SIAM J. CONTROL OPTIM.
Vol. 50, No. 2, pp. 888–899

THE HIGHER INTEGRABILITY AND THE VALIDITY OF THE EULER–LAGRANGE EQUATION FOR SOLUTIONS TO VARIATIONAL PROBLEMS*

GIOVANNI BONFANTI†, ARRIGO CELLINA†, AND MARCO MAZZOLA†

Abstract. We prove higher integrability properties of solutions to the problem of minimizing $\int_\Omega L(x,u(x),\nabla u(x))\mathrm{dx}$, where $\xi \mapsto L(x,u,\xi)$ is a convex function satisfying some additional conditions. As an application, we prove the validity of the Euler–Lagrange equation for a class of functionals with growth faster than exponential.

Key words. calculus of variations, necessary conditions, Euler–Lagrange equation

AMS subject classification. 49K10

DOI. 10.1137/110820890

1. Introduction. In this paper we consider a higher integrability property of a solution $\tilde{u}$ to the problem of minimizing

$$\int_\Omega L(x,u(x),\nabla u(x))\mathrm{dx}.$$

More precisely, our aim is to establish the local integrability of the map

$$|\nabla_\xi L\,(\cdot,\tilde{u}(\cdot),\nabla\tilde{u}(\cdot))\,||\nabla\tilde{u}(\cdot)|. \tag{1}$$

In fact, for Lagrangians $L(x,u,\cdot)$ growing faster than exponential, the integrability of $L\,(\cdot,\tilde{u}(\cdot),\nabla\tilde{u})$ does not, in general, imply the integrability of $|\nabla_\xi L\,(\cdot,\tilde{u}(\cdot),\nabla\tilde{u}(\cdot))\,|$ (see an example in [2]). However, the integrability of (1) is needed both to establish the validity of the Euler–Lagrange equation for the solution to this problem, i.e., in order to prove that the equation

$$\int_\Omega [\langle\nabla_\xi L(x,\tilde{u}(x),\nabla\tilde{u}(x)),\nabla\eta(x)\rangle + L_u(x,\tilde{u}(x),\nabla\tilde{u}(x))\eta(x)]\mathrm{dx} = 0$$

holds for every admissible variation η and to prove additional regularity properties (higher differentiability) of the solution, as in [3].

Clearly, a proof of regularity (∇u in L^∞) of the solution is also a proof of the higher integrability of the solution. In this sense, for the case $L(\xi) = e^{|\xi|^2}$, special cases of higher integrability have been obtained by Lieberman [6] and by Naito [7]; Lieberman, in the same paper, considers also a more general Lagrangian but assumes, among other regularity conditions, that the Euler–Lagrange equation admits a C^3 solution.

In [2], a Lagrangian of the kind $L = e^{f(|\nabla u|)} + g(x,u)$ was considered, where f and g are regular functions satisfying some growth assumptions and f is convex, and a higher integrability result was obtained. The purpose of the present paper

*Received by the editors January 12, 2011; accepted for publication (in revised form) December 19, 2011; published electronically April 17, 2012.
http://www.siam.org/journals/sicon/50-2/82089.html
†Dipartimento di Matematica e Applicazioni, Università degli Studi di Milano-Bicocca, Via R. Cozzi 53, I-20125 Milano, Italy (g.bonfanti3@campus.unimib.it, arrigo.cellina@unimib.it, m.mazzola7@campus.unimib.it).

is twofold: first we wish to present a more general result, suited for being used in the investigation of further regularity properties of the solution; second, we wish to use the higher integrability property to establish the validity of the Euler–Lagrange equation for a class of Lagrangians growing faster than exponential.

It is well known, in fact (see, e.g., [4]), that so far, the validity of the Euler–Lagrange equation for Lagrangians of general form has been established only for Lagrangians growing at most exponentially; Lieberman and Naito consider the case $L = e^{|\nabla u|^2}$; more recently, Degiovanni and Marzocchi in [5] consider functionals of the form $\int L(\nabla u(x))\mathrm{dx} + \varphi(\mathrm{u})$, where $\varphi \in W^{-1,p'}$, without any upper growth condition on L, and in [1], $\varphi(u)$ is replaced by a more general term $g(x,u)$ concave w.r.t. u. However, the few results proved so far for integrands having growth faster than exponential hold only for Lagrangians of a very special form.

The proof of the higher integrability result, which will be presented below, is independent on the validity of the Euler–Lagrange equation; this fact prompted us to try to use the higher integrability property to extend the validity of the Euler–Lagrange equation beyond exponential growth. A result along these lines is presented in the second part of the paper: in it, we allow the growth of L with respect to ξ to be approximately up to $|\xi|^{|\xi|} \equiv \exp(|\xi| \log |\xi|)$.

2. Assumptions and higher integrability results. Some results in this paper will depend on the properties of the *polar* or *Legendre–Fenchel transform* L^* of a convex function L, defined by

$$L^*(p) = \sup\{\langle p, \xi\rangle - L(\xi)\};$$

for its properties, we refer to [8].

We shall consider Lagrangians L satisfying the following convexity and regularity assumptions.

Assumption A. $L(x,u,\xi)$ is nonnegative and positive whenever $\xi \neq 0$, and the map $t \mapsto L(x,u,t\xi)$ is nondecreasing for $t \geq 0$. In addition, for every (x,u), the restriction to the set $|\xi| \geq 1$ of the mapping $\xi \mapsto L(x,u,\xi)$ is the restriction to the same set of a convex function. Moreover, $L(x,u,\xi)$ is $C^1(u \times \xi)$ for each fixed x and measurable in x for each fixed (u,ξ), and it is such that for every $\omega \subset\subset \Omega$ and U there exist constants $M = M(\omega,U)$, $K = K(\omega,U)$, and, for every R, a function $\alpha_{\omega,U,R}$ in $L^1(\omega)$ such that for almost every $x \in \omega$ and for every $|u| \leq U$ we have

(i) for every $\xi \in R^n$, $|\frac{\partial L(x,u,\xi)}{\partial u}| \leq KL(x,u,\xi)$;

(ii) $\sup\{|\nabla_\xi L(x,u,\xi)| : |u| \leq U; |\xi| \leq R\} \leq \alpha_{\omega,U,R}(x)$;

(iii) $\langle \nabla_\xi L(x,u,\xi), \xi\rangle \geq M|\nabla_\xi L(x,u,\xi)||\xi|$.

The higher integrability results will depend on the validity of the following condition. In it, and for the remainder of the paper, for an open $O \subset\subset \Omega$ and $\delta > 0$, we set $O_\delta = O + B(0,\delta)$. Explicit classes of Lagrangians satisfying Condition C will be provided by Theorem 2.

