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THE VALIDITY OF THE EULER-LAGRANGE EQUATION

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To Louis Nirenberg, with admiration

Abstract. We prove the validity of the Euler-Lagrange equation for a solution u to the problem of minimizing $\int_\Omega L(x,u(x),\nabla u(x))\,\mathrm{d}x$, where L is a Carathéodory function, convex in its last variable, without assuming differentiability with respect to this variable.

1. **Introduction.** This paper deals with the necessary conditions satisfied by a locally bounded solution u to the problem of minimizing

$$\int_\Omega L(x,v(x),\nabla v(x))\,\mathrm{d}x \qquad (1)$$

on $v_0+W_0^{1,1}(\Omega)$, where $L(x,v,\xi)$ is a Carathéodory function, differentiable with respect to v, and whose derivative L_v is also a Carathéodory function, and the map $\xi\mapsto L(x,v,\xi)$ is convex and defined on $\mathbb{R}^N$. We do not assume further regularity on L, with the exception of standard growth estimates, described below. For functionals of this form, it has been conjectured that the suitable form of the Euler-Lagrange equations satisfied by u should be

$$\exists\ p(\cdot)\in(\mathrm{L}^1(\Omega))^N,\ \text{a selection from } \partial L_\xi(\cdot,u(\cdot),\nabla u(\cdot)),\ \text{such that}$$
$$\operatorname{div}p(\cdot)=L_v(\cdot,u(\cdot),\nabla u(\cdot))$$

in the sense of distributions. This fact has been proved in a few special cases: in [2] for maps of the form $L(v,\xi)$, jointly convex in (v,ξ), and, more recently, in [1] for maps $L(x,v,\xi)=f(\|\xi\|)+g(x,v)$, depending on ξ through its norm. The proof introduced in [1] is elementary, and it is based on the Riesz representation Theorem and on the Hahn-Banach Theorem. This paper is a sequel to [1] and shows that a modification of the same elementary proof allows us to prove the conjecture in its full generality. The proofs we present are self-contained.

2. **Main results.** We consider $\mathbb{R}^N$ with the Euclidean norm $|\cdot|$ and unit ball $\mathbb{B}$. $\ell(A)$ is the N-dimensional Lebesgue measure of a set A. Given a closed convex $K\subset\mathbb{R}^N$, by m_K we mean the unique point of K of minimal norm and by $\|K\|$ we mean $\sup\{|k|:k\in K\}$; a set valued map K with values in the non-empty compact subsets of $\mathbb{R}^N$ is called *upper semicontinuous* at x_0 if $\forall\varepsilon\exists\delta$ such that $|x-x_0|<\delta$ implies $K(x)\subset K(x_0)+\varepsilon\mathbb{B}$. In this paper we shall also meet real valued upper and lower semicontinuous maps, with the usual definitions.

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Given a function $L(x, v, \xi)$, convex in ξ for each fixed (x, v), by $\partial_\xi L(x, v, \xi)$ we mean the *subdifferential* of L with respect to the variable ξ. Under the assumptions of the present paper, $\partial_\xi L(x, v, \xi)$ is a non-empty compact convex subset of $\mathbb{R}^N$ and the map $\xi \mapsto \partial_\xi L(x, v, \xi)$ is (for fixed (x, v)) an upper semicontinuous set valued map. We shall assume further properties of this map in Assumption A.

$I_A(\cdot)$ is the *indicator function* of the set A. f^* is the *polar* or *Fenchel transform* [4] of f. Ω is a bounded open subset of $\mathbb{R}^N$. Given a solution u, the shorthand notation $D_L(x)$ means the set $\partial_\xi L(x, u(x), \nabla u(x))$.

Assumption A.

i) *$L(x, v, \xi)$ is a Carathéodory function, differentiable with respect to v, and whose derivative L_v is also a Carathéodory function, and, for every pair (x, v), the map $\xi \mapsto L(x, v, \xi)$ is convex and defined on $\mathbb{R}^N$.*

ii) *There exist a convex non-negative function f and constants H_1 and H_2 such that*

$$\|\partial f(\xi)\| \leq H_1 f(\xi) + H_2 \tag{2}$$

and, for every U, there exist functions α_U, β_U and γ_U in $\mathrm{L}^1(\Omega)$ and positive constants h_U^1, h_U^2 and h_U^3, such that $|v| \leq U$ implies

$$\alpha_U(x) + h_U^1 f(\xi) \leq L(x, v, \xi) \tag{3}$$

$$\partial_\xi L(x, v, \xi) \leq \beta_U(x) + h_U^2 \partial f(\xi) \tag{4}$$

$$|L_u(x, v, \xi)| \leq \gamma_U(x) + h_U^3 f(\xi). \tag{5}$$

iii) *For every $\delta > 0$ there exists $\Omega_\delta \subset \Omega$, with $\ell(\Omega \setminus \Omega_\delta) < \delta$, such that the restriction of $\partial_\xi L(x, v, \xi)$ to $\Omega_\delta \times \mathbb{R} \times \mathbb{R}^N$ is upper semicontinuous.*

