infty$$
$$\lim_{t \to -\infty} v(t) = +\infty \quad \text{and} \quad \lim_{t \to +\infty} v(t) = 0.$$

These facts motivated Berestycki, Lin, Wei and Zhao to formulate in [12] the following conjectures.

Conjecture (De Giorgi-type conjecture)**.** At least up to the dimension $N = 8$, under the monotonicity condition

$$\frac{\partial u}{\partial x_N} > 0 \quad \text{and} \quad \frac{\partial v}{\partial x_N} < 0,$$

a solution (u, v) of (6.1) is 1-dimensional.

Here and in what follows we say that (u, v) is a 1-dimensional solution of (6.1) in $\mathbb{R}^N$ if there exists $\nu \in \mathbb{R}^N$ such that

$$u(x) = \bar{u}\left(\langle \nu, x \rangle\right) \quad \text{and} \quad v(x) = \bar{v}\left(\langle \nu, x \rangle\right),$$

that is, up to a rotation (u, v) is a solution of (6.1) depending only on one variable.

Conjecture (Gibbons-type conjecture)**.** Let $N \geq 2$, let (u, v) be a solution of (6.1) satisfying

$$\lim_{x_N \to -\infty} u(x', x_N) = 0 \quad \text{and} \quad \lim_{x_N \to +\infty} u(x', x_N) = +\infty$$
$$\lim_{x_N \to -\infty} v(x', x_N) = +\infty \quad \text{and} \quad \lim_{x_N \to +\infty} v(x', x_N) = 0,$$

the limits being uniform in $x' \in \mathbb{R}^{N-1}$. Then (u, v) is 1-dimensional.

In a successive paper, motivated by new available existence results, Berestycki, Terracini, Wang and Wei proposed another challenging issue.

Conjecture (Open problem 2 in [13])**.** For any $N \geq 2$, let (u,v) be a solution of (6.1), and assume that the function $N(r) := N(0,r)$, which will be defined by formula (6.5), is such that

$$\lim_{r\to+\infty} N(r) = 1. \tag{6.3}$$

Then (u,v) is 1-dimensional.

Assumption (6.3) is a growth condition. Indeed, as a consequence of the forthcoming Proposition 6.2.6, and Lemmas 6.2.8, 6.2.9, the reader can easily check that the previous statement is equivalent to the following.

Conjecture. For any $N \geq 2$, let (u,v) be a solution of (6.1) having linear growth. Then (u,v) is 1-dimensional.

Known results concerning the 1-dimensional symmetry of the solutions. The first contribution in this direction is contained in [12]: the authors proved that if $N = 2$, (u,v) has linear growth and is monotone in the e_N direction, in the sense that

$$\frac{\partial u}{\partial x_N} > 0 \quad \text{and} \quad \frac{\partial v}{\partial x_N} < 0 \quad \text{in } \mathbb{R}^N,$$

then (u,v) is 1-dimensional. An improvement of this results has been recently obtained by Farina in [35]; he replaced the linear growth condition with an arbitrary algebraic growth condition: there exist $p \geq 1$ and $C > 0$ such that

$$u(x) + v(x) \leq C(1 + |x|^p) \qquad \forall x \in \mathbb{R}^N. \tag{h1}$$

Moreover, he weakened the monotonicity assumption requiring that only one component between u and v is monotone in x_N. Always in case $N = 2$, Berestycki, Terracini, Wang and Wei [13] showed that if (u,v) has linear growth and is stable, then (u,v) is 1-dimensional. We recall that a solution (u,v) of (6.1) is stable if

$$\int_{\mathbb{R}^N} \left(|\nabla\varphi|^2 + |\nabla\psi|^2 + v^2\varphi^2 + u^2\psi^2 + 4uv\varphi\psi\right) \geq 0 \qquad \forall \varphi,\psi \in \mathcal{C}_c^\infty(\mathbb{R}^N).$$

As far as the higher dimensional case is concerned, we refer to the recent contribution by Wang [90], who proved that if $N \geq 2$, (u,v) has linear growth and is a local minimizer for the energy functional, then (u,v) is 1-dimensional. Saying that (u,v) is a local minimizer, we mean that for every smooth function $(\tilde{u},\tilde{v})$ such that $\tilde{u} = u$ and $\tilde{v} = v$ outside a ball $B_R(0)$, one has

$$\int_{\mathbb{R}^N} |\nabla u|^2 + |\nabla v|^2 + u^2v^2 \leq \int_{\mathbb{R}^N} |\nabla \tilde{u}|^2 + |\nabla \tilde{v}|^2 + \tilde{u}^2\tilde{v}^2.$$

Our contribution. The first main result is the proof of the validity of the the Gibbons-type conjecture for solutions having algebraic growth, in any dimension.

Theorem 6.1.1. *Let $N \geq 2$, let (u, v) be a solution of system* (6.1) *having algebraic growth (i.e. satisfying* (h1)*) and such that*

$$\begin{aligned} &\lim_{x_N \to -\infty} u(x', x_N) = 0 \quad \text{and} \quad \lim_{x_N \to +\infty} u(x', x_N) = +\infty \\ &\lim_{x_N \to -\infty} v(x', x_N) = +\infty \quad \text{and} \quad \lim_{x_N \to +\infty} v(x', x_N) = 0, \end{aligned} \tag{h2}$$

the limit being uniform in $x' \in \mathbb{R}^{N-1}$. Then (u, v) depends only on the x_N variable, and

$$\frac{\partial u}{\partial x_N} > 0 \quad \text{and} \quad \frac{\partial v}{\partial x_N} < 0 \quad \text{in } \mathbb{R}^N.$$

Some remarks are in order: the original conjecture was formulated without assumption (h1). Nevertheless, at this stage it seems really hard to deal without the algebraic growth condition, because, as already remarked in Chapter 4, most of the results which are present in the literature rest strongly on it; concerning symmetry results, we observe that, except the work [35], all the quoted achievements are obtained under the linear growth assumption. For us, the main problem to deal with solutions not satisfying the algebraic growth condition is the lack of convergence for the blow-down sequence (see the forthcoming Theorem 6.2.13); concerning this fact, we point out that in Chapter 4) we established a counterpart of this result for exponentially growing solutions, but our result requires more assumptions and is less flexible than Theorem 6.2.13.

On the other hand, in light of the coupled nature of system (6.1), we can weaken assumption (h2) obtaining again monotonicity and 1-dimensional symmetry.

Corollary 6.1.2. *Let $N \geq 2$, and let (u, v) be a solution of system* (6.1) *having algebraic growth (i.e. satisfying* (h1)*), and such that*

$$\lim_{x_N \to \pm\infty} \big(u(x', x_N) - v(x', x_N) \big) = \pm\infty, \tag{h3}$$

the limits being uniform in $x' \in \mathbb{R}^{N-1}$. Then (u, v) depends only on the x_N variable, and

$$\frac{\partial u}{\partial x_N} > 0 \quad \text{and} \quad \frac{\partial v}{\partial x_N} < 0 \quad \text{in } \mathbb{R}^N.$$

Strategy of the proofs. As we shall see, the 1-dimensional symmetry of the solution (u, v) follows from the monotonicity in the e_N direction. We wish to prove this monotonicity property by means of the moving planes method.

Since we are dealing with a system of equations instead of with a single equation, and with unbounded solutions, we have to face some relevant complications. We can overcome them thanks to a careful analysis of the properties of "local monotonicity" (cf. Proposition 6.5.1) and "local boundedness" (cf. Proposition 6.6.3) for the solutions.

Firstly, in Section 6.2, we review some known and a few new results which we use many times in the next chapter. We prefer to write down explicitly the statements which we will use (even if it has been already proved in some other work), because in the literature they do not always appear in the form which is more convenient to our aim, and because sometimes the proofs are missing. In such a case, we write them for the sake of completeness.

In Section 6.3, we provide some other estimates which we use in the rest of the chapter.

In Section 6.4 we make rigorous the intuitive fact that, under assumption (h2), x_N is the privileged variable of the solution (u,v): to be precise, by means of the blow-down technology, we show that, independently on the base point $x_0 \in \mathbb{R}^N$, the entire blow-down family converges to the same function $(\gamma x_N^+, \gamma x_N^-)$, with $\gamma > 0$.
In Section 6.5 we show that, under our assumptions, $\partial_N u(x) > 0$ in $\{x_N \gg 1\}$ and $\partial_N v(x) < 0$ in $\{x_N \ll 1\}$. This does not follow directly from the results of Section 6.4, because the quantitative information given by the convergence of the blow-down family get worse as $R \to +\infty$ (we refer to Section 6.5 for more details).

In Section 6.6 we use the moving planes method to prove that $\partial_N u > 0$ and $\partial_N v < 0$ in $\mathbb{R}^N$; firstly, by the fact that $\partial_N u > 0$ for $x_N \gg 1$ we deduce that in the same region $\partial_N v < 0$; this can be done thanks to a version of the maximum principle in unbounded domains, and allow us to start the moving planes method. We point out that it is not possible to proceed separately on u and on v (that is, it is not possible to show that $\partial_N u > 0$ and, in a second time, that $\partial_N v < 0$ in $\mathbb{R}^N$); this reflects the coupled nature of system (6.1).

In Section 6.7, we complete the proof of Theorem 6.1.1, passing from the monotonicity in the e_N direction to the monotonicity in all the directions of the upper hemisphere $\mathbb{S}^{N-1}_+ := \{\nu \in \mathbb{S}^{N-1} : \langle e_N, \nu \rangle > 0\}$; we follow the line of reasoning introduced by Farina in [33], taking advantage of the results of the previous sections in order to adapt it in the present situation.

Finally, in Section 6.8, we give the proof of Corollary 6.1.2; to be precise, we show that under (h1) and (h3), assumption (h2) is satisfied, so that Corollary 6.1.2 follows from our main theorem.

6.2 A brief review of some known results

The exponential decay. It is by now well known that, if (u,v) solves (4.1) and u is very large in a ball $B_{2r}(x_0)$, then v has to be exponentially small with respect to u in a smaller ball.

Lemma 6.2.1 (Lemma 4.4 in [26]). *Let $x_0 \in \mathbb{R}^N$ and $r > 0$. Let $u \in H^1(B_{2r}(x_0))$ be such that*

$$\begin{cases} -\Delta v \leq -Kv & \text{in } B_{2r}(x_0) \\ v \geq 0 & \text{in } B_{2r}(x_0) \\ v \leq A & \text{on } \partial B_{2r}(x_0), \end{cases}$$

where K and A are two positive constants. Then for every $\alpha \in (0,1)$ there exists $C_\alpha > 0$, not depending on A, K, R and x_0, such that

$$\sup_{x \in B_r(x_0)} v(x) \leq \alpha A e^{-C_\alpha K^{1/2} r}.$$

We always apply this result with a fixed choice of α (e.g. the reader may think to $\alpha = 1/2$), to simplify the notation.

A local segregation theorem. Let us consider problem

$$\begin{cases} -\Delta u_\beta = -\beta u_\beta v_\beta^2 \\ -\Delta v_\beta = -\beta u_\beta^2 v_\beta \\ u_\beta, v_\beta > 0, \end{cases} \tag{6.4}$$

where β is a positive parameter tending to $+\infty$. The following is the local version, proved in [90], of the uniform Hölder estimates obtained in [67].

Theorem 6.2.2. *Let $\{(u_\beta, v_\beta)\}$ be a family of solutions to (6.4) in a ball $B_{2r}(x_0) \subset \mathbb{R}^N$ (where $x_0 \in \mathbb{R}^N$ and $r > 0$). Assume that, as $\beta \to +\infty$, $\{(u_\beta, v_\beta)\}$ is uniformly bounded in $L^\infty(B_{2r}(x_0))$. Then $\{(u_\beta, v_\beta)\}$ is uniformly bounded in $\mathcal{C}^{0,\alpha}(B_r(x_0))$, for every $\alpha \in (0,1)$.*

As a consequence, one can easily adapt the proof of Theorem 1.2 of [67] and obtain a local segregation theorem; for point (iv) we refer to [29, 82].

Theorem 6.2.3. *Let $\{(u_\beta, v_\beta)\}$ be a family of solutions to (6.4) in a ball $B_{2r}(x_0) \subset \mathbb{R}^N$ (where $x_0 \in \mathbb{R}^N$ and $r > 0$). Assume that, as $\beta \to +\infty$, $\{(u_\beta, v_\beta)\}$ is uniformly bounded in $L^\infty(B_{2r}(x_0))$. Then there exists a pair (u_∞, v_∞) such that, up to a subsequence, it holds:*

(i) $u_\beta \to u_\infty$ *and* $v_\beta \to v_\infty$ *in* $\mathcal{C}^0(B_r(x_0)) \cap H^1(B_r(x_0))$*;*

(ii) $u_\infty v_\infty \equiv 0$ *in* $B_r(x_0)$ *and*

$$\lim_{\beta \to +\infty} \int_{B_r(x_0)} \beta u_\beta^2 v_\beta^2 = 0;$$

(iii) the limiting profile satisfies

$$\begin{cases} -\Delta u_\infty = 0 & \text{in } \{u_\infty > 0\} \cap B_r(x_0) \\ -\Delta v_\infty = 0 & \text{in } \{v_\infty > 0\} \cap B_r(x_0); \end{cases}$$

(iv) $u_\infty - v_\infty$ *is harmonic and both* u_∞ *and* v_∞ *are subharmonic in* $B_r(x_0)$*.*

Remark 6.2.4. In [67] it is considered a different system with some additional terms. In particular, the term u^3 appear in the equation for u, and v^3 in the equation for v. Since it is required that these powers are subcritical for the Sobolev embedding, this imposes a restriction on the dimension N. However, as explained in the introduction of the quoted paper, all the results are valid in any dimension provided u^3 and v^3 are replaced by subcritical terms; this is clearly the case of system (6.4), where such a term does not appear.

An Almgren monotonicity formula. For a solution (u, v) of problem (6.1), and for any $x_0 \in \mathbb{R}^N$ and $r > 0$, we define

$$\begin{aligned} E(x_0, r) &:= \frac{1}{r^{N-2}} \int_{B_r(x_0)} |\nabla u|^2 + |\nabla v|^2 + u^2 v^2 \\ H(x_0, r) &:= \frac{1}{r^{N-1}} \int_{\partial B_r(x_0)} u^2 + v^2 \\ N(x_0, r) &:= \frac{E(x_0, r)}{H(x_0, r)} = \frac{r \int_{B_r(x_0)} |\nabla u|^2 + |\nabla v|^2 + u^2 v^2}{\int_{\partial B_r(x_0)} u^2 + v^2}. \end{aligned} \tag{6.5}$$

The function N is called *Almgren frequency function* or *Almgren quotient*.

If we fix $x_0 \in \mathbb{R}^N$, these quantities can be considered as functions of the radius r.

Remark 6.2.5. A direct computation shows that

$$\frac{\partial}{\partial r} H(x_0, r) = 2r^{1-N} \int_{B_r(x_0)} |\nabla u|^2 + |\nabla v|^2 + 2u^2 v^2 \geq 0 :$$

for every $x_0 \in \mathbb{R}^N$ and $r > 0$ the function $H(x_0, r)$ is nondecreasing in r.

Proposition 5.2 of [13] says that also the Almgren quotient is nondecreasing.

Proposition 6.2.6 (Almgren monotonicity formula)**.** *Let* (u, v) *be a solution of* (4.1)*, let* $x_0 \in \mathbb{R}^N$*. The Almgren frequency function* $N(x_0, r)$ *is nondecreasing in* r*.*

A control on the Almgren frequency function gives useful information about the growth of the function H with respect to the radial variable. The proof of the following result is a straightforward modification of the proof of Proposition 5.3 in [13]

Corollary 6.2.7. *Let (u,v) be a solution of* (4.1) *and let $x_0 \in \mathbb{R}^N$.*

(i) If $N(x_0,r) \geq d_1$ for $r \geq R_1 > 0$, then

$$\frac{H(x_0,r_2)}{H(x_0,r_1)} \geq \frac{r_2^{2d_1}}{r_1^{2d_1}} \qquad \forall R_1 < r_1 < r_2;$$

(ii) if $N(x_0,r) \leq d_2$ for $r \leq R_2$, then

$$\frac{H(x_0,r_2)}{H(x_0,r_1)} \leq e^{d_2}\frac{r_2^{2d_2}}{r_1^{2d_2}} \qquad \forall 0 < r_1 < r_2 < R_2.$$

In light of the subharmonicity of (u,v), it is not difficult to deduce a pointwise estimate on the growth of the solution (u,v).

Corollary 6.2.8. *Let (u,v) be a solution of* (4.1)*, let $x_0 \in \mathbb{R}^N$ and $p \geq 1$, and assume that $N(x_0,r) \leq p$ for every $r > 0$. Then there exists $C > 0$ such that*

$$u(x) + v(x) \leq C(1 + |x|^p) \qquad \forall x \in \mathbb{R}^N.$$

Proof. The thesis follows if we show that there exists $C > 0$ such that

$$u(x) + v(x) \leq C(1 + |x - x_0|^p) \qquad \forall x \in \mathbb{R}^N.$$

Let us suppose by contradiction that our claim is not true. Then, without loss of generality we can assume that there exists $r_n \to +\infty$ such that

$$\lim_{n\to+\infty} \frac{u(x_0 + r_n x)}{r_n^p} = +\infty \tag{6.6}$$

for some $x \in \mathbb{S}^{N-1}$ and $r_n \to +\infty$. In light of Corollary 6.2.7, we have

$$\frac{H(x_0,2r_n)}{(2r_n)^{2p}} \leq e^p H(x_0,1) \quad \Longrightarrow \quad \int_{\partial B_{2r}(x_0)} u^2 + v^2 \leq C r_n^{2p+N-1}. \tag{6.7}$$

As u is subharmonic, $u \leq \varphi_n$ in $B_{2r_n}(x_0)$, where φ_n is the solution of

$$\begin{cases} -\Delta\varphi_n = 0 & \text{in } B_{2r_n}(x_0) \\ \varphi_n = u & \text{on } \partial B_{2r_n}(x_0). \end{cases}$$

By the representation formula for harmonic functions we know that for every $x \in \overline{B_{r_n}(x_0)}$

$$\begin{aligned} \varphi_n(x) &= \frac{4r_n^2 - |x-x_0|^2}{2N|\mathbb{S}^{N-1}|r_n} \int_{\partial B_{2r_n}(x_0)} \frac{u(y)}{|x-y|^N}\, d\sigma_y \\ &\leq C r_n \left(\int_{\partial B_{2r_n}(x_0)} \frac{d\sigma_y}{r_n^{2N}} \right)^{\frac{1}{2}} \left(\int_{\partial B_{2r_n}(x_0)} u^2 \right)^{\frac{1}{2}} \leq C r_n^{-\frac{N-1}{2}+p+\frac{N-1}{2}} = C r_n^p, \end{aligned}$$

where C depends only on the dimension N, and for the last inequality we used the (6.7). Thus, for every $x \in \mathbb{S}^{N-1}$ we obtain

$$u(x_0 + r_n x) \leq \varphi_n(x) \leq C r_n^p \qquad \forall n,$$

in contradiction with equation (6.6). □

As proved in [35], the converse holds true.