Condition C. *For every open $O \subset\subset \Omega$, $\delta^0 > 0$, and U there exist a constant $\delta \leq \delta^0$ such that $\overline{O}_\delta$ is in Ω, a Lipschitzian function $\eta \in C_c(O_\delta)$ with $\eta(x) \geq 0$ and $\eta(x) = 1$ on O, and constants $\tilde{K} = \tilde{K}(U,O_\delta) \geq 0$ and $\tilde{R} = \tilde{R}(U,O_\delta)$ such that for every ξ with $|\xi| \geq \tilde{R}$, for every u with $|u| \leq U$, for almost every $x \in O_\delta$, and for every $\varepsilon > 0$ sufficiently small, we have*

$$\log L(x, u - \varepsilon\eta u, \xi(1-\varepsilon\eta) - \varepsilon u\nabla\eta) - \log L(x,u,\xi) \leq \varepsilon\tilde{K}. \tag{2}$$

The next theorem infers the higher integrability result from the validity of Condition C.

THEOREM 1. *Let L satisfy Assumption* A *and Condition* C. *Let $\tilde{u}$ be a locally bounded solution to the problem of minimizing*

$$\int_\Omega L(x,u(x),\nabla u(x))\mathrm{d}x$$

on $u^0 + W_0^{1,1}(\Omega)$. Then,

$$|\nabla_\xi L(\cdot,\tilde{u}(\cdot),\nabla\tilde{u}(\cdot))||\nabla\tilde{u}(\cdot)| \in \mathrm{L}^1_{loc}(\Omega).$$

Proof. (a) Fix $O \subset\subset \Omega$. It is enough to prove the existence of H_1 such that

$$\int_O \langle\nabla_\xi L(x,\tilde{u}(x),\nabla\tilde{u}(x)),\nabla\tilde{u}(x)\rangle\,\mathrm{d}x \le H_1.$$

In fact, if this is true, taking O to be ω in Assumption A, point (iii) proves the claim.

Hence, let $O_{\delta^0} \subset\subset \Omega$, and let U be a bound for $|\tilde{u}|$ on O_{δ^0}. Let δ, η and the constants $\tilde{R}$ and $\tilde{K}$ be provided by Condition C (we assume $\tilde{R} \ge 1$).

Since $\tilde{u}$ is a solution, for the variation $-\varepsilon\eta\tilde{u}$, with $\varepsilon > 0$, we obtain

$$0 \le \frac{1}{\varepsilon}\int_{O_\delta}[L(x,\tilde{u}-\varepsilon\eta\tilde{u},\nabla\tilde{u}(1-\varepsilon\eta)-\varepsilon\tilde{u}\nabla\eta) - L(x,\tilde{u},\nabla\tilde{u})]\,\mathrm{d}x. \tag{3}$$

We have

$$\begin{aligned}&L(x,\tilde{u}-\varepsilon\eta\tilde{u},\nabla\tilde{u}(1-\varepsilon\eta)-\varepsilon\tilde{u}\nabla\eta) - L(x,\tilde{u},\nabla\tilde{u})\\ &\quad= \varepsilon\int_0^1\left[\frac{\partial L}{\partial u}(-\eta\tilde{u}) + \langle\nabla_\xi L, -\eta\nabla\tilde{u}-\tilde{u}\nabla\eta\rangle\right]\mathrm{d}s,\end{aligned}$$

where $\frac{\partial L}{\partial u}$ and $\nabla_\xi L$ are computed at $(x,\tilde{u}-s\varepsilon\eta\tilde{u},\nabla\tilde{u}(1-s\varepsilon\eta)-s\varepsilon\tilde{u}\nabla\eta)$; hence, as $\varepsilon \to 0$, by the continuity of the partial derivatives of L,

$$\begin{aligned}&\frac{L(x,\tilde{u}-\varepsilon\eta\tilde{u},\nabla\tilde{u}(1-\varepsilon\eta)-\varepsilon\tilde{u}\nabla\eta) - L(x,u,\nabla\tilde{u})}{\varepsilon}\\ &\qquad\to \frac{\partial L}{\partial u}(-\eta\tilde{u}) + \langle\nabla_\xi L, -\eta\nabla\tilde{u}-\tilde{u}\nabla\eta\rangle,\end{aligned} \tag{4}$$

pointwise in x, and with the right-hand side computed at $(x,\tilde{u}(x),\nabla\tilde{u}(x))$. Set ${O_\delta}^- = \{x \in O_\delta : |\nabla\tilde{u}(x)| < \tilde{R}\}$ and ${O_\delta}^+ = \{x \in O_\delta : |\nabla\tilde{u}(x)| \ge \tilde{R}\}$: on ${O_\delta}^-$, the left-hand side of (4) is uniformly bounded so that for every ε and for some $\tilde{M}$ we have

$$\left|\frac{1}{\varepsilon}\int_{{O_\delta}^-}[L(x,\tilde{u}-\varepsilon\eta\tilde{u},\nabla\tilde{u}(1-\varepsilon\eta)-\varepsilon\tilde{u}\nabla\eta) - L(x,\tilde{u}(x),\nabla\tilde{u}(x))]\mathrm{d}x\right| \le \tilde{M}.$$

(b) On ${O_\delta}^+$, consider the constant $\tilde{K}$: setting

$$\tilde{\ell}_\varepsilon(x) = \log L(x,\tilde{u}(x)-\varepsilon\eta(x)\tilde{u}(x),\nabla\tilde{u}(x)(1-\varepsilon\eta(x))-\varepsilon\tilde{u}(x)\nabla\eta(x)),$$

from (3) we have

$$\begin{aligned}-\tilde{M} &\le \int_{{O_\delta}^+}\left(\frac{e^{\tilde{\ell}_\varepsilon} - e^{\log L(x,\tilde{u}(x),\nabla\tilde{u}(x))}}{\varepsilon}\right)\mathrm{d}x\\ &= \int_{{O_\delta}^+}\left(\frac{e^{\tilde{\ell}_\varepsilon-\varepsilon\tilde{K}+\varepsilon\tilde{K}} - e^{\log L(x,\tilde{u}(x),\nabla\tilde{u}(x))}}{\varepsilon}\right)\mathrm{d}x\\ &= \int_{{O_\delta}^+} e^{\tilde{\ell}_\varepsilon-\varepsilon\tilde{K}}\left[\frac{e^{\varepsilon\tilde{K}} - 1 + 1 - e^{\log L(x,\tilde{u}(x),\nabla\tilde{u}(x))-\tilde{\ell}_\varepsilon+\varepsilon\tilde{K}}}{\varepsilon}\right]\mathrm{d}x,\end{aligned}$$

i.e.,

$$\tilde{M} + \int_{O_\delta{}^+} e^{\tilde{\ell}_\varepsilon - \varepsilon\tilde{K}} \left[\frac{e^{\varepsilon\tilde{K}} - 1}{\varepsilon}\right] \mathrm{d}x \geq \int_{O_\delta^+} e^{\tilde{\ell}_\varepsilon - \varepsilon\tilde{K}} \left[\frac{e^{\log L(x,\tilde{u}(x),\nabla\tilde{u}(x)) - \tilde{\ell}_\varepsilon + \varepsilon\tilde{K}} - 1}{\varepsilon}\right] \mathrm{d}x. \tag{5}$$

