Symmetries in Scalar Potential Scattering

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Abstract—

Motivation: The elastic scattering of the mass-less ($\omega^2 = c^2k^2$) complex scalar field by a potential, $q[\cdot]$, of bounded support is a prototype model which finds applications to acoustics and electromagnetics in a classical (i.e., non quantum mechanical) setting. Also, it provides the framework for direct and inverse obstacle scattering problems [1] from far-zone data, represented either by the complex-valued scattering amplitude, $A[\cdot]$, or by the modulus thereof (a.k.a the *scattering magnitude*). Of particular interest are the symmetries of the scattered wave, $u[\cdot]$ and of the quantities derived therefrom ($A[\cdot]$, $|A[\cdot]|^2$) caused by symmetry operations on $q[\cdot]$, such as translation, rotation, reflection, and scaling. An equally important problem is the separation of true from false symmetries, as it will be clarified below. This investigation is motivated by the analysis of scattering patterns of environmental interest, where the pattern "degree of symmetry" has been introduced on a heuristic basis [2].

Scattering by a Translated or Rotated Potential: Let k (> 0) and $\vec{\alpha} \in S^2$ respectively denote the wavenumber and incidence direction of a unit amplitude plane wave $u_{\mathrm{inc}}[\vec{x}; k, \vec{\alpha}] := \mathrm{e}^{\mathrm{i}k\vec{\alpha}\cdot\vec{x}}$. Let the potential, $q[\cdot]$, be a real-valued, bounded function, supported in the sphere of radius a (> 0) centered at the origin

$$q \in Q_a^\infty := \{q : q[\vec{x}] = q[\vec{x}]^*; \quad q[\vec{x}] = 0 \text{ for } |\vec{x}| > a; \quad q[\cdot] \in L^\infty(B_a[\mathbf{O}])\}. \tag{1}$$

Then the following operators can be defined.

Def. (*translation, rotation and reflection operators*)

Let $\vec{b} \in I\!R^3$, $\vec{\gamma} \in S^2$, $\mathbf{D}_\gamma \in \mathrm{SO}(3)$ and q_0 stand for the reference potential. Then $\mathcal{T}_b$, $\mathcal{D}_\gamma$ and $\mathcal{R}$ respectively are the operators of translation by $\vec{b}$, rotation by $\vec{\gamma}$ and reflection, which act on $q_0[\cdot]$ according to [3]

$$q_b[\vec{y}] = (q_0 \circ \mathcal{T}_b)[\vec{y}] = q_0[\vec{y} - \vec{b}]; \quad q_\gamma[\vec{y}] = (q_0 \circ \mathcal{D}_\gamma)[\vec{y}] = q_0[\mathbf{D}_\gamma^{-1} \cdot \vec{y}]; \quad q_r[\vec{y}] = (q_0 \circ \mathcal{R})[\vec{y}] = q_0[-\vec{y}]. \tag{2}$$

One further requires $q_b[\cdot],\ q_\gamma[\cdot] \in Q_a^\infty$.

If $\Phi[\vec{x} - \vec{y}]$, with $\vec{x}, \vec{y} \in I\!R^3$, stands for the fundamental solution to the Helmholtz equation $(\nabla^2 + k^2)\Phi[\vec{x} - \vec{y}] = \delta[\vec{y}]$, subject to the Sommerfeld radiation condition, then the scattered wave, $u_{\mathrm{sc}}[\vec{x}; \vec{\alpha}, k, q]$, is the solution to the integral equation

$$u_{\mathrm{sc}}[\vec{x}; \vec{\alpha}, k, q] = \int_{B_a[\mathbf{O}]} \Phi[\vec{x} - \vec{y}] \left(\mathrm{e}^{\mathrm{i}k\vec{\alpha}\cdot\vec{y}} + u_{\mathrm{sc}}[\vec{y}; \vec{\alpha}, k, q]\right) q[\vec{y}] \mathrm{d}^3y. \tag{3}$$

Since the solution to Eq. (3) is the limit of the (iterative) Born sequence, the following two results can be proved.

Thm. 1 (Born *sequence for the translated potential*).