Lemma 6.2.9 (Lemma 2.1 in [35])**.** *Let (u, v) be a solution of (4.1), let $x_0 \in \mathbb{R}^N$, and assume that there exist $p \geq 1$ and $C > 0$ such that*

$$u(x) + v(x) \leq C(1 + |x|^p) \qquad \forall x \in \mathbb{R}^N.$$

Then $N(x_0, r) \leq p$ for every $x_0 \in \mathbb{R}^N$ and for every $r > 0$.

Remark 6.2.10. Combining Corollary 6.2.8 and Lemma 6.2.9, we deduce that if there exists $x_0 \in \mathbb{R}^N$ such that $N(x_0, r) \leq p$ for every $r > 0$, then

$$u(x) + v(x) \leq C(1 + |x|^p) \qquad \forall x \in \mathbb{R}^N,$$

and hence $N(x, r) \leq p$ for every $x \in \mathbb{R}^N$. That is, a bound of the Almgren quotient centred in a point $x_0 \in \mathbb{R}^N$ provides the same bound for the quotients $N(x, \cdot)$ for every $x \in \mathbb{R}^N$.

Remark 6.2.11. We point out that all these results hold true for a solution (u_β, v_β) of (6.4), with $E(x_0, r)$ replaced by the corresponding energy function, that is,

$$\frac{1}{r^{N-2}} \int_{B_r(x_0)} |\nabla u_\beta|^2 + |\nabla v_\beta|^2 + \beta u_\beta^2 v_\beta^2.$$

The blow-down family. By means of the previous monotonicity formulae, in [13] it is proved that the asymptotic information about $\{(u_\beta, v_\beta)\}$ can be improved for particular sequences. Let (u, v) be a solution of (4.1). For every $x_0 \in \mathbb{R}^N$ and $R > 0$, we introduced the *blow-down family* (centred in x_0) as

$$(u_{x_0,R}(x), v_{x_0,R}(x)) := \left(\frac{1}{\sqrt{H(x_0,R)}} u(x_0 + Rx), \frac{1}{\sqrt{H(x_0,R)}} v(x_0 + Rx) \right). \tag{6.8}$$

By definition, $\int_{\partial B_1(0)} u_{x_0,R}^2 + v_{x_0,R}^2 = 1$ for every $x_0 \in \mathbb{R}^N$ and $R > 0$. Also, $(u_{x_0,R}, v_{x_0,R})$ solves

$$\begin{cases} -\Delta u_{x_0,R} = -H(x_0,R) R^2\, u_{x_0,R}\, v_{x_0,R}^2 & \text{in } \mathbb{R}^N \\ -\Delta v_{x_0,R} = -H(x_0,R) R^2\, u_{x_0,R}^2\, v_{x_0,R} & \text{in } \mathbb{R}^N \\ u_{x_0,R}, v_{x_0,R} > 0 & \text{in } \mathbb{R}^N. \end{cases}$$

Remark 6.2.12. A direct computation shows that if $N(x_0, r) \leq p$ for every $r \geq 1$, the same estimate holds true for the Almgren quotient associated to the function $(u_{x_0,R}, v_{x_0,R})$ (for every $x_0 \in \mathbb{R}^N$ and $R > 0$): for every $r \geq 1$

$$\frac{\frac{1}{r^{N-2}} \int_{B_r(0)} |\nabla u_{x_0,R}|^2 + |\nabla v_{x_0,R}|^2 + H(x_0,R)R^2\, u_{x_0,R}^2\, v_{x_0,R}^2}{\frac{1}{r^{N-1}} \int_{\partial B_r(0)} u_{x_0,R}^2 + v_{x_0,R}^2} = N(x_0, Rr) \leq p.$$

As a consequence, if we can bound $N(x_0, \cdot)$, we can apply Corollary 6.2.7 on any function of the sequence $\{(u_{x_0,R}, v_{x_0,R})\}$.

Theorem 1.4 in [13] says, roughly speaking, that if the Almgren frequency function is bounded, then the limit of $N(x_0, r)$ as $r \to +\infty$ (which exists by monotonicity) is a positive integer, and the limiting profile is a homogeneous harmonic polynomial. It is straightforward to check that, although therein it is considered the case $x_0 = 0$, the result holds true for any $x_0 \in \mathbb{R}^N$.

Theorem 6.2.13. *Let (u, v) be a solution of (4.1), let $x_0 \in \mathbb{R}^N$, and assume that*

$$\lim_{r\to+\infty} N(x_0, r) =: d_{x_0} < +\infty.$$

Then d_{x_0} is a positive integer. There exist a subsequence of the blow-down family $\{(u_{x_0,R}, v_{x_0,R}) : R > 0\}$, denoted by $\{(u_{x_0,R_n}, v_{x_0,R_n})\}$, and a homogeneous harmonic polynomial of degree d_{x_0}, denoted by Ψ_{x_0}, such that $(u_{x_0,R_n}, v_{x_0,R_n}) \to (\Psi_{x_0}^+, \Psi_{x_0}^-)$ as $R \to +\infty$ in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^N)$ and in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$. Moreover,

$$H(x_0, R)R^2\, u_{x_0,R_n}^2\, v_{x_0,R_n}^2 \to 0 \qquad \text{in } L^1_{\mathrm{loc}}(\mathbb{R}^N).$$

This achievement permits to say something more on the asymptotic of $H(x_0, \cdot)$ in case (u, v) has algebraic growth.

Corollary 6.2.14. *Let (u, v) be a solution of (4.1) with algebraic growth. For $x_0 \in \mathbb{R}^N$, let $d_{x_0} = \lim_{r\to+\infty} N(x_0, r)$, which is a positive integer by the previous statement. For every $\varepsilon > 0$ it results*

$$\lim_{r\to+\infty} \frac{H(x_0, r)}{r^{2d_{x_0}(1-\varepsilon)}} = +\infty.$$

Proof. As $d_{x_0} \geq 1$, using the Almgren monotonicity formula (Theorem 6.2.6) we deduce that for every $\varepsilon > 0$ there exists $r_\varepsilon > 0$ such that, if $r > r_\varepsilon$, then

$$N(x_0, r) \geq d_{x_0}\left(1 - \frac{\varepsilon}{2}\right).$$

Hence, we can use Corollary 6.2.7 to obtain

$$H(x_0,r) \geq C r^{2d_{x_0}\left(1-\frac{\varepsilon}{2}\right)} \qquad \forall r > r_\varepsilon,$$

with $C>0$. Therefore

$$\lim_{r\to+\infty} \frac{H(x_0,r)}{r^{2d_{x_0}(1-\varepsilon)}} \geq \lim_{r\to+\infty} C\frac{r^{2d_{x_0}\left(1-\frac{\varepsilon}{2}\right)}}{r^{2d_{x_0}(1-\varepsilon)}} = +\infty. \qquad \square$$

An Alt-Caffarelli-Friedman monotonicity formula. For a solution (u,v) to (4.1), we introduce the quantity

$$J(x_0,r) := \frac{1}{r^4}\int_{B_r(x_0)} \frac{|\nabla u(y)|^2+u^2(y)v^2(y)}{|y-x_0|^{N-2}}\,dy \int_{B_r(x_0)} \frac{|\nabla v(y)|^2+u^2(y)v^2(y)}{|y-x_0|^{N-2}}\,dy.$$

Note that when $N=2$ the denominator is equal to 1.

First of all, we report the useful formula (4.11) in [90].

Lemma 6.2.15. *There exists $C>0$ independent on $x_0\in\mathbb{R}^N$ and on $r\geq 1$ such that*

$$\frac{1}{r^2}\int_{B_r(x_0)} \frac{|\nabla u(y)|^2+u^2(y)v^2(y)}{|y-x_0|^{N-2}}\,dy \leq \frac{C}{r^{N+2}}\int_{B_{2r}(x_0)} u^2.$$

An analogue estimate holds true for v.

Proof. We consider the case $N\geq 3$. If $N=2$ the proof is simpler. Without loss of generality, we consider the case $x_0=0$, to ease the notation. For every $\eta\in\mathcal{C}_c^\infty(\mathbb{R}^N)$ we have

$$\begin{aligned}
\int_{\mathbb{R}^N} \frac{|\nabla u|^2+u^2v^2}{|y|^{N-2}}\eta^2 &= \int_{\mathbb{R}^N} \frac{\Delta\left(u^2\right)\eta^2}{2|y|^{N-2}} = \int_{\mathbb{R}^N} \frac{1}{2}\Delta\left(\eta^2|y|^{2-N}\right)u^2 \\
&= \int_{\mathbb{R}^N} \frac{1}{2}\Delta\left(|y|^{2-N}\right)\eta^2u^2 + \frac{1}{2}\int_{\mathbb{R}^N} u^2\left(2(2-N)|y|^{-N}\eta\langle y,\nabla\eta\rangle + |y|^{2-N}\Delta\left(\eta^2\right)\right) \\
&\leq \frac{1}{2}\int_{\mathbb{R}^N} u^2\left(2(2-N)|y|^{1-N}|\eta||\nabla\eta| + |y|^{2-N}|\Delta\left(\eta^2\right)|\right)
\end{aligned}$$

where we used the fact that

$$\Delta\left(|y|^{2-N}\right) = -\delta,$$

where δ denotes the Dirac delta centred in 0.

Now, let $r\geq 1$. Let $\varphi\in\mathcal{C}_c^\infty(\mathbb{R}^+)$ such that

$$\begin{cases} \varphi(t)=1 & t\in[0,1] \\ \varphi(t)=0 & t\in[2,+\infty) \\ 0\leq\varphi(t)\leq 1 & t\in(1,2) \end{cases}$$

We set, for any $r \geq 1$, $\eta_r(x) = \varphi(|x|/r)$. It is then clear that there exists $C > 0$ independent of r such that

$$|\nabla \eta_r|^2 \leq \frac{C}{r} \quad \text{and} \quad \Delta\left(\eta_r^2\right) \leq \frac{C}{r^2}$$

Using η_r as test function into the previous estimate we deduce that

$$\begin{aligned}
\int_{B_r(0)} \frac{|\nabla u|^2 + u^2 v^2}{|y|^{N-2}} \eta_r^2 &\leq \int_{\mathbb{R}^N} \frac{|\nabla u|^2 + u^2 v^2}{|y|^{N-2}} \eta_r^2 \\
&\leq \frac{1}{2} \int_{\mathbb{R}^N} u^2 \left(2(2-N)|y|^{1-N}|\eta_r||\nabla \eta_r| + |y|^{2-N}|\Delta\left(\eta_r^2\right)|\right) \\
&\leq \frac{1}{2} \int_{B_{2r}(0)\setminus B_r(0)} u^2 \left(2(2-N) r^{1-N}|\nabla \eta_r| + r^{2-N}|\Delta\left(\eta_r^2\right)|\right) \\
&\leq C r^{-N} \int_{B_{2r}(0)} u^2 .
\end{aligned}$$

We point out that C is a positive constant which is independent of r and also on the base point, which we fixed in 0. □

Recently, Wang proved an Alt-Caffarelli-Friedman monotonicity formula which enhances a previous similar result in [67].

Theorem 6.2.16 (Theorem 4.3 in [90])**.** *Let (u,v) be a solution of* (4.1) *having algebraic growth, and let $x_0 \in \mathbb{R}^N$. There exists $C(x_0) > 0$ such that*

$$r \mapsto e^{-C(x_0) r^{-1/2}} J(x_0, r) \quad \textit{is nondecreasing in } r$$

for every $r \geq 1$.

The proof of this statement, which we write below for the sake of completeness, is based upon the following lemma. As in the classical Alt-Caffarelli-Friedman result, the function

$$\Gamma(t) := \sqrt{\left(\frac{N-2}{2}\right)^2 + t} - \left(\frac{N-2}{2}\right)$$

plays an important role.

Lemma 6.2.17. *Let $(u_k), (v_k) \subset H^1\left(\mathbb{S}^{N-1}\right)$ be two sequences such that*

$$\int_{\mathbb{S}^{N-1}} u_k^2 = 1 \quad \textit{and} \quad \int_{\mathbb{S}^{N-1}} v_k^2 = \lambda_k^2,$$

and assume that there exists $\lambda > 1$ *such that*

$$\frac{1}{\lambda} \le \lambda_k \le \lambda \qquad \forall k.$$

Then there exists $C > 0$ *depending only on* λ *and* N *such that*

$$\Gamma\left(\frac{\int_{\partial B_1(0)} |\nabla_\theta u_k|^2 + k u_k^2 v_k^2}{\int_{\partial B_1(0)} u_k^2}\right) + \Gamma\left(\frac{\int_{\partial B_1(0)} |\nabla_\theta v_k|^2 + k^2 u_k^2 v_k^2}{\int_{\partial B_1(0)} \bar{v}_k^2}\right) \ge 2 - Ck^{-\frac{1}{4}}.$$

Proof of Theorem 6.2.16. It is convenient to introduce the following notation: for any $r > 0$

$$\Lambda_1(r) = \frac{r^2 \int_{\partial B_r(x_0)} |\nabla_\theta u|^2 + u^2 v^2}{\int_{\partial B_r(x_0)} u^2}$$

$$\Lambda_2(r) = \frac{r^2 \int_{\partial B_r(x_0)} |\nabla_\theta v|^2 + u^2 v^2}{\int_{\partial B_r(x_0)} v^2},$$

where ∇_θ denotes the tangential gradient: $|\nabla_\theta u|^2 = |\nabla u|^2 - (\partial_\nu u)^2$. Moreover, we set

$$J_1(r) = \int_{B_r(x_0)} \frac{|\nabla u(y)|^2 + u^2(y)v^2(y)}{|y - x_0|^{N-2}}\, dy$$

$$J_2(r) = \int_{B_r(x_0)} \frac{|\nabla v(y)|^2 + u^2(y)v^2(y)}{|y - x_0|^{N-2}}\, dy,$$

so that $J(x_0, r) = J_1(r)J_2(r)/r^4$; recall that we consider $x_0 \in \mathbb{R}^N$ fixed, so we do not write explicitly the dependence of Λ_1, Λ_2, J_1 and J_2 on it, to ease the notation. For this reason, we write B_r instead of $B_r(x_0)$, too.

By direct computations

$$\frac{\partial}{\partial r} \log J(x_0, r) = -\frac{4}{r} + \frac{\displaystyle\int_{\partial B_r} \frac{|\nabla u|^2 + u^2 v^2}{|x_0 - y|^{N-2}}}{J_1(r)} + \frac{\displaystyle\int_{\partial B_r} \frac{|\nabla v|^2 + u^2 v^2}{|x_0 - y|^{N-2}}}{J_2(r)}. \tag{6.9}$$

Let us test the differential equation for u with $u/|x_0 - y|^{N-2}$ in the ball B_r:

$$\begin{aligned}
0 &= \int_{B_r} \frac{1}{|x_0 - y|^{N-2}}(-\Delta u)u + \frac{1}{|x_0 - y|^{N-2}} u^2 v^2 \\
&= J_1(r) - \int_{B_r} \frac{1}{|x_0 - y|^{N-2}} \operatorname{div}\left(\nabla\left(\frac{u^2}{2}\right)\right) \\
&= J_1(r) + \int_{B_r} \langle \nabla\left(\frac{1}{|x_0 - y|^{N-2}}\right), \nabla\left(\frac{u^2}{2}\right)\rangle - \frac{1}{r^{N-2}} \int_{\partial B_r} u \partial_\nu u \\
&= J_1(r) + \int_{B_r} (-\Delta)\left(\frac{1}{|x_0 - y|^{N-2}}\right) \frac{u^2}{2} - \frac{1}{r^{N-2}} \int_{\partial B_r} u \partial_\nu u - \frac{N-2}{r^{N-1}} \int_{\partial B_r} \frac{u^2}{2}.
\end{aligned}$$

Recalling that

$$(-\Delta)\left(\frac{1}{|x_0-y|^{N-2}}\right)=\delta_{x_0},$$

where δ_{x_0} is the Dirac delta centred in x_0, we deduce

$$J_1(r)\leq -\frac{1}{r^{N-2}}\int_{\partial B_r} u\partial_\nu u-\frac{N-2}{r^{N-1}}\int_{\partial B_r}\frac{u^2}{2}, \tag{6.10}$$

for every $r>0$. Now, thanks to the Cauchy-Schwarz and the Young inequalities, for every $\delta\in\mathbb{R}$ and $r>0$ we have

$$\begin{aligned}\left|\int_{\partial B_r} u\partial_{nu}u\right| &\leq \left(\int_{\partial B_r} u^2\right)^{\frac{1}{2}}\left(\int_{\partial B_r}(\partial_\nu u)^2\right)^{\frac{1}{2}}\\ &\leq \frac{\sqrt{\Lambda_1(r)}}{2\delta^2 r}\int_{\partial B_r} u^2+\frac{\delta^2 r}{2\sqrt{\Lambda_1(r)}}\int_{\partial B_r}(\partial_\nu u)^2\\ &\leq \frac{1}{2}\left[\frac{1}{\delta^2}\int_{\partial B_r}\left(|\nabla_\theta u|^2+u^2v^2\right)+\delta^2\int_{\partial B_r}(\partial_\nu u)^2\right]\frac{r}{\sqrt{\Lambda_1(r)}},\end{aligned}$$

where we recall that Λ_1 has been defined at the beginning of the proof. Substituting into inequality (6.10), we obtain

$$J_1(r)\leq\frac{1}{2r^{N-3}}\left[\left(\frac{1}{\delta^2\sqrt{\Lambda_1(r)}}+\frac{N-2}{\Lambda_1(r)}\right)\int_{\partial B_r}\left(|\nabla_\theta u|^2+u^2v^2\right)+\frac{\delta^2}{\sqrt{\Lambda_1(r)}}\int_{\partial B_r}(\partial_\nu u)^2\right]$$

It is possible to choose δ such that

$$\frac{1}{\delta^2\sqrt{\Lambda_1(r)}}+\frac{N-2}{\Lambda_1(r)}=\frac{\delta^2}{\sqrt{\Lambda_1(r)}},$$

or equivalently

$$\frac{\sqrt{\Lambda_1(r)}}{\delta^2}=\Gamma(\Lambda_1(r)).$$

In this way, since $|x_0-y|^{N-2}=r^{N-2}$ on ∂B_r, it results

$$J_1(r)\leq\frac{r}{2\Gamma(\Lambda_1(r))}\int_{\partial B_r}\frac{|\nabla u|^2+u^2v^2}{|x_0-y|^{N-2}},$$

for every $r>0$. An analogue estimate holds true for J_2. So, coming back to (6.9), we obtain

$$\frac{\partial}{\partial r}\log J(x_0,r)\geq -\frac{4}{r}+\frac{2\Gamma(\Lambda_1(r))}{r}+\frac{2\Gamma(\Lambda_2(r))}{r}. \tag{6.11}$$