Assumption A ii) limits the growth of L in the variable ξ to be exponential. This growth limitation still holds, so far, for the proofs of the validity of the Euler-Lagrange equation for variational problems of general form, independently on whether there are additional differentiability assumptions or not. An exception to this statement is the recent paper [3], where no growth limitations are assumed, but for functionals of a special form.

It is our purpose to prove the following

Theorem 2.1. *Let L satisfy Assumption A. Let u be a locally bounded solution to Problem (1). Then,*

$$\exists\ p(\cdot) \in \mathrm{L}^1(\Omega),\ \text{a selection from } \partial_\xi L(\cdot, u(\cdot)\nabla u(\cdot)),$$

such that

$$\operatorname{div} p(\cdot) = L_u(\cdot, u(\cdot), \nabla u(\cdot))$$

in the sense of distributions.

We shall need the following variant of the Riesz Representation Theorem.

Lemma 2.2. *Let D be a map from Ω to the closed convex non-empty subsets of $R\mathbb{B}$, such that $v \in (\mathrm{L}^\infty(\Omega))^N$ implies that the map $x \mapsto m_{[D(x) - v(x)]}$ is measurable; let $T : (\mathrm{L}^1(\Omega))^N \to \mathbb{R}$ be a linear functional satisfying*

$$T(\xi) \leq \int_\Omega (I_{D(x)})^*(\xi(x))\ dx.$$

Then, there exists $\tilde{p} \in (\mathrm{L}^\infty(\Omega))^N$, $\tilde{p}(x)$ a.e. in $D(x)$, that represents T, i.e., such that

$$T(\xi) = \int_\Omega \langle \tilde{p}(x), \xi(x) \rangle\ dx. \tag{6}$$

Proof. a) Since $|(I_{D(x)})^*(\xi(x))| \leq \|D(x)\||\xi(x)|$ we have that T is a bounded linear functional on $(\mathrm{L}^1(\Omega))^N$. Writing ξ as $\xi_1(x)e_1 + \ldots + \xi_N(x)e_N$ and applying the standard Riesz representation Theorem, we infer the existence of a function $\tilde{p} \in (\mathrm{L}^\infty(\Omega))^N$ that satisfies (6). To show that $\tilde{p}(x)$ is in $D(x)$ a.e., assume that there exists a set $E \subset \Omega$ of positive measure such that, on E, $\tilde{p}(x) \notin D(x)$, i.e., $0 \notin D(x) - \tilde{p}(x)$. Setting $D^* := D(x) - \tilde{p}(x)$, we can equivalently say that $|m_{D^*(x)}| > 0$ on E.

Let $z(x)$ be the projection of minimal distance of $\tilde{p}(x)$ on $D(x)$, so that, $z(x) - \tilde{p}(x) = m_{D(x)-\tilde{p}(x)}$ or, $z(x) - \tilde{p}(x) = m_{D^*(x)}$. From the characterization of the projection of minimal distance, we obtain

$$\langle \tilde{p}(x) - z(x), z(x)\rangle \geq \langle \tilde{p}(x) - z(x), k\rangle, \quad \forall k \in D(x),$$

that can be rewritten as

$$\langle -m_{D^*}(x), \tilde{p}(x)\rangle \geq |m_{D^*}(x)|^2 + \langle -m_{D^*}(x), k\rangle, \quad \forall k \in D(x).$$

