Rapporto n. 240

Constructing a class of stochastic volatility models: empirical investigation with VIX data

A. Hitaj, L. Mercuri and E. Rroji

Agosto 2013

Dipartimento di Statistica e Metodi Quantitativi
Università degli Studi di Milano Bicocca
Via Bicocca degli Arcimboldi , 8 - 20126 Milano - Italia
Tel +39/02/64483103 - Fax +39/2/64483105
Segreteria di redazione: Andrea Bertolini

Constructing a class of stochastic volatility models: empirical investigation with VIX data

A. Hitaj, L. Mercuri and E. Rroji

August 30, 2013

Abstract

We propose a class of discrete-time stochastic volatility models that, in a parsimonious way, captures the time-varying higher moments observed in financial series. Three desirable results are obtained. First, we have a recursive procedure for the log-price characteristic function which allows a semi-analytical formula for option prices as in Heston and Nandi [2000]. Second, we reproduce some features of the VIX Index. Finally, we derive a simple formula for the VIX index and use it for option pricing.
Keywords: *Affine Stochastic Volatility; VIX; Implied Volatility Surface.*

The Black and Scholes model [see Black and Scholes, 1973] is probably the most famous model proposed for option pricing. Despite its success, the drawbacks in representing the real markets are well documented by an increasing empirical literature. Since Mandelbrot [1963], empirical results have shown that the process describing the log-returns is far from the Brownian motion one. Indeed the financial time series exhibit heavy tails, asymmetric distribution, persistence and clustering in volatility [see Embrechts et al., 1997].

Several models have been proposed in continuous and discrete time. Merton [1976] introduced jump diffusion model where the dynamics of log returns is a Lévy process given by the sum of a continuous diffusion process (Brownian motion with drift) with a pure jump one (compound Poisson). The Lévy processes have almost surely right-continuous sample paths with stationary and independent increments. Their marginal distribution can be derived using characteristic functions [see Schoutens, 2003, Cont and Tankov, 2003, for a general survey]. A special attention deserves the process whose distribution at time one is a normal variance-mean mixture. Particular cases widely applied in finance are the variance gamma process introduced by Madan and Seneta [1990], the normal inverse gaussian [see Barndorff-Nielsen and Shephard, 2001], the hyperbolic and the generalized hyperbolic [see Barndorff-Nielsen, 1977, Eberlein and Prause, 1998]. Although Lévy processes are able to represent some features of financial time series, the independence hypothesis makes them inadequate in capturing the time-dynamic of higher moments.

A way to overcome these limits is by using the stochastic volatility models for describing the log-returns dynamics. There are two sources of risk in these models: the first drives the volatility dynamics and the second directly the log-returns. The main problem is that the volatility process is not observable in the market.
In discrete-time the most commonly used class for modelling the financial time series is the family of Garch models. Despite the success in financial econometrics and risk management, their use for option pricing is not yet very well understood, as observed in Christoffersen et al. [2012]. Monte Carlo technique is often used to compute option prices in Garch models [see Duan, 1995, Duan and Simonato, 1998, for the efficiency of Monte Carlo estimator]. Another approach is using approximate formulas based on Edgeworth expansion [see Duan et al., 1999, 2006]. It is well known that the Monte Carlo procedure is time consuming when calibration exercise is considered, while the Edgeworth expansion seems to be less accurate for option prices with long-medium maturities.

A major breakthrough occurred with the paper of Heston and Nandi [2000] where the authors derive a recursive procedure for the characteristic function of the log-price at maturity, obtaining a semi-analytical formula for European call option based on inverse Fourier transform, as in Carr and Madan [1999]. Following the same idea a new class of Garch models, namely affine Garch, has been developed assuming different assumption for the innovations. In particular, Christoffersen et al. [2006] considered the Inverse Gaussian innovations while Bellini and Mercuri [2007] Gammma innovations. Later Mercuri [2008] generalized further the class of affine Garch models assuming that the log-returns are conditionally Tempered Stable distributed [see Ornthanalai, 2008, for more details on affine Garch models].

As observed in Christoffersen et al. [2006], the extreme asymmetry of the affine Garch models gives an advantage for options with very short maturity. However the fit is less accurate for options with medium maturity probably due to the fact that the medium term return distribution slowly converges to bell shaped one.

To overcome this limit, starting from the affine Garch model and assuming that the conditional distribution of log returns is a normal variance mean mixture, we construct a discrete time stochastic volatility model in a simple way. Indeed, substituting the mixing random variable with an affine Garch, we obtain a recursive procedure for computation of the characteristic function of log-price at maturity. Option prices are obtained via Fourier transform.

Feunou and Tedongap [2012] constructed a discrete time stochastic volatility model with time-varying conditional skewness. They decompose the joint distribution of returns and latent variables. As a first step they suppose that the distribution of returns, conditioned on latent variables and past information, is an inverse gaussian. The next step is the assumption that latent variables follow a multivariate gamma autoregressive process with mutually independent components.
Even in our construction methodology of discrete time stochastic volatility models we make two assumptions: the first is that the conditional distribution of returns is a normal variance mean mixture and the second is the autoregressive structure for the latent variable. Instead of assuming a particular process for the latent variable, we use the information obtained from the quoted VIX index to model it.

A desirable feature of our model is the possibility to obtain time-varying higher moments. Volatility [see Chicago Board Options Exchange, 2003] and Skew [see Chicago Board Options Exchange, 2011] indexes cannot exist in a world with constant higher moments since they would be useless. Time-dependence of these moments is coherent with price movements observed in the market making our approach more realistic.
In our model, it is possible to extrapolate information from the VIX data and use it in option pricing. Indeed we find a linear relation between the variance dynamics and the square of VIX (a similar result has been obtained by Zhang and Zhu 2006 under the Heston model and by Hao and Zhang 2013 under Garch assumption).

The VIX index was introduced by the Chicago Board Options Exchange (CBOE) in 1993 and was designed to measure the markets expectation of 30-day volatility of at-the money $S\&P100$ Index (OEX) option prices. In 2003, CBOE together with Goldmann Sachs substantially modified the VIX index. The OEX has been replaced by the SPX and a new methodology of evaluating the VIX was proposed (see the CBOE White Paper Chicago Board Options Exchange [2003] for details). Although the VIX index reflects only the market risk and doesn't take into account liquidity and systematic risk [see Dhaene et al., 2011], the markets participants use it as a Fear Index since they believe that the implied volatility reflects the sentiment of fear.
From empirical point of view, VIX's movements seem to be mean reverting. There is a negative correlation between VIX and the S&P500, therefore the practitioners take long position on VIX futures to hedge during crisis periods as an alternative to the classical straddle or strangle strategy. In addition there are some attempts in using this index to predict the start and the end of crisis by looking at the historical levels reached in different market phases. For example a level higher than 50 per cent was observed only during deep crisis.

The paper is organized as follows. Section 2, explains how we build the stochastic volatility model in discrete time. In Section 3 we prove that, in our setup, the VIX index is an autoregressive process with heteroscedastic innovations: we derive a linear relation between the unobservable variance and the current level of VIX index. In section 4 we derive explicit formulas specifying the conditional distribution of log returns. Section 5 is devoted to investigate the behavior of implied volatility surface in our framework. We outline the steps implied by our methodology and give some empirical results using the implied volatility surface obtained by Bloomberg data provider.

1 General Setup

The aim of this work is to propose a class of stochastic volatility models, in discrete time, through which we are able to price in a simple way options using the information extrapolated from the VIX index.
Given a filtered probability space $(\Omega, \mathcal{F}, \mathcal{F}_t, \mathbb{P})$, we consider a market with two assets:
- riskless with dynamics: $B_t = B_{t-1}\exp(r)$
- risky with price dynamics:

$$\begin{aligned} S_t &= S_{t-1}\exp(X_t) \\ X_t &= r + \lambda_0 h_t + \lambda_1 V_t + \sigma\sqrt{V_t}Z_t \end{aligned} \tag{1}$$

where: r is the deterministic free rate observed in the market; X_t is a discrete time stochastic process with continuous state space; $Z_t \sim N(0,1)$, $\forall t = 1, ..., T$ and is independent from V_t; V_t is a positive adapted process such that the conditional moment generating function exists and has the following form:

$$E[\exp(cV_t)|\mathcal{F}_{t-1}] = \exp(h_t f(c,\theta)) \tag{2}$$

$\forall$ fixed vector θ, $\exists\ \delta > 0$ such that $\forall c \in (-\delta, \delta)$ the function $f(c, \theta) \in C^\infty$ and $f(0, \theta) = 0$.
From (2) we have:

$$E[V_t|\mathcal{F}_{t-1}] = \left.\frac{\partial E[\exp(cV_t)|\mathcal{F}_{t-1}]}{\partial c}\right|_{c=0}$$

We define the function $g(\theta)$ as a partial derivative

$$g(\theta) := \left.\frac{\partial f(c, \theta)}{\partial c}\right|_{c=0} \tag{3}$$

and obtain an analytical expression for conditional mean of V_t:

$$E[V_t|\mathcal{F}_{t-1}] = h_t g(\theta) \tag{4}$$

In particular we define a dynamic for h_t so that it becomes a predictable process.

$$h_t = \alpha_0 + \alpha_1 V_{t-1} + \beta h_{t-1}.$$

By adding and subtracting the quantity $\alpha_1 g(\theta)$ we obtain a new representation

$$h_t = \alpha_0 + (\alpha_1 g(\theta) + \beta)h_{t-1} + \alpha_1(V_{t-1} - g(\theta)h_{t-1}). \tag{5}$$

Observe that h_t is an AR(1) with heteroscedastic error $V_{t-1} - g(\theta)h_{t-1}$. Therefore if we extrapolate from the market the time series of h_t, the generalized least square technique gives us estimates for the quantities α_0, α_1, and $\alpha_1 g(\theta) + \beta$. The process h_t is positive if the parameters α_0, α_1 and β are non negative.