Now, let us set

$$(\bar{u}_r(x), \bar{v}_r(x)) = (u(x_0 + rx), v(x_0 + rx)) \qquad \text{with } x \in \partial B_1(0).$$

With this notation

$$\Lambda_1(r) = \frac{\int_{\partial B_1(0)} |\nabla_\theta \bar{u}_r|^2 + r^2 \bar{u}_r^2 \bar{v}_r^2}{\int_{\partial B_1(0)} \bar{u}_r^2}$$
$$\Lambda_2(r) = \frac{\int_{\partial B_1(0)} |\nabla_\theta \bar{v}_r|^2 + r^2 \bar{u}_r^2 \bar{v}_r^2}{\int_{\partial B_1(0)} \bar{v}_r^2}.$$

We wish to introduce a suitable normalization in order to apply Lemma 6.2.17 to the family $\{(\bar{u}_r, \bar{v}_r)\}$. To do this, we have to prove that the quantity

$$\varphi(r) := \frac{\int_{\partial B_1(0)} \bar{u}_r^2}{\int_{\partial B_1(0)} \bar{v}_r^2}$$

is bounded from above and from below by positive constants, for $r \geq 1$. At first, the subharmonicity of u^2 and v^2 and the mean value inequality give

$$\int_{\partial B_1(0)} \bar{u}_r^2 \geq |\mathbb{S}^{N-1}| u(x_0) \quad \text{and} \quad \int_{\partial B_1(0)} \bar{v}_r^2 \geq |\mathbb{S}^{N-1}| v(x_0). \tag{6.12}$$

This estimate, together with the regularity of (u, v), implies that φ is continuous and well-defined for $r \in [1, +\infty)$. We observe also that, since (u, v) has algebraic growth, by Theorem 6.2.13 it follows that

$$\lim_{r \to +\infty} \frac{\int_{\partial B_1(0)} \bar{u}_r^2}{\int_{\partial B_1(0)} \bar{v}_r^2} = \frac{\int_{\partial B_1(0)} (\Psi^+)^2}{\int_{\partial B_1(0)} (\Psi^-)^2} = C_0 > 0,$$

where Ψ is a homogeneous harmonic polynomial of degree $d \in \mathbb{N}$. This says that there exists $\bar{r} > 1$ such that

$$\frac{C_0}{2} \leq \varphi(r) \leq \frac{3C_0}{2} \qquad \forall r \geq \bar{r}. \tag{6.13}$$

On the other hand, for every $r \geq 1$ we have

$$\int_{\partial B_1(0)} \bar{v}_r^2 = \frac{1}{r^{N-1}} \int_{\partial B_r} v^2 \leq H(x_0, r) \leq H(x_0, 1) e^d r^{2d}, \tag{6.14}$$

where we used Corollary 6.2.7, and similarly

$$\int_{\partial B_1(0)} \bar{u}_r^2 \leq H(x_0, 1) e^d r^{2d} \qquad \forall r \geq 1. \tag{6.15}$$

Combining (6.12), (6.14) and (6.15) we deduce that there exists $C_1 > 1$ such that

$$\frac{1}{C_1} \le \varphi(r) \le C_1 \qquad \forall r \in [1, \bar{r}],$$

which together with (6.13) proves the boundedness from above and from below of φ.

Now, let us consider the normalization

$$\hat{u}_r(x) := \frac{\bar{u}_r(x)}{\int_{\partial B_1(0)} \bar{u}_r^2} \quad \text{and} \quad \hat{v}_r(x) := \frac{\bar{v}_r(x)}{\int_{\partial B_1(0)} \bar{u}_r^2},$$

that is, we normalize both $\bar{u}_r$ and $\bar{v}_r$ with respect to the L^2 norm of $\bar{u}_r$ on the sphere $\partial B_1(0)$. In light of (6.12)

$$\begin{aligned}
\Lambda_1(r) &= \int_{\partial B_1(0)} |\nabla_\theta \hat{u}_r|^2 + r^2 \left(\int_{\partial B_1(0)} \bar{u}_r^2 \right) \hat{u}_r^2 \hat{v}_r^2 \\
&\ge \int_{\partial B_1(0)} |\nabla_\theta \hat{u}_r|^2 + |\mathbb{S}^{N-1}| u(x_0) r^2 \hat{u}_r^2 \hat{v}_r^2 \\
\Lambda_2(r) &= \int_{\partial B_1(0)} |\nabla_\theta \hat{v}_r|^2 + r^2 \left(\int_{\partial B_1(0)} \bar{u}_r^2 \right) \hat{u}_r^2 \hat{v}_r^2 \\
&\ge \int_{\partial B_1(0)} |\nabla_\theta \hat{v}_r|^2 + |\mathbb{S}^{N-1}| u(x_0) r^2 \hat{u}_r^2 \hat{v}_r^2.
\end{aligned}$$

As Γ is monotone nondecreasing, we deduce

$$\begin{aligned}
&\Gamma\left(\Lambda_1(r)\right) + \Gamma\left(\Lambda_2(r)\right) \\
&\qquad \ge \Gamma\left(\int_{\partial B_1(0)} |\nabla_\theta \hat{u}_r|^2 + C r^2 \hat{u}_r^2 \hat{v}_r^2 \right) + \Gamma\left(\int_{\partial B_1(0)} |\nabla_\theta \hat{v}_r|^2 + C r^2 \hat{u}_r^2 \hat{v}_r^2 \right).
\end{aligned}$$

Thanks to the first step, we are in position to apply Lemma 6.2.17 in order to obtain

$$\Gamma\left(\Lambda_1(r)\right) + \Gamma\left(\Lambda_2(r)\right) \ge 2 - \frac{C}{r^{\frac{1}{2}}},$$

where C is a positive constant independent on r. Coming back to (6.11), we deduce that there exists $C > 0$ such that

$$\frac{\partial}{\partial r} \log J(x_0, r) \ge -C r^{-\frac{3}{2}}$$

for every $r \ge 1$. An integration gives the desired result. □

Remark 6.2.18. Note that it is crucial, in order to apply Lemma 6.2.17, to show that the ratio φ is bounded from above and from below.

6.3 Preliminary estimates

The Alt-Caffarelli-Friedman monotonicity formula (Theorem 6.2.16) gives a lower bound for some integral quantities related to solutions having linear growth (cf. the results of Section 4 of [90]). In this section we prove some new results and we refine some estimates of the quoted paper, in order to use them in the next sections.

In Corollary 4.5 of [90], the author used the linear growth of the solution (u, v) to obtain a lower bound for the growth of the function

$$r \mapsto \int_{\partial B_r(0)} u^2 + v^2.$$

We think that it is interesting to note that an equivalent estimate holds true assuming only that (u, v) has algebraic growth. Clearly, this requires some extra-work.

Corollary 6.3.1. *Let* (u, v) *be a solution of* (6.1) *satisfying* (h1). *There exists* $C > 0$ *such that*

$$\int_{B_r(0)} u^2 + v^2 \geq C r^{N+2} \qquad \forall r \geq 1.$$

Proof. Assume by contradiction that the statement is not true: there exists $\varepsilon_n \to 0$ and $(r_n) \subset [1, +\infty)$ such that

$$\int_{B_{r_n}(0)} u^2 + v^2 \leq \varepsilon_n r_n^{N+2}. \tag{6.16}$$

Step 1) $\liminf_{n\to\infty} r_n = +\infty$.
If not, up to a subsequence $r_n \to \bar{r} \geq 1$. By the dominated convergence theorem, we have

$$\int_{B_{\bar{r}}(0)} u^2 + v^2 = 0,$$

and by subharmonicity we deduce that $(u, v) \equiv (0, 0)$, a contradiction.

Step 2) *Conclusion of the proof.*
To simplify the notation, we call $J_1(r)$ the quantity

$$\frac{1}{r^2} \int_{B_r(0)} \frac{|\nabla u(y)|^2 + u^2(y) v^2(y)}{|y|^{N-2}}\, dy,$$

and $J_2(r)$ the same quantity for the component v. Now, by Theorem 6.2.16 there exists $C > 0$ such that $J(0, r) \geq C$ for every $r \geq 1$, that is, $J_1(r) J_2(r) \geq C$ for every $r \geq 1$. In particular, this holds true for $r = r_n/2$. Up to a subsequence, we can assume

$J_1(r_n/2) \geq C$ for every n. By means of Lemma 6.2.15 (we remark that the constant appearing is independent on r) plus our absurd assumption (6.16), we obtain

$$0 < C \leq J_1\left(\frac{r_n}{2}\right) \leq \frac{C}{r_n^{N+2}} \int_{B_{r_n}(0)} u^2 \leq C\varepsilon_n \to 0$$

as $n \to \infty$, a contradiction. □

In case (u, v) has linear growth, we obtain a uniform (in both $x \in \mathbb{R}^N$ and $r \geq 1$) lower bound for the values $\{H(x, r)\}$.

Lemma 6.3.2. *Let (u, v) be a solution of* (6.1) *with linear growth. There exists $\bar{C}_1 > 0$ such that*

$$H(x, r) \geq \bar{C}_1$$

for every $x \in \mathbb{R}^N$ and $r \geq 1$.

Proof. By the monotonicity of $H(x, \cdot)$, it is sufficient to show that $H(x, 1) \geq C$ with C independent on $x \in \mathbb{R}^N$. By contradiction, assume that there exists $(x_i) \subset \mathbb{R}^N$ such that

$$\lim_{i\to\infty} \int_{\partial B_1(x_i)} u^2 + v^2 = 0. \tag{6.17}$$

By Corollary 6.3.1, we know that there exists $C > 0$ such that

$$\int_{B_r(0)} u^2 + v^2 \geq C r^{N+2} \qquad \forall r \geq 1,$$

Let $r \geq 1$; for every i we have

$$\int_{B_{r+|x_i|}(x_i)} u^2 + v^2 \geq \int_{B_r(0)} u^2 + v^2 \geq C r^{N+2}. \tag{6.18}$$

Note that

$$\int_{B_{r+|x_i|}(x_i)} u^2 + v^2 = \int_{B_{r+|x_i|}(x_i)\setminus B_1(x_i)} u^2 + v^2 + \int_{B_1(x_i)} u^2 + v^2;$$

thanks to Lemma 6.2.9 we know that $N(x_i, r) \leq 1$ for every $r \geq 1$, for every i. Hence, by means of Corollary 6.2.7, we deduce

$$\begin{aligned}\int_{B_{r+|x_i|}(x_i)\setminus B_1(x_i)} u^2 + v^2 &= \int_1^{r+|x_i|} \left(\int_{\partial B_s(x_i)} u^2 + v^2\right) ds \\ &\leq e \left(\int_{\partial B_1(x_i)} u^2 + v^2\right) \int_1^{r+|x_i|} s^{N+1}\, ds \\ &\leq e \left(\int_{\partial B_1(x_i)} u^2 + v^2\right) (r + |x_i|)^{N+2}.\end{aligned}$$

Therefore

$$\int_{B_{r+|x_i|}(x_i)} u^2+v^2 \le e\left(\int_{\partial B_1(x_i)} u^2+v^2\right)(r+|x_i|)^{N+2}+\int_{B_1(x_i)} u^2+v^2. \tag{6.19}$$

We observe that from the linear growth of (u,v) it follows also

$$\int_{B_1(x_i)} u^2+v^2 \le C(1+|x_i|)^2,$$

where C does not depend on i. Plugging into the (6.19) and choosing $r=r_i \ge |x_i|$, $r_i \to +\infty$ as $i\to\infty$ (here i is fixed, so this choice is possible), we deduce

$$\begin{aligned}\int_{B_{r_i+|x_i|}(x_i)} u^2+v^2 &\le e\left(\int_{\partial B_1(x_i)} u^2+v^2\right)(r_i+|x_i|)^{N+2}+C\left(1+|x_i|^2\right)\\ &\le C\left(\int_{\partial B_1(x_i)} u^2+v^2\right)r_i^{N+2}+C(1+r_i^2).\end{aligned}$$

A comparison with (6.18) yields

$$Cr_i^{N+2} \le C\left(\int_{\partial B_1(x_i)} u^2+v^2\right)r_i^{N+2}+Cr_i^2.$$

Dividing for r_i^{N+2} and passing to the limit as $i\to\infty$, we finally obtain a contradiction:

$$0<C\le C\lim_{i\to\infty}\int_{\partial B_1(x_i)} u^2+v^2=0,$$

where we used our absurd assumption, equation (6.17). □

Where $|u-v|$ is not too large, it is natural to expect that this provides a lower bound on the integrals of both u^2 and v^2. To be precise:

Lemma 6.3.3. *Let (u,v) be a solution of (6.1) having linear growth. For every $C_1 < \sqrt{\bar{C}_1/|\mathbb{S}^{N-1}|}$ (where $\bar{C}_1$ has been defined in Lemma 6.3.2) there exists $\bar{C}_2>0$ such that*

$$\int_{\partial B_1(x_0)} u^2 \ge \bar{C}_2 \quad \text{and} \quad \int_{\partial B_1(x_0)} v^2 \ge \bar{C}_2$$

for every $x_0 \in \{|u-v|<C_1\}$.

Proof. Without loss of generality, we can assume by contradiction that, for a sequence $(x_i) \subset \{|u - v| < C_1\}$, we have

$$\lim_{i\to\infty} \int_{\partial B_1(x_i)} u^2 = 0.$$

We claim that under this assumption

$$\lim_{i\to\infty} \int_{\partial B_1(x_i)} v^2 = 0.$$

If not, up to a subsequence there exists $\delta > 0$ such that $\lim_i \int_{\partial B_1(x_i)} v^2 \geq \delta^2$. We introduce the sequence

$$(u_i(x), v_i(x)) = \left(\frac{1}{\sqrt{H(x_i,1)}} u(x_i + x), \frac{1}{\sqrt{H(x_i,1)}} v(x_i + x) \right).$$

Note that $\int_{\partial B_1(0)} u_i^2 + v_i^2 = 1$ for every i. Each (u_i, v_i) solves

$$\begin{cases} -\Delta u_i = -H(x_i,1) u_i v_i^2 & \text{in } \mathbb{R}^N \\ -\Delta v_i = -H(x_i,1) u_i^2 v_i & \text{in } \mathbb{R}^N; \end{cases}$$

By Corollary 6.2.7 (which we can apply, see Remark 6.2.11), we deduce that

$$\int_{\partial B_r(0)} u_i^2 + v_i^2 \leq e r^{N+1} \qquad \forall r,\ \forall i. \tag{6.20}$$

As u_i and v_i are subharmonic, the (6.20) gives a uniform bound on the $L^\infty(B_{r/2}(0))$ norm of the family $\{(u_i, v_i)\}$, for every $r \geq 1$. Now, we have to distinguish between

(i) the sequence $\{H(x_i, 1)\}$ is bounded;

(ii) the sequence $\{H(x_i, 1)\}$ is unbounded.

In case (i), up to a subsequence $H(x_i, 1) \to H_\infty > 0$ (see Lemma 6.3.2). Also, $\{u_i\}, \{v_i\}, \{\Delta u_i\}, \{\Delta v_i\}$ are uniformly bounded in every compact subset K of $\mathbb{R}^N$. By standard gradient estimates for elliptic equations (see [45]) we deduce that $\{\nabla u_i\}, \{\nabla v_i\}$ are uniformly locally bounded in $\mathbb{R}^N$, so that up to a subsequence $(u_i, v_i) \to (u_\infty, v_\infty)$ in $\mathcal{C}^2_{\text{loc}}(\mathbb{R}^N)$ (to pass from the uniform convergence to the $\mathcal{C}^2$ convergence, we invoke the machinery of the regularity theory for elliptic equations, e.g. [45]). From the absurd assumption and our normalization it follows

$$\int_{\partial B_1(0)} u_\infty^2 = 0 \quad \text{and} \quad \int_{\partial B_1(0)} v_\infty^2 = 1. \tag{6.21}$$

Moreover, u_∞ and v_∞ are subharmonic and nonnegative. This implies $u_\infty \equiv 0$ in $B_1(0)$, which in turns yields (applying the strong maximum principle) $u_\infty \equiv 0$ in $\mathbb{R}^N$. Hence, v_∞ is harmonic and nonnegative in $\mathbb{R}^N$ (this follows by the $\mathcal{C}^2_{\text{loc}}(\mathbb{R}^N)$ convergence): by the Liouville theorem for harmonic functions, $v_\infty \equiv const$. Now, since $x_i \in \{|u - v| < C_1\}$ with $C_1 < \sqrt{\bar{C}_1 |\mathbb{S}^{N-1}|}$, and in light of Lemma 6.3.2, we deduce

$$\begin{aligned} v_\infty(0) &\leq \lim_{i\to\infty} \left(\frac{1}{\sqrt{H(x_i,1)}} |v(x_i) - u(x_i)| + u_i(0) \right) \\ &\leq \lim_{i\to\infty} \left(\frac{1}{\sqrt{\bar{C}_1}} C_1 + u_i(0) \right) < \sqrt{\frac{1}{|\mathbb{S}^{N-1}|}}. \end{aligned}$$

But since v_∞ is constant and (6.21) holds true, necessarily $v_\infty(0)^2 |\mathbb{S}^{N-1}| = 1$, a contradiction.

In case (ii), up to a subsequence $H(x_i, 1) \to +\infty$ as $i \to \infty$. Due to the fact the $\{(u_i, v_i)\}$ is uniformly bounded in every compact subset of $\mathbb{R}^N$, we are in position to apply Theorems 6.2.2 and 6.2.3: for every $K \subset\subset \mathbb{R}^N$, the sequence $\{(u_i, v_i)\}$ is uniformly bounded in $\mathcal{C}^{0,\alpha}(K)$ for every $\alpha \in (0, 1)$, and, up to a subsequence, $(u_i, v_i) \to (u_\infty, v_\infty)$ in $\mathcal{C}^0(K) \cap H^1(K)$, where $u_\infty - v_\infty$ is harmonic, u_∞ and v_∞ are subharmonic and (6.21) holds true. As in the previous case, by subharmonicity, nonnegativity, and the fact that $\int_{\partial B_1(0)} u_\infty^2 = 0$, we deduce that $u_\infty \equiv 0$ in $B_1(0)$. So, v_∞ is nonnegative and harmonic in $B_1(0)$; moreover,

$$\begin{aligned} v_\infty(0) &\leq \lim_{i\to\infty} \left(\frac{1}{\sqrt{H(x_i,1)}} |v(x_i) - u(x_i)| + u_i(0) \right) \\ &\leq \lim_{i\to\infty} \left(\frac{1}{\sqrt{H(x_i,1)}} C_1 + u_i(0) \right) = 0; \end{aligned}$$

this implies $v_\infty \equiv 0$ in $B_1(0)$, and gives a contradiction with (6.21).

We proved that if $\int_{\partial B_1(x_i)} u^2 \to 0$, then $H(x_i, 1) \to 0$ as $i \to \infty$. But this is contradiction with Lemma 6.3.2. □

Remark 6.3.4. From now on we will denote as $\bar{C}_3$ a fixed positive constant strictly smaller than $\sqrt{\bar{C}_1 |\mathbb{S}^{N-1}|}$.

