CONVEX BODIES APPEARING AS OKOUNKOV BODIES OF DIVISORS

ALEX KÜRONYA, VICTOR LOZOVANU, AND CATRIONA MACLEAN

ABSTRACT. Based on the work of Okounkov ([15], [16]), Lazarsfeld and Mustaţă ([13]) and Kaveh and Khovanskii ([10]) have independently associated a convex body, called the Okounkov body, to a big divisor on a smooth projective variety with respect to a complete flag. In this paper we consider the following question: what can be said about the set of convex bodies that appear as Okounkov bodies? We show first that the set of convex bodies appearing as Okounkov bodies of big line bundles on smooth projective varieties with respect to admissible flags is countable. We then give a complete characterisation of the set of convex bodies that arise as Okounkov bodies of $\mathbb{R}$-divisors on smooth projective surfaces. Such Okounkov bodies are always polygons, satisfying certain combinatorial criteria. Finally, we construct two examples of non-polyhedral Okounkov bodies. In the first one, the variety we deal with is Fano and the line bundle is ample. In the second one, we find a Mori dream space variety such that under small perturbations of the flag the Okounkov body remains non-polyhedral.

INTRODUCTION

Let X be a smooth complex projective variety of dimension n and let $\mathcal{L}$ be a big line bundle on X. Suppose given a flag

$$Y_\bullet \; : \; X = Y_0 \; \supseteq \; Y_1 \; \supseteq \; Y_2 \; \supseteq \; \ldots \; \supseteq \; Y_{n-1} \; \supseteq \; Y_n = \{\mathrm{pt}\}$$

of irreducible and smooth subvarieties on X with $\operatorname{codim}_X(Y_i) = i$. We call this an admissible flag. In [13], Lazarsfeld and Mustaţă, inspired by the work of Okounkov ([15], [16]), construct a convex body $\Delta_{Y_\bullet}(X; \mathcal{L})$ in $\mathbb{R}^n$, called the Okounkov body, associated to $\mathcal{L}$ and $Y_\bullet$. (The same body was constructed independently by Kaveh and Khovanskii in their paper [10].) This body encodes the asymptotic behaviour of the linear series $|\mathcal{L}^{\otimes m}|$. Lazarsfeld and Mustaţă link its properties to the geometry of $\mathcal{L}$. For example, because here $\mathcal{L}$ is big we have that

$$\operatorname{vol}_X(\mathcal{L}) \; = \; n! \cdot \operatorname{vol}_{\mathbb{R}^n}(\Delta_{Y_\bullet}(X; \mathcal{L}))$$

where the right hand side is the Euclidean volume of $\Delta_{Y_\bullet}(X; \mathcal{L})$. This enabled Lazarsfeld and Mustaţă to simplify the proofs of many basic properties of volumes of line bundles.

We recall the construction of $\Delta_{Y_\bullet}(X; \mathcal{L})$. To any effective divisor D on X we associate an integral vector

$$\nu_{Y_\bullet}(D) \; = \; (\nu_1(D), \ldots, \nu_n(D)) \in \mathbb{N}^n$$

defined as follows. We recursively construct numbers $\nu_i(D)$ and divisors D_i on Y_i in the following manner:

(1) $D_0 = D$,
(2) $\nu_i(D)$ is the coefficient of Y_i in D_{i-1},
(3) $D_i = (D_{i-1} - \nu_i Y_i)|_{Y_i}$.

We now set

$$\Gamma_{Y_\bullet}(X;\mathcal{L})_m = \left\{\nu_{Y_\bullet}(D) \mid D = \mathrm{zero}(s) \text{ for some } 0 \neq s \in H^0(X, \mathcal{L}^{\otimes m})\right\} \subseteq \mathbb{N}^n$$

for any $m \in \mathbb{N}$. The Okounkov body $\Delta_{Y_\bullet}(X;\mathcal{L})$ is then given by

$$\Delta_{Y_\bullet}(X;\mathcal{L}) = \text{ closed convex hull}\left(\bigcup_{m \geq 1} \frac{1}{m}\Gamma_{Y_\bullet}(X;\mathcal{L})_m\right) \subseteq \mathbb{R}^n .$$

If D is a (possibly non-rational) Cartier divisor on X then we define $\Delta_{Y_\bullet}(X;D)$ as follows:

$$\Delta_{Y_\bullet}(X;D) = \overline{\{\nu(D') \mid D' \geq 0, D' \sim_{\mathbb{R}} D\}},$$

which is simply $\Delta_{Y_\bullet}(X;\mathcal{O}_X(D))$ when D is an integral divisor.

In this paper we study the set of convex bodies appearing as Okounkov bodies of line bundles on smooth projective varieties with respect to some admissible flag. Our first result, proved in Section 1, shows that this set is countable.

Theorem A. *The collection of all Okounkov bodies is countable. That is, for any natural number $n \geq 1$, there exists a countable set of bounded convex bodies $(\Delta_i)_{i \in \mathbb{N}} \subset \mathbb{R}^n$ such that for any complex smooth projective variety X of dimension n, any big line bundle $\mathcal{L}$ on X and any admissible flag $Y_\bullet$ on X, the body $\Delta_{Y_\bullet}(X;\mathcal{L}) = \Delta_i$ for some $i \in \mathbb{N}$.*

The proof of Theorem A is similar to the proof of the countability of volume functions given in [11]. It was established in [13] that for a variety X equipped with a flag $Y_\bullet$ the Okounkov bodies of big real classes on X with respect to $Y_\bullet$ fit together in a convex cone, called the global Okounkov cone. We prove Theorem A by analysing the variation of global Okounkov cones in flat families.

The question then naturally arises whether this countable set of convex bodies can be characterised. We give an affirmative answer for surfaces. An explicit description of $\Delta(D)$ for any real divisor D on a smooth surface S with respect to a flag (C, x) based on the Zariski decomposition is given by Lazarsfeld and Mustaţă in [13, Theorem 6.4]. For comparison's sake, we include here a statement of their theorem.

Theorem (Lazarsfeld and Mustaţă). *The Okounkov body of an $\mathbb{R}$-divisor D on a smooth projective surface S with respect to a flag (C, p) is given as follows.*

$$\Delta = \{(t, y) \in \mathbb{R}^2 \mid \nu \leq t \leq \mu, \alpha(t) \leq y \leq \beta(t)\}$$

where

(1) $\nu = $ *the coefficient of C in the negative part of the Zariski decomposition of D,*
(2) $\mu = \max\{t | D - tC \text{ is effective }\}$
(3) $\alpha(t) = \mathrm{ord}_p(N_t \cdot C)$
(4) $\beta(t) = \mathrm{ord}_p(N_t \cdot C) + P_t \cdot C$

where $D - tC = P_t + N_t$ is a Zariski decomposition, P_t being the positive and N_t the negative part.

It was noted that it followed from this description that the Okounkov body was a possibly infinite polygon. We show that the Okounkov body is in fact a finite polygon and that the function α is in fact increasing : we show also that any polygon of the above form which satisfies these two additional conditions is in fact an Okounkov body, thereby giving a complete characterisation of Okounkov bodies on surfaces based on this work.

Theorem B. *The Okounkov body of an $\mathbb{R}$-divisor on a smooth projective surface with respect to some flag is a finite polygon. Up to translation, a real polygon $\Delta \subseteq \mathbb{R}_+^2$ is the Okounkov body of an $\mathbb{R}$-divisor D on a smooth projective surface S with respect to a complete flag (C, x) if and only if*

$$\Delta = \{(t, y) \in \mathbb{R}^2 \mid \nu \leq t \leq \mu, \alpha(t) \leq y \leq \beta(t)\}$$

for certain real numbers $0 \leq \nu \leq \mu$ and certain continuous piecewise linear functions $\alpha, \beta : [\nu, \mu] \to \mathbb{R}_+$ with rational slopes such that β is concave and α is increasing and convex.

When the divisor D is in fact a $\mathbb{Q}$-divisor, the break-points of the functions α and β occur at rational points and the number ν must be rational. We also show that the number μ might be irrational, but it satisfies a quadratic equation over $\mathbb{Q}$. We have not been able to establish which quadratic irrationals arise this way: Remark 2.3 links this problem to the irrationality of Seshadri constants.

Theorem B is proved using Zariski decomposition as in [13, Theorem 6.4]. More precisely, Theorem B is established via a detailed analysis of the variation of Zariski decomposition along a line segment. Conversely, we show that all convex bodies as in Theorem B are Okounkov bodies of divisors on smooth toric surfaces.