In our model, the conditional variance evolves according to the stochastic process h_t:

$$Var\left[X_t|\,\mathcal{F}_{t-1}\right] = h_t \left.\frac{\partial^2 f(c, \theta)}{(\partial c)^2}\right|_{c=0}.$$

An essential requirement, based on empirical evidence, is the negative correlation between the variable describing the returns and the volatility one. In this case we must have that:

$$Cov\left(V_t, X_t|\,\mathcal{F}_{t-1}\right) = \lambda_1 Var(V_t|\mathcal{F}_{t-1}) < 0 \tag{6}$$

i.e. we need $\lambda_1 < 0$.
We make the basic assumption that the constant term σ appearing in the dynamics of log-returns is non negative. In the special case when $\sigma = 0$ the process describing X_t is an affine Garch as in Christoffersen et al. [2006], Bellini and Mercuri [2007] and Mercuri [2008].

Our approach tries to generalize the Lévy processes built on the normal variance mean mixture since we introduce a dependence structure. Indeed the conditional distribution evolves through time due to the predictable process h_t.

In order to price at the reference time t an european call option with maturity T, we need the distribution of S_T given the information at the evaluation date. Here, we provide a simple recursive procedure that allows to obtain the conditional moment generating function using a similar approach as that introduced in Heston and Nandi [2000].

Proposition 1 *Under condition* (2)*, the moment generating function of the random variable* $\ln S_T$ *given the information at time t exists and is given by:*

$$E[\exp(c\ln(S_T))|\mathcal{F}_t] = S_t^c \exp[A(t; T, c) + B(t; T, c)h_{t+1}]$$

The time-dependent coefficients $A(t; T, c)$ *and* $B(t; T, c)$ *are:*

$$\begin{cases} A(t;T,c) &= cr + A(t+1;T,c) + \alpha_0 B(t+1;T,c) \\ B(t;T,c) &= c\lambda_0 + \beta B(t+1;T,c) + \\ & \quad f(c\lambda_1 + \alpha_1 B(t+1;T,c) + \frac{c^2\sigma^2}{2}, \theta) \end{cases} \tag{7}$$

with the following conditions:

$$\begin{aligned} A(T;T,c) &= 0 \\ B(T;T,c) &= 0. \end{aligned}$$

(see appendix 5.1)

The existence of m.g.f. allows to obtain the characteristic function since the latter is the former evaluated on the complex number and the distribution function is achieved by the inverse Fourier transform.

Our aim is to evaluate options prices and implied volatility indexes therefore we are interested in the distribution of the underlying asset under the Q measure. The following proposition is necessary to avoid arbitrage opportunities as stated in the first theorem of asset pricing.

Proposition 2 *Under the assumptions $E(S_t) < +\infty$ and $\lambda_0 = -f(\lambda_1 + \frac{\sigma^2}{2}, \theta)$, the discounted price is a martingale.*
(see Appendix 5.2)

We have obtained in prop.1 the m.g.f. for the underlying. The next step is the evaluation of European call option as in Heston [1993]

$$\begin{aligned} C(K,T) &= S_0 \Pi_1 - K e^{-rT} \Pi_2 \\ \Pi_1 &= \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \Re \left(\frac{K^{-iu} E_0^Q [S_T^{i(u-i)}]}{iu E_0^Q [S_T]} \right) du \\ \Pi_2 &= \frac{1}{2} + \frac{1}{\pi} \int_0^{+\infty} \Re \left(\frac{K^{-iu} E_0^Q [S_T^{iu}]}{iu} \right) du \end{aligned}$$

The exercise probabilities Π_1 and Π_2 can be computed following Feller [1968].

2 VIX Index

In this section we provide a linear relation between the current value of VIX squared and the dynamics of the h_t process defined above. A similar result have been proposed in Zhang and Zhu [2006] under the assumption that the SPX dynamics is described by Heston [1993]. The Methodology of computing the VIX index is based on the replication of a variance swap [see Demeterfi et al., 1999]. The current level of VIX is related to a portfolio composed by out-of-the money call/put options on the S&P500. Although the VIX depends on available options and can be considered a corridor implied volatility index, it is reasonable to assume that strike prices vary continuously from 0 to $+\infty$. Neglecting the discretization error and the VIX squared formula can be written as:

$$\begin{aligned} \left(\frac{VIX_t}{100} \right)^2 &= \frac{2e^{r(T-t)}}{T-t} \left[\int_0^{S^*} \frac{1}{K^2} P(S_t, K) dK + \right. \\ &+ \left. \int_{S^*}^{+\infty} \frac{1}{K^2} C(S_t, K) dK \right] = \\ &= \frac{2e^{r(T-t)}}{T-t} \left[E_t^Q \left(\frac{S_T - S^*}{S^*} - ln \left(\frac{S_T}{S^*} \right) \right) \right]. \end{aligned} \tag{8}$$

$C(S_t, K)$ and $P(S_t, K)$ are out-of-the money call and put option prices. S^* is the forward price of the SPX index.
The main result of our model is reported in the following proposition.

Proposition 3 *Under the conditions:*

$$\begin{aligned} &\alpha_1 g(\theta) + \beta < 1 \\ &\lambda_1 g(\theta) - f\left(\lambda_1 + \tfrac{\sigma^2}{2}, \theta\right) \leq 0 \\ &h_{t+1} > 0 \end{aligned} \tag{9}$$

the VIX squared is an affine linear function of the predictable process h_t:

$$\left(\frac{VIX_t}{100} \right)^2 = -\frac{2e^{r(T-t)}}{T-t} \left[C(t;T) + D(t;T) h_{t+1} \right] \tag{10}$$

where $C(t;T)$ and $D(t;T)$ are functions of the model parameters, given by

$$\begin{cases} C(t;T) = \alpha_0 \left[\lambda_1 g(\theta) + \lambda_0 \right] \left\{ \frac{T-t-1-[\alpha_1 g(\theta)+\beta] \frac{1-[\alpha_1 g(\theta)+\beta]^{(T-t)-1}}{1-[\alpha_1 g(\theta)+\beta]}}{1-[\alpha_1 g(\theta)+\beta]} \right\} \\ D(t;T) = \left[\lambda_1 g(\theta) + \lambda_0 \right] \frac{1-[\alpha_1 g(\theta)+\beta]^{T-t}}{1-[\alpha_1 g(\theta)+\beta]} \end{cases} \tag{11}$$

with $T - t = 30$ days.
(See Appendix 5.3)

Considering the fact that VIX is a measure of options on S&P500 implied volatility with time to maturity 30 days, equation (10) becomes:

$$\left(\frac{VIX_t}{100}\right)^2 = -\frac{2e^{r30}}{30}\left[C_{30} + D_{30}h_{t+1}\right]$$

where r is the one month libor rate on daily basis.
We define the adjusted VIX as:

$$VIX_t^{adj} = -\frac{30}{2e^{r30}}\frac{VIX_t^2}{10^4}$$

Notice that $VIX_t^{adj} < 0 \ \forall t$ since it is a decreasing linear transformation of the VIX squared.
Using proposition 3 we have:

$$VIX_t^{adj} = C_{30} + D_{30}h_{t+1} \ \Rightarrow \ h_{t+1} = \frac{VIX_t^{adj} - C_{30}}{D_{30}} \tag{12}$$

The requirement $h_{t+1} > 0$ implies that $0 > VIX_t^{adj} > C_{30} \ \forall t$.
Using the definition (5) of h_t, we have following proposition:

Proposition 4 *Under the same conditions of the prop 3, defining the heteroschedastic error term* $\tau_t := \alpha_1(V_t - g(\theta)h_t)D_{30}$*, the* VIX_t^{adj} *is an* $AR(1)$ *defined as:*

$$VIX_t^{adj} = int + slope VIX_{t-1}^{adj} + \tau_t$$

where

$$\begin{cases} int = 30\alpha_0\left(\lambda_1 g(\theta) + \lambda_0\right) \\ slope = \alpha_1 g(\theta) + \beta \end{cases}$$

(see Appendix 5.4)

Although the expression for τ_t may appear a little complex, in practice using this definition we can show that our model becomes a Garch one in the sense that a one-step distribution depends only on the previous VIX level. Given the model parameters, the current and one-day-ahead VIX level we have:

$$\tau_{t+1} = VIX_{t+1} - int - slope VIX_t.$$

From equation (12) we extract h_{t+1} and obtain the value of the main "unobservable" variable of our model, i.e V_{t+1}:

$$V_{t+1} = g(\theta) + \frac{\tau_{t+1}}{\alpha_1 D_{30}}.$$

The knowledge of V_{t+1} allows us to exploit the advantages of working with stochastic volatility models while preserving the low level of estimation difficulty as in Garch models.
Once estimated *int* and *slope* we can redefine D_{30} and C_{30} in order to extrapolate a multiple of h_{t+1} from the quoted VIX_t. In particular we get:

$$D_{30} = \frac{D_{30}^*}{\alpha_0} = \frac{int\left(1 - slope^{30}\right)}{30 * (1 - slope)}\frac{1}{\alpha_0}$$

$$C_{30} = \left[\frac{29 - slope\frac{1-slope^{29}}{1-slope}}{1 - slope}\right]\frac{int}{30}$$

$$\frac{VIX_t^{adj} - C_{30}}{D_{30}^*} = \frac{h_{t+1}}{\alpha_0} > 0$$

The quantity $\frac{h_{t+1}}{\alpha_0}$ can be used to compute the m.g.f. of $\ln(S_T)|\mathcal{F}_t$ needed in option pricing. If $slope < 1$, VIX_t^{adj} is mean reverting. The long term mean and the reverting speed are respectively:

$$\frac{int}{1 - slope}, \quad 1 - slope.$$

The conditional mean of the error term is zero but we are in presence of heteroskedasticity:

$$E\left[\tau_t|\,\mathcal{F}_{t-1}\right] = 0, \quad Var\left[\tau_t|\,\mathcal{F}_{t-1}\right] = \alpha_1^2 D_{30}^2 Var\left[V_t|\,\mathcal{F}_{t-1}\right].$$

Although $cov\left[\tau_{t+1},\tau_t\middle|\mathcal{F}_{t-1}\right]=0$ and $cov\left[\tau_{t+1},\tau_t^2\middle|\mathcal{F}_{t-1}\right]=0$, the error time-dependence structure is more complex than a linear one. The following quantities are different from zero and time dependent:

$$cov\left[\tau_{t+1}^2,\tau_t\middle|\mathcal{F}_{t-1}\right]=\alpha_1^3 D_{30}^3\left.\frac{\partial^2 f}{(\partial c)^2}\right|_{c=0}\left[\alpha_0+\alpha_1\left.\frac{\partial^2 f}{(\partial c)^2}\right|_{c=0}\right]h_t$$

$$cov\left[\tau_{t+1}^2,\tau_t^2\middle|\mathcal{F}_{t-1}\right]=\alpha_1^4 D_{30}\left.\frac{\partial^2 f(c,\theta)}{(\partial c)^2}\right|_{c=0}\left[\alpha_0+(\alpha_1 g(\theta)+\beta)\left.\frac{\partial^2 f(c,\theta)}{(\partial c)^2}\right|_{c=0}h_t^2+\alpha_1^2\mu_3\right]$$

where $\mu_3=E\left[(V_t-g(\theta))^3\middle|\mathcal{F}_{t-1}\right]$.
These observations give us the possibility to estimate the parameters that control the dynamics of h_t directly from VIX time series without any explicit distributional assumption on $V_t|\mathcal{F}_{t-1}$ [see J. and A., 1991, Campbell et al., 1997, for estimation techniques in autoregressive models with heteroskedastic errors].