Let us come back to the Alt-Caffarelli-Friedman monotonicity formula, see Theorem 6.2.16. In some cases it is possible to get rid of the dependence of the constant $C(x_0)$ on x_0. This is the purpose of the following general result, which holds true for solutions with arbitrary algebraic growth and allows x_0 to vary in a set of full measure.

Proposition 6.3.5. *Let* (u,v) *be a solution of* (6.1) *satisfying* (h1). *Assume that*

$$\int_{\partial B_1(x_0)} u^2 \geq C_1 \quad \textit{and} \quad \int_{\partial B_1(x_0)} v^2 \geq C_1 \quad \forall x_0 \in \{|u-v|<\delta\}, \tag{6.22}$$

where $C_1, \delta > 0$. *Then there exists* $C_2 > 0$ *such that*

$$r \mapsto e^{-C_2 r^{-1/2}} J(x_0, r) \quad \textit{is nondecreasing in } r$$

for every $r \geq 1$, *for every* $x_0 \in \{|u-v|<\delta\}$.

Proof (cf. proof of Theorem 4.3 and the observation before Corollary 4.8 in [90]). For any $x_0 \in \{|u-v|<\delta\}$ and $r \geq 1$, we denote

$$(\bar{u}_{x_0,r}(x), \bar{v}_{x_0,r}(x)) = (u(x_0+rx), v(x_0+rx)) \qquad \text{with } x \in \partial B_1(0).$$

As in the proof of Theorem 6.2.16, it results

$$\frac{\partial}{\partial r} \log J(x_0,r) \geq -\frac{4}{r} + \frac{2}{r}\left[\Gamma\left(\Lambda_1(x_0,r)\right) + \Gamma\left(\Lambda_2(x_0,r)\right)\right], \tag{6.23}$$

where $\Gamma(t) = \sqrt{\left(\frac{N-2}{2}\right)^2 + t} - \left(\frac{N-2}{2}\right)$,

$$\Lambda_1(x_0,r) = \frac{r^2 \int_{\partial B_r(x_0)} |\nabla_\theta u|^2 + u^2 v^2}{\int_{\partial B_r(x_0)} u^2} = \frac{\int_{\partial B_1(0)} |\nabla_\theta \bar{u}_{x_0,r}|^2 + r^2 \bar{u}_{x_0,r}^2 \bar{v}_{x_0,r}^2}{\int_{\partial B_1(0)} \bar{u}_{x_0,r}^2}$$

$$\Lambda_2(x_0,r) = \frac{r^2 \int_{\partial B_r(x_0)} |\nabla_\theta v|^2 + u^2 v^2}{\int_{\partial B_r(x_0)} v^2} = \frac{\int_{\partial B_1(0)} |\nabla_\theta \bar{v}_{x_0,r}|^2 + r^2 \bar{u}_{x_0,r}^2 \bar{v}_{x_0,r}^2}{\int_{\partial B_1(0)} \bar{v}_{x_0,r}^2},$$

and $|\nabla_\theta u|^2 = |\nabla u|^2 - (\partial_\nu u)^2$.

Step 1) *There exist* $\tilde{C}_1, \tilde{C}_2 > 0$ *such that*

$$\tilde{C}_1 \leq \frac{\int_{\partial B_1(0)} \bar{u}_{x_0,r}^2}{\int_{\partial B_1(0)} \bar{v}_{x_0,r}^2} \leq \tilde{C}_2$$

for every $x_0 \in \{|u-v|<\delta\}$ *and* $r \geq 1$.
By contradiction, there are sequences $(x_i) \subset \{|u-v|<\delta\}$ and $(r_i) \subset [1,+\infty)$ such that

$$\lim_{i\to\infty} \frac{\int_{\partial B_1(0)} \bar{u}_{x_i,r_i}^2}{\int_{\partial B_1(0)} \bar{v}_{x_i,r_i}^2} = +\infty$$

(if the limit were 0 we can argue in a similar way). By assumption (6.22), we have

$$\frac{\int_{\partial B_1(0)} \bar{u}^2_{x_i,r_i}}{\int_{\partial B_1(0)} \bar{v}^2_{x_i,r_i}} \le \frac{\int_{\partial B_1(0)} \bar{u}^2_{x_i,r_i}}{C_1}.$$

Consequently, $\int_{\partial B_1(0)} \bar{u}^2_{x_i,r_i} \to +\infty$ as $i \to \infty$, which in turns implies $H(x_i, r_i) \to +\infty$ as $i \to \infty$. Note that

$$\frac{\int_{\partial B_1(0)} \bar{u}^2_{x_i,r_i}}{\int_{\partial B_1(0)} \bar{v}^2_{x_i,r_i}} = \frac{\int_{\partial B_1(0)} u^2_{x_i,r_i}}{\int_{\partial B_1(0)} v^2_{x_i,r_i}} \quad \Longrightarrow \quad \lim_{i\to\infty} \frac{\int_{\partial B_1(0)} u^2_{x_i,r_i}}{\int_{\partial B_1(0)} v^2_{x_i,r_i}} = +\infty \tag{6.24}$$

where we recall that the notation $(u_{x,r}, v_{x,r})$ has been introduced in (6.8). We set $(u_i, v_i) := (u_{x_i,r_i}, v_{x_i,r_i})$. By definition

$$\begin{cases} -\Delta u_i = -H(x_i, r_i) r_i^2 u_i v_i^2 & \text{in } \mathbb{R}^N \\ -\Delta v_i = -H(x_i, r_i) r_i^2 u_i^2 v_i & \text{in } \mathbb{R}^N, \end{cases}$$

and

$$\int_{\partial B_1(0)} u_i^2 + v_i^2 = 1, \tag{6.25}$$

which, by means of Corollary 6.2.7, provides a uniform-in-i bound on $\int_{\partial B_r(0)} u_i^2 + v_i^2$ for every $r \ge 1$. In light of the subharmonicity of (u_i, v_i), this yields a uniform-in-i bound on the L^∞ norm of $\{(u_i, v_i)\}$ in every compact set of $\mathbb{R}^N$. As the competition parameter tends to $+\infty$, we are in position to apply the local segregation Theorem 6.2.3, deducing that up to a subsequence $(u_i, v_i) \to (u_\infty, v_\infty)$ in $\mathcal{C}^0_{\text{loc}}(\mathbb{R}^N)$, where $u_\infty - v_\infty$ is harmonic and both u_∞ and v_∞ are subharmonic. By (6.24)

$$\int_{\partial B_1(0)} v_\infty^2 = \lim_{i\to+\infty} \int_{\partial B_1(0)} v_i^2 = \lim_{i\to\infty} \frac{\int_{\partial B_1(0)} v_i^2}{\int_{\partial B_1(0)} u_i^2 + v_i^2} = 0.$$

As v_∞ is subharmonic and nonnegative, $v_\infty \equiv 0$. This implies that u_∞ is harmonic and nonnegative in $B_1(0)$. Also, from (6.25) it follows $\int_{\partial B_1(0)} u_\infty^2 = 1$. On the other hand, since $x_i \in \{|u - v| < \delta\}$ and $H(x_i, r_i) \to +\infty$ it results

$$u_\infty(0) \le \lim_{i\to\infty} \left(\frac{1}{\sqrt{H(x_i, r_i)}} |u(x_i) - v(x_i)| + v_i(0) \right) = 0$$

and by the strong maximum principle we obtain $u_\infty \equiv 0$, a contradiction.

Step 2) *Conclusion of the proof.*
For $x_0 \in \{|u-v| < \bar{\delta}\}$ and $r \geq 1$, we consider the functions

$$\hat{u}_{x_0,r}(y) := \frac{\bar{u}_{x_0,r}(y)}{\left(\int_{\partial B_1(0)} \bar{u}^2_{x_0,r}\right)^{\frac{1}{2}}} \quad \text{and} \quad \hat{v}_{x_0,r}(y) := \frac{\bar{v}_{x_0,r}(y)}{\left(\int_{\partial B_1(0)} \bar{u}^2_{x_0,r}\right)^{\frac{1}{2}}},$$

which are obtained by $\bar{u}_{x_0,r}$ and $\bar{v}_{x_0,r}$ after a normalization with respect to the L^2 norm of $\bar{u}_{x_0,r}$ on $\partial B_1(0)$. In light of assumption (6.22)

$$\begin{aligned}
\Lambda_1(x_0,r) &= \int_{\partial B_1(0)} |\nabla_\theta \hat{u}_{x_0,r}|^2 + r^2 \left(\int_{\partial B_1(0)} \bar{u}^2_{x_0,r}\right) \hat{u}^2_{x_0,r}\hat{v}^2_{x_0,r} \\
&\geq \int_{\partial B_1(0)} |\nabla_\theta \hat{u}_{x_0,r}|^2 + C_1 r^2 \hat{u}^2_{x_0,r}\hat{v}^2_{x_0,r} \\
\Lambda_2(x_0,r) &= \int_{\partial B_1(0)} |\nabla_\theta \hat{v}_{x_0,r}|^2 + r^2 \left(\int_{\partial B_1(0)} \bar{u}^2_{x_0,r}\right) \hat{u}^2_{x_0,r}\hat{v}^2_{x_0,r} \\
&\geq \int_{\partial B_1(0)} |\nabla_\theta \hat{v}_{x_0,r}|^2 + C_1 r^2 \hat{u}^2_{x_0,r}\hat{v}^2_{x_0,r}.
\end{aligned}$$

As Γ is monotone nondecreasing, we deduce

$$\begin{aligned}
&\Gamma\left(\Lambda_1(x_0,r)\right) + \Gamma\left(\Lambda_2(x_0,r)\right) \\
&\geq \Gamma\left(\int_{\partial B_1(0)} |\nabla_\theta \hat{u}_{x_0,r}|^2 + C_1 r^2 \hat{u}^2_{x_0,r}\hat{v}^2_{x_0,r}\right) + \Gamma\left(\int_{\partial B_1(0)} |\nabla_\theta \hat{v}_{x_0,r}|^2 + C_1 r^2 \hat{u}^2_{x_0,r}\hat{v}^2_{x_0,r}\right).
\end{aligned}$$

Thanks to the first step, we are in position to apply Lemma 6.2.17 in order to obtain

$$\Gamma\left(\Lambda_1(x_0,r)\right) + \Gamma\left(\Lambda_2(x_0,r)\right) \geq 2 - \frac{C}{r^{\frac{1}{2}}},$$

where C is a positive constant independent on $x_0 \in \{|u-v| < \delta\}$ and $r \geq 1$. Coming back to (6.23), we deduce that there exists $C > 0$ such that

$$\frac{\partial}{\partial r} \log J(x_0,r) \geq -C r^{-\frac{3}{2}}$$

for every $x_0 \in \{|u-v| < \delta\}$, for every $r \geq 1$. An integration gives the desired result. □

In light of Lemma 6.3.3, if (u,v) is a solution of (6.1) having linear growth then Proposition 6.3.5 holds true. By means of this uniform monotonicity formula, we deduce the following statement.

Corollary 6.3.6. *Let (u, v) be a solution of (6.1) having linear growth. Then there exists $\bar{C}_4 > 0$ such that*

$$\frac{1}{\bar{C}_4} \leq J(x_0, r) \leq \bar{C}_4, \qquad \int_{\partial B_1(x_0)} u^2 + v^2 \leq \bar{C}_4, \tag{6.26}$$

and

$$\sup_{x \in B_R(x_0)} u(x) + v(x) \leq \bar{C}_4(1 + R)$$

for every $x_0 \in \{|u - v| < \bar{C}_3\}$ and $r \geq 1$ (where $\bar{C}_3$ has been defined Remark 6.3.4).

Proof. It is possible to conveniently modify the proof of Corollaries 4.9 and 4.10 in [90] in our setting. We report here the proof for the sake of completeness.

Step 1) *There exists $C > 0$ such that $J(x_0, r) \leq C$ for every $x_0 \in \{|u - v| < \bar{C}_3\}$, for every $r \geq 1$.*
We note that as (u, v) has linear growth there exists $C > 0$ such that

$$\sup_{x \in B_r(0)} u(x) + v(x) \leq Cr.$$

Now, for every $x_0 \in \mathbb{R}^N$ and $r \geq |x_0|$ it results $B_r(x_0) \subset B_{2r}(0)$, so that

$$\sup_{x \in B_r(x_0)} u(x) + v(x) \leq 2Cr.$$

From Lemma 6.2.15, we deduce that

$$J(x_0, r) \leq Cr^{-2N-4} \int_{B_{2r}(x_0)} u^2 \int_{B_{2r}(x_0)} v^2 \leq C,$$

where C is a positive constant independent on $x_0 \in \mathbb{R}^N$ and $r \geq \max\{|x_0|, 1\}$. If $|x_0| > 1$, thanks to Lemma 6.3.3 we can combine the previous estimate with Proposition 6.3.5: there exists $C > 0$ independent on $x_0 \in \{|u - v| < \bar{C}_3\}$ such that

$$J(x_0, r) \leq e^C J(x_0, |x_0|) \leq C,$$

for every $x_0 \in \{|u - v| < \bar{C}_3\}$ and $r \in [1, |x_0|]$.

Step 2) *There exists $C > 0$ such that $J(x_0, r) \geq C$ for every $x_0 \in \{|u - v| < \bar{C}_3\}$, for every $r \geq 1$.*
In light of Proposition 6.3.5, it is sufficient to show that $\left\{J(x_0, 1) : x_0 \in \{|u - v| < \bar{C}_3\}\right\}$ is uniformly bounded below. By contradiction, assume that there exists a sequence $(x_i) \subset \{|u - v| < \bar{C}_3\}$ such that $J(x_i, 1) \to 0$ as $i \to \infty$. We set, as in the proof of Lemma 6.3.3,

$$(u_i(x), v_i(x)) = \left(\frac{1}{\sqrt{H(x_i, 1)}} u(x_i + x), \frac{1}{\sqrt{H(x_i, 1)}} v(x_i + x)\right).$$

It results $\int_{\partial B_1(0)} u_i^2 + v_i^2 = 1$ for every i, and (u_i, v_i) solves

$$\begin{cases} -\Delta u_i = -H(x_i, 1) u_i v_i^2 & \text{in } \mathbb{R}^N \\ -\Delta v_i = -H(x_i, 1) u_i^2 v_i & \text{in } \mathbb{R}^N; \end{cases}$$

By Corollary 6.2.7 (which we can apply, see Remark 6.2.11), we deduce that

$$\int_{\partial B_r(0)} u_i^2 + v_i^2 \leq e r^{N+1} \qquad \forall r, \ \forall i.$$

As u_i and v_i are subharmonic, the (6.20) gives a uniform bound on the $L^\infty(B_{r/2}(0))$ norm of the family $\{(u_i, v_i)\}$, for every $r \geq 1$. Now, we have to distinguish between

(*i*) the sequence $\{H(x_i, 1)\}$ is bounded;

(*ii*) the sequence $\{H(x_i, 1)\}$ is unbounded.

In case (*i*), up to a subsequence $H(x_i, 1) \to H_\infty > 0$. Repeating the argument already explained in Lemma 6.3.3, we deduce that up to a subsequence $(u_i, v_i) \to (u_\infty, v_\infty)$ in $\mathcal{C}^2_{\text{loc}}(\mathbb{R}^N)$, where (u_∞, v_∞) is a nonnegative solution to

$$\begin{cases} -\Delta u_\infty = -H_\infty u_\infty v_\infty^2 & \text{in } \mathbb{R}^N \\ -\Delta v_\infty = -H_\infty u_\infty^2 v_\infty & \text{in } \mathbb{R}^N, \end{cases}$$

such that

$$\int_{\partial B_1(0)} u_\infty^2 + v_\infty^2 = 1. \tag{6.27}$$

A direct computation yields

$$J(x_i, 1) = H(x_i, 1)^2 \int_{B_1(0)} \frac{|\nabla u_i|^2 + H(x_i, 1) u_i^2 v_i^2}{|y|^{N-2}} \int_{B_1(0)} \frac{|\nabla v_i|^2 + H(x_i, 1) u_i^2 v_i^2}{|y|^{N-2}}, \tag{6.28}$$

and thanks to the $\mathcal{C}^2_{\mathrm{loc}}(\mathbb{R}^N)$ convergence and the integrability of the kernel $|y|^{2-N}$, the absurd assumption $J(x_i,1)\to 0$ as $i\to\infty$ reads

$$\int_{B_1(0)}\frac{|\nabla u_\infty|^2+H_\infty^2u_\infty^2v_\infty^2}{|y|^{N-2}}\int_{B_1(0)}\frac{|\nabla v_\infty|^2+H_\infty^2u_\infty^2v_\infty^2}{|y|^{N-2}}=0.$$

Without loss of generality, suppose that the first term on the left hand side vanishes. Then u_∞ has to be constant, and, if $v_\infty\not\equiv 0$, then $u_\infty\equiv 0$ in $B_1(0)$ and, by subharmonicity, in $\mathbb{R}^N$. Therefore v_∞ is harmonic and nonnegative in $\mathbb{R}^N$, and by the Liouville theorem for harmonic function it has to be constant. Now, since $x_i\in\{|u-v|<\bar{C}_3\}$, and in light of Lemma 6.3.2, we deduce

$$\begin{aligned}v_\infty(0)&\le\lim_{i\to\infty}\left(\frac{1}{\sqrt{H(x_i,1)}}|v(x_i)-u(x_i)|+u_i(0)\right)\\&\le\lim_{i\to\infty}\left(\frac{1}{\sqrt{\bar{C}_1}}\bar{C}_3+u_i(0)\right)<\sqrt{\frac{1}{|\mathbb{S}^{N-1}|}}.\end{aligned}$$

But since v_∞ is constant and (6.27) holds true, necessarily $v_\infty(0)^2|\mathbb{S}^{N-1}|=1$, a contradiction.