An example of a non-polyhedral Okounkov body in higher dimensions was given in [13, Section 6.3], so no simple characterisation of Okounkov bodies along the lines of Theorem B can hold in higher dimensions. However, it is expected that polyhedral Okounkov bodies are related to finite generation of rings of sections. In [13], Lazarsfeld and Mustaţă asked if every Mori dream space admits a flag with respect to which the global Okounkov cone is polyhedral. In Section 3 we give two examples of Mori spaces (one of which is $\mathbb{P}^2 \times \mathbb{P}^2$) equipped with flags with respect to which most Okounkov bodies are not polyhedral. The second example has the advantage that the shape of the Okounkov body in question is stable under generic deformations of the flag.

1. Countability of Okounkov bodies

In this section we prove Theorem A using global Okounkov cones. Let X be a smooth projective complex variety of dimension n and let $Y_\bullet$ be an admissible flag on X. Let $N^1(X)$ be the Néron-Severi group of X, while $N^1(X)_\mathbb{R}$ will denote the (finite-dimensional) vector space of numerical equivalence classes of $\mathbb{R}$-divisors.

Consider the additive sub-semigroup of $\mathbb{N}^n \times N^1(X)$

$$\Gamma_{Y_\bullet}(X) \overset{\text{def}}{=} \Big\{ (\nu_{Y_\bullet}(D), [L]) \mid L \text{ a line bundle on X with } D \geq 0 \text{ and } \mathcal{O}_X(D) \simeq L \Big\}.$$

The global Okounkov cone of X with respect to $Y_\bullet$ is then given by

$$\Delta_{Y_\bullet}(X) \overset{\text{def}}{=} \text{ closed convex cone generated by } \Gamma_{Y_\bullet}(X) \text{ inside } \mathbb{R}^n \times N^1(X)_\mathbb{R} \ .$$

Theorem B of [13] says that for any big class $\xi \in N^1(X)_\mathbb{Q}$ we have that

$$\Delta_{Y_\bullet}(X) \cap (\mathbb{R}^n \times \{\xi\}) \ = \ \Delta_{Y_\bullet}(X; \xi) \ .$$

Thus to prove Theorem A, it is enough to show the following claim, which establishes the countability of the set of global Okounkov cones.

Theorem 1.1. *There exists a countable set of closed convex cones $\Delta_i \subseteq \mathbb{R}^n \times \mathbb{R}^\rho$ with $i \in \mathbb{N}$ with the property that for any smooth, irreducible, projective variety X of dimension n and Picard number ρ and any admissible flag $Y_\bullet$ on X, there is an integral linear isomorphism*

$$\psi_X : \mathbb{R}^\rho \ \to \ N^1(X)_\mathbb{R},$$

depending only on X, such that $(\mathrm{id}_{\mathbb{R}^n} \times \psi_X^{-1})(\Delta_{Y_\bullet}(X))$ is equal to Δ_i for some $i \in \mathbb{N}$.

We say that ψ_X is *integral* if $\psi_X(\mathbb{Z}^\rho) \subset N^1(X)$.

Remark 1.2. In [13], Okounkov bodies were defined in a more general setup. The subvarieties Y_i were not assumed to be smooth, but merely irreducible, and smooth at the point Y_n. The statement of Theorem A can easily be generalised to flags of this form. For this, suppose that Theorem A holds under the hypothesis that each element of $Y_\bullet$ is smooth.

Consider now a smooth variety X with a flag $Y_\bullet$ of irreducible varieties, smooth at the point Y_n. Choose a proper birational map $\mu : X' \to X$, which is an isomorphism in some neighbourhood of Y_n, such that the proper transform Y_i' of each Y_i is smooth and irreducible. The flag $Y_\bullet'$ is then admissible in our sense and hence for any line bundle $\mathcal{L}$ on X there is an $i \in \mathbb{N}$ such that $\Delta_{Y_\bullet'}(X'; \mu^*\mathcal{L}) = \Delta_i$. By Zariski's Main Theorem $\mu_*(\mathcal{O}_{X'}) = \mathcal{O}_X$ and hence

$$H^0(X, \mathcal{L}^{\otimes m}) \ = \ H^0(X', \mu^*(\mathcal{L}^{\otimes m}))$$

for any $m \in \mathbb{N}$. Since μ is an isomorphism in a neighbourhood of Y_n, it follows that $\Delta_{Y_\bullet}(X; \mathcal{L}) = \Delta_{Y_\bullet'}(X'; \mu^*(\mathcal{L})) = \Delta_i$.

We now give some definitions and technical prerequisites needed in the proof of Theorem 1.1. We set

$$W \ \overset{\text{def}}{=} \ \underbrace{\mathbb{P}^{2n+1} \times \ldots \times \mathbb{P}^{2n+1}}_{\rho \text{ times}} \ .$$

Note that every line bundle on W has the form

$$\mathcal{O}_W(\underline{m}) \stackrel{\text{def}}{=} p_1^*(\mathcal{O}_{\mathbb{P}^{2n+1}}(m_1)) \otimes \ldots \otimes p_\rho^*(\mathcal{O}_{\mathbb{P}^{2n+1}}(m_\rho))$$

for some $\underline{m} \stackrel{\text{def}}{=} (m_1, \ldots, m_\rho) \in \mathbb{Z}^\rho$, where $p_i : W \to \mathbb{P}^{2n+1}$ is the projection onto the i-th factor. For a projective subscheme $X \subseteq W$ we define its multigraded Hilbert function by

$$P_X(\underline{m}) \stackrel{\text{def}}{=} \chi(X, (\mathcal{O}_W(\underline{m}))|_X), \text{ for all } \underline{m} \in \mathbb{Z}^\rho.$$

For any projective smooth subvariety $X \subseteq W$ we denote by ψ_X the map

$$\psi_X : \mathbb{Z}^\rho \to N^1(X),$$

where $\psi_X(\underline{m}) = [(\mathcal{O}_W(\underline{m}))|_X]$. We also denote the induced map $\psi_X : \mathbb{R}^\rho \to N^1(X)_\mathbb{R}$ by ψ_X.

Proposition 1.3. *Suppose given an $(n+1)$-tuple of numerical functions $\mathbf{P} = (P_0, \ldots, P_n)$, where $P_i : \mathbb{Z}^\rho \to \mathbb{Z}$ for all i. There exist a quasi-projective scheme $H_\mathbf{P}$, a closed subscheme $\mathcal{X}_\mathbf{P} \subset W \times H_\mathbf{P}$ and a flag of closed subschemes $\mathcal{Y}_{\bullet,\mathbf{P}} : \mathcal{X}_\mathbf{P} = \mathcal{Y}_0 \supset \mathcal{Y}_1 \supset \ldots \supset \mathcal{Y}_n$ such that*

(1) *the induced projection map $\phi_i : \mathcal{Y}_i \to H_\mathbf{P}$ is flat and surjective for all i,*
(2) *for all i and all $t \in H_\mathbf{P}$ we have that $P_{\mathcal{Y}_{i,t}} = P_i$,*
(3) *for any projective subvariety $X \subseteq W$ of dimension n and any complete flag of subvarieties $X = Y_0 \supset Y_1 \supset Y_2 \supset \ldots \supset Y_n$ such that $P_{Y_i} = P_i$ there exist a closed point $t \in H_\mathbf{P}$ and an isomorphism $\beta : \mathcal{X}_t \to X$ with the property that $\beta(\mathcal{Y}_{i,t}) = Y_i$ for all i.*

Proof. For each i, [8, Corollary 1.2] says that there exists a multigraded Hilbert scheme H_{P_i}. This is equipped with a flat surjective family $\mathcal{Y}_i' \subset W \times H_{P_i}$ that has the property that for any $Y_i' \subset W$ with $P_{Y_i'} = P_i$ there is a t such that $\mathcal{Y}_{i,t}' = Y_i'$. We now consider the multi-Hilbert scheme

$$H_\mathbf{P} \subset H_{P_0} \times H_{P_1} \times \ldots \times H_{P_n}$$

defined by the equations

$$\mathcal{I}(\mathcal{Y}_{i,h_i}') \supset \mathcal{I}(\mathcal{Y}_{i-1,h_{i-1}}')$$

for all i, where $\mathcal{I}(Y)$ is the ideal of a subscheme Y. Each element $\mathcal{Y}_i$ of the flag $\mathcal{Y}_{\bullet,\mathbf{P}}$ is defined by $\mathcal{Y}_i = \pi_i^*(\mathcal{Y}_i')$, where $\pi_i : H_\mathbf{P} \to H_{P_i}$ is the projection onto the factor H_{P_i}. By definition, $\mathcal{Y}_i \subset \mathcal{Y}_{i-1}$ for all i and $\mathcal{Y}_i \to H_\mathbf{P}$ is surjective and flat because $\mathcal{Y}_i'$ is: condition (1) therefore holds. Condition (2) is immediate. By the universal property of the multigraded Hilbert schemes H_{P_i}, Condition (3) is also satisfied. This completes the proof of Proposition 1.3. $\square$