Since on $O_\delta{}^+$, $\tilde{\ell}_\varepsilon(x) - \varepsilon\tilde{K} \leq \log L(x,\tilde{u}(x),\nabla\tilde{u}(x))$ and also $\frac{e^{\varepsilon\tilde{K}}-1}{\varepsilon} \leq \tilde{K}e^{\tilde{K}}$, the left-hand side of (5) is bounded by

$$\tilde{M} + \tilde{K}e^{\tilde{K}} \int_\Omega L(x,\tilde{u}(x),\nabla\tilde{u}(x))\mathrm{d}x = H,$$

independent of ε. (c) Consider the right-hand side. For fixed x we have

$$\log L(x,\tilde{u}(x),\nabla\tilde{u}(x)) - \tilde{\ell}_\varepsilon(x) = -\varepsilon\left[\frac{1}{L}\frac{\partial L}{\partial u}(-\eta\tilde{u}) + \frac{1}{L}\left\langle \nabla_\xi L, -\eta\nabla\tilde{u} - \tilde{u}\nabla\eta\right\rangle\right] + o(\varepsilon)$$

so that, as $\varepsilon \to 0$, pointwise w.r.t. x,

$$\frac{e^{\log L(x,\tilde{u},\nabla\tilde{u}) - \tilde{\ell}_\varepsilon + \varepsilon\tilde{K}} - 1}{\varepsilon} \to \tilde{K} + \frac{1}{L}\frac{\partial L}{\partial u}\eta\tilde{u} + \frac{1}{L}\left\langle \nabla_\xi L, \eta\nabla\tilde{u} + \tilde{u}\nabla\eta\right\rangle. \tag{6}$$

In addition, by (2), $\log L(x,\tilde{u}(x),\nabla\tilde{u}(x)) - \tilde{\ell}_\varepsilon(x) + \varepsilon\tilde{K} \geq 0$ so that the left-hand side of (6) is nonnegative and so is its limit, $\tilde{K} + \frac{1}{L}\frac{\partial L}{\partial u}\eta\tilde{u} + \frac{1}{L}\left\langle \nabla_\xi L, \eta\nabla\tilde{u} + \tilde{u}\nabla\eta\right\rangle$. Finally, pointwise, $e^{\tilde{\ell}_\varepsilon - \varepsilon\tilde{K}} \to e^{\log L(x,\tilde{u},\nabla\tilde{u})}$. Hence, applying Fatou's lemma, we obtain

$$\int_{O_\delta^+} L(x,\tilde{u},\nabla\tilde{u})\left[\tilde{K} + \frac{\frac{\partial L}{\partial u}}{L}\eta\tilde{u} + \frac{1}{L}\left\langle \nabla_\xi L, \eta\nabla\tilde{u} + \tilde{u}\nabla\eta\right\rangle\right] \leq H,$$

i.e.,

$$\int_{O_\delta^+}\left[\tilde{K}L(x,\tilde{u},\nabla\tilde{u}) + \frac{\partial L}{\partial u}\eta\tilde{u} + \left\langle \nabla_\xi L, \eta\nabla\tilde{u} + \tilde{u}\nabla\eta\right\rangle\right] \leq H.$$

Since the integrand above is nonnegative, we have obtained, in particular, that

$$\int_{O\cap O_\delta^+}\left[\tilde{K}L(x,\tilde{u},\nabla\tilde{u}) + \frac{\partial L}{\partial u}\eta\tilde{u} + \left\langle \nabla_\xi L, \eta\nabla\tilde{u} + \tilde{u}\nabla\eta\right\rangle\right] \leq H.$$

On O we have that $\eta \equiv 1$, $\tilde{u}$ is bounded, and that, by (i) of Assumption A, there exists K such that

$$\left|\frac{\partial L(x,\tilde{u}(x),\nabla\tilde{u}(x))}{\partial u}\right| \leq KL(x,\tilde{u}(x),\nabla\tilde{u}(x));$$

hence there exists H^+ such that

$$\int_{O\cap O_\delta^+} \left\langle \nabla_\xi L\left(x,\tilde{u}(x),\nabla\tilde{u}(x)\right), \nabla\tilde{u}(x)\right\rangle \mathrm{d}x \leq H^+. \tag{7}$$

Consider $O \cap O_\delta^-$; by Assumption A(ii), we have that

$$|\langle \nabla_\xi L(x, \tilde{u}(x), \nabla \tilde{u}(x)), \nabla \tilde{u}(x) \rangle| \leq \tilde{R} \cdot \alpha_{\omega, U, \tilde{R}}(x)$$

on $O \cap O_\delta^-$. Hence we have obtained that the integral

$$\int_O \langle \nabla_\xi L(x, \tilde{u}(x), \nabla \tilde{u}(x)), \nabla \tilde{u}(x) \rangle \, \mathrm{d}x$$

is bounded, thus proving the theorem. □

It is easy to show that the Lagrangians of exponential growth satisfy Condition C. However, the following result shows that this condition is satisfied by a substantially larger class of functions. We shall need the following.

Definition 1. *A convex function $\Lambda \in C^1(\Re)$, $\Lambda(t) > 0$ for $t \neq 0$, is called a comparison function for L if for every U there exist constants K_0, K_1, and K_2 such that for almost every $x \in \omega$, $|u| \leq U$ and $|\xi| \geq 1$ imply*

(i) $\Lambda(|\xi|) \leq K_0 L(x, u, \xi)$;

(ii) $K_1 \Lambda'(|\xi|) \leq |\nabla_\xi L(x, u, \xi)| \leq K_2 \Lambda'(|\xi|)$.

We shall also refer to the following.

Exponential growth condition. *For every open $O \subset\subset \Omega$ and U there exists a constant c such that for almost every $x \in O$, $|u| \leq U$ and $|\xi| \geq 1$ imply $|\nabla_\xi L(x, u, \xi)| \leq c \cdot L(x, u, \xi)$.*

Theorem 2. *Let L satisfy Assumption* A. *Assume that either*

(i) *L satisfies the exponential growth condition or*

(ii) *for $|\xi| \geq 1$, the map $\log(L(x, u, \cdot))$ is the restriction to the set $|\xi| \geq 1$ of a convex function; there exists a comparison function Λ such that $\mathcal{L}(\cdot) = \log(\Lambda(\cdot))$ is convex and such that $\mathcal{L}^*$ is defined on $\Re$. Assume that*

$$\int^\infty \frac{1}{z \partial(\mathcal{L}^*)(z)} \mathrm{d}z < \infty. \tag{8}$$

Then Condition C *is satisfied.*

Remarks. For every sufficiently large z, $0 \notin \partial(\mathcal{L}^*)(z)$; if this was not the case, in fact, the map $z \mapsto (\mathcal{L}^*)(z)$ would be constant and $\mathcal{L}$, hence Λ, would be defined on a single point.