In case (ii), up to a subsequence $H(x_i,1)\to+\infty$ as $i\to\infty$. Due to the fact the $\{(u_i,v_i)\}$ is uniformly bounded in every compact subset of $\mathbb{R}^N$, we are in position to apply Theorems 6.2.2 and 6.2.3: up to a subsequence, $(u_i,v_i)\to(u_\infty,v_\infty)$ in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^N)\cap H^1_{\mathrm{loc}}(\mathbb{R}^N)$, where $u_\infty-v_\infty$ is harmonic, u_∞ and v_∞ are subharmonic, (6.27) holds true and $u_\infty v_\infty\equiv 0$. Moreover, $H(x_i,1)u_i^2v_i^2\to 0$ in $L^1_{\mathrm{loc}}(\mathbb{R}^N)$. Now, we recall the expression of $J(x_i,1)$ given by (6.28). We claim that

$$\lim_{i\to\infty}\int_{B_1(0)}\frac{|\nabla u_i|^2+H(x_i,1)u_i^2v_i^2}{|y|^{N-2}}=\int_{B_1(0)}\frac{|\nabla u_\infty|^2}{|y|^{N-2}}.\tag{6.29}$$

Indeed, for every $\varepsilon>0$ the $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^N)\cap H^1_{\mathrm{loc}}(\mathbb{R}^N)$ convergence implies that

$$\lim_{i\to\infty}\int_{B_1(0)\setminus B_\varepsilon(0)}\frac{|\nabla u_i|^2+H(x_i,1)u_i^2v_i^2}{|y|^{N-2}}=\int_{B_1(0)\setminus B_\varepsilon(0)}\frac{|\nabla u_\infty|^2}{|y|^{N-2}};$$

furthermore, thanks to Lemma 6.2.15 and using the linear growth of u we have

$$\int_{B_\varepsilon(0)}\frac{|\nabla u_i|^2+H(x_i,1)u_i^2v_i^2}{|y|^{N-2}}\le C\varepsilon^{-N}\int_{B_{2\varepsilon}(0)}u_i^2\le Cu_i^2(0)+C\varepsilon^2.$$

Since $u_\infty v_\infty \equiv 0$, there are two possibilities: if $u_\infty(0) = 0$ then $u_i^2(0) \to 0$ as $i \to \infty$, otherwise $v_\infty(0) = 0$ and

$$\begin{aligned}\lim_{i\to\infty} u_i(0) &\le \lim_{i\to\infty}\left(\frac{1}{\sqrt{H(x_i,1)}}|u(x_i)-v(x_i)| + v_i(0)\right)\\ &\le \lim_{i\to\infty}\left(\frac{1}{\sqrt{H(x_i,1)}}\bar{C}_3 + v_i(0)\right) = 0,\end{aligned}$$

since $H(x_i,1) \to +\infty$. In both cases we showed that for every $\varepsilon > 0$

$$\begin{aligned}\int_{B_1(0)\setminus B_\varepsilon(0)} \frac{|\nabla u_\infty|^2}{|y|^{N-2}} &\le \lim_{i\to\infty}\int_{B_1(0)} \frac{|\nabla u_i|^2 + H(x_i,1)u_i^2 v_i^2}{|y|^{N-2}}\\ &\le \int_{B_1(0)\setminus B_\varepsilon(0)} \frac{|\nabla u_\infty|^2}{|y|^{N-2}} + C\varepsilon^2,\end{aligned}$$

and this proves claim (6.29).

Arguing in the same way, and recalling that we are assuming $J(x_i,1) \to 0$ as $i \to \infty$, we deduce that

$$\int_{B_1(0)} \frac{|\nabla u_\infty|^2}{|y|^{N-2}} \int_{B_1(0)} \frac{|\nabla v_\infty|^2}{|y|^{N-2}} = 0.$$

Without loss of generality, we can then assume that u_∞ is constant in $B_1(0)$, and consequently in $\mathbb{R}^N$. Since $u_\infty v_\infty \equiv 0$, if $v_\infty \not\equiv 0$ then $u_\infty \equiv 0$. By the Liouville theorem for harmonic functions, also v_∞ is constant; moreover,

$$\begin{aligned}v_\infty(0) &\le \lim_{i\to\infty}\left(\frac{1}{\sqrt{H(x_i,1)}}|v(x_i)-u(x_i)| + u_i(0)\right)\\ &\le \lim_{i\to\infty}\left(\frac{1}{\sqrt{H(x_i,1)}}\bar{C}_3 + u_i(0)\right) = 0;\end{aligned}$$

this implies $v_\infty \equiv 0$ in $\mathbb{R}^N$, and gives a contradiction with (6.27).

This completes the proof of the existence of a uniform lower bound for $J(x_0,r)$ when $x_0 \in \{|u-v| < \bar{C}_3\}$ and $r \ge 1$.

Step 3) *There exists $C > 0$ such that $H(x_0,1) \le C$ for every $x_0 \in \{|u-v| < \bar{C}_3\}$.* Arguing as before and keeping the notation previously introduced, we deduce that, if there exists $(x_i) \subset \{|u-v| < \bar{C}_3\}$ such that $H(x_i,1) \to +\infty$, then necessarily

$$\int_{B_1(0)} \frac{|\nabla u_i|^2 + H(x_i,1)u_i^2 v_i^2}{|y|^{N-2}} \int_{B_1(0)} \frac{|\nabla v_i|^2 + H(x_i,1)u_i^2 v_i^2}{|y|^{N-2}} \ge C$$

for some $C > 0$. Indeed, if

$$\lim_{i\to\infty} \int_{B_1(0)} \frac{|\nabla u_i|^2 + H(x_i,1)u_i^2 v_i^2}{|y|^{N-2}} \int_{B_1(0)} \frac{|\nabla v_i|^2 + H(x_i,1)u_i^2 v_i^2}{|y|^{N-2}} = 0,$$

then we can proceed as in the previous step and reach a contradiction. As a consequence, by $H(x_i, 1) \to +\infty$ we deduce

$$\begin{aligned}\lim_{i\to+\infty} J(x_i,1) &= \lim_{i\to\infty} H(x_i,1)^2 \int_{B_1(0)} \frac{|\nabla u_i|^2 + H(x_i,1)u_i^2 v_i^2}{|y|^{N-2}} \int_{B_1(0)} \frac{|\nabla v_i|^2 + H(x_i,1)u_i^2 v_i^2}{|y|^{N-2}} \\ &= +\infty,\end{aligned}$$

which is in contradiction with the first step.

Step 4) *There exists $C > 0$ such that*

$$\sup_{x\in B_R(x_0)} u(x) + v(x) \leq C(1+R)$$

for every $x_0 \in \{|u - v| < \bar{C}_3\}$ and $r \geq 1$.
In light of the result of the third step, using Corollary 6.2.7 we deduce that there exists $C > 0$ such that

$$\int_{\partial B_R(x_0)} u^2 + v^2 \leq eCr^{N+1},$$

for every $x_0 \in \{|u - v| < \bar{C}_3\}$, for every $R \geq 1$. By subharmonicity, it is not difficult to obtain the desired estimate. □

6.4 Uniqueness of the asymptotic profile

In this section we show that, under assumptions (h1) and (h2) (in fact it is sufficient to assume much less), any solution to (6.1) having algebraic growth is a solution with linear growth. Moreover, we show that for every $x_0 \in \mathbb{R}^N$, the *entire* blow-down family $\{(u_{x_0,R}, v_{x_0,R}) : R > 0\}$ converges, as $R \to +\infty$, to the same harmonic function.

Proposition 6.4.1. *Let (u, v) be a solution of (6.1) satisfying assumptions (h1) and such that*

$$\lim_{x_N\to+\infty} v(x', x_N) = 0 \quad \textit{uniformly in } x' \in \mathbb{R}^{N-1}. \tag{6.30}$$

Then $N(x_0, r) \leq 1$ for every $r > 0$, and consequently (u, v) has linear growth. Furthermore, there exists a constant $\gamma > 0$ such that, for every $x_0 \in \mathbb{R}^N$, the blow-down family $\{(u_{x_0,R}, v_{x_0,R}) : R > 0\}$ converges to the pair $(\gamma x_N^+, \gamma x_N^-)$ as $R \to +\infty$, in $\mathcal{C}^0_{\mathrm{loc}}(\mathbb{R}^N)$ and in $H^1_{\mathrm{loc}}(\mathbb{R}^N)$.

Remark 6.4.2. It is possible to replace assumption (6.30) with

$$\lim_{x_N \to -\infty} u(x', x_N) = 0 \quad \text{uniformly in } x' \in \mathbb{R}^{N-1}.$$

Proof. As (u, v) has algebraic growth, thanks to Lemma 6.2.9 Theorem 6.2.13 applies: for every $x_0 \in \mathbb{R}^N$ there exists

$$\lim_{r \to +\infty} N(x_0, r) = d_{x_0} \in \mathbb{N} \setminus \{0\},$$

and there exists a subsequence $(u_{x_0,R_n}, v_{x_0,R_n})$ of the blow-down family which is convergent (in $\mathcal{C}^0_{\text{loc}}(\mathbb{R}^N)$ and in $H^1_{\text{loc}}(\mathbb{R}^N)$) to $(\Psi^+_{x_0}, \Psi^-_{x_0})$, where Ψ_{x_0} is a homogeneous harmonic polynomial of degree $d_{x_0} \geq 1$. As showed in Corollary 6.2.14, this implies that $\lim_{r \to \infty} H(x_0, r) = +\infty$.
Now, let $K \subset\subset \mathbb{R}^N_+$. Since

$$\inf\{x_N : x \in K\} > 0,$$

in light of assumption (6.30), it holds

$$\lim_{R \to +\infty} v_R(x) = 0 \quad \text{uniformly in } K.$$

As K has been arbitrarily chosen, it follows that $v_{x_0,R_n}(x) \to 0$ pointwise in $\mathbb{R}^N_+$. By the uniqueness of the limit, we deduce $\Psi^-_{x_0} = 0$ in $\mathbb{R}^N_+$. Thus, Ψ_{x_0} is a homogeneous harmonic polynomial (hence $\Psi_{x_0}(0) = 0$) which is nonnegative in $\mathbb{R}^N_+$ and is not identically 0 (this follows simply from the fact that $d_{x_0} \geq 1$):

$$\begin{cases} -\Delta \Psi_{x_0} = 0 & \text{in } \mathbb{R}^N_+ \\ \Psi_{x_0} \geq 0, \ \Psi_{x_0} \not\equiv 0 & \text{in } \mathbb{R}^N_+ \\ \Psi_{x_0}(0) = 0. \end{cases}$$

By the strong maximum principle, we deduce that $\Psi_{x_0} > 0$ in $\mathbb{R}^N_+$; hence, the Hopf lemma guarantees that $\nabla \Psi_{x_0}(0) \neq 0$. The unique (up to a constant factor) homogeneous harmonic polynomial satisfying these properties is the linear one: $\Psi_{x_0}(x) = C_{x_0} x_N$; but $C_{x_0} > 0$ is uniquely determined (independently on x_0) by the condition

$$\int_{\partial B_1(0)} C^2_{x_0} x_N^2 = \lim_{n \to \infty} \int_{\partial B_1(0)} u^2_{x_0,R_n} + v^2_{x_0,R_n} = 1.$$

Hence, for every x_0 the blow-down family converges (up to a subsequence) to the same pair $(\gamma x_N^+, \gamma x_N^-)$, for a constant $\gamma > 0$. By Theorem 6.2.13, the fact that the degree of the limiting profile is 1 means that $d_{x_0} = 1$ for every $x_0 \in \mathbb{R}^N$, and this gives the linear growth of (u, v), see Corollary 6.2.8.

It remains to show that, for every $x_0 \in \mathbb{R}^N$, the entire blow-down family converges to γx_N. Assume by contradiction that this is not true: there exist a compact $K \subset \mathbb{R}^N$, a $\bar{\varepsilon} > 0$ and a subsequence $\{(u_{x_0,R_m}, v_{x_0,R_m})\}$ with $R_m \to +\infty$ as $m \to \infty$, such that

$$\|u_{x_0,R_m} - \gamma x_N^+\|_{\mathcal{C}^0(K)} + \|u_{x_0,R_m} - \gamma x_N^+\|_{H^1(K)} \\ + \|v_{x_0,R_m} - \gamma x_N^-\|_{\mathcal{C}^0(K)} + \|v_{x_0,R_m} - \gamma x_N^-\|_{H^1(K)} \geq \bar{\varepsilon} \quad (6.31)$$

for every m. But now it is possible to repeat step by step the proof of Theorem 6.2.13 obtaining that, up to a subsequence, $\{(u_{x_0,R_m}, v_{x_0,R_m})\}$ converges, as $m \to +\infty$ to a homogeneous harmonic polynomial of degree $d_{x_0} \geq 1$. Following the above line of reasoning, we find that the limit is nothing but the function $(\gamma x_N^+, \gamma x_N^-)$, in contradiction with (6.31). □

6.5 Monotonicity at infinity

We aim at proving the following statement.

Proposition 6.5.1. *Let* (u, v) *be a solution of* (6.1) *satisfying* (h1) *and* (h2). *For every*

$$\nu \in \{\nu \in \mathbb{S}^{N-1} : \langle \nu, e_N \rangle > 0\},$$

there exists $M_\nu > 0$ *such that*

$$x \in \{x_N > M_\nu\} \Rightarrow \partial_\nu u(x) > 0 \quad \textit{and} \quad x \in \{x_N < -M_\nu\} \Rightarrow \partial_\nu v(x) < 0.$$

The achievement of Section 6.4 says that (u, v) behaves at infinity as $(\gamma x_N^+, \gamma x_N^-)$; thus, the idea is that u has to be increasing in the e_N direction for $x_N \gg 1$ and v has to be decreasing in the e_N direction for $x_N \ll -1$. In order to prove this conjecture, we wish to apply the standard gradient estimate for the Poisson equation (see e.g. [45]) on u minus "a suitable linear function" and on v minus "a suitable linear function": this idea is corroborated by the fact that Δu can be uniformly bounded by an exponentially decaying function for x_N sufficiently large. An analogous bound holds for Δv when x_N is sufficiently large and negative.

Lemma 6.5.2. *Let* (u, v) *be a solution of* (6.1) *satisfying* (h2). *For every* $p, q \geq 1$ *there exist* $M_1(p, q) > 0$ *and a positive constant* $C = C(p, q) > 0$ *such that*

$$u^p(x) v^q(x) \leq C e^{-C|x_N|} \qquad \forall x \in \{|x_N| > M_1(p, q)\}.$$

Proof. We consider the bound on $u^p v^q$ in $x_N \gg 1$, the same argument applies for $x_N \ll -1$.
Given $K > 0$ and $\delta > 0$, by (h2) there exists $M > 0$ such that

$$u(x) > K \quad \text{and} \quad v(x) < \delta \quad \text{if } x \in \{x_N > M/2\}.$$

For every $x \in \{x_N > M\}$ the ball $B_x := B_{x_N/4}(x)$ is contained in $\{x_N > M/2\}$. Consequently,

$$\begin{cases} u(y) \geq K_x := \inf_{B_x} u \geq K \\ v(y) \leq \delta \end{cases} \qquad \forall y \in B_x,\ \forall x \in \{x_N > M\},$$

so that

$$\begin{cases} -\Delta v \leq -K_x^2 v & \text{in } B_x \\ v \geq 0 & \text{in } B_x \\ v \leq \delta & \text{in } B_x. \end{cases}$$

We are in position to apply Lemma 6.2.1:

$$\sup_{B'_x} v \leq C\delta e^{-CK_x x_N}, \tag{6.32}$$

where B'_x denotes the ball $B_{x_N/8}(x)$. On the other hand, it is possible to apply the Harnack inequality (Theorem 8.20 in [45], see also the subsequent observation concerning the estimate on the constant) on u in B_x, with potential v^2:

$$\sup_{B_x} u \leq Ce^{C\delta x_N} K_x. \tag{6.33}$$

Inequalities (6.32) and (6.33) yields

$$u^p(x)v^q(x) \leq CK_x^p \delta^q e^{-C_1 q K_x x_N + C_2 p \delta x_N} \qquad \forall x \in \{x_N > M\}.$$

A suitable choice of $K \leq K_x$ and δ permits to obtain the desired result. □

Remark 6.5.3. From now on we will denote as $M_1 := \max\{M_1(1,2), M_1(2,1)\}$, where $M_1(1,2)$ and $M_1(2,1)$ have been defined in Lemma 6.5.2.

If we could show that the function u can be approximated in $\{x_N > M_1\}$ by a linear function with positive slope in the e_N direction, the gradient estimates for the Poisson equation would give the desired monotonicity for u. So far we showed that for given $x_0 \in \mathbb{R}^N$ and $\varepsilon > 0$ there exists $R_{x_0,\varepsilon} > 0$ such that

$$\sup_{x \in B_1(0)} |u_{x_0,R}(x) - \gamma x_N^+| + |v_{x_0,R}(x) - \gamma x_N^-| < \varepsilon \tag{6.34}$$

for every $R > R_{x_0,\varepsilon}$. This means that

$$\sup_{x \in B_R(x_0)} \left| u(x) - \gamma \frac{\sqrt{H(x_0,R)}}{R}(x_N - x_{0,N})^+ \right| + \left| v(x) - \gamma \frac{\sqrt{H(x_0,R)}}{R}(x_N - x_{0,N})^- \right| < \sqrt{H(x_0,R)}\varepsilon$$

whenever $R > R_{x_0,\varepsilon}$. This reveals that we have to face two problems: the first one is the fact that we have not a unique candidate to approximate u for $x_N \gg 1$ and v for $x_N \ll -1$, the second one is that this approximation, which holds for R sufficiently large, gets worse as R increases (recall that the function $H(x_0,\cdot)$ is nondecreasing and tends to $+\infty$ as $R \to +\infty$, see Corollary 6.2.14). In order to overcome the first problem, we wish to find a uniform estimate (in both x_0 and R) on the ratio $\sqrt{H(x_0,R)}/R$; in the forthcoming Lemma 6.5.6, we show that this is possible if $x_0 \in \{|u-v| < \bar{C}_3\}$, where $\bar{C}_3$ has been defined in Remark 6.3.4. Before, we deduce some useful information about this special set.

Lemma 6.5.4. *Under the assumption* (h2)*, the set* $\{|u-v| < \bar{C}_3\}$ *is bounded in the* e_N *direction and unbounded in all the other directions* $\{e_1,\dots,e_{N-1}\}$*. In particular, for every* $x' \in \mathbb{R}^{N-1}$ *there exists* $\tilde{x} \in \{|u-v| < \bar{C}_3\}$ *such that* $\tilde{x}' = x'$*.*

Proof. The properties follow easily by our main assumption (h2). Indeed

$$\lim_{x_N \to \pm\infty} (u(x',x_N) - v(x',x_N)) = \pm\infty,$$

uniformly in $x' \in \mathbb{R}^{N-1}$. This immediately implies that the sublevel set $\{|u-v| \le M\}$ is bounded in the e_N direction for every $M > 0$ (in particular, this holds for $\bar{C}_3$). On the other hand, for a given $x' \in \mathbb{R}^{N-1}$, we can consider the map $s \in \mathbb{R} \mapsto u(x',s) - v(x',s)$. This is a continuous function which tends to $\pm\infty$ as $s \to \pm\infty$, thus there exist $\tilde{s} \in \mathbb{R}$ such that $|u(x',\tilde{s}) - v(x',\tilde{s})| < \bar{C}_3$. □

Remark 6.5.5. From now on, we denote $\zeta := \sup\{|x_{0,N}| : x_0 \in \{|u-v| < \bar{C}_3\}\} < +\infty$.

In the next lemma we bound the ratio $\sqrt{H(x_0,R)}/R$ uniformly in $x_0 \in \{|u-v| < \bar{C}_3\}$ and $R \ge 1$.