Proof of Theorem 1.1. Let X be a smooth irreducible variety of dimension n and Picard number ρ, equipped with a flag $Y_\bullet$. We start by showing that X can then be embedded in W in such a way that the induced map of real vector spaces ψ_X is an integral isomorphism. Choose ρ very ample line bundles $\mathcal{L}_{1,X}, \ldots, \mathcal{L}_{\rho,X}$ on X forming a $\mathbb{Q}$-basis of $N^1(X)_\mathbb{Q}$. As X is smooth, [17, Theorem 5.4.9] says that for every i there is an embedding $\alpha_i : X \hookrightarrow \mathbb{P}^{2n+1}$ such that $\mathcal{L}_{i,X} = \alpha_i^*(\mathcal{O}_{\mathbb{P}^{2n+1}}(1))$. We can then embed X in W via

(1) $$X \xrightarrow{\Delta} \underbrace{X \times \ldots \times X}_{\rho \text{ times}} \xrightarrow{\alpha_1 \times \ldots \times \alpha_\rho} W$$

where Δ is the diagonal morphism. Note that $\psi_X : \mathbb{R}^\rho \to N^1(X)_\mathbb{R}$ is an integral linear isomorphism by construction.

Let us now consider the admissible flag $Y_\bullet$ on X; the associated multigraded Hilbert functions P_{Y_i} are polynomials with rational coefficients. There are therefore only countably many $(n + 1)$-tuples of numerical functions $\mathbf{P}$ which appear as the multigraded Hilbert function of a smooth n-dimensional subvariety of W, equipped with an admissible flag.

By Proposition 1.3, there exist countably many quasi-projective schemes T_j and closed subschemes $\mathcal{X}_j \subset W \times T_j$, each equipped with a flag $\mathcal{Y}_{\bullet,j}$ with the following property: for any smooth irreducible variety X of dimension n and Picard number ρ and any admissible flag $Y_\bullet$ on X there is a closed point $t \in T_j$ for some j such that the variety-flag pair $(\mathcal{X}_{j,t}, \{\mathcal{Y}_{\bullet,j,t}\})$ is isomorphic to the variety-flag pair $(X, \{Y_\bullet\})$ and the map $\psi_X : \mathbb{R}^\rho \to N^1(X)_\mathbb{R}$ is an integral linear isomorphism. Let T_j^{red} be T_j with its reduced scheme structure and let $\mathcal{X}_j^{\mathrm{red}}$ and $\mathcal{Y}_{i,j}^{\mathrm{red}}$ be the pull-backs of $\mathcal{X}_j$ and $\mathcal{Y}_{i,j}$ along the inclusion map $T_j^{\mathrm{red}} \to T_j$. (Note that this is not necessarily the reduced scheme structure on $\mathcal{X}_j$ and $\mathcal{Y}_{i,j}$.) We note that any closed fibre of $\mathcal{X}_j, \mathcal{Y}_{i,j} \to T_j$ is also a closed fibre of $\mathcal{X}_j^{\mathrm{red}}, \mathcal{Y}_{i,j}^{\mathrm{red}} \to T_j^{\mathrm{red}}$.

Through the rest of the proof of Theorem 1.1, T will be a reduced and irreducible quasi-projective scheme and $\mathcal{X} \subset W \times T$ will be a closed subscheme such that the induced projection map $\phi : \mathcal{X} \to T$ is surjective, flat and projective. We suppose given a flag of closed subschemes of $\mathcal{X}$

$$\mathcal{Y}_\bullet : \quad \mathcal{X} = \mathcal{Y}_0 \supseteq \mathcal{Y}_1 \supseteq \dots \supseteq \mathcal{Y}_{n-1} \supseteq \mathcal{Y}_n$$

such that the restriction maps $\phi_i \overset{\mathrm{def}}{=} \phi|_{\mathcal{Y}_i} : \mathcal{Y}_i \to T$ are flat, projective and surjective. We say that $t \in T$ has an *admissible fibre* if the fibre X_t is smooth and irreducible and the flag $Y_{\bullet,t}$ is admissible. We will say that $t \in T$ is *fully admissible* if it has an admissible fibre and the induced map $\psi_{X_t} : \mathbb{R}^\rho \to N^1(X_t)_\mathbb{R}$ is an isomorphism. We now prove the following proposition.

Proposition 1.4. *Given T, $\mathcal{X}$ and $\mathcal{Y}_i$ as above, there exists a countable set of convex cones $(\Delta_i)_{i \in \mathbb{N}} \subset \mathbb{R}^n \times \mathbb{R}^\rho$ such that for any fully admissible $t \in T$ the cone $(\mathrm{id}_{\mathbb{R}^n} \times \psi_{X_t}^{-1})(\Delta_{Y_{\bullet,t}}(X_t))$ is equal to Δ_i for some i.*

Proof of Proposition 1.4. Since the proposition is immediate if there is no fully admissible t we may assume at least one such t exists.

By induction on the dimension of T, it will be enough to prove the existence of a nontrivial open subset $U \subseteq T$ such that the conclusions of Proposition 1.4 hold for any fully admissible $t \in U$. We can therefore assume T is smooth and there exists a fully admissible closed point $t_0 \in T$. Since each ϕ_i is flat, by [7, Theorem 12.2.4] the set of points $t \in T$ such that $Y_{i,t}$ is smooth and irreducible for every i is open in T. We can therefore assume that all $t \in T$ have an admissible fibre.

Since X_t is smooth for all $t \in T$, the map ϕ is smooth by [9, Theorem III.10.2]. Therefore we can apply [11, Proposition 2.5] or Ehresmann's theorem to deduce that the map ψ_{X_t} is injective for all $t \in T$.

With this in mind, it will be enough to show that under the above hypotheses, the set

$$\{(\mathrm{id}_{\mathbb{R}^n} \times \psi_{X_t}^{-1})(\Delta_{Y_{\bullet,t}}(X_t)) \mid t \in T \}$$

is countable. We shall now consider for some t the set $(\mathrm{id}_{\mathbb{R}^n} \times \psi_{X_t}^{-1})(\Delta_{Y_\bullet,t}(X_t))$, or in other words, the subset of $\Delta_{Y_\bullet,t}(X_t)$ lying over $\mathrm{Im}(\psi_{X_t}) \subset N^1(X)_{\mathbb{R}}$. Since we know that the map ψ_{X_t} is injective for all $t \in T$, for any two points $t_1, t_2 \in T$ we have that

$$(\mathrm{id}_{\mathbb{R}^n} \times \psi_{X_{t_1}}^{-1})(\Delta_{Y_\bullet,t_1}(X_{t_1})) = (\mathrm{id}_{\mathbb{R}^n} \times \psi_{X_{t_2}}^{-1})(\Delta_{Y_\bullet,t_2}(X_{t_2}))$$

if and only if $\Delta_{Y_\bullet,t_1}(X_{t_1}; (\mathcal{O}_W(\underline{m}))|_{X_{t_1}})$ is equal to $\Delta_{Y_\bullet,t_2}(X_{t_2}; (\mathcal{O}_W(\underline{m}))|_{X_{t_2}})$ for every $\underline{m} \in \mathbb{Z}^\rho$. This will be the case whenever it happens that
(2)
$$\mathrm{Im}(\nu_{Y_\bullet,t_1} : \ H^0(X_{t_1}, (\mathcal{O}_W(\underline{m}))|_{X_{t_1}}) \ \to \ \mathbb{Z}^n) = \mathrm{Im}(\nu_{Y_\bullet,t_2} : \ H^0(X_{t_2}, (\mathcal{O}_W(\underline{m}))|_{X_{t_2}}) \ \to \ \mathbb{Z}^n)$$

for any $\underline{m} \in \mathbb{Z}^\rho$. It will then be enough to show the existence for every $\underline{m}$ of a proper Zariski-closed subset $F_{\underline{m}} \subset T$ such that $\mathrm{Im}(\nu_{Y_\bullet,t} : \ H^0(X_t, (\mathcal{O}_W(\underline{m}))|_{X_t})) \ \to \ \mathbb{Z}^n$ is constant on $T \setminus F_{\underline{m}}$. We would then have a subset $F = \cup F_{\underline{m}} \subseteq T$ consisting of a countable union of proper Zariski-closed subsets $F_{\underline{m}} \subsetneqq T$ such that (2) holds for every $\underline{m} \in \mathbb{Z}^\rho$ whenever $t_1, t_2 \in T \setminus F$. By induction on $\dim(T)$, this implies Proposition 1.4.