Hence, we have that, on E,

$$\langle -m_{D^*}(x), \tilde{p}(x)\rangle > \sup\{\langle -m_{D^*}(x), k\rangle : k \in D(x)\} = (I_{D(x)})^*(-m_{D^*}(x)).$$

b) Setting $\tilde{\xi} := -m_{D^*}\chi_E$, we have that $\tilde{\xi} \in \mathrm{L}^1(\Omega)$ and

$$T(\tilde{\xi}) = \int_\Omega \langle \tilde{p}, \tilde{\xi}\rangle = \int_E \langle \tilde{p}, -m_{D^*}\rangle > \int_\Omega (I_{D(x)})^*(\tilde{\xi}) \geq T(\tilde{\xi}),$$

a contradiction. □

Proposition 1. *Let $x \mapsto K(x)$ be an upper semicontinuous set-valued map. Then,* i) *the real valued map $x \mapsto |m_{K(x)}|$ is lower semicontinuous and the real valued map $x \mapsto \|K(x)\|$ is upper semicontinuous;* ii) *the real valued map $(x, \xi) \mapsto (I_{K(x)})^*(\xi)$ is continuous in ξ for each fixed x and upper semicontinuous in x for each fixed ξ.*

Proposition 2. *Let $x \mapsto K(x)$ be an upper semicontinuos set-valued map with values in the closed convex subsets of $\mathbb{R}^N$. Then, $|m_{K(\cdot)}|$ continuous at x_0 implies that $m_{K(\cdot)}$ is continuous at x_0.*

Proof. Fix x_0, a point of continuity of $|m_{K(\cdot)}|$, and consider two cases: i) $0 \notin K(x_0)$ and, ii), $0 \in K(x_0)$.

i) Fix $\varepsilon > 0$, with $\varepsilon < 2\sqrt{2}|m_{K(x_0)}|$. Let $\sigma > 0$ be such that $\left(|m_{K(x_0)}| - \sigma\right)^2 = |m_{K(x_0)}|^2 - \frac{\varepsilon^2}{8}$ and let η be such that $\frac{1}{2}\left(|m_{K(x_0)}|^2 + (|m_{K(x_0)}| + \eta)^2\right) = |m_{K(x_0)}|^2 + \frac{\varepsilon^2}{8}$. Let δ be such that $|x - x_0| < \delta$ implies that both $K(x) \subset K(x_0) + \sigma\mathbb{B}$ and $\big||m_{K(x_0)}| - |m_{K(x)}|\big| < \eta$. As a consequence, from the convexity of $K(x_0) + \sigma\mathbb{B}$, we obtain that $\frac{m_{K(x_0)} + m_{K(x)}}{2} \in K(x_0) + \sigma\mathbb{B}$, so that

$$\left|\frac{m_{K(x_0)} + m_{K(x)}}{2}\right| \geq |m_{K(x_0)}| - \sigma.$$

From the identity

$$\left|\frac{m_{K(x_0)} - m_{K(x)}}{2}\right|^2 = \frac{1}{2}\left(|m_{K(x_0)}|^2 + |m_{K(x)}|^2\right) - \left|\frac{m_{K(x_0)} + m_{K(x)}}{2}\right|^2$$

we obtain

$$\left|\frac{m_{K(x_0)} - m_{K(x)}}{2}\right|^2 \leq \frac{1}{2}\left(|m_{K(x_0)}|^2 + |m_{K(x_0)} + \eta|^2\right) - (|m_{K(x_0)}| - \sigma)^2$$
$$= |m_{K(x_0)}|^2 + \frac{\varepsilon^2}{8} - |m_{K(x_0)}|^2 + \frac{\varepsilon^2}{8} = \frac{\varepsilon^2}{4}.$$

ii) Fix $\varepsilon > 0$; for $\sigma > 0$ such that $|x - x_0| < \sigma$ implies $K(x) \subset K(x_0) + \varepsilon\mathbb{B}$, we have that $m_{K(x)} - 0 \in \varepsilon\mathbb{B}$. □

Lemma 2.3. i) $v \in (\mathrm{L}^\infty(\Omega))^N$ *implies that the map* $x \mapsto m_{[\frac{1}{\|D_L(x)\|}D_L(x)-v(x)]}$ *is in* $(\mathrm{L}^\infty(\Omega))^N$ *and,* ii)*, for* $\xi \in L^1(A)$*, the map* $x \mapsto (I_{[\frac{1}{\|D_L(x)\|}D_L(x)-v(x)]})^*(\xi(x))$ *is in* $L^1(\Omega)$.