3 Special cases

3.1 Normal Variance Mean Mixture

The conditional distribution of log returns belongs to the normal variance mean mixture family since Z_t in (1) is normally distributed. An univariate normal variance-mean mixture [see Barndorff-Nielsen et al., 1982] is a random variable defined as:

$$X\overset{d}{=}\mu+\lambda V+\sigma\sqrt{V}Z$$

where Z and V are independent univariate random variables, $Z\sim N(0,1)$, and V is defined on the positive real line. Below we introduce three special cases of our approach where the conditional distribution of log returns is respectively variance gamma [see Madan and Seneta, 1990], normal inverse gaussian [see Barndorff-Nielsen and Shephard, 2001] and normal tempered stable [see Barndorff-Nielsen and Shephard, 2001].

3.2 Dynamic Variance Gamma

Assuming that the affine Garch process V_t is conditionally gamma distributed [see Bellini and Mercuri, 2007] than X_t in (1) follows a Dynamic Variance Gamma model introduced by Bellini and Mercuri [2011].
The conditional moment generating function of the V_t is:

$$\begin{array}{ll} E\left[e^{cV_t}\middle|\mathcal{F}_{t-1}\right] & =\exp\left[-h_t\ln(1-c)\right] \\ f(c,\theta) & =-\ln(1-c) \\ g(\theta) & =1 \end{array}$$

The system 7 becomes:

$$\left\{\begin{array}{ll} A(t;T,c) & =cr+A(t+1;T,c)+\alpha_0 B(t+1;T,c) \\ B(t;T,c) & =c\lambda_0+\beta B(t+1;T,c)+ \\ & -\ln\left(1-c\lambda_1-\alpha_1 B(t+1;T,c)-\frac{c^2\sigma^2}{2}\right) \end{array}\right. \tag{13}$$

The system (11) becomes

$$\left\{\begin{array}{ll} C(t;T,c) & =\alpha_0(\lambda_1+\lambda_0)\left\{\frac{(T-t)-(\alpha_1+\beta)\frac{1-(\alpha_1+\beta)^{T-t-1}}{1-(\alpha_1+\beta)}}{1-(\alpha_1+\beta)}\right\} \\ D(t;T,c) & =(\lambda_1+\lambda_0)\frac{1-(\alpha_1+\beta)^{T-t}}{1-(\alpha_1+\beta)} \end{array}\right. \tag{14}$$

with final conditions $C(T;T,c)=0$ and $D(T;T,c)=0$. We have the following restrictions on parameters:

$$\left\{\begin{array}{l} \lambda_1\leq 0 \\ \lambda_0=\ln\left(1-\lambda_1-\frac{\sigma^2}{2}\right) \\ \alpha_1+\beta\leq 1 \\ \lambda_1+\ln\left(1-\lambda_1-\frac{\sigma^2}{2}\right)\leq 0 \end{array}\right. \tag{15}$$

For the last restriction a sufficient condition is $0\leq\sigma\leq\sqrt{2}$.

3.3 Dynamic Normal Inverse Gaussian

If the affine Garch process V_t is conditionally inverse gaussian distributed [see Christoffersen et al., 2006] than log-returns X_t, given the information at time $t-1$, have a normal inverse gaussian distribution [see Barndorff-Nielsen, 1997].
The density of inverse gaussian distribution is:

$$f_V(v) = \frac{h_t}{\sqrt{2\pi v^3}} \exp\left[-\frac{1}{2}\left(\sqrt{v} - \frac{h_t}{\sqrt{x}}\right)^2\right]$$

The conditional moment generating function of the V_t is:

$$\begin{array}{ll} E\left[e^{cV_t}\middle|\mathcal{F}_{t-1}\right] & = \exp\left[h_t\left(1-\sqrt{1-2c}\right)\right] \\ f(c,\theta) & = \left(1-\sqrt{1-2c}\right) \\ g(\theta) & = 1 \end{array}$$

The system 7 becomes:

$$\begin{cases} A(t;T,c) & = xr + A(t+1;T,c) + \alpha_0 B(t+1;T,c) \\ B(t;T,c) & = c\lambda_0 + \beta B(t+1;T,c) + \\ & \sqrt{1-2\left(c\lambda_1 + \alpha_1 B(t+1;T,c) + \frac{c^2\sigma^2}{2}\right)} \end{cases} \tag{16}$$

The system (11) becomes

$$\begin{cases} C(t;T,c) & = \alpha_0\left(\lambda_1+\lambda_0\right)\left\{\frac{(T-t)-(\alpha_1+\beta)\frac{1-(\alpha_1+\beta)^{T-t-1}}{1-(\alpha_1+\beta)}}{1-(\alpha_1+\beta)}\right\} \\ D(t;T,c) & = \left(\lambda_1+\lambda_0\right)\frac{1-(\alpha_1+\beta)^{T-t}}{1-(\alpha_1+\beta)} \end{cases} \tag{17}$$

with final conditions $C(T;T,c) = 0$ and $D(T;T,c) = 0$. We have the following restrictions on the parameters:

$$\begin{cases} \lambda_1 \leq 0 \\ \lambda_0 = -\left(1-\sqrt{1-2\left(\lambda_1+\frac{\sigma^2}{2}\right)}\right) \\ \lambda_1 - 1 + \sqrt{1-2\left(\lambda_1+\frac{\sigma^2}{2}\right)} < 0 \\ \alpha_1 + \beta < 0 \end{cases} \tag{18}$$

A sufficient condition for $\lambda_1 - 1 + \sqrt{1-2\left(\lambda_1+\frac{\sigma^2}{2}\right)} < 0$ is:

$$0 \leq \sigma^2 \leq 1$$

3.4 Dynamic Normal Tempered Stable

When the affine process V_t is the model proposed in Mercuri [2008] than log returns follow a conditional normal tempered stable as introduced in Barndorff-Nielsen and Shephard [2001]. We recall that the normal tempered stable is obtained as a normal variance mean mixture where the mixing density is a the tempered stable [see Tweedie, 1984] that is obtained by tempering the tail of a positively skewed $\alpha-$stable distribution with an exponential function. This distribution has all finite moments. The normal tempered stable has as special cases the variance gamma and the normal inverse gaussian.
The conditional moment generating function of $V_t|\mathcal{F}_{t-1}$ is:

$$E\left[e^{cV_t}\middle|\mathcal{F}_{t-1}\right] = \exp\left[h_t b\left(1-(1-2cb^{-1/\alpha})^{\alpha}\right)\right] \tag{19}$$

where $\alpha \in (0,1)$ and $b > 0$.
Comparing (19) with (2), we have:

$$f(c,\theta) = b\left(1-(1-2cb^{-1/\alpha})^{\alpha}\right)$$

and

$$g(\theta) = 2\alpha b^{(\alpha-1)/\alpha}.$$

Applying prop. 1, we obtain the recursive system for time dependent coefficients:

$$\begin{cases} A(t;T,c) & = cr + A(t+1;T,c) + \alpha_0 B(t+1;T,c) \\ B(t;T,c) & = c\lambda_0 + \beta B(t+1;T,c) + \\ & \quad b\left\{1 - \left[1 - 2b^{-\frac{1}{\alpha}}\left(c\lambda_1 + \alpha B(t+1;T,c) + \frac{c^2\sigma^2}{2}\right)\right]^{\alpha}\right\} \end{cases} \quad (20)$$

From prop. 2 we have the following constraint

$$\lambda_0 = -b\left[1 - \left(1 - 2\left(\lambda_1 + \frac{\sigma^2}{2}\right)b^{1/\alpha}\right)^{\alpha}\right]$$

and, implementing the fast Fourier transform, we evaluate an european call option.
Using prop. 3, we obtain the following time coefficients that allows us to extrapolate h_t from current level of VIX:

$$\begin{cases} C(t;T) & = \alpha_0\left(2\alpha b^{(\alpha-1)/\alpha}\lambda_1 + \lambda_0\right) * \\ & \quad * \left\{\frac{(T-t) - (2\alpha b^{(\alpha-1)/\alpha}\alpha_1 + \beta)\frac{1-\left(2\alpha b^{(\alpha-1)/\alpha}\alpha_1+\beta\right)^{T-t-1}}{1-\left(2\alpha b^{(\alpha-1)/\alpha}\alpha_1+\beta\right)}}{1-\left(2\alpha b^{(\alpha-1)/\alpha}\alpha_1+\beta\right)}\right\} \\ D(t;T) & = (2\alpha b^{(\alpha-1)/\alpha}\lambda_1 + \lambda_0)\frac{1-\left(2\alpha b^{(\alpha-1)/\alpha}\alpha_1+\beta\right)^{T-t}}{1-\left(2\alpha b^{(\alpha-1)/\alpha}\alpha_1+\beta\right)} \end{cases} . \quad (21)$$

In this case the condition (9) becomes:

$$\begin{cases} 2\alpha b^{(\alpha-1)/\alpha}\lambda_1 - b\left[1 - \left(1 - 2\left(\lambda_1 + \frac{\sigma^2}{2}\right)b^{-1/\alpha}\right)^{\alpha}\right] \leq 0. \\ \alpha_1 + \beta < 1 \end{cases} \quad (22)$$

4 Empirical Analysis

Nowadays studying and understanding the surface implied volatility is a central issue from both practical and theoretical point of view [see Gatheral, 2006, Alexander, 2008, for the relevance of the volatility surface in financial literature]. For this reason this section investigates the conditions under which our models are able to replicate the behavior of the volatility surface. We present as well a simple calibration exercise based on option volatilities whose underlying is the S&P500 Index.