We recall that each morphism $\phi_i : \mathcal{Y}_i \to T$ has smooth irreducible fibres, so by [9, Theorem III.10.2] ϕ_i is smooth. Since T is smooth each $\mathcal{Y}_i$ is smooth and hence $\mathcal{Y}_{i+1} \subseteq \mathcal{Y}_i$ is Cartier. Thus our family of flags satisfies the conditions of [13, Theorem 5.1], and consequently, for any $\underline{m}$ there exists a proper closed subset $F_{\underline{m}} \subseteq T$ such that the sets

(3)
$$\mathrm{Im}(\nu_{Y_\bullet,t} : \ H^0(X_t, (\mathcal{O}_W(\underline{m}))|_{X_t}) \ \to \ \mathbb{Z}^n)$$

coincide for all $t \notin F_{\underline{m}}$. Thus upon setting $F = \cup F_{\underline{m}}$, F has the properties we seek and

$$(\mathrm{id}_{\mathbb{R}^n} \times \psi_{X_t}^{-1})(\Delta_{Y_\bullet,t}(X_t)) \ \subseteq \ \mathbb{R}^n \times \mathbb{R}^\rho$$

is independent of $t \in T \setminus F$. When $t \in T$ is fully admissible ψ_{X_t} is an isomorphism and this completes the proof of Proposition 1.4. □

By the above, any smooth variety of dimension n and Picard number ρ, and any admissible flag on it, is a fibre in one of the countably many variety-flag pairs $(\mathcal{X}_{j,t}, \{\mathcal{Y}_{\bullet,j,t}\})$. Thus, by Proposition 1.4, we deduce the countability of global Okounkov cones. □

2. Conditions on Okounkov bodies on surfaces.

We turn our attention to Theorem B which characterises the convex bodies arising as Okounkov bodies of big $\mathbb{R}$-divisors on smooth surfaces. Whilst we do not characterise them completely, we also establish fairly strong conditions on the set of convex bodies which are Okounkov bodies of $\mathbb{Q}$-divisors. Our main technical tool will be Zariski decomposition of divisors.

Throughout the rest of this section, S will be a smooth surface equipped with an admissible flag (C, x), consisting of a smooth curve $C \subseteq S$ and a point $x \in C$, and D will be a pseudo-effective real (or rational) divisor on S.

Any pseudo-effective divisor D has a *Zariski decomposition*, (the effective case was treated in [18] and the pseudo-effective one in [6]; see also [1, Theorem 14.14] for an account of the proof of this fact). By a Zariski decomposition of D we mean that D can be uniquely written as a sum

$$D = P(D) + N(D)$$

of $\mathbb{R}$-divisors (or $\mathbb{Q}$-divisors whenever D is such) with the property that $P(D)$ is nef, $N(D)$ is either zero or effective with negative definite intersection matrix, and $(P(D).E) = 0$ for every irreducible component E of $N(D)$. $P(D)$ is called the *positive part* of D and $N(D)$ the *negative part.* Another important property of the Zariski decomposition is the minimality of the negative part (first proved in [18], cf. [1, Lemma 14.10]). This states that if $D = M + N$, where M is nef and N effective, then $N - N(D)$ is effective.

We prove Theorem B using Lazarsfeld and Mustaţă's description of the Okounkov body of a divisor on a surface [13, Theorem 6.4] via Zariski decomposition. Let ν be the coefficient of C in the negative part $N(D)$ and set

$$\mu = \mu(D; C) = \sup\{ t > 0 \mid D - tC \text{ is big } \}.$$

When there is no risk of confusion we will denote $\mu(D; C)$ by $\mu(D)$. For any $t \in [\nu, \mu]$ we set $D_t = D - tC$ and write $D_t = P_t + N_t$ for its Zariski decomposition. There then exist two continuous functions $\alpha, \beta : [\nu, \mu] \to \mathbb{R}_+$ defined as follows

$$\alpha(t) = \mathrm{ord}_x(N_t|_C), \quad \beta(t) = \mathrm{ord}_x(N_t|_C) + P_t \cdot C$$

such that the Okounkov body $\Delta_{(C,x)}(S; D) \subseteq \mathbb{R}^2$ is the region bounded by the graph of α and β:

$$\Delta_{(C,x)}(S; D) = \{(t, y) \in \mathbb{R}^2 \mid \nu \le t \le \mu, \alpha(t) \le y \le \beta(t)\} .$$

We now set $D' = D - \mu C$: the divisor D' is pseudo-effective by definition of μ. For any $t \in [\nu, \mu]$ we write $s = \mu - t$ and set

$$D'_s \stackrel{\mathrm{def}}{=} D' + sC = D' + (\mu - t)C = D - tC.$$

It turns out to be more useful to consider the line segment $\{D_t \mid t \in [\nu, \mu]\}$ in the form $\{D'_s \mid s \in [0, \mu - \nu]\}$. Let $D'_s = P'_s + N'_s$ be the Zariski decomposition of D'_s: the following proposition examines the variation N'_s as a function of $s \in [0, \mu - \nu]$.

Proposition 2.1. *With the definitions above we have the following:*

    (i) *The function $s \mapsto N'_s$ is decreasing on the interval $[0, \mu - \nu]$, i.e. for each $0 \le s' < s \le \mu - \nu$ the divisor $N'_{s'} - N'_s$ is effective.*

    (ii) *If n is the number of irreducible components of N'_0, then there is a partition $(p_i)_{0 \le i \le k}$ of the interval $[0, \mu - \nu]$, for some $k \le n$, and there exist divisors A_i and B_i with B_i rational such that $N'_s = A_i + sB_i$ for all $s \in [p_i, p_{i+1}]$.*

Proof. Let $C_1, \ldots, C_n$ be the irreducible components of $\mathrm{Supp}(N'_0)$. Choose real numbers s', s such that $0 \le s' < s \le \mu - \nu$. We can then write

$$P'_{s'} = D'_{s'} - N'_{s'} = (D'_s - (s - s')C) - N'_{s'} = D'_s - ((s - s')C + N'_{s'}).$$

As $P'_{s'}$ is nef and the negative part of the Zariski decomposition is minimal, the divisor $(s - s')C + N'_{s'} - N'_s$ is effective and it remains only to show that C is not in the support of N'_s for any $s \in [0, \mu - \nu]$. If C were in the support of N'_s for some $s \in [0, \mu - \nu]$, then for any $\lambda > 0$ the Zariski decomposition of $D'_{s+\lambda}$ would be $D'_{s+\lambda} = P'_s + (N'_s + \lambda C)$. In particular, C would be in the support of $N'_{\mu-\nu}$, contradicting the definition of ν.

Rearranging the C_i's, suppose that the support of $N'_{\mu-\nu}$ consists of $C_{k+1}, \ldots, C_n$. Let

$$p_i \stackrel{\mathrm{def}}{=} \sup\{s \mid C_i \subseteq \mathrm{Supp}(N'_s)\} \text{ for all } i = 1 \ldots k .$$

Without loss of generality, suppose $0 = p_0 < p_1 \leq \ldots \leq p_{k-1} \leq p_k \leq \mu - \nu$. We will show that N'_s is linear on $[p_i, p_{i+1}]$ for this choice of p_i's. By the continuity of the Zariski decomposition (see [2, Proposition 1.14]), it is enough to show that N'_s is linear on the open interval (p_i, p_{i+1}). If $s \in (p_i, p_{i+1})$ then the support of N'_s is contained in $\{C_{i+1}, \ldots, C_n\}$, and N'_s is determined uniquely by the equations

$$N'_s \cdot C_j = (D' + sC) \cdot C_j, \text{ for } i + 1 \leq j \leq n .$$

As the intersection matrix of the curves $C_{i+1}, \ldots, C_n$ is non-degenerate, there are unique divisors A_i and B_i supported on $\cup_{j=i+1}^n C_j$ such that

$$A_i \cdot C_j = D' \cdot C_j \text{ and } B_i \cdot C_j = C \cdot C_j \text{ for all } i + 1 \leq j \leq n .$$

Note that B_i is a rational divisor and it follows that $N'_s = A_i + sB_i$ for any $s \in (p_i, p_{i+1})$. $\square$

Proof of Theorem B. Theorem 6.4 of [13] implies that α is convex, β is concave and $\alpha \leq \beta$. It follows from Proposition 2.1 that α and β are piecewise linear with only finitely many breakpoints. And finally α is an increasing function of t by Proposition 2.1, because $N_t = N'_{\mu - s}$ and $\alpha(t) = \text{ord}_x(N_t|C)$. This proves that any Okounkov body has the required form.