The map $t \mapsto \exp(|t|^p)$ for $p > 1$ satisfies condition (ii) but not condition (i); the map $t \mapsto \exp(\exp(t))$ satisfies neither condition (i) nor condition (ii).

In the proof of Theorem 2, we shall need the following preliminary result (Lemma 1 in [2]).

Lemma 1. *Let $G : \Re \to 2^\Re$ be upper semicontinuous, strictly increasing, and such that $G(0) = \{0\}$. Assume that for a selection g from G*

$$\int^\infty g(1/s) \mathrm{d}s < \infty. \tag{9}$$

Then the implicit Cauchy problem

$$\begin{cases} x(t) \in G(x'(t)), \\ x(0) = 0 \end{cases}$$

admits a solution $\tilde{x}$, positive on some interval $(0, \tau]$.

Proof of Theorem 2. Fix O, δ^0, and U. Let $0 < \delta \leq \delta^0$ be such that $\overline{O}_\delta$ is in Ω. For a fixed variation η, we shall use the notation

$$\ell_\varepsilon(x,u,\xi) = \log L\left(x, u - \varepsilon\eta u, \xi(1-\varepsilon\eta) - \varepsilon u \nabla\eta\right).$$

(a) In case (i), choose any Lipschitz continuous function $\eta \in C_c(O_\delta)$ with $\eta(x) \geq 0$ and $\eta(x) = 1$ on O, and set $\mu = \sup|\nabla\eta|$. Choose $\tilde{R} = \max\{4, 2U\mu\}$ so that for $\varepsilon \leq \frac{1}{2}$, $|\xi| \geq \tilde{R}$ implies, for $s \in [0,1]$, $|\xi(1-\varepsilon\eta) - s\varepsilon u\nabla\eta| \geq 1$.

Fix $u, |u| \leq U$ and notice that for $s \in [0,1]$ we have $|u - s\varepsilon\eta u| \leq U$. We have

$$\begin{aligned}\ell_\varepsilon(x,u,\xi) - \log L(x,u,\xi) =& \int_0^1 \frac{\partial \log L(x, u - s\varepsilon\eta u, \xi(1-\varepsilon\eta) - \varepsilon u\nabla\eta)}{\partial u}(-\varepsilon\eta u)\mathrm{d}s \\ &+ \int_0^1 \langle \nabla_\xi \log L(x,u,\xi(1-\varepsilon\eta) - s\varepsilon u\nabla\eta), -\varepsilon u\nabla\eta\rangle\,\mathrm{d}s \\ &+ \log L(x,u,\xi(1-\varepsilon\eta)) - \log L(x,u,\xi).\end{aligned}$$

By Assumption A, $t \mapsto L(x,u,t\xi)$ is nondecreasing with respect to t on $\{t \geq 0\}$; hence the third term on the right-hand side is nonpositive. Moreover, $|\frac{\partial \log L}{\partial u}| = \frac{1}{L}|\frac{\partial L}{\partial u}| \leq K$ so that the first term is bounded by ε times a constant. By the exponential growth assumption, $|\nabla_\xi \log L| = \frac{|\nabla_\xi L|}{L} \leq c$, and hence the same is true for the second term.

(b) Consider case (ii). From

$$\ell_\varepsilon(x,u,\xi) - \log L(x,u,\xi) = \varepsilon(-\eta u)\int_0^1 \frac{1}{L}\frac{\partial L}{\partial u}\mathrm{d}s + \varepsilon\int_0^1 \left\langle \frac{\nabla_\xi L}{L}, -\eta\xi - u\nabla\eta\right\rangle \mathrm{d}s,$$

where the first integrand is evaluated at $(x, u - s\varepsilon\eta u, \xi(1-\varepsilon\eta) - \varepsilon u\nabla\eta)$ and the second at $(x, u, \xi(1 - s\varepsilon\eta) - s\varepsilon u\nabla\eta)$, we obtain

$$\ell_\varepsilon(x,u,\xi) - \log L(x,u,\xi) \leq \varepsilon\left[KU + \left\langle \frac{\nabla_\xi L}{L}(x,u,\xi_\varepsilon), -\eta\xi - u\nabla\eta\right\rangle\right], \tag{10}$$

where $\xi_\varepsilon = (1 - s_\varepsilon\varepsilon\eta)\,\xi - s_\varepsilon\varepsilon u\nabla\eta$ for some $0 \leq s_\varepsilon \leq 1$.

For $z \neq 0$, set

$$G(1/z) = \frac{1}{z}\frac{7U}{M\partial(\mathcal{L}^*)(z)}.$$

From the assumption of convexity, $\partial(\mathcal{L}^*)$ is nonincreasing as a function of $\frac{1}{z}$ and $\frac{7U}{M\partial(\mathcal{L}^*)(z)}$ is nondecreasing as a function of $\frac{1}{z}$ so that G satisfies the assumptions of Lemma 1.

Consider $\tilde{x}$, the solution to $\tilde{x} \in G(\tilde{x}')$, provided by Lemma 1, defined and positive on $(0,\tau]$. Possibly decreasing τ, we can assume, without loss of generality, that

$$x'(t) \leq 1 \text{ for all } t \in (0,\tau]. \tag{11}$$

Notice that from the inclusion $x(t) \in G(x'(t))$ we infer that $x' > 0$ on $(0,\tau]$, hence that x is strictly increasing, so that x' is strictly increasing as well. Set $\delta_\tau = \min\{\tau, \delta\}$ and define η as follows: let $d(x)$ be the distance from a point $x \in O_{\delta_\tau}$ to ∂O_{δ_τ} and set

$$\eta(x) = \inf\left\{\frac{1}{\tilde{x}(\delta_\tau)}\tilde{x}(d(x)), 1\right\}$$

so that, in particular, $\eta = 1$ on O. Almost everywhere, d is differentiable with $|\nabla d| = 1$ and, at a point of differentiability, we have

$$\nabla\eta(x) = \begin{cases} 0 & \text{if } d(x) > \delta_\tau, \\ \frac{1}{\tilde{x}(\delta_\tau)}\tilde{x}'(d(x))\nabla d(x) & \text{if } d(x) < \delta_\tau. \end{cases}$$

Hence, a.e., we have that $|\nabla\eta| \le \frac{1}{\tilde{x}(\delta_\tau)}\tilde{x}'(\delta_\tau)$ and that either $\nabla\eta = 0$ or

$$\begin{aligned}\eta(x) &= \frac{1}{\tilde{x}(\delta_\tau)}\tilde{x}(d(x)) = \frac{1}{\tilde{x}(\delta_\tau)}\tilde{x}'(d(x))\frac{7U}{M\partial(\mathcal{L}^*)(\frac{1}{\tilde{x}'(d(x))})} \\ &= h(\tilde{x}(\delta_\tau)|\nabla\eta(x)|)|\nabla\eta(x)|,\end{aligned} \tag{12}$$

where we have set

$$h(z) = \frac{7U}{M\partial(\mathcal{L}^*)(\frac{1}{z})}, \tag{13}$$

an increasing function.