Table 1 summarizes the steps going from the observation of VIX Index till the computation of the implied volatility.

Insert here Tab. 1.

At the reference date t we observe the market value of the VIX index. Then we easily obtain h_{t+1} necessary for getting the m.g.f. of the underlying. The inverse Fourier transformation allows us to compute the exercise probabilities appearing into the pricing formula of an European call option. The volatility surface is made up of points representing each the implied volatility of an option once fixed the strike and the time to maturity. These points are obtained by inverting the Black and Scholes formula for a call option.

We analyse the effect that each parameter has on the volatility surface. Keeping fixed the other parameters, we vary separately those that appear directly in log returns dynamics (λ_1 and σ), while the parameters (α_0, α_1 and β) that appear in the dynamics of h_t are moved together . We have observed that the parameters have the same effect for the three models therefore we report the plots only for the DVG one.

Insert here Fig. 1, 2 and 3.

Fixing $\sigma = 0.014$, $\alpha_0 = 0.033$, $\alpha_1 = 0.493$ and $\beta = 0.379$, we study the surface varying λ_1. A symmetric smile shape is observed when $\lambda_1 = 0$. As mentioned in Sec. 1 this is the case when the log-returns have a symmetric distribution. For negative values of the parameter λ_1, i.e. in the case of negative skewness, we observe a twist of the entire surface. By giving different values to this parameter we are able to reproduce the well-known smirk in the implied volatility surfaces.
Once fixed $\lambda_1 = 0$, $\alpha_0 = 0.033$,$\alpha_1 = 0.493$ and $\beta = 0.379$, we vary σ. The effect is only a parallel shift of the entire curve. It is interesting to notice that it affects only the implied volatility level but not the shape of the

surface. A similar phenomenon is achieved by simultaneously increasing the values of the parameters α_0, α_1 and β for fixed λ and σ.

We have undertaken a detailed study of the DTS model since through the additional parameters a and b it has the potential to become more flexible in dynamically capturing the main features of the observed volatility surface.

Insert here Fig. 4.

From figure 4 we observe that a higher level of the parameter a has a double effect: the first is the upward shift of the implied volatility surface whilst the second is a higher slope for any fixed strike. In particular for $a = 0.9751$ the implied volatility surface is less inclined than for $a = 0.99$. The higher the time to maturity, the higher is the sensitivity to the parameter a. Changes of the parameter b seem to influence only the slope for any fixed strike.

We also investigate in details the ability of our models to reproduce the behavior of European option prices on SPX index. We have two main objectives: to replicate the market option volatilities and to compare the theoretical VIX derived in our models with the observed one. The dataset is composed by the implied volatility surfaces observed each Wednesday going from May 2011 to April 2012 (the total number of observations is 1008). Our choice was influenced from the desire to avoid possible turn of week effects. From eqn. 1 we see that we need the term structure of the risk-free rate in order to compute the m.g.f of the variable $\ln S_T$. The Libor curve can be a possible choice though we know it is not the only one. We downloaded the needed curve from Bloomberg.

The first Wednesdays of each month are the in-sample data (231 observations), the remaining dataset (777 observations) is used for the out-of-sample analysis. We calibrate the model in each in-sample period. The values obtained for the parameters are the input for the out-of-sample analysis. The error measure considered is:

$$\sqrt{percMSE} = \sqrt{\frac{\sum_{k=1}^{K}\sum_{t=1}^{T}\left[\frac{\sigma^{mkt}(k,t)-\sigma^{theo}(k,t)}{\sigma^{mkt}(k,t)}\right]^2}{N_T * N_K}}$$

where $\sigma^{mkt}(k,t)$, $\sigma^{theo}(k,t)$ are respectively the implied volatilities observed in the market and those obtained by the models. N_T, N_K is the number of the available maturities and strikes.

Tables 4, 5 and 6 report the values of the calibrated parameters and the corresponding in-sample errors.

Insert here Tab. 4, 5 and 6.

Our calibration exercise takes into account the possibility of extrapolating the latent process h_t directly from the VIX index. We find that for the DNTS model the in-sample errors are the lowest except only in one case where the DNIG model has the best performance. This result strongly supports our initial guess that two additional parameters would allow to better capture the market dynamics. Observe that if $b = 2^a$ and $\alpha = \frac{1}{a}$ for $a \to 0$ we obtain the DVG model, while if $b = 1$ and $\alpha = \frac{1}{2}$ the model is the DNIG.

The out-of-sample results strongly support the supremacy of the DNTS model in the considered dataset. Indeed, computing the $\sqrt{percMSE}$ on the entire out-of-sample data, we find that the DNTS reaches an error level of 5.05% which is a reduction error of 21.10% with respect to DNIG (the second best model). To deeply analyse the out of sample error, Figure 8 reports the results obtained in 36 out-of-sample Wednesdays. In 72% of the cases the DNTS shows a lower error level than the other two while the DNIG has the lowest error level only in 14% of the cases.

Insert here Fig. 8.

We remark that in our model the square of the VIX is an autoregressive process. The conditional expected value of the VIX is not available in a closed form formula. However, using Jensen's inequality, we easily derive the following upper bound that we use in our analysis:

$$E\left[VIX_{t+1}\middle|\mathcal{F}_t\right] = E\left[\sqrt{VIX_{t+1}^2}\middle|\mathcal{F}_t\right] \leq \sqrt{E\left[VIX_{t+1}^2\middle|\mathcal{F}_t\right]} = VIX_{t+1}^{ub}.$$

Using the prop.4 and eqn. (12), our upper bound becomes:

$$VIX_{t+1}^{ub} = \sqrt{-\frac{2e^{30r} * 10^4}{30} int + slopeVIX_t^2}$$

where all quantities are on daily basis and the year conversion is necessary for comparison with its observed level.

We calibrate the model on the first Wednesday of each month (in total there are 12 calibration period). The resulting parameters are maintained fixed until the next in-sample day. From Figure 7 and Table 2 we observe that the DVG model is the one with the worst performance.

Insert here Fig. 6.

Insert here Tab. 3.

Instead of having fixed parameters for the entire month we can decide to make the recalibration period dynamic. Intuitively, if the market conditions change a lot (i.e. we observe a jump of the implied volatility from one observation to the other), it is reasonable to think that in order to have a better prediction for the VIX level we must update the model parameters. This update for us means to recalibrate the model using the option volatilities observed after the jump has been occurred.
We face the problem of defining the jump in terms of relative daily variation of the VIX Index level. If the observed VIX level is lower than 30 per cent we recalibrate if next day relative variation is higher than 30%. For example if the current level of VIX is 15% we recalibrate the model if the next day value is higher than 20% or lower than 10%. For higher levels of the VIX index (more than 30%) the required daily relative variation is fixed at 25%. This decision comes from the fact that VIX levels higher than 39% are rarely observed. In Figure 5 we report a comparison between the VIX and S&P500 for the considered dates.

Insert here Fig. 5.

We reduced the number of calibrations going from 12 (when we fixed the parameters for the entire month) to 9 (if the calibration decision is dependent on the VIX level). Based on this procedure we observe in Figure 7 and Table 2 that all the models predict better the Open values.

Insert here Fig. 7.

Insert here Tab. 2.
The supremacy of the DNTS showed in the calibration exercise seems to be weaker when we try to forecast the VIX index level. In particular, the DNIG seems to behave better in some extreme market conditions.

5 Appendix

5.1 Conditional Moment Generating Function

Following the approach proposed in Heston and Nandi [2000] we derive a recursive equations for the time dependent coefficient for the conditional m.g.f. of the random variable $\ln(S_T)$ given the available information at time t. We want to prove that the conditional m.g.f. is given by the following formula:

$$E_t\left[\exp\left(c\ln\left(S_T\right)\right)\middle|\,\mathcal{F}_t\right]=S_t^c\exp\left[A\left(t;T,c\right)+B\left(t;T,c\right)h_{t+1}\right]. \tag{23}$$

We use the mathematical induction method.

1. We observed that the relation (23) holds at time T since $A(T;T,c)=0$ and $B(T;T,c)=0$.
2. We suppose the relation holds at time $t+1$ and, by the law of iterated the conditional expectation, we prove it at time t.

$$\begin{aligned}
&E\left[E\left[S_T^c\middle|\,\mathcal{F}_{t+1}\right]\middle|\,\mathcal{F}_t\right]=E\left[\exp\left[A\left(t+1;T,c\right)+B\left(t+1;T,c\right)h_{t+2}\right]\middle|\,\mathcal{F}_t\right]\\
&=E\left[\exp\left[c\ln\left(S_T\right)+cr+A(t+1;T,c)\right.\right.\\
&+\ c\lambda_0h_{t+1}+c\lambda_1V_{t+1}+c\sigma\sqrt{V_{t+1}}Z_{t+1}+\\
&\left.\left.+\alpha_0B\left(t+1;T,c\right)+\alpha_1B\left(t+1;T,c\right)V_{t+1}+\beta B\left(t+1;T,c\right)h_{t+1}\ \right]\,\middle|\,\mathcal{F}_t\right]\\
&=S_t^c\exp\left[cr+A\left(t+1;T,c\right)+\alpha_0B\left(t+1;T,c\right)+\left(c\lambda_0+\beta B\left(t+1;T,c\right)\right)h_{t+1}\right]*\\
&*E\left[\exp\left[\left(c\lambda_1+\alpha_1B\left(t+1;T,c\right)+\tfrac{c^2\sigma^2}{2}\right)V_{t+1}\right]\middle|\,\mathcal{F}_t\right],
\end{aligned} \tag{24}$$

using the conditional m.g.f. of the r.v. V_{t+1} equation (24) becomes:

$$\begin{aligned}
E\left[E\left[S_T^c\middle|\,\mathcal{F}_{t+1}\right]\middle|\,\mathcal{F}_t\right] \;=\;& S_t^c\exp\left[cr+A\left(t+1;T,c\right)+\alpha_0B\left(t+1;T,c\right)+\right.\\
&\left.+\left(c\lambda_0+\beta B\left(t+1;T,c\right)+f\left(c\lambda_1+\alpha_1B\left(t+1;T,c\right)+\tfrac{c^2\sigma^2}{2},\theta\right)\right)h_{t+1}\right]
\end{aligned} \tag{25}$$

By comparing the expression obtained in equation (25) with (23) we obtain the following recursive system:

$$\begin{cases} A(t;T,c) & = cr + A(t+1;T,c) + \alpha_0 B(t+1;T,c) \\ B(t;T,c) & = c\lambda_0 + \beta B(t+1;T,c) + \\ & \quad f(c\lambda_1 + \alpha_1 B(t+1;T,c) + \frac{c^2\sigma^2}{2}, \theta) \end{cases} \tag{26}$$

with $A(T;T,c) = 0$ and $B(T;T,c) = 0$.