Let $\vec{x} \in I\!R^3 \backslash \overline{B_a[\mathbf{O}]}$ be fixed, $\vec{\eta} := \vec{y} - \vec{b}$, $u_{\mathrm{sc}}^{(0)} := 0$ and $n \geq 1$ (integer), then

$$u_{\mathrm{sc}}^{(n+1)}[\vec{x}; \vec{\alpha}, k, q_b] = \mathrm{e}^{\mathrm{i}k\vec{\alpha}\cdot\vec{b}} \int_{B_a[\mathbf{O}]} \Phi\left[\vec{x} - \vec{\eta} - \vec{b}\right] \left(\mathrm{e}^{\mathrm{i}k\vec{\alpha}\cdot\vec{\eta}} + u_{\mathrm{sc}}^{(n)}[\vec{\eta}; \vec{\alpha}, k, q_0]\right) q_0[\vec{\eta}] \mathrm{d}^3\eta. \tag{4}$$

Thm. 2 (Born *sequence for the rotated potential*).

Let $\vec{x}$, $u_{\mathrm{sc}}^{(0)}$ and n as above. Also, let $\vec{\eta} := \mathbf{D}_\gamma^{-1} \cdot \vec{y}$ and $\vec{\beta} := \mathbf{D}_\gamma^{-1} \cdot \vec{\alpha}$, then

$$u_{\mathrm{sc}}^{(n+1)}[\vec{x}; \vec{\alpha}, k, q_\gamma] = \int_{B_a[\mathbf{O}]} \Phi\left[\vec{x} - \mathbf{D} \cdot \vec{\eta}\right] \left(\mathrm{e}^{\mathrm{i}k\vec{\beta}\cdot\vec{\eta}} + u_{\mathrm{sc}}^{(n)}\left[\vec{\eta}; \vec{\beta}, k, q_0\right]\right) q_0[\vec{\eta}] \mathrm{d}^3\eta. \tag{5}$$

REM. The recursion formulas of Eqs. (4) and (5) are of non-local type, because $\mathit{\Phi}[\vec{x} - .]$ is NOT evaluated at the integration variable, $\vec{\eta}$. The reason for keeping $\vec{x}$ fixed is dictated by the analysis of experimental data [2].

Invariance Results for the Squared Scattering Magnitude:

DEF. (*the scattering amplitude from the first* BORN *approximation*)

Let $\vec{\alpha}' := \vec{x}/|\vec{x}|$, then the scattering amplitude derived from the first BORN approximation and due to the reference potential, q_0, is defined by

$$A_{1B}[\vec{\alpha}'; \vec{\alpha}, k, q_0] := -(1/4\pi) \int_{B_a[\mathbf{O}]} \mathrm{e}^{-\mathrm{i}k(\vec{\alpha}'-\vec{\alpha})\cdot\vec{y}} q_0[\vec{y}] \mathrm{d}^3 y. \tag{6}$$

With by now obvious notation, Eq. (2) in particular, the following properties can be easily shown to hold.

THM. 3 (*invariance properties*).

$$|A_{1B}[\vec{\alpha}'; \vec{\alpha}, k, q_b]|^2 := |A_{1B}[\vec{\alpha}'; \vec{\alpha}, k, q_0]|^2; \tag{7}$$

$$|A_{1B}[\vec{\alpha}'; \vec{\alpha}, k, q_r]|^2 := |A_{1B}[\vec{\alpha}'; \vec{\alpha}, k, q_0]|^2; \tag{8}$$

Discussion: THMS. 1 and 2 characterize the symmetries of the scattered wave exactly. Some symmetry properties of the scattering amplitude have been investigated before [4]. As it could be easily shown, the first BORN approximation, from which Eq. (6) follows, brings in an uncountably infinite set of false symmetries with respect to translation and rotation. False symmetries will be presented in detail.

ACKNOWLEDGMENT

The financial support by the following sources is gratefully acknowledged: *a*) Contract W911NF 11-1-0277 R&D 1449-BC-01 granted to Università Milan-Bicocca by the US Army RDECOM ACQ CTR, *b*) the home University's *Fondo di Ateneo per la ricerca*, from 2010 onwards, and *c*) the research agreement between the home University and *Digicom* S.p.A., Gallarate, IT.

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