Proof. i) Fix ε. Let Ω' be the subset of Ω provided by Assumption A, iii), with $\delta = \frac{\varepsilon}{4}$. Applying Lusin's theorem, there exists $E \subset \Omega'$ with $\ell(\Omega' \setminus E) \leq \frac{\varepsilon}{4}$, such that $u|_E$, $v|_E$ and $\nabla u|_E$ are continuous so that, on E, the set valued map D_L is upper semicontinuous and, by Proposition 1, the real valued map $\|D_L\|$ is upper semicontinuous. Hence, there exists $E' \subset E$, with $\ell(\Omega' \setminus E') \leq \frac{2}{4}\varepsilon$, such that the restriction of $\|D_L\|$ to E' is continuous. Then, on E', the set valued map $x \mapsto \frac{1}{\|D_L(x)\|}D_L(x)$ is upper semicontinuous: in fact, let $x_n \in E'$, $x_n \to x_*$ and $w_n \in \frac{1}{\|D_L(x_n)\|}D_L(x_n)$ with $w_n \to w_*$; then $\|D_L(x_n)\|w_n \to \|D_L(x_*)\|w_*$ that belongs to $D_L(x_*)$, i.e., $w_* \in \frac{1}{\|D_L(x_*)\|}D_L(x_*)$. We have obtained that the restriction to E' of the map $\frac{1}{\|D_L\|}D_L$ has closed graph, and it follows that it is u.s.c. Then, so is the the restriction to E' of the set valued map $\frac{1}{\|D_L\|}D_L - v$. Applying Proposition 1 i), we infer that the restriction to E' of $|m_{[\frac{1}{\|D_L\|}D_L-v]}|$ is lower semicontinuous, hence, for a suitable $E'' \subset E'$ with $\ell(\Omega' \setminus E'') \leq \frac{3}{4}\varepsilon$, its restriction to E'' is continuous. By Proposition 2, the restriction to E'' of $m_{[\frac{1}{\|D_L\|}D_L-v]}$ is continuous, and $\ell(\Omega \setminus E'') \leq \varepsilon$. Being ε arbitrary, $m_{\frac{1}{\|D_L\|}D_L-v}$ is measurable on Ω and belongs to $(\mathrm{L}^\infty(\Omega))^N$.

ii) Consider a simple function $\xi_s = \sum \alpha_i \chi_{A_i}$, with $\cup A_i = E'$; we have

$$(I_{[\frac{1}{\|D_L(x)\|}D_L-v(x)]})^*(\xi_s(x)) = \sum (I_{[\frac{1}{\|D_L(x)\|}D_L-v(x)}})^*(\alpha_i)\chi_{A_i}(x):$$

by Proposition 1 ii), it is upper semicontinuous in x on each A_i, hence measurable on E'. Let (ξ_ν) be a sequence of simple functions, converging to $\xi|_{E'}$. Fix $\tilde{x}$: again by Proposition 1,

$$(I_{[\frac{1}{\|D_L(\tilde{x})\|}D_L(\tilde{x})-v(\tilde{x})]})^*(\xi_\nu(\tilde{x})) \quad \text{converges to} \quad (I_{[\frac{1}{\|D_L(\tilde{x})\|}D_L(\tilde{x})-v(\tilde{x})]})^*(\xi(\tilde{x})).$$

Moreover, each of the functions $x \mapsto (I_{[\frac{1}{\|D_L(x)\|}D_L-v(x)]})^*(\xi_\nu(x))$ is measurable, and so is their pointwise limit $(I_{[\frac{1}{\|D_L(\cdot)\|}D_L(\cdot)-v(\cdot)]})^*(\xi(\cdot))$. Being ε arbitrary, we have that $x \mapsto (I_{[\frac{1}{\|D_L(x)\|}D_L(x)-v(x)]})^*(\xi(x))$ is measurable on Ω. Finally, $|(I_{[\frac{1}{\|D_L\|}D_L-v]})^*(\xi)| \leq |\xi|$. □

Proof of Theorem 1. a) Let u be a locally bounded solution to problem (1), let $\eta \in C_0^\infty(\Omega)$. Without loss of generality assume that $\sup|\eta| \leq 1$ and $\sup|\nabla\eta| \leq 1$. Set $\omega = supp(\eta)$, let U^* such that $|u(x)| \leq U^*$ on ω, and set $U = U^* + 1$. From (2) we infer that, for $|z| \leq 1$, $f(\xi + z) \leq f(\xi)e^H$. Recalling the notation $D_L(x) =$