Conversely, we show that a polygon as in Theorem B is the Okounkov body of a real T-invariant divisor on some toric surface[1]. This section of the proof is based on Proposition 6.1 in [13] which characterises the Okounkov body of a T-invariant divisor with respect to a T-invariant flag in a toric variety in terms of the polygon associated to T in the character lattice $M_\mathbb{Z}$ associated to S.

Let $\Delta \subseteq \mathbb{R}^2$ be a polygon of the form given in Theorem B. As α is increasing we can assume after translation that $(0, 0) \in \Delta \subseteq \mathbb{R}^2_+$. We identify $\mathbb{R}^2$ with the vector space $M_\mathbb{R}$ associated to a character lattice $M_\mathbb{Z} = \mathbb{Z}^2$. Let $E_1, \ldots, E_m$ be the edges of Δ. Considering that α and β have rational slopes, for each edge E_i choose a primitive vector $v_i \in N_\mathbb{Z}$ normal to E_i in the direction of the interior of Δ, where $N_\mathbb{R}$ is the dual of $M_\mathbb{R}$. We can then write

$$\Delta = \{ u \in M_\mathbb{R} \mid \langle u, v_i \rangle + a_i \geq 0 \text{ for all } i = 1 \ldots m \}$$

for some positive real a_i's. After adding additional vectors $v_{m+1}, \ldots, v_r$ we can assume that the set $\{v_1, \ldots, v_m\}$ has the following properties.

(1) The toric surface S associated to the complete fan Σ which is defined by the rays $\{\mathbb{R}_+ \cdot v_1, \ldots, \mathbb{R}_+ \cdot v_r\}$ is smooth.
(2) None of the v_i's lie in the interior of the first quadrant.
(3) For some $i_1, i_2 \in \{1, \ldots, m\}$ we have that $v_{i_1} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $v_{i_2} = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$

Condition (2) is possible because α is increasing. Since Δ is compact there exist real numbers $a_{m+1}, \ldots, a_r \in \mathbb{Q}_+$ such that

$$\Delta = \{ u \in M_\mathbb{R} \mid \langle u, v_i \rangle + a_i \geq 0 \text{ for all } i = 1 \ldots r \}$$

Condition (3) implies that we can choose $a_{i_1} = a_{i_2} = 0$. The general theory of toric surfaces now tells us that each v_i represents a T-invariant divisor D_i on S and on setting $D = \Sigma a_i D_i$

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[1]We thank Sebastien Boucksom, who suggested using toric surfaces, to replace a more complicated example using iterated blow-ups of $\mathbb{P}^2$.

the polytope $P(D) \subseteq M_{\mathbb{R}}$ associated to D is equal to Δ. We choose on S the flag consisting of the curve $C = D_{i_1}$ and the point $\{x\} = D_{i_1} \cap D_{i_2}$. The curve C is smooth and the intersection $D_{i_1} \cap D_{i_2}$ is a point because of conditions (1) and (2). By [13, Proposition 6.1], the Okounkov body $\Delta_{(C,x)}(S; D)$ of D with respect to the flag (C, x) is equal to $\psi_{\mathbb{R}}(P(D))$ where the map $\psi_{\mathbb{R}} : M_{\mathbb{R}} \to \mathbb{R}^2$ is defined as follows

$$\psi_{\mathbb{R}}(u) \;=\; (\langle u, v_{i_1} \rangle, \langle u, v_{i_2} \rangle) \text{ for any } u \in M_{\mathbb{R}}.$$

In our case $\psi_{\mathbb{R}} \equiv \mathrm{id}_{\mathbb{R}}$, so $\Delta_{(C,x)}(S; D) = P(D) = \Delta$ by construction. This completes the proof of Theorem B. $\qquad\square$

It is now natural to ask the following question: which of these polygons is the Okounkov body of a rational divisor? The above toric-surface construction implies that any polygon of the form considered in Theorem B which is given by rational data is the Okounkov body of a rational divisor. The next result provides a partial converse to the effect that the rationality of the divisor imposes strong rationality conditions on the points of the Okounkov body.

Proposition 2.2. *Let S be a smooth projective surface, D a big rational divisor on S and (C, x) be an admissible flag on S. Then:*

(1) *All the vertices of the polygon $\Delta(D)$ contained in the set $\{[\nu, \mu) \times \mathbb{R}\}$ have rational coordinates.*
(2) *$\mu(D)$ is either rational or satisfies a quadratic equation over $\mathbb{Q}$.*
(3) *If an irrational number $a > 0$ satisfies a quadratic equation over $\mathbb{Q}$ and the conjugate $\bar{a}$ of a over $\mathbb{Q}$ is strictly larger than a, then there exist a smooth, projective surface S, an ample $\mathbb{Q}$-divisor D and an admissible flag on S such that $\mu(D) = a$.*

Proof. The number ν is rational because the positive and negative parts of the Zariski decomposition of a $\mathbb{Q}$-divisor are rational: it follows that $\alpha(\nu)$ and $\beta(\nu)$ are rational. It follows from the proof of Proposition 2.1 that the break-points of α and β occur at points t_i which are intersection points between the line $D - tC$ and faces of the Zariski chamber decomposition of the cone of big divisors [2]. However, it is proved in [2, Theorem 1.1] that this decomposition is locally finite rational polyhedral, and hence the break-points of α and β occur at rational points.

For (2), notice that the volume $\mathrm{vol}_X(D)$, which is twice the area of the Okounkov polygon $\Delta(D)$, is rational (see [12, Corollary 2.3.22]). As the slopes and intermediate break-points of $\Delta(D)$ are rational, the equation computing the area of $\Delta(D)$ gives a quadratic equation for $\mu(D)$ with rational coefficients. Note that if μ is irrational then one edge of the polygon $\Delta(D)$ must sit on the vertical line $t = \mu$.

The final part of the proposition follows from a result of Morrison's [14] which states that any even integral quadratic form q of signature $(1, 2)$ occurs as the self-intersection form of a K3 surface S with Picard number 3. An argument of Cutkosky's [4, Section 3] shows that if the coefficients of the form are all divisible by 4, then the pseudo-effective and nef cones of S coincide and are given by

$$\{\alpha \in N^1(S) \,|\, (\alpha^2) \geq 0 \,,\, (h \cdot \alpha) > 0\}$$

for any ample divisor h on S. If D is an ample divisor and $C \subseteq S$ an irreducible curve (not in the same class as D), then the function $f(t) \stackrel{\text{def}}{=} ((D - tC)^2)$ has two positive roots and $\mu(D)$ with respect to C is equal to the smaller one, i.e.

$$\mu(D) \;=\; \frac{(D \cdot C) - \sqrt{(D \cdot C)^2 - (D^2)(C^2)}}{(C^2)} \;.$$

Since we are only interested in the roots of f we can start with any integral quadratic form of signature $(1, 2)$ and multiply it by 4. Hence we can exhibit any number with the required properties as $\mu(D)$ for suitable choices of the quadratic form, D, and C. $\qquad\square$

Remark 2.3. It was mentioned in passing in [13] that the knowledge of all Okounkov bodies determines Seshadri constants. In the surface case, there is a link between the irrationality of μ for certain special forms of the flag and that of Seshadri constants. Let D be an ample divisor on S, and let $\pi : \widetilde{S} \to S$ be the blow-up of a point $x \in S$ with exceptional divisor E. Then the Seshadri constant of D at x is defined by

$$\epsilon(D, x) \stackrel{\text{def}}{=} \sup\{t \in \mathbb{R} \,|\, \pi^*(D) - tE \text{ is nef in } \widetilde{S}\}.$$

We note that if $\epsilon(D, x)$ is irrational, then $\epsilon(D, x) = \mu(\pi^*(D))$ with respect to any flag of the form (E, y). Indeed, the Nakai–Moishezon criterion implies that either there is a curve $C \subseteq \widetilde{S}$ such that $C \cdot (\pi^*(D) - \epsilon E) = 0$ or $((\pi^*(D) - \epsilon E)^2) = 0$. But since $C \cdot \pi^*(D)$ and $C \cdot E$ are both rational, $C \cdot (\pi^*(D) - \epsilon E) = 0$ is impossible if ϵ is irrational. Therefore $(\pi^*(D) - \epsilon E)^2 = 0$, hence $\pi^*(D) - \epsilon E$ is not big and therefore $\epsilon = \mu \notin \mathbb{Q}$.