Lemma 6.5.6. *Let* (u,v) *be a solution of* (6.1) *satisfying* (h1) *and* (h2)*. There exists* $\bar{C}_5, \bar{C}_6 > 0$ *such that*

$$\bar{C}_5 \le \frac{\sqrt{H(x_0,R)}}{R} \le \bar{C}_6$$

for every $x_0 \in \{|u-v| < \bar{C}_3\}$ *and* $R \ge 1$*.*

Proof. By Proposition 6.4.1, we know that under (h1) and (h2) the solution (u,v) has linear growth. Hence, we can invoke Corollary 6.3.6; combining this result with Corollary 6.2.7 we deduce

$$\frac{H(x_0,R)}{R^2} \le eH(x_0,1) \le e\bar{C}_4 \qquad \forall x_0 \in \{|u-v| < \bar{C}_3\},\ R \ge 1.$$

For the lower bound, we show that the quantity

$$J_{x_0,R}(0,1) := \int_{B_1(0)} \frac{|\nabla u_{x_0,R}(y)|^2 + H(x_0,R)R^2 u_{x_0,R}^2(y) v_{x_0,R}^2(y)}{|y|^{N-2}}\, dy \cdot \int_{B_1(0)} \frac{|\nabla v_{x_0,R}(y)|^2 + H(x_0,R)R^2 u_{x_0,R}^2(y) v_{x_0,R}^2(y)}{|y|^{N-2}}\, dy$$

is bounded above by a positive constant C independent on $x_0 \in \mathbb{R}^N$ and $R \geq 1$. We use Lemma 6.2.15: there exists $C > 0$ independent on $x_0 \in \mathbb{R}^N$ and on $R \geq 1$ such that

$$\begin{aligned} \int_{B_1(0)} & \frac{|\nabla u_{x_0,R}(y)|^2 + H(x_0,R)R^2 u_{x_0,R}^2(y) v_{x_0,R}^2(y)}{|y|^{N-2}}\, dy \\ &= \frac{1}{H(x_0,R)} \int_{B_R(x_0)} \frac{|\nabla u(y)|^2 + u^2(y)v^2(y)}{|y-x_0|^{N-2}}\, dy \\ &\leq \frac{C}{H(x_0,R)R^N} \int_{B_{2R}(x_0)} u^2 = C \int_{B_2(0)} u_{x_0,R}^2. \end{aligned} \tag{6.35}$$

We point out that, as $N(x_0,r) \leq 1$ for every $x_0 \in \mathbb{R}^N$ and $r \geq 1$, the same estimate holds true for the Almgren quotient associated to $(u_{x_0,R}, v_{x_0,R})$, for every $x_0 \in \mathbb{R}^N$ and $R \geq 1$ (see Remark 6.2.12). As a consequence, the normalization $\int_{\partial B_1(0)} u_{x_0,R}^2 + v_{x_0,R}^2 = 1$ gives, by Corollary 6.2.7, a uniform (in both x_0 and R) upper bound for $\int_{\partial B_3(0)} u_{x_0,R}^2 + v_{x_0,R}^2$. Due to the subharmonicity of $(u_{x_0,R}, v_{x_0,R})$, we obtain a uniform bound for $\{(u_{x_0,R}, v_{x_0,R})\}$ in $L^\infty(B_2(0))$, so that we can estimate the right hand side of (6.35) obtaining

$$\int_{B_1(0)} \frac{|\nabla u_{x_0,R}(y)|^2 + H(x_0,R)R^2 u_{x_0,R}^2(y) v_{x_0,R}^2(y)}{|y|^{N-2}}\, dy \leq C$$

for every $x_0 \in \mathbb{R}^N$ and $R \geq 1$. Arguing in the same way on the second factor of $J_{x_0,R}(0,1)$ we obtain the desired upper bound: there exists $C > 0$ such that

$$J_{x_0,R}(0,1) \leq C \qquad \forall x_0 \in \mathbb{R}^N,\ \forall R \geq 1.$$

A simple change of variable shows that

$$J_{x_0,R}(0,1) = \frac{R^4}{H^2(x_0,R)} J(x_0,R),$$

so that

$$J(x_0,R) \leq C\frac{H^2(x_0,R)}{R^4} \qquad \forall x_0 \in \mathbb{R}^N,\ \forall R \geq 1. \tag{6.36}$$

A comparison between (6.36) and the uniform lower estimate of Corollary 6.3.6 provides the desired result:

$$\frac{H^2(x_0,R)}{R^4} \geq \frac{C}{\bar{C}_4} \qquad \forall x_0 \in \{|u-v| < \bar{C}_3\},\ \forall R \geq 1. \qquad \square$$

We are ready to improve the estimate given by (6.34). Firstly, we get rid of the dependence of $R_{x_0,\varepsilon}$ on x_0 for $x_0 \in \{|u-v| < \bar{C}_3\}$.

Lemma 6.5.7. *Let* (u,v) *be a solution of* (6.1) *satisfying* (h1) *and* (h2)*. For every* $\varepsilon > 0$ *there exists* $R_\varepsilon > 0$ *such that*

$$\sup_{x\in B_1(0)} |u_{x_0,R}(x) - \gamma x_N^+| + |v_{x_0,R}(x) - \gamma x_N^-| < \varepsilon$$

for every $R > R_\varepsilon$ *and* $x_0 \in \{|u-v| < \bar{C}_3\}$*, where* γ *and* $\bar{C}_3$ *have been defined in Proposition 6.4.1 and Remark 6.3.4 respectively.*

Proof. Assume by contradiction that there exist $\bar{\varepsilon} > 0$ and a sequence (x_j, R_j) with $x_j \in \{|u-v| < \bar{C}_3\}$ for every j, $R_j \to +\infty$, and

$$\sup_{x\in B_1(0)} |u_{x_j,R_j}(x) - \gamma x_N^+| + |v_{x_j,R_j}(x) - \gamma x_N^-| \geq \bar{\varepsilon} \tag{6.37}$$

for every j. Let us denote $(u_j, v_j) = (u_{x_j,R_j}, v_{x_j,R_j})$. We know that (u_j,v_j) solves

$$\begin{cases} -\Delta u_j = -H(x_j,R_j) R_j^2 u_j v_j^2 & \text{in } \mathbb{R}^N \\ -\Delta v_j = -H(x_j,R_j) R_j^2 u_j^2 v_j & \text{in } \mathbb{R}^N \end{cases} \qquad \forall j.$$

In light of Lemma 6.5.6, we know that

$$\lim_{j\to+\infty} H(x_j,R_j) \geq \lim_{j\to+\infty} \bar{C}_5 R_j^2 = +\infty; \tag{6.38}$$

a fortiori the competition parameter $H(x_j,R_j)R_j^2$ tends to $+\infty$ as $j \to +\infty$. Note that the normalization $\int_{\partial B_1(0)} u_j^2 + v_j^2 = 1$ implies, by means of Corollary 6.2.7 (which we can apply on (u_j,v_j), see Remark 6.2.12), that

$$\int_{\partial B_r(0)} u_j^2 + v_j^2 \leq e r^{N+1} \qquad \forall r > 1,\ \forall j.$$

By subharmonicity, the sequence $\{(u_j,v_j)\}$ is uniformly bounded in every compact set K of $\mathbb{R}^N$, and in light of Theorem 6.2.2 it is also uniformly bounded in $\mathcal{C}^{0,\alpha}(K)$, for every $\alpha \in (0,1)$. The local segregation Theorem 6.2.3 implies that, up to a subsequence, $(u_j,v_j) \to (u_\infty, v_\infty)$ in $\mathcal{C}^0_{\text{loc}}(\mathbb{R}^N) \cap H^1_{\text{loc}}(\mathbb{R}^N)$, and

(i) $u_\infty v_\infty \equiv 0$ in $\mathbb{R}^N$;

(ii) $H(x_j, R_j) R_j^2 u_j^2 v_j^2 \to 0$ as $j \to \infty$ in $L^1_{\rm loc}(\mathbb{R}^N)$;

(iii) $u_\infty - v_\infty$ is harmonic in $\mathbb{R}^N$;

(iv) by (6.38) and the fact that $x_j \in \{|u - v| < \bar{C}_3\}$

$$|u_\infty(0) - v_\infty(0)| = \lim_{j\to+\infty} \frac{1}{\sqrt{H(x_j, R_j)}} |u(x_j) - v(x_j)| = 0;$$

(v) by uniform convergence the normalization on $\partial B_1(0)$ pass to the limit:

$$\int_{\partial B_1(0)} u_\infty^2 + v_\infty^2 = 1; \tag{6.39}$$

(vi) by H^1 and uniform convergence and the point (ii)

$$\begin{aligned} \frac{r \int_{B_r(0)} |\nabla u_\infty|^2 + |\nabla v_\infty|^2}{\int_{\partial B_r(0)} u_\infty^2 + v_\infty^2} &= \lim_{j\to+\infty} \frac{r \int_{B_r(0)} |\nabla u_j|^2 + |\nabla v_j|^2 + H(x_j, R_j) R_j^2 u_j^2 v_j^2}{\int_{\partial B_r(0)} u_j^2 + v_j^2} \\ &= \lim_{j\to+\infty} N(x_j, R_j r) \le 1 \qquad \forall r \in (0,1), \end{aligned} \tag{6.40}$$

where the upper bound on N follows from the fact that, under assumptions (h1) and (h2), Proposition 6.4.1 applies and guarantees that (u, v) has linear growth.

Note that

$$N_\infty(0, r) := \frac{r \int_{B_r(0)} |\nabla u_\infty|^2 + |\nabla v_\infty|^2}{\int_{\partial B_r(0)} u_\infty^2 + v_\infty^2}$$

is the Almgren quotient of the harmonic function $u_\infty - v_\infty$, and it is nondecreasing. As $u_\infty(0) - v_\infty(0) = 0$, it results

$$N_\infty(0, r) \ge \lim_{s\to 0^+} N_\infty(0, s) = \deg(u_\infty - v_\infty, 0) \ge 1 \tag{6.41}$$

for every $r > 0$. Here, $\deg(u_\infty - v_\infty, 0)$ denotes the degree of vanishing of the harmonic function $u_\infty - v_\infty$ in 0, and is greater than 1 because it has to be a positive integer (this result is by now well known). By monotonicity, a comparison between (6.40) and (6.41) yields $N_\infty(0, r) = 1$ for every $r \in (0, 1)$, which implies (see Proposition 3.9 in [67], which we can apply, as explained in Remark 6.2.4) that $u_\infty - v_\infty$ is a linear function, that is, $(u_\infty(x), v_\infty(x)) = (\langle e, x\rangle^+, \langle e, x\rangle^-)$ for some $e \in \mathbb{R}^N$. We claim that

$$e = \gamma e_N, \tag{6.42}$$

which gives a contradiction with (6.37) and completes the proof of the statement. To prove the claim, we note that under our assumptions we have

$$v_j(x) = \frac{1}{\sqrt{H(x_j, R_j)}} v(x'_j + R_j x', x_{j,N} + R_j x_N) \to 0$$

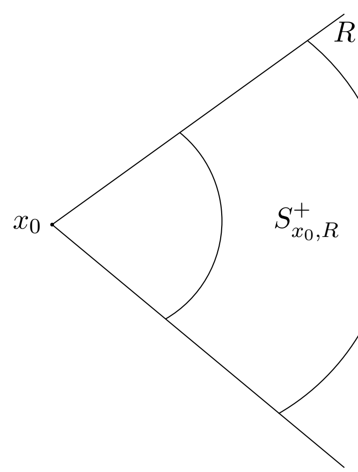
as $j \to +\infty$, uniformly in every compact subset of $B_1(0) \cap \mathbb{R}^N_+$; to pass to the limit, we used the fact that $H(x_j, R_j) \geq \bar{C}_1$ (see Lemma 6.3.2) and the boundedness of the set $\{|u - v| < \bar{C}_3\}$ in the e_N direction (see Lemma 6.5.4), which guarantees that $x_{j,N} + R_j x_N \to +\infty$ as $j \to +\infty$. By the uniqueness of the limit, we deduce $e = Ce_N$ for some $C > 0$. The normalization (6.39) yields $C = \gamma$, which concludes the proof of the claim (6.42). □

Definition 6.5.8. Let us fix $\tau > 0$ not too small (to be determined in the following Lemma). For a given $x_0 \in \mathbb{R}^N$ and $R > 0$ we introduce the conical sectors

$$S^+_{x_0,R} := \left\{ x = (x', x_N) \in \mathbb{R}^N : \frac{R}{2} < |x - x_0| < R, |x' - x'_0| < \tau(x_N - x_{0,N}) \right\}$$
$$S^-_{x_0,R} := \left\{ x = (x', x_N) \in \mathbb{R}^N : \frac{R}{2} < |x - x_0| < R, |x' - x'_0| < \tau(x_{0,N} - x_N) \right\},$$

and their union $S_{x_0,R}$.

The following picture represents the set $S^+_{x_0,R}$ for a given $x_0 \in \mathbb{R}^N$.



The geometry of the set $\{|u - v| < \bar{C}_3\}$ allows to show that the union of $S_{x_0,R}$ with R sufficiently large and $x_0 \in \{|u - v| < \bar{C}_3\}$ contains, and is contained in, the union of two half-spaces.

Lemma 6.5.9. *Let (u,v) be a solution of* (6.1) *satisfying* (h1) *and* (h2). *There exists $\bar{R}>0$ such that, for every $\widehat{R}\geq\bar{R}$ there exists $M_2=M_2\left(\widehat{R}\right)>\zeta$ such that*

$$\{|x_N|>M_2\}\subset\bigcup_{\substack{x_0\in\{|u-v|<\bar{C}_3\}\\ R>\widehat{R}}} S_{x_0,R}\subset\{|x_N|>\zeta\},$$

where ζ has been defined in Remark 6.5.5. Furthermore, for every $N\geq 2$ we can choose $\tau>0$ such that, if $x\in\{|x_N|>M_2\}$, there exist $\tilde{x}\in\{|u-v|<\bar{C}_3\}$ and $\tilde{R}>\widehat{R}$ such that

$$\overline{Q}_x\subset S_{\tilde{x},\tilde{R}},$$

where Q_x denotes the open cube centred in x with side $x_N/100$.

Proof. Thanks to Lemma 6.5.4, it is not difficult to see that, provided $\bar{R}$ is sufficiently large and $\widehat{R}>\bar{R}$, it results

$$\bigcup_{\substack{x_0\in\{|u-v|<\bar{C}_3\}\\ R>\widehat{R}}} S_{x_0,R}\subset\bigcup_{\substack{x_0\in\{|u-v|<\bar{C}_3\}\\ R>\bar{R}}} S_{x_0,R}\subset\{|x_N|>\zeta\}.$$

Now we argue in $\mathbb{R}^N_+$ showing that there exists $M_2=M_2(\widehat{R})>\zeta$ such that

$$\{x_N>M_2\}\subset\bigcup_{\substack{x_0\in\{|u-v|<\bar{C}_3\}\\ R>\widehat{R}}} S^+_{x_0,R},$$

and that for every $x\in\{x_N>M_2\}$ there exist the desired $\tilde{x}$ and $\tilde{R}$. For $x\gg 1$, let $\tilde{x}$ be the point of $\{|u-v|<\bar{C}_3\}$ such that $\tilde{x}'=x'$ ($\tilde{x}$ exists, see Lemma 6.5.4). Provided τ is not too small, the cube centred in x with side $x_N/100$ is contained in the conical sector $S^+_{\tilde{x},\tilde{R}}$ for $\tilde{R}:=3(x_N-\tilde{x}_N)/2$. Note that,

$$\frac{3}{2}(x_N-\tilde{x}_N)\geq\frac{3}{2}(x_N-\zeta)\geq\frac{5}{4}x_N>\widehat{R}.$$

whenever $x_N>M_2:=\max\left\{6\zeta,4\widehat{R}/5\right\}$. The same argument works in the half-space $\mathbb{R}^N_-$. □

Remark 6.5.10. From the previous proof we see that, fixed $\widehat{R}>\bar{R}$, it is possible to associate to every $x\in\{|x_N|>M_2\}$ the conical sector $S_{\tilde{x},\tilde{R}}$ which contains the cube Q_x; that is, $\tilde{x}$ is a point of $\{|u-v|<\bar{C}_3\}$ such that $\tilde{x}'=x'$ and

$$\tilde{R}=\begin{cases}\frac{3}{2}(x_N-\tilde{x}_N) & \text{if } x_N>M_2\\ \frac{3}{2}(\tilde{x}_N-x_N) & \text{if } x_N<-M_2.\end{cases}$$

In each $S_{x_0,R}$ we can obtain a further improvement, by means of Lemma 6.5.6, of the estimates of Lemma 6.5.7.

Lemma 6.5.11. *Let* (u,v) *be a solution of* (6.1) *satisfying* (h1) *and* (h2)*. For every* $\varepsilon > 0$*, if* $R > R_\varepsilon$ *and* $x_0 \in \{|u-v| < \bar{C}_3\}$ *then*

$$\sup_{x \in S_{x_0,R}} \left| \frac{u(x) - \gamma \frac{\sqrt{H(x_0,R)}}{R}(x_N - x_{0,N})^+}{|x-x_0|} \right| + \left| \frac{v(x) - \gamma \frac{\sqrt{H(x_0,R)}}{R}(x_N - x_{0,N})^-}{|x-x_0|} \right| < \varepsilon,$$

with $\bar{C}_5 \le \sqrt{H(x_0,R)}/R \le \bar{C}_6$*. We recall that* $\bar{C}_3, \bar{C}_5, \bar{C}_6$ *and* R_ε *have been defined in Remark 6.3.4, Lemma 6.5.6 and Lemma 6.5.7 respectively.*

Proof. Lemma 6.5.7 ensures that for every $R > R_\varepsilon$, for every $x_0 \in \{|u-v| < \bar{C}_3\}$

$$\sup_{x \in S_{0,1}} \left| \frac{u(x_0+Rx)}{\sqrt{H(x_0,R)}} - \gamma x_N^+ \right| + \left| \frac{v(x_0+Rx)}{\sqrt{H(x_0,R)}} - \gamma x_N^- \right| < \varepsilon,$$

that is,

$$\left| u(x_0+Rx) - \gamma\sqrt{H(x_0,R)}x_N^+ \right| + \left| v(x_0+Rx) - \gamma\sqrt{H(x_0,R)}x_N^- \right| < \sqrt{H(x_0,R)}\varepsilon$$

for every $x \in S_{0,1}$. Consequently, dividing both the sides for R we obtain

$$|x| \left(\left| \frac{u(x_0+Rx)}{|Rx|} - \gamma \frac{\sqrt{H(x_0,R)}}{R} \frac{Rx_N^+}{|Rx|} \right| + \left| \frac{v(x_0+Rx)}{|Rx|} - \gamma \frac{\sqrt{H(x_0,R)}}{R} \frac{Rx_N^-}{|Rx|} \right| \right) < \frac{\sqrt{H(x_0,R)}}{R}\varepsilon$$

for every $x \in S_{0,1}$, provided $R > R_\varepsilon$ and $x_0 \in \{|u-v| < \bar{C}_3\}$. In turns, this gives

$$\sup_{x \in S_{x_0,R}} \left| \frac{u(x) - \gamma \frac{\sqrt{H(x_0,R)}}{R}(x_N - x_{0,N})^+}{|x-x_0|} \right| + \left| \frac{v(x) - \gamma \frac{\sqrt{H(x_0,R)}}{R}(x_N - x_{0,N})^-}{|x-x_0|} \right| < 2\frac{\sqrt{H(x_0,R)}}{R}\varepsilon$$

for every $R > R_\varepsilon$ and $x_0 \in \{|u-v| < \bar{C}_3\}$. Finally, we can use the upper bound on $\sqrt{H(x_0,R)}/R$, see Lemma 6.5.6. □

We are ready to apply the gradient estimates for the Poisson equation in a half-space $x_N \gg 1$; we will show that if $x_N > 0$ is sufficiently large then there exists a linear function φ_x (depending on x) which approximates u in a $\mathcal{C}^1$-sense in x. In light of the uniform control given in Lemma 6.5.6, the slope of φ_x will turn to be uniformly bounded from below in an entire half-space (the same holds for v in $x_N \ll -1$), allowing to conclude the proof of Proposition 6.5.1. It is essential to work in conical sectors, because in this way we can control the quantity $|x - x_0|$ with the privileged component $|x_N - x_{0,N}|$.