5.2 Martingale condition

We want to prove that $\forall s \leq t$:

$$\lambda_0 = -f(\lambda_1 + \frac{\sigma^2}{2};\theta) \overset{(1)}{\Longrightarrow} E\left[\frac{S_t}{e^r}\middle| \mathcal{F}_{t-1}\right] = S_{t-1} \overset{(2)}{\Longrightarrow} E\left[\frac{S_t}{e^{r(t-s)}}\middle| \mathcal{F}_s\right] = S_s \tag{27}$$

$(\overset{(1)}{\Longrightarrow})$
We assume r constant but the proof holds even assuming r to be a predictable process. By simple calculus, we obtain

$$E\left[\frac{S_t}{e^r}\middle| \mathcal{F}_{t-1}\right] = S_{t-1}\exp\left[\left(\lambda_0 + f\left(\lambda_1 + \frac{\sigma^2}{2};\theta\right)\right)h_{t-1}\right] \tag{28}$$

substituting $\lambda_0 = -f(\lambda_1 + \frac{\sigma^2}{2};\theta)$ in (28) we obtain the result.
$(\overset{(2)}{\Longrightarrow})$
By the iterated law of conditional expectation we have:

$$\begin{aligned} E\left[\frac{S_t}{e^{r(t-s)}}\middle| \mathcal{F}_s\right] &= E\left[E\left[\frac{S_t}{e^{r(t-s)}}\middle| \mathcal{F}_{t-1}\right]\middle| \mathcal{F}_s\right] \\ &= E\left[\frac{1}{e^{r(t-s-1)}}\underbrace{E\left[\frac{S_t}{e^r}\middle| \mathcal{F}_{t-1}\right]}_{S_{t-1}}\middle| \mathcal{F}_s\right] \\ &= \ldots = E\left[\frac{S_{s+1}}{e^r}\middle| \mathcal{F}_s\right] = S_s \end{aligned}$$

5.3 VIX Index: derivation formula

We derive an analytical formula for the VIX index when the dynamics of S&P 500 belongs to our class. Defined S^* as the forward price with time-to-maturity $T-t$, we start from the VIX definition:

$$\left(\frac{VIX_t}{100}\right)^2 = \frac{2e^{r(T-t)}}{T-t}\left[\underbrace{E^Q\left[\frac{S_T - S^*}{S^*}\middle| \mathcal{F}_t\right]}_{(*)} - \underbrace{E^Q\left[\ln\left(\frac{S_T}{S^*}\right)\middle| \mathcal{F}_t\right]}_{(**)}\right].$$

The quantity in $(*)$ is 0 since:

$$E^Q\left[\frac{S_T - S^*}{S^*}\middle| \mathcal{F}_t\right] = \frac{1}{S_t e^{r(T-t)}}E^Q\left[S_T|\, \mathcal{F}_t\right] - 1 = 0.$$

Given the spot price S_t, we have $S_T = S_t \exp\left(\sum_{d=t+1}^T X_d\right)$ and by substituting in $(**)$ we get the following expression for VIX squared:

$$\left(\frac{VIX_t}{100}\right)^2 = -\frac{2e^{r(T-t)}}{T-t}\underbrace{E\left[\sum_{d=t+1}^{T}\lambda_1 V_d + \lambda_0 h_d\middle| \mathcal{F}_t\right]}_{(\Delta)} \tag{29}$$

In order to compute the quantity (Δ) in (29) we use the mathematical induction method. $\forall\, l = t, \ldots, T$ we assume that:

$$E\left[\sum_{d=t+1}^{T}\lambda_1 V_d + \lambda_0 h_d\middle| \mathcal{F}_l\right] = C(l;T) + D(l;T)h_{l+1} + \sum_{d=t+1}^{l}\lambda_1 V_d + \lambda_0 h_d \tag{30}$$

with $C(T;T) = 0$ and $D(T;T) = 0$. First, we notice that all the quantities on the right side of (30) are known given the information at time l.

1. Since V_t and h_t are respectively adapted and predictable process our assumption is true for $l = T$ if $C(T;T) = 0$ and $D(T;T) = 0$.

2. We suppose the relation hold at time $l+1$ and we prove for time l using the property of conditional expected value.

$$E\left[\sum_{d=t+1}^{T} \lambda_1 V_d + \lambda_0 h_d \middle| \mathcal{F}_l\right] = E\left[E\left[\sum_{d=t+1}^{T} \lambda_1 V_d + \lambda_0 h_d \middle| \mathcal{F}_{l+1}\right] \middle| \mathcal{F}_l\right]. \tag{31}$$

The quantity on the right hand of equation (31) is equal to:

$$E\left[C(l+1;T) + D(l+1;T)h_{l+2} + \sum_{d=t+1}^{l+1} \lambda_1 V_d + \lambda_0 h_d \middle| \mathcal{F}_l\right] \tag{32}$$

substituting in (32) the definition of h_{l+2} we have

$$\begin{aligned}&C(l+1;T) + \alpha_0 D(l+1;T) + (\beta D(l+1;T) + \lambda_0)h_{t+1} + \textstyle\sum_{d=t+1}^{l}(\lambda_1 V_d + \lambda_0 h_d)\\ &+E\left[\left(\alpha_1 D(l+1;T) + \lambda_1\right) V_{l+1} \middle| \mathcal{F}_l\right].\end{aligned}$$

From (4) we get:

$$\begin{aligned}&C(l+1;T) + \alpha_0 D(l+1;T) + \left[(\lambda_0 + \lambda_1 g(\theta)) + (\beta + \alpha_1 g(\theta))\, D(l+1;T)\right] h_{t+1}\\ &+\textstyle\sum_{d=t+1}^{l} \lambda_1 V_d + \lambda_0 h_d\end{aligned}$$

by comparison with (30) we get the following system:

$$\begin{cases} C(l;T) &= C(l+1;T) + D(l+1;T)\alpha_0 \\ D(l;T) &= \left[\lambda_1 g(\theta) + \lambda_0\right] + \left(\alpha_1 g(\theta) + \beta\right) D(l+1;T) \end{cases} \tag{33}$$

with final conditions $C(T;T) = 0$ and $D(T;T) = 0$.
We show that if the following two conditions are satisfied

- $\alpha_1 g(\theta) + \beta < 1$
- $\lambda_1 g(\theta) + \lambda_0 \leq 0$

the right hand of the equation (10) is positive, coherently with the fact of being equal to the squared VIX value. We notice that $D(l;T)$ is a linear difference equation whose solution at time $l = t,\ \forall t \leq T$ is given by

$$D(t;T) = \underbrace{\left[\lambda_1 g(\theta) + \lambda_0\right]}_{\leq 0} \underbrace{\frac{1 - \left[\alpha_1 g(\theta) + \beta\right]^{T-t}}{1 - \left[\alpha_1 g(\theta) + \beta\right]}}_{>0}$$

The solution of $D(l;T)$ and the positivity of α_0 ensure the negativity of $C(t;T)$:

$$\begin{aligned} C(t;T) &= \underbrace{C(T;T)}_{=0} + \underbrace{D(T;T)}_{=0} + \alpha_0 \sum_{l=t+1}^{T-1} \underbrace{D(l;T)}_{<0} \\ &= \alpha_0 \left[\lambda_1 g(\theta) + \lambda_0\right] \left\{ \frac{T - t - 1 - \left[\alpha_1 g(\theta) + \beta\right] \frac{1 - \left[\alpha_1 g(\theta) + \beta\right]^{(T-t)-1}}{1 - \left[\alpha_1 g(\theta) + \beta\right]}}{1 - \left[\alpha_1 g(\theta) + \beta\right]} \right\} \end{aligned}$$

5.4 VIX Index: autoregressive model

In equation (5), we substitute the expression for h_{t+1} and h_t using the VIX adjusted as in (12). We obtain

$$\frac{VIX_t^{adj} - C_{30}}{D_{30}} = \alpha_0 + (\alpha_1 g(\theta) + \beta) \frac{VIX_{t-1}^{adj} - C_{30}}{D_{30}} + \alpha_1 (V_t - g(\theta) h_t) \;\Rightarrow$$

$$VIX_t^{adj} = \alpha_0 D_{30} + C_{30}\left[1 - (\alpha_1 g(\theta) + \beta)\right] + (\alpha_1 g(\theta) + \beta)_{t-1}^{adj} + \alpha_1 D_{30}(V_t - g(\theta) h_t)$$

We can easily observe that VIX_t^{adj} is an $AR(1)$. Its expression can be written:

$$VIX_t^{adj} = int + slope VIX_{t-1}^{adj} + \tau_t.$$

Trivially we have:

$$\begin{aligned} int &= \alpha_0 D_{30} + C_{30}\left[1 - (\alpha_1 g(\theta) + \beta)\right] \\ slope &= (\alpha_1 g(\theta) + \beta) \\ \tau_t &= \alpha_1 D_{30}(V_t - g(\theta) h_t) \end{aligned}$$

Using the explicit solution (11) for C_{30} and D_{30} and by rearranging, we get a simple expression for int:

$$\begin{aligned} int &= \alpha_0 \left(\lambda_1 g(\theta) + \lambda_0\right) \tfrac{1-slope^{30}}{1-slope} + \alpha_0 \left(\lambda_1 g(\theta) + \lambda_0\right) \left(29 - slope \tfrac{1-slope^{29}}{1-slope}\right) \\ &= 30\alpha_0 \left(\lambda_1 + \lambda_0\right) \end{aligned}$$

References

C. Alexander. *Market Risk Analysis, Pricing, Hedging and Trading Financial Instruments.* Wiley & Sons, 2008.

O. Barndorff-Nielsen. Exponentially Decreasing Distributions for the Logarithm of Particle Size. *Royal Society of London Proceedings Series A*, 353:401–419, 1977.