3. Non-polyhedral Okounkov bodies

In this section we will give two examples of non-polyhedral Okounkov bodies of divisors on Mori dream space varieties, thereby showing in particular that ample divisors can nevertheless have non-polyhedral Okounkov bodies. The first example is Fano; the second one is not, but has the advantage that the non-polyhedral shape of Okounkov bodies is stable under generic perturbations of the flag.

Proposition 3.1. *Let X be a smooth projective variety of dimension n equipped with an admissible flag $Y_\bullet$. Suppose that D is a divisor such that $D - sY_1$ is ample. Then we have the following lifting property*

$$\Delta_{Y_\bullet}(X; D) \cap \left(\{s\} \times \mathbb{R}^{n-1}\right) = \Delta_{Y_\bullet}(Y_1; (D - sY_1)|_{Y_1}).$$

In particular, if $\overline{\mathrm{Eff}}(X)_\mathbb{R} = \mathrm{Nef}(X)_\mathbb{R}$ then on setting $\mu(D; Y_1) = \sup\{\, t > 0 \mid D - tY_1 \text{ ample }\}$ we have that the Okounkov body $\Delta_{Y_\bullet}(X; D)$ is the closure in $\mathbb{R}^n$ of the following set

$$\{(s, \underline{v}) \mid 0 \le s < \mu(D; Y_1), \underline{v} \in \Delta_{Y_\bullet}(Y_1; (D - sY_1)|_{Y_1})\}$$

Proof. In order to prove the lifting property we will use [13, Theorem 4.26], which in our context states that

$$\Delta_{Y_\bullet}(X; D) \cap \left(\{s\} \times \mathbb{R}^{n-1}\right) \;=\; \Delta_{Y_\bullet}(X|Y_1, D - sY_1)$$

where the second body is the restricted Okounkov body defined in [13, Section 2.4]. Hence it is enough to show that

$$(4) \qquad \Delta_{Y_\bullet}(X|Y_1, D - sY_1) = \Delta_{Y_\bullet}(Y_1, (D - sY_1)|_{Y_1}) .$$

We will prove this for $s \in \mathbb{Q}_+$, as the general case follows from the continuity of slices of Okounkov bodies. Combining [13, Theorem 4.26] and [13, Proposition 4.1] we obtain that the restricted Okounkov body satisfies the required homogeneity condition, i.e.

$$\Delta_{Y_\bullet}(X|Y_1, p(D - sY_1)) = p\Delta_{Y_\bullet}(X|Y_1, (D - sY_1)) \text{ for all } p \in \mathbb{N} .$$

The construction of restricted Okounkov bodies tells us that (4) will follow if one can check that $H^1(X, m(p(D - sY_1) - Y_1)) = 0$ for sufficiently large divisible $p, m \in \mathbb{N}$. As $D - sY_1$ is an ample divisor, this follows from Serre vanishing. $\qquad \square$

By combining Proposition 3.1 and the description of the Okounkov body of divisors on surfaces given in [13, Theorem 6.4], we can give an easy description of Okounkov bodies on certain three-folds.

Corollary 3.2. *Let X be a smooth three-fold and $Y_\bullet = (X, S, C, x)$ an admissible flag on X. Suppose that $\overline{\mathrm{Eff}}(X)_\mathbb{R} = \mathrm{Nef}(X)_\mathbb{R}$ and $\overline{\mathrm{Eff}}(S)_\mathbb{R} = \mathrm{Nef}(S)_\mathbb{R}$. The Okounkov body of any ample divisor D with respect to the admissible flag $Y_\bullet$ can be described as follows*

$$\Delta_{Y_\bullet}(X; D) = \{(r, t, y) \in \mathbb{R}^3 \mid 0 \le r \le \mu(D; S), 0 \le t \le f(r), 0 \le y \le g(r, t) \}$$

where $f(r) = \sup\{ s > 0 \mid (D - rS)|_S - sC \text{ is ample } \}$ and $g(r, t) = (C.(D - rS)|_S) - t(C^2)$. (All intersection numbers in the above formulae are defined with respect to the intersection form on S.)

This corollary follows from Proposition 3.1 on noting that:

(1) The function $f(r)$ is simply the function $f(r) = \mu((D - rS)|_S; C)$ where $\mu(D|_S, C)$ is the function defined in the discussion of Okounkov bodies on surfaces.
(2) Since all pseudo-effective lines bundles on S are nef we have that $N_{r,t} = N((D - rS)|_S - tC) = 0$ for all t such that $D - rS - tC$ is pseff - i.e. $\alpha_r(t) = \mathrm{ord}_p(N_{r,t} \cdot C) = 0$.
(3) Likewise, we have that $P_{r,t} = P((D - rS)|_S - tC) = ((D - rS)|_S - tC)$ and hence $\beta_r(t) = \mathrm{ord}_p(N_{r,t} \cdot C) + P_{r,t} \cdot C = g(r, t)$

Remark 3.3. In the context of Corollary 3.2 the data of the function $f : [0, \mu(D; S)] \to \mathbb{R}_+$ can force the associated Okounkov bodies to be non-polyhedral. Note that $f(r)$ is the real number such that $(D - rS)|_S - f(r)C$ lies on the boundary of the pseudo-effective cone of S, which under our assumptions coincides with the nef cone. The graph of $f(r)$ is therefore (an affine transformation of) the curve obtained by intersecting the boundary of $\mathrm{Nef}(S)_\mathbb{R}$ with the plane passing through $[D|_S], [(D - S)|_S]$ and $[D|_S - C]$ inside the vector space $N^1(S)_\mathbb{R}$. If the Picard group of S has dimension at least three and the boundary of the nef cone of S is defined by quadratic rather than linear equations then this intersection will typically be a conic curve, not piecewise linear.

Example 3.4 (Non-polyhedral Okounkov body on a Fano variety). We set $X = \mathbb{P}^2 \times \mathbb{P}^2$ and let D be a divisor in the linear series $\mathcal{O}_{\mathbb{P}^2 \times \mathbb{P}^2}(3, 1)$. We set

$$Y_\bullet : Y_0 = \mathbb{P}^2 \times \mathbb{P}^2 \supseteq Y_1 = \mathbb{P}^2 \times E \supseteq Y_2 = E \times E \supseteq Y_3 = C \supseteq Y_4 = \{\mathrm{pt}\}$$

where E is a general elliptic curve. Since E is general we have that

$$\overline{\mathrm{Eff}}(E \times E)_{\mathbb{R}} = \mathrm{Nef}(E \times E)_{\mathbb{R}} = \{(x, y, z) \in \mathbb{R}^3 \mid x + y + z \geq 0, xy + xz + yz \geq 0\}$$

under the identification

$$\mathbb{R}^3 \to N^1(E \times E)_{\mathbb{R}}, \ (x, y, z) \to xf_1 + yf_2 + z\Delta_E,$$

where $f_1 = \{\mathrm{pt}\} \times E$, $f_2 = E \times \{\mathrm{pt}\}$ and Δ_E is the diagonal divisor. Let $C \subseteq E \times E$ be a smooth general curve in the complete linear series $|f_1 + f_2 + \Delta_E|$ and let Y_4 be a general point on C. To prove that the Okounkov body $\Delta_{Y_\bullet}(X; D)$ is not polyhedral it will be enough to prove that the slice $\Delta_{Y_\bullet}(X; D) \cap \{0 \times \mathbb{R}^2\}$ is not polyhedral. Since $\overline{\mathrm{Eff}}(\mathbb{P}^2 \times \mathbb{P}^2)_{\mathbb{R}} = \mathrm{Nef}(\mathbb{P}^2 \times \mathbb{P}^2)_{\mathbb{R}}$, Proposition 3.1 applies and it will be enough to show that $\Delta_{Y_\bullet}(Y_1; \mathcal{O}_{Y_1}(D))$ is not polyhedral.