$\partial_\xi L(x, u(x), \nabla u(x))$, we have that

$$\frac{1}{\varepsilon}[L(x, u(x) + \varepsilon\eta(x), \nabla u(x) + \varepsilon\nabla\eta(x)) - L(x, u(x), \nabla u(x))]$$
$$\to \left[\sup_{k\in D_L(x)} \langle k, \nabla\eta(x)\rangle\right] + L_u(x, u(x), \nabla u(x))\eta(x)$$

pointwise w.r.t. x. Moreover,

$$\begin{aligned}
&\left|\frac{1}{\varepsilon}[L(x, u(x) + \varepsilon\eta(x), \nabla u(x) + \varepsilon\nabla\eta(x)) - L(x, u(x), \nabla u(x))]\right| \\
&\quad = \left|\frac{1}{\varepsilon}[L(x, u(x) + \varepsilon\eta(x), \nabla u(x) + \varepsilon\nabla\eta(x)) - L(x, u(x), \nabla u(x) + \varepsilon\nabla\eta(x))]\right| \\
&\qquad + \left|\frac{1}{\varepsilon}[L(x, u(x), \nabla u(x) + \varepsilon\nabla\eta(x)) - L(x, u(x), \nabla u(x))]\right| \\
&\quad \le |L_u(x, u(x) + \theta_1\varepsilon\eta(x), \nabla u(x) + \varepsilon\nabla\eta(x))\eta(x)| \\
&\qquad + |\sup\{\langle k, \nabla\eta(x)\rangle : k \in \partial_\xi L(x, u(x), \nabla u(x) + \theta_2\varepsilon\nabla\eta(x))\}| .
\end{aligned}$$

From (5), we have

$$\begin{aligned}
&|L_u(x, u(x) + \theta_1\varepsilon\eta(x), \nabla u(x) + \varepsilon\nabla\eta(x))\eta(x)| \\
&\quad \le \gamma_U(x) + h_U^3 f(\nabla u(x) + \varepsilon\nabla\eta(x)) \\
&\quad \le \gamma_U(x) + h_U^3 f(\nabla u(x))e^H.
\end{aligned} \tag{7}$$

Assumption (3) implies that $f(\nabla u)$ is integrable, so that the r.h.s. of (7) is an integrable function, independent of ε.

We also have:

$$\begin{aligned}
&|\sup\{\langle k, \nabla\eta(x)\rangle : k \in \partial_\xi L(x, u(x), \nabla u(x) + \theta_2\varepsilon\nabla\eta(x))\}| \\
&\quad \le |\nabla\eta(x)|[\beta_U(x) + h_U^2|\partial f(\nabla u(x) + \theta_2\varepsilon\nabla\eta(x))|] \\
&\quad \le |\nabla\eta(x)|[\beta_U(x) + h_U^2 K f(\nabla u(x) + \theta_2\varepsilon\nabla\eta(x))] \\
&\quad \le |\nabla\eta(x)|[\beta_U(x) + h_U^2 H f(\nabla u(x))e^H],
\end{aligned}$$

an integrable function, independent of ε.

Hence, by dominated convergence,

$$\begin{aligned}
&\frac{1}{\varepsilon}\left[\int_\Omega L(x, u(x) + \varepsilon\eta(x), \nabla u(x) + \varepsilon\nabla\eta(x))\,\mathrm{d}x - \int_\Omega L(x, u(x), \nabla u(x))\,\mathrm{d}x\right] \\
&\quad \to \int_\Omega \sup_{k\in D_L(x)} \langle k, \nabla\eta(x)\rangle\,\mathrm{d}x + \int_\Omega L_u(x, u(x), \nabla u(x))\eta(x)\,\mathrm{d}x \\
&\quad = \int_\Omega (I_{D_L(x)})^*(\nabla\eta(x))\,\mathrm{d}x + \int_\Omega L_u(x, u(x), \nabla u(x))\eta(x)\,\mathrm{d}x.
\end{aligned}$$

Hence, we obtain

$$0 \le \int_\Omega (I_{D_L(x)})^*(\nabla\eta(x))\,\mathrm{d}x + \int_\Omega L_u(x, u(x), \nabla u(x))\eta(x)\,\mathrm{d}x$$

or,

$$
\begin{aligned}
-\int_\Omega L_u(x,u(x),\nabla u(x))\eta(x)\ \mathrm{d}x &\le \int_\Omega (I_{D_L(x)})^*(\nabla\eta)\ \mathrm{d}x \\
&= \int_\Omega \sup_{\{k\in D_L(x)\}} \langle k,\nabla\eta\rangle = \int_\Omega \sup_{\{k\in D_L(x)\}} \langle \frac{k}{\|D_L(x)\|}, \|D_L(x)\|\nabla\eta\rangle.
\end{aligned} \tag{8}
$$