O. E. Barndorff-Nielsen. Normal inverse gaussian distributions and stochastic volatility modelling. *Scandinavian Journal of Statistics*, 24(1):1–13, 1997.

O. E. Barndorff-Nielsen and N. Shephard. Normal modified stable processes. Economics Series Working Papers 072, University of Oxford, Department of Economics, 2001.

O. E. Barndorff-Nielsen, J. Kent, and M. Sørensen. Normal variance-mean mixtures and z-distributions. *International Statistic Review*, 50:145–159, 1982.

F. Bellini and L. Mercuri. Option pricing in garch models. Technical Report 124, Universitá di Milano Bicocca, April 2007.

F. Bellini and L. Mercuri. Option pricing in a dynamic variance gamma model. *Journal of Financial Decision Making*, 7(1), 2011.

F. Black and M. S. Scholes. The pricing of options and corporate liabilities. *Journal of Political Economy*, 81 (3):637–54, 1973.

J. Y. Campbell, A. W. Lo, A. C. MacKinlay, and Lo A. Y. *The Econometrics of Financial Markets.* Princeton University Press, 1997.

P. Carr and D. B. Madan. Option valuation using the fast fourier transform. *Journal of Computational Finance*, 2:61–73, 1999.

Chicago Board Options Exchange. Vix-cboe volatility index. 2003. URL `http://www.cboe.com/micro/vix/vixwhite.pdf`.

Chicago Board Options Exchange. Cboe skew index - skew. 2011. URL `http://www.cboe.com/micro/skew/introduction.aspx`.

P. Christoffersen, S. Heston, and K. Jacobs. Option valuation with conditional skewness. *Journal of Econometrics*, 131(1-2):253–284, 2006.

P. Christoffersen, K. Jacobs, and C. Ornthanalai. Garch option valuation: Theory and evidence. Creates research papers, School of Economics and Management, University of Aarhus, May 2012.

R. Cont and P. Tankov. *Financial Modelling With Jump Processes.* Chapman & Hall/CRC Financial Mathematics Series, Boca Raton, 2003.

K. Demeterfi, E. Derman, M. Kamal, and J. Zou. A guide to volatility and variance swaps. *Journal of Derivatives*, 6:385–407, 1999.

J. M. L. Dhaene, J. Dony, M. Forys, D. Linders, and W. Schoutens. Fix - the fear index. measuring market fear. Open access publications from katholieke universiteit leuven, Katholieke Universiteit Leuven, 2011.

J. C. Duan. The garch option pricing model. *Mathematical Finance*, 5(1):13–32, 1995.

J.C. Duan and J.G. Simonato. Empirical martingale simulation for asset prices. *Management Science*, 44(9): 1218–1233, 1998.

J.C. Duan, G. Gauthier, C. Sasseville, and J.G. Simonato. An analytical approximation for the garch option pricing model. *Journal of Computational Finance*, 2(4):75–116, 1999.

J.C. Duan, G. Gauthier, C. Sasseville, and J.G. Simonato. Approximation the gjr-garch and the egarch option pricing models analytically. *Journal of Computational Finance*, 9(3), 2006.

E. Eberlein and K. Prause. *The Generalized Hyperbolic Model: Financial Derivatives and Risk Measures.* Springer, 1998.

P. Embrechts, C. Klüppelberg, and T. Mikosch. *Modelling Extremal Events for Insurance and Finance.* Springer, 1997.

W. Feller. *An Introduction to Probability Theory and Its Applications*, volume 3. Wiley, 1968.

B. Feunou and R. Tedongap. A stochastic volatility model with conditional skewness. *Journal of Business & Economic Statistics*, 30(4):576–591, 2012.

J. Gatheral. *The Volatility Surface (A practitioner's Guide).* Wiley Finance, 2006.

J. Hao and J. E. Zhang. Garch option pricing models, the cboe vix, and variance risk premium. *Journal of Financial Econometrics*, 11(3):556–580, 2013.

S. L. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6:327–343, 1993.

S. L. Heston and S. Nandi. A closed-form garch option valuation model. *The Review of Financial Studies*, 13 (3):585–625, 2000.

Brockwell P. J. and Davis R. A. *Time Series: Theory and Methods.* Springer-Verlag, 1991.

D. B. Madan and E. Seneta. The variance gamma (v.g.) model for share market returns. *The Journal of Business*, 63(4):511–524, 1990.

B. Mandelbrot. The variation of certain speculative prices. *The Journal of Business*, 36:394, 1963.

L. Mercuri. Option pricing in a garch model with tempered stable innovations. *Finance Research Letters*, 5(3): 172–182, 2008.

R. C. Merton. Option pricing when underlying stock returns are discontinuous. *Journal of Financial Economics*, 3(1-2):125–144, 1976.

C. Ornthanalai. A new class of asset pricing models with lvy processes: Theory and applications. Technical report, 2008.

W. Schoutens. *Lévy Processes in Finance.* Wiley, 2003.

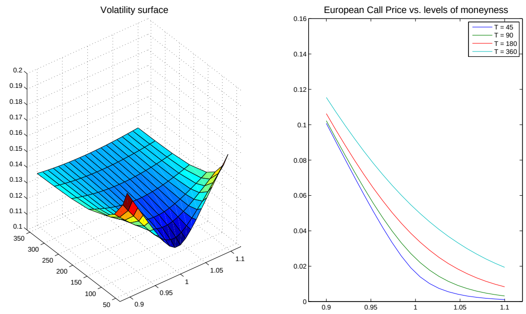
M. C. K. Tweedie. An index which distinguishes between some important exponential families. In J. K. Ghosh and J. Roy, editors, *Statistics: Applications and New Directions.* Proceedings of the Indian Statistical Institute Golden Jubilee International Conference, Calcutta: Indian Statistical Institute, 1984.

J. E. Zhang and Y. Zhu. Vix futures. *The Journal of Futures Markets*, 26(6):521–531, 2006.

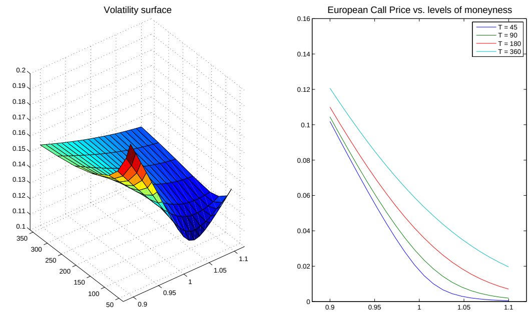
Steps for Theoretical Implied Volatility

1. $\left(\frac{VIX_t}{100}\right)^2 = -\frac{2e^{r(T-t)}}{T-t}\left[C(t;T) + D(t;T)\mathbf{h_{t+1}}\right].$
2. $\mathbf{E_t}\left[\exp(\mathbf{c}\ln \mathbf{S_T})\,|\mathcal{F}_\mathbf{t}\right] = S_t^c \exp[A(t;T,c) + B(t;T,c)h_{t+1}]$
3. $E\left[\exp(c\ln S_T)\,|\mathcal{F}_t\right] \overset{IFT}{\Longrightarrow} \mathbf{\Pi_1} \& \ \mathbf{\Pi_2}$
4. $\mathbf{C}(\mathbf{K},\mathbf{T}) = S_t\Pi_1 - Ke^{-r(T-t)}\Pi_2$
5. $C(K,T) \overset{B\&S}{\Longrightarrow} \sigma(\mathbf{K},\mathbf{T})$

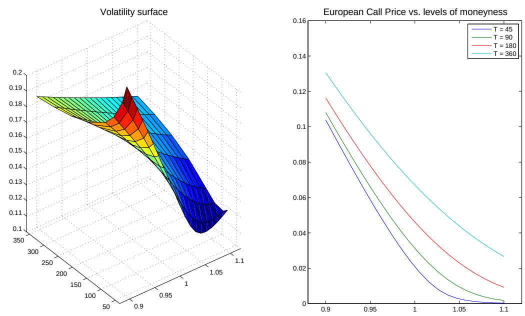
Table 1: We report the main steps necessary to obtain the volatility surface in our framework exploiting available informations from the VIX index.



(a) $\lambda_1 = 0$

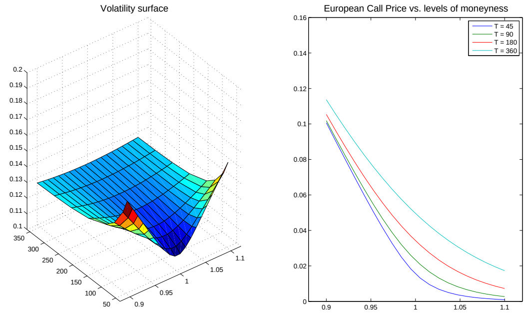


(b) $\lambda_1 = -0.006$

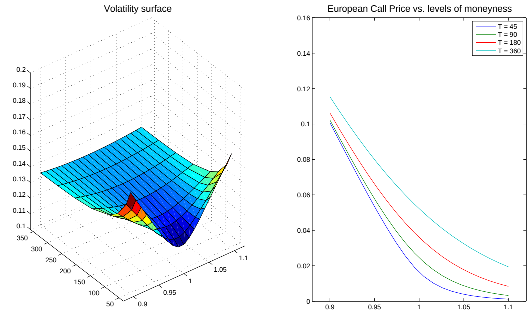


(c) $\lambda_1 = -0.012$

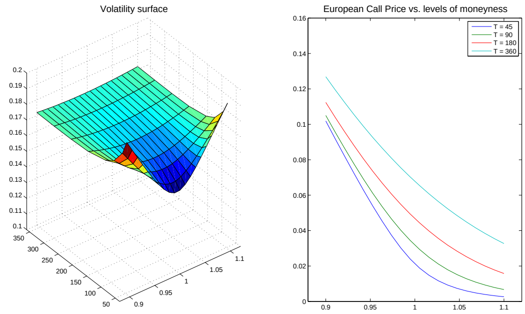
Figure 1: Implied volatility surfaces for $\sigma = 0.014$, $\alpha_0 = 0.033$, $\alpha_1 = 0.493$ and $\beta = 0.379$ when λ_1 varies.