The threefold $Y_1 = \mathbb{P}^2 \times E$ is homogeneous, so its nef cone is equal to its pseudo-effective cone: this cone is bounded by the rays $\mathbb{R}_+[\mathrm{line} \times E]$ and $\mathbb{R}_+[\mathbb{P}^2 \times \{\mathrm{pt}\}]$. We note that hypotheses of Corollary 3.2 therefore apply to Y_1 equipped with the flag (Y_2, Y_3, Y_4).

Using the explicit description given above of $\mathrm{Nef}(Y_1)_{\mathbb{R}}$, we see that $\mu(\mathcal{O}_{Y_1}(D), Y_2) = 1$. A simple calculation gives us

$$g(r, t) \ = \ (C.(D|_{Y_1} - rY_2)|_{Y_2}) - t(C^2) \ = \ 24 - 18r - 6t.$$

Let us now consider

$$f(r) \ = \ \sup\{s > 0 \mid (D - rY_2)|_{Y_2} - sC \text{ is ample}\}$$
$$= \ \sup\{s > 0 \mid (9 - 9r - s)f_1 + (3 - s)f_2 - s\Delta_E \text{ is ample }\}.$$

After calculation, we see that for positive s the divisor $(9 - 9r - s)f_1 + (3 - s)f_2 - s\Delta_E$ is ample if and only if $s < (4 - 3r - \sqrt{9r^2 - 15r + 7})$. Corollary 3.2 therefore tells us that the Okounkov body of D on Y_1, $\Delta_{Y_\bullet}(Y_1; D)$, has the following description

$$\{(r, t, y) \in \mathbb{R}^3 | 0 \leq r \leq 1, 0 \leq t \leq 4 - 3r - \sqrt{9r^2 - 15r + 7}, 0 \leq y \leq 24 - 18r - 6t\}.$$

As this body is non-polyhedral, the same can be said about the Okounkov body $\Delta_{Y_\bullet}(X; D)$.

In the following, we give an example of a Mori dream space such that the Okounkov body of a general ample divisor is non-polyhedral and remains so after generic deformations of the flag in its linear equivalence class. Our construction is based heavily on an example of Cutkosky's [4]. Cutkosky considers a K3 surface S whose Néron-Severi space $N^1(S)_{\mathbb{R}}$ is isomorphic to $\mathbb{R}^3$ with the lattice $\mathbb{Z}^3$ and the intersection form $q(x, y, z) = 4x^2 - 4y^2 - 4z^2$. Cutkosky shows that:

(1) The divisor class on S represented by the vector $(1, 0, 0)$ corresponds to the class of a very ample line bundle, which embeds S in $\mathbb{P}^3$ as a quartic surface.
(2) The nef and pseudo-effective cones of S coincide, and a vector $(x, y, z) \in \mathbb{R}^3$ represents a nef (pseudo-effective) class if it satisfies the inequalities

$$4x^2 - 4y^2 - 4z^2 \geq 0 \ , \ x \geq 0 \ .$$

We consider the surface $S \subset \mathbb{P}^3$, and the pseudo-effective classes on S given by $\alpha = (1, 1, 0)$ and $\beta = (1, 0, 1)$. By Riemann-Roch we have that $H^0(S, \alpha) \geq 2$ and $H^0(S, \beta) \geq 2$, so both α and β, being extremal rays in the effective cone, are classes of irreducible moving curves.

Since $\alpha^2 = \beta^2 = 0$, both these families are base-point free, and it follows from the base-point free Bertini theorem (see page 109 in [5]) that there are smooth irreducible curves C_1 and C_2 representing α and β respectively, which are elliptic by the adjunction formula. We may assume that C_1 and C_2 meet transversally in $C_1 \cdot C_2 = 4$ points.

Our threefold Z is constructed as follows. Let $\pi_1 : Z_1 \to \mathbb{P}^3$ be the blow-up along the curve $C_1 \subseteq \mathbb{P}^3$. We then define Z to be the blow up of the strict transform $\overline{C}_2 \subseteq Z_1$ of the curve C_2. Let $\pi_2 : Z \to Z_1$ be the second blow-up and π the composition $\pi_1 \circ \pi_2 : Z \to \mathbb{P}^3$. We denote by E_2 the exceptional divisor of π_2 and by E_1 the strict transform of the exceptional divisor of π_1 under π_2. We have the following proposition.

Proposition 3.5. *The variety Z defined above is a Mori dream space, and $-K_Z$ is effective. Given any two ample divisors on Z, L and D, such that the classes $[D], [L], [-K_Z]$ are linearly independent in $N^1(Z)_\mathbb{R}$, the Okounkov body $\Delta_{Y_\bullet}(X; D)$ is non-polyhedral with respect to any admissible flag (Y_1, Y_2, Y_3) such that $\mathcal{O}_Z(Y_1) = -K_Z$, $\mathrm{Pic}(Y_1) = \langle H, C_1, C_2 \rangle$ and $\mathcal{O}_{Y_1}(Y_2) = L|_{Y_1}$, where H is the pull-back of a hyperplane section of $\mathbb{P}^3$ by the map π.*

Remark 3.6. The advantage of this example over the previous one is that it does not depend on a choice of flag elements which are exceptional from a Noether–Lefschetz point of view. (Note that by standard Noether-Lefschetz arguments the condition that $\mathrm{Pic}(Y_1) = \langle H, C_1, C_2 \rangle$ holds for any very general Y_1 in $| - K_Z |$.)

In our first example of $X \supseteq Y_1 \supseteq Y_2 \supseteq Y_3 \supseteq Y_4$, the key point that enables us to construct a non-polyhedral Okounkov body is the following: the intersection of the effective cone of Y_2 with the subspace of $\mathrm{NS}(Y_2)$ generated by $\mathcal{O}_X(D)|_{Y_2}$, $\mathcal{O}_X(Y_1)|_{Y_2}$, $\mathcal{O}_{Y_1}(Y_2)|_{Y_2}$ and $\mathcal{O}_{Y_2}(Y_3)$ is a non-polyhedral cone.

Since any convex cone in a two-dimensional space is polyhedral, this requires that the space generated by these four elements should be at least three-dimensional, and since the Picard group of X is two-dimensional this requires that at least one of the inclusions $Y_1 \subset X$ and $Y_2 \subset Y_1$ should be Noether-Lefschetz exceptional (i.e. the induced map on Picard groups should be non-surjective). Moreover, at least one of the line bundles $\mathcal{O}_{Y_1}(Y_2)|_{Y_2}$ and $\mathcal{O}_{Y_2}(Y_3)$ must not be the restriction of a line bundle on X.

This is unsatisfying for two reasons. Firstly, it is to a certain extent less surprising that choosing flag elements in $\mathrm{Pic}(Y_i)$ that do not arise by restriction of elements in $\mathrm{Pic}(Y_{i-1})$ should lead to bad behaviour in the Okounkov body. There does not seem to be any reason why the fact that X is Fano should influence the geometry of the boundary of the part of the effective cone of Y_i which does not arise by restriction from X.

Secondly, if the construction of non-polyhedral Okounkov bodies depends on the choice of a flag with exceptional properties there is little hope of using such examples to construct a counter example to [13, Problem 7.1]. It is known for example that whenever Y_i ($i < n-1$) is general in an ample linear series on Y_{i-1}, then $\mathrm{Pic}(Y_i) = \mathrm{Pic}(Y_{i-1})$.

In the example of our variety $X = \mathbb{P}^2 \times \mathbb{P}^2$ it is fairly easy to see, using this remark, that if we construct a flag $Y_1 \ldots Y_4$ such that for any $i = 1, 2, 3$, the subvariety Y_i is a general element of the linear series $|\mathcal{O}_{Y_{i-1}}(1,1)|$ then the resulting global Okounkov body is polyhedral.

Proof of Proposition 3.5. We start by proving that Z is a Mori dream space ($-K_Z$ is immediately effective, since $\overline{S}$ is a section of $-K_Z$). By [3, Corollary 1.3.1], it is enough to find an effective big divisorial log terminal divisor Δ on Z such that $-K_Z - \Delta$ is ample. The existence of such a Δ will follow if we can show that $-K_Z$ is big and nef. Indeed, there then exists an effective divisor E such that $-K_Z - \epsilon E$ is ample for any sufficiently small ϵ. We are then done on setting $\Delta = \delta(-K_Z) + \epsilon E$ for any sufficiently small δ and ϵ.