Lemma 6.5.12. *Let* (u, v) *be a solution of* (6.1) *satisfying* (h1) *and* (h2)*. For every* $\varepsilon > 0$ *there exists* $M_\varepsilon > 0$ *such that*

$$\left|\nabla u(x) - \gamma \frac{\sqrt{H(\tilde{x}, \tilde{R})}}{\tilde{R}} e_N\right| < \varepsilon \qquad \forall x \in \{x_N > M_\varepsilon\},$$

where $\tilde{x}$ *and* $\tilde{R}$ *have been defined in Remark 6.5.10. Analogously,*

$$\left|\nabla v(x) - \gamma \frac{\sqrt{H(\tilde{x}, \tilde{R})}}{\tilde{R}} e_N\right| < \varepsilon \qquad \forall x \in \{x_N < -M_\varepsilon\}.$$

Proof. For every $\varepsilon > 0$, let R_ε be defined in Lemma 6.5.7. Let $M_{2,\varepsilon} := M_2(\max\{\bar{R}, R_\varepsilon\})$, where M_2 has been defined in Lemma 6.5.9. Let $M_\varepsilon := \max\{M_1, M_{2,\varepsilon}\}$, where M_1 has been defined in Remark 6.5.3. For $x \in \{x_N > M_\varepsilon\}$, there are $\tilde{R} > R_\varepsilon$ and $\tilde{x} \in \{|u - v| < \bar{C}_3\}$ such that $\overline{Q}_x \subset S^+_{\tilde{x}, \tilde{R}}$, see Lemma 6.5.9 and Remark 6.5.10. By the gradient estimates for the Poisson equation (see [45], Section 3.4) plus Lemmas 6.5.2 and 6.5.11, we deduce that

$$\begin{aligned}\left|\nabla u(x) - \gamma \frac{\sqrt{H(\tilde{x}, \tilde{R})}}{\tilde{R}} e_N\right| &\le \frac{C}{x_N} \sup_{y \in \overline{Q}_x} \left|u(y) - \gamma \frac{\sqrt{H(\tilde{x}, \tilde{R})}}{\tilde{R}} (y_N - \tilde{x}_N)\right| \\ &\quad + \frac{x_N}{2} \sup_{y \in \overline{Q}_x} v^2(y) u(y) \\ &\le \frac{C}{x_N} \sup_{y \in \overline{Q}_x} \varepsilon |y - \tilde{x}| + C x_N e^{-C x_N}.\end{aligned} \tag{6.43}$$

As $\overline{Q}_x \subset S^+_{\tilde{x}, \tilde{R}}$, for every $y \in \overline{Q}_x$ it results

$$\begin{aligned}|y - \tilde{x}| &< (\tau + 1)(y_N - \tilde{x}_N) \le (\tau + 1)(y_N - x_N) + (\tau + 1)(x_N - \tilde{x}_N) \\ &\le C x_N + (\tau + 1)(x_N + \zeta) \le C x_N,\end{aligned}$$

where we recall that $\zeta = \sup\{x_{0,N} : x_0 \in \{u = v\}\} < M_\varepsilon < x_N$. Plugging this estimate into the (6.43), we obtain

$$\left| \nabla u(x) - \gamma \frac{\sqrt{H(\tilde{x}, \tilde{R})}}{\tilde{R}} e_N \right| \leq C\varepsilon + C x_N e^{-C x_N}$$

whenever $x_N > M_\varepsilon$; if necessary, we can replace M_ε with a larger quantity, obtaining the thesis for u.
A similar argument can be carried on for v. □

Conclusion of the proof of Proposition 6.5.1. Given $\nu \in \{\nu \in \mathbb{S}^{N-1} : \langle \nu, e_N \rangle > 0\}$, we choose

$$0 < \varepsilon(\nu) \leq \frac{\gamma \bar{C}_5}{2} \langle e_N, \nu \rangle.$$

where $\bar{C}_5$ has been defined in Lemma 6.5.6. It results

$$\begin{aligned} \partial_\nu u(x) &= \left\langle \nabla u(x) - \gamma \frac{\sqrt{H(\tilde{x}, \tilde{R})}}{\tilde{R}} e_N, \nu \right\rangle + \gamma \frac{\sqrt{H(\tilde{x}, \tilde{R})}}{\tilde{R}} \langle e_N, \nu \rangle \\ &\geq -\varepsilon(\nu) + \gamma \bar{C}_5 \langle e_N, \nu \rangle > 0 \end{aligned}$$

for every $x \in \{x_N > M_\nu\}$, where $M_\nu := M_{\varepsilon(\nu)}$ has been defined in Lemma 6.5.12. The same argument gives the monotonicity of v for $x_N \ll 1$. □

With a slightly modification of the conclusion of the proof, we obtain also the

Corollary 6.5.13. *If we consider* $\Theta := \{\nu \in \mathbb{S}^{N-1} : \langle e_N, \nu \rangle \geq \hat{C}\}$ *with* $\hat{C} \in (0, 1]$, *then there exists* $M_\Theta > 0$ *such that*

$$\begin{aligned} x \in \{x_N > M_\Theta\} &\Rightarrow \partial_\nu u(x) > 0 \qquad \forall \nu \in \Theta \\ x \in \{x_N < -M_\Theta\} &\Rightarrow \partial_\nu v(x) < 0 \qquad \forall \nu \in \Theta. \end{aligned}$$

6.6 Monotonicity in the e_N direction

We are going to apply the moving planes method in order to show that u and v are monotone in the e_N direction in the whole $\mathbb{R}^N$. To be precise:

Proposition 6.6.1. *Let* (u, v) *be a solution of* (6.1) *satisfying* (h1) *and* (h2). *Then*

$$\frac{\partial u}{\partial x_N} > 0 \quad \textit{and} \quad \frac{\partial v}{\partial x_N} < 0 \quad \textit{in } \mathbb{R}^N.$$

In what follows we will use many times the following version of the maximum principle in unbounded domains, Lemma 2.1 in [10].

Lemma 6.6.2. *Let D be an open connected subset of $\mathbb{R}^N$, possibly unbounded. Assume that $\overline{D}$ is disjoint from the closure of an infinite open connected cone. Suppose that, for a function $c \in L^\infty_{\text{loc}}(D)$, $c \le 0$ a.e. in D, we have*

$$\begin{cases} \Delta v + c(x)v \ge 0 & \text{in } D \\ v \le 0 & \text{on } \partial D, \end{cases}$$

where $v \in \mathcal{C}^0(\overline{D}) \cap W^{2,N}_{\text{loc}}(D)$ and $v^+ \in L^\infty(D)$, that is, v is bounded above. Then $v \le 0$ in D.

We postpone the proof of Proposition 6.6.1 after the following lemma, which is a consequence of the uniform estimate given in Corollary 6.3.6.

Lemma 6.6.3. *Let (u, v) be a solution of* (6.1) *satisfying* (h1) *and* (h2). *Then for every $M > 0$ there exists $\bar{C}_M > 0$ such that*

$$u(x) + |\nabla u(x)| \le \bar{C}_M \quad \forall x \in \mathbb{R}^{N-1} \times (-\infty, M],$$
$$v(x) + |\nabla v(x)| \le \bar{C}_M \quad \forall x \in \mathbb{R}^{N-1} \times [-M, +\infty).$$

Proof. We prove only the first inequality. Under our assumptions, we know that (u, v) has linear growth (see Proposition 6.4.1). For any $x \in \mathbb{R}^N$, let $\tilde{x} \in \{|u - v| < \bar{C}_3\}$ such that $\tilde{x}' = x'$ and let $\tilde{R} = 3|x_N - \tilde{x}_N|/2$, so that $x \in B_{\tilde{R}}(\tilde{x})$ ($\tilde{x}$ exists, see Lemma 6.5.4). By means of Corollary 6.3.6 we deduce that

$$u(x) + v(x) \le \sup_{y \in B_{\tilde{R}}(\tilde{x})} \bar{C}_4 \left(1 + \frac{3}{2}|x_N - \tilde{x}_N|\right) \le \frac{3}{2}\bar{C}_4 \left(\frac{2}{3} + \zeta + |x_N|\right) \qquad \forall x \in \mathbb{R}^N, \tag{6.44}$$

where ζ has been defined in Remark 6.5.5. Now, let M_1 be defined in Remark 6.5.3, so that

$$uv^2 \le Ce^{-C|x_N|} \qquad \text{in } \{x_N < -M_1\}.$$

Moreover, by (h2) there exist $M_3 > 0$ such that $u \le 1$ in $\mathbb{R}^{N-1} \times (-\infty, -M_3 + 1/2]$. we set $M_4 := \max\{M_1, M_3\}$ and we take any $M > M_4$.

By (6.44), it results

$$u(x', x_N) \le \begin{cases} \frac{3}{2}\bar{C}_4 \left(\frac{2}{3} + \zeta + M\right) & \text{if } x \in \{|x_N| \le M\} \\ 1 & \text{if } x \in \{x_N \le -M\} \end{cases}$$
$$\le 1 + \frac{3}{2}\bar{C}_4 \left(\frac{2}{3} + \zeta + |x_N|\right) =: C_{1,M}$$

whenever $(x', x_N) \in \mathbb{R}^{N-1} \times (-\infty, M]$. Clearly, if $M \le M_4$ the same bound holds.

Now we pass to the estimate on the gradient. In $\mathbb{R}^{N-1} \times [-M-1/2, M+1/2]$ both u and uv^2 are uniformly bounded thanks to (6.44). Also, by definition of M_1 and M_3, both u and uv^2 are uniformly bounded in $\mathbb{R}^{N-1} \times (-\infty, -M]$. Altogether, this means that u and uv^2 are uniformly bounded in $\mathbb{R}^{N-1} \times (-\infty, M+1/2]$, so that we can apply the standard gradient estimates for the Poisson equation (see [45], Section 3.4) in cubes of side 1, obtaining the existence of $C_{2,M} > 0$ such that $|\nabla u(x)| \le C_{2,M}$ for every $x \in \mathbb{R}^{N-1} \times (-\infty, M]$.

The thesis is then satisfied with $\bar{C}_M := \max\{C_{1,M}, C_{2,M}\}$. □

Proof of Proposition 6.6.1. We introduce the classical notation for the moving planes method: for $\lambda \in \mathbb{R}$, we set

$$u_\lambda(x', x_N) := u(x', 2\lambda - x_N), \quad v_\lambda(x', x_N) := v(x', 2\lambda - x_N),$$

and $T_\lambda := \{x_N > \lambda\}$. We aim at proving that

$$u_\lambda(x) \le u(x) \quad \text{and} \quad v_\lambda(x) \ge v(x) \quad \forall x \in T_\lambda, \ \forall \lambda \in \mathbb{R}. \tag{6.45}$$

This and the strong maximum principle give the desired monotonicity.
To prove that (6.45) is satisfied, we show that

$$\Sigma := \{\lambda \in \mathbb{R} : u_\theta \le u \text{ and } v_\theta \ge v \text{ in } T_\theta \text{ for every } \theta \ge \lambda\} = \mathbb{R}.$$

Step 1) *There exists $\bar{M} > 0$ such that if $\lambda > \bar{M}$ then $u_\lambda \le u$ and $v_\lambda \ge v$ in T_λ.*
Let $M_N := M_{e_N}$, where M_{e_N} has been defined in Proposition 6.5.1. Let

$$K := \sup\{u(x) : x_N < M_N\} < +\infty.$$

By assumption (h2), for every $\delta > 0$ there exists $\bar{M} > 0$ such that

$$u(x) > K \quad \text{and} \quad v(x) < \delta \quad \text{in } \{x_N > 2\bar{M} - M_N\}. \tag{6.46}$$

Let $\lambda > \bar{M}$. If $x \in \{x_N \ge 2\lambda - M_N\}$, then $x_N \ge 2\bar{M} - M_N$ and $2\lambda - x_N \le M_N$, so that by definition

$$u_\lambda(x) = u(x', 2\lambda - x_N) \le K \le u(x).$$

To prove that $u_\lambda \le u$ in T_λ for every $\lambda > \bar{M}$, it remains to show that if $\lambda > \bar{M}$ then $u_\lambda \le u$ in $\{\lambda < x_N < 2\lambda - M_N\}$. If $x \in \{\lambda < x_N < 2\lambda - M_N\}$, then $x_N > 2\lambda - x_N > M_N$, so that the fact that $u_\lambda(x) \le u(x)$ follows directly from the monotonicity of u in the e_N

direction for $\{x_N > M_N\}$.
Now, let us show that if $\lambda > \bar{M}$ then $v_\lambda \geq v$ in T_λ. Since $u_\lambda \leq u$ in T_λ, we have

$$\begin{cases} \Delta(v - v_\lambda) - u_\lambda^2(v - v_\lambda) \geq 0 & \text{in } T_\lambda \\ v - v_\lambda = 0 & \text{on } \partial T_\lambda, \end{cases}$$

and $(v - v_\lambda)^+ \leq v \leq \delta$ in T_λ (see equation (6.46)). Consequently, we are in position to apply Lemma 6.6.2, obtaining $v - v_\lambda \leq 0$ in T_λ.

Step 2) $\Sigma = \mathbb{R}$.
In the first step we showed that $\Sigma \neq \emptyset$. Note that Σ is closed and contains the unbounded interval $(\bar{M}, +\infty)$. Assume by contradiction that $\Sigma \neq \mathbb{R}$, that is, $\Lambda := \inf \Sigma > -\infty$. Then there exist sequences $(\lambda_i) \subset \mathbb{R}$ and $(x^i) \subset T_{\lambda_i}$ such that $\lambda_i < \Lambda$ and $\lambda_i \to \Lambda$ as $i \to \infty$, and at least one between

$$u_{\lambda_i}(x^i) > u(x^i) \tag{6.47a}$$

$$v_{\lambda_i}(x^i) < v(x^i) \tag{6.47b}$$

holds true, for every i.

Assume that (6.47a) holds true. We claim that the sequence $(x_N^i) \subset \mathbb{R}$ is bounded. If not, as $x_N^i > \lambda_i$ and λ_i is bounded, up to a subsequence $x_N^i \to +\infty$ as $i \to \infty$. It follows that $2\lambda_i - x_N^i \to -\infty$, and in light of assumption (h2) we obtain

$$\lim_{i\to\infty} u_{\lambda_i}(x^i) = \lim_{i\to\infty} u((x^i)', 2\lambda_i - x_N^i) = 0 \quad \text{and} \quad \lim_{i\to\infty} u(x^i) = +\infty,$$

in contradiction with (6.47a) for i sufficiently large. Hence the claim is proved and, up to a subsequence, $x_N^i \to x_N^\infty$ as $i \to \infty$.

Let us set

$$u^i(x) := u((x^i)' + x', x_N) \quad \text{and} \quad v^i(x) := v((x^i)' + x', x_N).$$

From Lemma 6.6.3 it follows that $\{(u^i, v^i)\}$ is uniformly bounded and equi-Lipschitz-continuous in any compact subset of $\mathbb{R}^N$, so that the standard regularity theory for elliptic equations (see again [45]) implies that up to a subsequence (u^i, v^i) converges in $\mathcal{C}^2_{\text{loc}}(\mathbb{R}^N)$ to a pair (u^∞, v^∞), still solution of (6.1) in $\mathbb{R}^N$.

We wish to show that $x_N^\infty = \Lambda$. From the absurd assumption, equation (6.47a), we obtain

$$\begin{aligned} u_\Lambda^\infty(0', x_N^\infty) = u^\infty(0', 2\Lambda - x_N^\infty) &= \lim_{i\to\infty} u((x^i)', 2\lambda_i - x_N^i) \\ &= \lim_{i\to\infty} u_{\lambda_i}(x^i) \geq \lim_{i\to\infty} u(x^i) = u^\infty(0', x_N^\infty). \end{aligned} \tag{6.48}$$

Let us observe that $((x^i)' + x', x_N) \in T_\Lambda$ whenever $(x', x_N) \in T_\Lambda$. By definition of Λ, $u_\Lambda \le u$ in T_Λ. Consequently, by the convergence of u^i to u^∞ we deduce

$$\begin{aligned} u^\infty_\Lambda(x', x_N) &= \lim_{i\to\infty} u^i(x', 2\Lambda - x_N) = \lim_{i\to\infty} u((x^i)' + x', 2\Lambda - x_N) \\ &\le \lim_{i\to\infty} u((x^i)' + x', x_N) = \lim_{i\to\infty} u^i(x', x_N) = u^\infty(x', x_N) \end{aligned}$$

for every $(x', x_N) \in T_\Lambda$. Analogously, as $v_\Lambda \ge v$ in T_Λ, we have $v^\infty_\Lambda \ge v^\infty$ in T_Λ. Now,

$$\begin{cases} -\Delta(u^\infty - u^\infty_\Lambda) + (v^\infty)^2(u^\infty - u^\infty_\Lambda) = ((v^\infty_\Lambda)^2 - (v^\infty)^2)u^\infty_\Lambda \ge 0 & \text{in } T_\Lambda \\ u^\infty - u^\infty_\Lambda \ge 0 & \text{in } T_\Lambda \\ u^\infty - u^\infty_\Lambda = 0 & \text{on } \partial T_\lambda. \end{cases} \tag{6.49}$$

Furthermore, $u^\infty - u^\infty_\Lambda$ is not identically 0: indeed by assumption (h2)

$$\lim_{x_N\to+\infty} \big(u^\infty(x', x_N) - u^\infty_\Lambda(x', x_N)\big) = +\infty.$$

Hence, the strong maximum principle implies that necessarily $u^\infty - u^\infty_\Lambda > 0$ in T_Λ. A comparison with (6.48) reveals that

$$x^\infty_N = \Lambda.$$

Now, by the absurd assumption (6.47a) we deduce that for every i there exists $\xi^i \in (2\lambda_i - x^i_N, x^i_N)$ such that

$$0 < u_{\lambda_i}(x^i) - u(x^i) = u^i(x', 2\lambda_i - x^i_N) - u^i(x', x_N) = 2\partial_N u^i(x', \xi^i)(\lambda_i - x^i_N);$$

As $\lambda_i < x^i_N$, this implies $\partial_N u^i(x', \xi^i_N) < 0$ for every i. As $\lambda_i \to \Lambda$ and $x^i_N \to \Lambda$ as $i \to \infty$, passing to the limit as $i \to \infty$ we deduce

$$\partial_N u^\infty(0', \Lambda) \le 0. \tag{6.50}$$

On the other hand, thanks to the (6.49) and the fact that $u^\infty - u^\infty_\Lambda > 0$ in T_Λ, we are in position to apply the Hopf lemma:

$$\partial_\nu(u^\infty(0', \Lambda) - u^\infty_\Lambda(0', \Lambda)) < 0,$$

which means

$$2\partial_N u^\infty(0', \Lambda) > 0,$$

in contradiction with (6.50).