b) From (4), (3) and (2), we have that $\|D_L(x)\| \le \beta_U(x) + h_U^2 H f(\nabla u(x)) \le \beta_U(x) + h_U^2 H \frac{1}{h_U^1}(L(x,u,\nabla u(x)) - \alpha_U(x))$, so that $\|D_L\| \in \mathrm{L}^1(\Omega)$; for every $\eta \in C_0^\infty(\Omega)$ we have that $\|D_L\|\nabla\eta \in (\mathrm{L}^1(\Omega))^N$. Consider $\mathbb{L}$, the linear subspace of $(\mathrm{L}^1(\Omega))^N$ defined as

$$\mathbb{L} = \{\xi \in (L^1(\Omega))^N : \exists \eta \in C_0^\infty(\Omega) : \xi = \|D_L(x)\|\nabla\eta\}$$

and, on $\mathbb{L}$, the linear functional

$$T(\xi) = -\int_\Omega L_u(x,u(x),\nabla u(x))\eta(x)\ \mathrm{d}x.$$

We notice that T is well defined: assume that there exist η^1 and η^2 in $C_0^\infty(\Omega)$ such that $\xi = \|D_L\|\nabla\eta^1 = \|D_L\|\nabla\eta^2$: then, from (8), we have

$$|-\int_\Omega L_u(x,u(x),\nabla u(x))\eta^1(x)\ \mathrm{d}x + \int_\Omega L_u(x,u(x),\nabla u(x))\eta^2(x)\ \mathrm{d}x| = 0,$$

so that T is well defined.

The map

$$
\begin{aligned}
\varrho(\xi) :&= \int_\Omega \sup_{\{k\in D_L(x)\}} \langle \frac{k}{\|D_L(x)\|}, \|D_L(x)\|\nabla\eta\rangle\ \mathrm{d}x \\
&= \int_\Omega \sup_{\{h\in \frac{1}{\|D_L(x)\|}D_L(x)\}} \langle h, \|D_L(x)\|\nabla\eta\rangle\ \mathrm{d}x \\
&= \int_\Omega \left(I_{\frac{1}{\|D_L(x)\|}D_L(x)}\right)^* (\|D_L(x)\|\nabla\eta(x))\ \mathrm{d}x
\end{aligned}
$$

appearing at the r.h.s. of (8) is defined on $\mathbb{L}$ as a convex, positively homogeneous map. It can be extended, preserving these properties, to $(\mathrm{L}^1(\Omega))^N$, since $\left(I_{\frac{1}{\|D_L(x)\|}D_L(x)}\right)^* (\xi(x)) \le |\xi(x)|$.

Hence, by the Hahn Banach Theorem, the linear map T can be extended from $\mathbb{L}$ to the whole of $(\mathrm{L}^1(\Omega))^N$, still satisfying $|T(\xi)| \le \rho(\xi)$.

c) By Lemma 2, we can apply Lemma 1 to the map $D = \frac{1}{\|D_L\|}D_L$. Hence, we infer the existence of a $\tilde p \in (\mathrm{L}^\infty(\Omega))^N$, with $\tilde p(x) \in \frac{1}{\|D_L(x)\|}D_L(x)$ a.e. on Ω, i.e. $\tilde p(x) = \frac{1}{\|D_L(x)\|}p(x)$ with $p(x) \in D_L(x)$, representing the extension of T to $(\mathrm{L}^1(\Omega))^N$, in particular, representing T on $\mathbb{L}$. Hence, for every $\eta \in C_0^\infty(\Omega)$, we have

$$
\begin{aligned}
&-\int_\Omega L_u(x,u(x),\nabla u(x))\eta(x)\ \mathrm{d}x \\
&= \int_\Omega \langle \tilde p(x), \|D_L(x)\|\nabla\eta(x)\rangle\ \mathrm{d}x = \int_\Omega \langle p(x), \nabla\eta(x)\rangle\ \mathrm{d}x
\end{aligned}
$$

In other words, for every $\eta \in C_0^\infty(\Omega)$,

$$\int_\Omega \langle p(x), \nabla\eta(x)\rangle\ \mathrm{d}x + \int_\Omega L_u(x,u(x),\nabla u(x))\eta(x)\ \mathrm{d}x = 0.$$

The map $p(\cdot)$ is a selection from $\partial_\xi L(\cdot, u(\cdot), \nabla u(\cdot))$ defined on Ω, thus proving the Theorem. $\square$

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