Let's show that $-K_Z$ is nef. The first idea we need is to prove that any base point of $\mathcal{O}_Z(-K_Z)$ must be contained in $\pi^{-1}(C_1 \cap C_2)$. Note that $\pi_* \mathcal{O}_Z(-K_Z) = \mathcal{O}_{\mathbb{P}^3}(4) \otimes \mathcal{I}_{C_1+C_2}$. We will now show that each C_i is a complete intersection of two quadrics. Note first that $C_i \subseteq \mathbb{P}^3$ is non-degenerate, since $\mathcal{O}_S(H - C_i)$ has self-intersection on S equal to -4, and is therefore non-effective. We now consider the following exact sequence

$$0 \to H^0(\mathcal{I}_{C_i}(2)) \to H^0(\mathcal{O}_{\mathbb{P}^3}(2)) \to H^0(\mathcal{O}_{C_i}(2)) \to .$$

Note that $\dim(H^0(\mathcal{O}_{\mathbb{P}^3}(2))) = 10$ and $\dim(H^0(\mathcal{O}_{C_i}(2))) = 8$ by Riemann-Roch. It follows that $\dim(H^0(\mathcal{I}_{C_i}(2))) \geq 2$, so we can find linearly independent quadrics, P_i, Q_i, which vanish along C_i. As $C_i \subseteq \mathbb{P}^3$ is non-degenerate and of degree 4, it must be the complete intersection of P_i and Q_i. The pull-back to Z of any one of the polynomials $P_1 P_2, P_1 Q_2, Q_1 P_2, Q_1 Q_2$ gives a section of $\mathcal{O}_Z(-K_Z)$, so all the base points of $\mathcal{O}_Z(-K_Z)$ are included in $\pi^{-1}(C_1 \cap C_2)$.

To prove that $-K_Z$ is nef, it is therefore enough to check that the intersection of $-K_Z$ with any curve contained in $\pi^{-1}(C_1 \cap C_2)$ is positive. Set $C_1 \cap C_2 = \{p_1, p_2, p_3, p_4\}$, and let R_1 (resp. R_2) be the class of a curve in the ruling of E_1 (resp. E_2). For any i the set $\pi^{-1}(p_i)$ is then the union of two irreducible curves, one of class R_2 and the other of class $R_1 - R_2$. We have that $R_1 \cdot H = R_2 \cdot H = R_1 \cdot E_2 = R_2 \cdot E_1 = 0$ and $R_1 \cdot E_1 = -1$, $R_2 \cdot E_2 = -1$. In particular, $-K_Z \cdot R_2 = 1$ and $-K_Z \cdot (R_1 - R_2) = 0$, so $-K_Z$ is nef (but not ample).

It only remains to prove that $-K_Z$ is big. More explicitly, we show that the image of $\mathbb{P}^3$ under the rational map

$$\phi : \mathbb{P}^3 \dashrightarrow \mathbb{P}^4 \ , \phi = [F : P_1 P_2 : P_1 Q_2 : Q_1 P_2 : Q_1 Q_2]$$

is three-dimensional. Here F is the polynomial defining the surface $S \subseteq \mathbb{P}^3$ and is hence an element of $H^0(\mathcal{O}_{\mathbb{P}^3}(4) \otimes \mathcal{I}_{C_1+C_2})$.

We start by checking that the image of the restricted map $\phi|_S$ has dimension two. Observe that $\phi|_S$ can be factored as

$$f \circ (\phi_1 \times \phi_2) \ : \ S \dashrightarrow \mathbb{P}^1 \times \mathbb{P}^1 \dashrightarrow \mathbb{P}^4 \ ,$$

where $f([a : b], [c : d]) = [0 : ac : ad : bc : bd]$ and $\phi_i = [P_i : Q_i]$. The image of f is of dimension 2, thus it is enough to show that $\phi_1 \times \phi_2$ is generically surjective. Both P_1 and Q_1 vanish on S only along C_1, thus the general fibre of ϕ_1 is in the class $(2, 0, 0) - (1, 1, 0) = (1, -1, 0)$ and likewise the general fibre of ϕ_2 is $(1, 0, -1)$. Since $(1, 0, -1) \not\geq (1, -1, 0)$ and $(1, -1, 0) \not\geq (1, 0, -1)$ in $N^1(S)$, and ϕ_1 and ϕ_2 are individually generically surjective, $\phi_1 \times \phi_2$ is also generically surjective. The image of $\phi|_S$ is therefore two-dimensional.

It follows that either $\mathrm{Im}(\phi)$ is three-dimensional or $\mathrm{Im}(\phi) \subset \overline{\mathrm{Im}(\phi_S)}$. But if $p \notin S$ then $F(x) \neq 0$ so $\phi(p) \notin \overline{\mathrm{Im}(\phi_S)}$. Thus the image of ϕ is three-dimensional and $-K_Z$ is big.

We now show that if D and L satisfy the given independence condition and $\mathcal{O}_Z(Y_1) = -K_Z$ then $\Delta_{Y_\bullet}(Z; D)$ is non-polyhedral. We start by proving that the space spanned by

$\{H, C_1, C_2\}$ in $N^1(Y_1)_{\mathbb{R}}$ has the same properties as $N^1(S)_{\mathbb{R}}$. For this notice that Y_1 is the strict transform of a smooth K3 surface containing both C_1 and C_2 and by assumption $\text{Pic}(Y_1) = \langle H, C_1, C_2 \rangle$. Also, we have the following equalities of intersection numbers

$$\langle C_1, C_2 \rangle_{Y_1} = \langle C_1, C_2 \rangle_S, \langle H, C_1 \rangle_{Y_1} = \langle H, C_1 \rangle_S, \langle H, C_2 \rangle_{Y_1} = \langle H, C_2 \rangle_S$$

so for any integral class C on Y_1 we have that $4|C^2$. In particular, this implies there are no effective irreducible classes on Y_1 with negative self-intersection so

$$\overline{\text{Eff}}(Y_1) = \text{Nef}(Y_1) = \{v \in \text{Pic}(Y_1) \mid \langle v, v \rangle \geq 0, \langle v, H \rangle \geq 0\}.$$

This description of $\overline{\text{Eff}}(Y_1)$ implies that the (nonempty) intersection of this cone with an affine hyperplane is piecewise linear only if the hyperplane passes through 0.

Now, for small enough r we have that $D - rY_1$ is ample. It follows by Proposition 3.1 that for any small enough r

$$\Delta_{Y_\bullet}(Z; D) \cap \left(\{r\} \times \mathbb{R}^2\right) = \Delta_{Y_\bullet}(Y_1, (D - rY_1)|_{Y_1}).$$

We now set

$$f(r) = \max\{s \mid \Delta_{Y_\bullet}(Z; D) \cap \left((r, s) \times \mathbb{R}\right) \neq \emptyset\}$$

and note that f is piecewise linear if $\Delta_{Y_\bullet}(Z; D)$ is a polyhedron. We note that for small values of r we have, by the explicit description of Okounkov bodies of surfaces, that

$$f(r) = \sup\{\ s > 0 \mid (D - rY_1)|_{Y_1} - sY_2 \in \overline{\text{Eff}}(Y_1)_{\mathbb{R}}\}\ .$$

But, as explained in Remark 3.3, the graph of f is then an affine transformation of the intersection of the cone $\overline{\text{Eff}}(Y_1)$ with the plane passing through D, $D - Y_1$ and $D - Y_2$. By hypothesis this plane does not pass thorough 0, so its intersection with $\overline{\text{Eff}}(Y_1)$ is not piecewise linear. The Okounkov body $\Delta_{Y_\bullet}(Z; D)$ is therefore non-polyhedral. $\square$

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BUDAPEST UNIVERSITY OF TECHNOLOGY AND ECONOMICS, DEPARTMENT OF ALGEBRA, BUDAPEST P.O. BOX 91, H-1521 HUNGARY

E-mail address: `alex.kuronya@math.bme.hu`

UNIVERSITY OF MICHIGAN, DEPARTMENT OF MATHEMATICS, ANN ARBOR, MI 48109-1109, USA

E-mail address: `vicloz@umich.edu`

UNIVERSITÉ JOSEPH FOURIER, UFR DE MATHÉMATIQUES, 100 RUE DES MATHS, BP 74, 38402 ST MARTIN D'HÉRES, FRANCE

E-mail address: `Catriona.Maclean@ujf-grenoble.fr`