Consider the term $\varepsilon\langle\frac{\nabla_\xi L}{L}(x,u,\xi_\varepsilon), -\eta\xi - u\nabla\eta\rangle$ in (10). Set $\overline{\xi} = (1-\varepsilon\eta)\,\xi - \varepsilon u\nabla\eta$.

Set $\mu_1 = \sup|\nabla\eta|$ and $\tilde{R} = 2 + U\mu_1$ so that for $\varepsilon \le \frac{1}{2}$, $|\xi| \ge \tilde{R}$ implies that both $|\xi_\varepsilon| \ge 1$ and $|\overline{\xi}| \ge 1$.

For those x such that

$$\left\langle \frac{\nabla_\xi L}{L}(x,u,\xi_\varepsilon), -\eta\xi - u\nabla\eta \right\rangle \le 0, \tag{14}$$

any $\tilde{K} \ge KU$ will do to prove the result. Moreover, by Assumption A, we have $\frac{d}{ds}L(x,u,\xi(1-s\varepsilon\eta)) \le 0$, i.e., $\langle\nabla_\xi L(x,u,\xi(1-s\varepsilon\eta)), -\eta\xi\rangle \le 0$, so that

$$\left\langle \frac{\nabla_\xi L}{L}(x,u,\xi(1-s\varepsilon\eta)), -\eta\xi \right\rangle \le 0;$$

from this we infer that, when $\nabla\eta(x) = 0$, (14) holds.

Hence, we are left to consider those x such that, at once,

$$\left\langle \frac{\nabla_\xi L}{L}(x,u,\xi_\varepsilon), -\eta\xi - u\nabla\eta \right\rangle > 0$$

and $\eta(x) = |\nabla\eta(x)|h(\tilde{x}(\delta_\tau)|\nabla\eta(x)|)$.

Given any $v, w \in \Re^n$, from the assumption of convexity of $\log L(x,u,\cdot)$, we obtain that its gradient is monotonic, i.e., that

$$(s_1 - s_2)\left\langle \frac{\nabla_\xi L}{L}(x,u,v+s_1 w) - \frac{\nabla_\xi L}{L}(x,u,v+s_2 w), w \right\rangle \ge 0,$$

i.e., that the mapping $s \mapsto \langle\frac{\nabla_\xi L}{L}(x,u,v+sw), w\rangle$ is nondecreasing. Hence, from the inequality

$$\left\langle \frac{\nabla_\xi L}{L}(x,u,\xi_\varepsilon), -\eta\xi - u\nabla\eta \right\rangle > 0,$$

we obtain $\langle\frac{\nabla_\xi L}{L}(x,u,\overline{\xi}), -\eta\xi - u\nabla\eta\rangle > 0$. We infer that

$$\left\langle \frac{\nabla_\xi L}{L}(x,u,\overline{\xi}), \xi \right\rangle < \left\langle \frac{\nabla_\xi L}{L}(x,u,\overline{\xi}), -u\frac{\nabla\eta}{\eta} \right\rangle \le \left| \frac{\nabla_\xi L}{L}(x,u,\overline{\xi}) \right| U\frac{|\nabla\eta|}{\eta}. \tag{15}$$

Recalling Assumption A(iii), we have

$$\left\langle \frac{\nabla_\xi L}{L}\left(x,u,\overline{\xi}\right),\xi\right\rangle = \left\langle \frac{\nabla_\xi L}{L}\left(x,u,\overline{\xi}\right),\overline{\xi}\right\rangle - \varepsilon\left\langle \frac{\nabla_\xi L}{L}\left(x,u,\overline{\xi}\right),-\eta\xi-u\nabla\eta\right\rangle$$

$$\geq \left|\frac{\nabla_\xi L}{L}\left(x,u,\overline{\xi}\right)\right|\left[M|\overline{\xi}|-\varepsilon\eta|\xi|-\varepsilon U|\nabla\eta|\right]. \tag{16}$$

From inequalities (15) and (16), we infer

$$U\frac{|\nabla\eta|}{\eta} > M|\overline{\xi}|-\varepsilon\eta|\xi|-\varepsilon U|\nabla\eta| \geq M[(1-\varepsilon\eta)|\xi|-\varepsilon U|\nabla\eta|]-\varepsilon\eta|\xi|-\varepsilon U|\nabla\eta|,$$

i.e.,

$$U\frac{|\nabla\eta|}{\eta}+\varepsilon|\nabla\eta|U[M+1] > |\xi|[M(1-\varepsilon\eta)-\varepsilon\eta].$$

We are free to assume $M<1$; taking $\varepsilon<\frac{M}{4}$, we finally have

$$3U\frac{|\nabla\eta|}{\eta} > U\frac{|\nabla\eta|}{\eta}+\varepsilon|\nabla\eta|U[M+1] > |\xi|[M(1-\varepsilon\eta)-\varepsilon\eta] > \frac{1}{2}M|\xi| \tag{17}$$

and, recalling (12), we obtain

$$|\xi| < \frac{6U}{Mh(\tilde{x}(\delta_\tau)|\nabla\eta(x)|)}. \tag{18}$$

From Definition 1 we have

$$\left\langle \frac{\nabla_\xi L}{L}(x,u,\xi_\varepsilon),-\eta\xi-u\nabla\eta\right\rangle \leq (\eta|\xi|+U|\nabla\eta|)\,K_0K_2\mathcal{L}'\left(|\xi_\varepsilon|\right); \tag{19}$$

noticing that

$$|\xi_\varepsilon| \leq \frac{6U}{Mh(\tilde{x}(\delta_\tau)|\nabla\eta|)}+\varepsilon|\nabla\eta|U \tag{20}$$

and that $\mathcal{L}'$ is nondecreasing, from (10), (12), (17), (19), and (20) we obtain

$$\begin{aligned}
&\ell_\varepsilon(x,u,\xi)-\log L(x,u,\xi)\\
&\quad\leq \varepsilon KU+\varepsilon\left(\eta|\xi|+U|\nabla\eta|\right)K_0K_2\mathcal{L}'\left(|\xi_\varepsilon|\right)\\
&\quad\leq \varepsilon KU+\varepsilon|\nabla\eta|K_0K_2\mathcal{L}'\left(\frac{6U}{M\cdot h(\tilde{x}(\delta_\tau)|\nabla\eta|)}+\varepsilon|\nabla\eta|U\right)\left(\frac{6U}{M}+U\right).
\end{aligned}$$