(a) $\sigma = \mathbf{0.0133}$

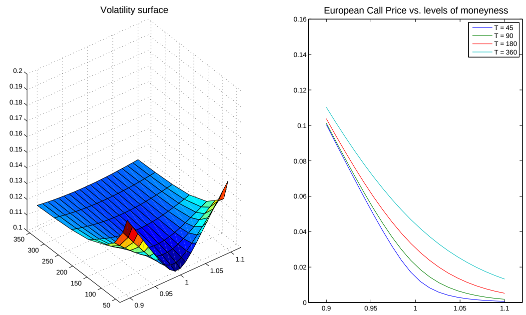


(b) $\sigma = \mathbf{0.014}$

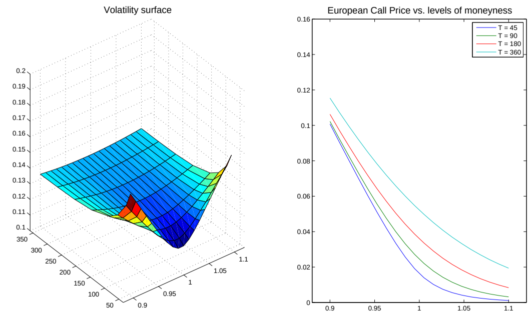


(c) $\sigma = \mathbf{0.0182}$

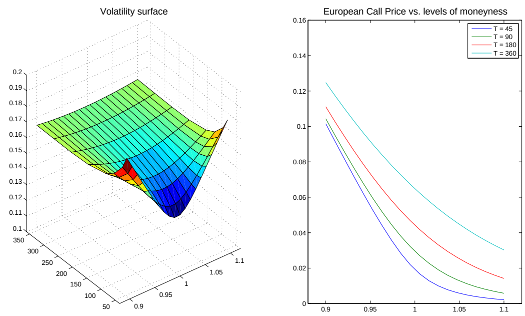
Figure 2: Implied volatility surfaces for $\lambda_1 = 0$, $\alpha_0 = 0.033$, $\alpha_1 = 0.493$ and $\beta = 0.379$ when σ varies.



(a) $\alpha_0 = \mathbf{0.031}$, $\alpha_1 = \mathbf{0.468}$ and $\beta = \mathbf{0.360}$



(b) $\alpha_0 = \mathbf{0.033}$, $\alpha_1 = \mathbf{0.493}$ and $\beta = \mathbf{0.379}$

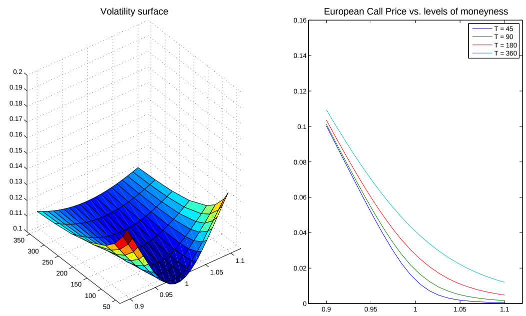


(c) $\alpha_0 = \mathbf{0.034}$, $\alpha_1 = \mathbf{0.517}$ and $\beta = \mathbf{0.397}$

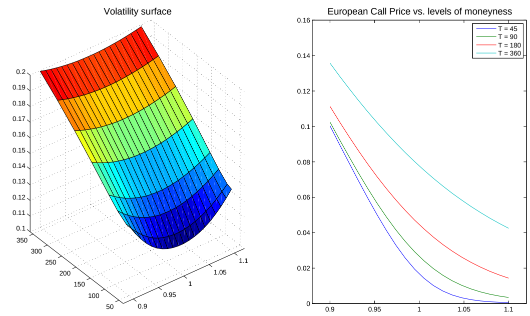
Figure 3: Implied volatility surfaces for $\lambda_1 = 0$, $\sigma = 0.014$ for varying α_0, α_1 and β vary.

	DVG	DNIG	DNTS
Open	0,589%	0,005%	0,080%
Closing	0,445%	0,139%	0,064%
High	1,665%	1,081%	1,156%
Low	0,550%	1,133%	1,059%

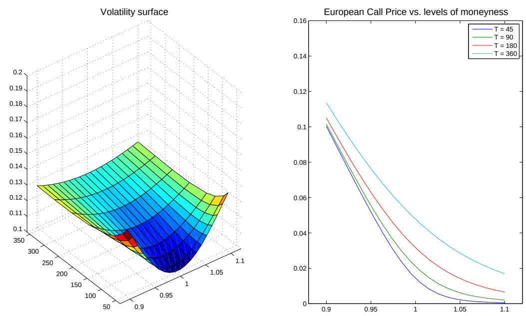
Table 2: Errors obtained when the calibration decision depends on the VIX index level.



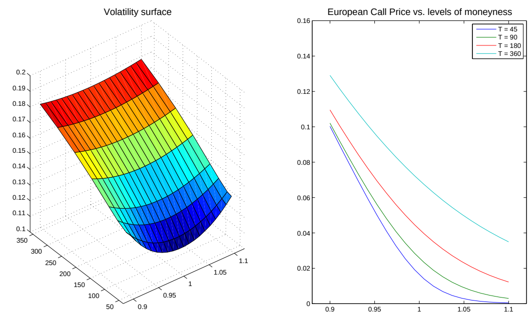
(a) $\mathbf{a = 0.975}$



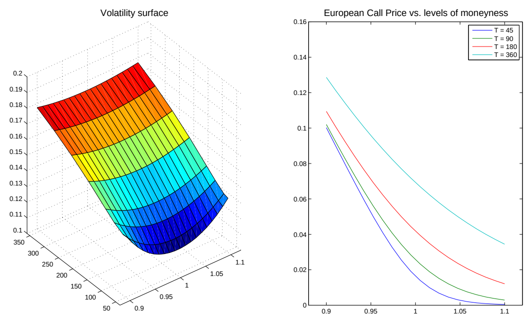
(b) $\mathbf{b = 0.569}$



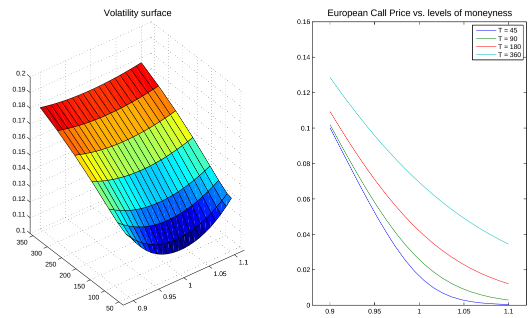
(c) $\mathbf{a = 0.985}$



(d) $\mathbf{b = 0.793}$



(e) $\mathbf{a = 1}$



(f) $\mathbf{b = 0.814}$

Figure 4: The plots show the change of the implied volatility surface for increasing values of a (left panel) and b (right panel).

	DVG	DNIG	DNTS
Open	1,111%	0,029%	0,140%
Closing	0,967%	0,173%	0,004%
High	2,187%	1,047%	1,216%
Low	0,028%	1,167%	0,999%

Table 3: Errors obtained when the calibration is done the first Wednesday of each month.

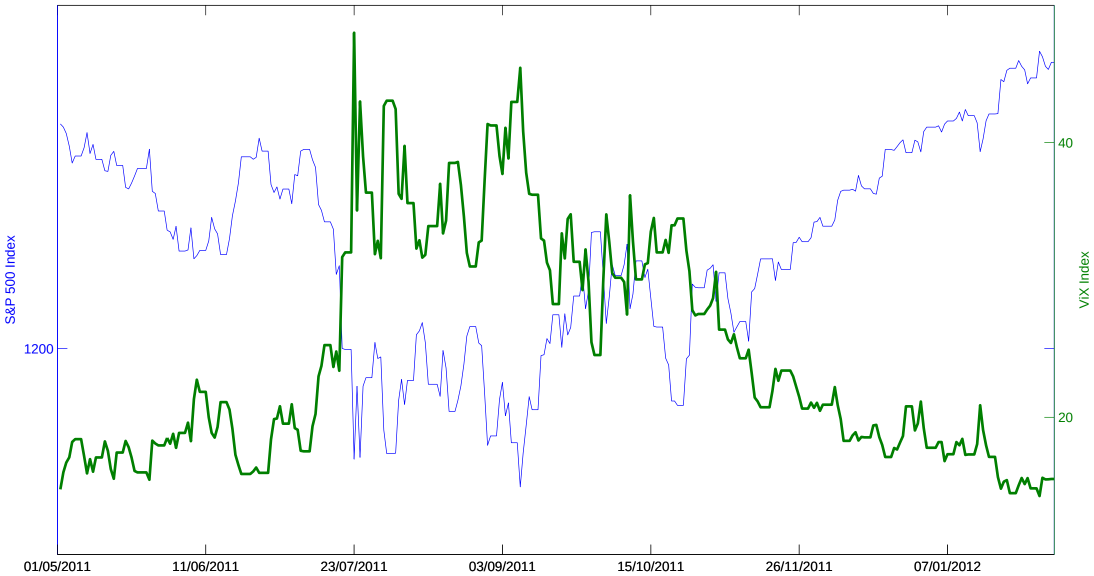


Figure 5: Comparison between the VIX and S&P500 Indices.