The above argument says that (6.47a) cannot occur. With minor changes, we can show that also (6.47b) is not verified, so that $\Sigma = \mathbb{R}$, which completes the proof. □

6.7 1-dimensional symmetry

In this section we complete the proof of our main result, Theorem 6.1.1. We follow the technique introduced in [33]: we show that, starting from Proposition 6.6.1, it is possible to prove that $\partial_\nu u > 0$ and $\partial_\nu v < 0$ for every $\nu \in \mathbb{S}^{N-1}_+ = \{\nu \in \mathbb{S}^{N-1} : \nu_N > 0\}$. The conclusion follows easily.

Proposition 6.7.1. *Let* (u, v) *be a solution of* (6.1) *satisfying* (h1) *and* (h2)*. Then* (u, v) *depends only on* x_N*.*

Proof. We divide the proof in several steps.

Step 1) *For every* $\sigma > 0$ *there exists* $\varepsilon = \varepsilon(\sigma) > 0$ *such that*

$$\partial_N u(x) \geq \varepsilon \quad \textit{and} \quad \partial_N v(x) \leq -\varepsilon \quad \forall x \in \overline{S}_\sigma,$$

where $S_\sigma := \mathbb{R}^{N-1} \times (-\sigma, \sigma)$.
By contradiction, fixed $\sigma > 0$, assume that there exists $(x^i) \subset S_\sigma$ such that at least one between

$$\lim_{i \to +\infty} \frac{\partial u}{\partial x_N}(x^i) = 0 \tag{6.51a}$$

$$\lim_{i \to +\infty} \frac{\partial v}{\partial x_N}(x^i) = 0 \tag{6.51b}$$

holds true. Only to fix our minds, assume that (6.51a) holds. We define

$$u^i(x) := u(x + x^i) \quad \text{and} \quad v^i(x) := v(x + x^i).$$

Note that $|x^i_N| \leq \sigma$ for every i, so that for any compact set $K \subset \mathbb{R}^N$ there exists $M > 0$ such that $x + x^i \in S_M$ for every $x \in K$. Lemma 6.6.3 and standard elliptic estimates say that, up to a subsequence, $(u^i, v^i) \to (u^\infty, v^\infty)$ in $\mathcal{C}^2_{\text{loc}}(\mathbb{R}^N)$, where (u^∞, v^∞) is still a solution to (6.1). By the convergence, we have

$$\frac{\partial u^\infty}{\partial x_N} \geq 0 \quad \text{and} \quad \frac{\partial v^\infty}{\partial x_N} \leq 0 \quad \text{in } \mathbb{R}^N,$$

and $\partial_N u^\infty(0) = 0$. Furthermore,

$$-\Delta\left(\partial_N u^\infty\right) + (v^\infty)^2\left(\partial_N u^\infty\right) = -2u^\infty v^\infty\left(\partial_N v^\infty\right) \geq 0 \quad \text{in } \mathbb{R}^N.$$

The strong maximum principle implies that either $\partial_N u^\infty > 0$ or $\partial_N u^\infty \equiv 0$. The former case is in contradiction with the fact that $\partial_N u^\infty(0) = 0$, the latter one is in contradiction with assumption (h2), which is also satisfied by the limiting profile (u^∞, v^∞). Thus, (6.51a) cannot occur. A similar argument shows that also (6.51b) does not hold.

Step 2) *For every $\sigma > 0$, the map $\nu \mapsto (\partial_\nu u, \partial_\nu v)$ is in $\mathcal{C}^{0,1}\left(\mathbb{S}^{N-1}, \left(\mathcal{C}^0(\overline{S}_\sigma)\right)^2\right)$.*
By Lemma 6.6.3, we know that $|\nabla u| + |\nabla v| \le \bar{C}_\sigma$ in $\overline{S}_\sigma$. Hence

$$\left|\frac{\partial u}{\partial \nu_1}(x) - \frac{\partial u}{\partial \nu_2}(x)\right| + \left|\frac{\partial v}{\partial \nu_1}(x) - \frac{\partial v}{\partial \nu_2}(x)\right| \le 2\bar{C}_\sigma |\nu_1 - \nu_2|$$

for every $x \in \overline{S}_\sigma$.

Step 3) *u is strictly increasing and v is strictly decreasing with respect to all the unit vectors of an open neighbourhood of e_N in $\mathbb{S}^{N-1}$.*
Let $\Theta := \left\{\nu \in \mathbb{S}^{N-1} : \langle e_N, \nu\rangle \ge \frac{1}{2}\right\}$. By Corollary 6.5.13, we know that there exists M_Θ such that

$$\frac{\partial u}{\partial \nu} > 0 \quad \text{in } \{x_N > M_\Theta\} \quad \text{and} \quad \frac{\partial v}{\partial \nu} < 0 \quad \text{in } \{x_N < -M_\Theta\},$$

for every $\nu \in \Theta$. Let $\sigma > M_\Theta$. Using steps 1) and 2), we deduce that there exists an open neighbourhood $\mathcal{O}_{e_N}$ of e_N in $\mathbb{S}^{N-1}$ such that

$$\frac{\partial u}{\partial \nu}(x) > 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu}(x) < 0 \quad \forall x \in S_\sigma,\ \forall \nu \in \mathcal{O}_{e_N}. \tag{6.52}$$

We can assume that $\mathcal{O}_{e_N} \subset \Theta$ (if not, we replace $\mathcal{O}_{e_N}$ with a smaller neighbourhood). This means that, for every $\nu \in \mathcal{O}_{e_N}$, it results

$$\frac{\partial u}{\partial \nu} > 0 \quad \text{in } \{x_N > -\sigma\} \quad \text{and} \quad \frac{\partial v}{\partial \nu} < 0 \quad \text{in } \{x_N < \sigma\},$$

Furthermore, for every $\nu \in \mathcal{O}_{e_N}$

$$\begin{cases} \Delta(-\partial_\nu u) - v^2(-\partial_\nu u) = -2uv\partial_\nu v \ge 0 & \text{in } \mathbb{R}^{N-1} \times (-\infty, -\sigma) \\ -\partial_\nu u \le 0 & \text{on } \partial\left(\mathbb{R}^{N-1} \times (-\infty, -\sigma)\right) \\ -\partial_\nu u \in L^\infty\left(\mathbb{R}^{N-1} \times (-\infty, -\sigma)\right), \end{cases}$$

where the last one follows from Lemma 6.6.3. We are then in position to apply Lemma 6.6.2, obtaining $\partial_\nu u \ge 0$ in $\mathbb{R}^{N-1} \times (-\infty, -\sigma)$. Together with (6.52), this gives $\partial_\nu u \ge 0$ in $\mathbb{R}^N$ for every $\nu \in \mathcal{O}_{e_N}$. Similarly, from

$$\begin{cases} \Delta(\partial_\nu v) - u^2(\partial_\nu v) = 2uv\partial_\nu u \ge 0 & \text{in } \mathbb{R}^{N-1} \times (\sigma, +\infty) \\ \partial_\nu v \le 0 & \text{on } \partial\left(\mathbb{R}^{N-1} \times (\sigma, +\infty)\right) \\ \partial_\nu v \in L^\infty\left(\mathbb{R}^{N-1} \times (\sigma, +\infty)\right), \end{cases}$$

we deduce $\partial_\nu v \le 0$ in $\mathbb{R}^N$ for every $\nu \in \mathcal{O}_{e_N}$. Finally, the strong maximum principle provides $\partial_\nu u > 0$ and $\partial_\nu v < 0$ in $\mathbb{R}^N$, for every $\nu \in \mathcal{O}_{e_N}$.

Step 4) *u is strictly increasing and v is strictly decreasing with respect to all the directions of the upper hemisphere* $\mathbb{S}_+^{N-1} = \{\nu \in \mathbb{S}^{N-1} : \langle e_N, \nu\rangle > 0\}$.
Let Ω be the set of $\nu \in \mathbb{S}_+^{N-1}$ for which there exists an open neighbourhood $\mathcal{O}_\nu \subset \mathbb{S}^{N-1}$ of ν such that

$$\frac{\partial u}{\partial \mu} > 0 \quad \text{and} \quad \frac{\partial v}{\partial \mu} < 0 \quad \text{in } \mathbb{R}^N, \ \forall \mu \in \mathcal{O}_\nu.$$

The set Ω is open by definition, and contains e_N for the previous step. If we show that it is closed with respect to the topology of $\mathbb{S}_+^{N-1}$, then $\Omega = \mathbb{S}_+^{N-1}$ and the claim is proved. Let $\bar\nu$ be a cluster point of Ω (note that $\langle e_N, \bar\nu\rangle > 0$, since we are considering the topology of $\mathbb{S}_+^{N-1}$), that is, there exists $(\nu_n) \subset \Omega$ such that $\nu_n \to \bar\nu$. As

$$\frac{\partial u}{\partial \nu_n} > 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu_n} < 0 \quad \text{in } \mathbb{R}^N, \ \forall n,$$

by continuity

$$\frac{\partial u}{\partial \bar\nu} \geq 0 \quad \text{and} \quad \frac{\partial v}{\partial \bar\nu} \leq 0 \quad \text{in } \mathbb{R}^N.$$

The strong maximum principle implies that either $\partial_{\bar\nu} u \equiv 0$ or $\partial_{\bar\nu} u > 0$ in $\mathbb{R}^N$; analogously, either $\partial_{\bar\nu} v \equiv 0$ or $\partial_{\bar\nu} v < 0$ in $\mathbb{R}^N$. As $\bar\nu$ is not orthogonal to e_N, assumption (h2) says that neither $\partial_{\bar\nu} u \equiv 0$ nor $\partial_{\bar\nu} v \equiv 0$ can be satisfied, thus $\partial_{\bar\nu} u > 0$ and $\partial_{\bar\nu} v < 0$ in $\mathbb{R}^N$. It remains to show that there exists an open neighbourhood $\mathcal{O}_{\bar\nu}$ of $\bar\nu$ in $\mathbb{S}_+^{N-1}$ such that for every $\mu \in \mathcal{O}_{\bar\nu}$

$$\frac{\partial u}{\partial \mu} > 0 \quad \text{and} \quad \frac{\partial v}{\partial \mu} < 0 \quad \text{in } \mathbb{R}^N.$$

It is possible to adapt the same proof of steps 1) to 3) with minor changes, in order to deduce the existence of $\mathcal{O}_{\bar\nu}$ (in the third step we replace Θ with $\{\nu \in \mathbb{S}^{N-1} : \langle e_N, \nu\rangle \geq \frac{1}{2}\langle e_N, \bar\nu\rangle > 0\}$). Consequently, $\bar\nu \in \Omega$ and Ω is closed with respect to the topology of $\mathbb{S}_+^{N-1}$.

Step 5) *Conclusion of the proof.*
Since $\Omega = \mathbb{S}_+^{N-1}$, by continuity we have

$$\frac{\partial u}{\partial \nu} \geq 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu} \leq 0 \quad \text{in } \mathbb{R}^N$$

for every ν which is orthogonal to e_N. But also $-\nu$ is orthogonal to e_N, so that

$$\frac{\partial u}{\partial \nu} \equiv 0 \quad \text{and} \quad \frac{\partial v}{\partial \nu} \equiv 0 \quad \text{in } \mathbb{R}^N$$

for every ν orthogonal to e_N. In particular

$$\frac{\partial u}{\partial x_i} \equiv 0 \quad \text{and} \quad \frac{\partial v}{\partial x_i} \equiv 0 \quad \text{in } \mathbb{R}^N, \text{for } i = 1, \dots, N-1. \qquad \square$$

6.8 Proof of Corollary 6.1.2

We will show that if (u, v) is a solution of (6.1) with algebraic growth and (h3) holds true, then (h2) is satisfied.

Proof of Corollary 6.1.2. Firstly, let us observe that, since $u, v > 0$, (h3) implies

$$\lim_{x_N \to +\infty} u(x', x_N) = +\infty \quad \text{and} \quad \lim_{x_N \to -\infty} v(x', x_N) = +\infty \tag{6.53}$$

uniformly in $x' \in \mathbb{R}^{N-1}$. Thus, in order to obtain the thesis it remains to show that under (h1) and (h3) we have

$$\lim_{x_N \to -\infty} u(x', x_N) = 0 \quad \text{and} \quad \lim_{x_N \to +\infty} v(x', x_N) = 0 \tag{6.54}$$

We prove only the second one in (6.54), for the first one it is possible to argue in the same way.

Step 1) *under* (h1) *and* (h3)*,* (u, v) *has linear growth.*
Given $K > 0$, by (h3) there exists $M > 0$ such that $u(x) > K$ if $x \in \{x_N > M/2\}$. For an arbitrary $\theta > 1$, if $x \in \{x_N > M,\ |x'| < \theta x_N\}$ the ball $B_x := B_{x_N/100}(x)$ is contained in $\{x_N > M/2,\ |x'| < 2\theta x_N\}$. Consequently, if $x \in \{x_N > M,\ |x'| < \theta x_N\}$ we have

$$u(y) \geq K_x := \inf_{z \in B_x} u(z) \geq K \qquad \forall y \in B_x,$$

and

$$v(y) \leq C(1 + |y|^p) \leq C\left(1 + (2\theta + 1)^p y_N^p)\right) \leq C(1 + x_N^p) \qquad \forall y \in B_x.$$

The latter one gives $\delta_x := \sup_{y \in B_x} v(y) \leq C(1 + x_N^p)$. Now,

$$\begin{cases} -\Delta v \leq -K^2 v & \text{in } B_x \\ v \geq 0 & \text{in } B_x \\ v \leq \delta_x & \text{in } B_x, \end{cases}$$

and we are in position to apply Lemma 6.2.1: it follows

$$v(x) \leq C\delta_x e^{-CKx_N} \leq C(1 + x_N^p)e^{-CKx_N} \qquad \forall x \in \left\{x_N > M,\ |x'| < \theta x_N\right\}. \tag{6.55}$$

Let us consider the blow-down family $(u_{0,R}, v_{0,R}) =: (u_R, v_R)$. In light of the algebraic growth of (u, v), Theorem 6.2.13 applies: there exists a homogeneous harmonic polynomial Ψ of degree $d \in \mathbb{N} \setminus \{0\}$ such that, up to a subsequence, (u_R, v_R) converges to (Ψ^+, Ψ^-) in $\mathcal{C}^0_{\text{loc}}(\mathbb{R}^N)$ as $R \to +\infty$. On the other hand, let $x \in \{|x'| < \theta x_N\}$; there exists $R_x > 0$ such that $Rx \in \{x_N > M, \; |x'| < \theta x_N\}$ for every $R > R_x$. By means of (6.55), we deduce that

$$\lim_{R\to+\infty} v_R(x) = \lim_{R\to+\infty} \frac{1}{\sqrt{H(0,R)}} v(Rx) = 0 \qquad \forall x \in \left\{|x'| < \theta x_N\right\},$$

where we used also Corollary 6.2.14 to ensure that $H(0,R)$ does not tend to 0. As θ has been arbitrarily chosen, we deduce that $v_R \to 0$ pointwise in $\mathbb{R}^N_+$. By the uniqueness of the limit, Ψ has to be a homogeneous harmonic polynomial which vanishes in the entire half-space $\mathbb{R}^N_+$: as showed in the proof of Proposition 6.4.1, necessarily Ψ is a linear function and $d = 1$. By means of Corollary 6.2.8, we deduce that (u, v) has linear growth.

Step 2) *Conclusion of the proof.*
As (u, v) has linear growth, we can choose $\bar{C}_3$ as in Remark 6.3.4. Assumption (h3) is sufficient to prove Lemma 6.5.4: $\{|u - v| < \bar{C}_3\}$ is bounded in the e_N direction and unbounded in all the other directions. Consequently, also Lemma 6.5.9 applies: for $\hat{R} \geq \bar{R}$, we can find M_2 as in the quoted statement.
Given $K > 0$, by (6.53) there exists $M > 0$ such that if $x \in \{x_N > M/2\}$ then $u(x) \geq K$. Let $M_5 := \max\{M, M_2\}$, so that

$$\{x_N > M_5\} \subset \bigcup_{\substack{x_0 \in \{|u-v|<\bar{C}_3\} \\ R > \hat{R}}} S^+_{x_0,R}.$$

If $x \in \{x_N > M_5\}$ then the ball $B_x := B_{x_N/100}(x)$ is contained in $\{x_N > M/2\}$, so that

$$\begin{cases} -\Delta v \leq -K^2 v & \text{in } B_x \\ v \geq 0 & \text{in } B_x \\ v \leq \delta_x & \text{in } B_x, \end{cases}$$

where $\delta_x := \sup_{B_x} v < +\infty$, because $v \in L^\infty_{\text{loc}}(\mathbb{R}^N)$. From Lemma 6.2.1 we obtain

$$v(x) \leq C \left(\sup_{y \in B_x} v(y) \right) e^{-CKx_N}. \tag{6.56}$$

To control $\sup_{B_x} v$, we consider $\tilde{x}$ and $\tilde{R}$ defined in Lemma 6.5.9 and Remark 6.5.10. As $B_x \subset Q_x$, a fortiori $B_x \subset S^+_{\tilde{x},\tilde{R}} \subset B_{\tilde{R}}(\tilde{x})$. We are then in position to apply Corollary 6.3.6:

$$\begin{aligned} \sup_{y \in B_x} v(y) &\leq \bar{C}_4(1 + \tilde{R}) = \bar{C}_4 \left(1 + \frac{3}{2}(x_N - \tilde{x}_N) \right) \\ &\leq \bar{C}_4 \left(1 + \frac{3}{2}\zeta + \frac{3}{2} x_N \right) \leq C x_N \end{aligned}$$

provided x_N is sufficiently large (recall the definition of ζ, Remark 6.5.5). Plugging into (6.56), we see that for every x such that $x_N \gg 1$ is sufficiently large it results

$$v(x) \leq C x_N e^{-CKx_N},$$

which gives the second limit in (6.54). □

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