By (11), we have that $\tilde{x}(\delta_\tau)|\nabla\eta|\leq 1$; there exists σ such that for $t<\sigma$ we have

$$h(1)\leq\frac{1}{MUt}$$

so that for $|\nabla\eta|<\sigma$, $h(\tilde{x}(\delta_\tau)|\nabla\eta|)\leq h(1)\leq\frac{1}{MU|\nabla\eta|}$. Then,

$$\frac{6}{M\cdot h(\tilde{x}(\delta_\tau)|\nabla\eta|)}+\varepsilon|\nabla\eta|U\leq\frac{7}{M\cdot h(\tilde{x}(\delta_\tau)|\nabla\eta|)}.$$

Hence, for those x such that $|\nabla\eta(x)| < \sigma$, recalling (13), we obtain

$$|\nabla\eta| L'\left(\frac{6U}{M \cdot h(\tilde{x}(\delta_\tau)|\nabla\eta|)} + \varepsilon|\nabla\eta|U\right) \leq |\nabla\eta|\mathcal{L}'\left(\frac{7U}{M \cdot h(\tilde{x}(\delta_\tau)|\nabla\eta|)}\right)$$
$$= |\nabla\eta|\mathcal{L}'\left(\partial(\mathcal{L}^*)\left(\frac{1}{\tilde{x}(\delta_\tau)|\nabla\eta|}\right)\right) = \frac{1}{\tilde{x}(\delta_\tau)},$$

a constant independent on ε, thus proving the result in this case.

It is left to consider those x such that $|\nabla\eta(x)| \geq \sigma$: in this case, from (18), we have $|\xi| \leq \frac{6U}{M \cdot h(\tilde{x}(\delta_\tau)\sigma)}$, and, from (10), the result follows from the boundedness of $|\nabla\eta|$. ∎

3. The validity of the Euler–Lagrange equation. The higher integrability property for a minimizer $\tilde{u}$ is independent on the validity of the Euler–Lagrange equation. In the next theorem we wish to use this result in order to establish the validity of the Euler–Lagrange equation for a class of problems including Lagrangians having growth faster than exponential.

THEOREM 3. *Let $L(x,u,\xi)$ satisfy Assumption* A *and assume that there exist a comparison function Λ and a constant $c > 0$ such that for $t \geq 1$ either*

(i) $\frac{d}{dt}\mathcal{L}(t) \leq c$ *or*

(ii) $\frac{d}{dt}\mathcal{L}(t) \leq c\,(1 + \log t)$, *$\mathcal{L}(\cdot)$ is convex, and Dom$(\mathcal{L}^*)$ is open,*

where $\mathcal{L}(\cdot) = \log\Lambda(\cdot)$. Then, a locally bounded solution $\tilde{u}$ to the problem of minimizing

$$\int_\Omega L\left(x, u\left(x\right), \nabla u\left(x\right)\right) \mathrm{d}x \quad \textit{for} \quad u \in u_0 + W_0^{1,1}\left(\Omega\right) \tag{21}$$

satisfies the Euler–Lagrange equation.

Lagrangians of exponential growth satisfy (i); the map $\Lambda(t) = t^t$ is not of exponential growth but satisfies (ii): in this case, $\mathrm{Dom}(\mathcal{L}^*) = \Re$.

In order to prove Theorem 3, we shall need the following lemma.

LEMMA 2. *Let $L : \Re \to \Re$ be convex and C^1 and such that Dom(∂L^*) is open; let δ^* be any selection from ∂L^*. Then, there exists a sequence of convex C^2 functions L_m such that*

(i) *Dom$(L'_m) \supset [-m+1, m-1]$; $\forall x \in$ Dom(L'_m), we have $|L'_m(x) - L'(x)| < \frac{1}{m}$;*

(ii) *$L^*_m \in C^1($Dom$(\partial L^*))$; for every $[a,b] \subset$ Dom(∂L^*) there exists a subsequence $m(j)$ such that $(L^*_{m(j)})' \to \delta^*$ pointwise a.e. on $[a,b]$.*

Proof. Ad (i). By assumption, L' is a single-valued, continuous, nondecreasing function; hence, its inverse, ∂L^*, is strictly increasing, possibly multivalued, and defined on the image of L'. The selection δ^* (discontinuous at most on a set of measure zero) is strictly increasing and bounded on sets compactly contained in its domain. Consider the interval $[-n,n]$. The interval $[L'(-n), L'(n)]$ is a compact subset of the open set $\mathrm{Dom}(\partial L^*)$. There exists a subsequence $n(m)$ such that both $L'(-n(m+1)) < L'(-n(m))$ and $L'(n(m+1)) > L'(n(m))$; there exists $N(n(m))$, with $N(n(m)) \geq n(m)$ and $\frac{1}{N(n(m))} \leq \frac{1}{4}\min\{L'(-n(m)) - L'(-n(m+1)), L'(n(m+1)) - L'(n(m))\}$, such that $[L'(-n(m)) - \frac{1}{N(n(m))}, L'(n(m)) + \frac{1}{N(n(m))}] \subset \mathrm{Dom}(\partial L^*)$, so that the map

$$(L^*_m)' = \rho_{N(n(m))} * \delta^*,$$

where $\rho_{N(n(m))}$ is a standard mollifier having support in $[-\frac{1}{N(n(m))}, \frac{1}{N(n(m))}]$, is well defined on $[L'(-n(m)), L'(n(m))]$ as a strictly increasing function. Its image is the interval $I(n(m)) = [(L^*_m)'(L'(-n(m)), (L^*_m)'(L'(n(m)))]$. We claim that $I(n(m+1)) \supset [-n(m), n(m)] \supset [-m, m]$. In fact, consider $\overline{p} = \frac{1}{2}(L'(-n(m+1)) + L'(-n(m)))$: for every p such that $|p - \overline{p}| \leq \frac{1}{N(n(m))}$, we have $\delta^*(p) < \delta^*(L'(-n(m))) = -n(m)$ so that $\rho_{N(n(m+1))} * \delta^*(\overline{p}) < -n(m)$, and analogously for $\frac{1}{2}(L'(n(m+1)) + L'(n(m)))$.