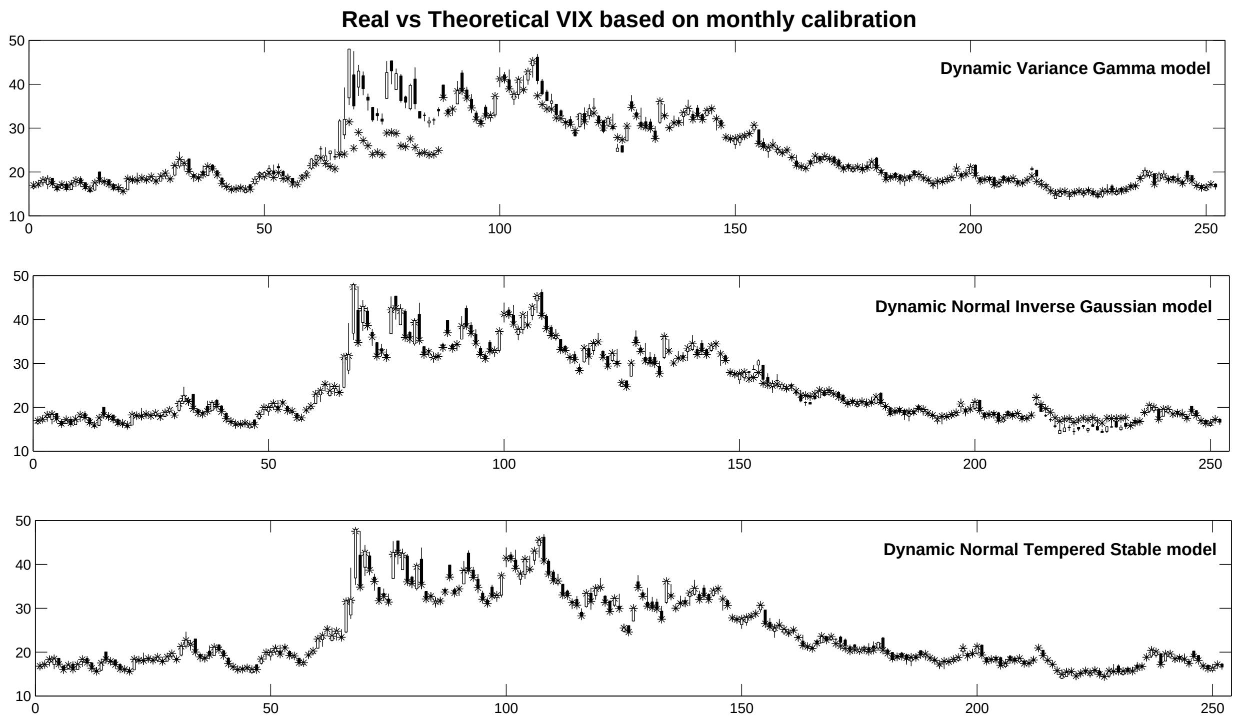


Figure 6: Comparison between the predict VIX (upper bound $*$) and next day open, closed, min, max VIX level using monthly calibration

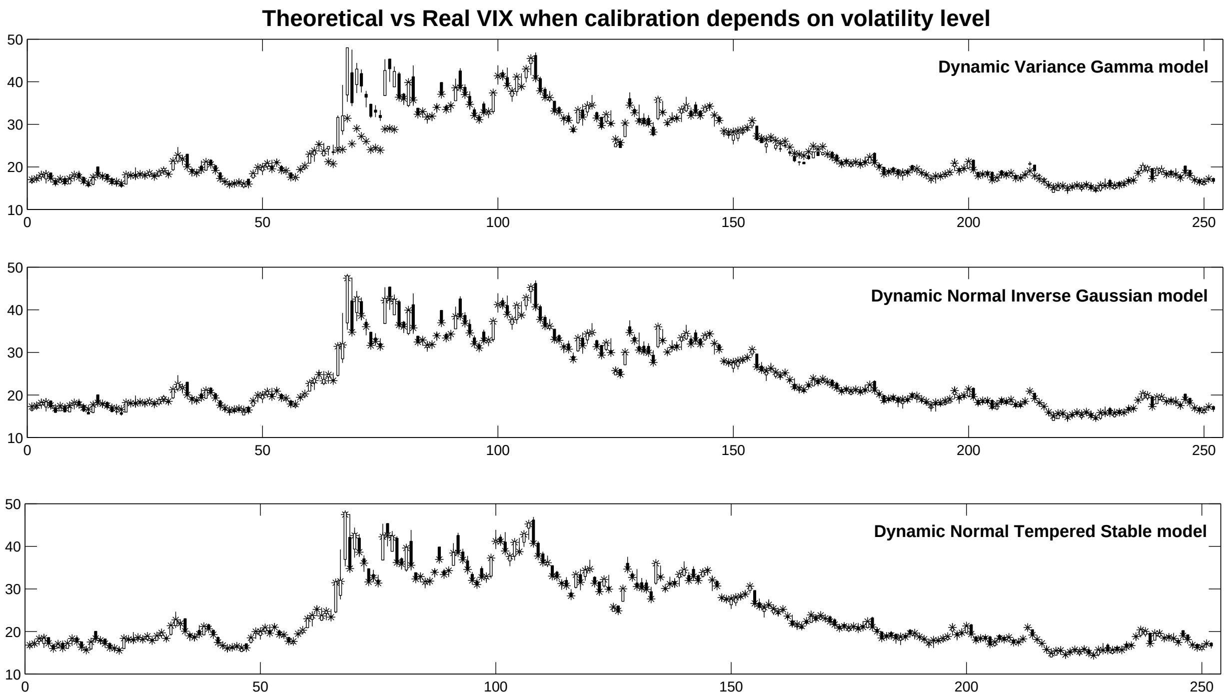


Figure 7: Comparison between the predict VIX (upper bound $*$) and next day open, closed, min, max VIX level using monthly calibration

In sample estimation for DVG							
date	λ_0	λ_1	σ	α_0	α_1	β	Perc. error
04-May-2011	0.012	-0.012	0.014	0.033	0.493	0.379	0.048
01-Jun-2011	0.036	-0.039	0.069	0.009	0.274	0.148	0.084
06-Jul-2011	0.005	-0.005	0.006	0.033	0.344	0.633	0.027
03-Aug-2011	0.034	-0.035	0.001	0.060	0.317	0.000	0.037
07-Sep-2011	0.008	-0.008	0.011	0.032	0.538	0.444	0.015
05-Oct-2011	0.051	-0.053	0.029	0.028	0.155	0.484	0.024
02-Nov-2011	0.095	-0.100	0.007	0.018	0.057	0.085	0.039
07-Dec-2011	0.060	-0.062	0.007	0.024	0.008	0.454	0.052
04-Jan-2012	0.019	-0.019	0.020	0.023	0.207	0.644	0.048
01-Feb-2012	0.036	-0.038	0.056	0.017	0.014	0.157	0.048
07-Mar-2012	0.042	-0.043	0.029	0.000	0.000	1.000	0.088

Table 4: Estimated parameters for DVG model in sample period

In sample estimation for DNIG							
date	λ_0	λ_1	σ	α_0	α_1	β	Perc. error
04-May-2011	0.049	-0.052	0.062	0.006	0.012	0.572	0.039
01-Jun-2011	0.047	-0.050	0.061	0.006	0.016	0.604	0.029
06-Jul-2011	0.009	-0.009	0.011	0.009	0.168	0.816	0.024
03-Aug-2011	0.035	-0.036	0.042	0.029	0.212	0.059	0.022
07-Sep-2011	0.067	-0.072	0.075	0.017	0.120	0.113	0.022
05-Oct-2011	0.007	-0.008	0.010	0.028	0.427	0.564	0.007
02-Nov-2011	0.060	-0.064	0.066	0.007	0.081	0.674	0.019
07-Dec-2011	0.046	-0.048	0.057	0.005	0.008	0.867	0.024
04-Jan-2012	0.029	-0.030	0.019	0.019	0.065	0.733	0.057
01-Feb-2012	0.030	-0.031	0.042	0.023	0.269	0.109	0.034
07-Mar-2012	0.013	-0.014	0.015	0.010	0.211	0.760	0.026

Table 5: Estimated parameters for DNIG model in sample period

In sample estimation for DNTS									
date	λ_0	λ_1	σ	α_0	α_1	β	b	a	Perc. error
04-May-2011	0.212	-0.107	0.013	0.002	0.363	0.276	0.814	0.990	0.009
01-Jun-2011	0.066	-0.042	0.039	0.011	0.296	0.000	0.800	0.750	0.019
06-Jul-2011	0.005	-0.005	0.006	0.039	0.380	0.596	1.000	0.500	0.025
03-Aug-2011	0.052	-0.027	0.008	0.012	0.510	0.000	0.897	0.955	0.007
07-Sep-2011	0.005	-0.006	0.009	0.051	0.658	0.409	0.962	0.413	0.015
05-Oct-2011	0.012	-0.016	0.026	0.001	0.116	0.910	0.946	0.345	0.004
02-Nov-2011	0.003	-0.003	0.006	0.133	0.806	0.233	0.863	0.351	0.011
07-Dec-2011	0.085	-0.043	0.010	0.004	0.473	0.072	0.854	0.975	0.005
04-Jan-2012	0.003	-0.005	0.007	0.105	0.772	0.449	1.000	0.341	0.018
01-Feb-2012	0.100	-0.053	0.021	0.001	0.104	0.803	0.800	0.934	0.014
07-Mar-2012	0.018	-0.011	0.007	0.032	0.539	0.090	0.965	0.803	0.024

Table 6: Estimated parameters for DNTS model in sample period

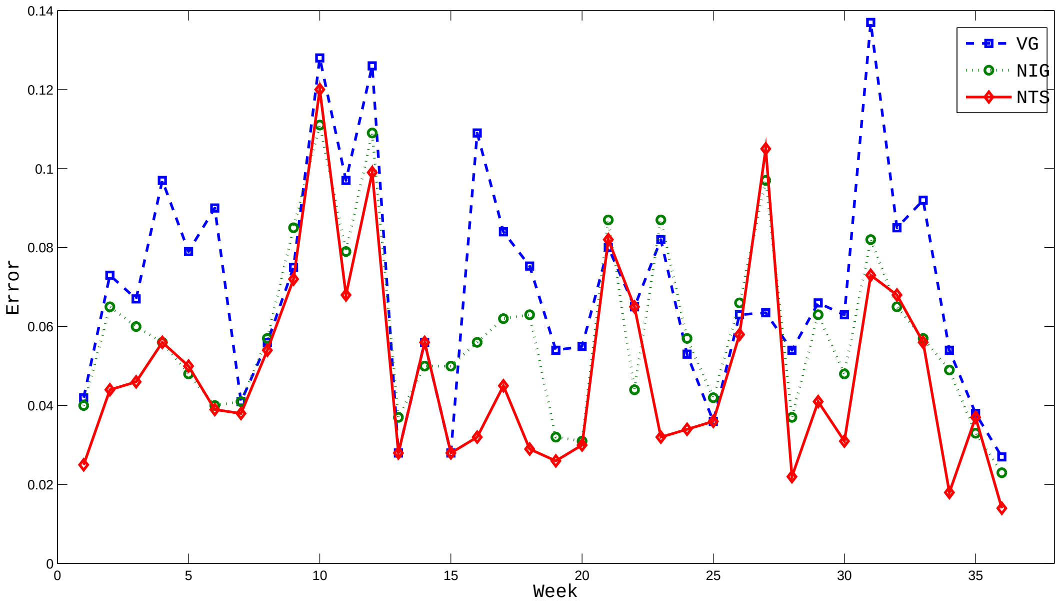


Figure 8: Out of sample weekly comparison.