The map $(L^*_m)'$ is a C^1 and strictly increasing, hence invertible, function: on the interval $I(n(m))$ we set $L'_m = ((L^*_m)')^{-1}$. Fix arbitrarily m and $\overline{x} \in I(n(m))$. Set $\overline{y}_m = L'_m(\overline{x})$ so that

$$\overline{x} = \left(\rho_{N(n(m))} * \delta^*\right)(\overline{y}_m) = \int_{[\overline{y}_m - \frac{1}{N(n(m))}, \overline{y}_m + \frac{1}{N(n(m))}]} \rho_{N(n(m))}(\overline{y}_m - y)\delta^*(y)\, dy.$$

We notice that $\overline{x} \in \mathrm{co}\{\delta^*(y) : |y - \overline{y}_m| \leq \frac{1}{N(n(m))}\}$. In fact, otherwise, there exists α such that for every $y \in \{|y - \overline{y}_m| \leq \frac{1}{N(n(m))}\}$ we have $\alpha\overline{x} > \alpha\delta^*(y)$; then,

$$\alpha\overline{x} = \int \alpha\overline{x} \cdot \rho_{N(n(m))}(\overline{y}_m - y) > \alpha \int \delta^*(y) \cdot \rho_{N(n(m))}(\overline{y}_m - y) = \alpha\overline{x}.$$

Hence, there are y_1 and y_2 such that $|y_i - \overline{y}_m| \leq \frac{1}{N(n(m))}$ and $\delta^*(y_1) \leq \overline{x} \leq \delta^*(y_2)$. By the monotonicity of L', the last inequality can be written as $y_1 \leq L'(\overline{x}) \leq y_2$ so that

$$\overline{y}_m - \frac{1}{N(n(m))} \leq y_1 \leq L'(\overline{x}) \leq y_2 \leq \overline{y}_m + \frac{1}{N(n(m))}.$$

We have obtained

$$|L'(\overline{x}) - L'_m(\overline{x})| \leq \frac{1}{n(m)} \leq \frac{1}{m}.$$

Ad (ii). Fix arbitrarily $[a, b]$. We have that $(L^*_m)' \to \delta^*$ in $L^1([a, b])$. Hence, there exists a sequence $(L^*_{m(j)})'$ converging to δ^* pointwise a.e. on $[a, b]$. ☐

The condition that $\mathrm{Dom}(\partial L^*)$ be open is not satisfied by a map like $L(t) = |t|$; it is satisfied by the minimal area functional $L(t) = \sqrt{1 + |t|^2}$ and a fortiori by any L of superlinear growth.

Proof of Theorem 3. Fix $0 < h_0 < \frac{1}{c}$; we claim that both in case (i) and in case (ii), there exists K such that $0 < h < h_0$ implies

$$\Lambda'(t + h) \leq K\left[1 + \Lambda(t) + t\Lambda'(t)\right]. \tag{22}$$

Set $\Lambda^* = \sup\{\Lambda'(s) : 0 \leq s \leq 1 + h_0\}$. In case (i), we have $\Lambda'(t + h) \leq \Lambda^*$ for $t \leq 1$ while for $t > 1$, $\Lambda(t + h) \leq c\Lambda(t)e^{ch}$, and we infer that (22) holds. In case (ii), again $\Lambda'(t + h) \leq \Lambda^*$ for $t \leq 1$, while for $t > 1$ we have

$$\begin{aligned}
\Lambda'(t + h) = L'(t + h)\frac{\Lambda(t + h)}{\Lambda(t)}\Lambda(t) &= \mathcal{L}'(t + h) \cdot \exp\left(\int_t^{t+h} \mathcal{L}'(s)\,\mathrm{d}s\right) \cdot \Lambda(t) \\
&\leq c\left[1 + \log(t + h_0)\right] \cdot \exp\left[c\int_t^{t+h_0} (1 + \log s)\,\mathrm{d}s\right] \cdot \Lambda(t) \\
&\leq c\left(1 + \log t + \frac{h_0}{t}\right) \cdot (t + h_0)^{ch_0} \cdot \exp\left[c\, t\left(\log(t + h_0) - \log t\right)\right]\Lambda(t) \\
&\leq c_1(1 + \log t) \cdot t^{ch_0}e^{ch_0}\Lambda(t) \leq c_2 \cdot t\Lambda(t).
\end{aligned}$$

By assumption, $\log(\Lambda)$ is convex for $t>1$ so that there exists c_3 such that $\frac{\Lambda'(t)}{\Lambda(t)} \geq c_3$, hence $\Lambda'(t+h) \leq c_2 c_3 \cdot t\Lambda'(t)$, and (22) is established.

Next, we claim that setting $t = |\nabla\tilde{u}(x)|$ in the right-hand side of (22), we obtain a function integrable on compact subsets of Ω. By (i) of the comparison assumption, we have that $\Lambda(|\nabla\tilde{u}|) \in L^1_{loc}(\Omega)$. By (ii) of the comparison assumption, to show that $|\nabla\tilde{u}|\Lambda'(|\nabla\tilde{u}|) \in L^1_{loc}(\Omega)$ it is enough to show that Theorem 1 holds, i.e., that the assumptions of Theorem 2 are satisfied. The assumptions are obviously satisfied in case (i), so we consider case (ii). We have to prove that

$$\int^{\infty} \frac{1}{z\partial(\mathcal{L}^*)(z)} \mathrm{d}z < \infty. \tag{23}$$

Since $\mathrm{Dom}(\partial\mathcal{L}^*)$ is open, Lemma 2 can be applied to $\mathcal{L}$. Consider the sequence $(\mathcal{L}_m)$; for any α, β $(\alpha \geq 1)$, by the change of variables $z = \mathcal{L}'_m(t)$, we have

$$\int_{\mathcal{L}'_m(\alpha)}^{\mathcal{L}'_m(\beta)} \frac{dz}{z(\mathcal{L}^*_m)'(z)} \mathrm{d}z = \int_\alpha^\beta \frac{\mathcal{L}''_m(t)}{t\cdot\mathcal{L}'_m(t)} \mathrm{d}t = \left.\frac{\log\mathcal{L}'_m(t)}{t}\right|_\alpha^\beta + \int_\alpha^\beta \frac{\log\mathcal{L}'_m(t)}{t^2}\mathrm{d}t. \tag{24}$$

By assumption (ii),

$$\left.\frac{\log\mathcal{L}'(t)}{t}\right|_\alpha^\beta + \int_\alpha^\beta \frac{\log\mathcal{L}'(t)}{t^2}\mathrm{d}t \leq \left.\frac{\log(c\,(1+\log t))}{t}\right|_\alpha^\beta + \int_\alpha^\beta \frac{\log(c\,(1+\log t))}{t^2}\,dt,$$

and there exists H such that for every $\alpha \geq 1$ and for every β, the right-hand side is bounded by H. Whenever $m-1 \geq \beta$, $\mathrm{Dom}(\mathcal{L}'_m) \supset [\alpha,\beta]$ and $|\mathcal{L}' - \mathcal{L}'_m| \leq \frac{1}{m}$, so that the right-hand side of (24) is bounded by $H+2$ (independent of α, β, and m). Consider the subsequence $m(j)$ provided by (ii) of Lemma 2. Fix any $a, b \in R$; let α, β such that for j sufficiently large, $[a,b] \subset [\mathcal{L}'_{m(j)}(\alpha), \mathcal{L}'_{m(j)}(\beta)]$. By (ii) of Lemma 2 and by Fatou's lemma,

$$\int_a^b \frac{1}{z\partial(\mathcal{L}^*)(z)}\mathrm{d}z = \int_a^b \frac{1}{z\delta^*(z)}\mathrm{d}z \leq \liminf \int_a^b \frac{dz}{z(\mathcal{L}^*_{m(j)})'(z)}\mathrm{d}z \leq H+2,$$

so that (8) is satisfied and the integrability claim holds.

To establish the validity of the Euler–Lagrange equation, fix $\eta \in C^1_c(\Omega)$ (we assume $\sup\eta \leq 1$) and set $\tilde{h} = \sup|\nabla\eta|$ and $S = \mathrm{spt}(\eta)$. Since $\tilde{u}$ is a solution, we have

$$\int_S \frac{L\left(x, \tilde{u}(x) + \varepsilon\eta(x), \nabla\tilde{u}(x) + \varepsilon\nabla\eta(x)\right) - L\left(x, \tilde{u}(x), \nabla\tilde{u}(x)\right)}{\varepsilon}\mathrm{d}x \geq 0; \tag{25}$$

the integrand converges pointwise to

$$\left\langle \nabla_\xi L\left(x,\tilde{u},\nabla\tilde{u}\right), \nabla\eta\right\rangle + \frac{\partial L}{\partial u}\left(x,\tilde{u},\nabla\tilde{u}\right)\cdot\eta$$

and we wish to dominate the integrand by an integrable function. We have

$$\begin{aligned}
&\left|\frac{L\left(x,\tilde{u}+\varepsilon\eta,\nabla\tilde{u}+\varepsilon\nabla\eta\right) - L\left(x,\tilde{u},\nabla\tilde{u}\right)}{\varepsilon}\right| \\
&\quad\leq \left|\frac{L\left(x,\tilde{u}+\varepsilon\eta,\nabla\tilde{u}\right) - L\left(x,\tilde{u},\nabla\tilde{u}\right)}{\varepsilon}\right| \\
&\qquad+ \left|\frac{L\left(x,\tilde{u}+\varepsilon\eta,\nabla\tilde{u}+\varepsilon\nabla\eta\right) - L\left(x,\tilde{u}+\varepsilon\eta,\nabla\tilde{u}\right)}{\varepsilon}\right| \\
&\quad\leq \left|\frac{\partial L}{\partial u}\left(x,\tilde{u}+\overline{s}\varepsilon\eta,\nabla\tilde{u}\right)\cdot\eta\right| + \left|\left\langle\nabla_\xi L\left(x,\tilde{u}+\varepsilon\eta,\nabla\tilde{u}+\overline{t}\varepsilon\nabla\eta\right),\nabla\eta\right\rangle\right|.
\end{aligned}$$

By Assumption A(i), the first term is bounded by $KL(x, \tilde{u}, \nabla\tilde{u}) \cdot e^K \cdot |\eta|$, an integrable function. Set $E = \{x : |\nabla\tilde{u}(x)| \geq 1 + \tilde{h}\}$ and write the second term as

$$\left|\left\langle \nabla_\xi L(x, \tilde{u} + \varepsilon\eta, \nabla\tilde{u} + \bar{t}\varepsilon\nabla\eta), \nabla\eta \right\rangle\right| \chi_{S\setminus E} + \left|\left\langle \nabla_\xi L\left(x, \tilde{u} + \varepsilon\eta, \nabla\tilde{u} + \bar{t}\varepsilon\nabla\eta\right), \nabla\eta \right\rangle\right| \chi_E.$$

Set $U = \sup\{|\tilde{u}(x)|\} + 1$. On $S \setminus E$, $|\nabla\tilde{u} + \varepsilon\nabla\eta| \leq 1 + 2\tilde{h} = R$, and hence by Assumption A(ii), the first term is bounded by an integrable function. On E, we have $|\nabla\tilde{u} + \varepsilon\nabla\eta| \geq 1$; hence by (ii) of the comparison assumption and (22), whenever $\varepsilon\tilde{h} < h_0$,

$$\begin{aligned}\left|\left\langle \nabla_\xi L\left(x, \tilde{u} + \varepsilon\eta, \nabla\tilde{u} + \bar{t}\varepsilon\nabla\eta\right), \nabla\eta \right\rangle\right| &\leq K_2\Lambda'\left(|\nabla\tilde{u}(x) + \bar{t}\varepsilon\nabla\eta|\right) \\ &\leq K_2K[1 + \Lambda\left(|\nabla\tilde{u}(x)|\right) + |\nabla\tilde{u}(x)|\Lambda'\left(|\nabla\tilde{u}(x)|\right)].\end{aligned}$$

The last term is integrable, by our previous claim, and is independent of ε, so that we can pass to the limit under the integral sign. Finally, considering also $-\eta$, we obtain that

$$\int_\Omega [\langle \nabla_\xi L(x, \tilde{u}(x), \nabla\tilde{u}(x)), \nabla\eta(x)\rangle + L_u(x, \tilde{u}(x), \nabla\tilde{u}(x))\eta(x)]\mathrm{dx} = 0$$

for every admissible variation η. ☐

Acknowledgment. This paper was improved by the sharp comments of an anonymous referee.

REFERENCES

[1] G. Bonfanti and M. Mazzola, *On the validity of the Euler-Lagrange equation in a nonlinear case*, Nonlinear Anal., 73 (2010), pp. 266–269.

[2] A. Cellina and M. Mazzola, *Higher integrability for solutions to variational problems with fast growth*, J. Convex Anal., 18 (2011), pp. 173–180.

[3] A. Cellina and M. Mazzola, *Higher differentiability of solutions to variational problems*, Calc. Var. Partial Differential Equations, to appear.

[4] F. H. Clarke, *Optimization and Nonsmooth Analysis*, 2nd ed., Classics Appl. Math. 5, SIAM, Philadelphia, 1990.

[5] M. Degiovanni and M. Marzocchi, *On the Euler-Lagrange equation for functionals of the calculus of variations without upper growth conditions*, SIAM J. Control Optim., 48 (2009), pp. 2857–2870.

[6] G. M. Lieberman, *On the regularity of the minimizer of a functional with exponential growth*, Comment. Math. Univ. Carolin., 33 (1992), pp. 45–49.

[7] H. Naito, *On a local Hölder continuity for a minimizer of the exponential energy functional*, Nagoya Math. J., 129 (1993), pp. 97–113.

[8] R. T. Rockafellar, *Convex Analysis*, Princeton University Press, Princeton, NJ, 1972.